CIVE707 – Theory of Transport Demand Modelling

Disaggregate Choice: Generalized Extreme Value (GEV)

Jason Hawkins

Outlines:

- Choice probability of Discrete Choice models
- Generating function
- Generalized Extreme Value (GEV) model
  - MNL: A Special Case of GEV
  - NL: A Special Case of GEV
  - Ordered GEV (OGEV) model
  - Paired Combinatorial Logit (PCL) model
  - Generalized Logit (GenL) model
  - Generalized Nested Logit/Cross-Nested Logit Model

Choice Probability of Discrete Choice Models

RUM-Based Discrete Choice Model

- RUM based discrete choice probability (**2 alternatives**):

$$\Pr(j|C_i) = Pr\left(\left(V_j + \epsilon_j\right) \geq (V_k + \epsilon_k)\right)$$

C_i is the choice set

$$\Pr(j|C_i) = Pr\left(\epsilon_k \leq (V_j - V_k + \epsilon_j)\right)$$

$$\Pr(j|C_i) = \int_{\epsilon_j=-\infty}^{+\infty} \left(\int_{\epsilon_k=-\infty}^{(V_j - V_k + \epsilon_j)} f(\epsilon_j, \epsilon_k)\, d\epsilon_k \right) d\epsilon_j \quad ; k \neq j, k, j \in C_i$$

$\rightarrow$ **multiple alternatives:**

$$\Pr(j|C_i) = \int_{\epsilon_j=-\infty}^{+\infty} \left(\int_{\epsilon_1=-\infty}^{(V_j - V_k + \epsilon_j)} \ldots . \int_{\epsilon_k=-\infty}^{(V_j - V_k + \epsilon_j)} f\left(\epsilon_1, \epsilon_2, \ldots, \epsilon_j, \ldots . \epsilon_k\right) d\epsilon_1 \ldots d\epsilon_k \right) d\epsilon_j$$

RUM-based Discrete Choice Model

$$\Pr(j|C_i) = \int_{\epsilon_j=-\infty}^{+\infty} \left(\int_{\epsilon_1=-\infty}^{(V_j - V_k + \epsilon_j)} \dots . \int_{\epsilon_k=-\infty}^{(V_j - V_k + \epsilon_j)} f\left(\epsilon_1, \epsilon_2, \dots, \epsilon_j, \dots . \epsilon_k\right) d\epsilon_1 \dots d\epsilon_k \right) d\epsilon_j$$

↘ Equivalently:

$$\Pr(j|C_i) = \left(\int_{\epsilon_j=-\infty}^{+\infty} F_j\left(\left(V_j - V_1 + \epsilon_j\right), \left(V_j - V_2 + \epsilon_j\right), \dots, \epsilon_j, .. \left(V_j - V_k + \epsilon_j\right)\right) d\epsilon_j \right)$$

- Joint CDF of all random utilities: $F\left(\epsilon_1, \epsilon_2, \dots, \epsilon_j, \dots . \epsilon_k\right)$

- Partial derivative of the joint CDF, F(….) with respect to ϵ_j:

$$F_j(\dots \dots \dots \dots ..) = \partial F(\dots \dots \dots \dots)/\partial \epsilon_j$$

Generating Function & GEV Model

McFadden (1978). Modelling the Choice of Residential Location," in Spatial Interaction Theory and Planning Models, ed. by Anders Karlqvist, et al. Amsterdam: North-Holland Publishing Company, pp. 75-96

- **Generating function**: A formal power series (polynomial function) that encodes an infinite sequence of x (place holder rather than a number) by treating them as the coefficient of the power series: especially powerful for recurring relationships
  - the function may not be a true function and the variables may be intermediate information units
  - Instead of dealing with infinite sequence, generating function gives a single function that encodes the sequence
  - A generating function expresses a sequence (the sequence of coefficients): the i^{th} term of the sequence is the coefficient of x^i in the generating function

GEV Generating Function, G(..)

- Define a generating function G(…) for the multivariate joint distribution of random utilities (ε) as

$$F\left(y_1, y_2, \dots, y_j, \dots . y_k\right) = exp\left(-G\left(e^{-y_1}, e^{-y_2}, \dots ., e^{-y_J}\right)\right)$$

- Following properties are key for the G() function:

1. $\mathrm{G(y)} \geq 0$ for any argument, y
2. $\mathrm{G(y)}$ is homogenous of degree D >0. This means $G\left(\alpha y_1, \alpha y_2, \dots \dots, \alpha y_J\right) = \alpha^D G\left(y_1, y_2, \dots \dots, y_J\right)$ for α>0
3. $Lim_{y_j \to \infty}\ G\left(y_1, y_2, \dots \dots, y_J\right) = +\infty$ for *j*=1, 2, 3…..*J*
4. Cross partial derivatives exists and are continuous and their signs change in a particular way:

$$G_j(..) = \frac{\partial G(..)}{\partial y_j} > 0\ for\ all\ j$$

$$G_{jk}(..) = \frac{\partial G_j(..)}{\partial y_k} \leq 0\ for\ j \neq k$$

$$G_{jkl}(..) = \frac{\partial G_{jk}(..)}{\partial y_l} \geq 0\ for\ distinctive\ j, k, l$$

Generalized Extreme Value (GEV) Model

$$\Pr(j|C_i) = \left(\int_{\epsilon_j=-\infty}^{+\infty} F_j\left((V_j - V_1 + \epsilon_j), (V_j - V_2 + \epsilon_j), \dots, \epsilon_j, \dots (V_j - V_k + \epsilon_j) \right) d\epsilon_j \right)$$

$\rightarrow$ Using GEV generating function

$$F\left(y_1, y_2, \dots, y_j, \dots . y_k\right) = exp\left(-G(e^{-y_1}, e^{-y_2}, \dots ., e^{-y_J})\right) \quad \rightarrow y_j \; is\ the\ argument$$

$$F_j\left(y_1, y_2, \dots, y_j, \dots . y_k\right) = \partial F(\dots)/\partial y_j$$

$$\boxed{= exp\left(-G(e^{-y_1}, e^{-y_2}, \dots ., e^{-y_J})\right) e^{-y_j} G_j(e^{-y_1}, e^{-y_2}, \dots ., e^{-y_J})}$$

$$where, G_j(\dots .) = \partial G(\dots)/\partial y_j$$

$\rightarrow$ Applying Euler's formula and homogeneity conditions in $F_j(..)$ of the integral of $\Pr(j)$, results in a closed form function:

$$\Pr(j|C_i) = \frac{e^{V_j} G_j\left(e^{V_1}, e^{V_2}, \dots e^{V_j}\right)}{G(e^{V_1}, e^{V_2}, \dots e^{V_j})}$$

Generalized Extreme Value (GEV) Model

- Choice probability of a GEV model based on the GEV generating function (for $G(..)$ of homogenous to degree 1)

$$\Pr(j|C_i) = \frac{e^{V_j} G_j\left(e^{V_1}, e^{V_2}, \dots e^{V_j}\right)}{G\left(e^{V_1}, e^{V_2}, \dots e^{V_j}\right)}$$

- The general formulation of Choice Probability of a GEV model based on the GEV generating function

$$\boxed{\Pr(j) = \frac{y_j\left(G_j(\dots)\right)}{DG(\dots)}}$$

$\rightarrow$ *D is the degree of homogeneity*

$\rightarrow$ *y_j is the argument, which is = e^{V_j} for the above*

- Expected Maximum Utility of the GEV model generated by the $G(..)$ function:

$$\boxed{\bar{U} = \frac{1}{D}\left(\ln\left(G(..)\right) + \gamma\right)}$$

γ is Euler's constant

D is the homogeneity degree defined by scale parameter

- Then:

$$\boxed{\Pr(j) = \frac{\partial \bar{U}}{\partial y_j} = \frac{y_j\left(\partial G(..)/\partial y_j\right)}{DG(..)}}$$

$\rightarrow y_j$ is the argument

Generalized Extreme Value (GEV) Model

$$\Pr(j|C_i) = \frac{y_j\left(G_j(y_1, y_2 \dots y_j \dots y_J)\right)}{DG(y_1, y_2 \dots y_j \dots y_J)}$$

- $D\ is\ degree\ of\ homogeneity$
- $y_j\ is\ the\ argument$

- GEV mode is the most flexible closed-form model that can capture a wide variety of choice models:
  - For different generating function, different choice model can arise.
  - MNL and NL are special cases of the GEV model
- Other GEV models: Network GEV, Cross-nested Logit, Paired-Combinatorial Logit etc. for the G(..) function with degree, μ

Estimation: GEV Model

- GEV models are of Closed form and so can be estimated by using the classical estimation technique: Classical Maximum Likelihood
- However, the conditions on scale (similarity or dissimilarity) parameters needs to be met.
- Constrained Maximum Likelihood (CML) method for the GEV model estimation that maintains restrictions on parameter value range while estimating
  - One can use mathematical expressions to ensure such conditions while using classical Maximum Likelihood approach
  - Classical Maximum Likelihood is less computationally intensive than the CML

Multinomial Logit as a GEV Special Case

Multinomial Logit Model

$$\Pr(j|C_i) = \frac{e^{V_j} G_j\left(e^{V_1}, e^{V_2}, \ldots e^{V_j}\right)}{G(e^{V_1}, e^{V_2}, \ldots e^{V_j})}$$

- Homogenous of degree D=1
- Argument is e^{V_j}, exponential of systematic utility with scale =1

- The case of MNL: Consider the following G(..) function as an additive function of the exponential of systematic utility functions

$$G(\ldots) = \sum_{k=1}^{J} (e^{\mu V_k})^{\frac{1}{\mu}}, \quad k\text{=1, 2, ...}J \in C_i$$

$$G_j(\ldots) = \frac{\partial}{\partial e^{V_j}} \left(\sum_{k=1}^{J} e^{V_k} \right) = 1$$

- An indirect CES structure
- Considering μ = 1

$$\therefore \Pr(j|C_i) = \frac{e^{V_j} G_j\left(e^{V_1}, e^{V_2}, \ldots e^{V_j}\right)}{G\left(e^{V_1}, e^{V_2}, \ldots e^{V_j}\right)} = \frac{\exp(V_j)}{\sum_k \exp(V_k)}$$

Nested Logit as a GEV Special Case

Nested Logit Model

- Consider J alternatives nested in k non-overlapping nests:

$$G(..) = \sum_{l=1}^{K}\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}$$

- K = total nests; B_l is the alt set in nest l
- μ_l is the nest l-specific scale & 0< μ_l <1
- $e^{\mu_l V_j}$ is the argument
- $G(..)$ is homogenous to degree $\frac{1}{\mu_l}$
- An indirect CES structure

