King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(2)

Methods of Proof

1-Direct Proof

2- Proof by Contraposition

3- Proof by Contradiction

4- Proof by Cases

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Direct Proofs: A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true

DEFINITION 1 The integer n is *even* if there exists an integer k such that $n = 2k$, and n is *odd* if there exists an integer k such that $n = 2k + 1$. (Note that every integer is either even or odd, and no integer is both even and odd.) Two integers have the *same parity* when both are even or both are odd; they have *opposite parity* when one is even and the other is odd.

EXAMPLE 1 Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: $n = 2k + 1 \ , \ k \in \mathbb{Z}$

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\,(2k^2 + 2k) + 1$$

$$n^2 = 2h + 1 \quad : \quad h = (2k^2 + 2k) \in \mathbb{Z}$$

$$\boxed{\therefore\ n^2 \text{ is odd}}$$

EXAMPLE 2 Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

Solution: We assume that m and n are both perfect squares. By the definition of a perfect square, $\exists\ s, t \in \mathbb{Z}$ such that $m = s^2$ and $n = t^2$. $mn = s^2t^2 = (st)^2 = h^2 \ : \ h = st \in \mathbb{Z}$

$\therefore mn = h^2$. $\boxed{\therefore mn \text{ is a perfect square}}$.

DEFINITION 3 The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. p and q have no common factors (so that the fraction p/q is in lowest terms.) A real number that is not rational is called ***irrational*.**

EXAMPLE 3 Prove that the sum of two rational numbers is rational.

Solution:

Let $r = p/q$ where *p and q are integers, with q ≠ 0, and* $s = t/u$ where *t and u are integers, with u ≠ 0 .*

$$r + s = {}^{p}\!/_{q} + {}^{t}\!/_{u} = {}^{pu+qt}\!/_{qu} \quad where\ pu + qt\ , qu \in \mathbb{Z} : qu \neq 0$$

$$\boxed{\therefore\ r + s \text{ is rational number}}$$

DEFINITION 4 Let $a\ , b\ \in \mathbb{Z} : a \neq 0.\ a\ \ is\ a\ divisor\ to\ b\ , a|b\ if\ there\ exist\ integer\ c\ such\ that\quad b = ac$

DEFINITION 5 Let $a\ , b\ \in \mathbb{Z}$ and $n \in \mathbb{Z}^{+}$. a is congruent to b modulo n , $a \equiv b(mod\ n)$ if $n|(a-b)$

EXAMPLE 4 : $25 \equiv 7(\text{mod}\ 9)$, $3 \equiv -15(\text{mod}\ 9)$

Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

Solution:

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2. Use a direct proof to show that the sum of two even integers is even.

Solution:

3. Show that the square of an even number is an even number using a direct proof.

Solution:

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4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

Solution:

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5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Solution:

6. Use a direct proof to show that the product of two odd numbers is odd.
Solution:

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7. Use a direct proof to show that every odd integer is the difference of two squares.
Solution:

$$\because (n \text{ is odd}) \Rightarrow \ n = 2k + 1 \quad : k \in \mathbb{Z}$$

$$(k + 1)^2 - \ k^2 = \ k^2 + 2k + 1 - k^2 = 2k + 1 = n$$

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8. Use a direct proof to show that $x^2 + \ x$ is an even number where x is integer.
Solution:

9. Use a direct proof to show that if x is an odd integer number, then $x^2 = 8m + 1$ Where m is integer .

Solution:

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10. Use a direct proof to show that if p is a prime number, then $p + 13$ is a composite number .

Solution:

11. Use a direct proof to show that if p is an odd prime number, then the number 4 is a divisor to $2p + 2$.

Solution:

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12. Use a direct proof to show that if n is an odd number, then $5n + 6$ is odd.

Solution:

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13. Use a direct proof to prove that if n is an odd number, then $n^2 \equiv 1(mod 4)$.

Solution:

14. Use a direct proof to prove that if n is an even number, then $n^2 \equiv 0(mod4)$.

Solution:

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15. Use a direct proof to prove that, if 3 is not a divisor of n, then $n^2 \equiv 1(mod3)$.

Solution:

16. If $n \in \mathbb{Z}^+$ and $a, b, c \in \mathbb{Z}$ where $a \equiv b(mod\ n)$ and $c \equiv d(mod\ n)$ prove that :

(i) $a + c \equiv b + d(mod\ n)$ (ii) $ac \equiv bd(mod\ n)$

Solution:

Proof by Contraposition Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true. In a proof by contraposition of $p \to q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow. the proof by contraposition can succeed when we cannot easily find a direct proof.

