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الملك سعود
King Saud University
1957

Theory of Computation
CSC 339 – Spring 2021

Chapter-7: part2
The Class P

King Saud University

Department of Computer Science

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Introduction

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- **Two main classes**
  - **Polynomial time algorithms (P)**
  - **Nondeterministic polynomial time algorithms (NP)**
- **In this part, we focus on the class P.**

The Class P

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- **It is crucial to understand the difference between <u>polynomial</u> and (<u>exponential</u> and <u>factorial</u>).**

The Class P: Polynomial vs. Exponential & Factorial

Polynomial

$O(1)$, $O(\log n)$,
$O(n \log n)$, $O(n^2)$,
$O(n^3)$, …

Exponential & Factorial

$O(2^n)$, $O(3^n)$, …,
$O(m^n)$, $O(n!)$, $O(n^n)$

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Polynomial time algorithms are
fast enough for many purposes

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Keep in mind that we're talking about decidable (solvable) problems!

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The Class P: Polynomial vs. Exponential & Factorial

Theoretically, ***exponential*** algorithms decide some problems, but they are so slow that they cannot be used in practice.

Poly
fast

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Let n = 10

$O(1)$	= 1
$O(\log_2 n)$	= 3.3
$O(n)$	= 10
$O(n \log_2 n)$	= 33
$O(n^2)$	= 100
$O(n^3)$	= 1000

$O(2^n)$	= 1024
$O(3^n)$	= 59049
$O(n!)$	= 3628800
$O(n^n)$	= 10000000000

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Let n = 20

$O(1)$	= 1
$O(\log_2 n)$	= 4.3
$O(n)$	= 20
$O(n \log_2 n)$	= 86
$O(n^2)$	= 400
$O(n^3)$	= 8000

$O(2^n)$	= 1048576
$O(3^n)$	= 3486784401
$O(n!)$	= 2432902008176640000
$O(n^n)$	= 104857600000000000000000000

Polynomial

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Let n = 100

$O(1)$	$= 1$
$O(\log_2 n)$	$= 6.64$
$O(n)$	$= 100$
$O(n \log_2 n)$	$= 664$
$O(n^2)$	$= 10000$
$O(n^3)$	$= 1000000$

Polynomial

$O(1)$, $O(\log n)$,
$O(n \log n)$, $O(n^2)$,
$O(n^3)$, …

$O(2^n)$ $=$ 1267650600228229401496703205376

$O(3^n)$ $=$ 515377520732011331036461129765621272702107522001

$O(n!)$ $= 9.33e+157$

$O(n^n)$ $= 1e+200$

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Exponential & Factorial

Number of atoms in the universe is ~10^{80}

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    - **“Exponential time algorithms typically arise when we solve problems by exhaustively searching a space of solutions (brute-force search).”**

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  - **“Exponential time algorithms typically arise when we solve problems by exhaustively searching a space of solutions (brute-force search).”**
  - **This means we try all possible combinations to reach a solution.**

The Class P

- **Can we avoid exponential time algorithms?**
  - **Yes**

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- **Can we avoid exponential time algorithms?**
  - **Yes, but not always…**
  - **A deep understanding of the problem can help us design a polynomial time algorithm.**

- **From now on, we will ignore polynomial differences in the running time of algorithms.**
  - **e.g., $O(n)$, $O(n^2)$, and $O(n^3)$ will be treated equally: they are all members of the Class P.**
  - **Just like we did when we ignored constants for *big-O*.**

The Class P

Definition 7.12

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k TIME(n^k).$$

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P corresponds to the class of problems that are *realistically* solvable on a computer.

The Class P: Analyzing Algorithms for Polynomiality

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- **Describe algorithms as a sequence of ordered stages.**

- **Doing so allows us to analyze these stages for polynomiality.**

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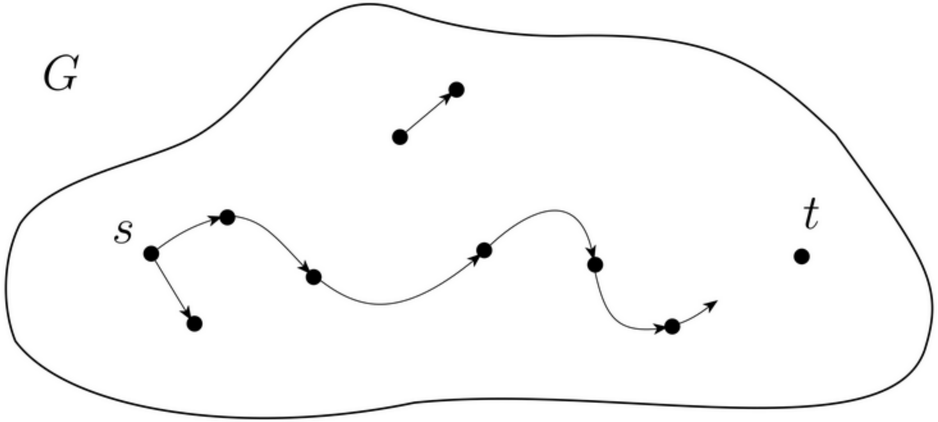

