

# Hawkes Process Presentation 1

Dong Shaocong,  
Wang Zexin

Hawkes Process in Finance

*Quantitative Finance*  
*National University of Singapore*

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# Today's Agenda

## 1 Preliminary

## 2 Univariate Hawkes Process

- Branching Structure
- Deterministic Intensity Functions
- Cluster of offspring
- Memory Kernels
- Marked Process
- Simulation
- Parameter Estimation

## 3 Bivariate Hawkes Process

- Branching Structure and Clustering Representation
- Memory Kernels
- Price Model
- Parameter Estimation

- Hawkes processes are a particularly interesting class of stochastic process that have been applied in diverse areas, from earthquake modelling to financial analysis.
- Events that are observed in time frequently cluster naturally.
- “self-exciting”: Event arrival can excite the process in the sense that the chance of a subsequent arrival is increased for some time period after the initial arrival.
- Simple and flexible for high frequency finance.

# Counting Process / Point Process

Counting process is the number of arrivals until a certain point of time. Arrivals are following certain prespecified distributions.

- Event times:  $T_i$  time of the  $i$ -th event
- $N_t = \sum_{i \geq 1} \mathbb{1}_{\{t \geq T_i\}}$
- $N_0 = 0$
- jump size = 1 at  $t = T_i \forall i$

Arrival time are distributed as exponential random variables.

- $\tau_i \sim \text{Exp}(\lambda)$
- pdf of  $\tau$  :  $f_\tau(t) = \lambda e^{-\lambda t}, \forall t \geq 0$
- expectation of  $\tau$  :  $\mathbb{E}(\tau) = \lambda^{-1}$
- $T_n = \sum_{j=1}^n \tau_j$
- $N_t = \sum_{i \geq 1} \mathbb{1}_{\{t \geq \tau_i\}}$
- Memoryless property:  $\mathbb{P}(\tau > t + m \mid \tau > m) = \mathbb{P}(\tau > t)$
- Homogeneous: all arrival times are distributed as exponential random variables with the same  $\lambda$
- Non-Homogeneous : intensity varies with time, more formally defined as:  $\lambda(t \mid H_t) = \lim_{h \rightarrow 0} \frac{\mathbb{P}\{N_{t+h} - N_t = 1 \mid H_t\}}{h}$

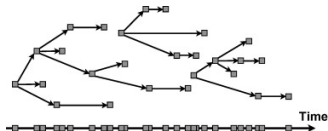
Formulation:

$$\lambda(t \mid F_t) = \lambda_0(t) + \sum_{i:t > T_i} \phi(t - T_i)$$

- $F_t$  should be recognized as a filtration, a welter of all the information known at time  $t$
- Self-exciting property: Arrivals of events increase the likelihood of future observations. Intensity at the current time depends on how many and how far away are the most recent arrivals in the past.
- Non-Homogeneous, as the intensity of the exponential random variables changes with time.
- Deterministic base intensity function:  $\lambda_0(t)$
- Memory Kernel:  $\phi(t - T_i)$  which links to past arrivals times
- Event Decay: monotonically decreasing kernel

# Branching Structure

- We can divide the events into two categories, immigrants and the offsprings.
- Immigrants : the events directly caused by the base intensity
- Offsprings : the events '**excited**' by immigrants or another offspring
- Self-exciting is the property which can be observed in the financial markets as momentum, in which investors following the trend in the prices tend to cause trend in the same direction.
- In econometrics, this property of markets is observed as reverse causality, as price movements drive demands, and demands drive prices as well.



# Deterministic Intensity Functions

Base intensity function  $\lambda_0(t) > 0$ , describes externally triggered events (immigrants)

- Independence: previous events within the process
- Base (or background) intensity
- In the multivariate/bivariate case, the intensity function can be a constant.
- For example,  $\lambda_i(t) = \mu_i$  for a formulation of  $\lambda_i(t) = \mu_i + \sum_{j: t > T_j} \phi(t - T_j)$



# Cluster of offspring

- The offsprings of one immigrant events, which are the immediate offsprings of it, and their immediate offsprings, and their immediate offsprings,  $\dots$ , can be grouped as one cluster.
- Branching factor is defined as the expected number of events generated by a parent event :  $|\Phi| = \int_0^\infty \phi(\tau) d\tau$
- Sub-Critical phase if  $|\Phi| < 1$ , meaning the number of events in one cluster is bounded.
- Super-Critical if  $|\Phi| > 1$ , meaning the number of events in one cluster is unbounded.
- The properties of Sub-Critical and Super-Critical mimics the stationarity property of stochastic processes.
- Expected number of events in one cluster =  $\frac{1}{1-|\Phi|}$

