King Saud University

College of Sciences

Department of Mathematics

106 Math Exercises

(15)

Improper Integrals

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Improper Integrals

التكاملات المعتلة

Def. $\int_a^b f(x)\,dx = ?$: $f(x)$ is cont. on $[a,b]$
for $\forall x \in [a,b]$

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improper integ.

$\boxed{1}$ $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx = \begin{cases} \text{Const} \Rightarrow I \text{ is Conv. متقارب} \\ \infty \Rightarrow I \text{ is div} \\ -\infty \end{cases}$

متباعد

f is cont on $[a,\infty)$

$\boxed{2}$ $\int_{-\infty}^{b} f(x)\,dx = \lim_{s\to-\infty}\int_s^b f(x)\,dx = \Big\{$

f is cont. on $(-\infty, b]$

#7) $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \underbrace{\ln x}_{u}\ \underbrace{\frac{dx}{x}}_{du}$ (Imp.)

$= \frac{1}{2}\lim_{t\to\infty}\Big[(\ln|x|)^2\Big]_1^t = \frac{1}{2}\lim_{t\to\infty}\big[\ln^2|t| - 0\big] = \infty$

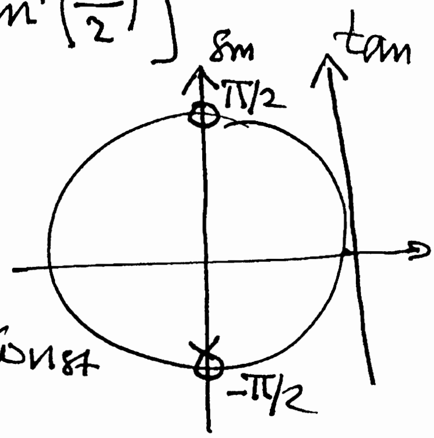
I is div. (متباعد)

#6] $\displaystyle\int_{-\infty}^{2} \frac{1}{x^2+4}\,dx$, $f(x) = \dfrac{1}{4+x^2}$ cont. on $(-\infty, 2]$

imp. $\nearrow$

$$= \lim_{s \to -\infty} \int_{s}^{2} \frac{1}{2^2+x^2}\,dx = \lim_{s \to -\infty} \left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_{s}^{2}$$

$$= \frac{1}{2} \lim_{s \to -\infty} \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{s}{2}\right) \right]$$



$$= \frac{1}{2} \left[\frac{\pi}{4} - \left(\frac{-\pi}{2}\right) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} + \frac{2\pi}{4} \right] = \frac{3\pi}{8} = \text{Const}$$

I is Conv. (متقارب)

$\boxed{3}$ $\int_{-\infty}^{\infty} f(x)\,dx$.

I_1 $\qquad$ I_2

$-\infty$ $f \in C(-\infty, a]$ a $f \in C[a, \infty)$ ∞

$$I_1 = \int_{-\infty}^{a} f(x)\,dx$$

$= c_1$ (متقارب) Conv.

div.
متباعد
Stop $\Rightarrow$ I is div.

$\Downarrow$

نحسب التكامل

$$I_2 = \int_{a}^{\infty} f(x)\,dx \; .$$

$= c_2$
Conv.
متقارب

div. $\Rightarrow$ I div. متباعد
متباعد

$\Downarrow$

متقارب $I = I_1 + I_2$ Conv.

$I = c_1 + c_2 = C$

متقارب Conv.

9] $\int_{-\infty}^{\infty} x e^{-x^2} \, dx$

هامش

$$I_1 = \frac{-1}{2}\int_{-\infty}^{0} e^{\overset{=u}{-x^2}} \underbrace{(-2)x\,dx}_{=du} = \frac{-1}{2}\lim_{s\to-\infty}\int_{s}^{0} e^{-x^2}\cdot(-2)x\,dx \quad \Big| \int e^u\,du = e^u$$

$$= \frac{-1}{2}\lim_{s\to-\infty}\left[e^{-x^2}\right]_s^0 = \frac{-1}{2}\lim_{s\to-\infty}\left[1 - e^{-s^2}\right]$$

$$= \frac{-1}{2}\left[1-0\right] = \frac{-1}{2} \Rightarrow I_1 \text{ Cont.}$$

☺ Go.on

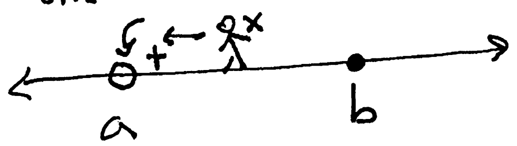
$$I_2 = \int_0^{\infty} \qquad = \lim_{t\to\infty}\int_0^t$$

