Name: Alexander Hawkins
ID: 105070938

CSCI 3104, Algorithms **Charlie Carlson & Ewan Davies**
Problem Set 2 – Due Sept. 10, 2020 **Fall 2020, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.

Advice 2: Informal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solutions:

- The solutions **should be typed using** LaTeXand we cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this template of at least 6 pages (or Gradescope has issues with it). **We will not accept submissions with fewer than 5 pages**.
- **You must CITE any outside sources you use, including websites and other people with whom you have collaborated. You do not need to cite a CA, TA, or course instructor.**
- **Posting questions to message boards or tutoring services including, but not limited to, Chegg, StackExchange, etc., is STRICTLY PROHIBITED. Doing so is a violation of the Honor Code.**

Quicklinks: 1 (2a) (2b) (2c) (2d) (2e) (2f)

Problem 1. Name (a) one advantage, (b) one disadvantage, and (c) one alternative to worst-case analysis. For (a) and (b) you should use full sentences.

Answer:

One advantage to worst-case analysis is it gives us a promise about performance which is the upper bound of the algorithm. One disadvantage to worst-case analysis is it is more difficult to understand our algorithms which is difficult to observe the useful distribution of the elements because its so large. One alternative to worst-case analysis is the average-case analysis because it will give us expected run-times on long term algorithms. Also, the average-case analysis could be a good fit because it can give us a precise average of overall outcomes which could include the expected outcomes from the lower and higher bound cases.

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Problem 2. Put the growth rates in order, from slowest-growing to fastest. That is, if your answer is $f_1(n), f_2(n), \ldots, f_k(n)$, then $f_i(n) \leq O(f_{i+1}(n))$ for all i. If two adjacent ones have the same order of growth (that is, $f_i(n) = \Theta(f_{i+1}(n))$), you must specify this as well. Justify your answer (show your work).

- You may assume transitivity: if $f(n) \leq O(g(n))$ and $g(n) \leq O(h(n))$, then $f(n) \leq O(h(n))$, and similarly for little-oh, etc. Note that the goal is to order the growth rates, so transitivity is very helpful. We encourage you to make use of transitivity rather than comparing all possible pairs of functions, as using transitivity will make your life easier.
- You may also use the Limit Comparison Test (see Michael's Calculus Notes on Canvas). However, you **MUST** show all limit computations at the same level of detail as in Calculus I-II. Should you choose to use Calculus tools, whether you use them correctly will count towards your mastery score.
- You may **NOT** use heuristic arguments, such as comparing degrees of polynomials or identifying the "high order term" in the function.
- If it is the case that $g(n) = c \cdot f(n)$ for some constant c, you may conclude that $f(n) = \Theta(g(n))$ without using Calculus tools. You must clearly identify the constant c (with any supporting work necessary to identify the constant- such as exponent or logarithm rules) and include a sentence to justify your reasoning.

(2a) *Polynomials.*

$$3n + 1, \qquad n^6, \qquad \frac{1}{n}, \qquad 1, \qquad n^2 + 3n - 5, \qquad n^2, \qquad \sqrt{n}, \qquad 10^{100}.$$

Through evaluation of each polynomial, I have gathered the slowest-growing to faster-growing order to be

$$\frac{1}{n} \leq 1 = 10^{100} \leq \sqrt{n} \leq 3n + 1 \leq n^2 \leq n^2 + 3n - 5 \leq n^6$$

for all elements of n. Since both 1 and 10^{100} are constants and $1 * 10^{100}$ equals 10^{100}, both constants can be set to each other. For cases such as n^6 and $n^2 + 3n - 5$,

$$\lim_{n\to\infty} \frac{n^2 + 3n - 5}{n^6} = \lim_{n\to\infty} \frac{n^2}{n^6} + \lim_{n\to\infty} \frac{3n}{n^6} - 5 \lim_{n\to\infty} \frac{1}{n^6} = 0$$

Each limit will grow to zero, which makes the limit result to zero. Thus, $n^2+3n-5 \leq O(n^6)$ for all elements of n. For cases such as n^2+3n-5 and n^2,

$$\lim_{n\to\infty} \frac{n^2+3n-5}{n^2} = \lim_{n\to\infty} \frac{n^2}{n^2} + \lim_{n\to\infty} \frac{3n}{n^2} - 5 \lim_{n\to\infty} \frac{1}{n^2} = 1$$

