Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Major Test

Date: 23 April 2011 Time: 2 hrs Marks: 60

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1. A plant has open loop transfer function given by

$$G(s) = \frac{1}{s(s+1)(s+5)}$$

Design a cascade compensator for unity negative feedback operation with the plant, such that

i. The insignificant pole at –5 is retained.

ii. The two significant closed loop poles respond with approximately 25% peak overshoot and 5s setting time to unit step input.

...(10)

2. At zero input, a certain second order system is described as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using the Lyapunov function $V(x_1, x_2) = x_1^2/2 + x_2^2/2 + x_1x_2$, obtain conditions in terms of the state variables that assure asymptotic stability for the system.

...(10)

3. A plant has equivalent discrete time transfer function (at some known sampling time T) is given by

$$G(z) = \frac{0.025z^2 + 0.06z + 0.008}{z^3 - 1.6z^2 + 0.73z - 0.1}$$

For this, we consider design of a proportional controller gain K_p. By considering the marginal stability condition (digital domain!), obtain the limiting value of K_p within which the closed loop system will remain stable.

...(23)

4. A typical electro-hydraulic valve control system adjusts pressure output at a valve opening based on control current input that has magnitude plot asymptotes in the usual dB/log-of-Hertz scale that may be described as

i. Constant 20dB from low frequencies up to 2Hz.

ii. –20dB/decade slope from 2Hz to 10Hz.

iii. Constant at the final value of "ii" up to 20Hz.

iv. –40dB/decade slope from 20Hz onwards.

Assuming a simple minimum order transfer function without all-pass terms, and the gain-to-asymptote error to be negligible at the gain crossover frequencies, calculate the approximate phase margin for the control system.

...(17)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Major Test

1. For the second order double poles to respond with 25% peak overshoot and 5s settling time,

$$e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.25 \Rightarrow \zeta = 0.4037$$

$$\omega_n \approx \frac{4}{\zeta t_{ss}} = \frac{4}{0.4037 \times 5} = 1.982\text{rad/s}$$

...(1+1)

Then the closed loop transfer function works out to

$$G_{\text{CL}}(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(s+5)}$$

$$= \frac{3.927}{(s^2 + 1.6s + 3.927)(s+5)}$$

...(3)

The transfer function required to be cascaded for unity negative feedback is then

$$H(s) = \frac{G_{\text{CL}}(s)}{[1 - G_{\text{CL}}(s)]G(s)}$$

$$= \left[\frac{3.927}{(s^2 + 1.6s + 3.927)(s+5)}\right]\left[1 - \frac{3.927}{(s^2 + 1.6s + 3.927)(s+5)}\right]^{-1}\left[\frac{1}{s(s+1)(s+5)}\right]^{-1}$$

...(5)

$$= \frac{3.927s(s+1)(s+5)}{s^3 + s^2(1.6+5) + s(3.927+8) + (4 \times 3.927)}$$

$$= \frac{3.927s(s+1)(s+5)}{s^3 + 6.6s^2 + 11.927s + 15.708}$$

(10 marks)

2. With the given Lyapunov function, we have

$$\frac{d}{dt}V(x_1, x_2) = x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} + x_1\frac{dx_2}{dt} + x_2\frac{dx_1}{dt} \quad \ldots(1)$$

$$= (x_1 + x_2)\left[\frac{dx_1}{dt} + \frac{dx_2}{dt}\right] \quad \ldots(1)$$

$$= (x_1 + x_2)[(-2x_1 + x_2) + (-4x_2)] \quad \ldots(1)$$

$$= -(x_1 + x_2)(2x_1 + 3x_2) \quad \ldots(1)$$

The system is therefore asymptotically stable if $dV/dt < 0$, that is

$$x_1 + x_2 > 0 \quad \text{and} \quad 2x_1 + 3x_2 > 0$$

$$\Rightarrow x_1 > -x_2 \quad \text{and} \quad x_1 > -\frac{3}{2}x_2$$

...(2+1)

or if

$$x_1 + x_2 < 0 \quad \text{and} \quad 2x_1 + 3x_2 < 0$$

$$\Rightarrow x_1 < -x_2 \quad \text{and} \quad x_1 < -\frac{3}{2}x_2$$

...(2+1)

(10 marks)

3. The closed loop system will retain the order three, so that it must have one real pole and a pair of complex conjugate poles. ...(1)

In a manner similar to the analog counterpart, the real pole in this case will move toward the origin (increasingly stable) as K_p increases.
...(2)

The complex pole pair will on the other hand, tend to move out of the unit circle, so that at the marginal stability condition, we expect the closed loop poles to be at some $p < 1.0$, and $e^{\pm j\theta}$ for some angle θ.
...(2)

Thus, writing the closed loop characteristic polynomial in two ways,

$$(z-p)(z-e^{j\theta})(z-e^{-j\theta}) = z^3 - 1.6z^2 + 0.73z - 0.1 + K_p(0.025z^2 + 0.06z + 0.008) \quad \ldots(2)$$

$$\Rightarrow z^3 - (p + e^{j\theta} + e^{-j\theta})z^2 + (pe^{j\theta} + pe^{-j\theta} + 1)z - p =$$
$$z^3 - (1.6 - 0.025K_p)z^2 + (0.73 + 0.06K_p)z - (0.1 - 0.008K_p) \quad \ldots(1)$$

$$\Rightarrow \left\{ \begin{array}{c} p + 2\cos\theta = 1.6 - 0.025K_p \\ 1 + 2p\cos\theta = 0.73 + 0.06K_p \\ p = 0.1 - 0.008K_p \end{array} \right\} \quad \ldots(2)$$

$$\Rightarrow \left\{ \begin{array}{c} 2\cos\theta = 1.5 - 0.017K_p \\ p = 0.1 - 0.008K_p \end{array} \right\} \quad \ldots(2)$$

$$\Rightarrow 1 + (0.1 - 0.008K_p)(1.5 - 0.017K_p) = 0.73 + 0.06K_p \quad \ldots(4)$$

$$\Rightarrow 0.000136K_p^2 - 0.0737K_p + 0.42 = 0 \quad \ldots(2)$$

$$\Rightarrow K_p = \frac{0.0737 \pm \sqrt{0.0737^2 - 4 \times 0.000136 \times 0.42}}{2 \times 0.000136} = \frac{0.0737 \pm 0.07213}{2 \times 0.000136} \quad \ldots(2)$$

which leads to 5.772 to 536.14 as the two values for K_p at marginal stability limit. ...(1)

The second is obviously absurd because it corresponds to $p = -4.189$ (by the expression for p obtained through coefficient comparison, as above) which is way beyond the unit circle. ...(1)

Therefore the limiting gain is 5.772, at which, $p = 0.053824$, and $\theta = \pm 45.5^{\circ}$.
...(1)

(23 marks)

4. The following features for the system can be concluded from the description of the problem:

i. The steady state gain for the transfer function is $10^{20/20} = 10.0$. ...(1)
ii. Single pole at 2Hz (12.57rad/s) ...(1)
iii. Single zero at 10Hz (62.83rad/s) ...(1)
iv. Double pole at 20Hz (125.66rad/s) ...(1)

The transfer function is thus given by

$$G(s) = \frac{10(1 + s/62.83)}{(1 + s/12.57)(1 + s/125.66)^2} \quad \ldots(1)$$

ignoring the possibility that we may have a ζ for the double pole that is not equal to unity, since we have no way to estimate it (this leads to some approximation in the phase margin). Tracking the gain plot asymptotes, we first obtain the magnitude at the 10Hz frequency by the –20dB/dec slope relation

$$\frac{M(10)-20}{\log 10-\log 2}=-20 \qquad \text{...(2+1)}$$

$$\Rightarrow M(10)=20-20\log 5=6.0206\text{dB}$$

Clearly, 10Hz is still more than the gain crossover frequency, since the magnitude is not negative in dB. We next move to the point 6.0206dB at 20Hz (constant up to that point), beyond which we have a plot of –40dB/dec slope relation

...(2)

$$\frac{M(f)-6.0206}{\log f-\log 20}=-40$$

for any frequency f. Then solving for gain crossover frequency as $M(f)$ = 0dB, we have

$$\log f=\log 20+6.0206/40 \qquad \text{...(1+1)}$$

$$\Rightarrow f=20\times 10^{6.0206/40}=28.28\text{Hz}=177.72\text{rad/s}$$

The phase at this frequency can be obtained from the transfer function expression as

$$\angle G(s)=\tan^{-1}(177.72/62.83)-\tan^{-1}(177.72/12.57)-2\tan^{-1}(177.72/125.66) \qquad \text{...(2+1)}$$

$$=-124.9^{o}$$

Thus the phase margin available is $(180–124.9)^{o}$ = 55.1^{o}. ...(2)

(17 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Minor Test I

Date: 11 February 2011 Time: 60 mins Marks: 20

1. For the plant transfer function given by

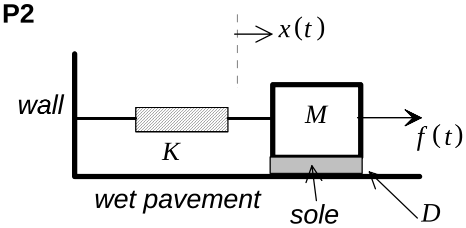
$$G(s) = \frac{s^2 + 4s + 6}{s^2 + 4s + 4}$$

a. Obtain the overall time-response when fed with an input step of magnitude 5.
b. From the result of part "a" obtain the steady state error between the input and final output. (Use of the *Final Value Theorem* for this part is not permitted.)

...(3+3)

2. Fig. **P2** shows an experiment conducted by engineers after construction of a pavement, so as to check if a pedestrian would be able to walk comfortably on it or not. The idea is to have a mass (roughly of the weight of a pedestrian) with the material of a typical shoe sole attached at the bottom, and to check if this mass has adequate damping coefficient on the wet pavement. The mass, which has a value 4kg, is attached to a wall across the pavement through an arbitrary spring, and a force of known value (say, 10N) is suddenly applied to it. In a particular experiment, the mass is found to move to maximum distance of 0.1163m from its initial rest position, and then settles at a distance of 0.1m from the same initial position after some oscillations.

P2



i. What is the damping constant D between the typical shoe and the pavement ?
ii. In practice of course, a "spring support" is hardly to be expected for any pedestrian, so that a more meaningful question can now be answered. If the spring had been missing, what will be the displacement response of the mass if subjected to an impulse of 10N (that is, how much will a pedestrian skid if given a push) ?

...(10+4)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Minor Test I

1.

a. We have

$$Y(s) = \frac{s^2+4s+6}{s^2+4s+4} \times \frac{5}{s} = \frac{5s^2+20s+30}{s(s+2)^2}$$
$$= \frac{15/2}{s} - \frac{5/2}{s+2} - \frac{5}{(s+2)^2}$$ ...(2)

By table of inverse Laplace transforms, we have ...(1)

$$y(t) = \frac{15}{2} - \frac{5}{2}e^{-2t} - 5te^{-2t}$$

b. From the above, the steady state value of the output is

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[\frac{15}{2} - \frac{5}{2}e^{-2t} - 5te^{-2t}\right]$$
$$= \frac{15}{2}$$ ...(2)

Therefore, steady state error to input = 5 – 15/2 = –5/2 ...(1)

(6 marks)

2. The dynamic equation to describe the experiment is a second order one, given by

$$M\frac{d^2}{dt^2}x(t) + D\frac{d}{dt}x(t) + Kx(t) = f(t)$$
$$\Rightarrow X(s) = \frac{1/M}{s^2+(D/M)s+(K/M)} \cdot F(s)$$ ...(1+1)

This has a standard form of the type

$$X(s) = \frac{G\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \cdot F(s)$$

i. If the input is a step so that $F(s) = F/s$, then the output is known to have

Steady state value $= GF$

Peak overshoot $= e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times$ steady state value

Here F = 10Nm, and $G = (1/M)/(K/M) = 1/K$. ...(1)

Now, from the experimental results we know that the steady state displacement is 0.1m, while the peak overshoot is 0.1163m. By simple substitution we have

$$10/K = 0.1 \Rightarrow K = 100\text{N/m}$$ …(1)

$$e^{-\pi\zeta/\sqrt{1-\zeta^2}} = (0.1163-0.1)/0.1 = 0.163$$
$$\Rightarrow \frac{\zeta^2}{1-\zeta^2} = 0.3334 \Rightarrow \zeta^2 = \frac{0.3334}{1.3334} = 0.25$$ …(1)

$$\Rightarrow \zeta = 0.5$$ …(1)

Then

$$\omega_n = \sqrt{K/M} = \sqrt{100/4} = 5\text{rad/s} \qquad \text{...(1+1)}$$

$$2\zeta\omega_n = 2 \times 0.5 \times 5 = D/M = D/4$$

which gives us D = 20N/m-s^{-1}. ...(2)

ii. Having found out the parameters, the displacement response (without the spring) will now be

$$X(s) = \frac{1}{Ms^2 + Ds} \cdot F \qquad \text{...(2)}$$

$$= \frac{0.25}{s(s+5)} \times 10 = \frac{2.5}{s(s+5)} = 0.5 \times \left[\frac{1}{s} - \frac{1}{s+5}\right] \qquad \text{...(1)}$$

so that in time domain,

$$x(t) = 0.5[1 - e^{-5t}]\text{m} \qquad \text{...(1)}$$

giving a steady state displacement of 0.5m.

(14 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Minor Test II

Date: 16 March 2011 Time: 60 mins Marks: 20

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1. The input-output relation describing a discrete time plant is given by
$$y(k+2) - 0.6y(k+1) + 0.05y(k) = 0.25u(k+1) + 0.2u(k)$$
Obtain the discrete time Lure state space representation for the system with all unity elements in the **C** matrix.
...(6)

2. If a proportional controller of gain K_P is designed for unity negative feedback closed loop operation of the plant
$$G(s) = \frac{10}{(s+5)(s+0.2)}$$
so as to have 20-30% peak overshoot on step response, what is the range of K_P required ?
...(8)

3. Obtain the value of K for which the root loci of
$$G(s) = \frac{K(s+1)}{(2s+1)(3s+1)}$$
break off from the real axis indicating identical real roots.
...(6)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Test II

1. The z-transform of the input-output relation is

$$[z^2 - 0.6z + 0.05]y(z) = [0.25z + 0.2]u(z) \qquad \ldots(1)$$

$$\Rightarrow G(z) = \frac{0.25z + 0.2}{z^2 - 0.6z + 0.05} = \frac{0.25z + 0.2}{(z-0.5)(z-0.1)} = \frac{0.8125}{z-0.5} - \frac{0.5625}{z-0.1} \qquad \ldots(2)$$

Defining x_1 and x_2 as the two Lure variables, we have

$$x_1(k+1) = 0.5x_1(k) + 0.8125u(k) \qquad \ldots(1)$$

$$x_2(k+1) = 0.1x_2(k) - 0.5675u(k) \qquad \ldots(1)$$

Then the state variable form is given as

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.8125 \\ -0.5675 \end{bmatrix}u(k) \qquad \ldots(1)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}\mathbf{x}(k)$$

(6 marks)

2. Since the system in closed loop will remain a second order system without zeros, the peak overshoot range of 20-30% can be related to the damping ratio by the relation

i. $e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.2 \Rightarrow \zeta = 0.4559$ …(0.5)

ii. $e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.3 \Rightarrow \zeta = 0.3579$ …(0.5)

The characteristic equation in either case takes the form

$$s^2 + 5.2s + 1 + 10K_P = 0 \qquad \ldots(1)$$

$$\Rightarrow \omega_n^2 = 1 + 10K_P \quad ; \quad 2\zeta\omega_n = 5.2 \qquad \ldots(0.5+0.5)$$

Corresponding to $\zeta = 0.4559$ and 0.3579, by the second relation we have natural frequencies of 5.703rad/s, and 7.265rad/s respectively. ...(1+1)

By the first relation, corresponding to the natural frequencies, we have K_P = 3.152, and 5.178 respectively. ...(1+1)

Therefore the required range of proportional controller gain is 3.152 to 5.178. ...(1)

(8 marks)

3. One could solve this by the conventional "derivative of K" method, but there is a short cut available for this simple second order system !

<u>Method 1:</u>

Simply assume that at each of the breakoffs, we encounter double closed loop poles at some negative real location (say, $-\sigma$). ...(1)

Then the closed loop characteristic equation can be written as

$$s^2 + 2\sigma s + \sigma^2 = \frac{(2s+1)(3s+1) + K(s+1)}{6} = 0$$ ...(1+1)

$$\Rightarrow s^2 + 2\sigma s + \sigma^2 = s^2 + \frac{5+K}{6}s + \frac{1+K}{6} = 0$$

By comparing coefficients, we have

$$\left[\frac{1}{2} \times \frac{5+K}{6}\right]^2 = \frac{1+K}{6}$$ …(1)

$$\Rightarrow K^2 + 10K + 25 = 24 + 24K$$

$$\Rightarrow K^2 - 14K + 1 = 0$$ …(1)

$$\Rightarrow K = \frac{14 \pm \sqrt{196-4}}{2} = 13.928,\ 0.0718$$ …(1)

which are the gains corresponding to breakoff points.

