Black-Litterman Model

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1 Introduction and Motivation

Ever since the introduction of portfolio theory by Harry Markowitz in his seminal 1952 paper *Portfolio Selection*[3], the Mean-Variance portfolio has become the cornerstone of quantitative portfolio selection, with Markowitz even being awarded the Nobel prize in economics for his research. While the model was hailed as a breakthrough - with some even describing it as the cornerstone of quantitative portfolio selection - it has not been used often in practice, largely due to two reasons as listed by Richard O. Michaud [4] :

1. The model produces very unintuitive portfolio selection
2. It creates portfolios with high transaction costs

As such, the famous economist Fischer Black - who had earlier developed the Black-Scholes formula for derivative pricing - came up with the Black-Litterman model while working at Goldman Sachs to deal with some of the downsides of the traditional Mean-Variance portfolio while also introducing a Bayesian approach to finding the optimal portfolio weights by including the subjective view of investors into the optimization problem [1] [2]. Furthermore, while the mean-variance portfolio builds on a "null portfolio", the Black-Litterman model builds on the equilibrium returns. The Black-Litterman model has been more widely used and has produced tangible results along with being more well served for industry needs, especially since it translates well with traditional investor practice such as including "score cards" when rating their opinion on different investment opportunities.

2 Markowitz Mean-Variance

The Markowitz mean-variance portfolio is the following optimization problem:

$$\max W^T\mu - \frac{\gamma}{2}\sigma_p^2$$

with the following constraints:

$$\sigma_p^2 = W^T\Sigma W$$

$$W^T\iota = 1$$

The solution to this optimization can be found by taking the derivative with respect to W and setting it equal to 0. Solving for W, we get

$$\frac{\Omega^{-1}\iota}{\iota^T\Omega^{-1}\iota} + \frac{\Omega^{-1}\mu}{\gamma}\frac{\iota^T\Omega^{-1}\mu}{\gamma}\frac{\Omega^{-1}\iota}{\iota^T\Omega^{-1}\iota}$$

The results of the mean-variance were computed using the R statistical software. However, the software was unable to find an optimal portfolio without the introduction of box constraints, as such it was set such that the maximum of each weight is 0.5 The three stocks used to calculate weights for and compare with the Black-Litterman model are the following:

1. Duolingo
2. Dollar General
3. Revolution Medicines Inc

This result from the mean-variance were very surprising (as seen in figure 1), as it puts most of the weights of the stocks in Duolingo, which is the worst performing stock of the 3,

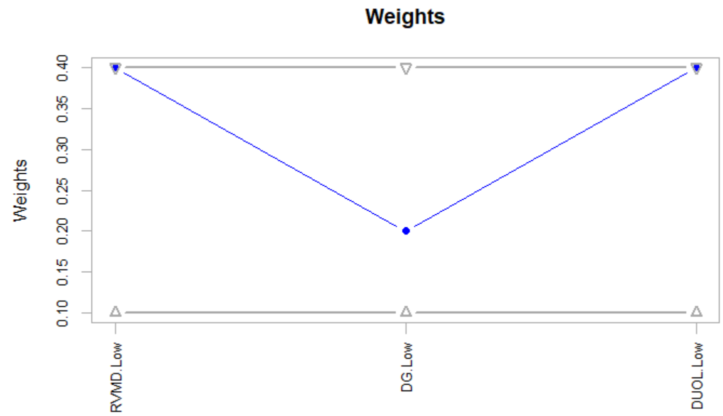

Figure 1: Mean-Variance Weights

and also puts almost half the portfolio into the RVMD stock, which is not as well performing as the Dollar General stock, the best performing of the 3. This highlights the original problem mentioned earlier of the Markowitz mean-variance portfolio is that it produces highly unintuitive portfolio selections. As such, we will develop the Black-Litterman theoretical model, and then compare its portfolio allocation to that of the Markowtiz mean-variance.

