King Saud University
College of Sciences
Department of Mathematics

106 Math Exercises

(19)

Parametric Equations

Malek Zein AL-Abidin

$\dfrac{1440\ H}{2019\ G}$

PARAMETRIC EQUATIONS

Parametric equations are used to describe and represent plane curves.
The parameter "t" is used to write x and y as functions of t .
$C : x = x(t)\ ,\ y = y(t)\ ;\ a \leq t \leq b$ is the general form of a parametric curve , where $a, b \in \mathbb{R}$.
Any point on the parametric curve is represented by $P(t) = (x(t), y(t))$.

Notes :

1. If the parametric curve does not intersect itself then it is called a simple curve.
2. If $P(a) = P(b)$ then the parametric curve is called a closed curve.
3. Parametric equation of a curve indicates its orientation (direction of the path).

Examples : Sketch the graph of the following parametric curves :

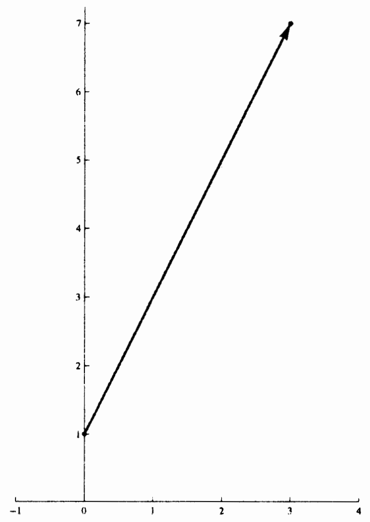
1. $C :\ x = t + 1\ ,\ y = 2t + 3\ ;\ -1 \leq t \leq 2$.

   $x = t + 1 \Rightarrow t = x - 1$

   $y = 2t + 3 \Rightarrow y = 2(x - 1) + 1 = 2x + 1$

t	-1	2
x	0	3
y	1	7

   The parametric equation represents a line segment from $(0, 1)$ to $(3, 7)$



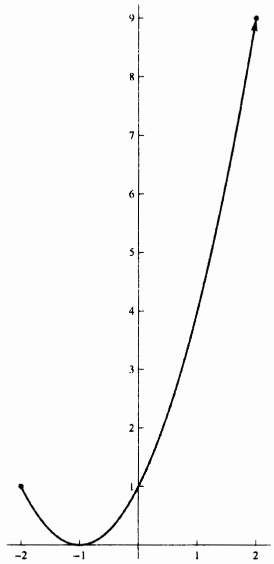
2. $C: \; x = t - 1 \; , \; y = t^2 \; ; \; -1 \leq t \leq 3$

$x = t - 1 \Rightarrow t = x + 1$

$y = t^2 \Rightarrow y = (x+1)^2$

t	-1	3
x	-2	2
y	1	9

The parametric equation represents a part of a parabola from $(-2, 1)$ to $(2, 9)$



3. $C: \; x = 1 + 3\cos t \; , \; y = -1 + 3\sin t \; ; \; 0 \leq t \leq 2\pi$

$x = 1 + 3\cos t \Rightarrow \cos t = \dfrac{x-1}{3}$

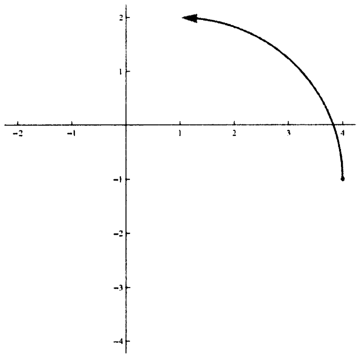
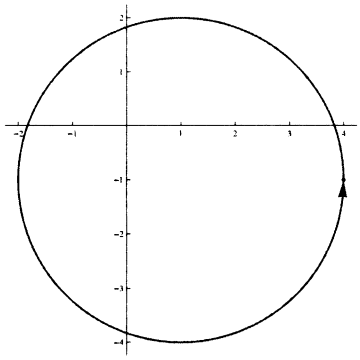
$y = -1 + 3\sin t \Rightarrow \sin t = \dfrac{y+1}{3}$

$\cos^2 t + \sin^2 t = 1 \Rightarrow \dfrac{(x-1)^2}{9} + \dfrac{(y+1)^2}{9} = 1 \Rightarrow (x-1)^2 + (y+1)^2 = 9$

t	0	$\frac{\pi}{2}$	2π
x	4	1	4
y	-1	2	-1

The parametric equation represents a circle with center $=(1, -1)$ and radius $= 3$.

