The Derivative of the Softmax Function

Non-Mandatory Homework

Consider the Softmax function:

$$\sigma(z_j) = \frac{\exp z_j}{\sum_{k=1}^{m} \exp z_k}$$

Prove that the derivative of $\sigma(z_j)$ w.r.t (with respect to) z_v can be formulated as:

$$\frac{\partial \sigma(z_j)}{\partial z_v} = \begin{cases} \sigma(z_j)\left(1 - \sigma(z_j)\right) & j = v \\ -\sigma(z_j)\sigma(z_v) & j \neq v \end{cases}$$

Solution

$$\frac{\partial \sigma(z_j)}{\partial z_v} = \frac{\frac{\partial \exp z_j}{\partial z_v}\Sigma - \frac{\partial \Sigma}{\partial z_v}\exp z_j}{\Sigma^2}$$

If $j = v$:

$$\frac{\partial \sigma(z_j)}{\partial z_v} = \frac{\exp(z_j)\,\Sigma - \exp(z_j)^2}{\Sigma^2} = \frac{\exp(z_j)\left(\Sigma - \exp(z_j)\right)}{\Sigma^2}$$

$$= \frac{\exp(z_j)}{\Sigma} \times \frac{\Sigma - \exp(z_j)}{\Sigma}$$

$$= \frac{\exp(z_j)}{\Sigma}\left(1 - \frac{\exp(z_j)}{\Sigma}\right) = \sigma(z_j)\left(1 - \sigma(z_j)\right)$$

If $j \neq v$:

$$\frac{\partial \sigma(z_j)}{\partial z_v} = \frac{0 \times \Sigma - \exp(z_v)\exp(z_j)}{\Sigma^2} = -\frac{\exp(z_v)\exp(z_j)}{\Sigma^2}$$

$$= -\frac{\exp(z_j)}{\Sigma} \times \frac{\exp(z_v)}{\Sigma} = -\sigma(z_j)\sigma(z_v)$$