3 Model

The main advantages of the Black-Litterman model is that it not only takes investor views into account, but also works from the CAPM model and equilibrium returns rather than starting from historical returns to calculate the weights. The derivation of the model is as follows. Firstly, it is important to remind ourselves of Bayes' rule, which is:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

and the multivariate normal distribution which is given by:

$$f(\mathbf{x}|\boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{k/2}\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

Suppose we have an investor with the following utility function:

$$U = W^T R - \frac{1}{2} A W^T \Sigma W$$

where W is the portfolio weights,A is the risk aversion factor, Σ is the variance-covariance matrix, and R is the vector of expected returns. If we maximize it with respect to the weights we get:

$$\frac{\partial U}{\partial W} = R - A\Sigma W = 0$$

solving for R we get

$$R = A\Sigma W$$

A can also be written as the excess return of the market divided by the variance of the market $\frac{E(r_m)-r_f}{\sigma_m^2}$. If we substitute the weights with the market capitalization weights and use the market risk aversion factor, R becomes the implied equilibrium expected excess returns, denoted by π. The market capitalization weights are found by dividing the market capitalization of each stock by the total market capitalization of the entire portfolio. Thus, we can write out π as

$$\pi = A_m \Sigma W_{mkt}$$

so the returns are $\sim \mathcal{N}(\pi, \Sigma)$

In order to include the subjective views of the investor, we create a vector that includes the views of the investor on how each stock is likely to perform. For example, if there are 3 assets and we believe that $r_A > r_b$ by 1% and $r_c > r_a$ by 0.005% (which expresses what is called *relative views* as they express our belief of the returns of two assets) we can include that in our model through an $NX1$ views vector, which, in the case of the example we listed earlier would be:

$$K = \begin{bmatrix} 0.01 \\ 0.005 \end{bmatrix}$$

In order to link these views to the rest of the model, we create what is called a *link matrix* and denote it by P, in our example the link matrix would be written as:

$$P = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

where the rows denote the different views and the columns are stocks a, b, and c accordingly. Since there is some uncertainty regarding the views, we introduce an error term into the views vector so it becomes $Q+\epsilon$ and we assume that $\epsilon \sim \mathcal{N}(0,\Omega)$ where Ω is a variance-covariance matrix. Black and Litterman suggest that this variance-covariance matrix can be computed as $\tau P \Sigma P^T$ where τ is a constant that is close to 0 and they suggest a value of 0.025 while other researchers suggest a value of 1. The inverse of this matrix would provide us with the confidence we have about our views.

What the B-L model does is provide us with an estimate of the excess returns by calculating a weighted average of π and our views, Q. The first weight is the confidence about π, which can be computed by $(\tau\Sigma)^{-1}$, the second weight is our confidence about our views, which can be computed by $P^T\Omega^{-1}$

We now go back to the Bayes rule and the multivariate normal distribution, using it, we can derive the rest of the model, since in essence the Black-Litterman model represents the expectation of the multivariate normal distribution of the expected returns given the equilibrium returns. Using Bayes' rule, we have that

$$P(E(r))|\pi) = \frac{P(\pi|E(r)PE((r))}{P(\pi)}$$

and the expectation of the returns represent the prior beliefs of the investor we had discussed earlier. Since the model assumes that the expected returns follow a normal distribution and that the distribution of the equilibrium returns given the expected returns also follow a normal distribution, we have

$$P(E(r)) = \frac{1}{\sqrt{(2\alpha^k)|\tau\Omega|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{K})^T\Omega^{-1}(\mathbf{x} - \boldsymbol{K})\right)$$

and that

$$P(\pi|E(r)) = \frac{1}{\sqrt{(2\alpha^k)|\tau\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{E(r)})^T(\tau\Sigma)^{-1}(\mathbf{x} - \boldsymbol{E(r)})\right)$$