It is a closed curve and its direction is counter-clockwise.

4. $C: \; x = 3 + 3\cos t \;,\; y = 2 + 2\sin t \;;\; 0 \le t \le 2\pi$

$$x = 3 + 3\cos t \Rightarrow \cos t = \frac{x-3}{3}$$

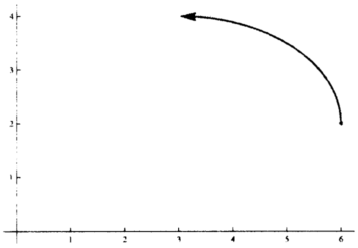
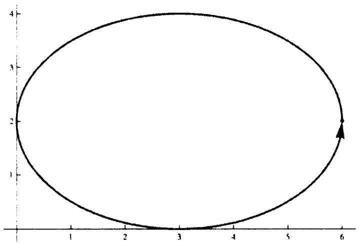
$$y = 2 + 2\sin t \Rightarrow \sin t = \frac{y-2}{2}$$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

t	0	$\frac{\pi}{2}$	2π
x	6	3	6
y	2	4	2

The parametric equation represents an ellipse with center $=(3,2)$,the endpoints of the major axis are $(0,2)$, $(6,2)$ (its length is 6) and the endpoints of the minor axis are $(3,0)$, $(3,4)$ (its length is 4).

it is a closed curve and its direction is counter-clockwise.

The slope of the tangent line to a parametric curve

If $C: \; x = x(t) \; , \; y = y(t) \; ; \; a \leq t \leq b$ is a differentiable parametric curve then the slope of the tangent line to C at $t_0 \in [a, b]$ is

$$m = \frac{dy}{dx}|_{t=t_0} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}|_{t=t_0}$$

Notes :

1. The tangent line to the parametric curve is horizontal if the slope equals zero , which means that $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
2. The tangent line to the parametric curve is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)}$, where $y' = \frac{dy}{dx}$

Examples :

1. The slope of the tangent line to $C: \; x = t^3 + 1 \; , \; y = t^4 - 1$ at $t = 1$ is

   $(a) \; \frac{3}{4} \qquad (b) \; 0 \qquad (c) \; \frac{4}{3} \qquad (d)$ None of these

   Answer : $m = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4t^3}{3t^2}$

   The slope at $t = 1$ is $m|_{t=1} = \frac{4}{3}$

   The right answer is (c) .

2. If $C: \; x = \sqrt{t} \; , \; y = \frac{1}{4}(t^2 - 1)$, find the first and second derivatives at $t = 4$.

   First derivative : $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{1}{2}t\right)}{\left(\frac{1}{2\sqrt{t}}\right)} = t^{\frac{3}{2}}$

   $\frac{dy}{dx}|_{t=4} = (4)^{\frac{3}{2}} = 8$.

   Second derivative : $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{3}{2}t^{\frac{1}{2}}\right)}{\left(\frac{1}{2\sqrt{t}}\right)} = 3t$

$\frac{d^2y}{dx^2}|_{t=4} = 3(4) = 12$.

3. If $C: \ x = 2\cos t$, $y = 2\sin t$, find the first and the second derivatives at $t = \frac{\pi}{4}$.

First derivative : $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2\cos t}{-2\sin t} = -\cot t$

$\frac{dy}{dx}|_{t=\frac{\pi}{4}} = -\cot\left(\frac{\pi}{4}\right) = -1$.

Second derivative : $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\csc^2 t}{-2\sin t} = \frac{-1}{2\sin^3 t}$

$\frac{d^2y}{dx^2}|_{t=\frac{\pi}{4}} = \frac{-1}{2\left(\frac{1}{\sqrt{2}}\right)^3} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$.

4. Find the equation of the tangent line to $C: \ x = t^3 - 3t$, $y = t^2 - 5t - 1$ at $t = 2$.

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t-5}{3t^2-3}$

The slope of the tangent line is $\frac{dy}{dx}|_{t=2} = \frac{2(2)-5}{3(4)-3} = \frac{-1}{9}$

At $t = 2: \ x = (2)^3 - 3(2) = 8 - 6 = 2$ and $y = (2)^2 - 5(2) - 1 = -7$

The tangent line to C at $t = 2$ passes through the point $(2, -7)$ and its slope is $-\frac{1}{9}$, therefore its equation is $\frac{y+7}{x-2} = -\frac{1}{9}$

5. Find the points on $C: \ x = e^t$, $y = e^{-t}$ at which the slope of the tangent line to C equals $-e^{-2}$

$m = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-e^{-t}}{e^t} = -e^{-2t}$

$m = -e^{-2} \Rightarrow -e^{-2t} = -e^{-2} \Rightarrow t = 1$.

