King Saud University

College of Sciences

Department of Mathematics

106 Math Exercises

(20)

Polar Coordinates

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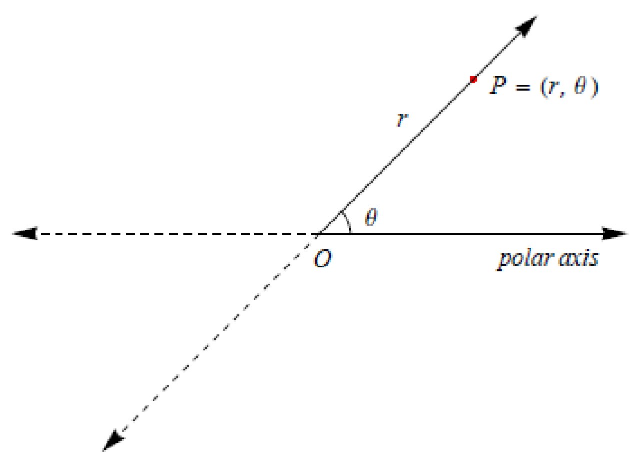
$$\frac{1439\ H}{2018\ G}$$

POLAR COORDINATES

In the recatangular coordinates system the ordered pair (a, b) represents a point , where "a" is the x-coordinat and "b" is the y-coordinate .

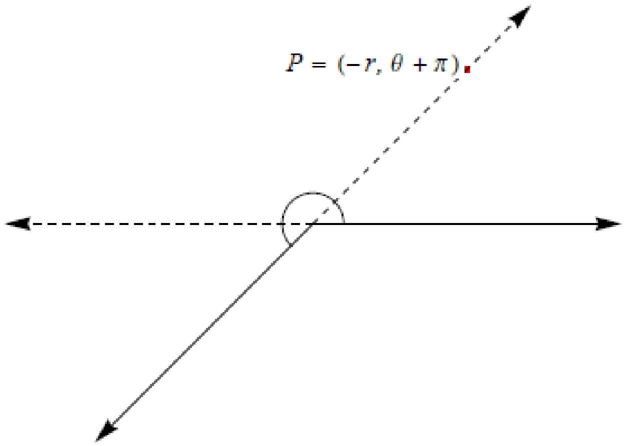
The polar coordinates system can be used also to represents points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system , and the **polar axis** is the directed half-line (the non-negative part of the x-axis).

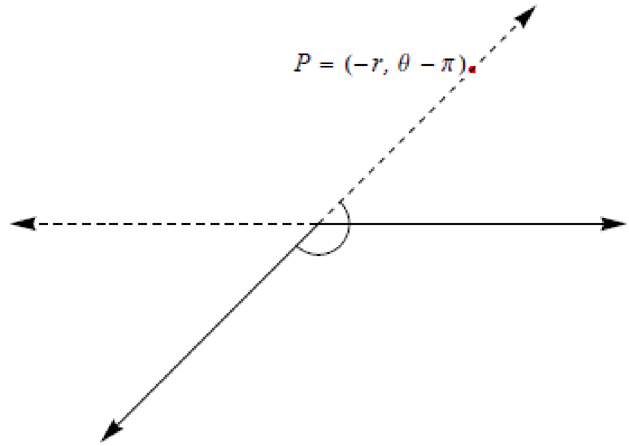
If P ia any point in the plane different from the origin, then its polar coordinates consists of two components r and θ , where r is the distance between P and the pole O , and θ is the measure of the angle determined by the polar axis and OP .



Note : The polar coordinates of a point is not unique , if $P = (r, \theta)$ then other representations are :

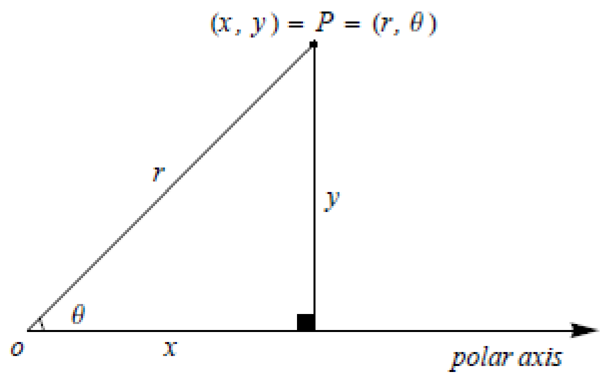
1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$.
2. $P = (-r, \theta + \pi)$.
3. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$.
4. $P = (-r, \theta - \pi)$
5. $P = (-r, \theta - \pi + 2n\pi)$, where $n \in \mathbb{Z}$.





Relationship between the polar and the rectangular coordinates
The polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point P are related as follows :

1. $x = r\cos\theta$ and $y = r\sin\theta$.
2. $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$.



Examples :

1. If $(r, \theta) = \left(2, \frac{\pi}{2}\right)$ then its other polar coordinates is

   a) $\left(-2, \frac{\pi}{2}\right)$ b) $\left(-2, \frac{3\pi}{2}\right)$ c) $\left(2, \frac{3\pi}{2}\right)$ d) $(2, \pi)$

   The answer : $(r, \theta) = \left(2, \frac{\pi}{2}\right) = \left(-2, \frac{\pi}{2} + \pi\right) = \left(-2, \frac{3\pi}{2}\right)$

   The right answer is (b) .

2. If $(r,\theta)=\left(-3,\dfrac{5\pi}{4}\right)$ then its other polar coordinates is

   a) $\left(-3,\dfrac{3\pi}{4}\right)$ b) $\left(3,\dfrac{7\pi}{4}\right)$ c) $\left(3,\frac{\pi}{4}\right)$ d) $\left(-3,\frac{\pi}{4}\right)$

   The answer : $(r,\theta)=\left(-3,\dfrac{5\pi}{4}\right)=\left(-(-3),\dfrac{5\pi}{4}-\pi\right)=\left(3,\frac{\pi}{4}\right)$

   The right answer is (c) .

3. If $(r,\theta)=(-5,\pi)$ then find its rectangular coordinates (x,y) .

   $x=-5\cos(\pi)=-5\ (-1)=5$ and $y=-5\sin(\pi)=-5\ (0)=0$

   $(x,y)=(5,0)$.

4. If $(x,y)=\left(2\sqrt{3},-2\right)$ then find its polar coordinates (r,θ) .

   $r^2=\left(2\sqrt{3}\right)^2+(-2)^2=12+4=16\Rightarrow r=4$

   $\tan\theta=\dfrac{-2}{2\sqrt{3}}=-\dfrac{1}{\sqrt{3}}\Rightarrow\theta=-\dfrac{\pi}{6}\ ,\ \theta=\dfrac{11\pi}{6}$

   $(r,\theta)=\left(4,-\frac{\pi}{6}\right)=\left(4,\dfrac{11\pi}{6}\right)$

Exercises :

1. If $(r,\theta)=\left(2,\frac{\pi}{2}\right)$ then find its rectangular coordinates (x,y) .

   Answer : $(x,y)=(0,2)$

2. If $(x,y)=\left(\sqrt{2},\sqrt{2}\right)$ then find its polar coordinates (r,θ) .

   Answer : $\left(2,\frac{\pi}{4}\right)$

POLAR CURVES

A polar curve is an equation in r and θ of the form $r = r(\theta)$.

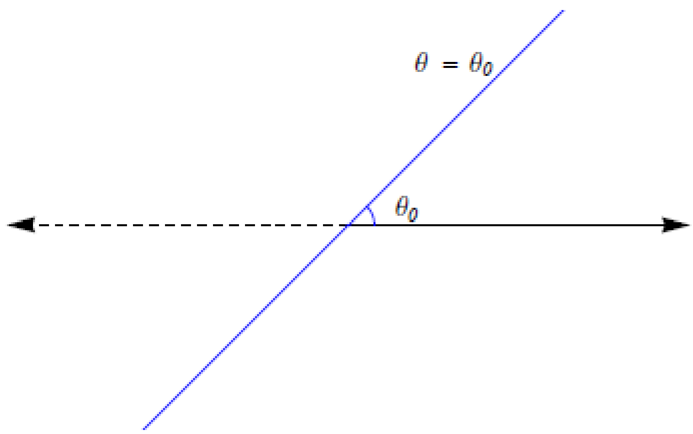
First - Straight Lines :

(1) Lines passing through the pole :

Any straight line passing through the pole has the form $\theta = \theta_0$, where θ_0 is the angle between the straight line and the polar axis .

$\theta = \theta_0 \Rightarrow \tan(\theta) = \tan(\theta_0) \Rightarrow \dfrac{y}{x} = \tan(\theta_0) \Rightarrow y = \tan(\theta_0)\ x$

The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan(\theta_0)$.

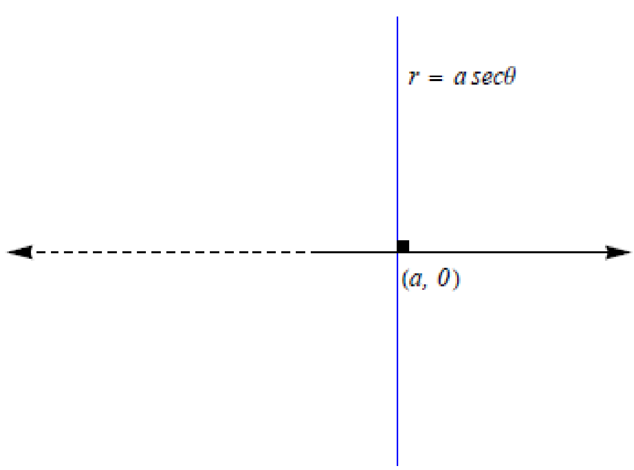


(2) Lines perpendicular to the polar axis :

Any straight line perpendicular to the polar axis has the form $r = a\ \sec\theta$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

$r = a\ \sec\theta \Rightarrow r = \dfrac{a}{\cos\theta} \Rightarrow r\cos\theta = a \Rightarrow x = a$.

The straight line $r = a\ \sec\theta$ is perpendicular to the polar axis at the point $(r, \theta) = (a, 0)$

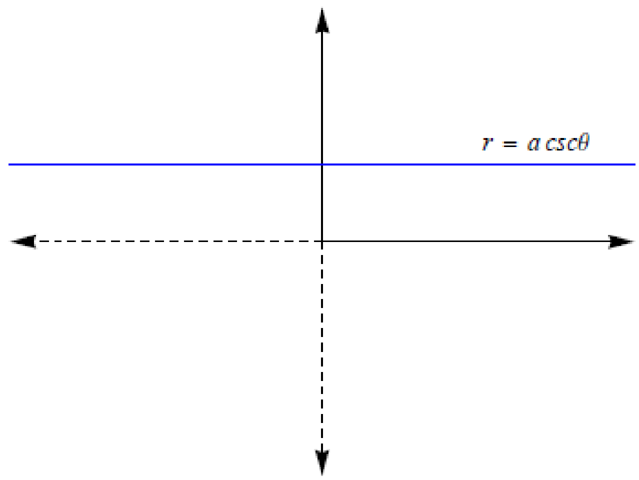


(3) Lines parallel to the polar axis :

Any straight line parallel to the polar axis has the form $r = a \ \csc\theta$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$..

$r = a \ \csc\theta \Rightarrow r = \dfrac{a}{\sin\theta} \Rightarrow r\sin\theta = a \Rightarrow y = a$.

The straight line $r = a \ \sec\theta$ is parallel to the polar axis and passing through the point $(r, \theta) = \left(a, \dfrac{\pi}{2}\right)$.



Examples :

1. $\theta = \dfrac{\pi}{4}$ is a straight line passing through the pole with a slope equals to $\tan\left(\frac{\pi}{4}\right) = 1$. Therefore its equation in $xy - form$ is $y = x$.

