King Saud University

College of Sciences

Department of Mathematics

106 Math Exercises

(21)

Arc length

&

Surface Area

(In Polar Coordinates)

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ARC LENGTH OF A POLAR CURVE

The arc length of the polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Examples : Find the arc length of the following polar curves :

1. $r = 1 + \cos\theta\ ,\ 0 \le \theta \le 2\pi$

$$\frac{dr}{d\theta} = -\sin\theta$$

Since $r = 1 + \cos\theta$ is symmetric with respect to the polar axis then

$$L = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\ d\theta$$

$$L = 2\int_0^{\pi} \sqrt{(1+2\cos\theta+\cos^2\theta) + \sin^2\theta}\ d\theta$$

$$L = 2\int_0^{\pi} \sqrt{2+2\cos\theta}\ d\theta$$

$$L = 2\int_0^{\pi} \sqrt{2(1+\cos\theta)}\ d\theta$$

Note that $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1+\cos\theta) \Rightarrow 2(1+\cos\theta) = 4\cos^2\left(\frac{\theta}{2}\right)$

$$L = 2\int_0^{\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\ d\theta = 2\int_0^{\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right|\ d\theta$$

$$L = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\ d\theta = 8\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 8(1-0) = 8$$

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2. $r = 2\cos\theta\ ,\ 0 \le \theta \le 2\pi$

$$\frac{dr}{d\theta} = -2\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\ d\theta$$

$$L = \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$L = \int_0^{2\pi} \sqrt{4}\, d\theta = \int_0^{2\pi} 2\, d\theta = [2\theta]_0^{2\pi} = 4\pi$$

Note that $r = 2\cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is a circle with center $= (1, 0)$ and radius equals 1 , therefore its circumference equals 2π , in this example $r = 2\cos\theta$, $0 \leq \theta \leq 2\pi$ which means that the curve is doubled , hence the circumference is also doubled .

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3. $r = e^{-\theta}$, $0 \leq \theta \leq \pi$

$$\frac{dr}{d\theta} = -e^{-\theta}$$

$$L = \int_0^{\pi} \sqrt{\left(e^{-\theta}\right)^2 + \left(-e^{-\theta}\right)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta = \int_0^{\pi} \sqrt{2e^{-2\theta}} d\theta$$

$$L = \int_0^{\pi} \sqrt{2}\left|e^{-\theta}\right|\, d\theta = \sqrt{2}\int_0^{\pi} e^{-\theta} d\theta$$

$$L = \sqrt{2}\left[-e^{-\theta}\right]_0^{\pi} = \sqrt{2}\left[-e^{-\pi} + e^0\right] = \sqrt{2}\left(1 - e^{-\pi}\right)$$

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4. $r = 2sin\theta \quad ,0 \leq \ \theta \ \leq \frac{\pi}{2}$

5. $r = {}^{1}/_{\theta}\ : from\ \theta = 1\ \ to\ \ \theta = 2$

6. $r = \theta \ \ from\ \theta = 0 \ \ to \ \ \theta = 4\pi$

7. $r = 2^{\theta}$ $from\ \theta = 0\ \ to\ \ \theta = \pi$

8. $r = cos^2\left(\frac{\theta}{2}\right)$ $from\ \theta = 0\ \ to\ \ \theta = \pi$

SURFACE AREA GENERATED BY REVOLVING A POLAR CURVE

The surface area generated by revolving the polar curve $r = r(\theta)$, $\theta_1 \le \theta \le \theta_2$ around the polar axis is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The surface area generated by revolving the polar curve $r = r(\theta)$, $\theta_1 \le \theta \le \theta_2$ around the line $\theta = \frac{\pi}{2}$ is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Examples : Find the surface area generated by revolving the following polar curves :

1. $r = e^{\frac{\theta}{2}}$, $0 \le \theta \le \pi$, around the polar axis .

$$\frac{dr}{d\theta} = \frac{1}{2} e^{\frac{\theta}{2}}$$

$$SA = 2\pi \int_0^{\pi} \left|e^{\frac{\theta}{2}}\sin\theta\right| \sqrt{\left(e^{\frac{\theta}{2}}\right)^2 + \left(\frac{1}{2}e^{\frac{\theta}{2}}\right)^2}\, d\theta$$

$$SA = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}}\sin\theta \sqrt{e^{\theta} + \frac{1}{4}e^{\theta}}\, d\theta = \int_0^{\pi} e^{\frac{\theta}{2}}\sin\theta \left|e^{\frac{\theta}{2}}\right| \sqrt{1 + \frac{1}{4}}\, d\theta$$

$$SA = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}}\sin\theta\ e^{\frac{\theta}{2}} \sqrt{\frac{5}{4}}\, d\theta = 2\pi \frac{\sqrt{5}}{2} \int_0^{\pi} e^{\theta}\sin\theta\, d\theta$$

Using integration by parts

$$SA = \sqrt{5}\pi \left[\frac{1}{2}e^{\theta}(\sin\theta - \cos\theta)\right]_0^{\pi} = \frac{\sqrt{5}\pi}{2}\left(e^{\pi} + 1\right)$$

2. $r = 2 + 2\cos\theta$, $0 \le \theta \le \frac{\pi}{2}$, around the polar axis .

$$\frac{dr}{d\theta} = -2\sin\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |(2 + 2\cos\theta)\sin\theta| \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)\sin\theta \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)\sin\theta\sqrt{8+8\cos\theta}\ d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)\sin\theta\sqrt{4(2+2\cos\theta)}\ d\theta$$

$$SA = 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)\sin\theta\sqrt{2+2\cos\theta}\ d\theta$$

$$SA = 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)^{\frac{3}{2}}\sin\theta\ d\theta$$

$$SA = -2\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)^{\frac{3}{2}}(-2\sin\theta)\ d\theta$$

$$SA = -2\pi\left[\frac{2}{5}(2+2\cos\theta)^{\frac{5}{2}}\right]_0^{\frac{\pi}{2}} = -2\pi\frac{2}{5}\left[4\sqrt{2}-32\right] = \frac{16\pi}{5}(8-\sqrt{2})$$

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3. $r=\cos\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, around the line $\theta=\frac{\pi}{2}$

$$\frac{dr}{d\theta} = -\sin\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta\ \ \cos\theta|\ \sqrt{(\cos\theta)^2+(-\sin\theta)^2}\ d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\cos^2\theta\right|\ \sqrt{\cos^2\theta+\sin^2\theta}\ d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\ d\theta$$

$$SA = \pi\left[\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi\left[\left(\frac{\pi}{2}+0\right)-\left(-\frac{\pi}{2}+0\right)\right] = \pi^2$$

4. $r = 2\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, around the line $\theta = \frac{\pi}{2}$

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |2\sin\theta \ \cos\theta| \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2} \ d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |\sin 2\theta| \sqrt{4\sin^2\theta + 4\cos^2\theta} \ d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} \sin 2\theta \ \sqrt{4} \ d\theta$$

$$SA = 4\pi \left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 4\pi$$

Note : it is the surface area of a sphere of radius 1.

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5. $r = 1 + cos\theta \ : \ \theta \in [0, \pi]$, around the polar axis .

6. $r = 4cos\theta \ : \ \theta \in \left[0, \frac{\pi}{2}\right]$, around the polar axis .

7. $r = e^{\theta} \quad from \quad \theta = 0 \;\; to \;\; \theta = 1$, around the line $\theta = \frac{\pi}{2}$