Each limit grows to zero except the $\lim_{n\to\infty} \frac{n^2}{n^2}$ limit which makes the limit result to 1. Thus, $n^2 \leq \theta(n^2+3n-5) \leq O(n^6)$. For cases such as n^2 and $3n+1$,

$$\lim_{n\to\infty} \frac{3n+1}{n^2} = \lim_{n\to\infty} \frac{3n}{n^2} + \lim_{n\to\infty} \frac{1}{n^2} = 0$$

Each limit grows to zero, which makes the limit result to zero. Thus, $3n+1 \leq O(n^2)$ and $n^2 \leq O(n^6)$ which means $3n+1 \leq O(n^6)$ for all elements of n. For cases such as $3n+1$ and $\sqrt{n}$,

$$\lim_{n\to\infty} \frac{\sqrt{n}}{3n+1} = 0$$

The limit grows to zero. Thus, $\sqrt{n} \leq O(3n+1)$ and $3n+1 \leq O(n^6)$ which means $\sqrt{n} \leq O(n^6)$ for all elements of n. As previously expressed, Since both 1 and 10^{100} are constants and $1*10^{100}$ equals 10^{100}, both constants can be set to each other. Thus, $1 = \theta(10^{100})$. For such cases such as $\frac{1}{n}$ and 1,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

The limit grows to zero. Thus, $\frac{1}{n} \leq O(1)$ and $1 \leq O(n^6)$ which makes $\frac{1}{n} \leq O(n^6)$ for all elements of n.

(2b) Prove that for any $a, b > 0$ where $a \neq 1$ and $b \neq 1$, that $\log_a(n) = \Theta(\log_b(n))$. Here, a and b do not depend on n. [**Hint:** Review the change of base formula.]

Proof.

Suppose the formal limit definition

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \quad => \quad f(n) < o(g(n)),$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \quad => \quad f(n) > w(g(n)),$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \quad => \quad f(n) = \theta(g(n)),$$

where c $>$ 0 is some strictly positive constant. In our case, for any a,b $>$ 0 where $a \neq 1$ and $b \neq 1$,

$$\lim_{n\to\infty} \frac{\log_a(n)}{\log_b(n)} =$$

$$\lim_{n\to\infty} \frac{\frac{\log(n)}{\log(a)}}{\frac{\log(n)}{\log(b)}} =$$

$$\lim_{n\to\infty} \frac{\log(b)}{\log(a)} =$$

$$\frac{\log(b)}{\log(a)} => \log_a(n) = \theta(\log_b(n))$$

$\square$

(2c) *Logarithms and related functions. [**Hint** Use part (2b).]*

$$(\log_3(n))^3 \qquad \log_5(n) \qquad \log_3(n) \qquad \sqrt[3]{n} \qquad \log_{2.5}(n) \qquad \log_5(n^2)$$

Suppose the formal limit definition

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0 \quad => \quad f(n) < o(g(n)),$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty \quad => \quad f(n) > w(g(n)),$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = c \quad => \quad f(n) = \theta(g(n)),$$

where $c > 0$ is some strictly positive constant.

Applying L'Hopitals rule,

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}}{(\log_3(n))^3} = \lim_{n\to\infty}\frac{\frac{1}{3\sqrt{n}}}{\frac{1}{n\log_{10}3}} = \lim_{n\to\infty}\frac{n\log_{10}3}{3\sqrt{n}} = \infty$$

Thus, $\sqrt[3]{n} > (\log_3(n))^3$. In such cases as $\log_5(n)$ and $\log_3(n)$,

$$\lim_{n\to\infty}\frac{\log_3(n)}{\log_5(n)} = \lim_{n\to\infty}\frac{n\log_{10}5}{n\log_{10}3} = 1.46497$$

Since 1.46497 is a constant, $\log_3(n)$ and $\log_5(n)$ are set equal to each other. In cases such as $\log_{2.5}(n)$ and $\log_3(n)$

$$\lim_{n\to\infty}\frac{\log_{2.5}(n)}{\log_3(n)} = \lim_{n\to\infty}\frac{n\log_{10}3}{n\log_{10}2.5} = 1.1989778$$