We then take the product of both PDFs and reach the following exponent

$$\exp\left(-\frac{1}{2}(\pi - \boldsymbol{E(r)})^T(\tau\Sigma)^{-1}(\pi - \boldsymbol{E(r)}) - \frac{1}{2}(\mathbf{E}(\mathbf{r}) - \mathbf{K})^T\boldsymbol{\Omega}^{\mathbf{-1}}(\mathbf{E}(\mathbf{r}) - \mathbf{K})\right)$$

we then need to rearrange this exponent so that it is the same as that of the multivariate normal distribution. After some rearranging, we have that:

$$\exp\left(-\frac{1}{2}(\mathbf{E}(\mathbf{r}) - \mathbf{H}^{-1}\mathbf{C})^T\mathbf{H}(\mathbf{E}(\mathbf{r}) - \mathbf{H}^{\mathbf{-1}}\mathbf{C})\right)$$

where

$$H = (\tau\Sigma)^{-1} + P^T\Omega^{-1}P$$

$$C = (\tau\Sigma)^{-1}\pi + P^T\Omega^{-1}K$$

which is the exponent of the multivariate normal distribution with mean $H^{-1}C$ and variance H^{-1}, which allows us to finally reach the Black-Litterman model, which can be expressed similarly to the CAPM model in measuring excess expected returns

$$E(r) - r_f = H^{-1}C = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\pi + P^T\Omega^{-1}K]$$

$$Var(r) = H^{-1} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}$$

After getting the previous CAPM formulation and calculting the excess returns of each stock, it becomes a simple optimization problem to determine the weights.

4 Results

The results of the Black Litterman model were computed using the R statistical software [5]. The same 3 stocks were used that were used earlier in the Markowtiz mean-variance in order to provide fair comparision about how each portfolio selection technique differs.

1. Duolingo

2. Dollar General

3. Revolution Medicines Inc.

The view that was included in the initial views vector is that the stock price for Duolingo will decrease while that of Dollar General will increase, which is not an unrealistic view since the stock price of Duolingo has been decreasing for quite a while and was listed as a loser stock by Bloomberg, while the stock price of Dollar General is generally pretty safe and was listed by Bloomberg as a gainer stock. The Data for their stock prices were taken starting from 2021 to 2022.

The Results were calculated as a function of the uncertainty one has about their views, which are listed on the x axis, the following was the result:

As can be seen from the graphs in figures 2 and 3, the Black-Litterman model would suggest that we put all our money into the Dollar General stock, which does make sense for a couple of reasons:

1. Dollar General is considered a very safe stock, as it does very well in times of recession in the American economy

2. There is no investment in the other stocks since their return is much lower than the Dollar General stock

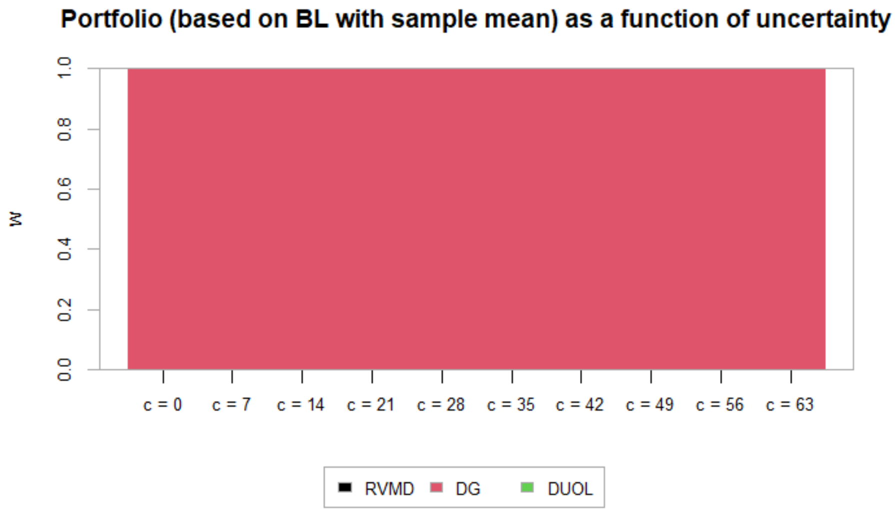

Figure 2: weights as a function of uncertainty

3. Our views listed earlier is that the Duolingo stock will decrease while the Dollar General stock will increase

However, if we take into account the CAPM formulation of excess returns listed earlier instead of using the sample mean,the weights change, which yield the following result