EXAMPLE 1 Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution:

Assume that n is even. Then, $n = 2k \ : k \in \mathbb{Z}$. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) = 2h \ : h = (3k + 1) \in \mathbb{Z}$.
This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem.

EXAMPLE 2 Prove that if $n = ab$, where a and b are positive integers, then

$$a \leq \sqrt{n} \quad or \quad b \leq \sqrt{n} \ .$$

Solution:

Assume that $a > \sqrt{n}$ and $b > \sqrt{n} \Rightarrow ab > \sqrt{n}\,.\sqrt{n} = n$

$\therefore \ ab > n \quad \Rightarrow \therefore \ ab \neq n$

EXAMPLE 3 Prove that if n is an integer and n^2 is odd, then n is odd.

Solution:

Assume that n is <u>even</u> integer, $n = 2k : k \in \mathbb{Z}$

$n^2 = (2k)^2 = 4k^2 = 2\,(2k^2) = 2h \quad ; h = (2k^2) \in \mathbb{Z}$

$\therefore \ n^2$ is even .

Exercises

1. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Solution:

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2. Prove that if m and n are integers and mn is even, then m is even or n is even.

Solution:

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3. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contraposition.

Solution:

4. Prove that if n is an integer and $3n + 2$ is even, then n is even ,using a proof by contraposition.

Solution:

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5. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

Solution:

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6. Prove that if a is an integer where $5 \nmid a$, then $5 \nmid (a + 20)$ using a proof by contraposition.

Solution:

7. Use a proof by contraposition to show that if $2|mn\ : m,n \in \mathbb{N}$, then $2|m\ or\ 2|n$.
Solution:

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8. Use a proof by contraposition to show that if xy is even number where $x,y \in \mathbb{Z}$, then x is even or y is even .
Solution:

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9. Use a proof by contraposition to show that if $x^2 + y^2 = z^2 : x,y,z \in \mathbb{Z}$, then one of these numbers x,y,z is at least should be even .
Solution:

10. Use a proof by contraposition to show that if $3mn + 2$ is irrational number, then m is irrational or n is irrational .

Solution:

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11. Use a proof by contraposition to show that if $a^2 + b^2$ is odd, then a is even or b is even .

Solution:

12. Use a proof by contraposition to show that if $x + y < 15$, where x and y are real numbers, then $x < 8$ or $y < 8$.

Solution:

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13. Prove that if m is an integer where $3 \nmid m$, then $3 \nmid (m+1)^2 + 2m^2 + 5$ using a proof by contraposition.

Solution:

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14. Use a proof by contraposition to show that if $3n^2 + 4n + 3$ is even, then n is odd.

Solution:

15. Let $6|m \ : m \in \mathbb{Z}$. Use a proof by contraposition to show that if $3 \nmid (m+n) \ : n \in \mathbb{Z}$, then $3 \nmid n$.

Solution:

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16. Prove that if n is a positive integer, then n is even <u>if and only if</u> $7n + 4$ is even .

Solution:

17. Prove that if n is a positive integer, then n is odd <u>if and only if</u> $5n + 6$ is odd.

<u>Solution:</u>

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18. Prove that if n is integer, then $3n + 2$ is odd <u>if and only if</u> $9n + 5$ is even .

<u>Solution:</u>

19. Use a proof by contraposition to show that if $3x - y = z$ where $x, y, z \in \mathbb{Z}$, then x is even or y is even or z is even .

Solution:

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20. Use a proof by contraposition to show that if $4n^2 + n - 3$ is even, then n is odd.

Solution:

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21. Use a proof by contraposition to show that if $a + b \leq 4$, where a and b are real numbers, then $a \leq 1$ or $b \leq 3$.

Solution:

22. Use a proof by contraposition to show that if $x + y < 9$, where x and y are real numbers, then $x < 4$ or $y < 5$.

Solution:

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23. Use a proof by contraposition to show that if $m \nmid n$, then $m \nmid l$ or $m \neq n + l$ where $l, m, n \in \mathbb{Z}$.

Solution:

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24. Use a proof by contraposition to show that if $a + b = 4$ where $a, b \in \mathbb{R}$, then $a \neq 2$ or $b \neq 3$.