2. $r = 3\sec\theta$ is a straight line perpendicular to the polar axis and passing through the point $(r, \theta) = (3, 0)$. Therefore its equation in $xy - form$ is $x = 3$.

3. $r = -2\csc\theta$ is a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(-2, \frac{\pi}{2}\right)$. Therefore its equation in the $xy - form$ is $y = -2$.

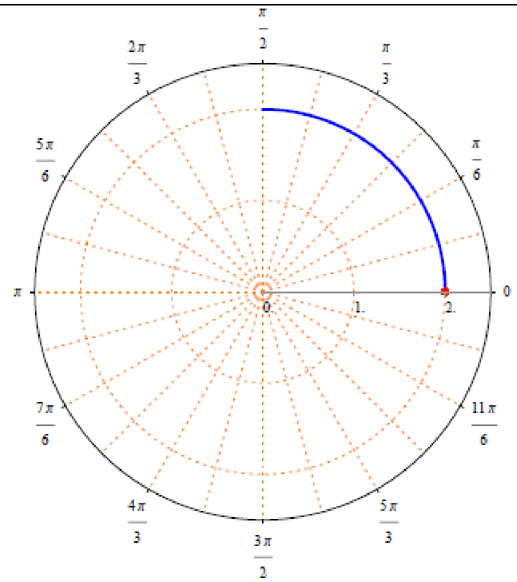
(1) Circles of the form $r = a$, where $a \in \mathbb{R}^*$

$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$

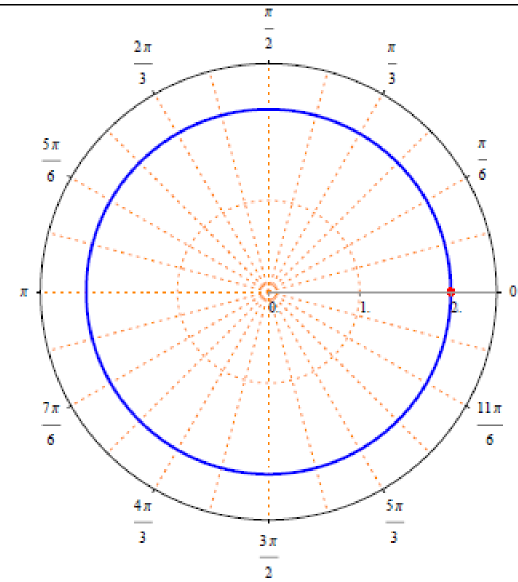
Therefore, $r = a$ represents a circle with center $= (0,0)$ and radius equals $|a|$.

Example :

1. $r = 2$ represents a circle with center $= (0,0)$ and radius equals to 2 .

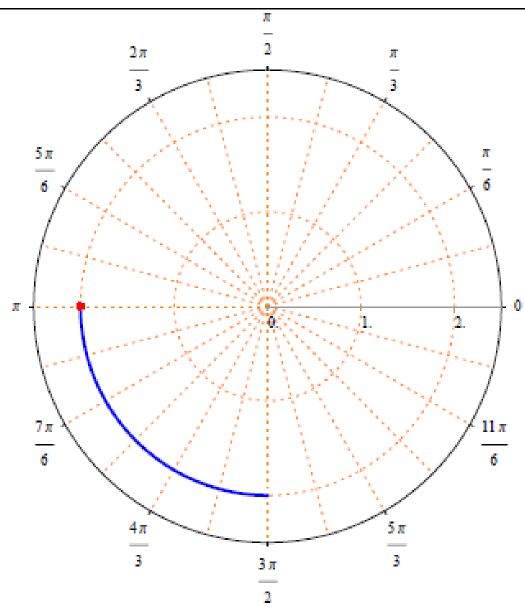


$r = 2 \ , \ 0 \le \theta \le \frac{\pi}{2}$

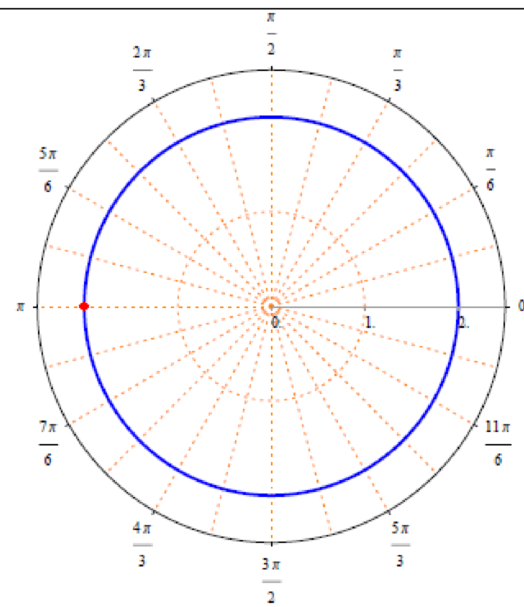


$r = 2 \ , \ 0 \le \theta \le 2\pi$

2. $r = -2$ represents a circle with center $= (0,0)$ and radius equals to 2 .



$r = -2 \ , \ 0 \le \theta \le \frac{\pi}{2}$



$r = -2 \ , \ 0 \le \theta \le 2\pi$

(2) Circles of the form $r = a\sin\theta$, where $a \in \mathbb{R}^*$ and $0 \le \theta \le \pi$

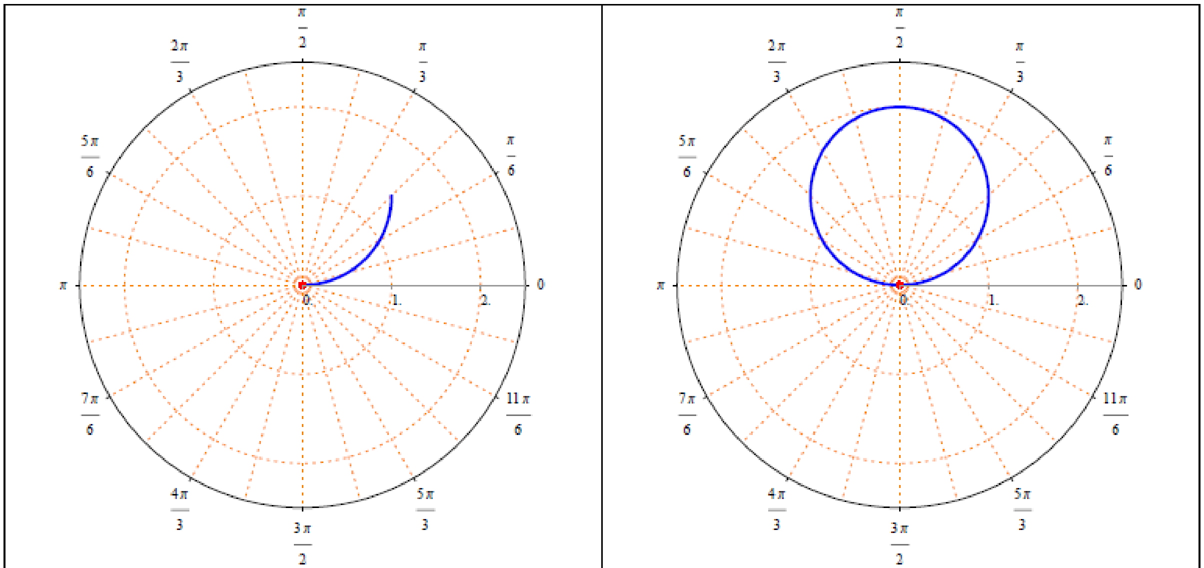
$r = a\sin\theta \Rightarrow r^2 = a\ r\sin\theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0$

$$\Rightarrow x^2 + \left(y^2 - ay + \frac{a^2}{4}\right) = \frac{a^2}{4} \Rightarrow x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

Therefore, $r = a\sin\theta$ represents a circle with center $= \left(0, \frac{a}{2}\right)$ and radius equals to $\frac{|a|}{2}$.

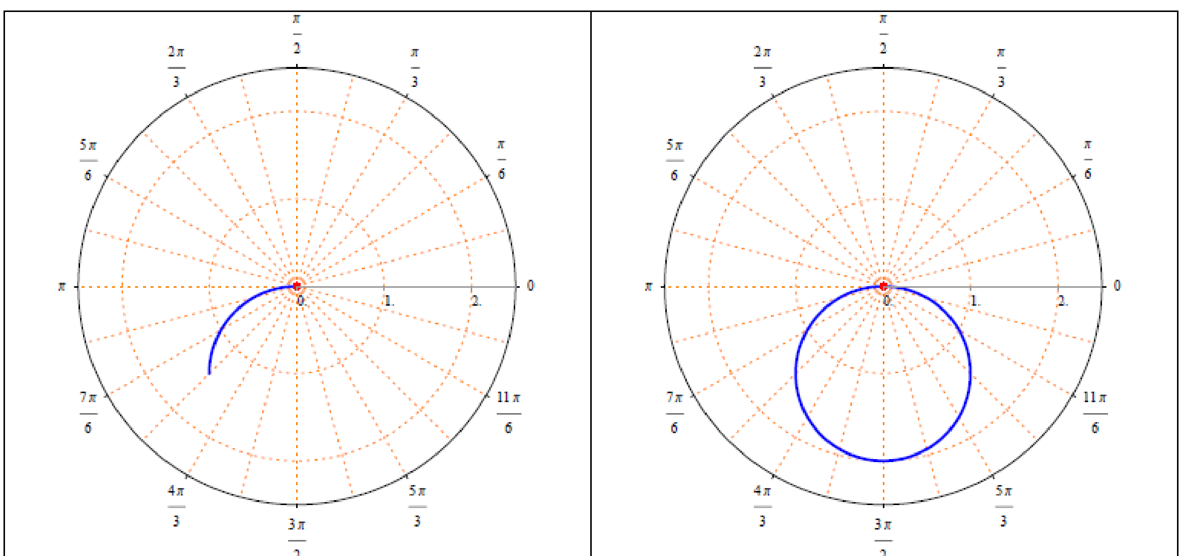
Examples :

1. $r = 2\sin\theta$ represents a circle with center $= (0,1)$ and radius equals to 1



$r = 2\sin\theta$, $0 \le \theta \le \frac{\pi}{4}$ $\qquad$ $r = 2\sin\theta$, $0 \le \theta \le \pi$

2. $r = -2\sin\theta$ represents a circle with center $= (0,-1)$ and radius equals to 1



$r = -2\sin\theta$, $0 \le \theta \le \frac{\pi}{4}$ $\qquad$ $r = -2\sin\theta$, $0 \le \theta \le \pi$

(3) Circles of the form $r = a\cos\theta$, where $a \in \mathbb{R}^*$ and $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$

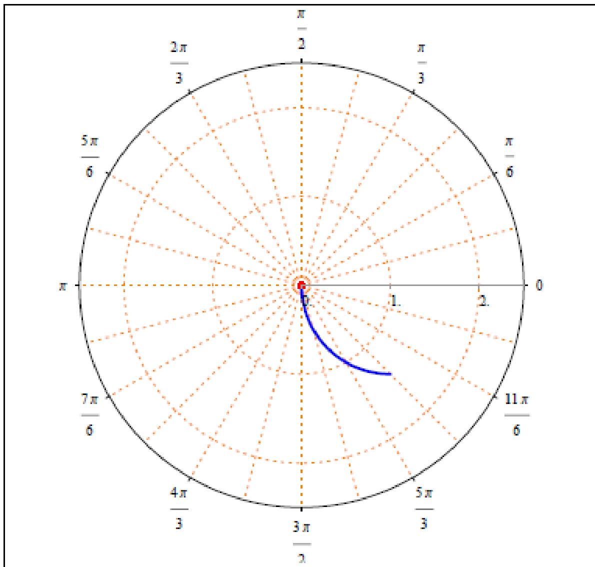
$r = a\cos\theta \Rightarrow r^2 = a\ r\cos\theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0$

$\Rightarrow \left(x^2 - ax + \dfrac{a^2}{4}\right) + y^2 = \dfrac{a^2}{4} \Rightarrow \left(x - \dfrac{a}{2}\right)^2 + y^2 = \dfrac{a^2}{4}$

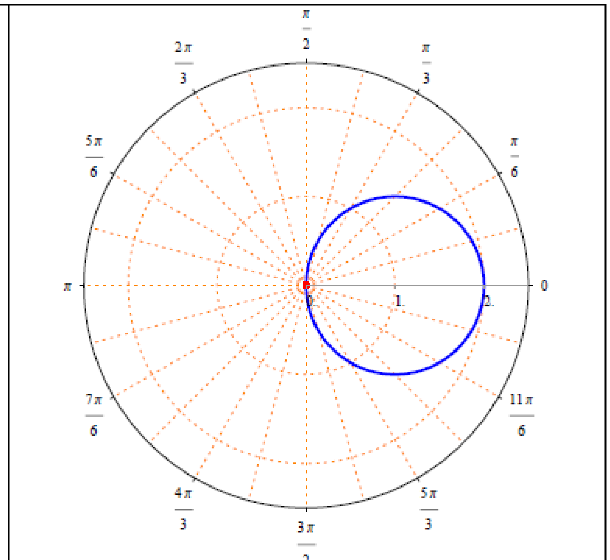
Therefore, $r = a\cos\theta$ represents a circle with center $= \left(\dfrac{a}{2}, 0\right)$ and radius equals to $\dfrac{|a|}{2}$.

