Market microstructure model with Hawkes Process

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Contents

Abstract

Introduction

1 Univariate Model

The likelihood function under univarite Hawkes process model:

$$\begin{aligned}
L(\Theta) &= \prod_{i=1}^{n} f(T_i) \prod_{i=1}^{n} (1 - F_{T_{i-1}}(T_i))(1 - F_{T_n}(T)) \\
&= \left[\prod_{i=1}^{n} \lambda(T_i)\right] \left[\prod_{i=1}^{n} \exp(-\int_{T_{i-1}}^{T_i} \lambda(s)ds)\right] \exp\left(-\int_{T_n}^{T} \lambda(t)dt\right) \\
&= \exp\left(-\int_{0}^{T} \lambda(t)dt\right) \prod_{i=1}^{n} \lambda(T_i)
\end{aligned}$$

1.1 Exponential Decay Model

Suppose we are going to use exponential decay kernel and constant base intensity function: $\phi(t) = \alpha e^{-\beta t}$, $\lambda_0(t) = \mu \rightarrow \lambda(t) = \mu + \sum_{i:t>T_i} \alpha e^{-\beta(t-T_i)}$

$$L(\Theta) = \exp\left(-\int_0^T \mu + \sum_{i:t>T_i} \alpha e^{-\beta(t-T_i)} dt\right) \prod_{i=1}^{n} (\mu + \sum_{j=1}^{i-1} \alpha e^{-\beta(T_i - T_j)})$$

$$\ln L(\Theta) = \sum_{i=1}^{n} \ln\left(\mu + \alpha \sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}\right) - \mu T - \alpha \int_0^T \sum_{i:t>T_i} e^{-\beta(t-T_i)} dt$$

Differentiate with respect to μ:

$$\frac{\partial}{\partial \mu} \ln L(\Theta) = \sum_{i=1}^{n} \frac{1}{\mu + \alpha \sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}} - T = 0$$

Differentiate with respect to α:

$$\frac{\partial}{\partial \alpha} \ln L(\Theta) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}}{\mu + \alpha \sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}} - \int_0^T \sum_{i:t>T_i} e^{-\beta(t-T_i)} dt = 0$$

Differentiate with respect to β:

$$\frac{\partial}{\partial \beta} \ln L(\Theta) = \frac{-\alpha \sum_{j=1}^{i-1} (T_i - T_j) e^{-\beta(T_i-T_j)}}{\mu + \alpha \sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}} - \alpha \int_0^T \sum_{i:t>T_i} -(t - T_i) e^{-\beta(t-T_i)} dt = 0$$

Below are the three equations we need to solve to arrive at the values of μ, α and β.

$$\sum_{i=1}^{n} \frac{1}{\mu + \alpha \sum_{j=0}^{i} e^{-\beta(T_i-T_j)}} = T \tag{1}$$

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}}{\mu + \alpha \sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}} = \int_0^T \sum_{i:t>T_i} e^{-\beta(t-T_i)} dt \tag{2}$$

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i-1} (T_i - T_j) e^{-\beta(T_i-T_j)}}{\mu + \alpha \sum_{j=1}^{i-1} e^{-\beta(T_i-T_j)}} = \int_0^T \sum_{i:t>T_i} (t - T_i) e^{-\beta(t-T_i)} dt \tag{3}$$

1.2 Univariate Power Law Kernel

Substituting $\phi(t) = \frac{\alpha}{(1+\beta t)^{1+\gamma}}$ in:

$$
\begin{aligned}
L(\Theta) &= \exp(-\int_0^T \lambda(t \mid F_t)dt)\prod_{i=1}^{n}\lambda(T_i \mid F_{T_i}) \\
&= \exp(-\int_0^T \mu + \sum_{i:t>T_i}\frac{\alpha}{(1+\beta(t-T_i))^{1+\gamma}}dt)\prod_{i=1}^{n}(\mu + \sum_{j=0}^{i}\frac{\alpha}{(1+\beta(T_i-T_j))^{1+\gamma}})
\end{aligned}
$$

$$
\ln L(\Theta) = \sum_{i=1}^{n}\ln(\mu + \alpha\sum_{j=0}^{i}\frac{1}{(1+\beta(T_i-T_j))^{1+\gamma}}) - \mu T - \alpha\int_0^T\sum_{i:t>T_i}\frac{1}{(1+\beta(t-T_i))^{1+\gamma}}dt
$$

Differentiate with respect to μ:

$$
\frac{\partial}{\partial\mu}\ln L(\Theta) = \sum_{i=1}^{n}\frac{1}{\mu + \alpha\sum_{j=0}^{i}\frac{1}{(1+\beta(T_i-T_j))^{1+\gamma}}} - T = 0
$$

Differentiate with respect to α:

$$
\frac{\partial}{\partial\alpha}\ln L(\Theta) = \sum_{i=1}^{n}\frac{\sum_{j=0}^{i}\frac{1}{(1+\beta(T_i-T_j))^{1+\gamma}}}{\mu + \alpha\sum_{j=0}^{i}\frac{1}{(1+\beta(T_i-T_j))^{1+\gamma}}} - \int_0^T\sum_{i:t>T_i}\frac{1}{(1+\beta(t-T_i))^{1+\gamma}}dt = 0
$$

Differentiate with respect to β:

$$
\frac{\partial}{\partial\beta}\ln L(\Theta) = \sum_{i=1}^{n}\frac{\alpha\sum_{j=0}^{i}\frac{T_i-T_j}{(1+\beta(T_i-T_j))^{2+\gamma}}}{\mu + \alpha\sum_{j=0}^{i}\frac{1}{(1+\beta(T_i-T_j))^{1+\gamma}}} - \alpha\int_0^T\sum_{i:t>T_i}\frac{t-T_i}{(1+\beta(t-T_i))^{2+\gamma}}dt = 0
$$

According to Bacry[1]'s research, the empirical kernel is closed to the power law kernel when the parameter γ is closed to 0. Hence we can reduce the equations to these three below to solve for μ, α and β.

$$
\sum_{i=1}^{n}\frac{1}{\mu + \alpha\sum_{j=0}^{i}\frac{1}{1+\beta(T_i-T_j)}} = T \tag{4}
$$

$$
\sum_{i=1}^{n}\frac{\sum_{j=0}^{i}\frac{1}{1+\beta(T_i-T_j)}}{\mu + \alpha\sum_{j=0}^{i}\frac{1}{1+\beta(T_i-T_j)}} = \int_0^T\sum_{i:t>T_i}\frac{1}{1+\beta(t-T_i)}dt \tag{5}
$$

$$
\sum_{i=1}^{n}\frac{\sum_{j=0}^{i}\frac{T_i-T_j}{(1+\beta(T_i-T_j))^2}}{\mu + \alpha\sum_{j=0}^{i}\frac{1}{1+\beta(T_i-T_j)}} = \int_0^T\sum_{i:t>T_i}\frac{t-T_i}{(1+\beta(t-T_i))^2}dt \tag{6}
$$

2 Bivariate Model

Likelihood function of the multivariate Hawkes process model with D variables:

$$
\begin{aligned}
L(\Theta) &= \prod_{i=1}^{D} \left[\prod_{j=1}^{n_i} f(T_{i,j}) \prod_{j=1}^{n_i} (1 - F_{T_{i,j-1}}(T_{i,j}))(1 - F_{T_{i,n_i}}(T)) \right] \\
&= \prod_{i=1}^{D} \left[\prod_{j=1}^{n_i} \lambda_i(T_{i,j}) \left(\prod_{j=1}^{n_i} \exp(- \int_{T_{i,j-1}}^{T_{i,j}} \lambda_i(s) ds) \right) \exp\left(- \int_{T_{i,n}}^{T} \lambda_i(s) ds \right) \right] \\
&= \prod_{i=1}^{D} \left[\exp\left(- \int_{0}^{T} \lambda_i(s) ds \right) \prod_{j=1}^{n_i} \lambda_i(T_{i,j}) \right] \\
\ln L(\Theta) &= \sum_{i=1}^{D} \left[\sum_{j=1}^{n_i} \ln \lambda_i(T_{i,j}) - \int_{0}^{T} \lambda_i(s) ds \right]
\end{aligned}
$$

Likelihood function of the bivariate Hawkes process model is one with $D = 2$.