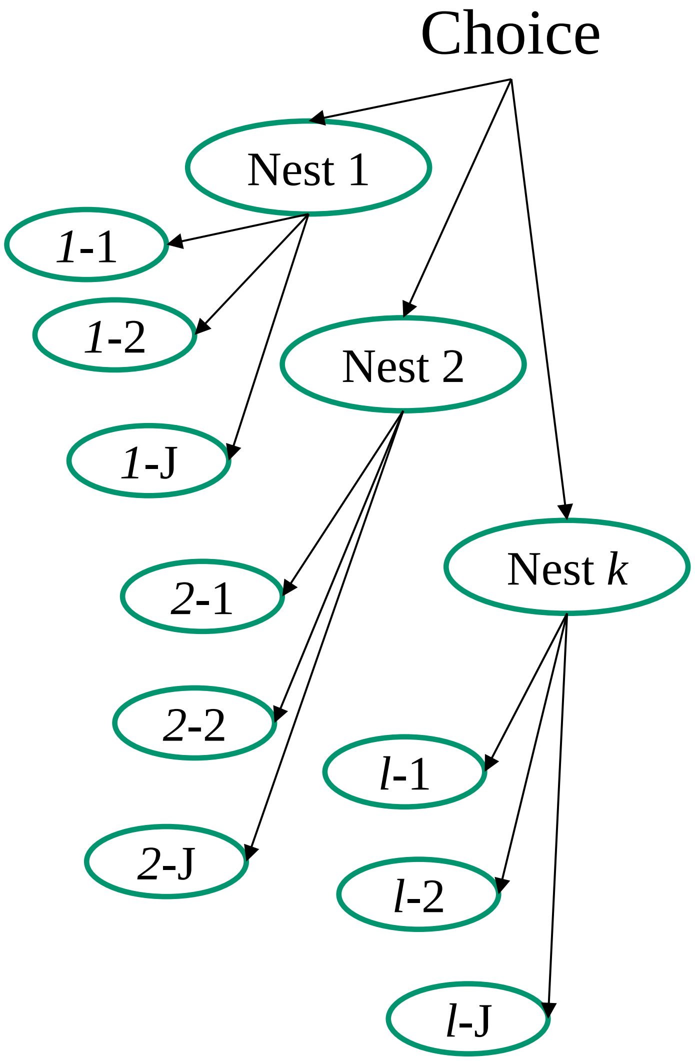
- Partial derivative

$$G_j(..) = \frac{\partial}{\partial e^{V_j}}\left(\sum_{l=1}^{K}\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}\right) \qquad = \frac{\mu_l e^{\mu_l V_j}}{\mu_l}\left(\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l - 1}\right)$$

$$\therefore \Pr(j|C_i) = \frac{e^{V_j} G_j(\ldots)}{G(\ldots)}$$

$$= \frac{e^{\mu_l V_j}\left(\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l - 1}\right)}{\sum_{l=1}^{K}\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}} \qquad = \frac{e^{\mu_l V_j}}{\sum_{j\in B_l} e^{\mu_l V_j}} \frac{\left(\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}\right)}{\sum_{l=1}^{K}\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}}$$

Nested Logit Model



$$\Pr(j|C_i) = \frac{e^{\mu_l V_j}}{\sum_{j\in B_l} e^{\mu_l V_j}} \frac{\left(\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}\right)}{\sum_{l=1}^{K}\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}}$$

$$= \frac{e^{\mu_l V_j}}{\sum_{j\in B_l} e^{\mu_l V_j}} \frac{exp\left(\ln\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}\right)}{\sum_{l=1}^{K} exp\left(\ln\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)^{1/\mu_l}\right)}$$

$$= \frac{e^{\mu_l V_j}}{\sum_{j\in B_l} e^{\mu_l V_j}} \frac{exp\left(\frac{1}{\mu_l}\ln\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)\right)}{\sum_{l=1}^{K} exp\left(\frac{1}{\mu_l}\ln\left(\sum_{j\in B_l} e^{\mu_l V_j}\right)\right)}$$

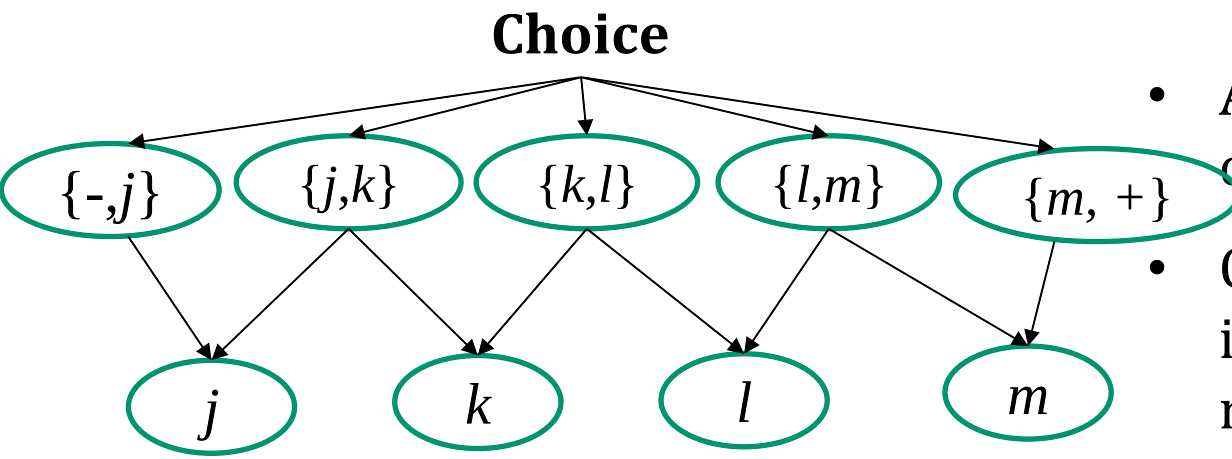
$$= \Pr(j|nest\ l)\Pr(nest\ l\ out\ of\ k\ nests)$$

$$\Pr(1-j|C_i) = \frac{e^{\mu_1 V_1}}{\sum_{j\in Nest1} e^{\mu_1 V_j}} \frac{exp\left(\frac{1}{\mu_1}\ln\left(\sum_{j\in Nest1} e^{\mu_1 V_j}\right)\right)}{exp\left(\frac{1}{\mu_1}\ln\left(\sum_{j\in Nest1} e^{\mu_1 V_j}\right)\right) + \cdots + exp\left(\frac{1}{\mu_k}\ln\left(\sum_{j\in Nestk} e^{\mu_k V_j}\right)\right)}$$

Ordered GEV Model (OGEV) Model

A Discrete Choice Model for Ordered Alternatives
Author(s): Kenneth A. Small
Source: *Econometrica*, Vol. 55, No. 2 (Mar., 1987), pp. 409-424

Standard OGEV Model



- Alternatives have natural ordering
- OGEV allows alternatives in successive overlapping nests along the order

- Classical OGEV G(.) function: $G(..) = \sum_{j=1}^{J+1} \left(\frac{1}{2} e^{\mu V_{j-1}} + \frac{1}{2} e^{\mu V_j} \right)^{1/\mu}$

  $\rightarrow$ An indirect CES structure

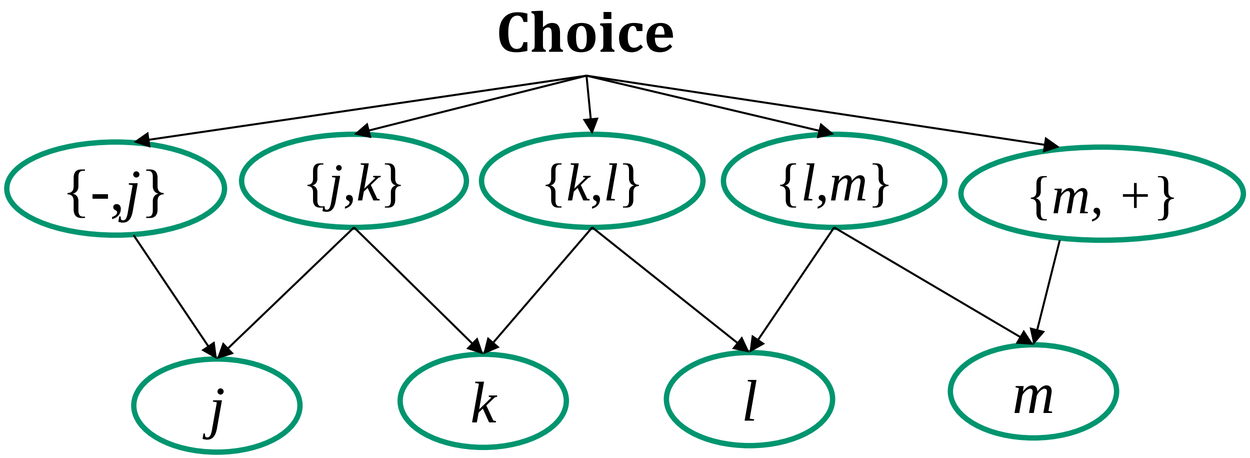
$$\Pr(j) = \frac{e^{\mu V_j} \left[\left(e^{\mu V_{j-1}} + e^{\mu V_j} \right)^{\frac{1}{\mu}-1} + \left(e^{\mu V_j} + e^{\mu V_{j+1}} \right)^{\frac{1}{\mu}-1} \right]}{\sum_{k=1}^{J+1} \left(e^{\mu V_{k-1}} + e^{\mu V_k} \right)^{1/\mu}}$$

$$e^{\mu V_k} = 0 \; for \; k < 1 \; \& \; k > J$$

- For ($\mu \rightarrow 1$) it collapses into an MNL model

- Approximate correlations between utility functions in a nest/pair: $\rho = 1 - (1/\mu)^2; \quad 0 \leq \rho \leq 1$

Standard OGEV Model: Elasticity



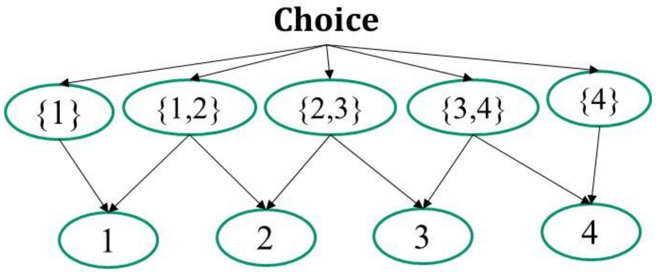
- Direct elasticity for alternative *j*:
- *k is all alternatives other than j*

$$E_D = \frac{\beta x_j}{\Pr(j)} \left(\sum_{j \neq m} \Pr(jk) \Pr(j|jk)[(1 - \Pr(j)) + (\mu - 1)(1 - \Pr(j|jk))] \right)$$

- Cross elasticity of between a pair of alternatives *j* and *k*:

$$E_C = -\beta x_j \left(\Pr(j) + \frac{(\mu - 1)\Pr(jm) \Pr(j|jm) \Pr(m|jm)}{\Pr(m)} \right)$$

OGEV Example



$$\Pr(1) = \frac{e^{\mu V_1}\left[(0 + e^{\mu V_1})^{\frac{1}{\mu}-1} + (e^{\mu V_1} + e^{\mu V_2})^{\frac{1}{\mu}-1}\right]}{\begin{pmatrix}(0 + e^{\mu V_1})^{1/\mu} + (e^{\mu V_1} + e^{\mu V_2})^{1/\mu} + (e^{\mu V_2} + e^{\mu V_3})^{1/\mu} \\ +(e^{\mu V_3} + e^{\mu V_4})^{1/\mu} + (e^{\mu V_4} + 0)^{1/\mu}\end{pmatrix}}$$

$$\Pr(3) = \frac{e^{\mu V_3}\left[(e^{\mu V_2} + e^{\mu V_3})^{\frac{1}{\mu}-1} + (e^{\mu V_3} + e^{\mu V_4})^{\frac{1}{\mu}-1}\right]}{\begin{pmatrix}(0 + e^{\mu V_1})^{1/\mu} + (e^{\mu V_1} + e^{\mu V_2})^{1/\mu} + (e^{\mu V_2} + e^{\mu V_3})^{1/\mu} \\ +(e^{\mu V_3} + e^{\mu V_4})^{1/\mu} + (e^{\mu V_4} + 0)^{1/\mu}\end{pmatrix}}$$

$$\Pr(4) = \frac{e^{\mu V_4}\left[(e^{\mu V_3} + e^{\mu V_4})^{\frac{1}{\mu}-1} + (e^{\mu V_4} + 0)^{\frac{1}{\mu}-1}\right]}{\begin{pmatrix}(0 + e^{\mu V_1})^{1/\mu} + (e^{\mu V_1} + e^{\mu V_2})^{1/\mu} + (e^{\mu V_2} + e^{\mu V_3})^{1/\mu} \\ +(e^{\mu V_3} + e^{\mu V_4})^{1/\mu} + (e^{\mu V_4} + 0)^{1/\mu}\end{pmatrix}}$$