Memory Kernels can be in any form, with two popular forms:

- Exponential decay kernel:  $\phi(t) = \alpha e^{-\delta t}$
- Power law kernel:  $\phi(t) = \frac{\alpha}{(1+\beta t)^{1+\gamma}}$
- By the self-exciting property, the kernel should possess decay property since we expect the intensity to be higher when there are more arrivals in the near past.
- The multiplier  $\beta$  to  $\alpha$  is left out because a variable  $\alpha$  is able to capture the changes by itself.

Each event has a corresponding mark / magnitude  $m_i$  at event time  $T_i$

- Event lies in domain  $S \times M$
- power-law kernel  $\phi_m(\tau) = \kappa m^\beta (\tau + c)^{-(1+\theta)}$
- Event marks  $m_i$  can be i.i.d. samples from a power law distribution  $P(m) = (\alpha - 1)m^{-\alpha}$
- Four parameters  $\theta = \{\kappa, \beta, c, \theta\}$ 
  - ①  $\kappa$ : “event quality”, scales the subsequent event rate
  - ②  $c > 0$ : temporal shift to keep  $\phi_m(\tau)$  bounded
  - ③  $\theta$ : power-law exponent, describing how fast an event is forgotten
- $\kappa m^\beta$  accounts for magnitude of influences
- $(\tau + c)^{-(1+\theta)}$  models the memory over time

# Simulating events from Hawkes processes

- **Goal:** simulate inter-arrival times  $\tau_i$ ,  $i = 1, 2, \dots$ , according to intensity function  $\lambda_t$
- **Poisson Process:**  $f_\tau(t) = \lambda e^{-\lambda t}$ ,  $F_\tau(t) = 1 - e^{-\lambda t}$ ,  $t > 0$
- *inverse transform sampling:*  $Y = F_X(X) \sim U(0, 1) \Rightarrow$  Sample  $u \sim U(0, 1)$ , then compute  $\tau = \frac{-\ln u}{\lambda}$

# Thinning Algorithm - rejection sampling

Applies to all non-homogeneous Poisson processes.

- *thinning property*: Poisson process with intensity  $\lambda$  can be split to two independent processes with intensities  $\lambda_1$  and  $\lambda_2$ , where  $\lambda = \lambda_1 + \lambda_2$
- Monotonically decreasing kernel:  $[T_i, T_{i+1})$ ,  $\lambda(T_i)$  is the upper bound of event intensity.
- Sampling procedure:
  - ①  $T = T_i$ , sample  $\tau$  (inverse transform)
  - ②  $\lambda^* = \lambda(T)$ , update  $T = T + \tau$  (thinning property)
  - ③  $s \sim U(0, 1)$ ,  $T_i = T$  if  $s < \frac{\lambda(T)}{\lambda^*}$ , otherwise repeat process
  - ④ Repeat until reach to  $i = N$

# Decomposition Algorithm

Efficient sampling for Hawkes process with exponential kernel

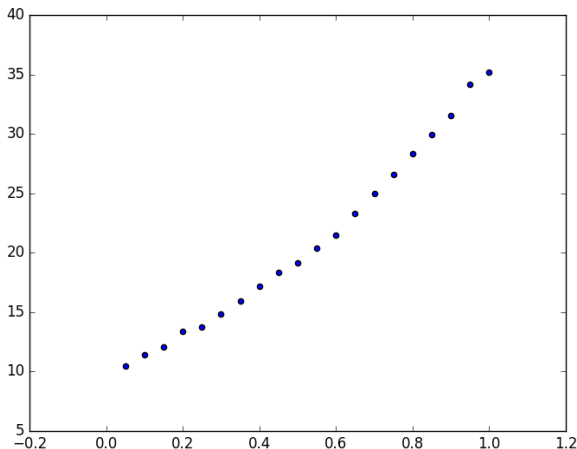
$$\lambda(t) = \underbrace{a + (\lambda_0 - a)e^{-\delta t}}_{\text{immigrant rate}} + \underbrace{\sum_{T_i < t} \gamma e^{-\delta(t-T_i)}}_{\text{jump by event}}, t > 0$$

- Markov (process) decomposition when  $\Phi$  is exponential
- Split to two independent parts:
  - Part 1: sample  $s_0 = -\frac{1}{a} \ln u_0$  (inverse transform)
  - Part 2:  $s_1 = -\frac{1}{\delta} \ln d$ ,  $d = 1 + \frac{\delta \ln u_1}{\lambda(T_{i-1}^+) - a}$  (Markov property)
- Inter Arrival time =  $\min\{s_0, s_1\}$  to get the first event occurring time

# Univariate Exponential Kernel Simulation

Expected number of events with  $\alpha$  varying from 0.05 to 1.