Solution:

25. Use a proof by contraposition to show that if $mn = l$, then $l \geq 0$ or $m \geq 0$ or $n \geq 0$: $l, m, n \in \mathbb{Z}$

Solution:

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26. Use a proof by contraposition to show that if $a + b - c$ is even then a is even or b is even or c is even, where $a, b, c \in \mathbb{Z}$.

Solution:

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27. Use a proof by contraposition to show that if $n^2 - 1$ is odd, then n is even : $n \in \mathbb{Z}$.

Solution:

Proofs by Contradiction Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true. Because q is false, but $\neg p \to q$ is true, we can conclude that $\neg p$ is false, which means that p is true. How can we find a contradiction q that might help us prove that p is true in this way? Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called **proofs by contradiction**

EXAMPLE 1 Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Proof: Let p be the proposition "$\sqrt{2}$ is irrational." We will show that assuming that $\neg p$ is true leads to a contradiction. If $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = {}^{a}/_{b}$ where $b \neq 0$ and $\gcd(a, b) = 1$; a and b have no common factors (so that the fraction a/b is in lowest terms.)

$$\sqrt{2} = {}^{a}/_{b} \overset{(\,)^2}{\Rightarrow} 2 = {}^{a^2}/_{b^2} \Rightarrow a^2 = 2b^2 \quad (1)$$

$$\Rightarrow a^2 \text{ is even} \Rightarrow 2|a^2 \Rightarrow 2|a \Rightarrow a = 2k : k \in \mathbb{Z} \xRightarrow{by\ subist.into(1)} (2k)^2 = 2b^2$$

$$\Rightarrow 4k^2 = 2b^2 \overset{\div 2}{\Rightarrow} 2k^2 = b^2 \Rightarrow \therefore b^2 \text{ is even too} \Rightarrow 2|b^2 \Rightarrow 2|b \Rightarrow \therefore 2|a \text{ and } 2|b$$

This is contradiction with the assumption that $\gcd(a, b) = 1$. $\therefore p$ is true .

$$\boxed{\therefore \text{ "}\sqrt{2} \text{ is irrational."}}$$

EXAMPLE 2 Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

Solution: Let p be "$3n + 2$ is odd" and q be "n is odd." To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n + 2$ is odd and that n is not odd.

Because n is not odd, we know that it is even. Because n is even, there is an integer k such That $n = 2k$. This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Because $3n + 2$ is $2t$, where $t = 3k + 1$, $\therefore 3n + 2$ is even. Note that the statement "$3n + 2$ is even" is equivalent to the statement $\neg p$, because an integer is even if and only if it is not odd. Because both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd.

Exercises

1. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Solution:

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2. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contradiction.

Solution:

3. Prove that if n is an integer and $3n + 2$ is even, then n is even using a proof by contradiction.

Solution:

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4. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$.

[*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by $b3$. Then look at whether a and b are each odd or even.]

Solution:

5. Prove that $\sqrt{3}$ is irrational by giving a proof by contradiction.

Solution:

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6. Prove that $\sqrt{5}$ is irrational by giving a proof by contradiction.

Solution:

7. Let $\sqrt{5}$ is irrational, prove that $2 - 3\sqrt{5}$ is irrational number.

Solution:

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8. Let $\sqrt{2}$ is irrational, prove that $1 + 3\sqrt{2}$ is irrational number.

Solution:

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9. Let $\sqrt{6}$ is irrational, prove that $-3 + 2\sqrt{6}$ is irrational number.

Solution:

10. Let $\sqrt{5}$ is irrational, prove that $\frac{\sqrt{5}-3}{2}$ is irrational number.

Solution:

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11. Let $m\ ,n$ are rational numbers : $n \neq 0$ and $\sqrt{5}$ is irrational, prove that $\frac{\sqrt{5}}{2n} - m$ is irrational number.

Solution:

12. Let $x, y, z \in \mathbb{R}\ : x + y + z = 20$, use a proof by contradiction to show that $x \geq 10$ or $y \geq 6$ or $z \geq 4$.

Solution:

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13. Let $x, y, z \in \mathbb{R}\ : x + y + z = 21$, use a proof by contradiction to show that $x \geq 8$ or $y \geq 7$ or $z \geq 6$.

Solution:

14. Use a proof by contradiction to prove that the product of an irrational number and a non-zero rational number is irrational.

Solution:

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15. Let p is a prime number . use a proof by contradiction to prove that if $p|n$, then $p \nmid n+1$.

Solution:

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16. Use a proof by contradiction to prove that $\nexists\ m, n \in \mathbb{Z}$, such that $4n + 6m = 11$

Solution:

17. Prove or disprove that the product of two irrational numbers is irrational.

Solution:

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18. Prove or disprove that the sum of two irrational numbers is irrational.

