King Saud University
College of Sciences
Department of Mathematics

106 Math Exercises

(7)

Hyperbolic Functions
&
Inverse Hyperbolic Functions

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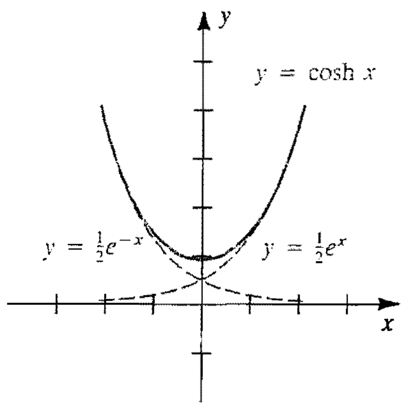
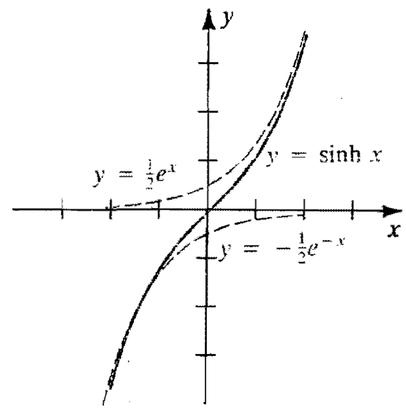
Hyperbolic Functions

Definition:

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \quad , \quad \sinh x = \frac{e^{x}-e^{-x}}{2} \quad : x \in \mathbb{R}$$





$$\cosh u + \sinh u = e^{u} \quad , \quad \cosh u - \sinh u = e^{-u}$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

PROOF By Definition (6.41),

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} \\ &= \frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4} \\ &= \frac{4}{4} = 1.\end{aligned}$$

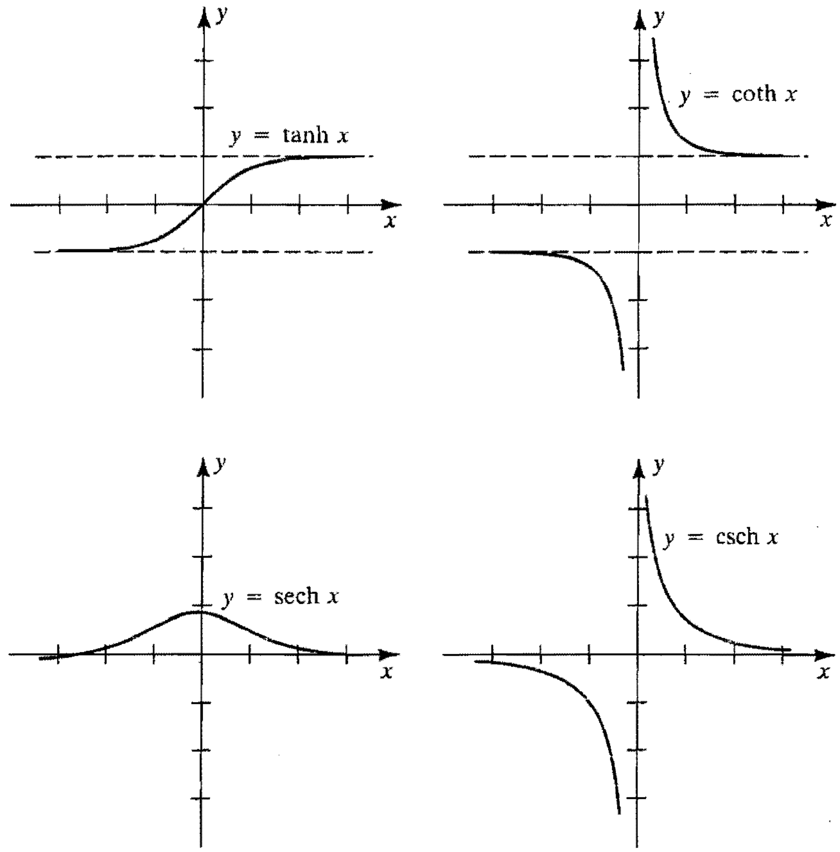
Definition

(i) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

(ii) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}, \quad x \neq 0$

(iii) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x}+e^{-x}}$

(iv) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x}-e^{-x}}, \quad x \neq 0$



Formulas:

$$1 - tanh^2x = sech^2x \quad , \quad coth^2x - 1 = csch^2x$$

Differentiation & Integration

(i) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(ii) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(iv) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(v) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(vi) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

(i) $\int \sinh u \, du = \cosh u + C$

(ii) $\int \cosh u \, du = \sinh u + C$

(iii) $\int \operatorname{sech}^2 u \, du = \tanh u + C$

(iv) $\int \operatorname{csch}^2 u \, du = -\coth u + C$

(v) $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

(vi) $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

Exercises

Q(1): Find $f'(x)$ for the following :

1) $f(x) = e^{sinhx} + \cosh(3^x)$

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2) $f(x) = \operatorname{sech}(x^2+1) + 3^{tanhx}$

