King Saud University

College of Sciences

Department of Mathematics

106 Math Exercises

(18)

Arc length & Surface Area

Malek Zein AL-Abidin

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ARC LENGTH

If $f(x)$ is continuous function on the interval $[a,b]$, then the arc length of $f(x)$ from $x=a$ to $x=b$ is $L=\int_a^b \sqrt{1+[f'(x)]^2}\, dx$

If $g(y)$ is continuous function on the interval $[c,d]$, then the arc length of $g(y)$ from $y=c$ to $y=d$ is $L=\int_c^d \sqrt{1+[g'(y)]^2}\, dy$

Examples : Find the arc length of the following :

1. $y=\dfrac{x^3}{12}+\dfrac{1}{x}$ from $A=(1,\frac{13}{12})$ to $B=(2,\frac{7}{6})$.

$$f(x)=\frac{x^3}{12}+\frac{1}{x} \Rightarrow f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$L=\int_1^2 \sqrt{1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2}\, dx=\int_1^2 \sqrt{1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}}\, dx$$

$$=\int_1^2 \sqrt{\frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}}\, dx=\int_1^2 \sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\, dx=\int_1^2 \left|\frac{x^2}{4}+\frac{1}{x^2}\right|\, dx$$

$$L=\int_1^2 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)\, dx=\left[\frac{x^3}{12}-\frac{1}{x}\right]_1^2=\frac{13}{12}$$

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2. $y=\dfrac{1}{2}\left(e^x+e^{-x}\right)$, $x\in[0,2]$

$$f(x)=\frac{e^x+e^{-x}}{2}=\cosh x \Rightarrow f'(x)=\sinh x$$

$$L=\int_0^2 \sqrt{1+\sinh^2 x}\, dx=\int_0^2 \sqrt{\cosh^2 x}\, dx$$

$$=\int_0^2 |\cosh x|\, dx=\int_0^2 \cosh x\, dx$$

$$L=[\sinh x]_0^2=\sinh(2)-\sinh(0)=\frac{e^2-e^{-2}}{2}-0=\frac{e^2-e^{-2}}{2}$$

3. $x^2 + y^2 = 25$, $-5 \leq y \leq 5$

Note : In this problem the arc length is equal to half of the perimeter of the circle $x^2 + y^2 = 25$, the arc length is equal to 5π .

$x^2 + y^2 = 25 \Rightarrow x^2 = 25 - y^2 \Rightarrow x = \pm\sqrt{25 - y^2}$, in this problem $x = \sqrt{25 - y^2}$

$$g(y) = \sqrt{25 - y^2} \Rightarrow g'(y) = \frac{-y}{\sqrt{25 - y^2}}$$

$$L = \int_{-5}^{5} \sqrt{1 + \left(\frac{-y}{\sqrt{25 - y^2}}\right)^2}\, dy = \int_{-5}^{5} \sqrt{1 + \frac{y^2}{25 - y^2}}\, dy$$

$$= \int_{-5}^{5} \sqrt{\frac{25 - y^2 + y^2}{25 - y^2}}\, dy = 5\int_{-5}^{5} \frac{1}{\sqrt{25 - y^2}}\, dy$$

$$L = 5\left[\sin^{-1}\left(\frac{y}{5}\right)\right]_{-5}^{5} = 5\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 5\left[\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right] = 5\pi \ .$$

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4. Find the arc length of the graph of the graph $y = x$, *from* $x = 0$ *to* $x = 3$.

Solution:

5. Find the arc length of the graph of the curve $y = x^{3/2} + 4\ ,\ 0 \leq x \leq 1$.

Solution:

6. Find the arc length of the graph of the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}\ , 2 \leq x \leq 4$

Solution:

7. Find the arc length of the graph of the curve $f(x) = 3x^{2/3} - 10$, from the point $A(8,2)\ to\ B(27,17)$.

Solution:

8. Find the arc length of the graph of the curve $y = \frac{x^3}{3} + \frac{1}{4x} \quad , \; 1 \le x \le 3$

Solution:

9. Find the arc length of the graph of the curve $x = \frac{y^4}{16} + \frac{1}{2y^2}$ from the point $A\left(\frac{9}{8}, -2\right)$ to $B(\frac{9}{16}, 17)$.

