Q Prove that $1.1! + 2.2! + \cdots + n.n!$
$= (n+1)! - 1$
whenever n is a positive integer.
[Use Mathematical Induction].

Let $P(n)$ be $1.1! + 2.2! + \cdots + n.n! = (n+1)! - 1$

Basis step $n = 1$

$1.1! + 2.2! + \cdots + n.n! = 1.1! = 1.1 = 1$

$(n+1)! - 1 = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

So, $P(1)$ is true.

Induction step: Let $P(k)$ be true.

$1.1! + 2.2! + \cdots + k.k! = (k+1)! - 1$

We need to prove that $P(k+1)$ is also true

$1.1! + 2.2! + \cdots + k.k! + (k+1).(k+1)!$

$= (k+1)! - 1 + (k+1).(k+1)!$

$= 1.(k+1)! + (k+1).(k+1)! - 1$

$= (1 + k+1)(k+1)! - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$

$= ((k+1)+1)! - 1$

So, $P(k+1)$ is also true. Proved.

Q Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

(a) Show statements $P(18)$, $P(19)$, $P(20)$, $P(21)$ are true completing the basis step of proof.

(b) what is the inductive hypothesis of the proof.

(c) what do you need to prove in the inductive step.

(d) Complete the Inductive step for $K \geq 21$.

(e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Sol:

(a) Basis step $n=18$, $n=19$, $n=20$, $n=21$

$P(18)$ is true because $2 \cdot 7 + 4 = 18$

$P(19)$ is true because $7 + 3 \cdot 4 = 19$

$P(20)$ is true because $5 \cdot 4 = 20$

$P(21)$ is true because $7 \cdot 3 = 21$

(b) Inductive hypothesis: We assume that $P(18), P(19), \ldots, P(k)$ are all true, thus any postage between 18 and k cents can be formed with 4 and 7 cents.

(c) Inductive step: We need to prove that $P(k+1)$ is true, so we need to prove that $k+1$ cents can be formed using 4 and 7 cents only.

(d) By Inductive hypothesis we know that $P(k-3)$ is true and a postage of $k-3$ cents can be formed using 4 and 7 cent stamps.

Since $k+1=(k-3)+4$, $P(k+1)$ is then true, because the number of 7 cents stamps are same for $k+1$ as for $k-3$, while 4 cents stamps is 1 more for $k+1$ than for $k-3$.

By the principle of Strong Induction $P(n)$ is true for all positive

(e) By the principle of Strong Induction $P(n)$ is true for all positive integers n with $n \geq 18$.

Q Find the prime factorization of the following:

a) 39
b) 81
c) 101
d) 1001
e) 289
f) 899

Sol

a) $3 \cdot 13$
b) 3^4
c) $1 \cdot 101$
d) $7 \cdot 11 \cdot 13$
e) 17^2
f) $29 \cdot 31$

Q Find prime factorization of $10!$.

Sol

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$\Rightarrow 2 \cdot 5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 1$

$\Rightarrow 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$

Q Which positive integers less than 30 are relatively prime to 30.

Sol

1, 7, 11, 13, 17, 19, 23, 29

because two numbers are relatively prime if their gcd is 1.

Q Find gcd of the following:

a) $2^2, 3^3, 5^5$, $2^5, 3^3, 5^2$

b) $2.3.5.7.11.13$, $2^{11}.3^9.11.17^{14}$

c) $17, 17^{17}$.

d) $2^2.7$, $5^3.13$

e) $2.3.5.7$, $2.3.5.7$

Sol

a) $2^2.3^3.5^2$

b) $2.3.11$

c) 17

d) 1

e) $2.3.5.7$

Q what is the LCM in the previous exercise.

Sol

a) $2^5, 3^3.5^5$

b) $2^{11}, 3^9.5.7.11.13.17^{14}$

c) 17^{17}

d) $2^2.7.5^3.13$

e) $2.3.5.$

Q If product of 2 integers is $2^7 \, 3^2 . 5^2 . 7^{11}$. If their gcd is $2^3 . 3^4 . 5$ then what is the LCM? فيه خطأ بالسؤال بأحد الأسس، المفروض يكون أكبر.

Sol

$$2^4 . 3^4 . 5 . 7^{11}$$