At $t = 1: \ x = e^1 = e$ and $y = e^{-1} = \frac{1}{e}$.

Hence, the point at which the slope of the tangent line to C equals $-e^{-2}$ is $\left(e, \frac{1}{e}\right)$.

6. Find the points on $C: \ x = 4 + 4\cos t \ , \ y = -1 + \sin t \ ; \ 0 \leq t \leq 2\pi$ at which the tangent line is : (a) Vertical , (b) Horizontal .

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-4\sin t}$$

(a) The tangent line is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$\frac{dx}{dt} = 0 \Rightarrow -4\sin t = 0 \Rightarrow t = 0 \ , \ t = \pi$

Note that $0, \pi \in [0, 2\pi]$ and $\frac{dy}{dt} \neq 0$ at $t = 0$ or $t = \pi$.

At $t = 0 : x = 4 + 4(1) = 8$ and $y = -1 + 0 = -1$.

At $t = \pi : x = 4 + 4(-1) = 0$ and $y = -1 + 0 = -1$.

Hence, The tangent line to C is vertical at the points $(8, -1)$ and $(0, -1)$.

(b) The tangent line is horizontal if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2} \ , \ t = \frac{3\pi}{2}$

Note that $\frac{\pi}{2}, \frac{3\pi}{2} \in [0, 2\pi]$ and $\frac{dx}{dt} \neq 0$ at $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$.

At $t = \frac{\pi}{2} : x = 4 + 4(0) = 4$ and $y = -1 + 1 = 0$.

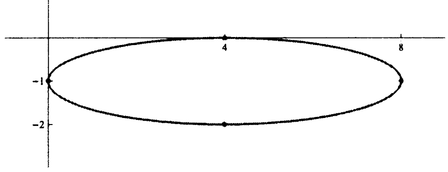
At $t = \frac{3\pi}{2} : x = 4 + 4(0) = 4$ and $y = -1 + (-1) = -2$.

Hence, The tangent line to C is horizontal at the points $(4, 0)$ and $(4, -2)$.

Note : $C: \ x = 4 + 4\cos t \ , \ y = -1 + \sin t \ ; \ 0 \leq t \leq 2\pi$ represents the ellipse $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{1} = 1$, with center $= (4 - 1)$, the endpoints of the major axis are $(0, -1)$ and $(8, -1)$, the endpoints of the minor axis are $(4, 0)$ and $(4, -2)$.

Clearly, there are two vertical tangent lines to C , one passes through $(-1, 0)$ and the other passes through $(8, -1)$.

Also, there are two horizontal tangent lines to C , one passes through $(4, 0)$ and the other passes through $(4, -2)$



Exercises :

1. If $C: \ x = t \ , \ y = t^2$, find the slope of the tangent line to C at $t = 1$.

2. The point at which the curve $C: \ x = 3\cos t \ , \ y = 3\sin t \ ; \ 0 \leq t \leq \pi$ has horizontal tangent line is

   (a) $(0,3)$ (b) $(3,3)$ (c) $(3,0)$ (d) None of these

   (Hint : the parametric curve is the upper half of the circle with center = $(0,0)$ and radius = 3) .

ARC LENGTH OF A PARAMETRIC CURVE

If $C: \ x = x(t) \ , \ y = y(t) \ ; \ a \le t \le b$ is a differentiable parametric curve ,then its arc length equals $L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$.

Examples : Find the arc length of the following parametric curves :

1. $C: \ x = \frac{1}{3}t^3 + 1 \ , \ y = \frac{1}{2}t^2 + 2 \ ; \ \ 0 \le t \le 2$

$$\frac{dx}{dt} = t^2 \text{ and } \frac{dy}{dt} = t$$

$$L = \int_0^2 \sqrt{(t^2)^2 + (t)^2} \, dt = \int_0^2 \sqrt{t^4 + t^2} \, dt = \int_0^2 \sqrt{t^2(t^2+1)} \, dt$$

$$L = \int_0^2 |t|\sqrt{t^2+1} \, dt = \frac{1}{2}\int_0^2 (t^2+1)^{\frac{1}{2}}(2t) \, dt$$

$$L = \frac{1}{2}\left[\frac{2}{3}(t^2+1)^{\frac{3}{2}}\right]_0^2 = \frac{1}{3}\left(5\sqrt{5} - 1\right) .$$