Transportation (2007) 34:161–175
DOI 10.1007/s11116-006-0008-2

An analysis of weekend work activity patterns in the San Francisco Bay Area

Elizabeth A. Sall · Chandra R. Bhat

Table 3 Weekend work start time of day results[4]

Log-likelihood values: at convergence: –1,373.0; Constants only: –1,638.9; Equal shares: –1749.5
Rho-squared with respect to constant model: 0.162
Rho-squared with respect to equal shares model: 0.215 Number of observations: 1,087

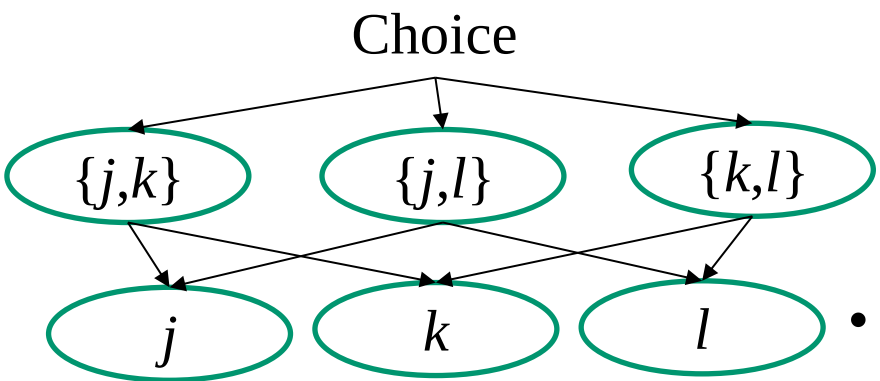
Variable	Regular morning (7–8:59 AM)		Late morning (9–11:59 AM)		Afternoon (12–3:59 PM)		Evening (4 PM-2:59 AM)	
	Coeff.	*t*-stat	Coeff.	*t*-stat	Coeff.	*t*-stat	Coeff.	*t*-stat
Constant	2.2911	5.447	3.9599	8.825	3.4795	6.548	3.8549	6.795
Work location and duration								
Work at home	–	–	0.5567	2.392	0.9213	3.471	1.5802	4.967
Work duration	–0.0029	–4.141	–0.0068	–8.966	–0.0091	–10.901	–0.0136	–12.216
Day-of-week								
Sunday	–	–	–	–	0.3048	1.671	1.2033	4.660
Individual demographic variables								
Female	0.4482	2.428	0.4103	2.200	0.4115	1.914	–	–
Race								
African-American	–	–	0.6833	2.140	–	–	–	–
Asian	0.5427	2.492	–	–	–	–	–	–
Son/daughter of household head	–	–	–	–	–	–	2.020	5.183
Work-related variables								
Private, non-profit, firm	0.6391	2.685	–	–	–	–	–	–
Self-employed	0.9155	3.216	0.7648	2.729	0.6369	2.079	–	–
Location variables								
County of residence								
San Mateo	–	–	0.9183	3.700	0.7305	1.889	–	–
Santa Clara	–	–	0.5517	2.575	0.8872	2.847	0.620	1.992
Alameda	–0.3089	–1.481	–	–	0.5974	1.970	–	–
Contra Costa	–	–	0.6979	2.806	0.7705	2.150	–	–
Solano	–1.0550	–3.194	–1.0453	–2.471	–	–	0.9671	2.028
Napa	–0.6427	–1.573	–	–	–	–	–	–
Sonoma	–0.6668	–2.362	–	–	0.5826	1.620	–	–
Density-based area type of residence								
Suburban/rural	–	–	–0.6776	–3.499	–0.4447	–1.698	–0.6470	–1.914

- Work start time-of-day choice is modelled as an OGEV model
- Time-of-day is divided into 4 discrete choices
- The model does not have any time-of-day specific attributes though
- Authors should consider variables (e.g. congestion level, store opening hours, etc.) that distinguishes alternatives of time-of-day

Paired Combinatorial Logit (PCL) Model

Koppelman and Wen. (2000) The paired combinatorial logit model: properties, estimation and application. Transportation Research Part B 34 (2000) 75-89

Paired Combinatorial Logit (PCL)



- Pair-wise nests
- Discrete choice

Allows overlapping nests: one alternative in more than 2 nests

- Generating function: $G(..) = \sum_{j=1}^{J-1} \sum_{k=j+1}^{J} \left(e^{\mu_{jk}V_j} + e^{\mu_{jk}V_k}\right)^{1/\mu_{jk}}$

$$\Pr(j) = \sum_{j \neq k} \Pr(j|\{j,k\}) \Pr\{j,k\}$$

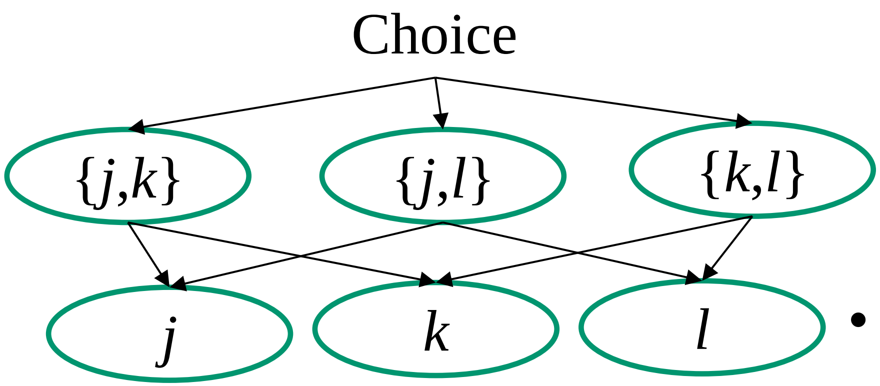
→ *An indirect CES structure*

$$\Pr(j|\{j,k\}) = \frac{\exp(\mu_{jk}V_j)}{\exp(\mu_{jk}V_j) + \exp(\mu_{jk}V_k)}$$

$$\Pr\{j,k\} = \frac{\left(\exp(\mu_{jk}V_j) + \exp(\mu_{jk}V_k)\right)^{1/\mu_{jk}}}{\sum_{m=1}^{J-1} \sum_{p=m+1}^{J} \left(\exp(\mu_{mp}V_m) + \exp(\mu_{mp}V_p)\right)^{1/\mu_{mp}}}$$

- Approximate correlations between utility functions in a nest/pair: $\rho_{jk} = 1 - \left(1/\mu_{jk}\right)^2; \qquad 0 \leq \rho_{jk} \leq 1$

PCL Model: Elasticities



- Pair-wise nests
- Discrete choice

Allows overlapping nests: one alternative in more than 2 nests

- Direct elasticity for alternative *j*:

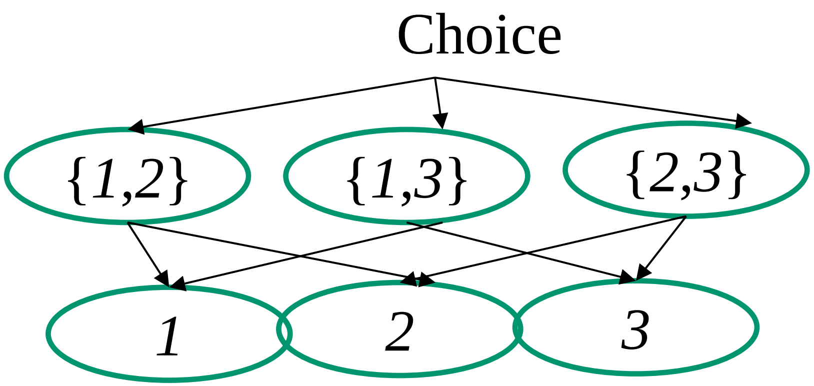
- *k is all alternatives other than j*

$$E_D = \left\{ \frac{\sum_{j \neq k} \Pr(jk) \Pr(j|jk) \left[(1 - \Pr(j)) + (\mu_{jk} - 1)(1 - \Pr(j|jk)) \right]}{\Pr(j)} \right\} \beta x_j$$

- Cross elasticity of between a pair of alternatives *j* and *k*:

$$E_C = -\left\{ \Pr(j) + \frac{(\mu_{jk} - 1)\Pr(jk) \Pr(j|jk) \Pr(k|jk)}{\Pr(k)} \right\} \beta x_j$$

PCL Model-Example



$$\Pr(1) = \Pr(1|\{1,2\})\Pr\{1,2\} + \Pr(1|\{1,3\})\Pr\{1,3\}$$

$$\Pr(2) = \Pr(2|\{1,2\})\Pr\{1,2\} + \Pr(2|\{2,3\})\Pr\{2,3\}$$

$$\Pr(3) = \Pr(3|\{1,3\})\Pr\{1,3\} + \Pr(3|\{2,3\})\Pr\{2,3\}$$

$$\Pr(1|\{1,2\}) = \frac{\exp(\mu_{12}V_1)}{\exp(\mu_{12}V_1) + \exp(\mu_{12}V_2)}$$

$$\Pr\{1,2\} = \frac{\left(\exp(\mu_{12}V_1) + \exp(\mu_{12}V_2)\right)^{1/\mu_{12}}}{\left(\begin{array}{c}\left(\exp(\mu_{12}V_1) + \exp(\mu_{12}V_2)\right)^{1/\mu_{12}} + \left(\exp(\mu_{13}V_1) + \exp(\mu_{13}V_2)\right)^{1/\mu_{13}} \\ +\left(\exp(\mu_{23}V_2) + \exp(\mu_{23}V_3)\right)^{1/\mu_{23}}\end{array}\right)}$$

Transportation Research Part B 34 (2000) 75–89

The paired combinatorial logit model: properties, estimation and application

Frank S. Koppelman [a,b,*], Chieh-Hua Wen [c,1]

The estimation results of the MNL, NL and PCL models

Variables	Estimated parameters (standard errors)					
	MNL model	PCL models			NL models	
		Train–car similarity	Air–car similarity	Train–car and air–car similarity	Train–car nested	Air–car nested
	1	2	3	4	5	6
Mode constants						
Air	3.6643	2.9361	2.7558	1.9603	0.2148	2.6830
	(0.431)	(0.321)	(0.398)	(0.396)	(0.416)	(0.496)
Train	1.6728	1.7782	1.2588	1.2942	1.6385	1.2909
	(0.225)	(0.135)	(0.204)	(0.164)	(0.180)	(0.240)
Car (base)						
Frequency	0.0944	0.0926	0.0683	0.0651	0.0935	0.0733
	(0.005)	(0.004)	(0.005)	(0.005)	(0.004)	(0.007)
Travel cost	−0.0461	−0.0388	−0.0311	−0.0242	−0.0428	−0.0317
	(0.004)	(0.003)	(0.004)	(0.004)	(0.004)	(0.006)
In-vehicle time	−0.0099	−0.0084	−0.0083	−0.0076	−0.0093	−0.0093
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Out-of-vehicle time	−0.0426	−0.0387	−0.0358	−0.0321	−0.0400	−0.0400
	(0.003)	(0.002)	(0.003)	(0.002)	(0.003)	(0.003)
Similarity parameters						
Train–car	–	0.8788	–	0.5791	0.1768	–
		(0.055)		(0.087)	(0.061)	
Air–car	–	–	07057	0.7342	–	0.2585
			(0.082)	(0.081)		(0.069)
Log-likelihood						
At convergence	−1919.8	−1912.1	−1911.7	−1903.9	−1917.4	−1914.5
At market share	−2837.1	−2837.1	−2837.1	−2837.1	−2837.1	−2837.1
At zero	−3042.1	−3042.1	−3042.1	−3042.1	−3042.1	−3042.1
L'hood ratio index						
versus market share	0.3233	0.3260	0.3262	0.3289	0.3242	0.3252
versus zero	0.3689	0.3715	0.3716	0.3741	0.3697	0.3707
Value of time						
In-vehicle time	C$13	C$13	C$16	C$19	C$13	C$16
Out-of-vehicle time	C$55	C$60	C$69	C$80	C$56	C$69
Log likelihood test versus MNL		15.4 > 3.8	16.2 > 3.8	31.8 > 6.0	4.8 > 3.8	10.6 > 3.8
Implied utility correlation						
Train–car	0.000	0.448	0.000	0.385	0.322	0.000
Air–car	0.000	0.000	0.422	0.429	0.000	0.450