Solution:

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19. Use a proof by contradiction to prove that $x + \frac{1}{x} \geq 2$ for $\forall\, x \in \mathbb{R}^*$.

Solution:

20. Use a proof by contradiction to prove that $\nexists\, a , b \in \mathbb{Z}^{+}$, such that

$$a + b = 500 \quad \text{and} \quad \gcd(a, b) = 7$$

Solution:

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21. Prove or disprove that if $a^2 | b^3$ then $a | b$: $a , b \in \mathbb{N}$.

Solution:

PROOF BY CASES A proof by cases must cover all possible cases that arise in a theorem. We illustrate proof by cases with a couple of examples. In each example, you should check that all possible cases are covered.

EXAMPLE 1 Prove that if n is an integer, then $n^2 \geq n$.

Solution: We can prove that $n2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -1$.We split the proof into three cases because it is straight forward to prove the result by considering zero, positive integers, and negative integers separately.

Case (i): When $n = 0$, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.

Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive Integer n, we obtain $n \cdot n \geq n \cdot 1$. This implies that $n^2 \geq n$ for $n \geq 1$.

Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.

Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$.

EXAMPLE 4 Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|a| = \begin{cases} a & : a \geq 0 \\ -a: & a < 0 \end{cases}$)

Solution: In our proof of this theorem, we remove absolute values using the fact that $|a| = a$ when $a \geq 0$ and $|a| = -a$ when $a < 0$. Because both $|x|$ and $|y|$ occur in our formula, we will need four cases: *(i)* x and y both nonnegative,

(ii) x nonnegative and y is negative,

(iii) x negative and y nonnegative

(iv) x negative and y negative.

We denote by $p1$, $p2$, $p3$, and $p4$, the proposition stating the assumption for each of these four cases, respectively.

(Note that we can remove the absolute value signs by making the appropriate choice of signs within each case.)

*Case (i):*We see that $p1 \rightarrow q$ because $xy \geq 0$ when $x \geq 0$ and $y \geq 0$,

so that $|xy| = xy = |x||y|$.

Case (ii): To see that $p2 \rightarrow q$, note that if $x \geq 0$ and $y < 0$, then $xy \leq 0$,

so that $|xy| = -xy = x(-y) = |x||y|$. (Here, because $y < 0$, we have $|y| = -y$.)

Case (iii): To see that $p3 \to q$, we follow the same reasoning as the previous case with the roles of x and y reversed.

Case (iv): To see that $p4 \to q$, note that when $x < 0$ and $y < 0$, it follows that $xy > 0$.

Hence, $|xy| = xy = (-x)(-y) = |x||y|$.

Because $|xy| = |x||y|$ holds in each of the four cases and these cases exhaust all possibilities, we can conclude that $|xy| = |x||y|$, whenever x and y are real numbers.

EXAMPLE 7 Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even.

Solution: We will use proof by contraposition, the notion of without loss of generality, and proof by cases. First, suppose that x and y are not both even. That is, assume that x is odd or that y is odd (or both).Without loss of generality, we assume that x is odd, so that

$x = 2m + 1$ for some integer k. To complete the proof, we need to show that

xy is odd or $x + y$ is odd. Consider two cases:

(i) y even

(ii) y odd.

In (i), $y = 2n$ for some integer n, so that $x + y = (2m + 1) + 2n = 2(m + n) + 1$ is odd.

In (ii), $y = 2n + 1$ for some integer n, so that $xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$ is odd. This completes the proof by contraposition.

(Note that our use of without loss of generality within the proof is justified because the proof when y is odd can be obtained by simply interchanging the roles of x and y in the proof we have given.)

Exercises

1. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$.

[*Hint:* Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.]

Solution: If $x \leq y$, then $\max(x, y) + \min(x, y) = y + x = x + y$.

If $x \geq y$, then $\max(x, y) + \min(x, y) = x + y$.

Because these are the only two cases, the equality always holds .

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2. Use a proof by cases, to prove that $n^2 + 1 \geq 2n$ when n is a positive integer with $1 \leq n \leq 4$.

Solution: $1^2 + 1 = 2 \geq 2 = 2(1)$;

$2^2 + 1 = 5 \geq 4 = 2(2)$;

$3^2 + 1 = 10 \geq 8 = 2(3)$;

$4^2 + 1 = 17 \geq 16 = 2(4)$