2. $C: \ x = \sin t \ , \ y = \cos t \ ; \ \ 0 \le t \le \frac{\pi}{2}$

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = -\sin t$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t)^2 + (-\sin t)^2} \, dt = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t + \sin^2 t} \, dt$$

$$L = \int_0^{\frac{\pi}{2}} dt = [t]_0^{\frac{\pi}{2}} = \frac{\pi}{2} .$$

Note : The parametric curve represents the first quarter of the unit circle, therefore its arc length equals $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

3. $C: \ x = e^t \cos t \ , \ y = e^t \sin t \ ; \ \ 0 \le t \le \pi$

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$$

$$\frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$$

$$L = \int_0^{\pi} \sqrt{[e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2} \, dt$$

$$L = \int_0^{\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} \, dt$$

$$L = \int_0^{\pi} \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t)} \, dt$$

$$L = \int_0^{\pi} \sqrt{2e^{2t}}\ dt = \int_0^{\pi} \sqrt{2}|e^t|\ dt = \sqrt{2}\int_0^{\pi} e^t\ dt$$

$$L = \sqrt{2}\left[e^t\right]_0^{\pi} = \sqrt{2}(e^{\pi} - 1)\ .$$

SURFACE AREA GENERATED BY REVOLVING A PARAMETRIC CURVE

If $C: \ x = x(t) \ , \ y = y(t) \ ; \ a \le t \le b$ is a differentiable parametric curve ,then the surface area generated by revolving C around the x-axis is

$$SA = 2\pi \int_a^b |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt \ .$$

The surface area generated by revolving C around the y-axis is

$$SA = 2\pi \int_a^b |x(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt \ .$$

Examples : Find the surface area generated by revolving the following parametric curves :

1. $C: \ x = t \ , \ y = \frac{1}{3}t^3 + \frac{1}{4}t^{-1} \ ; \ 1 \le t \le 2$, around the x-axis .

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = t^2 - \frac{t^{-2}}{4}$$

$$SA = 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{(1)^2 + \left(t^2 - \frac{t^{-2}}{4}\right)^2} \ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{1 + \left(t^4 - \frac{1}{2} + \frac{t^{-4}}{16}\right)} \ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{t^4 + \frac{1}{2} + \frac{t^{-4}}{16}} \ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{\left(t^2 + \frac{t^{-2}}{4}\right)^2} \ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \left|t^2 + \frac{t^{-2}}{4}\right| \ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \left(t^2 + \frac{t^{-2}}{4}\right) \ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^5}{3} + \frac{t}{2} + \frac{t^{-3}}{16}\right) \ dt$$

$$SA = 2\pi \left[\frac{t^6}{18} + \frac{t^2}{4} - \frac{t^{-2}}{32}\right]_1^2 = \frac{547\pi}{64}$$

2. $C: \ x = 4t^{\frac{1}{2}} \ , \ y = \frac{1}{2}t^2 + t^{-1} \ ; \ 1 \le t \le 4$, around the y-axis .

$$\frac{dx}{dt} = 2t^{-\frac{1}{2}}$$

$$\frac{dy}{dt} = t - t^{-2}$$

$$SA = 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{\left(2t^{-\frac{1}{2}}\right)^2 + (t - t^{-2})^2}\, dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{4t^{-1} + (t^2 - 2t^{-1} + t^{-4})}\, dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{t^2 + 2t^{-1} + t^{-4}}\, dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{(t + t^{-2})^2}\, dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \left|t + t^{-2}\right|\, dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \left(t + t^{-2}\right)\, dt$$

$$= 8\pi \int_1^4 \left(t^{\frac{3}{2}} + t^{-\frac{3}{2}}\right)\, dt$$

$$SA = 8\pi \left[\frac{2}{5}t^{\frac{5}{2}} - 2t^{-\frac{1}{2}}\right]_1^4 = \frac{536\pi}{5}$$

Exercises : Find the surface area generated by revolving the following parametric curves :

1. $C:\ x = 3t\ ,\ y = 4t\ \ ,0 \leq t \leq 2$, around the x-axis .
2. $C:\ x = t\ ,\ y = 2t\ \ ,0 \leq t \leq 4$, around the y-axis .