(6 marks)

Method 2:

By the conventional "rules" for root-loci, the breakoff points are obtained as solutions to

$$\frac{\partial K}{\partial s} = 0$$

$$\Rightarrow \frac{\partial}{\partial s}\left[-\frac{6s^2+5s+1}{s+1}\right] = 0$$ …(1)

$$\Rightarrow \frac{(s+1)(12s+5) - (6s^2+5s+1)}{(s+1)^2} = 0$$ …(1)

$$\Rightarrow 6s^2 + 12s + 4 = 0$$ …(1)

$$\Rightarrow s = \frac{-6 \pm \sqrt{36-24}}{6} = -1.5773, \quad -0.4226$$ …(1)

Substitution gives us

$$\text{For } s = -1.5773, K = -\frac{6 \times 1.5773^2 - 5 \times 1.5773 + 1}{-1.5773+1} = 13.928$$ …(1)

$$\text{For } s = -0.4226, K = -\frac{6 \times 0.4226^2 - 5 \times 0.4226 + 1}{-0.4226+1} = 0.0718$$ …(1)

(6 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Minor Test III

Date: 19 April 2011 Time: 60 mins Marks: 20

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1. A unity negative feedback system uses a cascade controller

$$H(s) = \frac{5}{5s+1}$$

on an analog plant of open-loop transfer function

$$G(s) = \frac{2(s+1)}{(2s+1)}.$$

Calculate the exact gain crossover frequency for the closed loop system.

...(10)

2. An analog hydraulic speed control system has a two-state representation with

$$\mathbf{A}=\begin{bmatrix} -3 & 10 \\ 0 & 0 \end{bmatrix} \;;\; \mathbf{B}=\begin{bmatrix} 0 \\ 5 \end{bmatrix} \;;\; \mathbf{C}=\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Design a state-feedback gain row vector **K** with two elements k_1 and k_2, such that the closed-loop system eigenvalues are located corresponding to a natural frequency of 29.7rad/s and damping ratio of 0.7.

...(10)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Test III

1. The open-loop transfer function of the system and controller is given by

$$H(s)G(s) = \frac{5}{5s+1}\frac{2(s+1)}{(2s+1)} = \frac{10(s+1)}{(5s+1)(2s+1)} \qquad \text{...(1)}$$

At the gain crossover frequency, we have

$$|H(j\omega)G(j\omega)| = 1$$

$$\Rightarrow \frac{10\sqrt{\omega^2+1}}{\sqrt{25\omega^2+1}\,\sqrt{4\omega^2+1}} = 1 \qquad \text{…(2)}$$

$$\Rightarrow 100(\omega^2+1) = (25\omega^2+1)(4\omega^2+1) \qquad \text{…(2)}$$

$$\Rightarrow 100\omega^4 - 71\omega^2 - 99 = 0 \qquad \text{…(1)}$$

$$\Rightarrow \omega^2 = \frac{71 \pm \sqrt{71^2 + 4\times 100 \times 99}}{2\times 100} = \frac{71 \pm 211.2842}{200} \qquad \text{…(1)}$$

The only feasible value to the above corresponds to the addition in the numerator, so that $\omega^2 = 1.4114$ $(\text{rad/s})^2$, that is $\omega = 1.188$rad/s. The gain crossover frequency is therefore $1.188/2\pi = 0.189$Hz....(2+1)

(10 marks)

2. The state-space system with feedback is given by

$$\frac{d}{dt}\mathbf{x} = \mathbf{Ax} + \mathbf{B}u = \mathbf{Ax} + \mathbf{B}[r + \mathbf{Kx}] = [\mathbf{A} + \mathbf{BK}]\mathbf{x} + \mathbf{B}r$$

$$= \left\{\begin{bmatrix} -3 & 10 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right\}\mathbf{x} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r \qquad \text{…(1)}$$

$$= \begin{bmatrix} -3 & 10 \\ 5k_1 & 5k_2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r \qquad \text{…(1)}$$

so that the characteristic equation is given by

$$\det\left\{ s\mathbf{I} - \begin{bmatrix} -3 & 10 \\ 5k_1 & 5k_2 \end{bmatrix}\right\} = 0 \qquad \text{…(1)}$$

$$\Rightarrow \det\left\{\begin{bmatrix} s+3 & -10 \\ -5k_1 & s-5k_2 \end{bmatrix}\right\} = 0 \qquad \text{…(1)}$$

$$\Rightarrow (s+3)(s-5k_2) - 50k_1 = 0 \qquad \text{…(1)}$$

$$\Rightarrow s^2 + (3-5k_2)s - (50k_1 + 15k_2) = 0 \qquad \text{…(1)}$$

If this is to correspond to a natural frequency of 29.7rad/s, and a damping constant of 0.7, then by the standard form of the second order characteristic equation, we conclude that

$$-(50k_1 + 15k_2) = 29.7^2 \Rightarrow 50k_1 + 15k_2 = -882.09 \qquad \text{…(1)}$$

$$(3-5k_2) = 2\times 0.7 \times 29.7 = 41.58 \qquad \text{…(1)}$$

Thus by a pair of simple calculations,

$$k_2 = \frac{3 - 41.58}{5} = -7.716 \qquad \ldots(1)$$

$$k_1 = \frac{15 \times 7.716 - 882.09}{50} = -15.327 \qquad \ldots(1)$$

which are the required gains.

(10 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Major Test

<u>Date:</u> 7 May 2013 <u>Time:</u> 3 hrs <u>Marks:</u> 60

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1. A plant with digital transfer function

$$G(z) = \frac{z+1}{z^2 - 4z + 9}$$

is provided with an output feedback through a feedback gain K, which is adjustable. Obtain an expression for the closed loop transfer function. Find the values of K for which the closed loop plant becomes

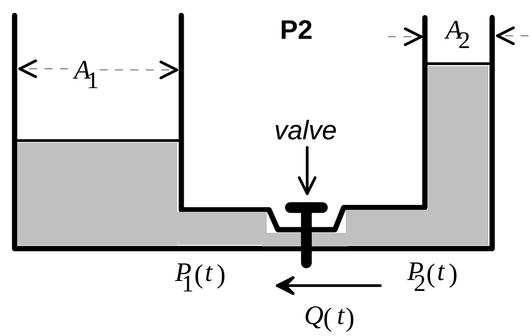
a. Marginally stable (or on the verge of being stable).
b. Stable with critical damping.

...(10)

2. Fig. **P2** shows a hydraulic problem involving two tanks. For a tank with an opening at the bottom, it can be shown that the flow rate $Q(t)$ is related to the pressure at the bottom as

$$Q(t) = (A/\rho g) \cdot dP(t)/dt$$

where ρ is the density of liquid in a tank, A is its cross section area, and $P(t)$ is the pressure at the bottom (g: acceleration due to gravity).



Consider two tanks as shown with initial pressures P_{1o} and P_{2o} ($P_{2o} > P_{1o}$) at the bottoms, cross section areas A_1 and A_2, and connected initially by a closed valve. When open, the valve offers a hydraulic resistance R to the flow $Q(t)$. Obtain the time variation of $Q(t)$ from the instant at which the valve is opened.

...(10)

3. A designer is contemplating variation of the phase crossover frequency for a plant

$$G(s) = \frac{s+1}{(s+2)(s+5)}$$

by cascading it with different all-pass transfer functions. What are the values for gain margin available to him when the crossover is set at the different corner frequencies of the plant ?

...(10)

4. The digital input-output relation describing a plant is given by

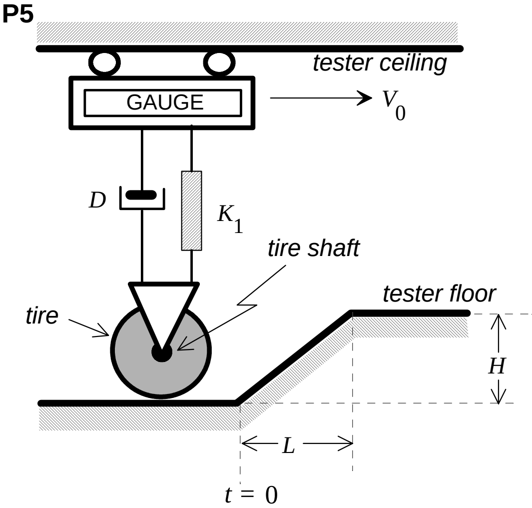
$$y(k+2) - 0.6y(k+1) + 0.05y(k) = 0.25u(k+1) + 0.2u(k)$$

Obtain the discrete time Lure state space representation for the system.

...(10)

5. Fig. **P5** shows the basic configuration of a tire tester, that is used by automotive tire manufacturers to study their product at the factory. Elasticity of the tire and the pressure of air pumped into it together decide an equivalent spring constant K_2 between the tire shaft and the point of contact on the floor. The support for the force gauge, which runs against the ceiling by wheels, has an equivalent damping constant D and spring constant K_1.

P5



As the tester is pulled over a ramp at constant horizontal speed V_0, the vertical force recorded by the gauge is used as the *output signal* $f(t)$ by the manufacturer to assess the tire. The *input* to the system is obviously the height of the contact point of the tire with the floor, which may be assumed to be approximately equal to the floor height $h(t)$, starting from $t = 0$ at the ramp base where all variables have zero value. For convenience, let the height of the wheel shaft above its initial position be $h_s(t)$ (which too is zero initially). Thus the entire system is to be analysed in terms of $h(t)$, $h_s(t)$ and $f(t)$.

Obtain the force time response $f(t)$ recorded by the gauge as the tester moves on the ramp (only up to the horizontal distance L), starting at the ramp base at $t = 0$.

...(20)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Major Test

1. Assuming the negative feedback convention (K positive for negative feedback), the closed loop transfer function is given by

$$G_{\mathrm{CL}}(z) = \frac{G(z)}{1+KG(z)}$$
$$= \frac{z+1}{z^2+(K-4)z+(K+9)}$$

so that

$$2e^{\sigma T}\cdot\cos\omega T = K-4$$
$$e^{2\sigma T} = K+9$$

i. For marginally stable case, the roots of the characteristic polynomial must lie on the unit circle, so that $e^{\sigma T}$=1

This requires $K + 9 = 1$, that is $K = -8$ However, corresponding to this, we get cos $\omega T = -6$, which is impossible. Therefore, no value of the closed loop gain satisfies the marginal stability problem.

ii. For critical damping case we require that $\omega = 0$, so that.

$$\left(\frac{K-4}{2}\right)^2 = K+9$$
$$\Rightarrow K^2-12K-20=0$$
$$\Rightarrow K = \frac{12\pm\sqrt{144+80}}{2} = 6\pm\sqrt{56} = 13.48,\ \ -1.48$$

Thus two gains are possible these being a negative feedback of 13.48, or a positive feedback of 1.48. However, neither solution is stable, since in each case,

$$e^{\sigma T} = \frac{K-4}{2}$$
$$= 4.74, -2.74$$

which have magnitudes greater than unity (where feasible !). Therefore, critically damped closed loop solutions can not be stable for this system.

(10 marks)

2. It is clear from the statement of the problem that for any tank $\boldsymbol{A/\rho g}$ **is essentially its hydraulic capacitance *C***.

Thus from the instance that the valve is opened, the following dynamics apply

$$Q(t) = \frac{P_2(t) - P_1(t)}{R} = C_1 \frac{dP_1(t)}{dt} = -C_2 \frac{dP_2(t)}{dt}$$

$$\Rightarrow \left[\frac{1}{C_1} + \frac{1}{C_2}\right] \cdot Q(t) = -\frac{d}{dt}[P_2(t) - P_1(t)] = -R \cdot \frac{dQ(t)}{dt}$$

$$\Rightarrow \frac{RC_1C_2}{C_1 + C_2} \cdot \frac{dQ(t)}{dt} + Q(t) = 0$$

$$\Rightarrow \frac{RA_1A_2}{\rho g(A_1 + A_2)} \cdot \frac{dQ(t)}{dt} + Q(t) = 0$$

Now the initial flow immediately after the valve is opened is obviously

$$Q_o = (P_{2o} - P_{1o})/R$$

so that by conversion to Laplace domain we get

$$\left[\frac{RA_1A_2}{\rho g(A_1 + A_2)} \cdot s + 1\right] Q(s) = \frac{RA_1A_2}{\rho g(A_1 + A_2)} \cdot \frac{P_{2o} - P_{1o}}{R}$$

$$\Rightarrow Q(t) = \frac{P_{2o} - P_{1o}}{R} \cdot \exp\left[-\frac{\rho g(A_1 + A_2)}{RA_1A_2} \cdot t\right]$$

which is the expected flow dynamics.

(10 marks)

3. The gain margin at a particular phase crossover frequency ω is given by the expression

$$\mathrm{GM} = -20\log\sqrt{\omega^2 + 1} + 20\log\sqrt{\omega^2 + 4} + 20\log\sqrt{\omega^2 + 25}$$

The corner frequencies of the transfer function are at 1rad/s, 2rad/s, and 5rad/s

If these are set as the phase crossover frequencies, we obtain gain margins as

$$\mathrm{GM}(1) = -20\log\sqrt{2} + 20\log\sqrt{5} + 20\log\sqrt{26} = 18.13\mathrm{dB}$$

$$\mathrm{GM}(2) = -20\log\sqrt{5} + 20\log\sqrt{8} + 20\log\sqrt{29} = 16.67\mathrm{dB}$$

$$\mathrm{GM}(5) = -20\log\sqrt{26} + 20\log\sqrt{29} + 20\log\sqrt{50} = 17.46\mathrm{dB}$$

(10 marks)

4. The z-transform of the input-output relation is

$$[z^2 - 0.6z + 0.05]y(z) = [0.25z + 0.2]u(z)$$

$$\Rightarrow G(z) = \frac{0.25z + 0.2}{z^2 - 0.6z + 0.05} = \frac{0.25z + 0.2}{(z - 0.5)(z - 0.1)} = \frac{0.8125}{z - 0.5} - \frac{0.5625}{z - 0.1}$$

Defining x_1 and x_2 as the two Lure variables, we have

$$x_1(k+1) = 0.5x_1(k) + 0.8125u(k)$$

$$x_2(k+1) = 0.1x_2(k) - 0.5675u(k)$$

Then the state variable form is given as

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.8125 \\ -0.5675 \end{bmatrix}\mathbf{u}(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}\mathbf{x}(k)$$

(10 marks)

5. Equating forces at the shaft, we have

$$K_2[h(t) - h_s(t)] = K_1 h_s(t) + D\frac{dh_s(t)}{dt}$$

$$\Rightarrow K_2 H(s) = [K_1 + K_2]H_s(s) + DsH_s(s)$$

$$\Rightarrow \frac{H_s(s)}{H(s)} = \frac{K_2}{[K_1 + K_2] + Ds}$$

At the gauge,

$$f(t) = K_1 h_s(t) + D\frac{dh_s(t)}{dt}$$

$$\Rightarrow F(s) = K_1 H_s(s) + DsH_s(s)$$

$$\therefore\ G(s) = \frac{F(s)}{H(s)} = \frac{F(s)}{H_s(s)} \cdot \frac{H_s(s)}{H(s)} = \frac{K_2[K_1 + Ds]}{[K_1 + K_2] + Ds}$$

While on the ramp, the input to this transfer function is $h(t) = H(V_0 t/L)$, so that the gauge record can be obtained as

$$F(s) = \frac{K_2[K_1 + Ds]}{[K_1 + K_2] + Ds} \cdot \frac{HV_0}{Ls^2} = \frac{K_2 HV_0}{L} \cdot \frac{s + K_1/D}{s^2[s + (K_1 + K_2)/D]}$$

$$= \frac{K_2 V_0 H}{L}\left[\frac{K_2 D}{(K_1 + K_2)^2}\frac{1}{s} + \frac{K_1}{K_1 + K_2}\frac{1}{s^2} - \frac{K_2 D}{(K_1 + K_2)^2}\frac{1}{[s + (K_1 + K_2)/D]}\right]$$

Then the record of the gauge works out to

$$f(t) = \frac{K_2 V_0 H}{L}\left[\frac{K_2 D}{(K_1 + K_2)^2} + \frac{K_1}{K_1 + K_2} \cdot t - \frac{K_2 D}{(K_1 + K_2)^2} \cdot e^{-(K_1 + K_2)t/D}\right]$$

(20 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Minor Test I

Date: 1 March 2013 Time: 2 hrs Marks: 20

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1 When tested independently, the unit step response of a first order system $H(s)$ reaches a value of 6 in 2s finally attaining a steady state value of 8. $H(s)$ is now configured as a negative feedback path to a second order system $G(s)$ with unity steady state gain and both poles at -4.

i. Obtain the transfer functions for $G(s)$ and $H(s)$.

ii. What will be the steady state error for the feedback system when subjected to a step input of $5u(t)$?

...(5+5)

2 When an inpulse input

$$u(k) = 1 \text{ for } k = 0$$
$$= 0 \text{ for all other values of } k.$$

is applied to a certain digital system, the response is

$$\begin{aligned} y(k) &= 1 && \text{for } k = 0 \\ &= -5 && \text{for } k = 1 \\ &= 6 && \text{for } k = 2 \\ &= 0 && \text{for all other values of } k. \end{aligned}$$

i. Obtain the transfer function for the system.

ii. Obtain the poles and zeros of the system.

iii. Briefly reason out whether the system is stable or unstable.

...(2+2+1)

3. Find the phase variable representation for a second order single-input/single-output plant that has the following particulars:

a. Poles at -1 and -3.

b. Zero at -2

c. Steady state gain of 2/3.

Write down the **A**, **B**, **C**, **D** matrices clearly to conclude your answer.