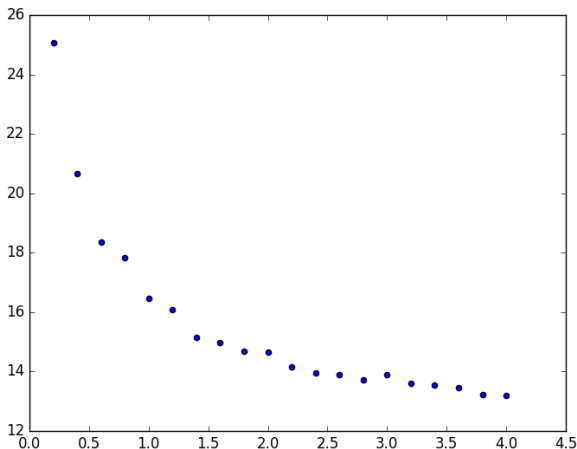
$\beta = 5, T = 10, \mu = 1$



# Univariate Exponential Kernel Simulation

Expected number of events with  $\beta$  varying from 0.2 to 4.

$\alpha = 0.2, T = 10, \mu = 1$

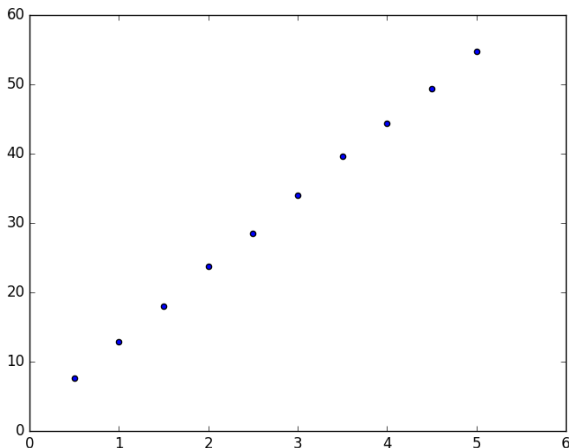




# Univariate Exponential Kernel Simulation

Expected number of events with  $\mu$  varying from 0.5 to 5.

$\alpha = 0.2, \beta = 5, T = 10$



# Likelihood functions for Exponential Kernels

In this section we put some efforts into writing out the likelihood function of Univariate Hawkes Process, and solving for its parameters if possible and not too computationally costly.

- Ideally the likelihood function should be a function of the probability density functions at the event times, and the cumulative density functions between event times.
- $L(\Theta) = \prod_{i=1}^n f(T_i) \prod_{i=1}^n (1 - F_{T_{i-1}}(T_i))(1 - F_{T_n}(T))$
- Cumulative density function for non-homogeneous Poisson Process can be written as :  $F(t) = 1 - \exp(-\int_0^t \lambda(s)ds)$ , where the intensity  $\lambda(s)$  is a function of time
- Probability density function for Hawkes Process, as we described in the previous slides, can be written as :  $f(t) = \lambda(t) \exp(-t\lambda(t))$ , however when we let it participate in the likelihood function,  $f(t) = \lambda(t)$  since the duration of event is negligible.

# Likelihood functions for Exponential Kernels

$$\begin{aligned} L(\Theta) &= \prod_{i=1}^n f(T_i) \prod_{i=1}^n (1 - F_{T_{i-1}}(T_i))(1 - F_{T_n}(T)) \\ &= \left[ \prod_{i=1}^n \lambda(T_i) \right] \left[ \prod_{i=1}^n \exp\left(-\int_{T_{i-1}}^{T_i} \lambda(s) ds\right) \right] \left( \exp\left(-\int_{T_n}^T \lambda(s) ds\right) \right) \\ &= \exp\left(-\int_0^T \lambda(s) ds\right) \prod_{i=1}^n \lambda(T_i) \end{aligned}$$