2.1 Bivariate Exponential Kernel

$\phi^{1,1}(t) = \phi^{2,2}(t) = \phi^s(t)$, $\phi^{1,2}(t) = \phi^{2,1}(t) = \phi^c(t)$, $\lambda_{i,0}(t) = \mu_i$
$\rightarrow \lambda_i(t) = \mu_i + \sum_{j=1}^{2} \sum_{k:t>T_{j,k}} \phi^{i,j}(t)$ where $\phi^s(t) = \alpha^s e^{-\beta^s t}$, $\phi^c(t) = \alpha^c e^{-\beta^c t}$

Partial differentiations of ϕ are: $\frac{\partial\phi(t)}{\partial\alpha} = e^{-\beta t}$, $\frac{\partial\phi(t)}{\partial\beta} = \alpha e^{-\beta t} t = \phi(t)t$

$$
\begin{aligned}
\ln L(\Theta) &= \sum_{i=1}^{2} \left[\sum_{j=1}^{n_i} \ln \lambda_i(T_{i,j}) - \int_0^T \lambda_i(s)ds \right] \\
&= \sum_{j=1}^{n_1} \ln \lambda_1(T_{1,j}) - \int_0^T \lambda_1(t)dt + \sum_{j=1}^{n_2} \ln \lambda_2(T_{2,j}) - \int_0^T \lambda_2(t)dt \\
\ln L(\Theta) &= \sum_{j=1}^{n_1} \ln \left(\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j} - T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j} - T_{2,k}) \right) \\
&\quad -\mu_1 T - \int_0^T \sum_{k:t>T_{1,k}} \phi^s(t - T_{1,k}) + \sum_{k:t>T_{2,k}} \phi^c(t - T_{2,k})dt \\
&\quad + \sum_{j=1}^{n_2} \ln \left(\mu_2 + \sum_{k=1}^{j-1} \phi^s(T_{2,j} - T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}} \phi^c(T_{2,j} - T_{1,k}) \right) \\
&\quad -\mu_2 T - \int_0^T \sum_{k:t>T_{2,k}} \phi^s(t - T_{2,k}) + \sum_{k:t>T_{1,k}} \phi^c(t - T_{1,k})dt \\
&\quad \text{where } \phi^s(t) = \alpha^s e^{-\beta^s t}, \phi^c(t) = \alpha^c e^{-\beta^c t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\mu_1} \ln L(\Theta) &= \sum_{j=1}^{n_1} \left(\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j} - T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j} - T_{2,k}) \right)^{-1} - T = 0 \\
\rightarrow T &= \sum_{j=1}^{n_1} \left(\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j} - T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j} - T_{2,k}) \right)^{-1} \\
\rightarrow T &= \sum_{j=1}^{n_1} \left(\mu_1 + \sum_{k=1}^{j-1} \alpha^s e^{-\beta^s(T_{1,j} - T_{1,k})} + \sum_{k:T_{1,j}>T_{2,k}} \alpha^c e^{-\beta^c(T_{1,j} - T_{2,k})} \right)^{-1}
\end{aligned}
$$

$$\frac{\partial}{\partial\mu_2}\ln L(\Theta)=\sum_{j=1}^{n_2}\left(\mu_2+\sum_{k=1}^{j-1}\phi^s(T_{2,j}-T_{2,k})+\sum_{k:T_{2,j}>T_{1,k}}\phi^c(T_{2,j}-T_{1,k})\right)^{-1}-T=0$$

$$\rightarrow T=\sum_{j=1}^{n_2}\left(\mu_2+\sum_{k=1}^{j-1}\phi^s(T_{2,j}-T_{2,k})+\sum_{k:T_{2,j}>T_{1,k}}\phi^c(T_{2,j}-T_{1,k})\right)^{-1}$$

$$\rightarrow T=\sum_{j=1}^{n_2}\left(\mu_2+\sum_{k=1}^{j-1}\alpha^s e^{-\beta^s(T_{2,j}-T_{2,k})}+\sum_{k:T_{2,j}>T_{1,k}}\alpha^c e^{-\beta^c(T_{2,j}-T_{1,k})}\right)^{-1}$$