...(2+3)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Minor Test I

1

i. $G(s)$ has unity gain with two poles at -4, so it takes the form of

$$G(s) = \frac{4^2}{(s+4)^2} = \frac{16}{(s+4)^2}$$

$H(s)$ gives a unit-step response of output 8, so that its steady state gain is 8/1 = 8. It therefore takes a form

$$H(s) = \frac{8}{1+\tau s}$$

with a unit-step response given by

$$Y(s) = \frac{1}{s} \times \frac{8}{1+\tau s} = 8\left[\frac{1}{s} - \frac{\tau}{1+\tau s}\right]$$

$$\Rightarrow y(t) = 8[1 - e^{-t/\tau}]$$

At t = 2s, $y(t)$ = 6, so that

$$6 = 8[1 - e^{-2/\tau}]$$

$$\Rightarrow e^{-2/\tau} = 0.25 \Rightarrow \tau = \frac{-2}{\ln 0.25} = 1.4427\text{s}$$

Therefore

$$H(s) = \frac{8}{1+1.4427s}$$

ii. If $G(s)$ is provided negative feedback through $H(s)$, then the closed-loop transfer function is

$$\begin{aligned} G_{\mathrm{CL}}(s) &= \frac{G(s)}{1+G(s)H(s)} \\ &= \frac{16/(s+4)^2}{1+\left[16/(s+4)^2\right][8/(1+1.4427s)]} \\ &= \frac{16(1+1.4427s)}{(s+4)^2(1+1.4427s)+16\times 8} \end{aligned}$$

The steady state gain for the closed-loop system is obtained in the limit as

$$G_{\mathrm{CL}ss} = \frac{16}{16+16\times 8} = \frac{1}{9}$$

Then steady state error to a step of 5 is obtained as

$$\begin{aligned} e_{ss}\left(\frac{5}{s}\right) &= \lim_{s\to 0} s\left[\frac{1}{9}\times\frac{5}{s} - \frac{16(1+1.4427s)}{(s+4)^2(1+1.4427s)+16\times 8}\times\frac{5}{s}\right] \\ &= \left[\frac{5}{9} - \frac{16\times 5}{4^2+16\times 8}\right] \\ &= 0 \end{aligned}$$

(5+5 marks)

2

i. In terms of generic time instant k, we have

$$y(k) = 1.u(k)$$
$$y(k+1) = -5.u(k)$$
$$y(k+2) = 6.u(k)$$

The impulse response is the summation of all for $y(k)$ - which gives us the convolution - as

$$y(k) = [\,1 - 5z^{-1} + 6z^{-2}\,].u(k)$$

so that the transfer function becomes $G(z) = 1 - 5z^{-1} + 6z^{-2}$

ii. We can write the transfer function as

$$G(z) = \frac{z^2 - 5z + 6}{z^2} = \frac{(z-3)(z-2)}{z^2}$$

which is seen to have two poles at the origin, and zeros at +2 and +3.

iii. The system is obviously stable, because both poles are within the unit circle.

(2+2+1 marks)

3

The transfer function for the plant is

$$G(s) = \frac{s+2}{(s+1)(s+3)} = \frac{s+2}{s^2+4s+3}$$

Define

$$x_1(s) = \frac{1}{s^2+4s+3}u(s)$$
$$x_2(s) = s x_1(s)$$

In time domain these work out to

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{d^2x_1(t)}{dt^2} + 4\frac{dx_1(t)}{dt} + 3x_1(t) = u(t) \Rightarrow \frac{dx_2(t)}{dt} = -3x_1(t) - 4x_2(t) + u(t)$$
$$\therefore \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

Then the output is obtained as

$$y(s) = (s+2)x_1(s)$$
$$\Rightarrow y(t) = \frac{dx_1(t)}{dt} + 2x_1(t) = x_2(t) + 2x_1(t) = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Finally

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \;;\quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;;\quad \mathbf{C} = \begin{bmatrix} 2 & 1 \end{bmatrix} \;;\quad \mathbf{D} = [0]$$

(2+3 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Minor Test II

Date: 27 April 2013 Time: 2 hrs Marks: 20

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1. Two known strains of bacteria are mutually supportive of each other's growth according to the equations

$$dx_1/dt = -x_1 + x_1 x_2$$

$$dx_2/dt = -2x_2 + x_1 x_2$$

where x_1 and x_2 are concentration of the two strains in number per-unit-volume of a sample substrate. Note that if each population is sufficiently small, the linear terms dominate over the nonlinear terms, and each concentration would decay to zero over time. However if the individual concentrations are high, the nonlinear terms make each growth rate positive, so that the concentrations may go up without limit ! Clearly the desirable equilibrium has both concentrations is zero, and we are interested to find out the condition under which this will be achieved over time.

Using the total bacterial concentration (x_1 + x_2) as a Liapunov function, obtain the condition under which the zero concentration equilibrium is asymptotically stable.

...(5)

2. A proportional controller gain K_P is to be designed for unity feedback operation of a plant with open-loop transfer function

$$G(s) = \frac{10}{(s+5)(s+0.2)}$$

By studying the coefficients of the closed loop characteristic polynomial, obtain the value of K_P at which the root-locus based design will result in a closed loop response of 20% overshoot to unit step input.

...(7)

3. A plant has open loop poles at –1 and –5, and steady state gain of unity. What is the cascade compensator transfer function $H(s)$, which in closed loop with the plant will ensure a critically damped second order system with unity steady state gain. Your answer will be in terms of the desired natural frequency ω_n of the closed loop system.

For a particular desired natural frequency of the closed loop system, where must the poles of $H(s)$ lie?

...(8)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Minor Test II

1. With the given Liapunov function, we have

$$V(x_1, x_2) = x_1 + x_2$$

$$\frac{d}{dt}V(x_1, x_2) = \frac{dx_1}{dt} + \frac{dx_2}{dt}$$

$$= -x_1 + x_1x_2 + -2x_2 + x_1x_2$$

$$= -x_1 - 2x_2 + 2x_1x_2$$

With both x_1 and x_2 as greater than zero, the system is therefore asymptotically stable if

$$x_1 + 2x_2 > 2x_1x_2$$

$$\Rightarrow \frac{1}{x_2} + \frac{2}{x_1} > 2$$

(5 marks)

2. The closed loop characteristic polynomial of the plant with controller is given by

$$(s+5)(s+0.2) + 10K_P = 0$$

$$\Rightarrow s^2 + 5.2s + [1 + 10K_P] = 0$$

$$\Rightarrow \omega_n^2 = 1 + 10K_P \quad ; \quad 2\zeta\omega_n = 5.2$$

$$\Rightarrow 2\zeta\sqrt{1 + 10K_P} = 5.2$$

Now, for a 20% overshoot to step input,

$$e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.2$$

$$\Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = 0.5123$$

$$\Rightarrow \zeta = 0.4559$$

Substituting this value in the above relation,

$$2 \times 0.4559\sqrt{1 + 10K_P} = 5.2 \Rightarrow K_P = 3.1517$$

which is the required value that will be obtained by the loci at the appropriate ζ-line.

(7 marks)

3. The plant transfer function is described as

$$G(s) = \frac{5}{(s+1)(s+5)}$$

$$= \frac{5}{s^2 + 6s + 5}$$

With $H(s)$ as a cascade controller, the closed loop transfer function must have second order, critical damping, and unity steady state gain. It must therefore have the form

$$\frac{G(s)H(s)}{1+G(s)H(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow \frac{5H(s)}{s^2 + 6s + 5 + 5H(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow 5H(s)\left[1 - \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}\right] = \frac{\omega_n^2(s^2 + 6s + 5)}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow H(s) = \frac{\omega_n^2(s^2 + 6s + 5)}{5(s^2 + 2\omega_n s)}$$

$H(s)$ has two poles, of which one must be located at the origin, and the other at $-2\omega_n$ depending on the desired natural frequency.

(8 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Major Test

Time: 3 hrs | (Open book) | Marks: 60

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1. The digital transfer function of a second order system in open loop has positive real poles at *a* and *b*, both located within the unit circle. In closed loop the system has a cascaded controller of proportional gain K_P. Obtain an equation for the digital root loci once they leave the real axis.

...(15)

2. An analog system has open loop negative real poles at $-a$, $-b$, and $-c$. If the loop is closed through a gain *K*, what will be its value at the point of marginal stability?

...(10)

3. The Lyapunov function defined as

$$V(x_1, x_2) \triangleq \frac{1}{2}x_2^2 + \int_0^{x_1} x^5 \cdot dx$$

is to be used to examine stability of

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1^5 - x_2^2 \end{bmatrix}$$

at its equilibrium points. Are the equilibria stable, asymptotically stable, or unstable ?

...(10)

4. The non-linear system given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1.1x_1 + g_2(x_2) \\ g_1(x_1) - 1.1x_2 \end{bmatrix}$$

where $g_i(x_i) = \frac{2}{\pi} \cdot \tan^{-1}(0.7\pi x_i)$

has equilibria at (0.454, 0.454), (–0.454, –0.454), and (0, 0). By eigenvalue analysis, examine the stability to small changes around each of the equilibrium point.

...(15)

5. If $\zeta = 0.5$ for a complex conjugate pole pair, by how much dB does the gain response differ from the critically damped case at the corresponding corner frequency ?.

...(10)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Major Test

1. The closed loop characteristic equation is given by

$$(z-a)(z-b)+K_P=0$$

Once the loci leave the real axis, the closed loop poles will always occur as complex conjugate pole pairs. Let these be represented as $z = \rho.e^{\pm j\theta}$. Then

$$(\rho e^{\pm j\theta}-a)(\rho e^{\pm j\theta}-b)+K_P=0$$

$$\Rightarrow \rho^2 e^{\pm j2\theta}-(a+b)\rho e^{\pm j\theta}+ab+K_P=0$$

Since the equation holds separately for positive and negative signs of θ, we obtain real and imaginary component equations as

$$\rho^2\cos 2\theta-(a+b)\rho\cos\theta+(ab+K_P)=0$$

$$\rho^2\sin 2\theta-(a+b)\rho\sin\theta=0$$

Since neither ρ nor $\sin\theta$ can be zero for the complex conjugate poles, the second equation reduces to

$$\rho\cos\theta-(a+b)/2=0$$

Since this expression is free of K_P, it gives a relation for the locus. We observe that once the locus leaves the real axis, it follows the perpendicular bisector of the real section between a and b.

(15 marks)

2. From our knowledge of third order transfer function at the point of marginal stability, we expect a single real closed loop pole, together with a complex conjugate pair of imaginary poles. Thus in closed loop,

$$(s+\sigma)(s^2+\omega_n^2)=(s+a)(s+b)(s+c)+K$$

By comparison of closed loop coefficients, we get relations to be simultaneously satisfied

$$\sigma=a+b+c$$

$$\omega_n^2=ab+bc+ca$$

$$\sigma\cdot\omega_n^2=abc+K$$

Thus the required value of K is

$$K=\sigma\cdot\omega_n^2-abc$$

$$=(a+b+c)(ab+bc+ca)-abc$$

(10 marks)

3. From the state equation we observe that time derivative of the first state variable can be zero only if $x_2 = 0$. Time variation of the second derivative can therefore be zero only if $x_1 = 0$ as well, because at any other x_1 the derivative assumes negative values. Thus the only equilibrium point to be examined is $(x_1, x_2) = (0, 0)$.

Further we have

$$V(x_1,x_2)\triangleq\frac{1}{2}x_2^2+\int_0^{x_1}x^5\cdot dx$$

$$=\frac{1}{2}x_2^2+\frac{1}{6}x_1^6$$

which has a value of zero at the origin and positive values at all other points, indicating that it is positive definite.

Next,

$$\frac{d}{dt}V(x_1,x_2)=[\nabla V(x_1,x_2)]^T\cdot\frac{d}{dt}\mathbf{x}$$

$$=\begin{bmatrix} x_1^5 & x_2 \end{bmatrix}\cdot\begin{bmatrix} x_2 \\ -x_1^5-x_2^2 \end{bmatrix}$$

$$=x_1^5x_2-x_2x_1^5-x_2^3$$

$$=-x_2^3$$

Since the time derivative is negative only for positive values of x_2, we conclude that the point (0, 0) is unstable.

(10 marks)

4. About either of the steady state points, we can obtain the incremental linear state space system by the Jacobian of the nonlinear functions $\mathbf{f}(\mathbf{x})$ in the state space equation. We get

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}=\begin{bmatrix} -1.1 & 1.4/\left[1+(0.7\pi x_2)^2\right] \\ 1.4/\left[1+(0.7\pi x_1)^2\right] & -1.1 \end{bmatrix}_{\mathbf{x}_0}\cdot\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

At $\mathbf{x}_0$ = (0.454, 0.454), or $\mathbf{x}_0$ = (–0.454, –0.454),

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}=\begin{bmatrix} -1.1 & 0.7073 \\ 0.7073 & -1.1 \end{bmatrix}\cdot\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

In either case, the characteristic equation emerges as

$$(s+1.1)^2-0.7073^2=0$$

$$\Rightarrow s=-1.1\pm 0.7073=-1.8073,-0.3927$$

With the poles as negative real, both points are asymptotically stable.

At $\mathbf{x}_0$ = (0, 0),

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}=\begin{bmatrix} -1.1 & 1.4 \\ 1.4 & -1.1 \end{bmatrix}\cdot\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

so that the characteristic equation is

$$(s+1.1)^2-1.4^2=0$$

$$\Rightarrow s=-1.1\pm 1.4=-2.5,0.3$$

This point has a positive real pole, and is therefore unstable.

(15 marks)

5. The ratio of the transfer function terms is given by

$$g(s)=\frac{s^2+2\omega_n s+\omega_n^2}{s^2+2\times 0.5\omega_n s+\omega_n^2}$$

$$\Rightarrow g(j\omega_n)=\frac{-\omega_n^2+2j\omega_n^2+\omega_n^2}{-\omega_n^2+j\omega_n^2=+\omega_n^2}$$

$$\Rightarrow |g(j\omega_n)|=\frac{2j\omega_n^2}{j\omega_n^2}=2\Leftrightarrow 20\log_{10}2=6.02\text{dB}$$

(10 marks)

EEL 205: Control Engineering

Minor Test I

Time: 2 hrs (Open Book) Marks: 20

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1 The Ziegler-Nichols gain setting is to be done for an analog system, whose dominant dynamics transfer function is known to be

$$G(s) = \frac{2}{s(\tau s + 1)^2}$$

A closed-loop response test is done with a proportional controller of gain K_p, which is progressively increased until sustained oscillations are obtained at an *ultimate period*. At that point, how does the *ultimate gain* K_p relate to the parameter τ?

...(10)

2 An analog system has a state space form that is known to be

$$\frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \cdot \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \cdot \mathbf{u}(t)$$

An embedded controller that samples at 0.1s intervals, is to be set up for closed loop control exclusively using the third state variable for feedback. Obtain the digital difference equation that can be used by the controller to generate $x_3(k+1)$ from state and input values at the k-th instant; all of which are measurable.

...(5)

3. A chemical solution tank is heated by a current carrying coil at located at one end, while its temperature is sensed by a thermistor located at the other end. Starting initially at steady state, when the current in the coil is changed by a step of +1A, the temperature sensor is found to record a rise in temperature by 1.2^{o}C after 0.5s, and by 2^{o}C once steady state has been resumed. Assuming the solution tank to have first order thermal dynamics, obtain a first order state equation for temperature T in terms of coil current I.

...(5)

EEL 205: Control Engineering

Solutions to Minor Test I

1 The closed loop transfer function of the given system, through the controller gain K_P, is given by

$$G_{\mathrm{CL}}(s) = \frac{2K_P}{s(\tau s+1)^2 + 2K_P}$$

With the *ultimate gain*, there must be some closed-loop pole that is *purely imaginary*, since this must correspond to sustained oscillations with the *ultimate period*. Let this imaginary pole be $j\omega$ (note that this occurs with a complex conjugate pair).

Being a pole value, it must reduce the denominator to zero. Therefore

$$j\omega(j\tau\omega+1)^2 + 2K_P = 0$$

$$\Rightarrow j\omega(1-\tau^2\omega^2) + 2(K_P - \tau\omega^2) = 0$$

Both the real and imaginary parts of this equation must equate to zero simultaneously, for the same value of ω.

Since ω itself can not be zero, the imaginary part of the equation gives us $\omega = 1/\tau$. For this value of ω, the real part of the equation gives $K_P = 1/\tau$, which the the condition for ultimate gain.

(10 marks)

2 Since the matrix **A** for the given system is diagonal, its higher power are also diagonal, and so is the expansion $e^{\mathbf{A}T}$. The only non-zero elements of $e^{\mathbf{A}T}$ are therefore the diagonal elements, each of which can be obtained as the exponential of the corresponding term of **A**.