# Likelihood functions for Exponential Kernels

Suppose we are going to use exponential decay kernel representation and constant base intensity function:

$$\phi(t) = \alpha e^{\delta t}, \lambda_0(t) = \mu \rightarrow \lambda(t | F_t) = \mu + \sum_{i:t > T_i} \alpha e^{\delta(t-T_i)}$$

$$\begin{aligned} L(\Theta) &= \exp\left(-\int_0^T \lambda(t | F_t) dt\right) \prod_{i=1}^n \lambda(T_i | F_{T_i}) \\ &= \exp\left(-\int_0^T \mu + \sum_{i:t > T_i} \alpha e^{\delta(t-T_i)} dt\right) \prod_{i=1}^n \left(\mu + \sum_{j=0}^i \alpha e^{\delta(T_i-T_j)}\right) \end{aligned}$$

$$\ln L(\Theta) = \sum_{i=1}^n \ln\left(\mu + \alpha \sum_{j=0}^i e^{\delta(T_i-T_j)}\right) - \mu T - \alpha \int_0^T \sum_{i:t > T_i} e^{\delta(t-T_i)} dt$$

# Maximum Likelihood Estimates for Exponential Kernels

Differentiate with respect to  $\mu$ :

$$\ln L(\Theta) = \sum_{i=1}^n \ln(\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}) - \mu T - \alpha \int_0^T \sum_{i:t > T_i} e^{\delta(t - T_i)} dt$$

$$\frac{\partial}{\partial \mu} \ln L(\Theta) = \sum_{i=1}^n \frac{1}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} - T = 0$$

$$\rightarrow \sum_{i=1}^n \frac{1}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} = T$$

# Maximum Likelihood Estimates for Exponential Kernels

Differentiate with respect to  $\alpha$ :

$$\begin{aligned}\ln L(\Theta) &= \sum_{i=1}^n \ln(\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}) - \mu T - \alpha \int_0^T \sum_{i:t > T_i} e^{\delta(t - T_i)} dt \\ \frac{\partial}{\partial \alpha} \ln L(\Theta) &= \sum_{i=1}^n \frac{e^{\delta(T_i - T_j)}}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} - \int_0^T \sum_{i:t > T_i} e^{\delta(t - T_i)} dt = 0 \\ &\rightarrow \sum_{i=1}^n \frac{e^{\delta(T_i - T_j)}}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} = \int_0^T \sum_{i:t > T_i} e^{\delta(t - T_i)} dt\end{aligned}$$

# Maximum Likelihood Estimates for Exponential Kernels

Differentiate with respect to  $\delta$ :

$$\begin{aligned}\ln L(\Theta) &= \sum_{i=1}^n \ln(\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}) - \mu T - \alpha \int_0^T \sum_{i:t > T_i} e^{\delta(t - T_i)} dt \\ \frac{\partial}{\partial \delta} \ln L(\Theta) &= \sum_{i=1}^n \frac{(T_i - T_j) e^{\delta(T_i - T_j)}}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} - \int_0^T \sum_{i:t > T_i} (t - T_i) e^{\delta(t - T_i)} dt = 0 \\ &\rightarrow \sum_{i=1}^n \frac{(T_i - T_j) e^{\delta(T_i - T_j)}}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} = \int_0^T \sum_{i:t > T_i} (t - T_i) e^{\delta(t - T_i)} dt\end{aligned}$$

# Maximum Likelihood Estimates for Exponential Kernels

$$\sum_{i=1}^n \frac{1}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} = T \quad (1)$$

$$\sum_{i=1}^n \frac{e^{\delta(T_i - T_j)}}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} = \int_0^T \sum_{i:t > T_i} e^{\delta(t - T_i)} dt \quad (2)$$

$$\sum_{i=1}^n \frac{(T_i - T_j) e^{\delta(T_i - T_j)}}{\mu + \alpha \sum_{j=0}^i e^{\delta(T_i - T_j)}} = \int_0^T \sum_{i:t > T_i} (t - T_i) e^{\delta(t - T_i)} dt \quad (3)$$