Examples :

1. $r = 2\cos\theta$ represents a circle with center $= (1,0)$ and radius equals to 1

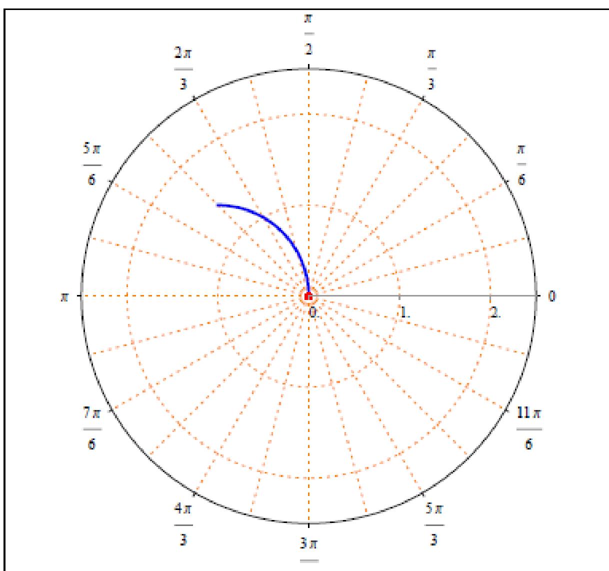


$r = 2\cos\theta$, $-\dfrac{\pi}{2} \leq \theta \leq -\dfrac{\pi}{4}$

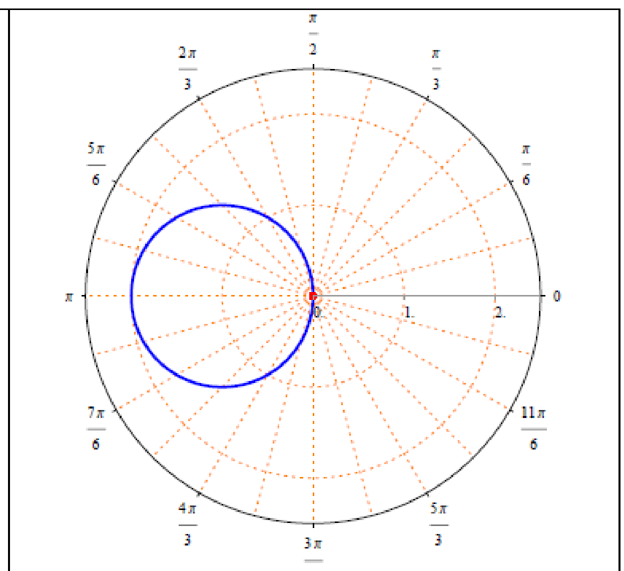


$r = 2\cos\theta$, $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$

2. $r = -2\cos\theta$ represents a circle with center $= (-1,0)$ and radius equals to 1



$r = -2\cos\theta$, $-\dfrac{\pi}{2} \leq \theta \leq -\dfrac{\pi}{4}$



$r = -2\cos\theta$, $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$

Third - Limaçon curves :

The general form of a Limaçon curve is

$r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

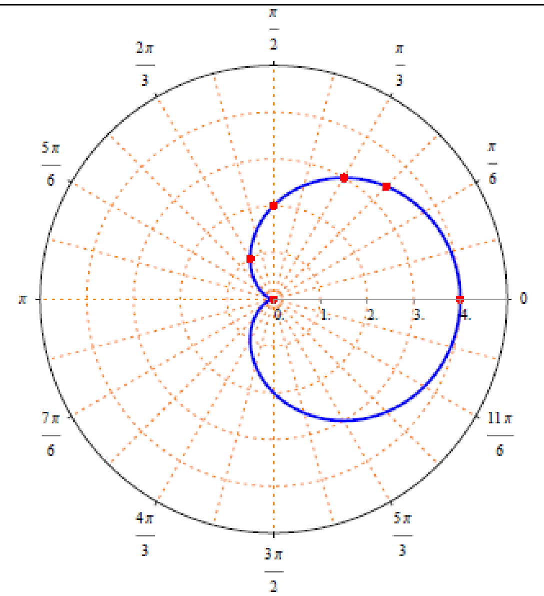
(1) Cardioid (Heart-shaped) :

It has the form $r(\theta) = a + a\sin\theta$ or $r(\theta) = a + a\cos\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

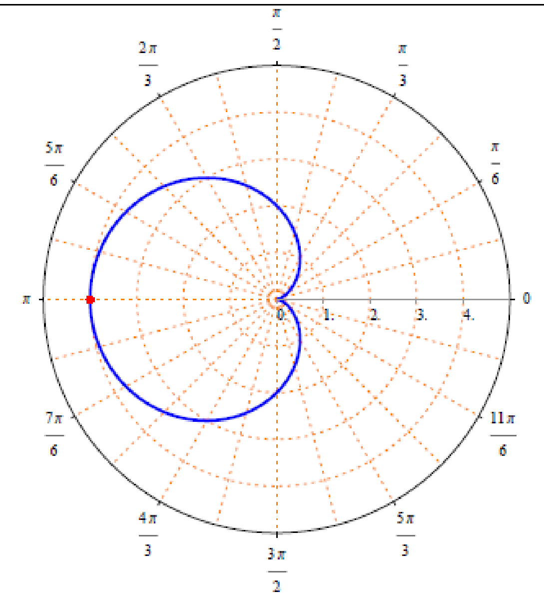
Examples :

1. $r(\theta) = 2 + 2\cos\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	$2+\sqrt{2}$	3	2	1	0

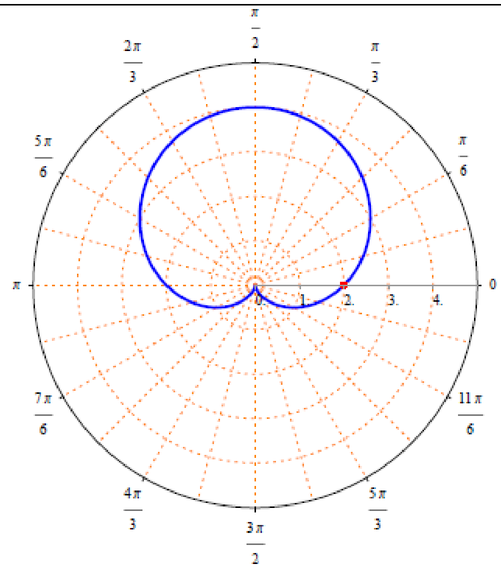


$r(\theta) = 2 + 2\cos\theta$, $0 \leq \theta \leq 2\pi$

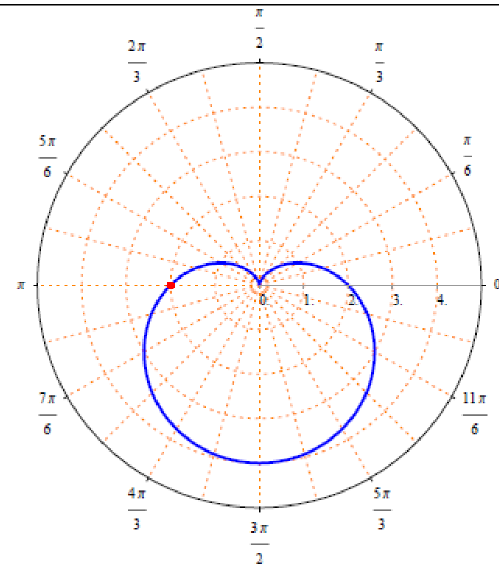


$r(\theta) = 2 - 2\cos\theta$, $0 \leq \theta \leq 2\pi$

2. $r(\theta) = 2 + 2\sin\theta$ and $r(\theta) = \ 2 - 2\sin\theta$



$r(\theta) = 2 + 2\sin\theta \ , \ \ 0 \le \theta \le 2\pi$



$r(\theta) = \ 2 - 2\sin\theta \ , \ \ 0 \le \theta \le 2\pi$

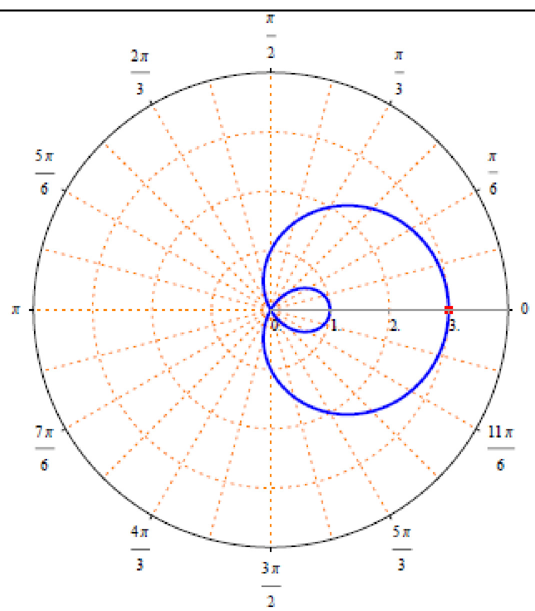
(2) Limaçon with inner loop :

It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \le \theta \le 2\pi$

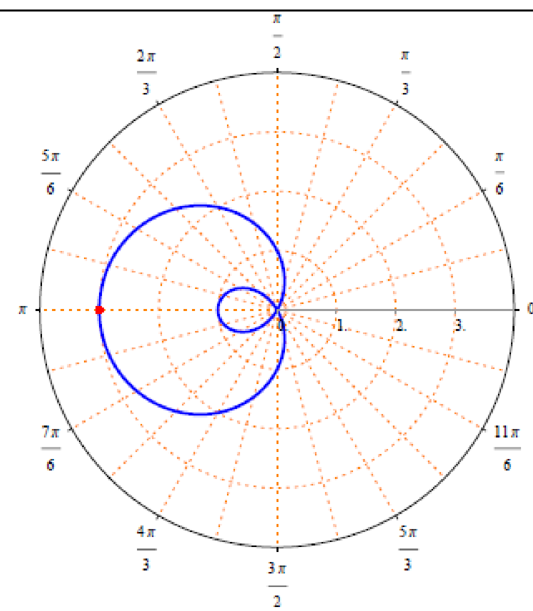
Note : Note that $|a| < |b|$ in this case .