We can therefore evaluate

$$\mathbf{F}(T) = e^{\mathbf{A}T} = \begin{bmatrix} e^{1\times 0.1} & 0 & 0 \\ 0 & e^{-1\times 0.1} & 0 \\ 0 & 0 & e^{-4\times 0.1} \end{bmatrix} = \begin{bmatrix} 1.1051 & 0 & 0 \\ 0 & 0.9048 & 0 \\ 0 & 0 & 0.6703 \end{bmatrix}$$

and

$$\mathbf{G}(T) = \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\cdot\mathbf{B}\cdot d\tau = -\mathbf{A}^{-1}\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\cdot\mathbf{B}\cdot d\mathbf{A}[(k+1)T-\tau] = -\mathbf{A}^{-1}\cdot e^{\mathbf{A}T}\cdot\mathbf{B}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}^{-1}\cdot\begin{bmatrix} 1.1051 & 0 & 0 \\ 0 & 0.9048 & 0 \\ 0 & 0 & 0.6703 \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1.1051 & 0 & 0 \\ 0 & -0.9048 & 0 \\ 0 & 0 & -0.1676 \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1.1051 & 1.1051 \\ -0.9048 & 0.9048 \\ 0.1676 & 0 \end{bmatrix} = \begin{bmatrix} -1.1051 & -1.1051 \\ 0.9048 & -0.9048 \\ -0.1676 & 0 \end{bmatrix}$$

From the third row of the vector digital relation

$$\mathbf{x}(k+1) = \mathbf{F}(T)\cdot\mathbf{x}(k) + \mathbf{G}(T)\cdot \mathrm{u}(k),$$

we obtain the required answer as

$$x_3(k+1) = 0.6703\cdot x_3(k) - 0.1676\cdot u_2(k)$$

(5 marks)

3 Assume the first order state equation to be

$$\frac{d}{dt}T(t) = -aT(t) + bi(t)$$

$$\Rightarrow (s+a)T(s) = bI(s)$$

$$\Rightarrow T(s) = \frac{b}{s+a} \cdot \frac{1}{s} = \frac{b}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right]$$

$$\Rightarrow T(t) = \frac{b}{a}[1 - e^{-at}]$$

as a response to the +1A step change in $I(t)$.
At steady state, this must have a value b/a which is 2°C
Equating values at 0.5s,

$$1.2 = 2[1 - e^{-0.5a}]$$

$$\Rightarrow e^{-0.5a} = 1 - \frac{1.2}{2} = 0.4$$

$$\Rightarrow a = 1.8326\text{s}$$

so that the state equation is

$$\frac{d}{dt}T(t) = -1.8326T(t) + 3.6652I(t)$$

(5 marks)

EEL 205: Control Engineering

Minor Test II

Time: 2 hrs (Open Book) Marks: 20

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1 Find the phase variable representation for a second order single-input/single-output plant that has the following particulars:

a. Poles at –1 and –3.

b. Zero at –2 and –4.

c. Steady state gain of 8/3.

Write down the **A**, **B**, **C**, **D** matrices clearly to conclude your answer..

...(10)

2, What is the phase margin of the transfer function

$$G(s) = \frac{Ks}{s^2 + \omega_n^2}$$

at its gain cross-over frequency ?

...(5)

3. For the unstable system

$$\frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \cdot \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \cdot \mathbf{u}(t)$$

obtain the response at time $t > t_0$, starting with initial time at t_0.

...(5)

EEL 205: Control Engineering

Solutions to Minor Test II

1 The transfer function for the plant is

$$G(s) = \frac{(s+4)(s+2)}{(s+1)(s+3)}$$
$$= \frac{s^2+6s+8}{s^2+4s+3}$$
$$= 1 + \frac{2s+5}{s^2+4s+3}$$

Define

$$x_1(s) = \frac{1}{s^2+4s+3}u(s)$$
$$x_2(s) = sx_1(s)$$

In time domain these work out to

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{d^2x_1(t)}{dt^2} + 4\frac{dx_1(t)}{dt} + 3x_1(t) = u(t) \Rightarrow \frac{dx_2(t)}{dt} = -3x_1(t) - 4x_2(t) + u(t)$$
$$\therefore \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

Then the output is obtained a

$$y(s) = (2s+5)x_1(s) + u(s)$$
$$\Rightarrow y(t) = 2\frac{dx_1(t)}{dt} + 5x_1(t) + u(t) = 2x_2(t) + 5x_1(t) + u(t) = \begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [1]u(t)$$

Finally

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \;;\; \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;;\; \mathbf{C} = \begin{bmatrix} 5 & 2 \end{bmatrix} \;;\; \mathbf{D} = [1]\,.$$

(10 marks)

2 Frequency response of the given transfer function is

$$G(j\omega) = \frac{jK\omega}{-\omega^2 + \omega_n^2}$$
$$\Rightarrow \angle G(j\omega) = \tan^{-1}\infty = \frac{\pi}{2}$$

Regardless of the frequency. Thus whatever the gain cross-over frequency, the phase margin is $-\pi/2$.

(5 marks)

3

The resolvent matrix is given by

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+4 \end{bmatrix}^{-1} = \begin{bmatrix} (s-1)^{-1} & 0 & 0 \\ 0 & (s+1)^{-1} & 0 \\ 0 & 0 & (s+4)^{-1} \end{bmatrix}$$

The state transition matrix between t_0 and t is then given by

$$\left|Ł^{-1}(s\mathbf{I}-\mathbf{A})^{-1}\right|_{t-t_0} = Ł^{-1}\begin{bmatrix} (s-1)^{-1} & 0 & 0 \\ 0 & (s+1)^{-1} & 0 \\ 0 & 0 & (s+4)^{-1} \end{bmatrix}_{t-t_0} = \begin{bmatrix} e^{(t-t_0)} & 0 & 0 \\ 0 & e^{-(t-t_0)} & 0 \\ 0 & 0 & e^{-4(t-t_0)} \end{bmatrix}$$

The response at t, starting with t_0, is then given by

$$\mathbf{x}(t) = \begin{bmatrix} e^{(t-t_0)} & 0 & 0 \\ 0 & e^{-(t-t_0)} & 0 \\ 0 & 0 & e^{-4(t-t_0)} \end{bmatrix} \cdot \mathbf{x}(t_0) + \int_{t_0}^{t} \left\{ \begin{bmatrix} e^{(t-\tau)} & 0 & 0 \\ 0 & e^{-(t-\tau)} & 0 \\ 0 & 0 & e^{-4(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \cdot \mathbf{u}(\tau) \right\} d\tau$$

$$= \begin{bmatrix} e^{(t-t_0)} & 0 & 0 \\ 0 & e^{-(t-t_0)} & 0 \\ 0 & 0 & e^{-4(t-t_0)} \end{bmatrix} \cdot \mathbf{x}(t_0) + \int_{t_0}^{t} \left\{ \begin{bmatrix} e^{(t-\tau)} & e^{(t-\tau)} \\ e^{-(t-\tau)} & -e^{-(t-\tau)} \\ -e^{-4(t-\tau)} & 0 \end{bmatrix} \cdot \mathbf{u}(\tau) \right\} d\tau$$

(5 marks)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Major Test

Time: 3 hrs (Open book) Marks: 60

(Please note that no credit will be awarded for any figure. They can be included for your own convenience only.)

1. For the linear dynamical system model

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \cdot \mathbf{u} \quad ;$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Design a state feedback cascade controller that stabilises the system about the operating state corresponding to the constant input $u = 2$. The desired closed loop poles are at –2, –3.

...(20)

2. How much time does it take an unexcited system:

$$\dot{x}_1 = x_2 \quad ; \quad \dot{x}_2 = -x_1$$

to move from state $[0 \;\; 1]^T$ to state $[1 \;\; 0]^T$?

...(10)

3. Under what conditions is the nonlinear state space system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \cdot u \\ -x_1 \cdot u \end{bmatrix}$$

stable at the origin? Assume the Lyapunov function $V(\mathbf{x}) = x_1^2 + x_2^2$ to be applicable.

...(10)

4. A second order digital transfer function has open loop poles at the origin and unity, and no open-loop zeros. What is its overall gain constant when it is marginally unstable in closed loop ?

...(10)

5. A second order analog transfer function has the same natural frequency for the numerator and denominator polynomials, but the damping ratio of the two are $\sqrt{1-\zeta^2}$ and ζ, respectively. If the gain constant of the transfer function is unity, what is the magnitude and phase values at the natural frequency ?

...(10)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Major Test

1. The state corresponding to $u = 2$ is obtained by putting the state derivative equal to null at that input, and this will be the reference vector for the system.

$$\mathbf{0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 \quad \Rightarrow \quad \mathbf{x} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Let the cascade gain vector be some $\begin{bmatrix} k_1 & k_2 \end{bmatrix}$, since it will convert a second order error vector to a scalar input. That is, in closed loop,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix} \cdot \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \mathbf{x} \right\}$$

$$= \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right\} \cdot \mathbf{x} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 - k_1 & -k_2 \end{bmatrix} \cdot \mathbf{x} + \begin{bmatrix} 0 \\ 2 \cdot k_1 \end{bmatrix}$$

The characteristic equation of the closed loop system must have roots at –2 and –3. That is

$$s \cdot (s + k_2) - (1 - k_1) = (s + 2) \cdot (s + 3) = 0$$

$$\Rightarrow s^2 + k_2 \cdot s + (k_1 - 1) = s^2 + 5s + 6 = 0$$

$$\Rightarrow k_1 = 7 \quad ; \quad k_2 = 5$$

The feedback gain matrix should therefore be [7   5] for required performance.

(20 marks)

2. In state space, the system is represented as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = £^{-1} \left\{ \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} \right\} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = £^{-1} \left\{ \frac{1}{s^2 + 1} \cdot \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \right\} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Substituting the values of initial and final state vectors:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow 1 = \sin t \quad ; \quad 0 = \cos t$$

$$\Rightarrow t = \pi/2 \text{ s}$$

(10 marks)

3. For stability at the origin,

$$\frac{d}{dt}V(x_1,x_2)=[\nabla V(\mathbf{x})]^T\cdot\frac{d}{dt}\mathbf{x}$$
$$=\begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}\cdot\begin{bmatrix} -2x_1+x_2\cdot u \\ -x_1\cdot u \end{bmatrix}$$
$$=-4x_1^2+2x_1x_2u-2x_1x_2u$$
$$=-4x_1^2$$

which makes the origin always stable, since at all values other than the origin the time derivative of the Lyapunov function is negative, *regardless of the value of input* !

Thus as long as the origin is an equilibrium point, it will always be stable. To check if the origin is an equilibrium, examine

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\begin{bmatrix} -2x_1+x_2\cdot u \\ -x_1\cdot u \end{bmatrix}$$

The condition for this is obviously that *u must be finite* (bounded), which is the required condition.

(10 marks)

4. Since the plotting rules for the two-pole transfer function are well known, we realise that once the root loci leave the real axis, the real component of roots will maintain a value of (0 + 1)/2 = 0.5.

   Now at the point of marginal stability, the loci must cross the unit circle, so the imaginary components of the CL poles must be $\pm j(1-0.5^2)^{1/2}=\pm j0.8660$. Thus the closed loop characteristic equation becomes

$$(z-0.5)^2+0.8660^2=z(z-1)+K=0$$
$$\Rightarrow z^2-z+1=z^2-z+K=0$$

   which occurs when the gain is unity.

(10 marks)

5. Magnitude:

$$20\log_{10}\sqrt{\frac{(-\omega_n^2+\omega_n^2)^2+4(1-\zeta^2)\omega_n^4}{(-\omega_n^2+\omega_n^2)^2+4(\zeta^2)\omega_n^4}}=20\log_{10}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

   Phase:

$$\tan^{-1}\left[\frac{2\sqrt{1-\zeta^2}\ \omega_n^2}{0}\right]-\tan^{-1}\left[\frac{2\zeta\omega_n^2}{0}\right]=0$$

(10 marks)

EEL 205: Control Engineering

Minor Test I

Time: 2 hrs (Open Book) Marks: 20

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1 An oven temperature control system has an analog transfer function given by

$$G(s) = \frac{K}{s^2 + 3s + 10}$$

For what value of open loop gain K will the closed loop system have peak overshoots less than 5% to step inputs ?

...(7)

2 For an internal combustion engine, the transfer function with injected fuel flow rate as input and fuel flow rate into the cylinder as output is given by[1]

$$G(s) = \frac{\varepsilon\tau s + 1}{\tau s + 1}$$

where τ is a time constant, and ε is known as the *fuel split parameter*. Starting with the transfer matrix expression of $\{\mathbf{C}.[s\mathbf{I} - \mathbf{A}]^{-1}.\mathbf{B} + \mathbf{D}\}$, obtain the state space representation for the system in terms of **A**, **B**, **C**, **D**.

(Note that use of standard forms such as *phase variable form* or *Lure canonical form* lifted from the text will be given no credit. You must only follow the procedure indicated in the problem statement.)

...(8)

3 Obtain the difference equation by which a computer will represent the engine of Problem 2, if the input *fuel flow* and the output *speed* are both sampled at time step T. Do not include the zero-order hold at either end, since the A/D and D/A converters are not to be included in your answer.

...(5)

EEL 205: Control Engineering

Solutions to Minor Test I

1 From the peak overshoot condition,

$$\exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right] \le 0.05$$

$$\Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} \ge 0.9536$$

$$\Rightarrow \zeta \ge 0.6901$$

The closed loop transfer function has a denominator polynomial

$$s^2 + 3s + (10+K),$$

from which, by comparison of coefficients, we obtain

$$\omega_n^2 = 10 + K \quad ; \quad 2\zeta\omega_n = 3$$

$$\Rightarrow \zeta = \frac{3}{2\sqrt{10+K}} \ge 0.6901$$

$$\Rightarrow 10 + K \le 4.7245$$

$$\Rightarrow K \le -5.2754$$

The upper limit of ζ is of course, unity (the critical damping limit). Corresponding to it, ω_n = 1.5rad/s; and hence $K = -7.75$.

The required range of gain is therefore −7.75 to −5.2754.

(7 marks)

2 Since the transfer function represents a SISO system, we have input and output vectors reduced to scalars $u(t)$ and $y(t)$, both being of unity dimension.

Further, the denominator polynomial, which is equal to $\|s\mathbf{I}-\mathbf{A}\|$, is also of first order. Thie therefore reduces the matrix **A** to a scalar constant, and number of states to unity. Thus **A**, **B**, **C**, **D** are all of order 1x1.

Working with the transfer function, we begin by taking out **D**, which would make the numerator polynomial of an order strictly less than that of the denominator polynomial (one reducde dto zero)

$$G(s) = \frac{\varepsilon\tau s + 1}{\tau s + 1}$$

$$= \frac{1-\varepsilon}{\tau s + 1} + \varepsilon$$

that is, $\mathbf{D} = [\varepsilon]$. Further, dividing both the numerator and denominator of the remaining transfer function by τ, an equivalent for the first order $\|s\mathbf{I}-\mathbf{A}\|$ can be obtained

$$G(s) = \frac{(1-\varepsilon)/\tau}{s + 1/\tau} + \varepsilon$$

$$\Rightarrow \|s\mathbf{I} - \mathbf{A}\| = s + 1/\tau$$

$$\Rightarrow \mathbf{A} = [-1/\tau]$$

The numerator of the transfer function, after removal of **D** is equivalent to $\mathbf{C}.\mathrm{adj}[s\mathbf{I}-\mathbf{A}].\mathbf{B}$. The adjoint, since $[s\mathbf{I}-\mathbf{A}]$ is of order one, is unity. The factor $(1-\varepsilon)/\tau$ can be factorised between **B** and **C** in a variety of ways. A good choice (though not the only one) is $\mathbf{B} = [1/\tau]$ and $\mathbf{C} = (1-\varepsilon)$, because this keeps the state equation and output equation - each dependent on *exactly one constant*. This makes the state space representation as

$$\dot{\mathbf{x}} = [-1/\tau] \cdot \mathbf{x} + [1/\tau] \cdot \mathbf{u}$$

$$\mathbf{y} = [1-\varepsilon] \cdot \mathbf{x} + [\varepsilon] \cdot \mathbf{u}$$

(8 marks)

3 The given transfer function has an analog pole at $-1/\tau$ and an analog zero at $-1/(\varepsilon\tau)$, the DC gain (at s = 0) being $1/\varepsilon$.

By pole-zero map, the digital transfer function must have a pole at $e^{-T/\tau}$ and a zero at $e^{-T/(\varepsilon\tau)}$ while the DC gain must remain the same. The overall digital transfer function (without holds) would therefore be

$$\begin{aligned} G(z) &= \left[\frac{1-e^{-T/\tau}}{1-e^{-T/(\varepsilon\tau)}} \right] \cdot \frac{z-e^{-T/(\varepsilon\tau)}}{z-e^{-T/\tau}} \\ &= \left[\frac{1-e^{-T/\tau}}{1-e^{-T/(\varepsilon\tau)}} \right] \cdot \frac{1-e^{-T/(\varepsilon\tau)} \cdot z^{-1}}{1-e^{-T/\tau} \cdot z^{-1}} \end{aligned}$$

The difference equation therefore provides the values of output in terms of past values of output, and past and present values of input as

$$y(k) = e^{-T/\tau} \cdot y(k-1) + \left[\frac{1-e^{-T/\tau}}{1-e^{-T/(\varepsilon\tau)}} \right] \cdot [u(k) - e^{-T/(\varepsilon\tau)} \cdot u(k-1)]$$

(5 marks)

EEL 205: Control Engineering

Minor Test 2

Time: 2 hrs (Open Book) Marks: 20

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1 An open loop analog transfer function has real poles at a, b, and complex conjugate zeros at $c \pm jd$. If the transfer function is closed with a negative feedback and the overall gain of the loop is K, at what values or range of the gain will the closed loop transfer function be unstable ?

...(10)

2 What is the decibel contribution to the magnitude response of a transfer function by complex conjugate pole pairs located at $\rho\underline{/\pm\theta}$ at a frequency of ω rads/s ? Assume the sampling time of all signals to be some T in seconds.

...(5)

3 A second order SISO state space system has a Jordan form with double eigenvalue at some non-zero λ, and an input matrix $[b_1 \quad b_2]^T$. Under what condition is this system uncontrollable ?

...(5)

EEL 205: Control Engineering

Solutions to Minor Test I

1 There are two possibilities within the solution depending on the values of a, b, c, and d.

a. Note that part of the locus lies on the real axis between a and b, and there could be unstable conditions within this, if either or both are positive real.

b. Additionally depending on c, the complex conjugate branches may or may not cross the imaginary axis.

