Q: Prove that:

IF: $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$ (1)

Then: $f(n) \in \Theta(h(n))$ (2)

Answer:

From (1):

$\exists\ c_1, c_2 > 0,\ n_{01}, n_{02} \geq 0$ s.t.

$f(n) \leq c_1\, g(n) \quad , \forall\ n \geq n_{01}$ (3)

$g(n) \leq c_2\, h(n) \quad , \forall\ n \geq n_{02}$ (4)

multiply (4) $* c_1 \Rightarrow$

$c_1\, g(n) \leq c_1 \cdot c_2 \cdot h(n) \quad , \forall\ n \geq n_{02}$ (5)

From (3) and (5):

$f(n) \leq c_1\, g(n) \leq c_1 \cdot c_2 \cdot h(n), \forall\ n \geq n_{03}$

where $n_{03} = \text{Max}(n_{01}, n_{02})$

So $f(n) \leq \underbrace{c_1 \cdot c_2}_{c} \cdot h(n), \forall\ n \geq \underbrace{n_{03}}_{n_0}$

Prove that $n^2 \in O(2^n)$:

Using Mathematical Induction:

<u>Base Case</u>:

for $n=5$, $n^2 \leqslant 2^n$ because $5^2 \leqslant 2^5$

<u>Induction Step</u>:

Induction Hypothesis:

$$n^2 \leqslant 2^n$$

Let's Check for $(n+1)$:

$(n+1)^2$	$2^{(n+1)}$
$= \underbrace{n^2} + \underbrace{2n+1}$	$= \underbrace{2^n} + \underbrace{2^n}$

We know from the Induction Hypothesis that $\boxed{n^2 \leqslant 2^n \quad 1}$

We also know that for $n \geqslant 5$:

$$2n+1 < n^2$$

So, $\boxed{2n+1 < 2^n \quad 2}$ as well

$1 + 2$: $n^2+2n+1 \leqslant 2^n + 2^n$

i.e., $(n+1)^2 \leqslant 2^{(n+1)}$