- Inter-city mode choice for Toronto-Montreal
- PCL model better captures pair-wise competition between different modes

Direct-elasticity of train ridership in response to improvements in train service

Train level of service attribute	MNL model	NL model with train–car nested	NL model with air–car nested	Preferred PCL model
Part a: Elasticity values				
Frequency	0.333	0.401	0.260	0.312
Cost	2.068	2.331	1.427	1.473
In-vehicle time	1.788	2.039	1.685	1.862
Out-of-vehicle time	2.927	3.336	2.757	2.993
Part b: Ratio of MNL and NL elasticities to PCL elasticities				
Frequency	1.067	1.285	0.833	1.000
Cost	1.404	1.582	0.969	1.000
In-vehicle time	0.960	1.095	0.905	1.000
Out-of-vehicle time	0.978	1.115	0.921	1.000

Transportation Research Record 1857
Paper No. 03-3395

Solving the Overlapping Problem in Route Choice with Paired Combinatorial Logit Model

Anthony Chen, Panatda Kasikitwiwat, and Zhaowang Ji

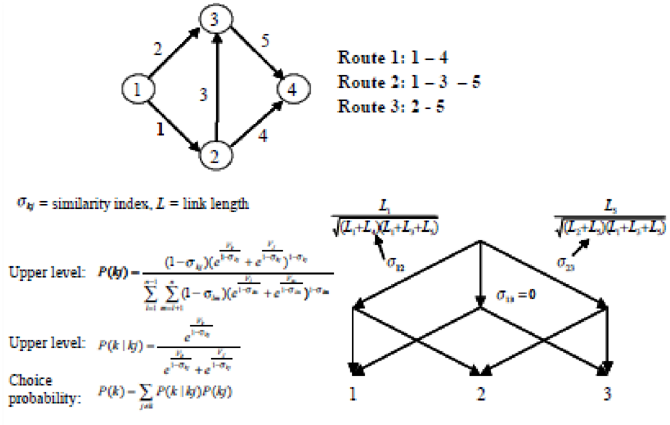


FIGURE 1 Tree representation of PCL for a simple network (see Equations 4 and 5 (L = link length).

PARTIAL LINEARIZATION ALGORITHM

Notations

n = iteration counter,
x_a = flow on link a,
y_a = auxiliary flow on link a,
$t_a(x_a)$ = travel time on link a,
q_{rs} = demand between origin r and destination s,
δ_{ka}^{rs} = path-link incidence indicator,
$\bar{k}_{rs}$ = shortest path between origin r and destination s,
K_{rs} = path set between origin r and destination s,
f_k^{rs} = flow on route k from origin r and destination s,
h_k^{rs} = auxiliary flow on route k from origin r and destination s,
α = step size,
ϵ = tolerance, and
E = maximum percentage change in link flow.

Initialization

Generate an initial path for each O-D pair

1. Set $x_a(0)=0$, $t_a(0)=t_a[x_a(0)]$, $\forall a$, and $K_{rs}(0)=\phi$, $\forall r, s$.
2. Set iteration counter $\Rightarrow n=1$.
3. Solve the shortest path problem for all origins $\bar{k}_{rs}(n)$

$$K_{rs}(n) = K_{rs}(n-1) \cup \bar{k}_{rs}(n), \forall r, s.$$

4. Perform all-or-nothing traffic assignment $\Rightarrow f_{\bar{k}rs}^{rs}(n) = q_{rs}$, $\forall \bar{k}_{rs}(n), r, s$.
5. Assign path flows to links $\Rightarrow x_a(n) = \sum_{r\in R}\sum_{s\in S}\sum_{k\in K_{rs}(n)} f_k^{rs}(n)\delta_{ka}^{rs}, \forall a$.

PCL SUE Solver

6. Increment iteration counter $\Rightarrow n = n+1$.
7. Update link travel time $\Rightarrow t_a(n) = t_a[x_a(n-1)]$, $\forall a$.
8. Solve shortest path problem $\Rightarrow \bar{k}_{rs}(n)$, $\forall r, s$.
9. Determine whether $\bar{k}_{rs}(n)$ exists in the path set $K_{rs}(n-1)$ or not $\Rightarrow$.

If $\bar{k}_{rs}(n) \notin K_{rs}(n-1)$, then $K_{rs}(n) = \bar{k}_{rs}(n) \cup K_{rs}(n-1)$.

Otherwise, tag the shortest path among the paths in $K_{rs}(n-1)$ as $\bar{k}_{rs}(n)$ and set $K_{rs}(n) = K_{rs}(n-1)$.

10. Update route costs $c_k^{rs}(n)\sum_a t_a(n)\delta_{ka}^{rs}$, $\forall k \in K_{rs}(n), r, s$.
11. Compute similarity index between route k and j,

$$\sigma_{kj}^{rs} = \frac{L_{kj}^{rs}}{\sqrt{L_k^{rs}\times L_j^{rs}}} \qquad \forall k = 1, \ldots, K_{rs}(n)-1, \quad j = k+1, \ldots, K_{rs}(n), r, s$$

where L_{kj}^{rs} is the length of the common part of route k and j, then set $\beta_{kj}^{rs} = 1-\sigma_{kj}^{rs}$.

12. Compute route choice probabilities and auxiliary route flows.

$$P_{kj}^{rs}(n) = \frac{\beta_{kj}^{rs}\left\{\exp\left[-\frac{\theta c_k^{rs}(n)}{\beta_{kj}}\right] + \exp\left[-\frac{\theta c_j^{rs}(n)}{\beta_{kj}}\right]\right\}^{\beta_{kj}^{rs}}}{\sum_{m=1}^{|K_{rs}(n)|-1}\sum_{l=m+1}^{|K_{rs}(n)|}\left\{\exp\left[-\frac{\theta c_m^{rs}(n)}{\beta_{ml}^{rs}}\right] + \exp\left[-\frac{\theta c_l^{rs}(n)}{\beta_{ml}^{rs}}\right]\right\}^{\beta_{ml}^{rs}}}$$

$$P_{k|(j)}^{rs}(n) = \frac{\exp\left(-\frac{\theta c_k^{rs}(n)}{\beta_{kj}^{rs}}\right)}{\exp\left(-\frac{\theta c_k^{rs}(n)}{\beta_{kj}^{rs}}\right) + \exp\left(-\frac{\theta c_j^{rs}(n)}{\beta_{kj}^{rs}}\right)}$$

$$P_k^{rs}(n) = \sum_{j\neq k} P_{kj}^{rs}(n) \times P_{k|(kj)}^{rs}(n) \qquad \forall k, r, s$$

$$h_k^{rs}(n) = P_k^{rs}(n) \times q_{rs} \qquad \forall k, r, s$$

13. Assign path flows to links $\Rightarrow y_a(n) = \sum_{r\in R}\sum_{s\in S}\sum_{k\in K_{rs}(n)} h_k^{rs}\delta_{ka}^{rs}, \forall a$.
14. Perform line search.
   - Set $\alpha(n) = 1/n$
   - $\min_{0\le\alpha\le1} Z[x+\alpha(y-x)]$
   - $\min_{0\le\alpha\le1} Z_1[x+\alpha(y-x)] + Z_2^{fisk}[f+\alpha(h-f)]$
   - $\min_{0\le\alpha\le1} Z_1[x+\alpha(y-x)] + Z_2[f+\alpha(h-f)]$
     $+ Z_3[f+\alpha(h-f)]$
15. Update path flow $\Rightarrow f_k^{rs}(n) = f_k^{rs}(n-1) + \alpha(n)[h_k^{rs}(n) - f_k^{rs}(n-1)]$, $\forall k, r, s$.
16. Assign route flows to links $\Rightarrow x_a(n) = \sum_{r\in R}\sum_{s\in S}\sum_{k\in K_{rs}(n)} f_k^{rs}(n)\delta_{ka}^{rs}$, $\forall a$.
17. Determine maximum percent change of link flow difference $\Rightarrow$

$$E = \max_a\left|\frac{x_a(n)-x_a(n-1)}{x_a(n-1)}\right| \times 100.$$

18. Check convergence $\Rightarrow E \le \epsilon$, then stop, otherwise go to Step 6.

- PCL model is used in a Stochastic User Equilibrium (SUE) Assignment using this algorithm

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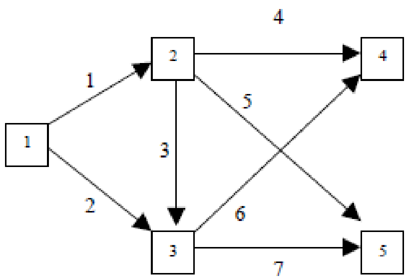


	Link	Link Length	Capacity
1	12	4	25
2	13	5.2	25
3	23	1	15
4	24	5	15
5	25	5	15
6	34	4	15
7	35	4	15

$$t_a(x_a) = \alpha_a + \beta_a x_a$$

TABLE 2 Route Results of UE, MNL, and PCL Models for Network 1

O-D	Path	Link Sequence	Average Similarity Index	Route Choice Probability		Route Flow			Route Travel Time		
				MNL	PCL	UE	MNL	PCL	UE	MNL	PCL
(1-4)	1	1-4	0.2222	0.3474	0.3993	9.723	6.948	7.983	9.685	9.896	9.772
	2	**1-3-6**	**0.442**	**0.2455**	**0.1696**	**0.947**	**4.909**	**3.396**	**9.685**	**10.243**	**9.967**
	3	2-6	0.2198	0.4072	0.4311	9.330	8.143	8.621	9.685	9.737	9.702
(1-5)	1	2-7	0.2222	0.3915	0.4295	12.152	9.787	10.737	9.876	9.997	9.907
	2	**1-3-7**	**0.442**	**0.2289**	**0.1583**	**1.672**	**5.722**	**3.955**	**9.876**	**10.534**	**10.029**
	3	1-5	0.2198	0.3797	0.4122	11.176	9.491	10.308	9.876	10.028	9.945