The characteristic equation in terms of s is given for gain K as

$$(s^2 - [a+b]s + ab) + K \cdot (s^2 - 2c \cdot s + [c^2 + d^2]) = 0$$

$$\Rightarrow [1+K] \cdot s^2 - [a+b+2cK] \cdot s + (ab + K[c^2 + d^2]) = 0$$

Once the loci leave the real axis, let the roots have a general form $x \pm jy$. By substitution,

$$[1+K] \cdot (x^2 - y^2 \pm j2xy) - [a+b+2cK] \cdot (x \pm jy) + (ab + K[c^2 + d^2]) = 0$$

$$\Rightarrow \left\{ \begin{array}{c} [1+K] \cdot (x^2 - y^2) - [a+b+2cK] \cdot x + (ab + K[c^2 + d^2]) = 0 \\ [1+K] \cdot 2xy - [a+b+2cK] \cdot y = 0 \end{array} \right\}$$

To examine instability conditions on the *real pole branch*, we substitute $y = 0$ in both equations. This makes the second equation valid for any gain, while the first one reduces to

$$[1+K] \cdot x^2 - [a+b+2cK] \cdot x + (ab + K[c^2 + d^2]) = 0$$

If the *real pole branch at all* crosses the origin at any gain, then for such K,

$$ab + K[c^2 + d^2] = 0$$

$$\Rightarrow K = -\frac{ab}{c^2 + d^2}$$

That is, the real pole branch can have instability conditions if

a. Only one out of the two open loop real poles is negative, and

b. $K \geq -ab/(c^2+d^2)$

To examine instability conditions on *complex conjugate branches*, since y is non-zero at the crossover point, we can eliminate it from the second equation by division. Further, $x = 0$ can be substituted in both equations, thereby reducing them to

$$[1+K] \cdot y^2 = (ab + K[c^2 + d^2])$$
$$[a+b+2cK] = 0$$

That is, the complex branches move from stability to instability at $K = -(a+b)/(2c)$ if the value works out to be positive (otherwise the complex branches are entirely stable or entirely unstable). In case the complex branches do cross the imaginary axis,

i. The system is unstable for $K \leq -(a+b)/(2c)$ if the complex branches tilt left to reach the OL zeros at high gains.

ii. The system is unstable for $K \geq -(a+b)/(2c)$ if the complex branches tilt right to reach the OL zeros at high gains.

(10 marks)

2 The decibel contribution by the complex conjugate pole pair is obtained as

$$
\begin{aligned}
-20\log_{10}|[z-\rho e^{j\theta}][z-\rho e^{-j\theta}]| &= -20\log_{10}|z^2-2z\rho\cos\theta+\rho^2| \\
&= -20\log_{10}\left|(e^{j\omega T})^2-2e^{j\omega T}\rho\cos\theta+\rho^2\right| \\
&= -20\log_{10}\sqrt{[\cos(2\omega T)+\rho^2-2\rho\cos\theta\cos(\omega T)]^2+[\sin(2\omega T)-2\rho\cos\theta\sin(\omega T)]^2} \\
&= -20\log_{10}\sqrt{1+\rho^4+4\rho^2\cos^2\theta+2\rho^2\cos(2\omega T)-4\rho\cos\theta\cos(\omega T)-4\rho^3\cos\theta\cos(\omega T)} \\
&= -20\log_{10}\sqrt{[1+\rho^4+4\rho^2\cos^2\theta-4\rho\cos\theta]+[2\rho^2\cos(2\omega T)-4\rho^3\cos\theta\cos(\omega T)]}
\end{aligned}
$$

(5 marks)

3 The system as defined has

$$\mathbf{A}=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}\ ;\quad \mathbf{b}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

The controllability matrix is then computed as

$$\begin{bmatrix}\mathbf{b} & \mathbf{A}\cdot\mathbf{b}\end{bmatrix}=\begin{bmatrix}b_1 & (\lambda b_1+b_2)\\ b_2 & (\lambda b_2)\end{bmatrix}$$

For the system to be uncontrollable, the controllability matrix must not have full rank, that is, the determinant of the controllability matrix must be zero. For this to be so,

$$
\begin{aligned}
b_1\cdot(\lambda b_2)-b_2\cdot(\lambda b_1+b_2)&=0\\
\Rightarrow -b_2^2&=0\\
\Rightarrow b_2&=0
\end{aligned}
$$

(5 marks)

Indian Institute of Technology, Ropar.

EEL205: Control Engineering

Major Test

Time: 3 hrs (Open book) Marks: 60

(Please note that no credit will be awarded for any figure. They can be included for your own convenience only.)

1. For a *gas flow valve* that is being controlled by a dedicated servomechanism, the *speed* and *position* of the *valve gate* are linearly related. At the position of 3cm from the "fully open" position, the *gate* speed is 6cm/s; while at a *gate* position of 6cm, its speed is 2cm/s. How long does the *valve gate* take to traverse between the 3cm and 6cm positions, one way ?

...(10)

2. If a particular analog transfer function has a k-th order complex conjugate pole-pair at a natural frequency ω_n (total of $2k$ poles at ω_n), by how many dB does the magnitude plot differ at ω_n from the critically damped case ?

...(10)

3. The characteristic equation of a linear digital control system is

$$z^3 + z^2 + 1.5Kz - (K + 0.5) = 0$$

For what values of K will the system be asymptotically stable ? (A well reasoned approach, and not some algorithmic answer is expected for this problem.)

...(10)

4. A system has an analog open loop transfer function $G_{OL}(s) = 100/s^2$. In closed loop, it is to be controlled by a *P-D* controller of transfer function $C(s) = K_P + K_D s$. If the maximum peak overshoot and maximum 2% settling time to step input are available as design specifications 10% and 5s respectively, establish the design relations between K_P and K_D.

...(10)

5. Find out the range of state vector across which the following system is stable:

$$\dot{x}_1 = (1 - x_1 - 0.5x_2)x_1$$

$$\dot{x}_2 = (1 - 0.5x_1 - x_2)x_2$$

Use the square of the state vector magnitude as the Liapunov function.

(10)

6. A second order, single input system, when digitised has

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \; ; \; \mathbf{g} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Design the gain matrix for a *negative state feedback controller*, so that the closed loop eigenvalues are located at 0.5 and 0.7.

...(10)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Major Test

1. We can obtain the linear relation between the position (x) and speed as

$$\frac{\dot{x}-6}{x-3} = \frac{2-6}{6-3} = -\frac{4}{3}$$

$$\Rightarrow \dot{x}-6 = -\frac{4}{3}x+4$$

$$\Rightarrow \dot{x} = -\frac{4}{3}x+10$$

This essentially becomes a state equation of order one, and the state variation as a function of time is obtained as

$$x(t) = e^{-4(t-t_0)/3}\cdot x_0 + \int_{t_0}^{t} e^{-4(t-\tau)/3}\cdot 10\cdot d\tau$$

$$= e^{-4(t-t_0)/3}\cdot x_0 + \frac{3}{4}\times 10\times [e^{-4(t-\tau)/3}]_{t_0}^{t}$$

$$= e^{-4(t-t_0)/3}\cdot x_0 + \frac{15}{2}(1-e^{-4(t-t_0)/3})$$

$$= \frac{15}{2} + \left(x_0 - \frac{15}{2}\right)e^{-4(t-t_0)/3}$$

For the total time, simply equate $x(t)$ = 6cm and x_0 = 3cm, so that

$$(t-t_0) = -\frac{3}{4}\times \ln\left[\frac{6-15/2}{3-15/2}\right] = 0.8240\text{s}$$

(10 marks)

2. The complex conjugate pole pair at damping ratio ζ contributes a factor to the transfer function given by

$$1/(s^2+2\zeta\omega_n s+\omega_n^2)^k$$

with a corresponding critically damped factor of

$$1/(s^2+2\omega_n s+\omega_n^2)^k$$

The difference in magnitude plots due to the two terms at any ω will be

$$-10k\log_{10}\left[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right]+10k\log_{10}\left[(\omega_n^2-\omega^2)^2+(2\omega_n\omega)^2\right]$$

At $\omega = \omega_n$, the difference becomes

$$-10k\log_{10}\left[(2\zeta\omega_n^2)^2\right]+10k\log_{10}\left[(2\omega_n^2)^2\right]$$

$$= -20k\cdot\log_{10}\zeta$$

(10 marks)

3. Since the characteristic equation is of order three, at any value of K it must always have either
   i. An isolated real pole together with a complex conjugate pair, say σ and $\rho\underline{/\pm\theta}$ with $|\sigma|$, $\rho < 1$; or
   ii. Three isolated real poles with $|\sigma_1|$, $|\sigma_2|$, $|\sigma_3| < 1$.

   In case "i", the characteristic polynomial has a form:

$$(z-\sigma)(z^2-2\rho z\cos\theta+\rho^2)=z^3+z^2+1.5Kz-(K+0.5)=0$$

$$\Rightarrow \left\{ \begin{array}{c} \sigma+2\rho\cos\theta=-1 \\ \rho^2+2\rho\sigma\cos\theta=1.5K \\ \sigma\rho^2=K+0.5 \end{array} \right\}$$

where the three relations above have been obtained by comparison of coefficients.

For asymptotic stability, the conditions to be satisfied simultaneously are $\sigma > -1$, $\sigma < +1$ and $\rho < 1$. The loci can be examined for limiting conditions of $\sigma = -1$, $\sigma = +1$ or $\rho = 1$, either of which need to be satisfied at the limit of asymptotic stability. We consider each of these in sequence.

a. If $\sigma = -1$ at any K, then

$$\begin{array}{c} -1+2\rho\cos\theta=-1 \quad\Rightarrow\quad \rho\cos\theta=0 \\ \rho^2-2\rho\cos\theta=1.5K \quad\Rightarrow\quad \rho^2=1.5K \\ -\rho^2=K+0.5 \quad\Rightarrow\quad K=-0.2 \end{array}$$

which is invalid since ρ^2 must be positive.

b. If $\sigma = 1$ at any K, then

$$\begin{array}{c} 1+2\rho\cos\theta=-1 \quad\Rightarrow\quad \rho\cos\theta=-1 \\ \rho^2-2=1.5K \quad\Rightarrow\quad \rho^2=1.5K+2 \\ \rho^2=K+0.5 \quad\Rightarrow\quad K=-3 \end{array}$$

which is invalid since ρ^2 must be positive.

c. If $\rho = 1$ at any K, then

$$\begin{array}{c} \sigma+2\cos\theta=-1 \\ 1+2\sigma\cos\theta=1.5K \\ \sigma=K+0.5 \end{array}$$

Elimination of σ and $\rho\cos\theta$ from the three relations gives

$$1+(K+0.5)(-1-K-0.5)=1.5K$$

$$\Rightarrow (K+0.5)(K+1.5)+1.5K-1=0$$

$$\Rightarrow K^2+3.5K-0.25=0$$

$$\Rightarrow K=\frac{-3.5\pm\sqrt{12.25+1}}{4}=-1.785 \text{ or } 0.07$$

Thus the possibilities are $-1.785 < K < 0.07$, since $\rho = 1$ has been taken as a precondition.

In case "ii", the loci can be examined for either $|\sigma_1|$, $|\sigma_2|$, $|\sigma_3| < 1$. All three poles must be $< +1$ or > -1, so that for each of them,

$$\sigma^3+\sigma^2+1.5K\times\sigma-(K+0.5)=0<1+1+1.5K\ -(K+0.5) \quad\Rightarrow\quad K>-3 \qquad \text{or}$$

$$\sigma^3+\sigma^2+1.5K\times\sigma-(K+0.5)=0>(-1)^3+(-1)^2+1.5K(-1)-(K+0.5) \quad\Rightarrow\quad K>-0.2$$

which are satisfied if K is greater than arbitrary negative values.

Between "i-ii", the intersection range is $-0.2 < K < 0.07$, which is the required answer.

(10 marks)

4. The closed loop polynomial for the *P-D* controlled system is

$$s^2+100K_Ds+100K_P=0$$

$$\Rightarrow \left\{ \begin{array}{c} \omega_n=\sqrt{100K_P} \\ \zeta=100K_D/\left(2\sqrt{100K_P}\right) \end{array} \right\}$$

From the 10% peak overshoot condition, the minimum value of ζ can be obtained as

$$\exp\left[-\pi\zeta/\sqrt{1-\zeta^2} \le 0.1\right] \quad \Rightarrow \quad \zeta \ge 0.5912$$

From the 2% settling time of 5s,

$$5 \ge -\ln 0.02/(\zeta\omega_n) = 3.912/(\zeta\omega_n) \quad \Rightarrow \zeta\omega_n \ge 0.7824 \quad \Rightarrow \omega_n \ge 1.3235\text{rad/s}$$

The design rules thus become:

$$\begin{aligned} \sqrt{100K_P} &\ge 1.3235 \quad \Rightarrow \quad K_P \ge 0.01751 \\ 100K_D/\left(2\sqrt{100K_P}\right) &\ge 0.5912 \quad \Rightarrow \quad K_D \ge 0.01565 \end{aligned}.$$

(10 marks)

5. The equilibria for x_1 must have $x_1 = 0$ or $x_1 + 0.5x_2 = 1$, while for x_2 require $x_2 = 0$ or $0.5x_1 + x_2 = 1$. The system therefore has four equilibrium points at (0, 0), (0, 1), (1, 0), (2/3, 2/3).

Now, $V(\mathbf{x})$ as the square of state vector magnitude has value 0, 1, 1, and 8/9 at the four points. Therefore only the origin needs to be examined for stability, since it is the only one with zero value of the Liapunov function.

Evaluating the time derivative of the Liapunov function,

$$\begin{aligned} \frac{d}{dt}V(x_1, x_2) &= [\nabla V(\mathbf{x})]^T \cdot \frac{d}{dt}\mathbf{x} \\ &= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \cdot \begin{bmatrix} (1 - x_1 - 0.5x_2)x_1 \\ (1 - 0.5x_1 - x_2)x_2 \end{bmatrix} \\ &= 2(x_1^2 - x_1^3 - 0.5x_1^2x_2) + 2(x_2^2 - 0.5x_1x_2^2 - 2x_2^3) \\ &= 2(x_1^2 + x_2^2) - x_1x_2(x_1 + x_2) - 2(x_1^3 + x_2^3) \end{aligned}$$

At all four equilibrium points: (0, 0), (0, 1), (1, 0), and (2/3, 2/3), the rate of change of the Liapunov function is zero. However, for it to be negative in the neighborhood of the origin, note that

$$\begin{aligned} &\frac{d}{dt}V(x_1, x_2) < 0 \\ \Rightarrow\ &2(x_1^2 + x_2^2) < x_1x_2(x_1 + x_2) + 2(x_1^3 + x_2^3) \end{aligned}$$

The last inequality is the condition for asymptotic stability around the origin. But notably, the time derivative is zero at *all four equilibria*, which proves that it is not negative within the entire vicinity of the origin. Thus the origin is stable, but not asymptotically stable !

(10 marks)

6. The closed loop system matrix is given by

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - k_1 & -1 - k_2 \end{bmatrix}$$

The characteristic equation is given by

$$\det\begin{bmatrix} z & -1 \\ 1 + k_1 & z + (1 + k_2) \end{bmatrix} = 0$$

$$\Rightarrow z[z + (1 + k_2)] + (1 + k_1) = 0$$

The roots of the characteristic equation must be at 0.5 and 0.7, so that

$$0.5[0.5 + (1 + k_2)] + (1 + k_1) = 0 \quad \Rightarrow \quad k_1 + 0.5k_2 = -1.75$$

$$0.7[0.7 + (1 + k_2)] + (1 + k_1) = 0 \quad \Rightarrow \quad k_1 + 0.7k_2 = -2.19$$

The required values are $k_1 = -0.65$, and $k_2 = -2.2$

(10 marks)

EEL 205: Control Engineering

Minor Test I

Time: 2 hrs (Open Book) Marks: 20

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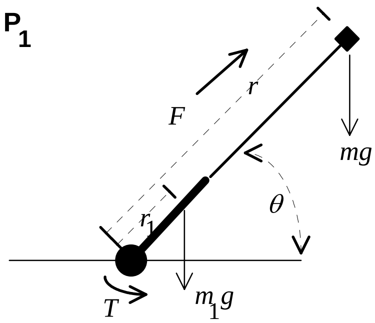
1 The lumped model of a *robot arm manipulator* is shown in Fig. **P1**. The heavy arm base has an effective mass m_1 that can be assumed to be concentrated at a distance r_1 from the axis of rotation, both quantities assumed constant and known by design. At a distance r from the axis, the arm carries a load of mass m. The mass of load can assume arbitrary positive values, but is constant for a particular experiment.

The system has two displacements variables r and θ, each with its forcing inputs: force F and torque T, respectively.

Obtain the nonlinear state space system that describes the given manipulator.

P1



...(6)

2 For known values $r = r^*$ and $\theta = \theta^*$ at which the *robot arm manipulator* is initially at standstill with $F = F^*$ and $T = T^*$, obtain the linearised state space system for the system in Problem 1. In a more restrictive experiment, if $r = r^*$ is held constant (with no Δr changes allowed), under what condition will $\Delta\theta$ show oscillatory dynamics about $\theta = \theta^*$? What will be the natural frequency and damping ratio of the oscillations (as and when they occur) ?

...(8)

3 For a first order system with time constant τ, what would be the approximate settling times (in terms of τ) for step response settlement to 2% and 5% of steady state.

