King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(1)

Sentential (Propositional) Calculus

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Algebraic Properties of Connectives

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بفرض p,q,r تقارير (Statements) :
                                                                                  : ( Commutative Rules) قاعدتا الإبدال (1
                                                                                                   p \wedge q \equiv q \wedge p \quad (i)
                            p \lor q \equiv q \lor p \ (\because)
                                                                                   : (Associative Rules) قاعدتا التجميع (2)
       (p \lor q) \lor r \equiv p \lor (q \lor r) ( \hookrightarrow )
                                                                                  (p \land q) \land r \equiv p \land (q \land r)^{\binom{1}{2}}
                                                                                  : (Distributive Rules) قاعدتا التوزيع (3)
p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) ( \rightarrow )
                                                                      p \land (q \lor r) \equiv (p \land q) \lor (p \land r) (i)
                                                                                : (Identity Rules) قاعدتا العنصر المحايد (4)
                                                                                                             p \wedge T \equiv P(1)
                                       p \lor F \equiv p ( \hookrightarrow )
                                                                                       : (Negation Rules) قاعدتا النفي (5)
                                 p \land \neg p \equiv F ( \hookrightarrow )
                                                                                                        p \vee \neg p \equiv T (1)
                                       \neg (\neg p) \equiv p
                                                                      : ( Double Negation Rule ) قاعدة نفى النفى ( 6 )
                                                                                 : ( Idempotent Rules ) قاعدتا الجمود ( 7 )
                                                                                                          p \lor p \equiv p \ (\ )
                                       p \land p \equiv p ( \hookrightarrow )
                                                                           : ( DeMorgan's Rules ) قاعدتا ديمورجان ( 8 )
                   \neg (p \lor q) \equiv \neg p \land \neg q ( \hookrightarrow )
                                                                                        \neg (p \land q) \equiv \neg p \lor \neg q(^{\dagger})
                                                                                  : ( Universal Rules ) قاعدتا الشمول ( 9 ) قاعدتا
                                                                                                           p \vee T \equiv T (1)
                                       p \wedge F \equiv F ( \hookrightarrow )
                                                                           : ( Absorption Rules ) قاعدتا الإمتصاص (10)
                                                                                                  p \lor (p \land q) \equiv p(f)
                             p \land (p \lor q) \equiv p ( \hookrightarrow )
                                                                : ( Alternative proof Rules ) قاعدتا البرهان البديل ( 11)
                                            p \to (q \lor r) \equiv (p \land \neg q) \to r \equiv (p \land \neg r) \to q (i)
                                                                 (p \lor q) \rightarrow r \equiv (p \rightarrow r) \land (q \rightarrow r) ( \rightarrow )
                                                                                : ( Conditional Rules ) قاعدتا الشرط ( 12)
                                                                                                 p \rightarrow q \equiv \neg p \lor q  ( ^{\dagger} )
                     \neg(p \to q) \equiv p \land \neg q \quad (\because)
                                                                       : ( Biconditional Rules ) قواعد ثنائي الشرط ( 13)
                                                                            p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) (1)
                                                                        p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q) ( \because )
                                                                         p \leftrightarrow q \equiv (\neg p \lor q) \land (p \lor \neg q) (\tau)
                          p \rightarrow q \equiv \neg q \rightarrow \neg p : ( Rule of Contrapositive ) قاعدة المكافئ العكسى (14)
                                                (15) قاعدة الإنطلاق والوصول (Exportation – Importation Rule ):
                          p \rightarrow (q \rightarrow r) \equiv (p \land q) \rightarrow r
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في التقرير الشرطي p o q ، يسمى التقرير p (المقدمة Antecedent)، بينما يسمى التقرير p o q).

يقترن بالتقرير الشرطي q o q تقارير شرطية أخرى هي :

 $q \rightarrow p : (Converse)$

 $\neg p \rightarrow \neg q$: (Inverse) المعكوس المكافئ العكسي (Contrapositive) المكافئ

p	q	$p \wedge q$	$p \lor q$	p o q	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
1		-		-	

يكون صواباً T إذا كان كل منهما صائباً T، عدا ذلك يكون خاطئاً . $p \wedge q$

يكون خاطئاً F إذا كان كل منهما خاطئاً F، عدا ذلك يكون صواباً $p \vee q$

یکون خاطئاً F اِذا کان p صائباً T و کان p خاطئاً F ، عدا ذلك یکون صواباً .