TABLE 3 Link Results of UE, MNL, and PCL Models for Network 1

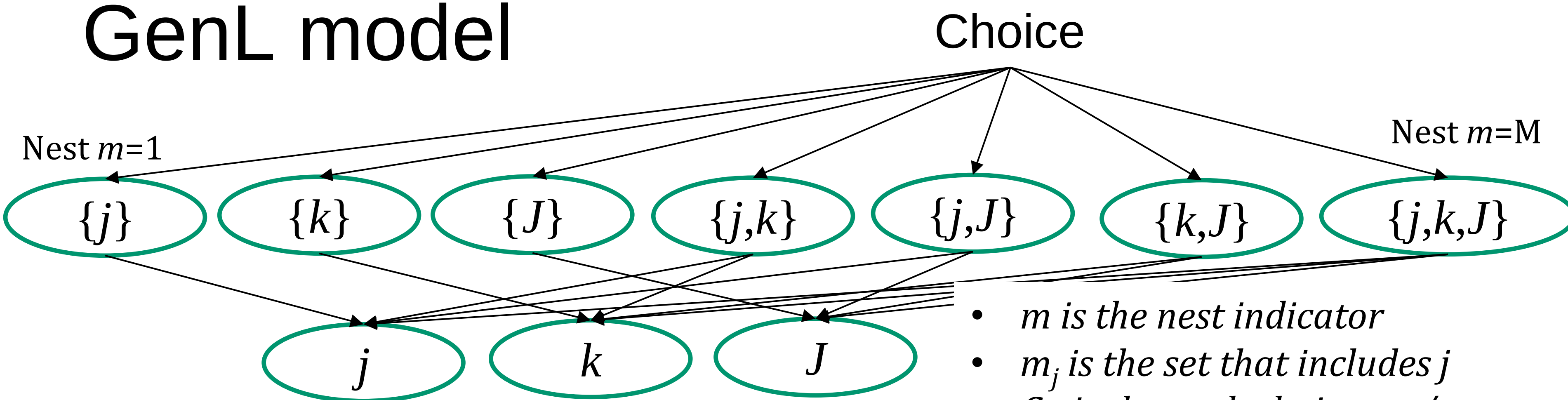
Link Number	Link Flow			Link Cost			V/C Ratio		
	UE	MNL	PCL	UE	MNL	PCL	UE	MNL	PCL
1	24.494	27.365	26.085	4.553	4.861	4.711	0.980	1.095	1.043
2	20.506	17.635	18.914	5.553	5.393	5.456	0.820	0.705	0.757
3	2.620	10.631	7.353	1.000	1.038	1.009	0.175	0.709	0.490
4	9.722	6.948	7.988	5.132	5.035	5.060	0.648	0.463	0.533
5	12.152	9.787	10.745	5.323	5.136	5.197	0.810	0.652	0.716
6	10.278	13.052	12.011	4.132	4.344	4.247	0.685	0.870	0.801
7	12.848	15.213	14.255	4.323	4.635	4.489	0.856	1.014	0.950

- PCL model allows handling complicated correlated route choice in congested situation that outperforms MNL-based SUE

Generalized Logit (GenL) Model

Joffre Swait (2001). Choice set generation within GEV discrete choice model. Transport Research Part B 35: 643-666

GenL model



- *m is the nest indicator*
- *m_j is the set that includes j*
- *C_m is the m-th choice-set/nest*

Generating function: $$G(..) = \sum_{m=1}^{M}\left(\sum_{j\in C_m}\left(e^{\mu_m V_j}\right)\right)^{\mu/\mu_m}$$

condition: $\mu_m > \mu$

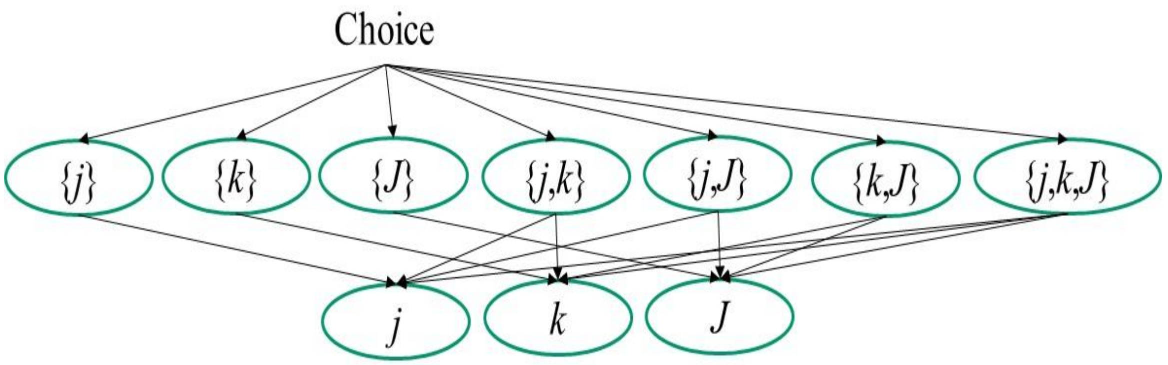
$$\Pr(j) = \sum_{m\in m_j} \Pr(j|C_m)\Pr(C_m)$$

- μ is the root scale
- μ_m is the nest-specific scale

$$\Pr(j|C_m) = \frac{\exp(\mu_m V_j)}{\sum_{j\in C_m}\left(e^{\mu_m V_j}\right)} \quad \Pr(C_m) = \frac{\exp(\mu I_m)}{\sum_{r=1}^{M}\exp(\mu I_r)} \quad I_m = \frac{1}{\mu_m}\ln\left(\sum_{j\in C_m}\left(e^{\mu_m V_j}\right)\right)$$

- Approximate nest-specific correlations: $$\rho_m = 1 - \left({}^{\mu}/_{\mu_m}\right)^2; \quad 0 \le \rho_m \le 1$$

GenL Model Elasticities



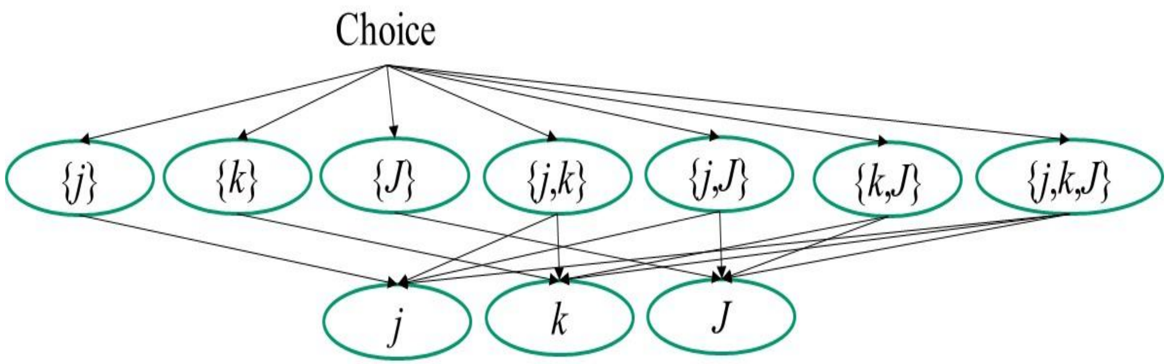
- Direct elasticity for alternative *j* (when *j* is not in any nest:

$$E_D = (\Pr(j))\beta x_j$$

- Cross elasticity between *j* & *k* when they not in any nests:

$$E_C = -\Pr(j)\,\beta x_j$$

GenL Model Elasticities



- *j,k are alternatives*
- *m is the nest indicator*
- *C_m is the m-th choice-set/nest*
- *k indicates all alternative <u>excluding</u> j*

- *Subscript m_j means all nests that have j*
- *Subscript m_k means all nests that have k*
- *Subscript m_{jk} means all nests that have j & k*

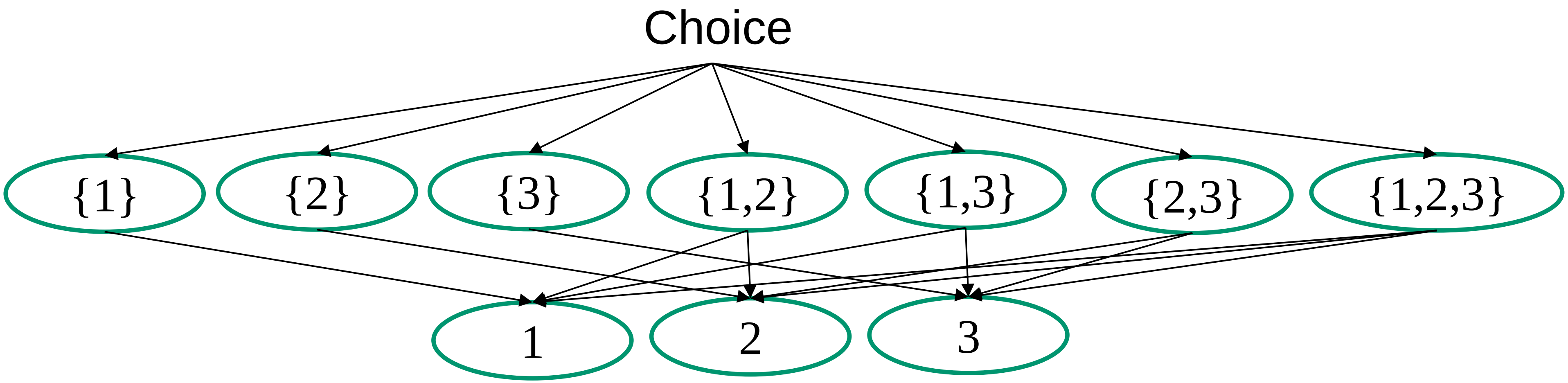
- Direct elasticity for alternative *j* (when j in one or more nests):

$$E_D = \mu\beta x_j \sum_{m\in m_j}\left(\frac{\Pr(C_m)\Pr(j|C_m)}{\Pr(j)}\left[\frac{\mu_m}{\mu} + \Pr(j|C_m)\left(\frac{\mu-\mu_m}{\mu}\right) - \sum_{r\in m_k}\Pr(j|C_r)\Pr(C_r)\right]\right)$$

- Cross elasticity of choice of *j* with respect to any variable *(x_k)* in other alternatives, *k* that is in one or more nests with *j*:

$$E_C = -\mu\beta x_k \sum_{m\in m_{jk}}\left(\frac{\Pr(C_m)\Pr(j|C_m)}{\Pr(j)}\left[\Pr(k|C_m)\left(\frac{\mu_m-\mu}{\mu}\right) + \sum_{r\in m_k}\Pr(k|C_r)\Pr(C_r)\right]\right)$$

GenL model-Example



$$\Pr(1) = \Pr(1|\{1\})\Pr(\{1\}) + \Pr(1|\{1,2\})\Pr(\{1,2\}) + \Pr(1|\{1,3\})\Pr(\{1,3\}) + \Pr(1|\{1,2,3\})\Pr(\{1,2,3\})$$

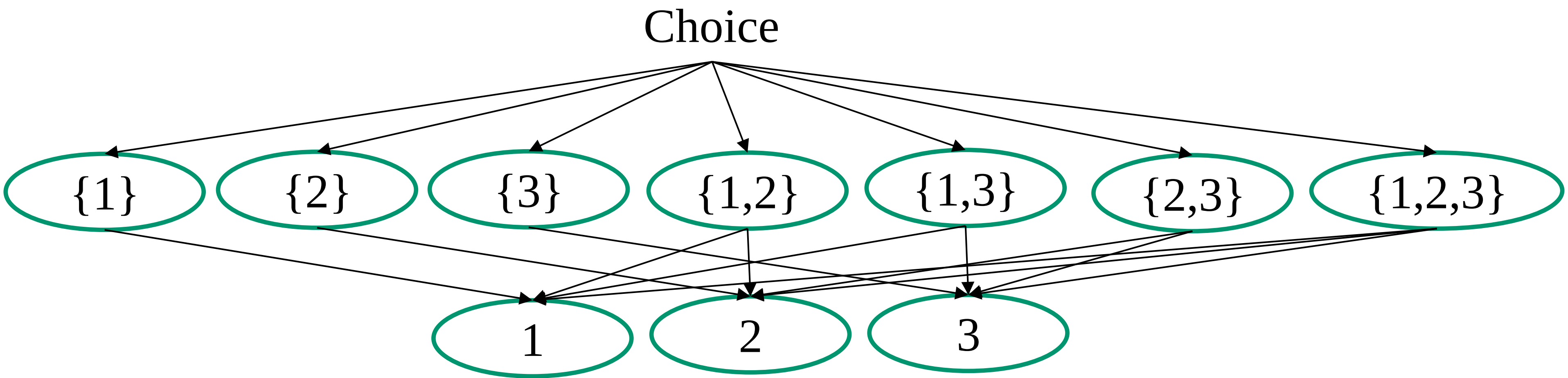
$$\Pr(1|\{1\}) = 1$$

$$\Pr(1|\{1,2\}) = \frac{\exp(\mu_{12}V_1)}{\exp(\mu_{12}V_1) + \exp(\mu_{12}V_2)}$$

$$\Pr(1|\{1,3\}) = \frac{\exp(\mu_{13}V_1)}{\exp(\mu_{13}V_1) + \exp(\mu_{13}V_3)}$$

$$\Pr(1|\{1,2,3\}) = \frac{\exp(\mu_{123}V_1)}{\exp(\mu_{123}V_1) + \exp(\mu_{123}V_2) + \exp(\mu_{123}V_3)}$$