...(6)

EEL 205: Control Engineering

Solutions to Minor Test I

1 The force F is balanced by the weight components of the two masses, as well as the acceleration force component for m (note that m_1 can not have an inertial component, since r_1 is constant !). We get

$$m \cdot \frac{d^2 r}{dt^2} + (m+m_1)g \cdot \sin\theta = F$$

$$\Rightarrow \frac{d^2 r}{dt^2} = -\left(\frac{m+m_1}{m}\right)g \cdot \sin\theta + \frac{1}{m}F$$

The torque T is balanced by the weight torque components of the two masses, as well as the angular acceleration torque for both masses:

$$(mr^2 + m_1 r_1^2) \cdot \frac{d^2\theta}{dt^2} + (mr + m_1 r_1)g \cdot \cos\theta = T$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{mr+m_1 r_1}{mr^2 + m_1 r_1^2}\right)g \cdot \cos\theta + \frac{T}{mr^2 + m_1 r_1^2}$$

By choosing the four state variables as $x_1 = r$, $x_2 = dr/dt$, $x_3 = \theta$, and $x_4 = d\theta/dt$, we obtain a fourth order nonlinear state space form as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -([m+m_1]/m)g \cdot \sin x_3 + (1/m)\cdot F \\ x_4 \\ -([mx_1 + m_1 r_1]/[mx_1^2 + m_1 r_1^2])g \cdot \cos x_3 + T/[mx_1^2 + m_1 r_1^2] \end{bmatrix}$$

(6 marks)

2 We note that the initial value of both speeds is zero. The Jacobians are obtained as

$$\mathbf{A} \triangleq [d\boldsymbol{f}/d\mathbf{x}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -([m+m_1]/m)g \cdot \cos x_3 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{m+m_1 r_1}{mx_1^2+m_1 r_1^2} - \frac{2mx_1}{[mx_1^2+m_1 r_1^2]^2}\right)g \cdot \cos x_3 + \frac{2mx_1}{[mx_1^2+m_1 r_1^2]^2}T^* & 0 & ([mx_1 + m_1 r_1]/[mx_1^2 + m_1 r_1^2])g \cdot \sin x_3 & 0 \end{bmatrix}$$

$$\mathbf{B} \triangleq [d\boldsymbol{f}/d\mathrm{u}] = \begin{bmatrix} 0 & 0 \\ (1/m) & 0 \\ 0 & 0 \\ 0 & (mx_1^2 + m_1 r_1^2)^{-1} \end{bmatrix}$$

Substitution of equilibrium values gives us the specific matrices **A** and **B** as:

$$\mathbf{A} \triangleq [d\boldsymbol{f}/d\mathbf{x}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -(\frac{m+m_1}{m})g \cdot \cos\theta^* & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{m+m_1 r_1}{m\cdot r^{*2}+m_1 r_1^2} - \frac{2m\cdot r^*}{[m\cdot r^{*2}+m_1 r_1^2]^2}\right)g \cdot \cos\theta^* + \frac{2m\cdot r^*}{[mr^{*2}+m_1 r_1^2]^2}T^* & 0 & \left(\frac{m\cdot r^* + m_1 r_1}{m\cdot r^{*2}+m_1 r_1^2}\right)g \cdot \sin\theta^* & 0 \end{bmatrix}$$

$$\mathbf{B} \triangleq [d\boldsymbol{f}/d\mathrm{u}] = \begin{bmatrix} 0 & 0 \\ (1/m) & 0 \\ 0 & 0 \\ 0 & (m \cdot r^{*2} + m_1 r_1^2)^{-1} \end{bmatrix}$$

In the restricted system with $\Delta r = 0$, the system reduces to a second order one with **A** modified to $\mathbf{A}'$ given by

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ \left(\frac{m\cdot r^* + m_1 r_1}{m\cdot r^{*2}+m_1 r_1^2}\right)g \cdot \sin\theta^* & 0 \end{bmatrix}$$

The transfer function for $\Delta\theta$ dynamics has a denominator polynomial given by

$$\det\begin{bmatrix} s & -1 \\ -\left(\frac{m\cdot r^* + m_1 r_1}{m\cdot r^{*2} + m_1 r_1^2}\right) g\cdot\sin\theta^* & s \end{bmatrix} = s^2 - \left(\frac{m\cdot r^* + m_1 r_1}{m\cdot r^{*2} + m_1 r_1^2}\right) g\cdot\sin\theta^*$$

The dynamics will be oscillatory only if sin θ^* is negative (that is, the steady value θ^* is between $\pi/2$ and $3\pi/2$), in which case

$$\omega_n = \sqrt{\left(\frac{m\cdot r^* + m_1 r_1}{m\cdot r^{*2} + m_1 r_1^2}\right)\cdot g\cdot|\sin\theta^*|}$$

$$\zeta = 0$$

(8 marks)

3 The unit step response for the first order system in terms of time constant τ is obtained as

$$y(t) = A\cdot(1 - e^{-t/\tau})$$

which has a steady state output of value A. For 2% setting time,

$$0.98\cdot A \le A\cdot(1 - e^{-t/\tau}) \quad \Rightarrow e^{-t/\tau} \le 1 - 0.98 = 0.02$$

$$\Rightarrow t/\tau \ge -\ln 0.02 \quad \Rightarrow t \ge 3.91\cdot\tau \approx 4\cdot\tau$$

Likewise for 5% settling time,

$$0.95\cdot A \le A\cdot(1 - e^{-t/\tau}) \quad \Rightarrow e^{-t/\tau} \le 1 - 0.95 = 0.05$$

$$\Rightarrow t/\tau \ge -\ln 0.05 \quad \Rightarrow t \ge 2.99\cdot\tau \approx 3\cdot\tau$$

(6 marks)

EEL 205: Control Engineering

Minor Test II

Time: 2 hrs (Open Book) Marks: 20

(Please note that no credit will be awarded for any figure. They can be included for your own convenience only.)

1 Upon linearisation, a digital system in open loop has a state equation:

$$x(k+1) = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot u(k)$$

The single order input is to be generated as a feedback of the state vector through a gain **k**, as $u(k) = r(k) - \mathbf{k}^T.\mathbf{x}(k)$, where $r(k)$ is the one dimensional reference vector. Obtain the relation between the elements of **k** that would make the closed loop states always controllable.

...(6)

2 The product of an open-loop system transfer function and its cascaded controller is given by

$$C(s) \cdot G(s) = \frac{K}{s(Ts+1)(0.5s+1)}$$

Obtain the design rule between K and T which will ensure that the negative unity feedback, closed loop system is always stable.

...(6)

3 A unity feedback, closed loop system has the output *error sensed* through a transducer of transfer function $T(s)$, after which the sensed error is used to generate a *proportional-derivative control* through a block of transfer function $C(s)$. The system being controlled has an open-loop transfer function $G(s)$. All three transfer function blocks are defined as follows:

$$T(s) = \frac{1}{s+1} \quad ; \quad C(s) = K(s+a) \quad ; \quad G(s) = \frac{1}{s(s+2)(s+3)}$$

In closed loop, the system is required to have a gain margin of 20dB at a frequency of 5rad/s. Obtain the values of a and K.

...(8)

EEL 205: Control Engineering

Solutions to Minor Test II

1 The gain vector obviously has two elements, and may be defined as $\begin{bmatrix} k_1 & k_2 \end{bmatrix}^T$. The design condition is required as a relation between the two elements.

The state equation is modified to

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \left\{ r(k) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \cdot \mathbf{x}(k) \right\}$$

$$= \begin{bmatrix} 1-2k_1 & -1-2k_2 \\ 1-k_1 & -2-k_2 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot r(k)$$

The controllability matrix can be constructed as:

$$\begin{bmatrix} \mathbf{g} & \mathbf{Fg} \end{bmatrix} = \begin{bmatrix} 2 & 2(1-2k_1)+1(-1-2k_2) \\ 1 & 2(1-k_1)+1(-2-k_2) \end{bmatrix} = \begin{bmatrix} 2 & (1-2k_1-2k_2) \\ 1 & (-2k_1-k_2) \end{bmatrix}$$

The closed loop system is always controllable if the *controllability matrix* has a full rank of two. Being a square matrix, for fuul rank the *controllability matrix* needs to be non-singular. Thus the design condition required is:

$$2(-2k_1-k_2)-(1-2k_1-2k_2) \neq 0$$

$$\Rightarrow k_1 \neq -1/2$$

(6 marks)

2 The system has open loop poles at the origin (marginally unstable), $-1/T$, and -2. To ensure closed loop stability, the root locus should not have any marginally stable solution for non zero values of K.

We begin by substituting the marginal stability condition in the closed loop characteristic polynomial:

$$j\omega(jT\omega+1)(j0.5\omega+1)+K=0$$

$$\Rightarrow \begin{array}{rl} \omega - 0.5T\omega^3 = 0 & ; \quad \text{from imaginary component balance} \\ \omega^2(0.5+T) = K & ; \quad \text{from real component balance} \end{array}$$

$$\Rightarrow \begin{array}{ll} \omega^2 = 1/(0.5T) & ; \text{since the origin is ruled out } (\omega \neq 0) \\ \omega^2(0.5+T) = K & \end{array}$$

$$\Rightarrow K = \frac{0.5+T}{0.5T} = 2 + 1/T$$

which becomes the required rule.

(6 marks)

3 The overall open loop transfer function is given by

$$T(s)C(s)G(s) = \frac{K(s+a)}{s(s+1)(s+2)(s+3)}$$

The magnitude and phase plots are then given by

$$\text{dB} = 20\log_{10}K + 10\log_{10}(\omega^2+a^2) - 20\log_{10}\omega - 10\log_{10}(\omega^2+1) - 10\log_{10}(\omega^2+4) - 10\log_{10}(\omega^2+9)$$

$$\text{rads} = -\frac{\pi}{2} + \tan^{-1}\frac{\omega}{a} - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{3}$$

Since it is desired to have a PCO at 5 rad/s, the phase relation can be rewritten as

$$\pi = -\frac{\pi}{2} + \tan^{-1}\frac{5}{a} - \tan^{-1}5 - \tan^{-1}\frac{5}{2} - \tan^{-1}\frac{5}{3}$$

$$\Rightarrow a = \frac{5}{\tan[3\pi/2 + \tan^{-1}5 + \tan^{-1}2.5 + \tan^{-1}1.67]}$$

$$\Rightarrow a = -2.436$$

Similarly, the GM condition makes the first relation:

$$20 = 20\log_{10}K + 10\log_{10}(25 + 2.436^2) - 20\log_{10}5 - 10\log_{10}(25+1) - 10\log_{10}(25+4) - 10\log_{10}(25+9)$$

$$\Rightarrow 1 = \log_{10}K + 0.7452 - 0.6990 - 0.7075 - 0.7312 - 0.7657$$

$$\Rightarrow \log_{10}K = 3.1582$$

$$\Rightarrow K = 1439.46$$

(8 marks)

Indian Institute of Technology, Ropar.

EEL205: Control Engineering

Major Test

Time: 3 hrs (Open book) Marks: 60

(Please note that no credit will be awarded for any figure. They may be included solely for your own convenience.)

1. In many high pressure, steam based thermal systems, oscillations in pressure are discouraged so as to prevent fractures, leaks, and consequent contingencies. Post design, a particular boiler is found to have normalised (or dimensionless) dominant second order dynamics of *natural frequency* and *damping ratio* measured as ω_n and ζ, respectively in open loop. The transfer function may be assumed to have no zeros. If the analog closed loop control to the boiler (between **steam valve position** and **boiler pressure** as *input* and *output*, respectively) uses an effective gain constant K, what range is permissible for K to obtain satisfactory operation with no oscillations of pressure for any step change of steam valve position?

...(10)

2. It is known that when digitised, a particular nonlinear system has dominant behaviour of a second order digital transfer function with no zeros. Of the two open loop poles, one is always at the digital unity, while the second has a stable, real value that may change according to the nonlinearities (we will refer to this as the *sensitive pole*). When operating in unity feedback, closed loop, how must the effective closed loop gain change so that the dominant dynamics are critically damped regardless of the *sensitive pole*? Does the answer depend on the digitisation time step ?

...(10)

3. The forward path transfer function of a unity feedback analog control system with a P-D controller is

$$G(s) = \frac{10(K_P + K_D \cdot s)}{s^2}$$

What is the difference between the GCO and PCO frequencies for this system?

...(10)

4. A digital state space system has the following matrices in open loop

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \quad ; \quad \mathbf{g} = \begin{bmatrix} 1 \\ b \end{bmatrix}$$

Is the system controllable and stable in open loop ?

...(10)

5. Find all the possible equilibrium states of

$$\dot{x}_1 = 4x_1 - 2x_1^2 \cdot x_2 \quad ; \quad \dot{x}_2 = -16x_2 + 4x_1^2 \cdot x_2$$

and the state matrix **A** for the system linearised about each equilibrium point.

(10)

6. An n-th order analog state space system is transformed to a different one by an nxn transformation matrix **T**. Obtain the *state transition matrix* for the "new" (or transformed) state space system for time difference $(t - t_0)$. How does the *forward gain matrix* **F** of the corresponding digitised system change by the transformation, assuming some realistic digitisation time step τ for either system ?

...(10)

Indian Institute of Technology, Ropar.

EEL 205: Control Engineering

Solutions to Major Test

1. The closed loop characteristic equation for the system becomes

$$s^2 + 2\zeta\omega_n s + \omega_n^2 + K = 0$$

To eliminate oscillations to step input, $\zeta \geq 1$ essentially replacing the underdamped roots by overdamped ones. This implies that K must be negative, with a minimum magnitude governed by

$$\omega_n^2 + K \leq (\zeta\omega_n)^2$$

$$\Rightarrow K \leq -(1-\zeta^2)\cdot\omega_n^2$$

However as K is progressively reduced, one of the loci moves to the positive real s-plane whereafter the closed loop system becomes unstable (obviously not permissible). This restriction demands that

$$\omega_n^2 + K \geq 0$$

$$\Rightarrow K \geq -\omega_n^2$$

The range of negative gain admissible is therefore

$$-\omega_n^2 \leq K \leq -(1-\zeta^2)\cdot\omega_n^2$$

(10 marks)

2. The closed loop characteristic equation for the digitised system is

$$(z-1)(z-a) + K = 0 \quad ; \quad a < 1$$

$$\Rightarrow z^2 - (1+a)\cdot z + (a+K) = 0$$

For critical damping $\zeta = 1$, which leads to identical closed loop real poles as

$$\rho\angle \pm\theta = e^{-\zeta\omega_n T}\angle\left(\pm\omega_n\sqrt{1-\zeta^2}\;T\right)$$

$$= e^{-\omega_n T} = \rho$$

This immediately gives

$$z^2 - (1+a)\cdot z + (a+K) = 0$$

$$\Rightarrow 2\rho = 1 + a \quad ; \quad \rho^2 = a + K$$

$$\Rightarrow K = \left(\frac{1+a}{2}\right)^2 - a = \left(\frac{1-a}{2}\right)^2$$

as the required design rule.

Clearly, this is completely independent of T.

(10 marks)

3. The frequency response of the forward transfer function is obtained as

$$G(j\omega) = \frac{10(K_P + jK_D\cdot\omega)}{-\omega^2}$$

$$= \frac{10}{\omega^2}\sqrt{K_P^2 + (K_D\cdot\omega)^2}\;\cdot\exp[\pi + \tan^{-1}(K_D\cdot\omega/K_P)]$$

The GCO is obtained by solving (realistic values only)

$$\frac{10}{\omega_{\mathrm{GCO}}^2}\sqrt{K_P^2+(K_D\cdot\omega_{\mathrm{GCO}})^2}=1$$

$$\Rightarrow \omega_{\mathrm{GCO}}^4-(100K_D^2)\cdot\omega_{\mathrm{GCO}}^2-100K_P^2=0$$

$$\Rightarrow \omega_{\mathrm{GCO}}^2=\frac{(100K_D^2)\pm\sqrt{(100K_D^2)^2+400K_P^2}}{2}$$

$$\Rightarrow \omega_{\mathrm{GCO}}=\sqrt{\frac{(100K_D^2)+\sqrt{(100K_D^2)^2+400K_P^2}}{2}}$$

Similarly, the PCO can be obtained by solving

$$\pi=(2n+1)\pi+\tan^{-1}(K_D\cdot\omega_{\mathrm{PCO}}/K_P)$$

$$\Rightarrow \tan^{-1}(K_D\cdot\omega_{\mathrm{PCO}}/K_P)=2n\pi$$

$$\Rightarrow \omega_{\mathrm{PCO}}=0$$

Difference between the two is the same as the GCO, that is

$$\omega_{\mathrm{GCO}}=\sqrt{\frac{(100K_D^2)+\sqrt{(100K_D^2)^2+400K_P^2}}{2}}\quad .$$

(10 marks)

4. For controllability condition, we have

$$\mathbf{g}=\begin{bmatrix}1\\b\end{bmatrix}\quad;\quad \mathbf{F}\cdot\mathbf{g}=\begin{bmatrix}0&1\\-1&a\end{bmatrix}\cdot\begin{bmatrix}1\\b\end{bmatrix}=\begin{bmatrix}b\\ab-1\end{bmatrix}$$

For the system to be controllable, the controllability matrix must be full rank, that is

$$\det\begin{bmatrix}1&b\\b&ab-1\end{bmatrix}\neq 0\quad\Rightarrow\quad ab-1-b^2\neq 0$$

The open loop system will be controllable subject to ($ab - 1 - b^2$) working out to a non-zero value.