Since 1.1989778 is a constant, $\log_{2.5}(n)$ and $\log_3(n)$ are set equal to each other. In such cases as $\log_{2.5}(n)$ and $\log_5(n^2)$,

$$\lim_{n\to\infty}\frac{\log_5(n^2)}{\log_{2.5}(n)} = 2\lim_{n\to\infty}\frac{\log_{2.5}n}{\log_5 n} = 3.5129415$$

Since 3.5129415 is a constant, $\log_{2.5}(n)$ and $\log_5(n^2)$ are set equal to each other. In such cases as $(\log_3(n))^3$ and $\log_5(n^2)$,

$$\lim_{n\to\infty} \frac{(\log_3{(n)})^3}{\log_5{(n^2)}} = \infty$$

Thus, $(\log_3(n))^3 > (\log_5{(n^2)})$. In conclusion, the ordering of the logarithms and related function are $\log_5(n) = \log_3(n) = \log_{2.5}(n) = \log_5(n^2) \leq (\log_3(n))^3 \leq \sqrt[3]{n}$.

(2d) Construct specific functions $f(n)$ and $g(n)$ such that $f(n) = \Theta(g(n))$ but $2^{f(n)} \neq \Theta(2^{g(n)})$. Formally show that $2^{f(n)} \neq \Theta(2^{g(n)})$ here.

Suppose f(n) = n and g(n) = (n / 2)

$$\lim_{n\to\infty} \frac{n}{(n/2)} = 2$$

Suppose f(n) = $\theta(g(n))$ and n = $\theta(g(n))$

$$\lim_{n\to\infty} \frac{2^n}{2^{(n/2)}} = \lim_{n\to\infty} 2^{(n/2)} = \infty$$

To conclude, $2^n \neq \Theta(2^{n/2})$ are not equal sets as both demonstrations are not equal to each other.

(2e) *Logarithms in exponents. [**Hint:** Review the logarithm change of base formula, as well as the rules of logarithms.]*

$$n^{\log_4(n)} \qquad n^{\log_5(n)} \qquad n^{1/\log_3(n)} \qquad n \qquad 1$$

Suppose the formal limit definition

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \quad => \quad f(n) < o(g(n)),$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \quad => \quad f(n) > w(g(n)),$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \quad => \quad f(n) = \theta(g(n)),$$

where c > 0 is some strictly positive constant.

Through evaluation of each logarithm in exponents, the resulting order is

$$1 = n^{1/\log_3(n)} \leq n \leq n^{\log_5(n)} \leq n^{\log_4(n)}$$

For such cases as 1 and $n^{1/\log_3(n)}$,

$$\lim_{n\to\infty} \frac{1}{n^{1/\log_3(n)}} = 0.333$$

Since 0.333 is a constant, 1 and $n^{1/\log_3(n)}$ are set equal to each other. For cases such as $n^{1/\log_3(n)}$ and n,

$$\lim_{n\to\infty} \frac{n^{1/\log_3(n)}}{n} = 0$$

The limit grows to zero and thus, $n^{1/\log_3(n)} \leq O(n)$. For cases such as n and $n^{\log_5(n)}$,

$$\lim_{n\to\infty} \frac{n^{\log_5(n)}}{n} = \infty$$

The limit grows to infinity and thus, $n^{\log_5(n)} > n$ which is also $n \leq O(n^{\log_5(n)})$. If $n \leq n^{\log_5(n)}$ and $\log_5(n) \leq \log_4(n)$, then $n^{\log_5(n)} \leq n^{\log_4(n)}$. Thus, $n^{\log_5(n)} \leq O(n^{\log_4(n)})$.

(2f) *Exponentials. [**Hint:** Recall the Ratio and Root Tests from Michael's Calculus Notes.]*

$$n! \qquad 3^n \qquad 3^{5n} \qquad 3^{n\log_4(n)} \qquad 3^{n+13}$$

From intuition of each exponential, the resulting order of each exponential is

$$n! \leq 3^{n\log_4(n)} \leq 3^n \leq 3^{5n} \leq 3^{n+13}$$