$$\begin{aligned}
\frac{\partial}{\partial\alpha^s}\ln L(\Theta)=&\sum_{j=1}^{n_1}\frac{\sum_{k=1}^{j-1}\frac{\partial\phi^s(T_{1,j}-T_{1,k})}{\partial\alpha^s}}{\mu_1+\sum_{k=1}^{j-1}\phi^s(T_{1,j}-T_{1,k})+\sum_{k:T_{1,j}>T_{2,k}}\phi^c(T_{1,j}-T_{2,k})}\\
&-\int_0^T\sum_{k:t>T_{1,k}}\frac{\partial\phi^s(t-T_{1,k})}{\partial\alpha^s}dt-\int_0^T\sum_{k:t>T_{2,k}}\frac{\partial\phi^s(t-T_{2,k})}{\partial\alpha^s}dt\\
&+\sum_{j=1}^{n_2}\frac{\sum_{k=1}^{j-1}\frac{\partial\phi^s(T_{2,j}-T_{2,k})}{\partial\alpha^s}}{\mu_2+\sum_{k=1}^{j-1}\phi^s(T_{2,j}-T_{2,k})+\sum_{k:T_{2,j}>T_{1,k}}\phi^c(T_{2,j}-T_{1,k})}=0\\
\rightarrow&\sum_{j=1}^{n_1}\frac{\sum_{k=1}^{j-1}e^{-\beta^s(T_{1,j}-T_{1,k})}}{\mu_1+\sum_{k=1}^{j-1}\phi^s(T_{1,j}-T_{1,k})+\sum_{k:T_{1,j}>T_{2,k}}\phi^c(T_{1,j}-T_{2,k})}\\
&+\sum_{j=1}^{n_2}\frac{\sum_{k=1}^{j-1}e^{-\beta^s(T_{2,j}-T_{2,k})}}{\mu_2+\sum_{k=1}^{j-1}\phi^s(T_{2,j}-T_{2,k})+\sum_{k:T_{2,j}>T_{1,k}}\phi^c(T_{2,j}-T_{1,k})}\\
&=\int_0^T\sum_{k:t>T_{1,k}}e^{-\beta^s(t-T_{1,k})}dt+\int_0^T\sum_{k:t>T_{2,k}}e^{-\beta^s(t-T_{2,k})}dt
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \alpha^c} \ln L(\Theta) =& \sum_{j=1}^{n_1} \frac{\sum_{k:T_{1,j}>T_{2,k}} \frac{\partial \phi^c(T_{1,j}-T_{2,k})}{\partial \alpha^c}}{\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j}-T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j}-T_{2,k})} \\
&- \int_0^T \sum_{k:t>T_{2,k}} \frac{\partial \phi^c(t-T_{2,k})}{\partial \alpha^c} dt - \int_0^T \sum_{k:t>T_{1,k}} \frac{\partial \phi^c(t-T_{1,k})}{\partial \alpha^c} dt \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{k:T_{2,j}>T_{1,k}} \frac{\partial \phi^c(T_{2,j}-T_{1,k})}{\partial \alpha^c}}{\mu_2 + \sum_{k=1}^{j-1} \phi^s(T_{2,j}-T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}} \phi^c(T_{2,j}-T_{1,k})} = 0 \\
\rightarrow& \sum_{j=1}^{n_1} \frac{\sum_{k:T_{1,j}>T_{2,k}} e^{-\beta^c(T_{1,j}-T_{2,k})}}{\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j}-T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j}-T_{2,k})} \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{k:T_{2,j}>T_{1,k}} e^{-\beta^c(T_{2,j}-T_{1,k})}}{\mu_2 + \sum_{k=1}^{j-1} \phi^s(T_{2,j}-T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}} \phi^c(T_{2,j}-T_{1,k})} \\
&= \int_0^T \sum_{k:t>T_{2,k}} e^{-\beta^c(t-T_{2,k})} dt + \int_0^T \sum_{k:t>T_{1,k}} e^{-\beta^c(t-T_{1,k})} dt
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \beta^s} \ln L(\Theta) =& \sum_{j=1}^{n_1} \frac{\sum_{k=1}^{j-1} \frac{\partial \phi^s(T_{1,j}-T_{1,k})}{\partial \beta^s}}{\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j}-T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j}-T_{2,k})} \\