Examples :

1. $r(\theta) = 1 + 2\cos\theta$ and $r(\theta) = \ 1 - 2\cos\theta$

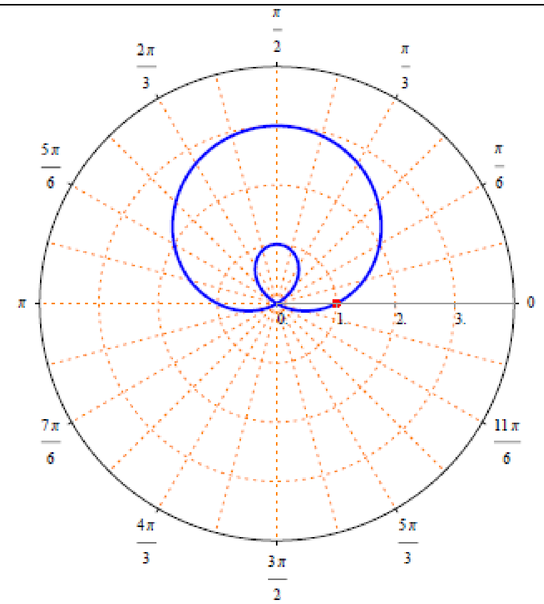


$r(\theta) = 1 + 2\cos\theta \ , \ \ 0 \le \theta \le 2\pi$

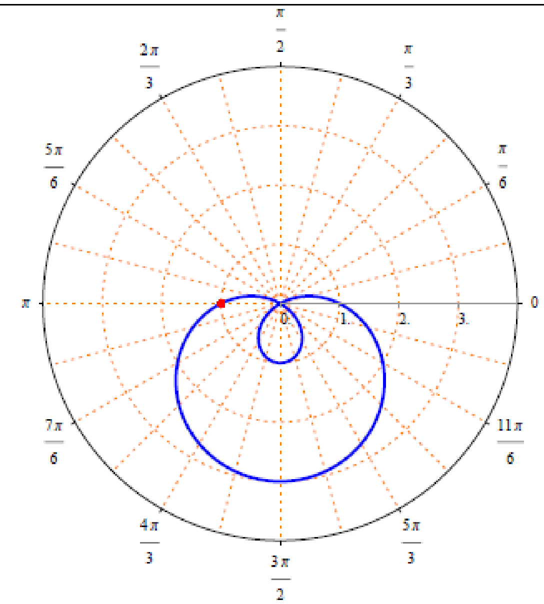


$r(\theta) = \ 1 - 2\cos\theta \ , \ \ 0 \le \theta \le 2\pi$

2. $r(\theta) = 1 + 2\sin\theta$ and $r(\theta) = 1 - 2\sin\theta$



$r(\theta) = 1 + 2\sin\theta \ , \quad 0 \le \theta \le 2\pi$



$r(\theta) = 1 - 2\sin\theta \ , \quad 0 \le \theta \le 2\pi$

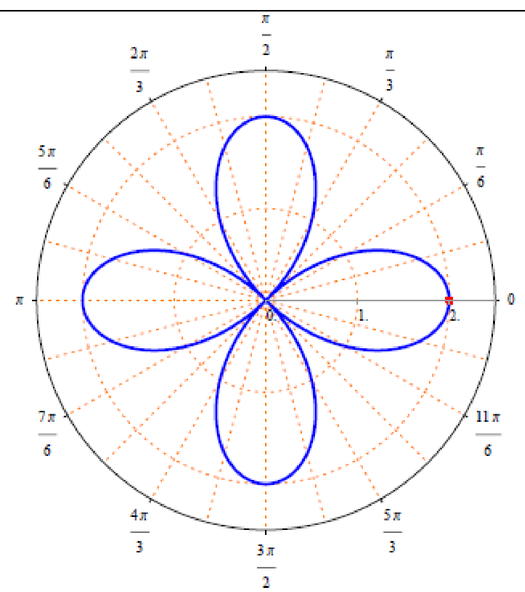
Fourth - Rose curves :

It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*$, $n \in \mathbb{N}$ and $n \ge 2$

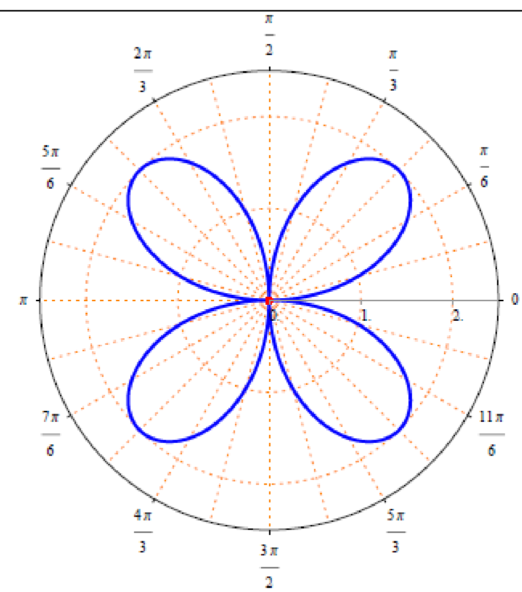
1. **n is even :** In this case the number of loops (or leaves) is $2n$.

   Examples : $r(\theta) = 2\cos(2\theta)$ or $r(\theta) = 2\sin(2\theta)$, $0 \le \theta \le 2\pi$

   The number of loops (or leaves) equals 4 .



$r(\theta) = 2\cos(2\theta) \ , \quad 0 \le \theta \le 2\pi$

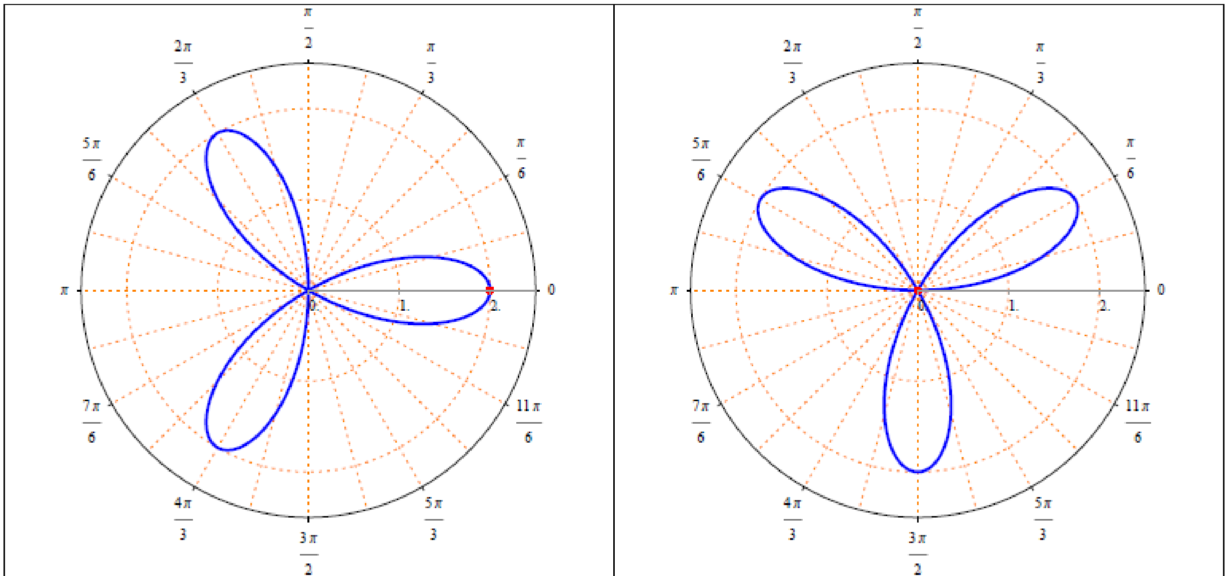


$r(\theta) = 2\sin(2\theta) \ , \quad 0 \le \theta \le 2\pi$

2. **n is odd** : In this case the number of loops (or leaves) is n .

Examples : $r(\theta) = 2\cos(3\theta)$ or $r(\theta) = 2\sin(3\theta)$, $0 \le \theta \le \pi$

The number of loops (or leaves) equals 3 .



$r(\theta) = 2\cos(3\theta)$, $0 \le \theta \le \pi$ $\qquad$ $r(\theta) = 2\sin(3\theta)$, $0 \le \theta \le \pi$

Examples :

1. $r = \dfrac{2}{\cos\theta}$ represents
   a) a straight line b) a circle c) a cardioid d) a rose curve

   Answer : $r = \dfrac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2.$

   Hence , $r = \dfrac{2}{\cos\theta}$ represents a straigh line .

   The right answer is (a) .

2. The polar equation $r = 2\cos\theta - 2$ represents
   a) a straight line b) a circle c) a cardioid d) a rose curve

   $r = 2\cos\theta - 2$ is a Limaçon curve with $a = b = 2$.

   Therefore , $r = 2\cos\theta - 2$ represents a cardioid .

   The right answer is (c) .

3. The number of leaves in the rose curve $r = \sin 2\theta$ is
   a) 6 b) 4 c) 2 d) None of these

   Since $n = 2$ is an even number then the number of leaves in the rose curve $r = \sin 2\theta$ equals $2n = 2(2) = 4$

   The right answer is (b)

4. Write the polar equation $r = 2\cos\theta + 2\sin\theta$ in terms of x and y (or cartesian equation) .

   $r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2\ r\cos\theta + 2\ r\sin\theta \Rightarrow x^2 + y^2 = 2x + 2y$

   $\Rightarrow (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2 \Rightarrow (x-1)^2 + (y-1)^2 = 2$

   It is a circle with center $= (1, 1)$ and radius equals $\sqrt{2}$

Test of symmetry

1. The graph of $r = r(\theta)$ is symmetric with repect to the polar axis if

   $r(\theta) = r(-\theta)$

   Examples : The circle $r = 4\cos\theta$ and the cardioid $r = 2 + 2\cos\theta$ are both symmetric with respect to the polar axis .

2. The graph of $r = r(\theta)$ is symmetric with repect to the line $\theta = \frac{\pi}{2}$ if

   (a) $r(\theta) = -r(-\theta)$

   (b) $r(\theta) = r(\pi - \theta)$

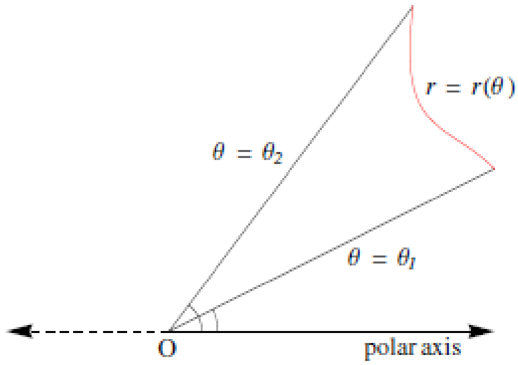
   Examples : The circle $r = 4\sin\theta$ and the cardioid $r = 2 + 2\sin\theta$ are both symmetric with respect to the line $\theta = \frac{\pi}{2}$.

3. The graph of $r = r(\theta)$ is symmetric with repect to the pole if

   $r(\theta) = r(\pi + \theta)$

   Example : The rose curve $r = \sin 2\theta$ is symmetric with respect to the pole .