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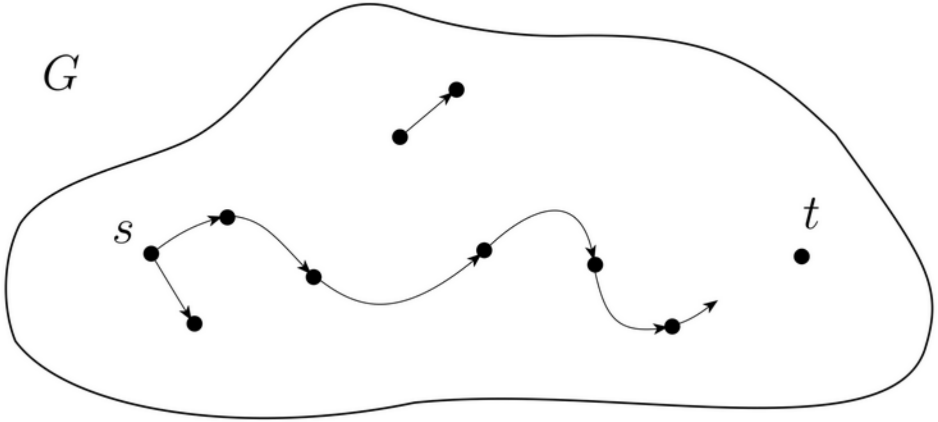
$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}.$



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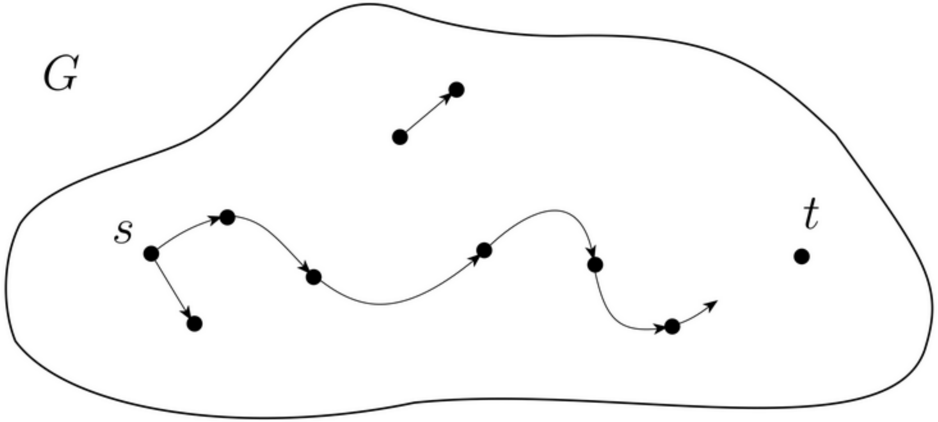
Theorem 7.14

$PATH \in P$



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  - **Examine ALL potential paths in** G**, and determine whether there is a path from** s **to** t**.**

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YES

The Class P: Example Problems (PATH)

Algorithm M for PATH

M = "On input $\langle G, s, t\rangle$, where G is a directed graph with nodes s and t:

1. Place a mark on node s.
2. Repeat the following until no additional nodes are marked:
3. Scan all the edges of *G*. If an edge (*a, b*) is found going from a marked node *a* to an unmarked node *b*, mark node *b*.
4. If *t* is marked, *accept*. Otherwise, *reject*."

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Executed only once

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Runs at most *m* times

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- **Clearly, this approach goes through an exponential number of potential divisors, and has an exponential running time.**

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See page 289 for algorithm

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  - **A decider can try all possible derivations of length *k* (*2n-1*)**

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- **One idea:**
  - **A decider can try all possible derivations of length *k* (*2n-1*)**
  - **If any of these is a derivation of *w*, the decider *accepts*. Otherwise, it *rejects*.**
  - **But, the number of derivations with *k* steps may be <u>exponential</u> in *k*.**

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- **To solve this in polynomial time, we leverage <u>dynamic programming</u>.**

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- **We achieve this by recording the solution to any smaller subproblem so that we solve it only once.**

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See page 291 for algorithm

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