# Likelihood functions for Power Law Kernels

Substituting  $\phi(t) = \frac{\alpha}{(1+\beta t)^{1+\gamma}}$  in:

$$\begin{aligned} L(\Theta) &= \exp\left(-\int_0^T \lambda(t \mid F_t) dt\right) \prod_{i=1}^n \lambda(T_i \mid F_{T_i}) \\ &= \exp\left(-\int_0^T \mu + \sum_{i:t>T_i} \frac{\alpha}{(1+\beta(t-T_i))^{1+\gamma}} dt\right) \\ &\quad \prod_{i=1}^n \left(\mu + \sum_{j=0}^i \frac{\alpha}{(1+\beta(T_i-T_j))^{1+\gamma}}\right) \end{aligned}$$

$$\begin{aligned} \ln L(\Theta) &= \sum_{i=1}^n \ln\left(\mu + \alpha \sum_{j=0}^i \frac{1}{(1+\beta(T_i-T_j))^{1+\gamma}}\right) - \mu T \\ &\quad - \alpha \int_0^T \sum_{i:t>T_i} \frac{1}{(1+\beta(t-T_i))^{1+\gamma}} dt \end{aligned}$$

# Maximum Likelihood Estimates for Power Law Kernels

$$\ln L(\Theta) = \sum_{i=1}^n \ln(\mu + \alpha \sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}) - \mu T \\ - \alpha \int_0^T \sum_{i:t > T_i} \frac{1}{(1 + \beta(t - T_i))^{1+\gamma}} dt$$

$$\frac{\partial}{\partial \mu} \ln L(\Theta) = \sum_{i=1}^n \frac{1}{\mu + \alpha \sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}} - T = 0$$

# Maximum Likelihood Estimates for Power Law Kernels

$$\ln L(\Theta) = \sum_{i=1}^n \ln(\mu + \alpha \sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}) - \mu T \\ - \alpha \int_0^T \sum_{i:t > T_i} \frac{1}{(1 + \beta(t - T_i))^{1+\gamma}} dt$$

$$\frac{\partial}{\partial \alpha} \ln L(\Theta) = \sum_{i=1}^n \frac{\sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}}{\mu + \alpha \sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}} \\ - \int_0^T \sum_{i:t > T_i} \frac{1}{(1 + \beta(t - T_i))^{1+\gamma}} dt = 0$$

# Maximum Likelihood Estimates for Power Law Kernels

$$\ln L(\Theta) = \sum_{i=1}^n \ln(\mu + \alpha \sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}) - \mu T \\ - \alpha \int_0^T \sum_{i:t > T_i} \frac{1}{(1 + \beta(t - T_i))^{1+\gamma}} dt$$

$$\frac{\partial}{\partial \beta} \ln L(\Theta) = \sum_{i=1}^n \frac{\alpha \sum_{j=0}^i \frac{T_i - T_j}{(1 + \beta(T_i - T_j))^{2+\gamma}}}{\mu + \alpha \sum_{j=0}^i \frac{1}{(1 + \beta(T_i - T_j))^{1+\gamma}}} \\ - \int_0^T \sum_{i:t > T_i} \frac{t - T_i}{(1 + \beta(t - T_i))^{2+\gamma}} dt = 0$$

# Univariate Model for market activity

As tested on the 10 years Euro-Bond future front contract by Bacry, when  $\gamma$  is closed to 0, the empirical kernel is well described by the power-law kernel.

$$\begin{aligned}\sum_{i=1}^n \frac{1}{\mu + \alpha \sum_{j=0}^i \frac{1}{1+\beta(\bar{T}_i - \bar{T}_j)}} &= T \\ \sum_{i=1}^n \frac{\sum_{j=0}^i \frac{1}{(1+\beta(\bar{T}_i - \bar{T}_j))}}{\mu + \alpha \sum_{j=0}^i \frac{1}{1+\beta(\bar{T}_i - \bar{T}_j)}} &= \int_0^T \sum_{i:t>\bar{T}_i} \frac{1}{1 + \beta(t - \bar{T}_i)} dt \\ \sum_{i=1}^n \frac{\alpha \sum_{j=0}^i \frac{\bar{T}_i - \bar{T}_j}{(1+\beta(\bar{T}_i - \bar{T}_j))^2}}{\mu + \alpha \sum_{j=0}^i \frac{1}{1+\beta(\bar{T}_i - \bar{T}_j)}} &= \int_0^T \sum_{i:t>\bar{T}_i} \frac{t - \bar{T}_i}{(1 + \beta(t - \bar{T}_i))^2} dt\end{aligned}$$