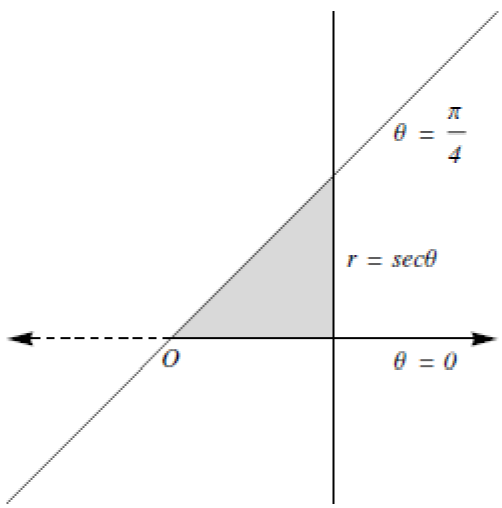
AREA INSIDE-BETWEEN POLAR CURVES



The area of the region bounded by the graphs of the polar curves $r = r(\theta)$, $\theta = \theta_1$ and $\theta = \theta_2$ is $A = \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} [r(\theta)]^2\, d\theta$

Examples :

1. Find the area of the region bounded by the graph of the polar curves $r = \sec\theta$, $\theta = 0$ and $\theta = \frac{\pi}{4}$.

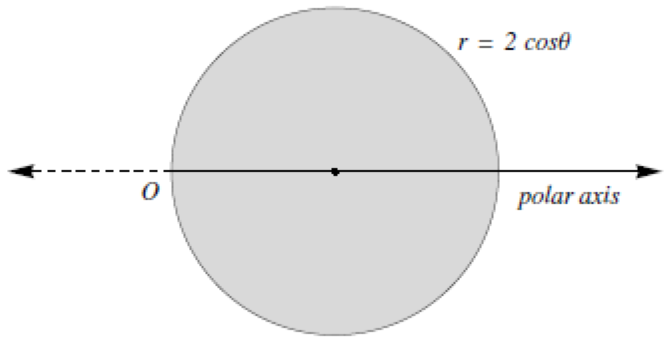


Note that $r = \sec\theta$ is a straight line perpendicular to the polar axis at the point $(r, \theta) = (1, 0)$, $\theta = 0$ is the polar axis and $\theta = \frac{\pi}{4}$ is a straight line passing the pole with a slope equals 1 (in fact it is the line $y = x$) .

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sec\theta)^2 d\theta = \frac{1}{2}\left[\tan\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{2}[1 - 0] = \frac{1}{2}$$

Note : In fact it is the area of the triangle of base equals 1 and height equals also 1 .

2. Find the area inside the polar curve $r = 2\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



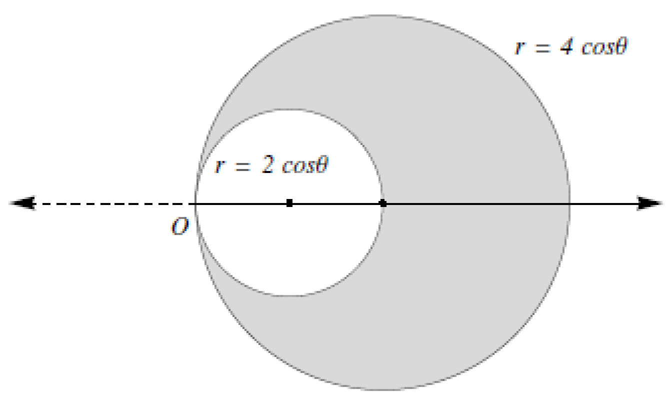
Note that $r = 2\cos\theta$ is a circle with center $= (1,0)$ and radius equals 1 .

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2\theta\ d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2\theta]\ d\theta$$

$$A = \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left[\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right] = \pi\ .$$

Note : In fact it is the area of a circle of radius equals 1 and in this case $A = \pi(1)^2 = \pi$.

3. Find the area inside the polar curve $r = 4\cos\theta$ and outside the curve $r = 2\cos\theta$.



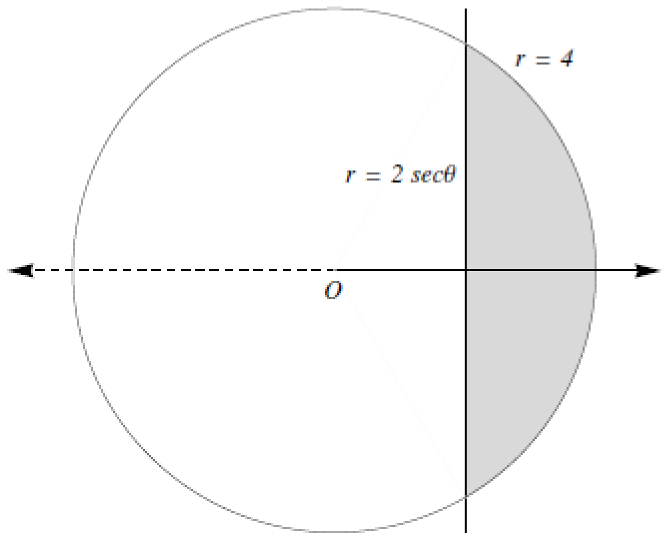
Note that $r = 4\cos\theta$ is a circel with center $= (2,0)$ and radius equals to 2 , also $r = 2\cos\theta$ is another circle with center $= (1,0)$ and radius equals 1 .

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12\cos^2\theta\ d\theta$$

$$A = 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2\theta]\ d\theta = 3\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\pi$$

Note : In fact it is the difference between the area of a circle with radius 2 and the area of a circle of radius 1 , so the desired area is $A = \pi(2)^2 - \pi(1)^2 = 3\pi$.

4. Find the area inside $r = 4$ and to the right of $r = 2\sec\theta$



Note that $r = 4$ is a circle with center $= (0, 0)$ and radius equals 4 , $r = 2\sec\theta$ is a straight line perpendicular to the polar axis (it is the line $x = 2$)

Angles of intersection between $r = 4$ and $r = 2\sec\theta$:

$$2\sec\theta = 4 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}\ ,\ \theta = -\frac{\pi}{3}$$

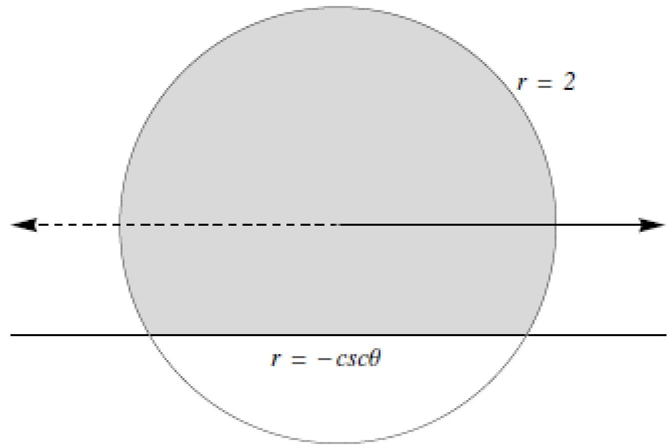
Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} (4)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} (2\sec\theta)^2 d\theta\right)$$

$$A = 16\int_0^{\frac{\pi}{3}} d\theta - 4\int_0^{\frac{\pi}{3}} \sec^2\theta\ d\theta$$

$$A = 16[\theta]_0^{\frac{\pi}{3}} - 4[\tan\theta]_0^{\frac{\pi}{3}} = 16\left(\frac{\pi}{3} - 0\right) - 4(\sqrt{3} - 0) = \frac{16\pi}{3} - 4\sqrt{3}$$

5. Find the area inside $r = 2$ and above $r = -\csc\theta$.



Note that $r = 2$ is a circle with center $= (0,0)$ and radius equals 2 , $r = -\csc\theta$ is a straight line parallel to the polar axis (it is the line $y = -1$)

Angles of intersection between $r = 2$ and $r = -\csc\theta$:

$$-\csc\theta = 2 \Rightarrow \csc\theta = -2 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6} \ , \ \theta = -\frac{5\pi}{6}$$

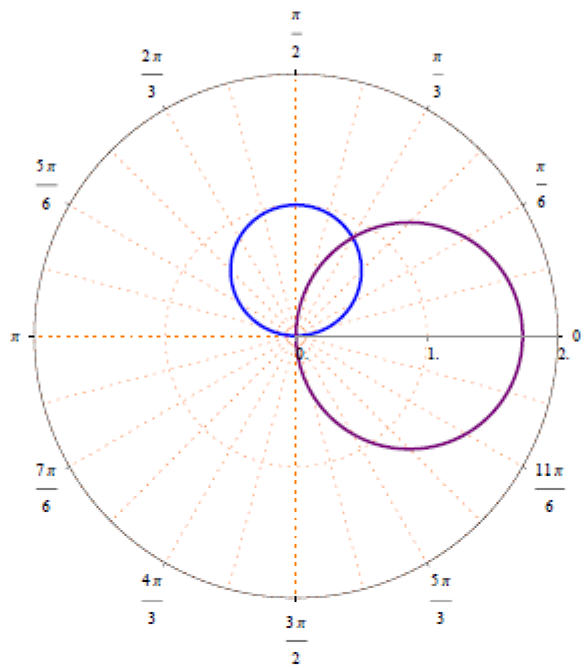
Since the desired area is symmetric with respect to the line $\theta = \frac{\pi}{2}$, then

$$A = 2\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} (-\csc\theta)^2 d\theta + \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (2)^2 d\theta\right)$$

$$A = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \csc^2\theta \ d\theta + 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta$$

$$A = [-\cot\theta]_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + 4[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3} + \frac{2\pi}{3}$$

6. Find the area of the common region between $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$



Note that $r = \sqrt{3}\cos\theta$ is a circle with center $= \left(\frac{\sqrt{3}}{2}, 0\right)$ and radius equals $\frac{\sqrt{3}}{2}$, also $r = \sin\theta$ is a circle with center $= \left(0, \frac{1}{2}\right)$ and radius equals $\frac{1}{2}$.

Angle of intersection between $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$

$$\sqrt{3}\cos\theta = \sin\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\theta)^2 d\theta$$

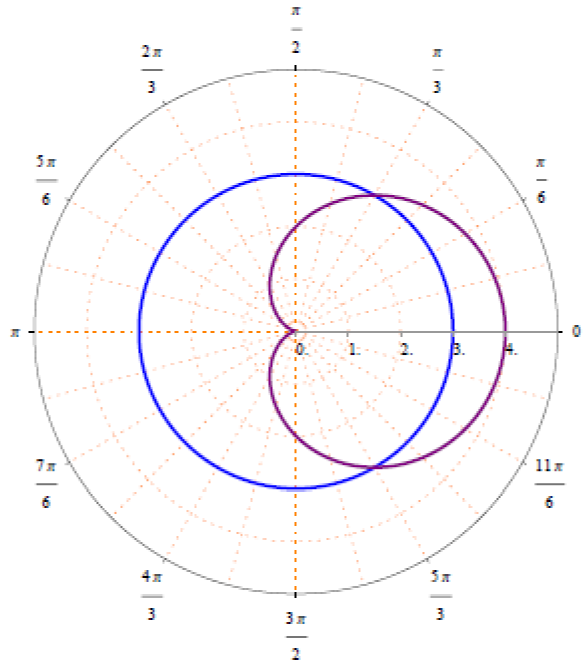
$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1}{2}[1 - \cos 2\theta]\, d\theta + \frac{3}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}[1 + \cos 2\theta]\, d\theta$$

$$A = \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}} + \frac{3}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$A = \frac{1}{4}\left(\frac{\pi}{3} - \frac{1}{2}\frac{\sqrt{3}}{2}\right) + \frac{3}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2}\right)\right]$$

$$A = \frac{5\pi}{24} - \frac{\sqrt{3}}{4} .$$

7. Find the area inside $r = 3$ and outside $r = 2 + 2\cos\theta$.



Note that $r = 3$ is a circle with center $= (0,0)$ and radius equals 3 , $r = 2 + 2\cos\theta$ is a cardioid .