GenL model-Example



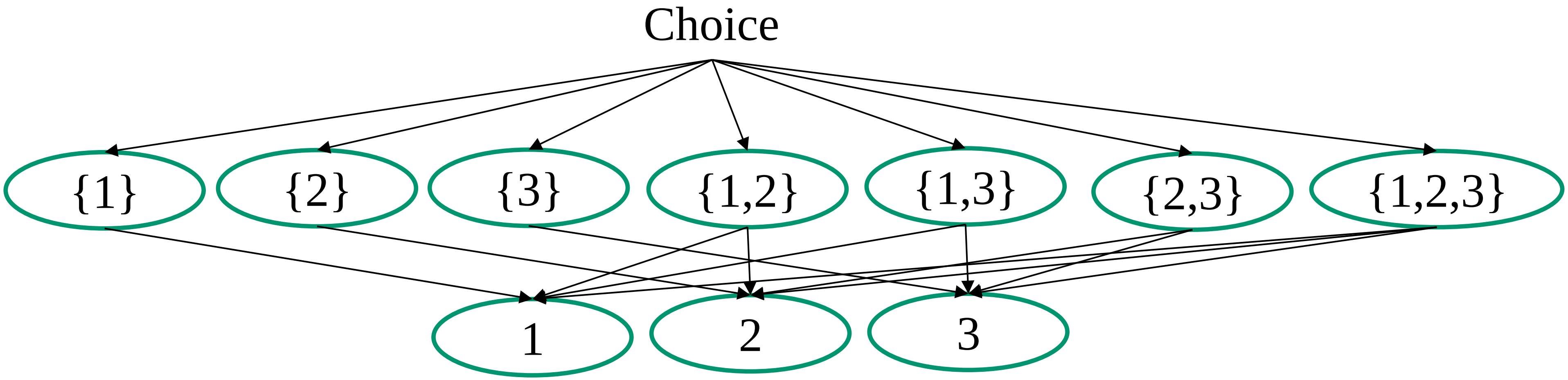
$$I_1 = \frac{1}{\mu}\ln(\exp(\mu V_1)) = V_1$$

$$I_{1,2} = \frac{1}{\mu_{12}}\ln(\exp(\mu_{12}V_1) + \exp(\mu_{12}V_2))$$

$$I_{1,3} = \frac{1}{\mu_{13}}\ln(\exp(\mu_{13}V_1) + \exp(\mu_{13}V_3))$$

$$I_{1,2,3} = \frac{1}{\mu_{123}}\ln(\exp(\mu_{123}V_1) + \exp(\mu_{123}V_2) + \exp(\mu_{123}V_3))$$

GenL model-Example



$$\Pr(\{1\}) = \frac{\exp(\mu I_1)}{\exp(\mu I_1) + \exp(\mu I_2) + \exp(\mu I_3) + \exp(\mu I_{1,2}) + exp(\mu I_{2,3}) + \exp(\mu I_{1,3}) + \exp(\mu I_{1,2,3})}$$

$$\Pr(\{1,2\}) = \frac{\exp(\mu I_{1,2})}{\exp(\mu I_1) + \exp(\mu I_2) + \exp(\mu I_3) + \exp(\mu I_{1,2}) + exp(\mu I_{2,3}) + \exp(\mu I_{1,3}) + \exp(\mu I_{1,2,3})}$$

$$\Pr(\{1,3\}) = \frac{\exp(\mu I_{1,3})}{\exp(\mu I_1) + \exp(\mu I_2) + \exp(\mu I_3) + \exp(\mu I_{1,2}) + exp(\mu I_{2,3}) + \exp(\mu I_{1,3}) + \exp(\mu I_{1,2,3})}$$

$$\Pr(\{1,2,3\}) = \frac{\exp(\mu I_{1,2,3})}{\exp(\mu I_1) + \exp(\mu I_2) + \exp(\mu I_3) + \exp(\mu I_{1,2}) + exp(\mu I_{2,3}) + \exp(\mu I_{1,3}) + \exp(\mu I_{1,2,3})}$$

Transportation Research Part A 50 (2013) 15–32

Modelling departure time choices by a Heteroskedastic Generalized Logit (Het-GenL) model: An investigation on home-based commuting trips in the Greater Toronto and Hamilton Area (GTHA)

Ana Sasic, Khandker Nurul Habib *

Department of Civil Engineering, University of Toronto, Canada

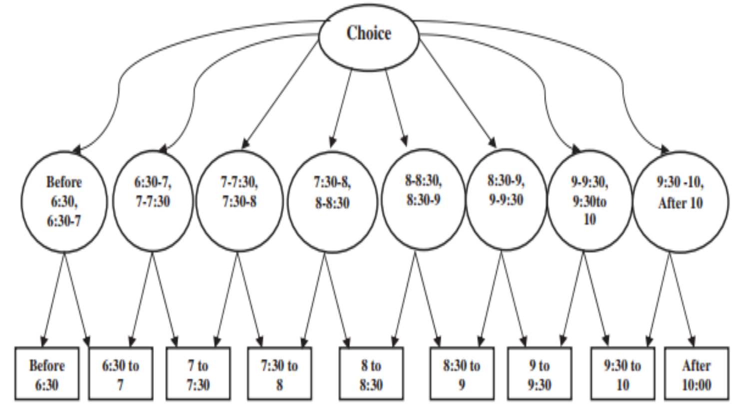
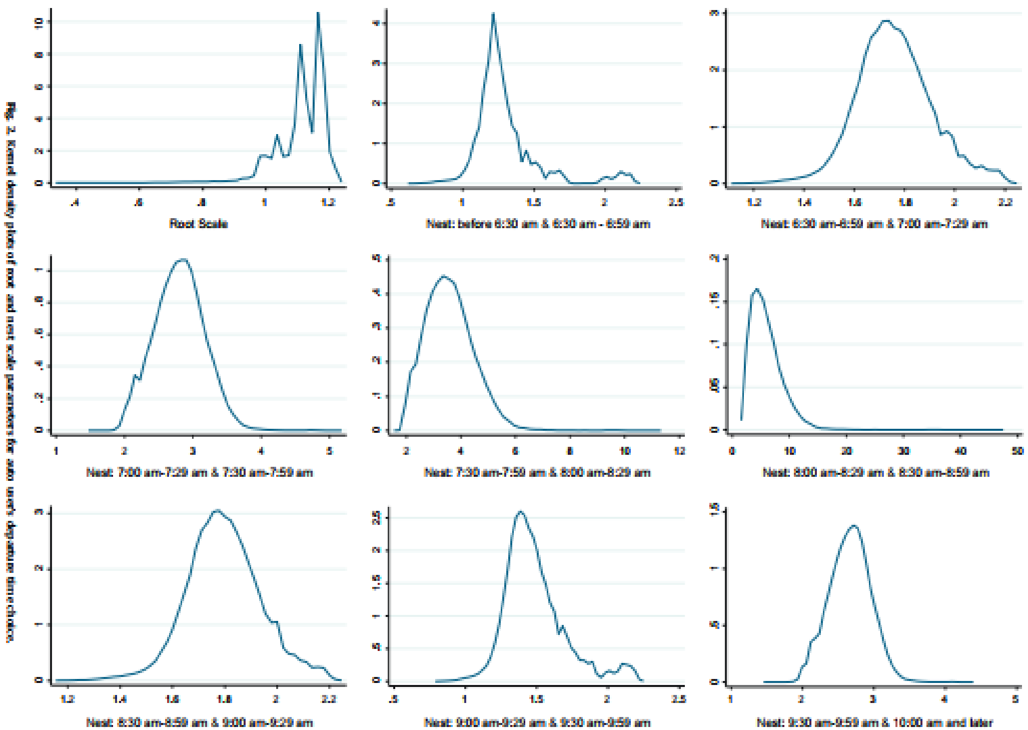
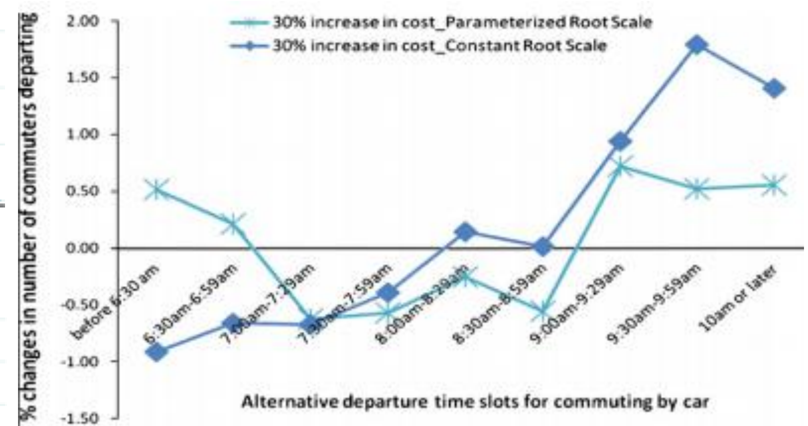
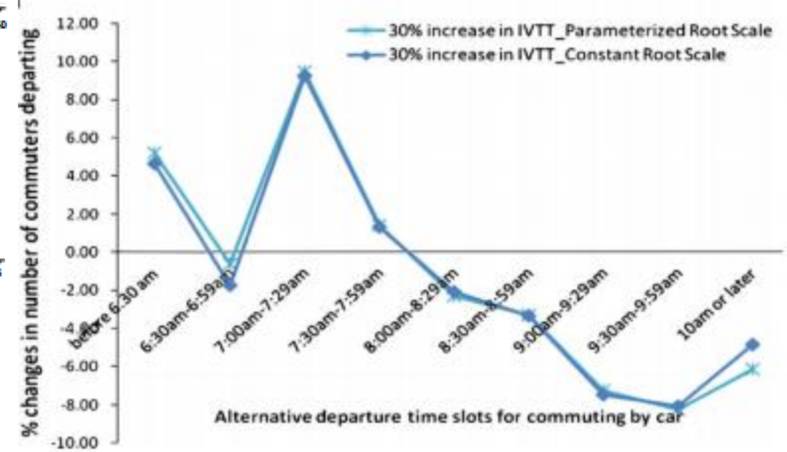


Fig. 7. Departure time choice framework in the Het-GEV model (Sasic and Habib, 2013).