For stability condition, we examine the characteristic polynomial in z-domain as

$$\det(z\mathbf{I}-\mathbf{F})=\det\begin{bmatrix}z&-1\\1&z-a\end{bmatrix}=0$$

$$\Rightarrow z(z-a)+1=0$$

$$\Rightarrow z^2-az+1=0$$

Since magnitude product of the two poles is the constant term in the characteristic equation, clearly at least one of them must occur outside the unit circle. The system can therefore not be open loop stable regardless of the value of a.

(10 marks)

5. At any equilibrium point, time derivative of either variable should be zero; that is,

$$4x_1-2x_1^2\cdot x_2=0\quad\Rightarrow\quad x_1=0 \text{ or } x_1x_2=2$$

$$-16x_2+4x_1^2\cdot x_2=0\quad\Rightarrow\quad x_2=0 \text{ or } x_1^2=4$$

The possible equilibria are at [0 0], [–2 –1] and [+2 +1].

We obtain the **A** matrix as the Jacobian given by

$$\mathbf{A} = \begin{bmatrix} \partial\dot{x}_1/\partial x_1 & \partial\dot{x}_1/\partial x_2 \\ \partial\dot{x}_2/\partial x_1 & \partial\dot{x}_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} (4-4x_1x_2) & -2x_1^2 \\ 8x_1x_2 & -16+4x_1^2 \end{bmatrix}$$

$$\therefore \text{ At } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}.$$

$$\text{At } \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -4 & -8 \\ 16 & 0 \end{bmatrix}.$$

$$\text{At } \begin{bmatrix} +2 \\ +1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -4 & -8 \\ 16 & 0 \end{bmatrix}.$$

(10 marks)

6. The analog system matrix **A** converts to $\mathbf{T}^{-1}\mathbf{AT}$ for the transformed system. The state transition matrix for the transformed system works out to

$$\begin{aligned} \Phi[(\mathbf{T}^{-1}\mathbf{AT}),(t-t_0)] &= \sum_{i=0}^{\infty}\left\{(\mathbf{T}^{-1}\mathbf{AT})^i \cdot \frac{(t-t_0)^i}{i!}\right\} \\ &= \sum_{i=0}^{\infty}\left\{\mathbf{T}^{-1}(\mathbf{A}^i)\mathbf{T} \cdot \frac{(t-t_0)^i}{i!}\right\} \\ &= \mathbf{T}^{-1} \cdot \sum_{i=0}^{\infty}\left\{(\mathbf{A}^i) \cdot \frac{(t-t_0)^i}{i!}\right\} \cdot \mathbf{T} \\ &= \mathbf{T}^{-1} \cdot \Phi[\mathbf{A},(t-t_0)] \cdot \mathbf{T} \end{aligned}$$

Since **F** for the digitised system is simply a special case of $\Phi[(\mathbf{T}^{-1}\mathbf{AT}),(t-t_0)]$ with $(t - t_0) = T$, the forward gain matrix of the transformed, digitised system, is simply $\mathbf{T}^{-1}\mathbf{FT}$ where F is the forward gain matrix of the original digitised system.

(10 marks)

EEL 205: Control Engineering

Minor Test I

Time: 2 hrs (Open Book) Marks: 20

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1 The *angular position* of a solar collector can be changed by a *motor torque*, the effective transfer function between the two being $G_A(s) = 10/[s(s+5)]$ following calibration. The *motor and its controller* together have a transfer function $G_M(s) = (s+5)/(s+10)$ between the *angular position error* and the *motor torque*.

Once installed, a *wind gust* disturbance $\omega(s)$ is found to continuously change the *angular position* from what it should have been in the "gust free" environment. To take care of this problem, the *gust* is sensed through a *sensor-compensator* $G_d(s)$, the output of which is subtracted from the *angular position error*.

What should be the transfer function $G_d(s)$ so as to eliminate the effect of gusts altogether ?

...(8)

2 A second order, linear, analog state space system has

$$\mathbf{A} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$$

Obtain the *state transition matrix* corresponding to **A**. By a clear set of points that discuss properties of the *state transition matrix*, establish the condition under which any initial state vector $\mathbf{x}_0$ would always converge to the null vector (or the origin) in absence of input signals.

...(6)

3 A digital control system is described by the state equation

$$x(k+1) = (0.368 - 0.632K) \cdot x(k) + K \cdot r(k)$$

Obtain the values of K that will lead to stable performance.

...(6)

EEL 205: Control Engineering

Solutions to Minor Test I

1 Between the reference *angular position reference* θ_{ref} and actual *angular position* $\theta(s)$, the transfer function in absence of gusts is given by

$$\frac{\theta(s)}{\theta_{ref}(s)} = \frac{G_M(s)\cdot G_A(s)}{1+G_M(s)\cdot G_A(s)}$$

$$= \frac{10/[s\cdot(s+10)]}{1+10/[s\cdot(s+10)]} = \frac{10}{s^2+10s+10}$$

Between the *gust* and the angular position, without the *sensor-compensator*, the *disturbance transfer function* is given by

$$\frac{\theta(s)}{\omega(s)} = \frac{1}{1+G_M(s)\cdot G_A(s)}$$

$$= \frac{1}{1+10/[s\cdot(s+10)]} = \frac{s^2+10s}{s^2+10s+10}$$

Once the *sensor-compensator* is installed, the *gust* has a second path to the *angular position* through $G_d(s)$, with an effective (additional) transfer function:

$$\left[\frac{\theta(s)}{\omega(s)}\right]_{G_d(s)} = -G_d(s)\cdot\frac{G_M(s)\cdot G_A(s)}{1+G_M(s)\cdot G_A(s)}$$

$$= -G_d(s)\cdot\frac{10}{s^2+10s+10}$$

The overall expression for the angular position then becomes

$$\theta(s) = \frac{10}{s^2+10s+10}\cdot\theta_{ref}(s) + \frac{s^2+10s}{s^2+10s+10}\cdot\omega(s) - G_d(s)\cdot\frac{10}{s^2+10s+10}\cdot\omega(s)$$

$$= \frac{10}{s^2+10s+10}\cdot\theta_{ref}(s) + \left[\frac{s^2+10s}{s^2+10s+10} - G_d(s)\cdot\frac{10}{s^2+10s+10}\right]\cdot\omega(s)$$

The effect of gust can be totally eliminated if $G_d(s) = (s^2+10s)/10$.

(8 marks)

2 From the given system matrix, we have (what is referred to as the *resolvent matrix* by many texts)

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix}^{-1}$$

$$= \frac{1}{(s-\sigma)^2+\omega^2}\cdot\begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix} = \begin{bmatrix} \frac{s-\sigma}{(s-\sigma)^2+\omega^2} & \frac{-\omega}{(s-\sigma)^2+\omega^2} \\ \frac{\omega}{(s-\sigma)^2+\omega^2} & \frac{s-\sigma}{(s-\sigma)^2+\omega^2} \end{bmatrix}$$

The state transition matrix is then obtained as

$$e^{\mathbf{A}t} = £^{-1}[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} £^{-1}\left[\frac{s-\sigma}{(s-\sigma)^2+\omega^2}\right] & £^{-1}\left[\frac{-\omega}{(s-\sigma)^2+\omega^2}\right] \\ £^{-1}\left[\frac{\omega}{(s-\sigma)^2+\omega^2}\right] & £^{-1}\left[\frac{s-\sigma}{(s-\sigma)^2+\omega^2}\right] \end{bmatrix}$$

$$= \begin{bmatrix} e^{\sigma t}\cos\omega t & -e^{\sigma t}\sin\omega t \\ e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix} = e^{\sigma t}\cdot\begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

Since the matrix elements of $e^{\mathbf{A}t}$ have undamped sine and cosine variation, the convergence of any initial vector to the origin is entirely decided by the value of σ. If $\sigma \le 0$, then the convergence is assured, otherwise not.

(6 marks)

3. Converting to z-transforms

$$z \cdot X(z) = (0.368 - 0.632K) \cdot X(z) + K \cdot R(z)$$

$$\Rightarrow \frac{X(z)}{R(z)} = \frac{K}{z - (0.368 - 0.632K)}$$

Assuming real coefficients, this will be stable if

$$|0.368 - 0.632K| < 1$$

$$\Rightarrow -1 < 0.368 - 0.632K < 1$$

$$\Rightarrow 2.1646 > K > -1$$

which is the required range of stable operation.

(6 marks)

EEL 205: Control Engineering

Minor Test II

Time: 2 hrs (Open Book) Marks: 20

(Please note that no credit will be awarded for any figure. They may be included solely for your own convenience.)

1 Upon linearisation, a digital system in open loop has a state equation:

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(k)$$

The single order input is to be generated as a feedback of the state vector through a gain $\mathbf{k}$, as $u(k) = r(k) - \mathbf{k}^T.\mathbf{x}(k)$, where $r(k)$ is the one dimensional reference vector. Obtain the relation between the elements of $\mathbf{k}$ that would make the closed loop states always stable. For what values of gain matrix elements will the closed loop poles locate at 0.5 and 0.7 ?

...(6)

2 A system that to operate with unity feedback has a system and cascaded controller transfer functions given respectively by

$$G(s) = \frac{1000}{s(s+10)}$$

$$G_c(s) = \frac{1+aTs}{1+Ts} \quad ; \quad a > 1$$

Design the controller so as to eliminate the open loop pole at −10, while the closed loop performance should be critically damped.

...(6)

3 The inventory stock of a product $x_1(t)$, its rate of sale $x_2(t)$, and production rate $u(t)$ are related by an analog production management model:

$$\frac{dx_1(t)}{dt} = -2x_2(t)$$

$$\frac{dx_2(t)}{dt} = -2u(t)$$

$$y(t) = x_1(t)$$

the time being in units of days. Design a P-D controller of the form $(K_P + K_D.s)$ so as to realise a closed-loop natural frequency of 1rad/s, and damping ratio of 0.707.

...(8)

EEL 205: Control Engineering

Solutions to Minor Test II

1 Since the state space is two-dimensional and the input one-dimensional, we expect the gain matrix **k** to be 1x2. With **k** = [k_1 k_2], the closed loop state equation becomes

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \left\{ r(k) - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \cdot \mathbf{x}(k) \right\}$$

$$\Rightarrow \mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -(1+k_1) & -(1+k_2) \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot r(k)$$

The closed loop state equation can be converted to z-domain as

$$\begin{bmatrix} z & -1 \\ (1+k_1) & z+(1+k_2) \end{bmatrix} \cdot \mathbf{X}(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot r(k)$$

The characteristic equation of the closed loop system is therefore given by

$$z \cdot [z + (1+k_2)] + (1+k_1) = 0$$

$$\Rightarrow z^2 + (1+k_2) \cdot z + (1+k_1) = 0$$

$$\Rightarrow z = \frac{-(1+k_2) \pm \sqrt{(1+k_2)^2 - 4 \cdot (1+k_1)}}{2}$$

This gives us two design rules:

a. *Real poles*:
If $(1+k_2)^2 \geq 4 \cdot (1+k_1)$, the

$$\left| \frac{-(1+k_2) \pm \sqrt{(1+k_2)^2 - 4 \cdot (1+k_1)}}{2} \right| < 1.0$$

b. *Complex conjugate pole pairs:*
If $(1+k_2)^2 < 4 \cdot (1+k_1)$, the

$$(1+k_2)^2 - 2 \cdot (1+k_1) < 1.0$$

$$\Rightarrow k_2^2 + 2 \cdot k_2 < 2.0 + 2k_1$$

If the roots are to be at 0.5 and 0.7, then by comparison of characteristic equation coefficients,

$$z^2 + (1+k_2) \cdot z + (1+k_1) = (z-0.5) \cdot (z-0.7) = 0$$

$$\Rightarrow \left\{ \begin{array}{c} 0.35 = 1 + k_1 \Rightarrow k_1 = -0.65 \\ -1.2 = 1 + k_2 \Rightarrow k_2 = -2.2 \end{array} \right\} \Rightarrow \mathbf{k} = \begin{bmatrix} -0.65 & -2.2 \end{bmatrix}$$

(6 marks)

2 The overall transfer function in open loop becomes:

$$G_c(s) \cdot G(s) = \frac{1 + aTs}{1 + Ts} \times \frac{1000}{s(s+10)}$$

$$= a \cdot \frac{1000 \cdot (s + 1/aT)}{s \cdot (s + 1/T) \cdot (s + 10)}$$

To cancel the open loop pole at −10, we must have $1/aT = 10$, that is $aT = 0.1$.

With the cancellation, the open loop transfer function reduces to

$$G_c(s)\cdot G(s)=\frac{1000\cdot a}{s\cdot(s+1/T)}$$

The closed loop characteristic polynomial is accordingly

$$s^2+s/T+1000a$$

Critically damped condition requires this polynomial to be a perfect square, in which case,

$$(1/T)^2-4\times 1000a=0$$

$$\Rightarrow aT^2=1/4000$$

This gives T = 1/(4000x0.1) = 1/400 = 0.0025s.
Accordingly, a = 0.1/0.0025 = 40.0.

(6 marks)

3 It is easy to see that, the system can be reduced to SISO form as

$$y(t)=x_1(t)$$

$$x_2(t)=-\frac{1}{2}\frac{dx_1(t)}{dt}=-\frac{1}{2}\frac{dy(t)}{dt}$$

$$\frac{dx_2(t)}{dt}=-\frac{1}{2}\frac{d}{dt}\frac{dy(t)}{dt}=-2u(t)$$

$$\therefore\ s^2\cdot Y(s)=4\cdot U(s)\quad\Rightarrow\quad G(s)=\frac{4}{s^2}$$

With the P-D control as stated, the closed loop transfer function becomes

$$G_{CL}(s)=\frac{4\cdot(K_P+K_D\cdot s)}{s^2+4\cdot(K_P+K_D\cdot s)}$$

$$=\frac{4\cdot(K_P+K_D\cdot s)}{s^2+4K_D\cdot s+4K_P}$$

The closed loop characteristic polynomial is noted as

$$s^2+4K_D\cdot s+4K_P=s^2+2\zeta\omega_n\cdot s+\omega_n^2$$

A natural frequency of 1 rad/s implies = (24 x 60 x 60) x 1 rad/day = 86400 rad/day. This leads to

$$4K_P=86400^2$$

$$\Rightarrow K_P=1.86624\times 10^9 \text{ units of inventory stock,}$$

and

$$4K_D=2\times 0.707\times 86400$$

$$\Rightarrow K_D=30542.4 \text{ units of inventory stock-day}$$

(8 marks)

EEL 205: Control Engineering

Minor Test I

Time: 2 hrs (Open Book) Marks: 20

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1 The dominant unit step response of an unknown system has steady output value of unity. The first dynamic peak of 1.25, occurs at 0.01s from the instant that the step input is applied. After how much time from the latter instant, must the dynamics decay within the range 1.1 to 0.9, for a second order transfer function (without any zero) to be a valid representation for the dominant-dynamics ? ...(8)

2 In open-loop, an aircraft turboprop engine is controlled by *fuel rate* and *propeller blade angle* to set two outputs, namely *engine speed* and *turbine inlet temperature*. Upon linearisation and conversion to *s*-domain, the turboprop engine has an open-loop transfer matrix

$$[\mathbf{G}_{OL}(s)] = \begin{bmatrix} 2/[s(s+2)] & 10 \\ 5/s & 1/(s+1) \end{bmatrix}$$

If the outputs are fed back to the inputs through an identity matrix in the negative feedback path, find the resulting closed loop transfer matrix. ...(6)

3 A discrete time SISO system has an open-loop transfer function given by

$$G_{OL}(z) = \frac{0.6321z^{-1}}{1 - 0.3276z^{-1}}$$

is controlled in closed-loop negative feedback through a P-I controller:

$$u(k) = u(k-1) + q_0 \cdot e(k) + q_1 \cdot e(k-1)$$

Establish relations for parameters q_0 and q_1, required to maintain stability of the closed-loop system. ...(6)

EEL 205: Control Engineering

Solutions to Minor Test I

1 The peak overshoot, with steady state value of output as unity, is obtained as 1.25 − 1.0 = 0.25. For an assumed second oder representation of dominant dynamics,

$$\exp\left[-\pi\zeta/\sqrt{1-\zeta^2}\right] = 0.25$$

$$\Rightarrow \zeta/\sqrt{1-\zeta^2} = 0.4413$$

$$\Rightarrow \zeta = 0.4413/\sqrt{1+0.4413^2} = 0.4037$$

Since the first peak of the step response occurs at 0.01s

$$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.01$$

$$\Rightarrow \omega_n = \frac{\pi}{0.01\times\sqrt{1-0.4037^2}} = 343.38\text{rad/s}$$

Expecting the step-response oscillations to confine in the range of 1.1 to 0.9 (with steady state value as unity), essentially amounts to a 10% settling time, which is

$$T_{10\%} = \frac{-\ln 0.1}{\zeta\cdot\omega_n} = \frac{-\ln 0.1}{0.4037\times 343.38}$$

$$= 0.01661\text{s}$$

(8 marks)

2 This problem can be solve dfrom fundamentals of feedback reduction (similar to what we derived in the lectures for the SISO block), but the more confident can go for the direct expression as follows.