Angles of intersection between $r = 3$ and $r = 2 + 2\cos\theta$:

$$2 + 2\cos\theta = 3 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \ , \ \theta = \frac{5\pi}{3} = -\frac{\pi}{3}$$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (3)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (2 + 2\cos\theta)^2 d\theta\right)$$

$$A = \int_{\frac{\pi}{3}}^{\pi} [9 - (4 + 8\cos\theta + 4\cos^2\theta)] \ d\theta$$

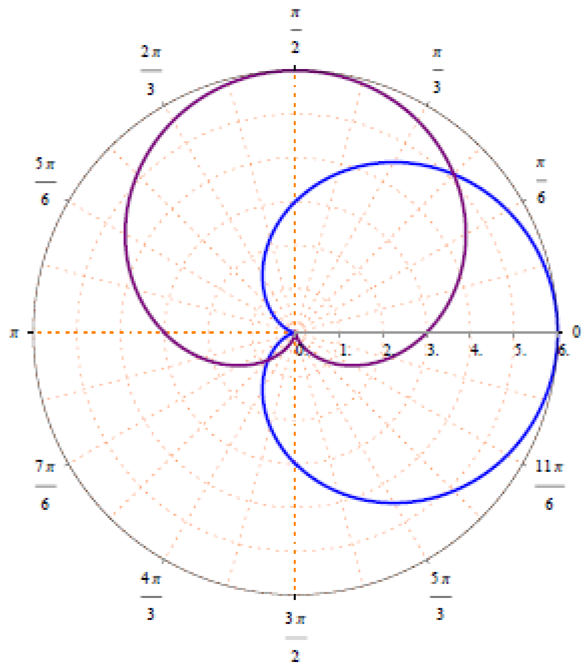
$$A = \int_{\frac{\pi}{3}}^{\pi} [5 - 8\cos\theta - 2(1 + \cos 2\theta)] \ d\theta$$

$$A = \int_{\frac{\pi}{3}}^{\pi} [3 - 8\cos\theta - 2\cos 2\theta] \ d\theta$$

$$A = [3\theta - 8\sin\theta - \sin 2\theta]_{\frac{\pi}{3}}^{\pi}$$

$$A = \left[(3\pi - 0 - 0) - \left(\pi - 8\ \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right] \qquad A = 2\pi + \frac{9\sqrt{3}}{2}$$

8. Find the area inside $r = 3 + 3\cos\theta$, outside $r = 3 + 3\sin\theta$ and at the first quadrant.



Angles of intersection between $r = 3 + 3\cos\theta$ and $r = 3 + 3\sin\theta$:

$$3 + 3\cos\theta = 3 + 3\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \;,\; \theta = \frac{5\pi}{4}$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (3 + 3\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} (3 + 3\sin\theta)^2 d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[(9 + 18\cos\theta + 9\cos^2\theta) - (9 + 18\sin\theta + 9\sin^2\theta)\right] d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[18\cos\theta - 18\sin\theta + 9\cos^2\theta - 9\sin^2\theta\right] d\theta$$

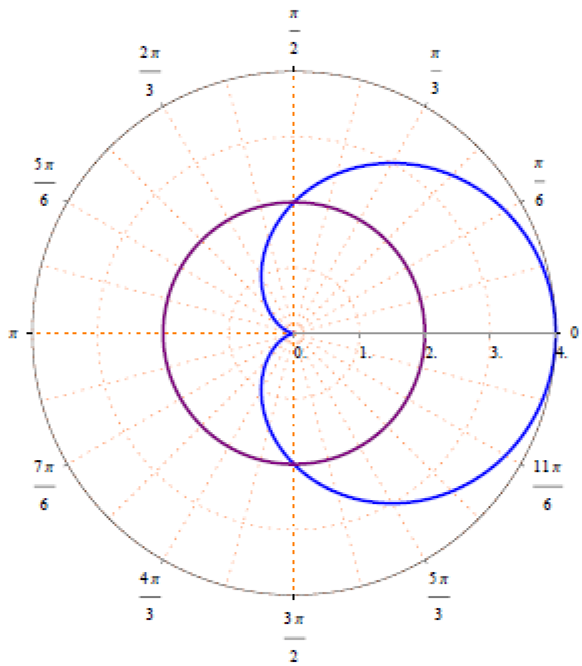
$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[18\cos\theta - 18\sin\theta + \frac{9}{2}(1 + \cos 2\theta) - \frac{9}{2}(1 - \cos 2\theta)\right] d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} [18\cos\theta - 18\sin\theta + 9\cos 2\theta]\, d\theta$$

$$A = \frac{1}{2}\left[18\sin\theta + 18\cos\theta + \frac{9}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$A = \frac{1}{2}\left[\left(\frac{18}{\sqrt{2}} + \frac{18}{\sqrt{2}} + \frac{9}{2}\right) - (0 + 18 + 0)\right] = \frac{18}{\sqrt{2}} - \frac{27}{4}$$

9. Find the area inside $r = 2 + 2\cos\theta$ and outside $r = 2$.



Note that $r = 2$ is a circle with center $= (0,0)$ and radius equals 2 , $r = 2 + 2\cos\theta$ is a cardioid .

Angles of intersection between $r = 2$ and $r = 2 + 2\cos\theta$:

$$2 + 2\cos\theta = 2 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \ , \ \theta = \frac{3\pi}{2}$$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2+2\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2 d\theta\right)$$

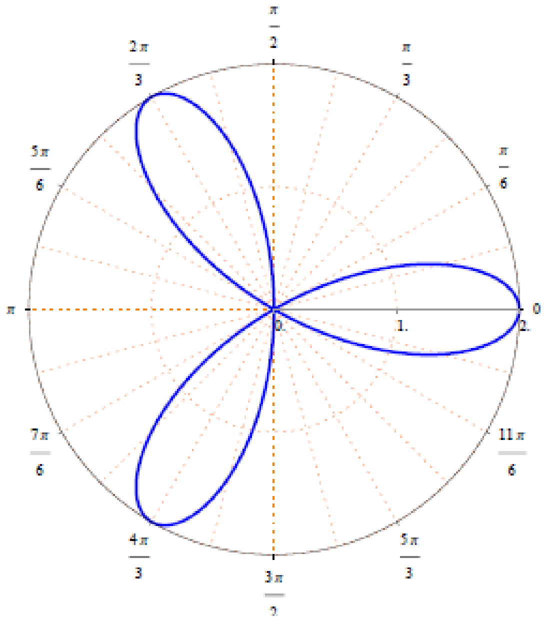
$$A = \int_0^{\frac{\pi}{2}} (4 + 8\cos\theta + 4\cos^2\theta - 4) \ d\theta$$

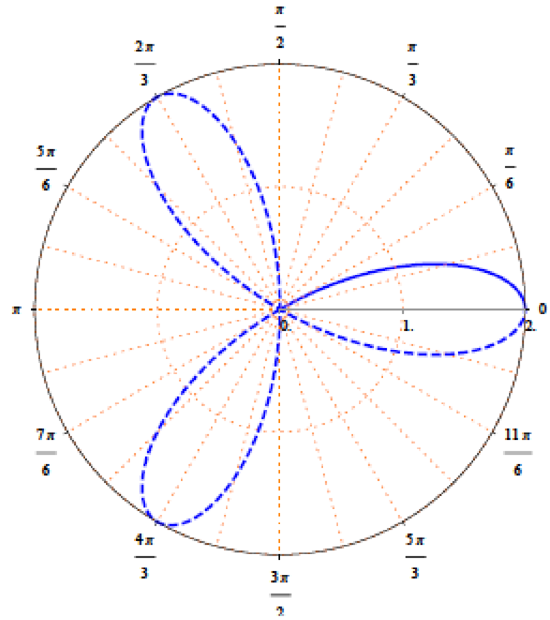
$$A = \int_0^{\frac{\pi}{2}} (8\cos\theta + 2(1+\cos 2\theta)) \ d\theta$$

$$A = \int_0^{\frac{\pi}{2}} (2 + 8\cos\theta + 2\cos 2\theta) \ d\theta)$$

$$A = [2\theta + 8\sin\theta + \sin 2\theta]_0^{\frac{\pi}{2}} = \pi + 8$$

10. Find the area inside one leaf of the rose curve $r = 2\cos 3\theta$.





The rose curve $r = 2\cos 3\theta$, $0 \le \theta \le \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$

$$r = 0 \Rightarrow 2\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Since the desired area is symmetric with respect to the polar axis , then

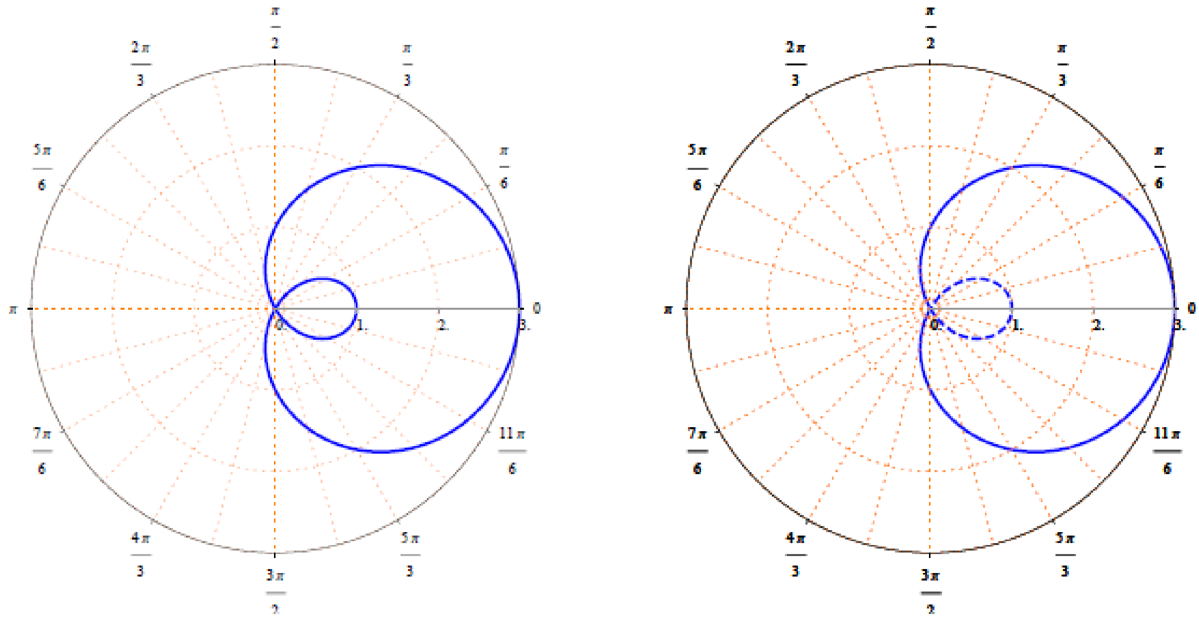
$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos 3\theta)^2 d\theta\right) = 4\int_0^{\frac{\pi}{6}} \cos^2 3\theta \; d\theta$$

$$A = 4\int_0^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos 6\theta) \; d\theta = 2\int_0^{\frac{\pi}{6}} (1 + \cos 6\theta) \; d\theta$$

$$A = 2\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{6}} = \frac{\pi}{3}$$

#

11. Find the area betwen the loops of the curve $r = 1 + 2\cos\theta$



$$r = 0 \Rightarrow 1 + 2\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \ , \ \theta = \frac{4\pi}{3}$$

The interior loop starts at $\theta = \dfrac{2\pi}{3}$ and ends at $\theta = \dfrac{4\pi}{3}$

$$A = \frac{1}{2}\int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 d\theta + \int_{\frac{4\pi}{3}}^{2\pi} (1 + 2\cos\theta)^2 d\theta - \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos\theta)^2 d\theta$$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 d\theta - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (1 + 2\cos\theta)^2 d\theta\right)$$

$$A = \int_0^{\frac{2\pi}{3}} \left(1 + 4\cos\theta + 4\cos^2\theta\right)\, d\theta - \int_{\frac{2\pi}{3}}^{\pi} \left(1 + 4\cos\theta + 4\cos^2\theta\right)\, d\theta$$

$$A = \int_0^{\frac{2\pi}{3}} \left(3 + 4\cos\theta + 2\cos 2\theta\right)\, d\theta - \int_{\frac{2\pi}{3}}^{\pi} \left(3 + 4\cos\theta + 2\cos 2\theta\right)\, d\theta$$

$$A = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{\frac{2\pi}{3}} - \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{\frac{2\pi}{3}}^{\pi}$$

$$A = \left[\left(2\pi + \tfrac{3\sqrt{3}}{2}\right) - 0\right] - \left[3\pi - \left(2\pi + \tfrac{3\sqrt{3}}{2}\right)\right] = \pi + 3\sqrt{3}$$

12. Sketch the region R that lies inside the graph the equation $r = 2 + 2cos\theta$ and outside the graph of the graph of the equation $r = 2cos\theta$ and set up (Do not evaluate) an integral the can be used to find its area .