Transportation Research Part A 94 (2016) 411–430

Time-dependent congestion pricing system for large networks Integrating departure time choice, dynamic traffic assignment and regional travel surveys in the Greater Toronto Area

Aya Aboudina Ph.D. [a,b,*], Hossam Abdelgawad Ph.D., P.Eng. [a,b,*], Baher Abdulhai
Khandker Nurul Habib Ph.D, P.Eng [a]

[a] Department of Civil Engineering, University of Toronto, M5S 1A4, Canada
[b] Cairo University, Faculty of Engineering, 12631 Giza, Egypt

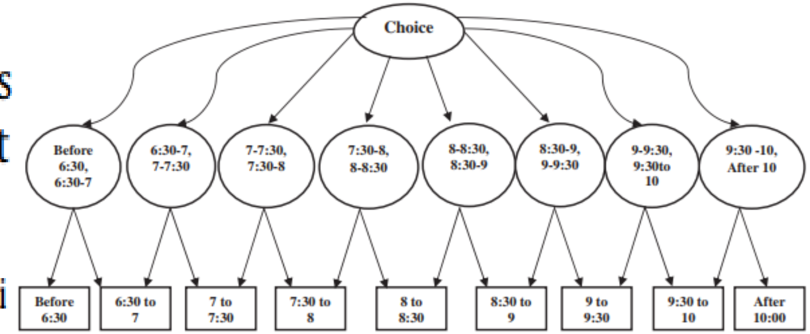
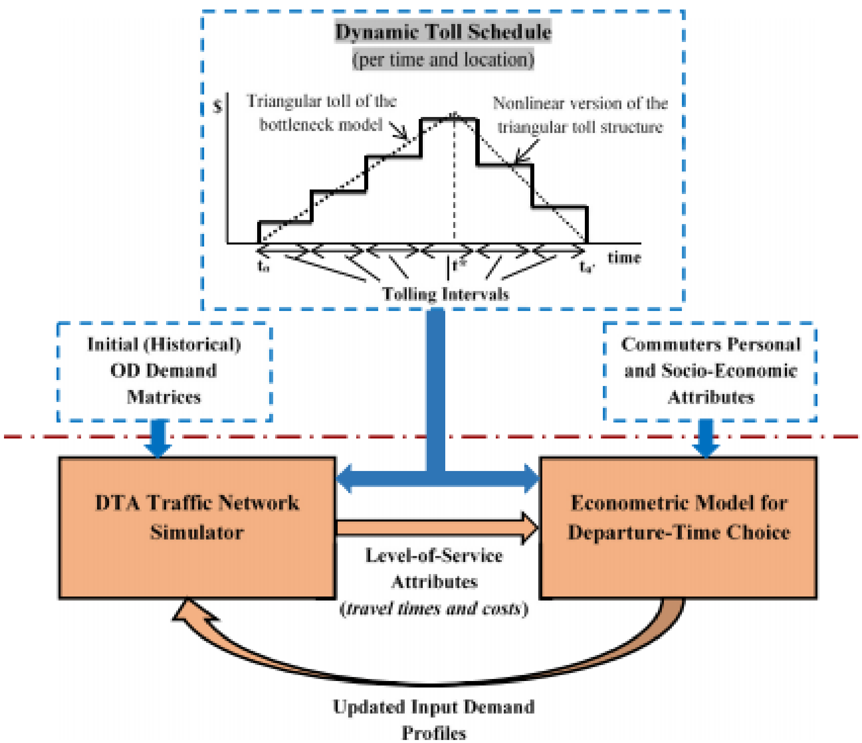
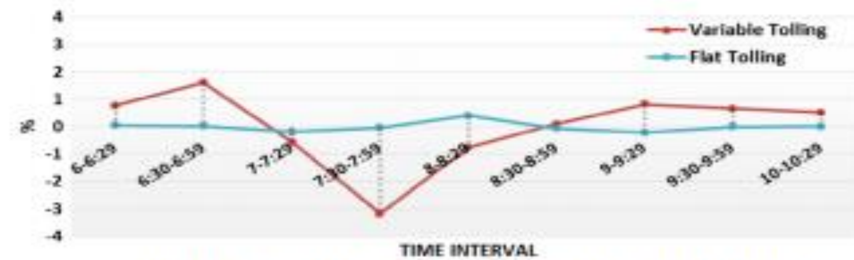
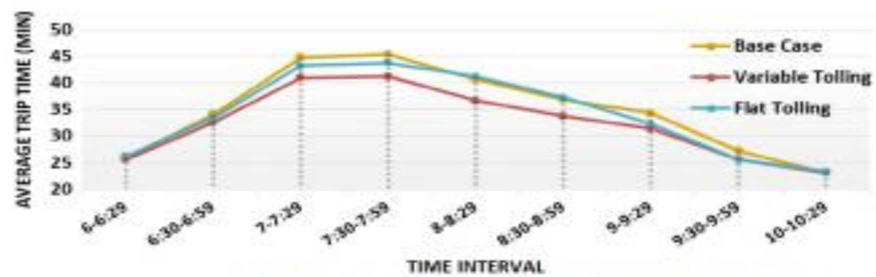


Fig. 7. Departure time choice framework in the Het-GEV model (Sasic and Habib, 2013).

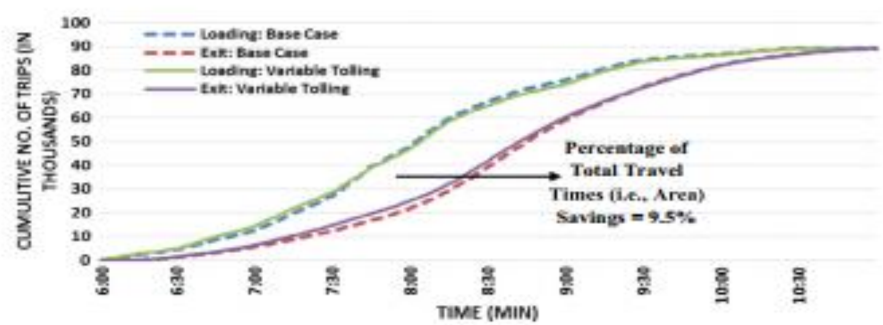




a) The Percentage of Trips Shifted (from or to) Each Time Interval



b) Average Travel Time among Trips Started at Each Time Interval

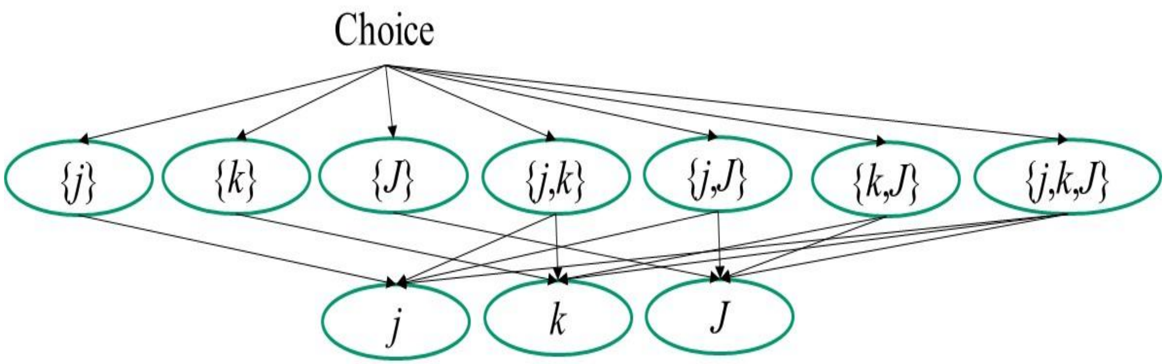


c) Loading and Exit Curves of Trips Travelling Through the GE Corridor after Variable Tolling

Generalized Nested Logit (GNL) Model
Cross-Nested Logit (CNL) Model

Koppelman and Wen. (2001) The generalized nested logit model. Transportation Research Part B 35 (2001) 627-641

Generalized Nested Logit (GNL) Model



- *All feasible nests*
- *However, alternatives may have different weights (allocation parameter,α) in different nests*

- Generating function: $G(..) = \sum_{nest,\, m=1}^{total\ nests, M} \left(\sum_{j \in m} \left(\alpha_{jm} e^{\mu_m V_j} \right)^{1/\mu_m} \right)$

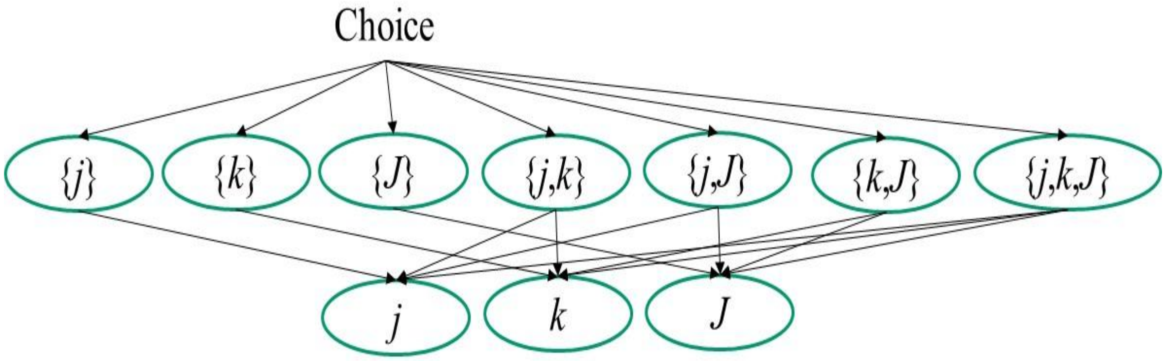
Allocation parameter: $\alpha_{jm} \geq 0$ & $\sum_{m=1}^{M} \alpha_{jm} = 1$

$$\Pr(j) = \sum_{j \in m} \Pr(j|m) \Pr(m)$$

$$\Pr(j|m) = \frac{\alpha_{jm} \exp(\mu_m V_j)}{\sum_{k \in m} \alpha_{km} \exp(\mu_m V_k)}$$

$$\Pr(m) = \frac{\left(\sum_{j \in m} \alpha_{jm} \exp(\mu_m V_k) \right)^{1/\mu_m}}{\sum_{m=1}^{M} \left(\left(\sum_{j \in m} \alpha_{jm} \exp(\mu_m V_k) \right)^{1/\mu_m} \right)}$$

Generalized Nested Logit (GNL) Model



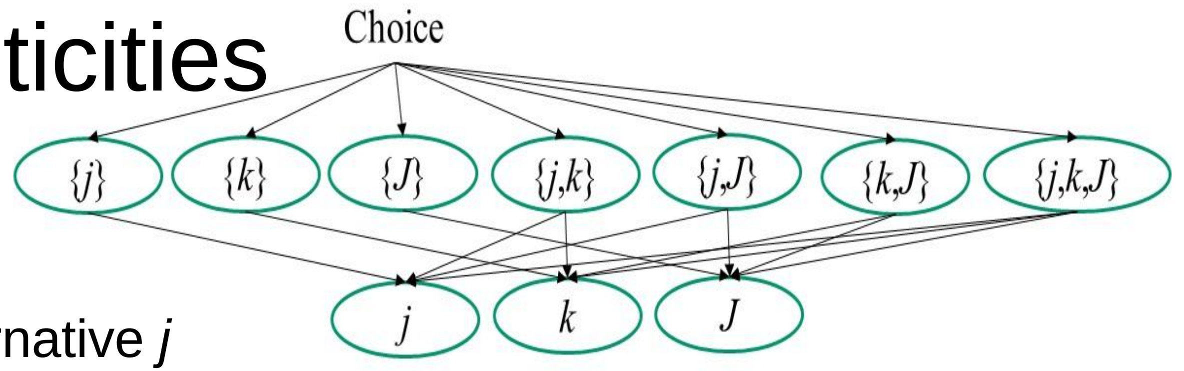
- *All feasible nests*
- *However, alternatives may have different weights (allocation parameter,α) in different nests*

- Only GNL provides a general correlation stricture as: the correlations between U_j & U_k:

$$\rho_{jk} = \sum_{m=1}^{M} \alpha_{jm}^{1/2} \alpha_{km}^{1/2} \left(1 - \left({}^{\mu}/_{\mu_m} \right)^2 \right); \qquad 0 \leq \rho_{jk} \leq 1$$

  - OGEV, PCl and GenL models can derived from the GNL formulation with equal allocation (the absence of α parameter) of all alternatives in all nests

GNL Model Elasticities



- Direct elasticity for alternative *j* (when *j* is not in any nest: $E_D = (\Pr(j))\beta x_j$

- Direct elasticity for alternative *j* (when *j* is in one or more nests):