<u>Solution:</u>

10. Find the arc length of the graph of the equation $30xy^4 - y^8 = 15$ from the point $A\left(\frac{8}{15}, 1\right)$ to $B(\frac{271}{240}, 2)$.

Solution:

11. Find the arc length of the graph of the equation $y = 5 - \sqrt{x^3}$ from the point $A(1,4)\ to\ B(4,-3)$.

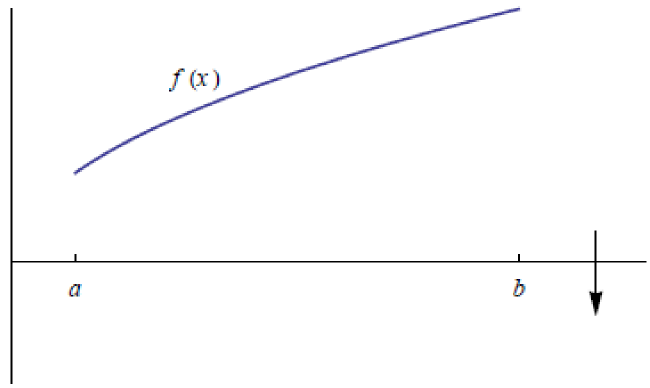
Solution:

12. Find the arc length of the graph of the curve $y = \ln(cosx) \,, 0 \leq x \leq \frac{\pi}{4}$.

Solution:

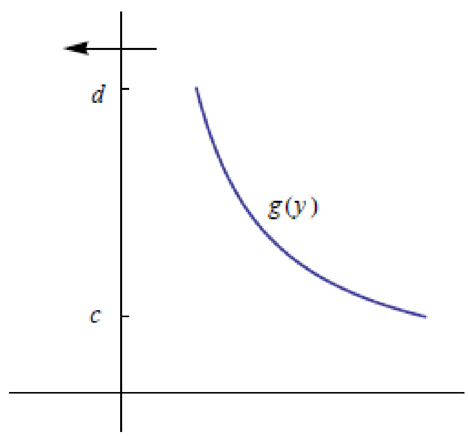
SURFACE AREA
(SURFACE OF REVOLUTION)

Definition:



If $f(x)$ is a continuous function on the interval $[a, b]$, then the surface area generated by revolving the graph of the function $f(x)$ around the x-axis is

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$



If $g(y)$ is a continuous function on the interval $[c, d]$, then the surface area generated by revolving the graph of the function $g(y)$ around the y-axis is

$$SA = 2\pi \int_c^d g(y)\sqrt{1 + [g'(y)]^2}\, dy$$

Examples : Find the surface area generated by revolving the following functions around the given axis :

1. $4x = y^2$, from $A = (0, 0)$ to $B = (1, 2)$, around the x-axis .

   Solution:

$$4x = y^2 \Rightarrow y = \pm 2\sqrt{x}$$

$$f(x) = 2\sqrt{x} \Rightarrow f'(x) = \frac{1}{\sqrt{x}}$$

$$SA = 2\pi \int_0^1 2\sqrt{x}\sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2}\, dx = 4\pi \int_0^1 \sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$$

$$SA = 4\pi \int_0^1 \sqrt{x+1}\, dx = 4\pi \left[2\frac{(x+1)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{8\pi}{3}\left(2\sqrt{2} - 1\right)$$

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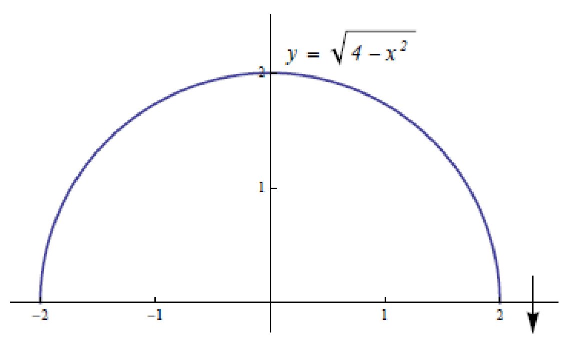
2. $y = \sqrt{4 - x^2}$, $x \in [-2, 2]$, around the x-axis .