13. Sketch the region R that lies inside the graph the equation $r = 3sin\theta$ and outside the graph of the equation $r = 1 + sin\theta$ and set up (Do not evaluate) an integral the can be used to find its area .

14. Sketch the graph of the equation $r = 2 + 2cos\theta$ and the graph of the equation $r = 6cos\theta$ and set up (Do not evaluate) an integral the can be used to find its area of the region R that lies :

(i) Inside $r = 6cos\theta$ and outside $r = 2 + 2cos\theta$.

(ii) outside $r = 6cos\theta$ and inside $r = 2 + 2cos\theta$.

(iii) Inside $r = 6cos\theta$ and inside $r = 2 + 2cos\theta$. (common region)

15. Sketch the region R that lies inside the graph the equation $r = 2sin\theta$ and outside the graph of the graph of the equation $r = 1$ and set up (Do not evaluate) an integral the can be used to find its area .

16. Sketch the region R that lies inside the graph the equation $r = 2cos\theta$ and outside the graph of the graph of the equation $r = cos\theta$ and set up (Do not evaluate) an integral the can be used to find its area .

17. Sketch the region R that lies inside the graph the equation $r = 2sin\theta$ and outside the graph of the graph of the equation $r = sin\theta$ and set up (Do not evaluate) an integral the can be used to find its area .

18. Sketch the region R that lies inside the graph the equation $r = 4cos\theta$ and outside the graph of the graph of the equation $r = 2$ and set up (Do not evaluate) an integral the can be used to find its area .

19. Sketch the region R that lies inside the graph the equation $r = 2cos\theta$ and inside the graph of the graph of the equation $r = 1$ and set up (Do not evaluate) an integral the can be used to find its area .

20. Sketch the region R that lies inside the graph the equation $r = 1 + cos\theta$ and inside the graph of the graph of the equation $r = 1 - cos\theta$ and set up (Do not evaluate) an integral the can be used to find its area .

21. Sketch the region R that lies inside the graph the equation $r = 1 - cos\theta$ and outside the graph of the graph of the equation $r = 1$ and set up (Do not evaluate) an integral the can be used to find its area .

23. Sketch the region R that lies inside the graph the equation $r = 1 + sin\theta$ and inside the graph of the graph of the equation $r = 1$ and set up (Do not evaluate) an integral the can be used to find its area .

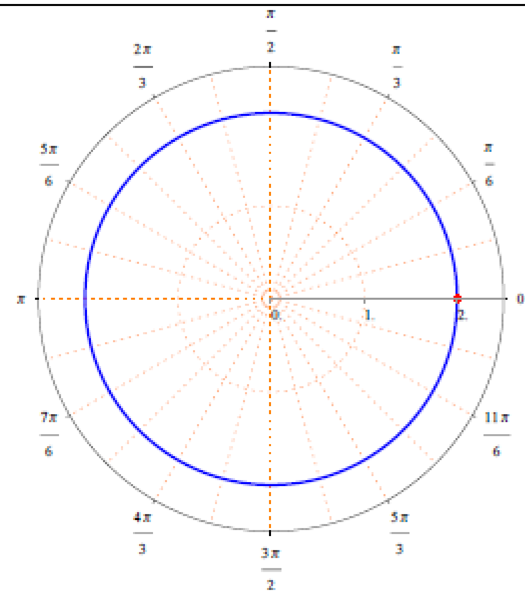
24. Sketch the region R that is inside $r = sin\theta$ and outside $r = cos\theta$ for $\theta \in \left[0, \frac{\pi}{2}\right]$.

Set up (Don't evaluate) an integral that can be used to find the area of R .

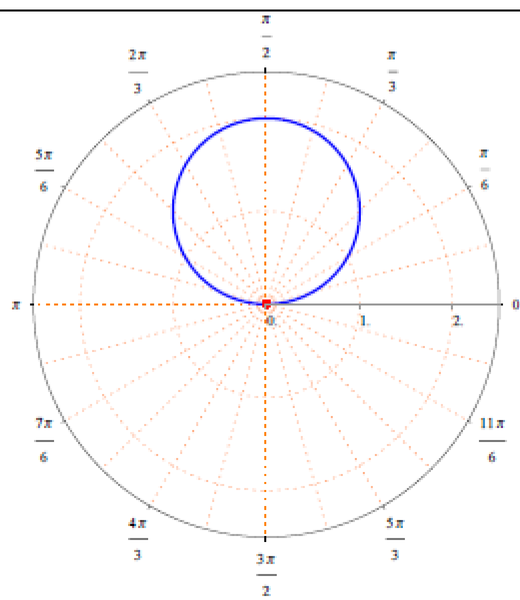
25. Find the area of the common region between the curve $r = 2sin\theta$ and

$r = 2cos\theta$

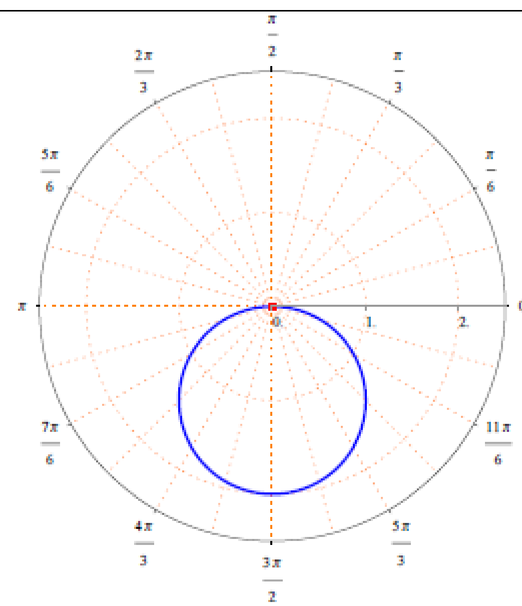
26. Find the area inside $r = cos\theta$ and outside the curve $r = 1 - cos\theta$



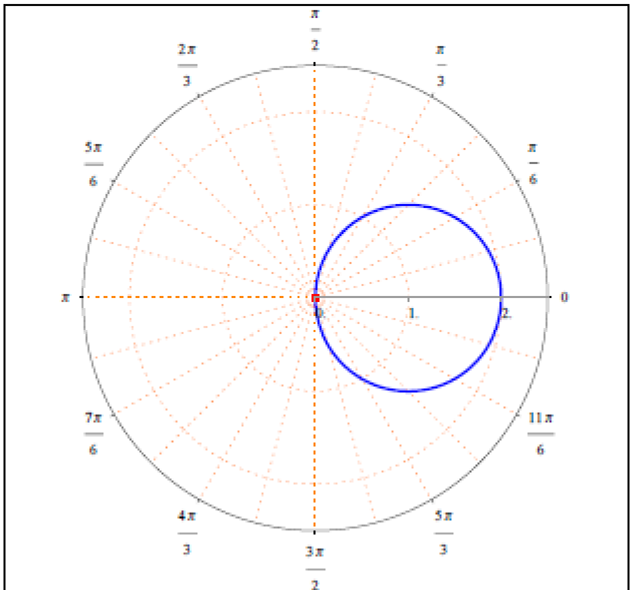
$$r = 2 \quad , \quad 0 \le \theta \le 2\pi$$



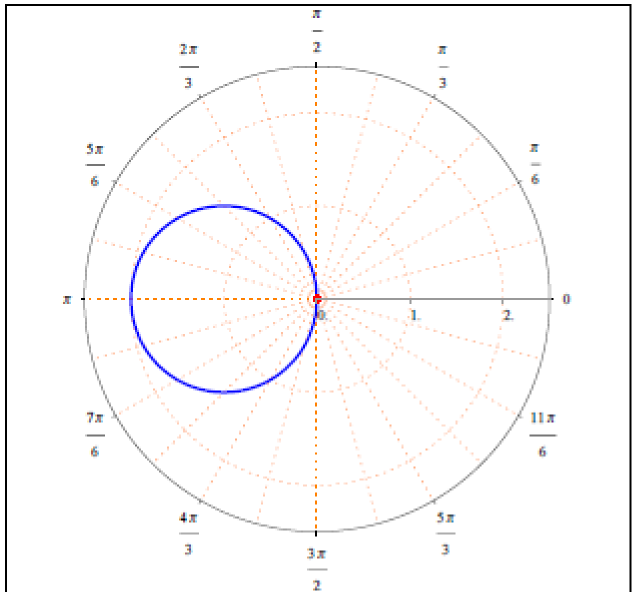
$$r = 2\sin\theta \quad , \quad 0 \le \theta \le \pi$$



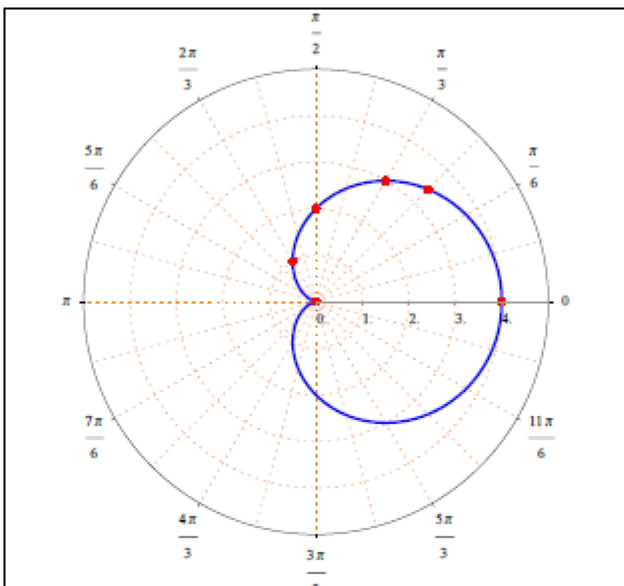
$$r = -2\sin\theta \quad , \quad 0 \le \theta \le \pi$$



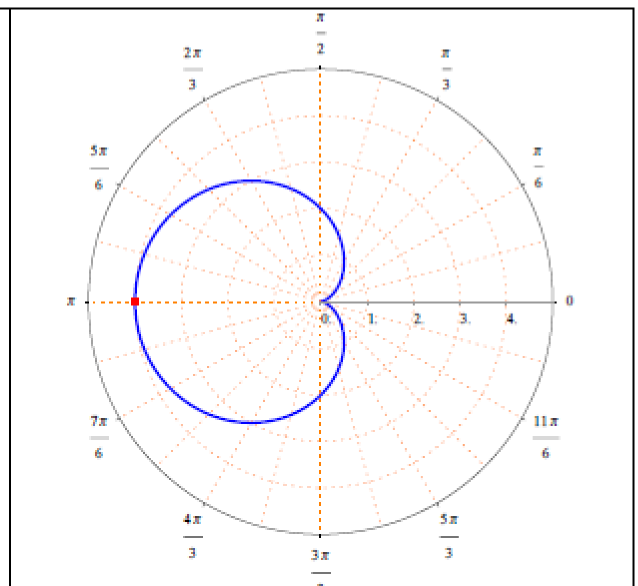
$r = 2\cos\theta \quad , \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$