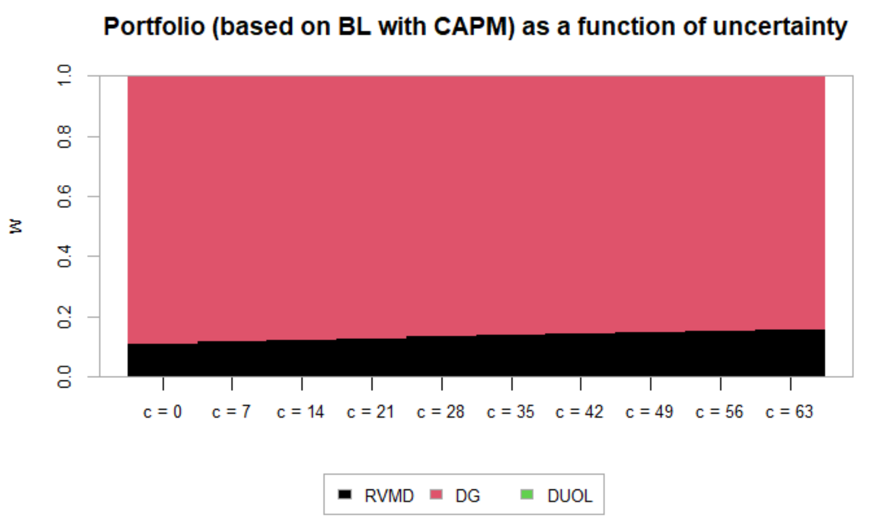

Figure 3: weights over time as a function of uncertainty (CAPM)

This then reinvests some of the portfolio into the RVMD stock, which as can be seen the weight put into the RVMD increases as the uncertainty in our views increase, which makes sense since the more uncertain we are that the Dollar General stock will increase, the more we will invest into the other gainer stock, which is RVMD. Finally, in figure 4 we can see the return of the portfolio as a function of both time and the uncertainty, with a higher uncertainty corresponding to a lower return.

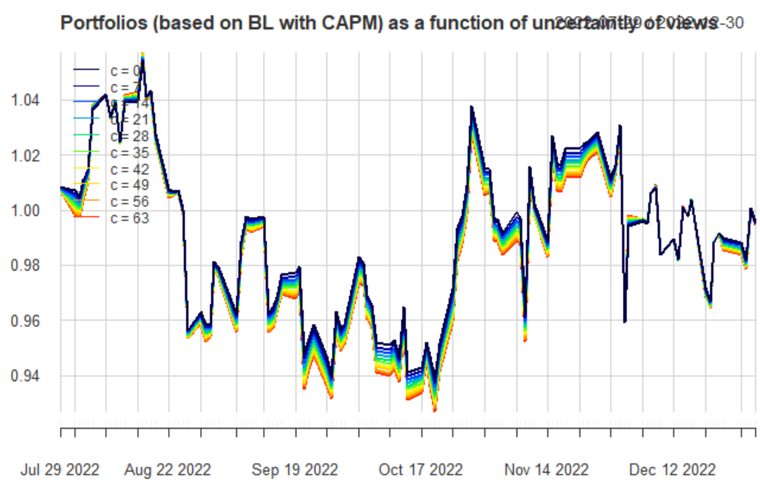

Figure 4: Return of Portfolio Over Time as a Function of Uncertainty

5 Comparision with mean-variance

As can be seen, this model provides vastly different results to the Markowitz mean-variance portfolio. Firstly, it completely inverses the weights of each stock, so while the mean-variance puts most of the weight of the portfolio in the Duolingo stock, the Black-Litterman puts nothing into the Duolingo stock, as the initial views that were fed to the model was that the stock of Duolingo will decrease while that of dollar general will increase, and hence the model optimized accordingly. Secondly, the Black-Litterman model puts the vast majority of the portfolio weights in the Dollar General stock rather than the RVMD stock, since it was told in the initial views that we expect the stock price of Dollar General to increase while remaining neutral on whether the RVMD stock will increase or decrease, hence why much less of the weights are on RVMD.

6 Conclusion

To conclude, the Black-Litterman model provides a new way to allocate portfolio weights. In contrast to the traditional mean-variance approach, it allows investors to include not only their own personal views or "intuition" into the optimization problem, but how confident they are in their views, providing more intuitive portfolio selection than the traditional mean-variance approach and one that is more suited for practice.

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