&- \int_0^T \sum_{k:t>T_{1,k}} \frac{\partial \phi^s(t-T_{1,k})}{\partial \beta^s} dt - \int_0^T \sum_{k:t>T_{2,k}} \frac{\partial \phi^s(t-T_{2,k})}{\partial \beta^s} dt \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{k=1}^{j-1} \frac{\partial \phi^s(T_{2,j}-T_{2,k})}{\partial \beta^s}}{\mu_2 + \sum_{k=1}^{j-1} \phi^s(T_{2,j}-T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}} \phi^c(T_{2,j}-T_{1,k})} = 0 \\
\rightarrow& \sum_{j=1}^{n_1} \frac{\sum_{k=1}^{j-1} (T_{1,j}-T_{1,k}) \phi^s(T_{1,j}-T_{1,k})}{\mu_1 + \sum_{k=1}^{j-1} \phi^s(T_{1,j}-T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}} \phi^c(T_{1,j}-T_{2,k})} \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{k=1}^{j-1} (T_{2,j}-T_{2,k}) \phi^s(T_{2,j}-T_{2,k})}{\mu_2 + \sum_{k=1}^{j-1} \phi^s(T_{2,j}-T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}} \phi^c(T_{2,j}-T_{1,k})} \\
&= \int_0^T \sum_{k:t>T_{1,k}} (t-T_{1,k}) \phi^s(t-T_{1,k}) dt + \int_0^T \sum_{k:t>T_{2,k}} (t-T_{2,k}) \phi^s(t-T_{2,k}) dt
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\beta^c}\ln L(\Theta) =& \sum_{j=1}^{n_1} \frac{\sum_{k:T_{1,j}>T_{2,k}} \frac{\partial\phi^c(T_{1,j}-T_{2,k})}{\partial\beta^c}}{\mu_1 + \sum_{k=1}^{j-1}\phi^s(T_{1,j}-T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}}\phi^c(T_{1,j}-T_{2,k})} \\
&- \int_0^T \sum_{k:t>T_{2,k}} \frac{\partial\phi^c(t-T_{2,k})}{\partial\beta^c}dt - \int_0^T \sum_{k:t>T_{1,k}} \frac{\partial\phi^c(t-T_{1,k})}{\partial\beta^c}dt \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{k:T_{2,j}>T_{1,k}} \frac{\partial\phi^c(T_{2,j}-T_{1,k})}{\partial\beta^c}}{\mu_2 + \sum_{k=1}^{j-1}\phi^s(T_{2,j}-T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}}\phi^c(T_{2,j}-T_{1,k})} = 0 \\
&\to \sum_{j=1}^{n_1} \frac{\sum_{k:T_{1,j}>T_{2,k}} (T_{1,j}-T_{2,k})\phi^c(T_{1,j}-T_{2,k})}{\mu_1 + \sum_{k=1}^{j-1}\phi^s(T_{1,j}-T_{1,k}) + \sum_{k:T_{1,j}>T_{2,k}}\phi^c(T_{1,j}-T_{2,k})} \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{k:T_{2,j}>T_{1,k}} (T_{2,j}-T_{1,k})\phi^c(T_{2,j}-T_{1,k})}{\mu_2 + \sum_{k=1}^{j-1}\phi^s(T_{2,j}-T_{2,k}) + \sum_{k:T_{2,j}>T_{1,k}}\phi^c(T_{2,j}-T_{1,k})} \\
&= \int_0^T \sum_{k:t>T_{2,k}} (t-T_{2,k})\phi^c(t-T_{2,k})dt + \int_0^T \sum_{k:t>T_{1,k}} (t-T_{1,k})\phi^c(t-T_{1,k})dt
\end{aligned}
$$

References

[1] E. Bacry, K. Dayri, and J.F. Muzy, *Non-parametric kernel estimation fore symmetric hawkes processes application to high frequency financial data*. Eur Physics, J. B, 85:157, 2012.