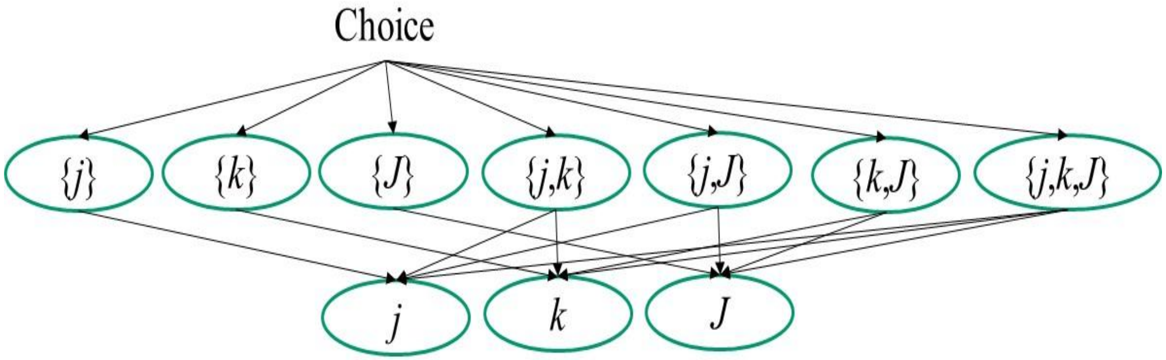
$$E_D = \frac{\beta x_j}{\Pr(j)} \sum_m \Pr(m)\Pr(j|m)[(1 - \Pr(j)) + (\mu_m - 1)\,(1 - \Pr(j|m))]$$

- Cross elasticity between *j* & *k* when they not in any nests: $E_C = -\Pr(j)\,\beta x_j$

- Cross elasticity between *j* & *k* when they in one or more nests:

$$E_C = -\beta x_j \left[\Pr(j) + \frac{\sum_m (\mu_m - 1)\Pr(m)\Pr(j|m)\Pr(k|m)}{\Pr(k)} \right]$$

Cross Nested Logit (CNL) Model



- *All feasible nests*
- *However, alternatives may have different weights (allocation parameter,α) in different nests*

- Generating function: $G(..) = \sum_{nest,\, m=1}^{total\ nests, M} \left(\sum_{j \in m} \left(\alpha_j e^{\mu V_j} \right)^{1/\mu} \right)$

Allocation parameter: $\alpha_j \geq 0$ & $\sum_{m=1}^{M} \alpha_{jm} = 1$

$$\Pr(j) = \sum_{j \in m} \Pr(j|m) \Pr(m)$$

$$\Pr(j|m) = \frac{\alpha_{jm} \exp(\mu V_j)}{\sum_{k \in m} \alpha_{km} \exp(\mu V_k)}$$

$$\Pr(m) = \frac{\left(\sum_{j \in m} \alpha_{jm} \exp(\mu V_k) \right)^{1/\mu}}{\sum_{m=1}^{M} \left(\left(\sum_{j \in m} \alpha_{jm} \exp(\mu V_k) \right)^{1/\mu} \right)}$$

- CNL is a special case of GNL where all nest scale parameters are equal

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A comparison of flight routes in a dual-airport region using overlapping error components and a cross-nested structure in GEV models

Chih-Wen Yang *, Hsiao-Chun Wang

Model		MNL coef. (t val.)	GNL[a] coef. (t val.)
Variables	Alternative		
TPE-SHA constant (base)	TS	–	–
TPE-PVG constant	TP	−1.78 (−8.0)	−1.23 (−4.6)
TSA-SHA constant	SS	−0.77 (−5.7)	−0.58 (−4.6)
TSA-PVG constant	SP	−1.39 (−7.3)	−1.49 (−8.3)
Flying time	All	−6.60 (−2.2)	−4.51 (−1.9)
Airfare	All	−0.09 (−2.5)	−0.08 (−2.9)
Flight inertia	TP, SS, SP	2.01 (16.4)	1.49 (6.8)
Frequency-BT + FF	All	0.05 (1.8)	0.03 (1.4)
Fast check in-HPI	SS, SP	0.31 (1.9)	0.26 (2.0)
Access time	All	−20.76 (−9.0)	−14.79 (−5.4)
Access cost	All	−0.49 (−2.2)	−0.34 (−2.6)
Egress time	TP, SP	−4.23 (−3.1)	−3.28 (−2.9)
AF-route diversity	SS, SP	0.86 (5.7)	0.63 (4.4)
RF-near destination	TP, SP	0.75 (5.5)	0.59 (4.4)
Similarity parameter			
-Logsum(N1)/EC1	(TP, SP)		0.77 (1.7)
-Logsum(N2)/EC2	(TS,TP, SS)		0.67 (2.7)
Allocation parameter			
-TP(N1)	(TP, SP)		0.18 (1.0)
-TP(N2)	(TS, TP, SS)		0.82 (–)
Samples		1206	1206
$LL(0)$		−1671.87	−1671.87
$LL(ms)$		−1599.88	−1599.88
$LL(\beta)$		−1296.74	−1293.44
ρ^2		0.2244	0.2264
$\bar{\rho}^2$		0.2166	0.2168
Computation time (s)[c]		4	17

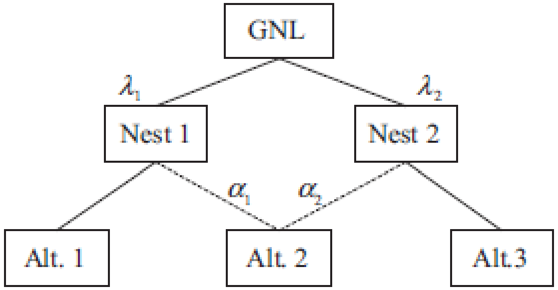

Fig. 1. The cross-nested structure of GNL model.

- The model is highly non-linear
- Estimating allocation as well as scale parameter can be difficult

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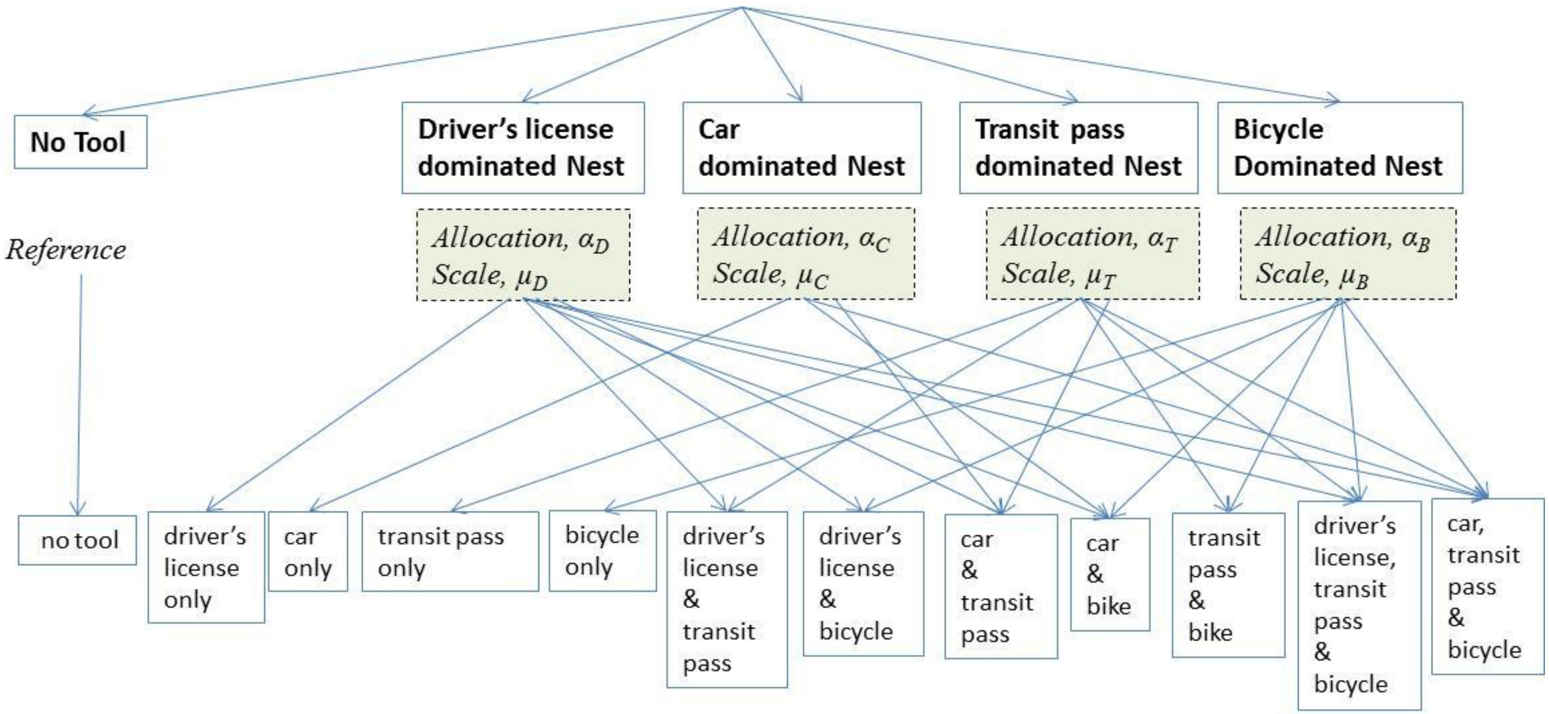
On the Heterogeneity and Substitution Patterns in Mobility Tool Ownership Choices of Post-Secondary Students: The Case of Toronto

Khandker Nurul Habib



- *Allocation parameter, α explains the relative weight of a nest and thereby relative importance of alternatives within*
- *Scale, μ explains the degree of substitution*

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Table 2: Summary statistics of final specifications of two alternative formulations

	Detailed formulation of GEV Model	Multinomial Logit Model	Mixed Multinomial Logit Model	Parsimonious formulation of GEV Model
Total number of observation	13542	13542	13542	13542
Log-likelihood at convergence	-28957	-28920	-28844	-28652
Log-likelihood of null model	-33651	-33651	-33651	-33651
Number of estimated parameters	140	132	136	69
Adjusted Rho-squared value	0.14	0.14	0.15	0.15
AIC value	58193	57567	57960	57441
BIC value	59245	58559	58981	57960
Chi-squared test	----			610
Parameter difference:	(Detailed and parsimonious GEV)			71

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$$\sum_{g'} \alpha_{g'j} = 1 \quad So,\ \alpha_{gj} = \frac{e^{(\sum \eta v)_g}}{\sum_{g'} e^{(\sum \eta v)_{g'j}}}$$

$$\mu_g \geq 1 \qquad So,\ \mu_g = 1 + \exp(\sum \gamma z)_g$$

	Detailed formulation			Parsimonious formulation		
Nest specific scale parameter						
No tool nest: Reference alternative			1.00			1.00
Additional exponential component						
	Parameter	*t*-stat		Parameter	*t*-stat	
Driver's license common nest	0.26	5.30	2.30	-4.51	-2.41	1.01
Car common nest	-1.58	-2.64	1.20	-6.08	-1.30	1.00
Transit pass common nest	-1.01	-3.94	1.36	-2.05	-1.75	1.12
Bicycle common nest	0.31	7.18	2.37	-7.29	-2.15	1.00

Allocation parameter functions (Logit function)

	Parameter	*t*-stat	
Driver's license and transit pass	-1.34	-1.63	for transit pass dominated nest
Driver's license and bicycle	0.05	1.15	for bicycle dominated nest
Car and transit pass	0.51	0.79	for car dominated nest
Car and bicycle	-0.28	-4.35	for car dominated nest

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- The empirical model clearly explains that age, gender, household size and household car ownership have a multitude of influences on multimodality of post-secondary students in Toronto.
- It is clear that older and male students are more multimodal that younger and female students
- Large household size seems to decrease multimodality
- Students living outside downtown Toronto and mostly in the inner or outer suburbs of Toronto are more likely to own bicycles as their only mobility tool than those living in downtown or outside Toronto