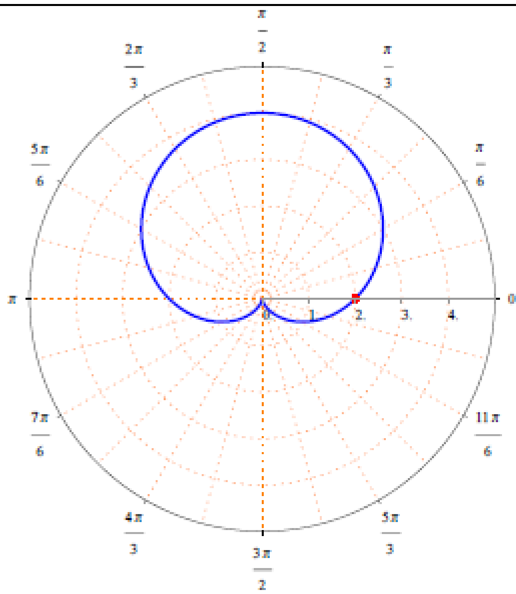
$r = -2\cos\theta \quad , \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$



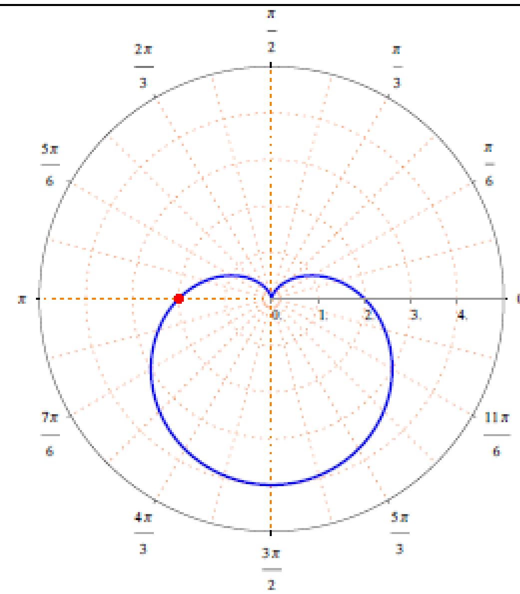
$r(\theta) = 2 + 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$



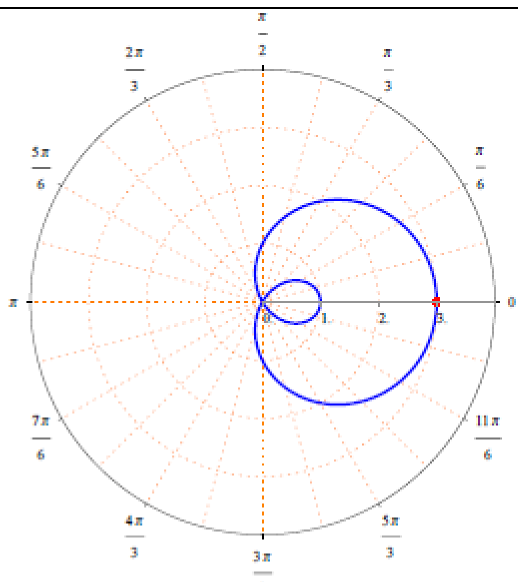
$r(\theta) = \quad 2 - 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$



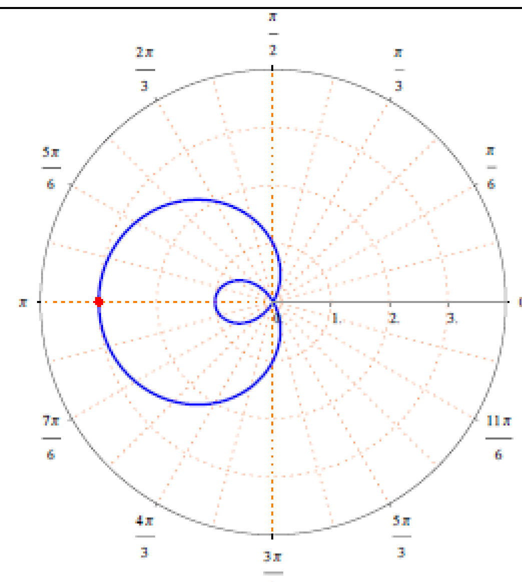
$r(\theta) = 2 + 2\sin\theta \quad , \quad 0 \le \theta \le 2\pi$



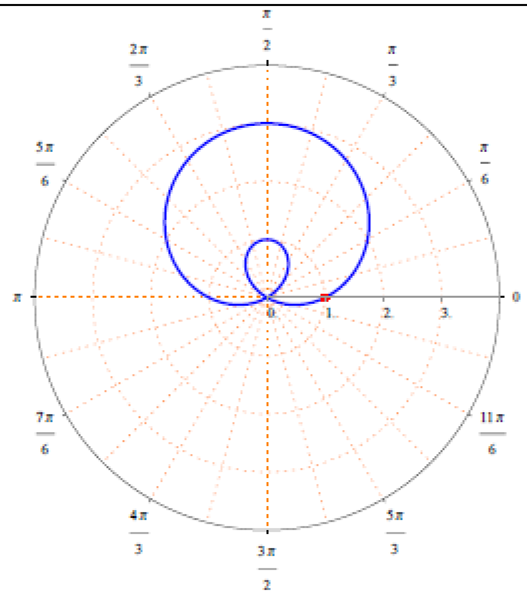
$r(\theta) = 2 - 2\sin\theta \quad , \quad 0 \le \theta \le 2\pi$



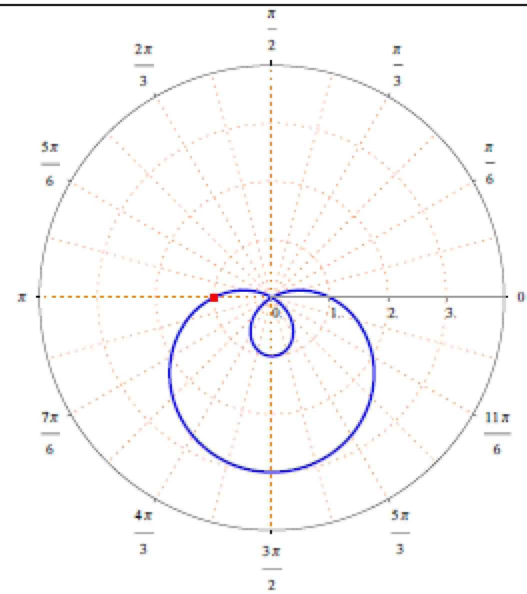
$r(\theta) = 1 + 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$



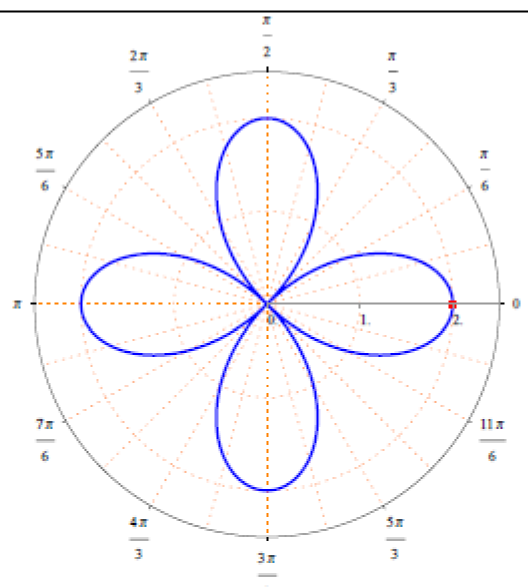
$r(\theta) = 1 - 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$



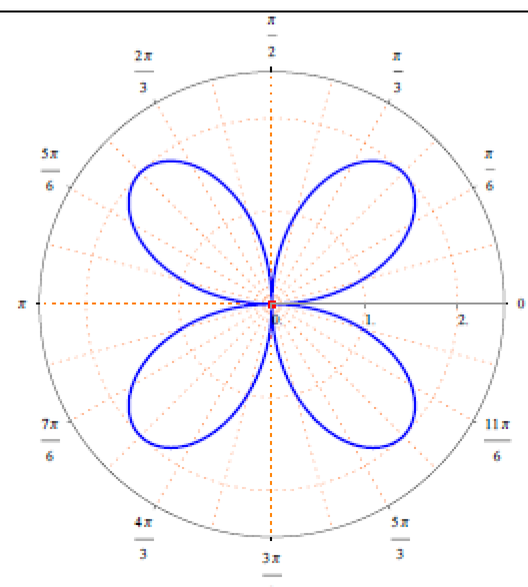
$r(\theta) = 1 + 2\sin\theta \quad , \quad 0 \le \theta \le 2\pi$



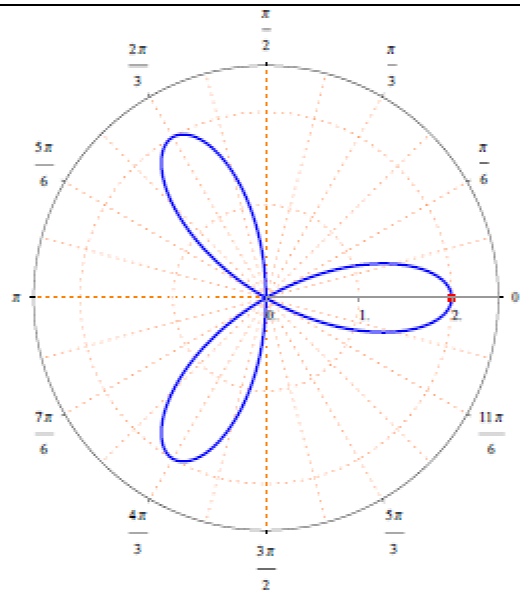
$r(\theta) = \quad 1 - 2\sin\theta \quad , \quad 0 \le \theta \le 2\pi$



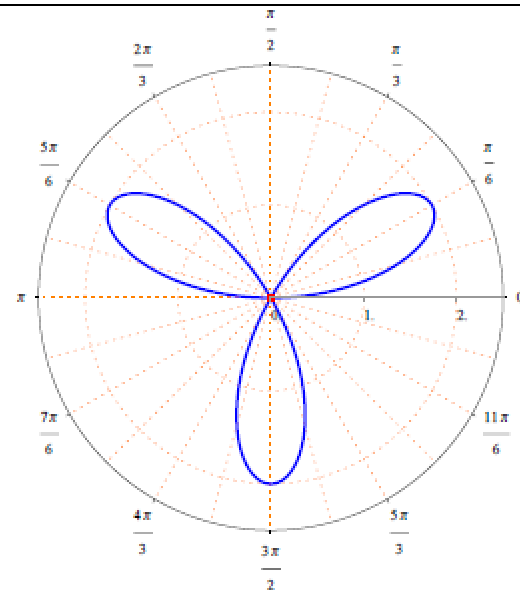
$r(\theta) = 2\cos(2\theta) \quad , \quad 0 \le \theta \le 2\pi$



$r(\theta) = 2\sin(2\theta) \quad , \quad 0 \le \theta \le 2\pi$



$r(\theta) = 2\cos(3\theta)$, $0 \le \theta \le \pi$



$r(\theta) = 2\sin(3\theta)$, $0 \le \theta \le \pi$