$$[\mathbf{G}_{CL}(s)] = \{\mathbf{I}+\mathbf{I}\cdot[\mathbf{G}_{OL}(s)]\}^{-1}\cdot[\mathbf{G}_{OL}(s)]$$

$$= \left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}2/[s(s+2)] & 10\\ 5/s & 1/(s+1)\end{bmatrix}\right\}^{-1}\cdot\begin{bmatrix}2/[s(s+2)] & 10\\ 5/s & 1/(s+1)\end{bmatrix}$$

$$= \begin{bmatrix}(s^2+2s+2)/(s^2+2s) & 10\\ 5/s & (s+2)/(s+1)\end{bmatrix}^{-1}\cdot\begin{bmatrix}2/[s(s+2)] & 10\\ 5/s & 1/(s+1)\end{bmatrix}$$

$$= \frac{1}{(s^2+2s+2)/(s^2+s)-50/s}\cdot\begin{bmatrix}(s+2)/(s+1) & -10\\ -5/s & (s^2+2s+2)/(s^2+2s)\end{bmatrix}\cdot\begin{bmatrix}2/[s(s+2)] & 10\\ 5/s & 1/(s+1)\end{bmatrix}$$

$$= \frac{s(s+1)}{s^2-48s-48}\begin{bmatrix}2/[s(s+1)]-50/s & 10(s+2)/(s+1)-10/(s+1)\\ -10/[s^2(s+2)]+5(s^2+2s+2)/[s^2(s+2)] & -50/s+(s^2+2s+2)/[s^2(s+2)]\end{bmatrix}$$

$$= \frac{1}{s^2-48s-48}\begin{bmatrix}-(48+50s) & 10s(s+1)\\ 5(s+1) & -(49s^2+148s+98)/(s+2)\end{bmatrix}$$

(6 marks)

3. The P-I controller can be converted to z-domain as

$$u(k) = u(k-1)+q_0\cdot e(k)+q_1\cdot e(k-1)$$

$$\Rightarrow [1-z^{-1}]\cdot U(z) = [q_0+q_1\cdot z^{-1}]\cdot E(z)$$

$$\Rightarrow \frac{U(z)}{E(z)} \triangleq G_C(z) = \frac{q_0+q_1\cdot z^{-1}}{1-z^{-1}}$$

The closed loop transfer function has a characteristic equation given by

$$1+\left[\frac{q_0+q_1\cdot z^{-1}}{1-z^{-1}}\right]\cdot\left[\frac{0.6321z^{-1}}{1-0.3276z^{-1}}\right]=0$$

$$\Rightarrow (1-z^{-1})(1-0.3276z^{-1})+(q_0+q_1\cdot z^{-1})(0.6321z^{-1})=0$$

$$\Rightarrow (z-1)(z-0.3276)+0.6321(q_0\cdot z+q_1)=0$$

$$\Rightarrow z^2+(0.6321q_0-1.3276)z+(0.3276+0.6321q_1)=0$$

The coefficients must satisfy the conditions

$$2\rho\cos\theta = 0.6321q_0-1.3276$$

$$\rho^2 = 0.3276+0.6321q_1$$

For stability, ρ must have a magnitude less than unity, and since by its very nature (cos θ) must assume a magnitude less than unity,

$$\sqrt{0.3276+0.6321q_1}\ \le\ 1 \quad\Rightarrow\quad 0.3276+0.6321q_1\le 1 \quad\Rightarrow\quad q_1\le 1.06376\quad ;$$

$$\left|\frac{0.6321q_0-1.3276}{2\cdot\sqrt{0.3276+0.6321q_1}}\right|\ \le\ 1 \quad\Rightarrow\quad (0.6321q_0-1.3276)^2\le 4\cdot(0.3276+0.6321q_1)$$

$$\Rightarrow -2\le 0.6321q_0-1.3276\le 2 \quad\Rightarrow\quad -1.06376\le q_0\le 5.2644$$

(6 marks)

EEL 205: Control Engineering

Minor Test 2

<u>Time:</u> 2 hrs (Open Book) <u>Marks:</u> 20

Note:

- For problems that have **stated method for solution**, use of alternative methods will not secure any credit.
- There is no credit for any graphic, or graphic solution. Figures may be used by students only for ease of understanding problems, with no mark expected for them.

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1 A digitised system has an open loop transfer function given by

$$G(z) = \frac{1.1(z-1)}{(z-a)(z-0.8)}$$

From properties of the digital root-loci, obtain range of the real parameter "*a*" for unity feedback, closed loop, stable operation. ...(8)

2 The forward path transfer function of a unity-feedback analog control system is

$$G(s) = \frac{K(s+\alpha)}{s^2(s+3)}$$

For $-\infty < K < \infty$, determine value of α for which the closed loop root-loci will have zero, one, and two breakaway points, *excluding the one at the origin*. ...(8)

3 Obtain the expression for dB magnitude plot for the transfer function of Problem #1. Your answer should be in terms of the parameter "*a*". ...(4)

EEL 205: Control Engineering

Solutions to Minor Test 2

1 The closed loop transfer function is obtained as

$$(z-a)(z-0.8)+1.1(z-1)=0$$

$$\Rightarrow z^2-(a-0.3)z-1.1=0$$

$$\Rightarrow z=\frac{(a-0.3)\pm\sqrt{(a-0.3)^2+4\times 1.1}}{2}=\frac{(a-0.3)\pm\sqrt{a^2-0.6a+5.3}}{2}$$

The closed loop system can either have two real poles or a complex conjugate pole pair. If both poles are real, then for stable range,

$$-1<\frac{(a-0.3)\pm\sqrt{a^2-0.6a+5.3}}{2}<1$$

$$\Rightarrow -1.7-a<\pm\sqrt{a^2-0.6a+5.3}<2.3-a$$

$$\Rightarrow 2.89+3.4a+a^2<a^2-0.6a+5.3<5.29-4.6a+a^2$$

$$\Rightarrow \left\{\begin{array}{c} 4a<2.41 \\ 4a<-0.01 \end{array}\right\}\Rightarrow a<-0.0025$$

If the poles are complex conjugate pairs, then

$$\left(\frac{a-3}{2}\right)^2+\frac{a^2-0.6a+5.3}{4}<1$$

$$\Rightarrow 2a^2-6.6a+14.3<4$$

$$\Rightarrow a^2-3.3a+5.15<0$$

$$\Rightarrow a<\frac{3.3\pm\sqrt{10.89-20.6}}{2}$$

Since this requires "a" to be complex, the possibility is not admissible. Thus $a < -0.0025$ defines the design range for stable operation.

(8 marks)

2 The closed loop characteristic equation for all three cases, has the form:

$$K(s+\alpha)+s^2(s+3)=0$$

$$\Rightarrow K=-\frac{s^2(s+3)}{(s+\alpha)}$$

Thus all breakaway points must lie on the real axis. Differentiating throughout with respect to s:

$$\frac{\partial K}{\partial s}=-\frac{(s+\alpha)\cdot(3s^2+6s)-s^2(s+3)}{(s+\alpha)^2}=0$$

$$\Rightarrow (s+\alpha)\cdot(3s^2+6s)-s^2(s+3)=0$$

$$\Rightarrow s\cdot[3s^2+3\alpha s+6s+6\alpha-s^2-3s]=0$$

$$\Rightarrow s\cdot[2s^2+3s(1+\alpha)+6\alpha]=0$$

Other than the origin, the breakaway points can be located at

$$s=\frac{-3(1+\alpha)\pm\sqrt{9(1+\alpha)^2-48\alpha}}{4}$$

$$=\frac{-3(1+\alpha)\pm\sqrt{9+9\alpha^2-30\alpha}}{4}$$

i. For no breakaway point other than the origin, the above values for s must be complex. That is

$$9a^2 - 30a + 9 < 0$$

$$\Rightarrow \frac{10 - \sqrt{100-36}}{6} < a < \frac{10 + \sqrt{100-36}}{6}$$

$$\Rightarrow \frac{1}{3} < a < 3$$

ii. For one breakaway point other than the origin,

$$9a^2 - 30a + 9 = 0$$

$$\Rightarrow a = \frac{1}{3} \text{ or } a = 3$$

iii. For two breakaway points other than the origin,

$$9a^2 - 30a + 9 > 0$$

$$\Rightarrow a < \frac{10 - \sqrt{100-36}}{6} = \frac{1}{3} \quad \text{or} \quad a > \frac{10 + \sqrt{100-36}}{6} = 3$$

(8 marks)

3. This is simply obtained as

$$20\log_{10}\left|\frac{1.1(e^{j\omega T} - 1)}{(e^{j\omega T} - a)(e^{j\omega T} - 0.8)}\right|$$

$$= 20\log_{10}1.1 + 10\log_{10}(2 - 2\cos\omega T) - 10\log_{10}(1 + a^2 - 2a\cos\omega T) - 10\log_{10}(1.64 - 2a\cos\omega T)$$

The intermediate steps for each pole and zero term will include the *square* of "*cosine term added to the real pole*" added to *square* of the "*sine term*", but those you can figure out easily. I am skipping those steps here.

(4 marks)

EEL 205: Control Engineering

Major Test

Time: 3 hrs (Open Book) Marks: 60

Note:

- For problems that have **stated method for solution**, use of alternative methods will not secure any credit.
- There is no credit for any graphic, or graphic solution. Figures may be used by students only for ease of understanding problems, with no mark expected for them.

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1 Using a Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$, estimate the region around all equilibria points, for asymptotic stability:

$$\dot{x}_1 = (2 - x_1 - x_2)x_1$$

$$\dot{x}_2 = (2 - x_1 - x_2)x_2$$

...(10)

2 A digital system in open loop is described by the matrices:

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \;;\; \mathbf{G} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;;\; \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The input is obtained as state feedback through gain matrix **K**, such that $u(k) = -\mathbf{K}^T.\mathbf{x}(k)$. Determine if and how controllability and observability of the closed loop system are affected by elements of **K**. ...(10)

3 In open loop, a gas operated furnace has a second order transfer function

$$G(s) = \frac{1}{(1+10s)(1+25s)}$$

between *burner temperature* as input, and *furnace temperature* (as sensed at the furnace wall remote to the burner). A fixed delay of 2s exists between the *burner gas flow* and *burner temperature*, that has a transfer function of e^{-2s}, so that the overall system open loop transfer function has a form

$$G'(s) = \frac{e^{-2s}}{(1+10s)(1+25s)}$$

What will be the dB and phase plot expressions for $G'(s)$? ...(10)

4 A SISO digital system has a CL characteristic equation given by

$$z^2 + (0.15K - 1.5)z + 1 = 0$$

Given that K is a positive gain parameter, determine its values at conditions of marginal stability. ...(10)

5 When plotted against linear scale of ω in rad/s, the $|G(j\omega)|$ for a second order analog transfer function has values 0.9 at $\omega = 0$ rad/s, from where it rises to a maximum of 1.4 at $\omega = 3$ rad/s, falling progressively thereafter at higher frequencies. How much will be the peak overshoot in a step response of $G(s)$? ...(10)

6 The loop transfer function of a single-loop analog control system is

$$G(s)H(s) = \frac{K(s+5)}{s(s+2)(1+Ts)}$$

Obtain the relation between gain K and time constant T that represents the boundary for marginal stability in closed loop ....(10)

EEL 205: Control Engineering

Solutions to Major Test

1 Setting time derivative for both variables to zero, we immediately obtain two conditions for equilibria:

a. All points on the line $x_1 + x_2 = 2$.

b. The origin, that is $[0\ 0]^T$.

To examine Liapunov stability for the equilibria "a", we note that

$$V(x_1, x_2) = x_1^2 + (2 - x_1)^2$$
$$= 2x_1^2 + 4 - 4x_1$$

which assumes the the mandatory zero value for further checks of Liapunov stability only if

$$2x_1^2 - 4x_1 + 4 = 0$$
$$\Rightarrow x_1 = \frac{4 \pm \sqrt{16 - 32}}{4} = 1 \pm j$$
$$\Rightarrow x_2 = 1 \mp j$$

Since complex values for any state variable are not admissible, the given Liapunov function can not be used to check stability of any point satisfying condition "a".

The origin is the only equilibrium corresponding to condition "b", at which $V(0, 0) = 0$.

Further, at any other point $V(x_1, x_2) > 0$, so stability conditions can be examined further. We have

$$\frac{\partial V(x_1, x_2)}{\partial t} = [\nabla V(x_1, x_2)]^T \cdot \mathbf{f}(x_1, x_2)$$
$$= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \cdot \begin{bmatrix} (2 - x_1 - x_2)x_1 \\ (2 - x_1 - x_2)x_2 \end{bmatrix}$$
$$= 2(2 - x_1 - x_2)(x_1^2 + x_2^2)$$

The time derivative is negative conditional to $x_1+x_2 > 2$. So the origin is not asymptotically stable.

(10 marks)

2 The closed loop has the same gain and output matrices, but the forward matrix changes to

$$\mathbf{F}_{\text{CL}} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} (-k_1) & (1 - k_2) \\ (-1 - 2k_1) & (-3 - 2k_2) \end{bmatrix}$$

Accordingly, the closed loop controllability matrix is obtained as

$$\begin{bmatrix} 1 & -k_1 + 2(1 - k_2) \\ 2 & (-1 - 2k_1) + 2(-3 - 2k_2) \end{bmatrix} = \begin{bmatrix} 1 & 2 - k_1 - 2k_2 \\ 2 & -7 - 2k_1 - 4k_2 \end{bmatrix}$$

For the CL system to be controllable, the controllability matrix should be full rank, that is

$$\left\| \begin{matrix} 1 & 2 - k_1 - 2k_2 \\ 2 & -7 - 2k_1 - 4k_2 \end{matrix} \right\| \neq 0$$
$$\Rightarrow (-7 - 2k_1 - 4k_2) - 2(2 - k_1 - 2k_2) \neq 0$$
$$\Rightarrow 11 \neq 0$$

We may therefore conclude that the CL system is controllable for all settings of matrix **K**.

Similarly, the closed loop observability matrix is obtained as

$$\begin{bmatrix} 1 & 1 \\ (-1 - 3k_1) & (-2 - 3k_2) \end{bmatrix}$$

For the CL system to be observable, the observability matrix should be full rank, that is

$$\left\| \begin{matrix} 1 & 1 \\ (-1-3k_1) & (-2-3k_2) \end{matrix} \right\| \neq 0$$

$$\Rightarrow (-2-3k_2)+(1+3k_1) \neq 0$$

$$\Rightarrow k_1 - k_2 - 1/3 \neq 0$$

(10 marks)

3. The dB response is obtained as

$$\begin{aligned} 20\log_{10}|G'(j\omega)| &= 20\log_{10}\left|\frac{e^{-j2\omega}}{(1+j10\omega)(1+j25\omega)}\right| \\ &= 20\log_{10}|e^{-j2\omega}| - 20\log_{10}|1+j10\omega| - 20\log_{10}|1+j25\omega| \\ &= 0 - 10\log_{10}\left[1+(10\omega)^2\right] - 10\log_{10}\left[1+(25\omega)^2\right] \\ &= -10\log_{10}\left[1+(10\omega)^2\right] - 10\log_{10}\left[1+(25\omega)^2\right] \end{aligned}$$

The phase response is similarly obtained as

$$\begin{aligned} \angle G'(j\omega) &= \angle\left[\frac{e^{-j2\omega}}{(1+j10\omega)(1+j25\omega)}\right] \\ &= -2\omega - \tan^{-1}(10\omega) - -\tan^{-1}(25\omega) \end{aligned}$$

(10 marks)

4. The OLTF corresponding to the problem has OL poles

$$z^2 - 1.5z + 1 = 0$$

$$\Rightarrow z = \frac{1.5 \pm \sqrt{2.25-4}}{2} = 0.75 \pm j0.6614$$

which are on the unit circle, and therefore one of the conditions of marginal instability is K=0.

It is further noted that there is only one OL zero at the origin (corresponding to infinite K), so that the closed loop poles must always be a complex conjugate pair. Their magnitude must be unity at any point of marginal stability, so we may assume them to be of the form $e^{\pm j\theta}$. Then comparing coefficients with the CL characteristic polynomial given,

$$0.15K - 1.5 = -2\cos\theta$$

$$\Rightarrow \cos\theta = 0.75 - 0.075K$$

$$\Rightarrow -1 \le 0.75 - 0.075K \le 1$$

$$\Rightarrow -1.75 \le -0.075K \le 0.25$$

$$\Rightarrow 23.33 \ge K \ge -3.33$$

of which the positive range of K is admissible.

(10 marks)

5. Let the second order transfer function be represented as

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The description of the TF provided indicates:

- There is no zero.
- K = 0.9
- ω_n = 3rad/s

Equating the magnitude at 3rad/s,

$$\frac{0.9\times 3^2}{2\zeta\times 3^2} = 1.4$$

$$\Rightarrow \zeta = \frac{0.9}{2\times 1.4} = 0.3214$$

The peak overshoot for a step response will then be

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = \exp\left[-0.3214\times\pi/\sqrt{1-0.3214^2}\,\right] = 0.34428 = 34.428\%$$

(10 marks)

6. The CL characteristic equation is given by

$$K(s+5)+s(s+2)(1+Ts)=0$$

At points of marginal stability, $s = \pm j\omega$, so that

$$Ts^3+(2T+1)s^2+(K+2)s+5K=0$$

$$\Rightarrow -jT\omega^3-(2T+1)\omega^2+j(K+2)\omega+5K=0$$

$$\Rightarrow \left\{\begin{array}{c} -T\omega^3+(K+2)\omega=0 \\ -(2T+1)\omega^2+5K=0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \omega^2=(K+2)/T \\ \omega^2=5K/(2T+1) \end{array}\right\}$$

$$\Rightarrow (K+2)/T = 5K/(2T+1) \Rightarrow 2KT+4T+K+2=5KT$$

$$\Rightarrow 4T+K-3KT+2=0$$

(10 marks)