یکون صواباً T اِذا کان کل منهما صائباً T ، أو اِذا کان کل منهما خاطئاً T ، عدا ذلك یکون خاطئاً . $p \leftrightarrow q$

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

TABLE 1 Examples of a Tautology and a Contradiction.						
p	$\neg p$	$p \vee \neg p$	$p \land \neg p$			
T F	F T	T T	F F			

TABLE 2 De Morgan's Laws.				
$\neg(p \land q) \equiv \neg p \lor \neg q$				
$\neg(p \vee q) \equiv \neg p \wedge \neg q$				

EXAMPLE 2 Show that $\neg (p \lor q)$ and $\neg p \land \neg q$ are logically equivalent.

TABI	TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.									
p	\boldsymbol{q}	$p \vee q$	$\neg (p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$				
T	T	Т	F	F	F	F				
T	F	T	F	F	T	F				
F	T	T	F	T	F	F				
F	F	F	Т	T	T	Т				

EXAMPLE 3 Show that $p \to q$ and $\neg p \lor q$ are logically equivalent.

TABLE 4 Truth Tables for $\neg p \lor q$ and $p \to q$.								
p	p q $\neg p$ $\neg p \lor q$ $p \to q$							
T	T	F	Т	Т				
T	F	F	F	F				
F	T	T	T	T				
F	F	T	Т	Т				

EXAMPLE 4 Show that $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

	TABLE 5 A Demonstration That $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ Are Logically Equivalent.										
p	\boldsymbol{q}	r	$q \wedge r$	p	∨ (q ∧ ı	r)	$p \vee q$	$p \vee r$	(p v	q) ^ (p	∨ <i>r</i>)
T	T	T	Т		T		Т	T		T	
T	T	F	F		T		T	T		T	
T	F	T	F		T		T	T		T	
T	F	F	F		T		T	T		T	
F	T	T	T		T		T	T		T	
F	T	F	F		F		T	F		F	
F	F	T	F		F		F	T		F	
F	F	F	F		F		F	F		F	

TABLE 6 Logical Equivalences.							
Equivalence	Name						
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws						
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws						
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws						
$\neg(\neg p) \equiv p$	Double negation law						
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws						
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws						
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws						
$\neg (p \land q) \equiv \neg p \lor \neg q$ $\neg (p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws						
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws						
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws						

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \lor q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \lor q \equiv \neg p \rightarrow q$$

$$p \land q \equiv \neg (p \rightarrow \neg q)$$

$$\neg (p \rightarrow q) \equiv p \land \neg q$$

$$(p \rightarrow q) \land (p \rightarrow r) \equiv p \rightarrow (q \land r)$$

$$(p \rightarrow r) \land (q \rightarrow r) \equiv (p \lor q) \rightarrow r$$

$$(p \rightarrow q) \lor (p \rightarrow r) \equiv p \rightarrow (q \lor r)$$

$$(p \rightarrow r) \lor (q \rightarrow r) \equiv (p \land q) \rightarrow r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$$\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Exercises

Q1- Show whether the following statement is a tautology or a contradiction or a contingency:

1)
$$(p \land q) \rightarrow (\neg p \rightarrow q)$$

Solution:

p	q	$\neg p$	$p \wedge q$	$\neg p \rightarrow q$	$(p \land q) \rightarrow (\neg p \rightarrow q)$
T	T				
T	F				
F	T				
F	F				

(By rules "without using the truth tables")

$$2) \qquad [\neg p \land (p \lor q)] \rightarrow q$$

Solution:

p	q	$\neg p$	$p \lor q$	$\neg p \land (p \lor q)$	$[\neg p \land (p \lor q)] \rightarrow q$
T	T				
T	F				
F	T				
F	F				

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$$[\neg p \land (p \lor q)] \rightarrow q \equiv \neg [\neg p \land (p \lor q)] \lor q \qquad (Conditional Rule)$$

$$\equiv p \lor \neg (p \lor q) \lor q \qquad (DeMorgan's Rule)$$

$$\equiv (p \lor q) \lor \neg (p \lor q) \qquad (Commutative and Associative Rules)$$

$$\equiv x \lor \neg x \equiv t \qquad (Negation Rule)$$

$$: x \equiv p \lor q \qquad (Substitution Rule for Logical Equivalence)$$

3)
$$\neg (p \rightarrow q) \rightarrow \neg q$$

Solution:

p	q	$\neg q$	$p \rightarrow q$	$\neg (p \rightarrow q)$	$\neg (p \rightarrow q) \rightarrow \neg q$
T	T				
T	F				
F	T				
F	F				

