CSC 311: Design and Analysis of Algorithms1
Tutorial 1:Mathematical essentials

Limits:

- $\lim_{x \to c} f(x) = L \Rightarrow \forall \epsilon \, \exists \delta (\forall x : 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon)$.

- L'Hsopital's rule:
  $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ OR } \pm\infty \Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Logarithms and exponents:

- $a^b = c \Leftrightarrow \log_a c = b$.
- $a^{\log_a b} = b$.
- For any $a, b, x \in \mathbb{R}^+$, $\log_a b = \dfrac{\log_x b}{\log_x a}$.
- $\log(ab) = \log a + \log b$.
- $\log(\frac{a}{b}) = \log a - \log b$.
- $\log(x^a) = a \log x$.
- $x^a \cdot x^b = x^{a+b}$.

- $\frac{x^a}{x^b} = x^{a-b}$.
- $(x^a)^b = (x^b)^a = x^{ab}$.
- $x^0 = 1$.

Summations:

$$\sum_{i=1}^{n2} i = 1 + 2 + 3 + \cdots + n2$$

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$
- $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$
- $\displaystyle\sum_{i=a}^{b} r^i = \frac{r^{b+1} - r^a}{r-1}, r \neq 1.$
- $\displaystyle\sum_{i=a}^{\infty} r^i = \frac{r^a}{1-r}, r < 1.$

  A special case for $a = 0$: $\displaystyle\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, r < 1.$

$$\sum_{i=1}^{n}\sum_{j=1}^{n+1} j = \sum_{i=1}^{n}\frac{(n+i)(n+i+1)}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n}(n2 + 2in + i2 + n + i)$$

$$= \frac{1}{2}[n^3 + 2n\sum_{i=1}^{n} i + \sum_{i=1}^{n} i^2 + n^2 + \sum_{i=1}^{n} i]$$

$$= \frac{1}{2}\left[n^3 + 2n.\frac{n(n+1)}{2} + \frac{n(n+1(2n+1)}{6} + n^2 + \frac{n(n+1)}{2}\right]$$

$$= \left[n^3 + n^2(n+1) + \frac{n(n+1(2n+1)}{6} + n^2 + \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{2}\left[n^3 + n^3 + n^2 + \frac{1}{6}(2n^3 + 3n^2 + n) + n^2 + \frac{1}{2}(n^2 + n)\right]$$

$$= \frac{1}{2}\left[\frac{7}{3}n^3 + 3n^2 + \frac{2}{3}n\right]$$

$$= \frac{n}{2}\left[\frac{7}{3}n^2 + 3n + \frac{2}{3}\right]$$

Big-Oh notation

$O(g(n))$ is the set of all functions with a smaller or same order of growth as $g(n)$.
$f(n) \in O(g(n)) \Leftrightarrow \exists c > 0, n_0 \geq 0 | f(n) \leq cg(n)$ for all $n \geq n_0$.

- $\sum_{i=0}^{m} a_i n^i \in O(n^m)$.

- $f(n) \in O(g_1(n))$ and $h(n) \in O(g_2(n)) \Rightarrow f(n) + h(n) \in O(MAX(g_1(n), g_2(n)))$.
- $f(n) \in O(g_1(n))$ and $h(n) \in O(g_2(n)) \Rightarrow f(n) \cdot h(n) \in O(g_1(n) \cdot g_2(n))$.

Basic asymptotic classes:

1 (constant).

$\log n$ (logarithmic).

n (linear).

$n \log n$ (n-log-n).

n^2 (quadratic).

n^3 (cubic).

2^n (exponential).

$n!$ (factorial).