Formulation in our original paper:

$$\lambda_t^i = \mu^i + \sum_{j=1}^D \int dN_{t'}^j \phi^{i,j}(t - t')$$

- $D$  : the number of variables following Hawkes Process
- $\phi^{i,j}$  : the effect of variable  $j$ 's arrival on variable  $i$ 's intensity
- $\mu^i$  : the constant base intensity function of variable  $i$
- $\lambda_t^i$  : the intensity of variable  $i$  at time  $t$
- $T_{j,k}$  : the  $k$ th event time of variable  $j$

Formulation after reconciliation with notation in univariate case:

$$\lambda_t^i = \mu^i + \sum_{j=1}^2 \sum_{k:t > T_{j,k}} \phi^{i,j}(t - T_{j,k})$$

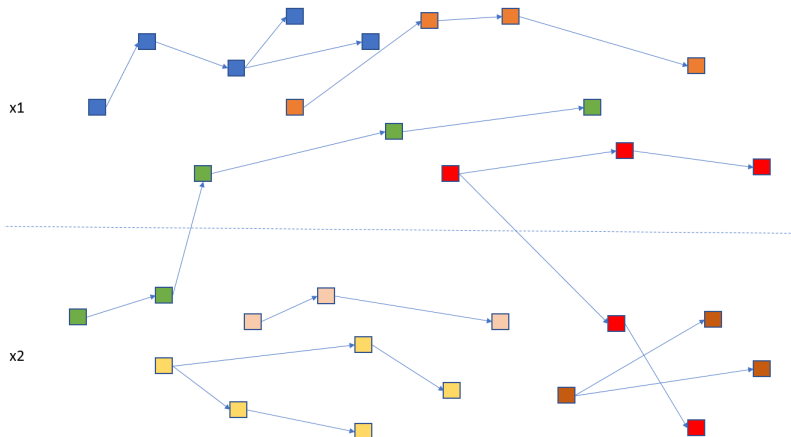
$$\lambda_1(t | F_t) = \lambda_{1,0}(t) + \sum_{i:t > T_{1,i}} \phi^{1,1}(t - T_{1,i}) + \sum_{i:t > T_{2,i}} \phi^{1,2}(t - T_{2,i})$$

$$\lambda_2(t | F_t) = \lambda_{2,0}(t) + \sum_{i:t > T_{1,i}} \phi^{2,1}(t - T_{1,i}) + \sum_{i:t > T_{2,i}} \phi^{2,2}(t - T_{2,i})$$

- Base intensity function  $\lambda_{i,0}(t) > 0$  describes externally triggered events (immigrants) of variable  $i$
- The memory kernels will appear in matrix of functions, in which the functions are positive and causal
- $\phi^{i,j}(t) = 0 \ \forall t < 0$
- $\phi^{i,j}(t) \geq 0 \ \forall t$

# Branching Structure and Clustering Representation

- The different variables share the clusters, as each variable have the ability to excite other variables and own events which parent the offsprings events in other variables





# Bivariate Exponential Kernels

$$\Phi(t) = \begin{pmatrix} \phi^s(t) & \phi^c(t) \\ \phi^c(t) & \phi^s(t) \end{pmatrix}.$$

- $\phi^s(t) = \alpha^s \beta^s \exp(-\beta^s t), \forall t > 0$
- $\phi^c(t) = \alpha^c \beta^c \exp(-\beta^c t), \forall t > 0$
- This is one special case, in which two variables share the same function / parameter which determine the effect of them exciting themselves or exciting the other variable.
- $\phi^{1,2}(t) = \phi^{2,1}(t) = \phi^s(t)$  : **self-effect**
- $\phi^{1,1}(t) = \phi^{2,2}(t) = \phi^c(t)$  : **cross-effect**

# Bivariate Price Model

- The upward and downward price movements can be modelled separately using two variables following Hawkes Process.
- $P_t = P_0 + N_t^1 - N_t^2$
- In the two dimensional model, if we assume that upward and downward movements are the same,  $\phi^s(t) = \phi^c(t)$ , i.e. share the same parameters
- With mean-reverting assumption,  $\phi^s(t) = 0$  and  $\phi^c(t)$  follows exponential shape