Solution:



Note : It is the surface area of the sphere with radius 2 , and it is equal to $4\pi(2)^2 = 16\pi$

$$f(x) = \sqrt{4 - x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{4 - x^2}}$$

$$SA = 2\pi \int_{-2}^{2} \sqrt{4-x^2}\,\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\; dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{4-x^2}\sqrt{\frac{(4-x^2)+x^2}{4-x^2}}\; dx = 2\pi \int_{-2}^{2} \sqrt{4-x^2}\frac{2}{\sqrt{4-x^2}}\; dx$$

$$SA = 4\pi \int_{-2}^{2} dx = 4\pi\,[x]_{-2}^{2} = 16\pi$$

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3. $y = 2\sqrt[3]{x}$, from $A=(1,2)$ to $B=(8,4)$, around the y-axis .

Solution:

$$y = 2\sqrt[3]{x} \Rightarrow \sqrt[3]{x} = \frac{y}{2} \Rightarrow x = \frac{y^3}{8}$$

$$g(y) = \frac{y^3}{8} \Rightarrow g'(y) = \frac{3}{8}y^2$$

$$SA = 2\pi \int_{2}^{4} \frac{y^3}{8}\sqrt{1+\left(\frac{3}{8}y^2\right)^2}\; dy = 2\pi \int_{2}^{4} \frac{y^3}{8}\sqrt{1+\frac{9}{64}y^4}\; dy$$

$$= 2\pi\frac{1}{8}\frac{16}{9}\int_{2}^{4}\left(1+\frac{9}{64}y^4\right)^{\frac{1}{2}}\left(\frac{9}{16}y^3\right)\; dy$$

$$SA = \frac{4\pi}{9}\left[2\frac{(1+\frac{9}{64}y^4)^{\frac{3}{2}}}{3}\right]_2^4$$

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4. $y = x^2$, $0 \le x \le 2$, around the y-axis .

$$y = x^2 \Rightarrow x = \pm\sqrt{y} \Rightarrow x = \sqrt{y}\ , \ \text{since } 0 \le x \le 2$$

$$0 \le x \le 2 \Rightarrow 0 \le y \le 4$$

$$g(y) = \sqrt{y} \Rightarrow g'(y) = \frac{1}{2\sqrt{y}}$$

$$SA = 2\pi \int_0^4 \sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy = 2\pi \int_0^4 \sqrt{y}\sqrt{1+\frac{1}{4y}}\, dy$$

$$SA = 2\pi \int_0^4 \sqrt{y+\frac{1}{4}}\, dy = 2\pi \left[\frac{2\left(y+\frac{1}{4}\right)^{\frac{3}{2}}}{3}\right]_0^4$$

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5. Find the surface area generated by revolving the curve of the function $f(x) = \sqrt{x}$, $1 \leq x \leq 4$ around the $x - axis$.

Solution:

6. Find the surface area generated by revolving the curve of the function $g(y) = \frac{1}{3}\, y^3 \quad , 2 \leq y \leq 4$ around the $y - axis$.

Solution:

7. Find the surface area generated by revolving the curve of the function $f(x) = 1 - x \quad , -1 \leq x \leq 1$ around the $-axis$.

Solution:

8. Find the surface area generated by revolving the curve of the function $f(x) = 2x^{1/2}$, $1 \leq x \leq 2$ around the $-axis$.

Solution: