Задача 1

а). $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}) = \frac{1}{n}\sum_{i=1}^{n}(X_i^2 + \bar{X}^2 - 2\bar{X}X_i) = \overline{X_i^2} - \bar{X}^2$

$S^2 = \overline{X^2} - \bar{X}^2 \xrightarrow{\text{по ЗБЧ}} EX^2 - (EX)^2 = \sigma^2 \Rightarrow S$ со σ^2

б) $ES^2 = E\overline{X^2} - E(\bar{X}^2) = E\overline{X^2} - (E(\bar{X}))^2 - \sigma^2(\bar{X}) = EX_1^2 - EX_1^2 -$

$- \frac{n}{n^2}\sigma^2(X_1) = \sigma^2(X_1) - \frac{1}{n}\sigma^2(X_1) = \frac{n-1}{n}\sigma^2(X_1) \Rightarrow S^2$ не явл.

несмещ. оценкой σ^2.

в) $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - a - (\bar{X} - a))^2 =$

$= \overline{(X-a)^2} - (\bar{X} - a)^2 \quad a = EX_1$

$\sqrt{n}(S^2 - \sigma^2) = \sqrt{n}\left(\overline{(X-a)^2} - (\bar{X} - a)^2 - \sigma^2\right) =$

$= \sqrt{n}\left(\overline{(X-a)^2} - E(X_1 - a)^2\right) - \sqrt{n}(\bar{X} - a)^2 =$

$= \frac{\sum_{i=1}^{n}(X_i - a)^2 - nE(X_1 - a)^2}{\sqrt{n}} - (\bar{X} - a)\sqrt{n}(\bar{X} - a) \xrightarrow{\text{ц.п.т.}}$

$\to \mathcal{N}(0, D(X_1 - EX_1)^2)$

Задача 2

$$l_x(a,\sigma^2) = \sum_{i=1}^{n} \ln \rho(x_i) = -\sum_{i=1}^{n} \frac{(x_i-a)^2}{2\sigma^2} - \ln\sigma - \ln\sqrt{2\pi}.$$

$$l_x(a,\sigma^2)'_a = \sum_{i=1}^{n} \frac{x_i - a}{\sigma^2} = 0 \Rightarrow a = \overline{x}_i$$

$$l_x(a,\sigma^2)'_{\sigma} = +\sum_{i=1}^{n} \frac{(x_i-a)^2}{\sigma^3} - \frac{1}{\sigma} = \frac{\overline{x^2} - 2\overline{x}a + a^2 - \sigma^2}{\sigma^3} = 0 \Rightarrow$$

$$\Rightarrow \sigma^2 = \overline{x^2} - 2\overline{x}a + a^2 = \{a = \overline{x}\} = \overline{x^2} - \overline{x}^2.$$

$$d^2 l_x(a,\sigma^2) = \begin{bmatrix} \frac{\partial^2 l_x(a,\sigma^2)}{\partial a^2} & \frac{\partial^2 l_x}{\partial a \partial \sigma} \\ \frac{\partial^2 l_x}{\partial \sigma \partial a} & \frac{\partial^2 l_x(a,\sigma^2)}{\partial^2 \sigma} \end{bmatrix} = \begin{bmatrix} -\frac{n}{\sigma^2} & -2\sum_{i=1}^{n} \frac{x_i - a}{\sigma^3} \\ -2\sum_{i=1}^{n} \frac{x_i - a}{\sigma^3} & -3\sum_{i=1}^{n} \frac{(x_i-a)^2}{\sigma^4} + \frac{1}{\sigma^2} \end{bmatrix} =$$

$a = \overline{x}, \sigma^2 = \overline{x^2} - \overline{x}^2$

$$= \cancel{\frac{n}{\overline{x^2} - \overline{x}}} \begin{bmatrix} \frac{-n}{\sigma^2} & 0 \\ 0 & -\frac{3n}{\sigma^2} + \frac{1}{\sigma^2} \end{bmatrix} < 0.$$

Ответ: $\hat{\theta} = (\overline{x};\ \overline{x^2} - \overline{x}^2)$.

~~Задача 3~~

~~а) $P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$; $P(x_i) = \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$~~ ;

~~$l_x(\theta) =$~~

Задача № 3

а) 1). $L_x(\Theta) = \frac{\Theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i} e^{-\Theta n}$

$L_x(\Theta)' = \frac{-ne^{-\Theta n}\Theta^{\sum_{i=1}^{n} x_i} + \sum_{i=1}^{n} x_i e^{-\Theta n}\Theta^{\sum_{i=1}^{n} x_i - 1}}{\prod_{i=1}^{n} x_i} = 0 \Rightarrow$

$\Rightarrow \Theta = \sum_{i=1}^{n} \frac{x_i}{n} = \cancel{n}\,\bar{x}.$

2). $i(\Theta) = E_\Theta\left(\frac{\partial \ell_x(\Theta)}{\partial\Theta}\right)^2 = E_\Theta\left(\frac{\partial\left(\sum_{i=1}^{n} x_i \ln\Theta - n\Theta - \sum_{i=1}^{n} \ln x_i!\right)}{\partial\Theta}\right)^2 =$

$= E_\Theta\left(\frac{\bar{x}n}{\Theta} - n\right)^2 = n^2\left(\frac{E_\Theta(\bar{x}^2)}{\Theta^2} + 1 - \frac{2\Theta E(\bar{x})}{\Theta^2}\right) = n^2\frac{D(\bar{x})}{\Theta^2} = \frac{n}{\Theta} \Rightarrow$

$\Rightarrow \sigma^2(\Theta) = \frac{\Theta}{n}.$

б) 1). $p_\Theta(x) = \frac{\Theta^\beta x^{\beta-1}}{\Gamma(\beta)} e^{-\Theta x}$, β фикс.

$\ell_x(\Theta) = \sum_{i=1}^{n}\left(\beta\ln\Theta - \Theta x_i + (\beta-1)\ln x_i - \ln(\Gamma(\beta))\right)$

$\ell_x(\Theta)' = \sum_{i=1}^{n}\left(\frac{\beta}{\Theta} - x_i\right) = \frac{n\beta}{\Theta} - n\bar{x} = 0 \Rightarrow \Theta = \frac{\beta}{\bar{x}}$

2) $i(\Theta) = E_\Theta\left(\frac{\partial \ell_x(\Theta)}{\partial\Theta}\right)^2 = E_\Theta\left(\frac{n\beta}{\Theta} - n\bar{x}\right)^2 = n^2\left(E_\Theta\bar{x}^2 + \frac{\beta^2}{\Theta^2} - 2\frac{E\bar{x}\,\beta}{\Theta}\right) =$

$= \{E\bar{x} = \beta\Theta\} = n^2\left(D\bar{x} + \beta^2\Theta^2 + \frac{\beta^2}{\Theta^2} - \beta^2\right) = \left\{D\bar{x} = \frac{\beta\Theta}{n^2}\right\} =$

$= n^2\left(\frac{\beta^2\Theta}{n^2} + \beta^2\Theta^2 + \frac{\beta^2}{\Theta^2} - \beta^2\right) \Rightarrow \sigma^2(\Theta) = \frac{1}{i(\Theta)}.$

2). а). $p_\theta = \frac{\theta^\beta \, x^{\beta-1}}{\Gamma(\beta)} e^{-\theta x}$

$l_x(\theta) = \beta \cdot n \ln\theta + (\beta - 1) \cancel{\ln x} \ln \prod_{i=1}^{n} x_i - \theta \sum_{i=1}^{n} x_i - \ln \Gamma(\beta)$

$l'_x(\theta) = \frac{\beta \cdot n}{\theta} - n\bar{X} = 0 \Rightarrow \theta = \frac{\beta}{\bar{X}}$

б). $i(\theta) = E_\theta \left(\frac{\partial l_x(\theta)}{\partial \theta} \right)^2 = n^2 E_\theta \left(\frac{\beta}{\theta} - \bar{X} \right)^2 = n^2 \left[\left(\frac{\beta}{\theta} \right)^2 + E_\theta(\bar{X}^2) - \frac{2\beta}{\theta} E\bar{X} \right] =$

$= \left\{ E\bar{X} = \frac{\beta}{\theta} \right\} = n^2 \left(\left(\frac{\beta}{\theta} \right)^2 - 2\frac{\beta^2}{\theta^2} + \frac{\beta^2}{\theta^2} + D(\bar{X}) \right) = n^2 \cdot D(\bar{X}) =$

$= \left\{ D(\bar{X}) = \frac{1}{n} \cdot \frac{\beta}{\theta^2} \right\} = \frac{n\beta}{\theta^2} \Rightarrow \sigma^2(\theta) = \frac{\theta^2}{n\beta} = \frac{\beta^2}{\bar{X}^2 n}$.