.....

$$4) \qquad [p \land (p \rightarrow q)] \rightarrow q$$

Solution:

p	q	$p \rightarrow q$	$p \land (p \rightarrow q)$	$[p \land (p \to q)] \to q$
T	T			
T	F			
F	T			
F	F			

$$5) \quad (p \land q) \rightarrow (p \rightarrow q)$$

Solution:

p	q	$p \wedge q$	$p \rightarrow q$	$(p \land q) \to (p \to q)$
T	T			
Т	F			
F	T			
F	F			

6)
$$(p \lor \neg q) \rightarrow (p \land q)$$

p	q	$\neg q$	$p \lor \neg q$	$p \wedge q$	$(p \lor \neg q) \to (p \land q)$
T	T				
T	F				
F	T				
F	F				

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7)
$$p \land \neg [q \rightarrow (p \lor r)]$$

p	q	r	$p \lor r$	$q \rightarrow (p \lor r)$	$\neg[\ q \rightarrow (p \lor r)]$	$p \land \neg [q \rightarrow (p \lor r)]$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	Т				
F	T	F				
F	F	Т				
F	F	F				

(By rules " without using the truth tables)

8)
$$\neg u \rightarrow [(u \land v) \rightarrow w]$$

u	v	w	$\neg u$	$u \wedge v$	$(u \wedge v) \rightarrow w$	$\neg u \rightarrow [(u \land v) \rightarrow w]$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

(By rules " without using the truth tables)

9) Show whether the following statement is a tautology or a contradiction?

$$[\,(\,p\,\rightarrow q\,)\,\vee(\,q\,\rightarrow r\,)]\rightarrow(\,p\,\rightarrow\,\neg r\,)$$

10) Show that the following statement is a tautology $[(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r)$?

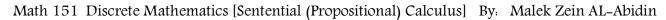
11) Show that the following statement is a tautology:

$$(p \land q) \rightarrow (r \rightarrow q)$$
 ?

12) Prove the following statement is a tautology, without using the truth table:

$$(p \land q) \rightarrow [(q \lor r) \rightarrow p]$$

$$\begin{array}{l} \begin{array}{l} \underline{Frooj:} \\ \hline (p \land q) \rightarrow [(q \lor r) \rightarrow p] \equiv \neg (p \land q) \lor [\neg (q \lor r) \lor p] \\ \hline \equiv (\neg p \lor \neg q) \lor [(\neg q \land \neg r) \lor p] \\ \hline \equiv (\neg p \lor \neg q) \lor [(\neg q \lor p) \land (\neg r \lor p)] \\ \hline \equiv [(\neg p \lor \neg q) \lor (\neg q \lor p)] \land [(\neg p \lor \neg q) \lor (\neg r \lor p)] \\ \hline \equiv [\neg p \lor \neg q \lor \neg q \lor p] \land [\neg p \lor \neg q \lor \neg r \lor p] \\ \hline \equiv [(\neg p \lor p) \lor \neg q] \land [(\neg p \lor p) \lor (\neg q \lor \neg r)] \\ \hline \equiv [T \lor \neg q] \land [T \lor (\neg q \lor \neg r)] \\ \hline \equiv T \land T \equiv T \end{array}$$

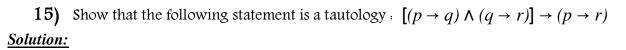


13) Without using the truth table, show whether the following statement is a tautology or not :

$$(p \land q) \rightarrow [r \rightarrow (p \lor q)]$$

Solution:

14) Show that the following statement is a tautology : $[(p \lor q) \land (p \to r) \land (q \to r)] \to r$ Solution:



16) Show that the following statement is a tautology: $[p \leftrightarrow (q \lor r)] \rightarrow [(\neg q \land \neg r) \lor p]$ **Solution:**

Q2: 1) Show that $\neg (p \rightarrow q)$ and $p \land \neg q$ are logically equivalent?

Solution:

p	q	$\neg q$	$p \rightarrow q$	$\neg (p \rightarrow q)$	$p \land \neg q$
T	T				
T	F				
F	T				
F	F				

$$\neg(p \to q) \equiv \neg(\neg p \lor q) \qquad \text{by condition law}$$

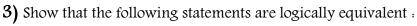
$$\equiv \neg(\neg p) \land \neg q \qquad \text{by the second De Morgan law}$$

$$\equiv p \land \neg q \qquad \text{by the double negation law}$$

2) Show that $\neg (p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences.

p	q	¬p	$\neg q$	$\neg p \land q$	$p \lor (\lnot p \land q)$	$\neg (p \lor (\neg p \land q))$	$\neg p \land \neg q$
T	T						
T	F						
F	T						
F	F						

$$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$$
 by the second De Morgan law
$$\equiv \neg p \land [\neg(\neg p) \lor \neg q]$$
 by the first De Morgan law
$$\equiv \neg p \land (p \lor \neg q)$$
 by the double negation law
$$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$$
 by the second distributive law
$$\equiv \mathbf{F} \lor (\neg p \land \neg q)$$
 because $\neg p \land p \equiv \mathbf{F}$ by the commutative law for disjunction
$$\equiv \neg p \land \neg q$$
 by the identity law for \mathbf{F}



$$(p \rightarrow q) \rightarrow q \equiv (p \lor q)$$

Solution:

4) Show that the following statements are logically equivalent:

$$(p \rightarrow q) \rightarrow (\neg p \rightarrow r) \equiv p \lor r$$

5) Show that the following statements are logically equivalent:

$$(p \rightarrow q) \land (p \rightarrow r) \equiv (\neg q \lor \neg r) \rightarrow \neg p$$

Solution:

6) Show that the following statements are logically equivalent:

$$(p \to q) \lor (p \to r) \equiv (\neg p \lor q) \lor (\neg p \lor r) \qquad (Conditional Rule)$$

$$\equiv \neg p \lor q \lor \neg p \lor r$$

$$\equiv [(\neg p \lor \neg p) \lor q] \lor r \qquad (Commutative and Associative Rules)$$

$$\equiv (\neg p \lor q) \lor r \qquad (Idempotent Rule)$$

$$\equiv (p \to q) \lor r \qquad (Conditional Rule)$$

7) Show that the following statements are logically equivalent:

$$(\neg p \lor \neg r) \rightarrow (p \land q) \equiv p \land (q \lor r)$$

Solution:

8) Show that the following statements are logically equivalent:

$$(p \lor q) \rightarrow r \equiv \neg (p \lor q) \lor r \qquad (Conditional Rule)$$

$$\equiv (\neg p \land \neg q) \lor r \qquad (DeMorgan's Rule)$$

$$\equiv (\neg p \lor r) \land (\neg q \lor r) \qquad (Distributive Rule)$$

$$\equiv (p \rightarrow r) \land (q \rightarrow r) \qquad (Conditional Rule)$$

9) Show that the following statements are logically equivalent:

$$(p \lor q) \rightarrow (\neg p \land r) \equiv \neg p \land (q \rightarrow r)$$

Solution:

$$(p \lor q) \to (\neg p \land r) \equiv \neg (p \lor q) \lor (\neg p \land r) \qquad (Conditional Rule)$$

$$\equiv (\neg p \land \neg q) \lor (\neg p \land r) \qquad (DeMorgan's Rule)$$

$$\equiv \neg p \land (\neg q \lor r) \qquad (Distributive Rule)$$

$$\equiv \neg p \land (q \to r) \qquad (Conditional Rule)$$

10) Show that the following statements are logically equivalent:

$$(\neg p \lor \neg r) \rightarrow (p \land q) \equiv p \land (q \lor r)$$

11) Show that the following statements are logically equivalent

$$(p \rightarrow q) \land (q \lor \neg r) \equiv (p \lor r) \rightarrow q$$

Then use it to prove that

$$[(u \lor v) \to w] \land [w \lor \neg(x \land y)] \equiv [(u \lor v) \lor (x \land y)] \to w$$

12) Show that $(p \to q) \land (p \to r)$ and $p \to (q \land r)$ are logically equivalent? Solution:

¹³⁾ Show that $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$ are logically equivalent ? Solution:

14) Show that $(p \to q) \lor (p \to r)$ and $p \to (q \lor r)$ are logically equivalent ? Solution:

15) Show that $(p \to r) \lor (q \to r)$ and $(p \land q) \to r$ are logically equivalent? Solution:

.

16) Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \lor r)$ are logically equivalent? Solution:

¹⁷⁾ Show that $p \leftrightarrow q$ and $(p \to q) \land (q \to p)$ are logically equivalent? Solution:



18) Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent? Solution:

19) Show that the following statements are logically equivalent?

$$\neg p \to (q \to r) \equiv q \to (p \lor r)$$

20) Show that the following statements are logically equivalent?

Solution: $(p \land \neg r) \rightarrow q \equiv (p \land \neg q) \rightarrow r$

21) Show that the following statements are logically equivalent?

Solution: $\neg q \lor \neg [\neg p \lor (p \land q)] \equiv \neg q$

22) Show that the following statements are logically equivalent?

Solution:
$$\neg [p \land (q \lor r)] \equiv (p \rightarrow \neg q) \land (p \rightarrow \neg r)$$

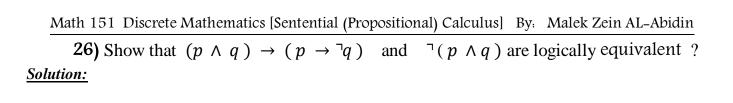
23) Show that the following statements are logically equivalent

24) Show that the following statements are logically equivalent

Solution:
$$(p \land q) \rightarrow r \equiv (p \land \neg r) \rightarrow \neg q$$

25) Show that the following statements are logically equivalent

Solution: $[p \rightarrow (q \rightarrow p)] \land (p \rightarrow r) \land (p \rightarrow \neg r) \equiv \neg p$



27) Show that the following statements are logically equivalent, or not?

$$(p \land q) \rightarrow r \equiv (p \rightarrow r) \land (p \rightarrow r)$$

28) Show that the following statements are logically equivalent, or not?

$$(p \rightarrow q) \lor (p \rightarrow \neg r) \equiv p \rightarrow (r \rightarrow q)$$

Solution:

29) Show that the following statements are logically equivalent?

$$\neg p \lor (q \rightarrow r) \equiv (p \rightarrow r) \lor (q \rightarrow r)$$

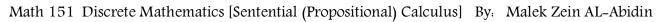


30) Show that the following statements are logically equivalent, without using the truth table.

$$p \leftrightarrow (\neg q \land \neg r) \equiv \neg p \leftrightarrow (q \lor r)$$

Solution:

³¹⁾ Show that $(p \land r) \leftrightarrow (q \land r)$ and $(p \leftrightarrow q) \land r$ are logically equivalent? Solution:



32) Show that the contrapositive of $(p \land q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$

Solution:

33) Show that the contrapositive of $(p \lor q) \to r$ is logically equivalent to $\neg r \to (\neg p \lor \neg q)$

- Q3 Write the contrapositive of the following statements :
 - 1) If mn is an odd number, then m is an odd number and also n is an odd number.

2) If the number 3 divided the numbers m and n, then 3 divided m+n

3) If $m \cdot n = l$, then $m \ge 0$ or $n \ge 0$ or $l \ge 0 : m, n, l \in \mathbb{Z}$

4) If the number a+b-c is an even number, then a is an even or b is an even or c is an even, where $a,b,c\in\mathbb{Z}$

 6) I	If χ is an integer number, then χ is an odd number or χ is an even number .
 6) I	If χ is an integer number, then χ is an odd number or χ is an even number .
 6) I	If x is an integer number, then x is an odd number or x is an even number .
 6) I	If x is an integer number, then x is an odd number or x is an even number .
 6) I	If χ is an integer number, then χ is an odd number or χ is an even number .
 6) I	If χ is an integer number, then χ is an odd number or χ is an even number .
 6) I	If x is an integer number, then x is an odd number or x is an even number.
 6) I	If x is an integer number, then x is an odd number or x is an even number.
6) I	If x is an integer number, then x is an odd number or x is an even number.
U) I	if χ is an integer number, then χ is an odd number of χ is an even number.
,	
7) I	If a is an odd number and b is to an odd number, then $a + b$ is an even number.
8) 1	Find the inverse, converse and also the contrapositive of the following statement:
	"If you study the course 151 Math, and didn't absence from tutorial classes, you will pass course with high grade"