$$= \frac{-1}{2}\lim_{t\to\infty}\left[e^{-x^2}\right]_0^t = \frac{-1}{2}\lim_{t\to\infty}\left[e^{-t^2} - 1\right]$$

$$= \frac{-1}{2}\left[0-1\right] = \frac{1}{2} \Rightarrow I_2 \text{ Conv.}$$

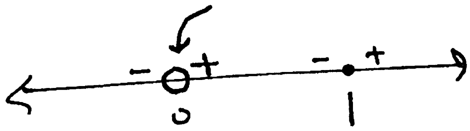
$$I = I_1 + I_2 = -\frac{1}{2} + \frac{1}{2} = \boxed{0} \text{ Conv.}$$

$\boxed{4}$ $\int_a^b f(x)\,dx$, f disc. at a



$$= \lim_{s\to a^+} \int_s^b f(x)\,dx = ?$$

\# 10] $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$



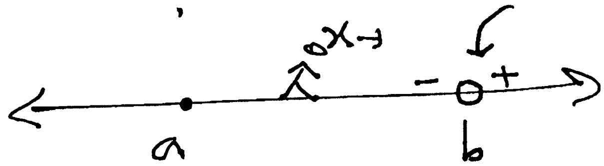
$$= 2\lim_{s\to 0^+} \int_s^1 e^{\sqrt{x}}\,\frac{dx}{2\sqrt{x}}$$

$$= 2\lim_{s\to 0^+} \left[e^{\sqrt{x}}\right]_s^1 = 2\lim_{s\to 0^+}\left[e - e^{\sqrt{s}}\right]$$

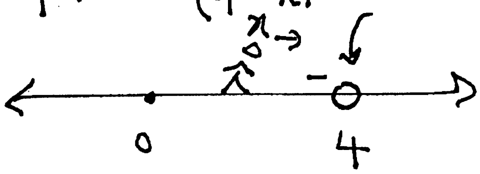
$$= 2(e-1) \Rightarrow I \text{ conv.}$$

5] $\int_a^b f(x)\,dx$: f(x) discont. at b

$$= \lim_{t\to b^-} \int_a^t f(x)\,dx = ?$$



#12 $\displaystyle\int_0^{4} \frac{1}{(4-x)^{2/3}}\,dx$, $f(x) = \dfrac{1}{(4-x)^{2/3}}$ discont. at 4.



$$= \lim_{t \to 4^-} \int_0^t \frac{1}{(4-x)^{2/3}}\,dx = ?$$

$$= -3 \lim_{t \to 4^-} \left[(4-x)^{1/3}\right]_0^t$$

$$= -3 \lim_{t \to 4^-} \left[(4-t)^{1/3} - 4^{1/3}\right]$$

$$= 3(4)^{1/3} \quad \Rightarrow \quad I \text{ conv. متقارب}$$

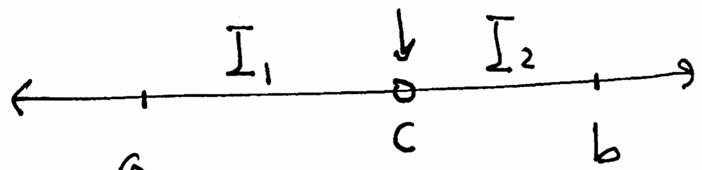
هامش :

$$\int (4-x)^{-2/3}\,dx$$

$u = 4-x \Rightarrow -du = dx$

$$I = -\int u^{-2/3}\,du = -3u^{1/3} = -3(4-x)^{1/3}$$

6] $\int_a^b f(x)\,dx$: $f(x)$ discont. at $x=c\in(a,b)$



$$I_1=\int_a^c f(x)\,dx$$

$= c_1$

متقارب conv.

$\Downarrow$

نستمر ونحسب قيمة التكامل

div. متباعد

Stop

$\Downarrow$ I div.

متباعد

$$I_2=\int_c^b f(x)\,dx$$

$= c_2$

متقارب conv.

$\Downarrow$

$I = I_1 + I_2$

div متباعد

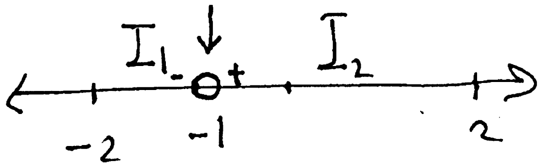
$\Downarrow$

I div.

متباعد

$I = c_1 + c_2 = c_3 \Rightarrow I$ conv. متقارب

#14 $\displaystyle\int_{-2}^{2} \frac{1}{(x+1)^3}\,dx$



$f(x) = \dfrac{1}{(x+1)^3}$ disc. at $x=-1 \in (-2,2)$.

$I_1 = \displaystyle\int_{-2}^{\text{[redacted]}} \frac{1}{(x+1)^3}\,dx = \lim_{t \to -1^-} \int_{-2}^{t} (x+1)^{-3}\,dx$

$= \dfrac{-1}{2} \displaystyle\lim_{t\to -1^-} \left[\frac{1}{(x+1)^2}\right]_{-2}^{t} = \frac{-1}{2} \lim_{t \to -1^-} \left[\frac{1}{(t+1)^2} - 1\right]$

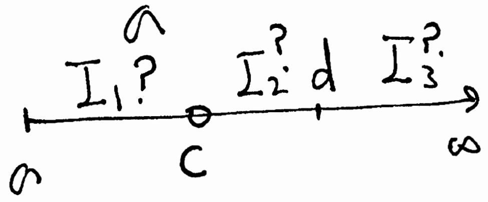
$(t+1)^2 \to 0$

$= \dfrac{-1}{2}\left[\infty - 1\right] = -\infty \to I_1$ div.

stop $\Rightarrow I$ div

متباعد

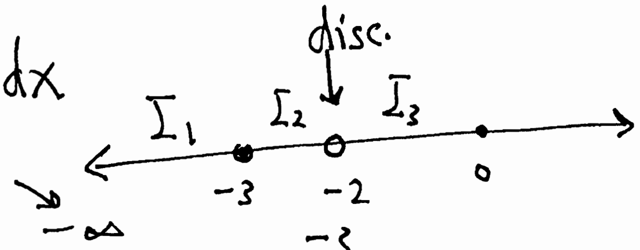
$\boxed{7}$ $\displaystyle\int_a^{\infty} f(x)\,dx$ or $\displaystyle\int_{-\infty}^{b} f(x)\,dx$ or $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$



f disc. at $c \in [a, \infty)$

ونعالجه كما ذُكر في ط2 و ط1 السابقة

#17 $\displaystyle\int_{-\infty}^{0} \frac{1}{x+2}\,dx$



$$I_1 = \int_{-\infty}^{-3} \frac{1}{x+2}\,dx = \lim_{s \to -\infty} \int_{s}^{-3} \frac{1}{x+1}\,dx$$

$$= \lim_{s \to -\infty} \Big[\ln|x+1| \Big]_s^{-3} = \lim_{s \to -\infty} \big[\ln 2 - \ln|s+1| \big]$$

$$= -\infty \Rightarrow I_1 \text{ div.}$$

Stop $\Rightarrow I$ div

Q . Determine whether the following improper integral converges or diverges and if it converges, find its value?

1)

$$\int_0^1 \frac{e^{\frac{1}{x}}}{x^2}\, dx$$

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2)

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$$

3)

$$\int_{0}^{\infty} \frac{1}{4+x^2}\,dx$$

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4)

$$\int_{e}^{\infty} \frac{1}{x\,(lnx)^2}\,dx$$

5)

$$\int_{0}^{-1} \frac{1}{\sqrt[3]{x+1}}\,dx$$

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6)

$$\int_{0}^{\infty} \frac{1}{x+1}\,dx$$

7)

$$\int_0^{\infty} xe^{-x^2}\ dx$$

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8)

$$\int_0^{\pi/2} \frac{2}{1+cos2x}\ dx$$

9)

$$\int_{0}^{3} \frac{2}{(x-3)^3}\, dx$$

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10)

$$\int_{\pi/2}^{\pi} \frac{1}{1+cosx}\, dx$$

11)

$$\int_{0}^{4} \frac{1}{x^2 - x - 2}\, dx$$

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12)

$$\int_{-\infty}^{0} \frac{1}{x + 2}\, dx$$

13)

$$\int_{0}^{\infty} \frac{1}{(x-4)^2}\, dx$$

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14)

$$\int_{1}^{\infty} \frac{1}{x^{4/3}}\, dx$$

15)

$$\int_{1}^{\infty} \frac{1}{x^{3/4}}\,dx$$