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3) $f(x) = x^{coshx}$

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4) $f(x) = tan^{-1}(sinhx)$

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5) $f(x) = \ln[\, cosh\,(x^3+1)]$

Q(2) Evaluate the following integrals :

1)

$$\int x^2 \cosh\left(x^3\right) dx$$

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2)

$$\int \frac{1}{cosh^2(3x)} dx$$

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3)

$$\int \sinh\left(1-3x\right) dx$$

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4)

$$\int \frac{csch\left(\frac{1}{x}\right) coth\left(\frac{1}{x}\right)}{x^2} dx$$

---

5)

$$\int \frac{cosh\sqrt{x}}{\sqrt{x}} dx$$

6)

$$\int sech^3x\ tanhx\ dx$$

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7)

$$\int cosh^2(x-1)\sinh(x-1)\,dx$$

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8)

$$\int \frac{sinhx}{1+sinh^2x}dx$$

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9)

$$\int sinhx\ sech^2x\ dx$$

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10)

$$\int coshx\ csch^2x\ dx$$

11)

$$\int \cosh^2 x \, dx$$

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12)

$$\int \sinh^2 x \, dx$$

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13)

$$\int e^x \cosh x \, dx$$

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14)

$$\int e^x \sinh^2 x \, dx$$

15)

$$\int \frac{e^x}{(coshx - sinhx)^2} dx$$

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16)

$$\int \frac{e^x}{(coshx + sinhx)^3} dx$$

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17)

$$\int e^x (coshx - sinhx)^4 \, dx$$

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18)

$$\int \frac{5^{coshx}}{cschx} dx$$

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19)

$$\int \frac{sinh^2 x}{e^{-x}} dx$$

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20)

$$\int \frac{\cosh{(lnx)}}{x} dx$$

21)

$$\int cosh^2(2x)\,dx$$

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22)

$$\int \frac{e^x}{coshx}\,dx$$

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23)

$$\int e^{2x} sinhx\,dx$$

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24)

$$\int 2^x \sinh(2^x)\,dx$$

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25)

$$\int \frac{sinhx}{coshx + sinhx}\,dx$$

26)

$$\int \frac{\sinh x}{\cosh x + \sinh x}\, dx$$

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27)

$$\int \frac{e^{5x}}{(\cosh 2x + \sinh 2x)^2}\, dx$$

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28)

$$\int_{-1}^{1} \sinh(x)\, dx$$

Q(3) Find the domain of the function $f(x) = \ln(\cosh x - 1)$

Inverse Hyperbolic Functions

Definition:

(i) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), \quad x \geq 1$

(iii) $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1$

(iv) $\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x}, \quad 0 < x \leq 1$

Differentiation & Integration

(i) $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

(ii) $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$

(iii) $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$

(iv) $\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$

(i) $\int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\frac{u}{a} + C, \quad a > 0$

(ii) $\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\frac{u}{a} + C, \quad 0 < a < u$

(iii) $\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C, \quad |u| < a$

(iv) $\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + C, \quad 0 < |u| < a$

(v)

(a) $\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1} = \tanh^{-1}\left(\frac{1}{x}\right), \quad |x| > 1$

(b) $\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$

(c) $\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\coth^{-1}\frac{u}{a} + C, \qquad |u| > a$

(vi)

(a) $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) = \sinh^{-1}\left(\frac{1}{x}\right), \quad x \neq 0$

(b) $\frac{d}{dx}(\operatorname{csch}^{-1} u) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \qquad |u| \neq 0$

(c) $\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|u|}{a} + C, \quad u \neq 0$

Exercises

Q(1) Find $f'(x)$ for the following :

(1) $f(x) = 3^{sinh^{-1}x} + tanh^{-1}(e^{2x})$

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(2) $f(x) = ln[cosh^{-1}(4x)] - cosh^{-1}(ln4x)$

---

(3) $f(x) = tanh^{-1}(sin3x) + sech^{-1}\sqrt{1-x}$

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(4) $f(x) = (sech^{-1}x)^{-3} + csch^{-1}(3x)$

Q(2) Evaluate the following integrals :

(1)

$$\int \frac{1}{\sqrt{64+25x^2}}\,dx$$

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(2)

$$\int \frac{1}{\sqrt{16x^2-9}}\,dx$$

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(3)

$$\int \frac{1}{49-9x^2}\,dx$$

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(4)

$$\int \frac{e^x}{\sqrt{e^{2x}-16}}\,dx$$

(5)

$$\int \frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx$$

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(6)

$$\int \frac{2}{5-3x^2}\,dx$$

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(7)

$$\int \frac{1}{\sqrt{16-e^{2x}}}\,dx$$

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(8)

$$\int_4^8 \frac{x}{x^4-16}\,dx$$

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(9)

$$\int \frac{1}{\sqrt{x}\sqrt{9+x}}\,dx$$

(10)

$$\int \frac{1}{\sqrt{1+e^{2x}}}dx$$

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(11)

$$\int \frac{e^{x}}{\sqrt{16+9e^{2x}}}dx$$

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(12)

$$\int \frac{1}{x\sqrt{1-x}}dx$$