# Likelihood functions for Bivariate Exponential Kernels

Since price models using two variables following Hawkes Process with exponential kernels, we present its likelihood functions here:

$$\begin{aligned} L(\Theta) &= \prod_{i=1}^2 \left[ \prod_{j=1}^{n_i} f(T_{i,j}) \prod_{j=1}^{n_i} (1 - F_{T_{i,j-1}}(T_{i,j}))(1 - F_{T_{i,n_i}}(T)) \right] \\ &= \prod_{i=1}^2 \left[ \prod_{j=1}^{n_i} \lambda_i(T_{i,j}) \left( \prod_{j=1}^{n_i} \exp\left(-\int_{T_{i,j-1}}^{T_{i,j}} \lambda_i(s) ds\right) \right) \exp\left(-\int_{T_{i,n_i}}^T \lambda_i(s) ds\right) \right] \\ &= \prod_{i=1}^2 \left[ \exp\left(-\int_0^T \lambda_i(s) ds\right) \prod_{j=1}^{n_i} \lambda_i(T_{i,j}) \right] \\ \ln L(\Theta) &= \sum_{i=1}^2 \left[ \sum_{j=1}^{n_i} \ln \lambda_i(T_{i,j}) - \int_0^T \lambda_i(s) ds \right] \end{aligned}$$

# Likelihood functions for Bivariate Exponential Kernels

$$\phi^{1,1}(t) = \phi^{2,2}(t) = \phi^s(t), \phi^{1,2}(t) = \phi^{2,1}(t) = \phi^c(t), \lambda_{i,0}(t) = \mu_i, \\ \rightarrow \lambda_i(t \mid F_t) = \mu_i + \sum_{j=1}^2 \sum_{k:t > T_{j,k}} \phi^{i,j}(t)$$

$$\ln L(\Theta) = \sum_{i=1}^2 \left[ \sum_{j=1}^{n_i} \ln \lambda_i(T_{i,j}) - \int_0^T \lambda_i(s) ds \right] \\ = \sum_{j=1}^{n_1} \ln \lambda_1(T_{1,j}) - \int_0^T \lambda_1(t) dt + \sum_{j=1}^{n_2} \ln \lambda_2(T_{2,j}) - \int_0^T \lambda_2(t) dt$$

# Likelihood functions for Bivariate Exponential Kernels

$$\begin{aligned}\ln L(\Theta) = & \sum_{j=1}^{n_1} \ln \left( \mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j} - T_{1,k}) + \sum_{k: T_{1,j} > T_{2,k}} \phi^c(T_{1,j} - T_{2,k}) \right) \\ & - \mu_1 T - \int_0^T \sum_{k: t > T_{1,k}} \phi^s(t - T_{1,k}) + \sum_{k: t > T_{2,k}} \phi^c(t - T_{2,k}) dt \\ & + \sum_{j=1}^{n_2} \ln \left( \mu_2 + \sum_{k=1}^{j-1} \phi^s(T_{2,j} - T_{2,k}) + \sum_{k: T_{2,j} > T_{1,k}} \phi^c(T_{2,j} - T_{1,k}) \right) \\ & - \mu_2 T - \int_0^T \sum_{k: t > T_{2,k}} \phi^s(t - T_{2,k}) + \sum_{k: t > T_{1,k}} \phi^c(t - T_{1,k}) dt\end{aligned}$$

# Likelihood functions for Bivariate Exponential Kernels

Since it is too computationally costly to expand  $\phi^s$  and  $\phi^c$ , we are going to leave the derivation out in this presentation.

$$\begin{aligned} \frac{\partial}{\partial \mu_1} \ln L(\Theta) &= \sum_{j=1}^{n_1} \left( \mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j} - T_{1,k}) + \sum_{k: T_{1,j} > T_{2,k}} \phi^c(T_{1,j} - T_{2,k}) \right)^{-1} \\ &\quad - T = 0 \\ \rightarrow T &= \sum_{j=1}^{n_1} \left( \mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j} - T_{1,k}) + \sum_{k: T_{1,j} > T_{2,k}} \phi^c(T_{1,j} - T_{2,k}) \right)^{-1} \end{aligned}$$

A result of literature review shows that Hawkes Process models can be applied for:

- prices of equity index, bond futures, foreign exchange rates
- peculiar non-stationarities in market microstructures such as intraday seasonalities and overnight gaps
- volatility clustering phenomenon modelling (correlated nature of volatility fluctuations)

Hawkes Process models can also be applied to model the market impact which is:

- the impact on prices caused by placing order with significant amount
- extra cost induced per transaction which needs to be added to the transaction costs charged by market
- mechanism enforcing efficiency of market, allowing prices to reflectg fundamental information



# Thank You

*E0007424@u.nus.edu*

*E0012680@u.nus.edu*