King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(2)

The Universal Quantifiers

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The Universal Quantifier

DEFINITION 1 The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x\ P(x)$ denotes the universal quantification of $P(x)$.

Here $\forall$ is called the ***universal quantifier.***

We read $\forall x\ P(x)$ as "for all x $P(x)$" or "for every x $P(x)$"

An element for which $P(x)$ is false is called a ***counterexample*** of $\forall x\ P(x)$.

Note that When all the elements in the domain can be listed —say, $x_1, x_2, \dots, x_n$—

it follows that the universal quantification $\forall x\ P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$$

because this conjunction is true if and only if $P(x_1), P(x_2), \dots, P(x_n)$ are all true.

The Existential Quantifier

DEFINITION 2 The *existential quantification* of *P(x)* is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x\ P(x)$ for the existential quantification of $P(x)$.

Here $\exists$ is called the ***existential quantifier.***

Note that when all the elements in the domain can be listed —say, $x_1, x_2, \dots, x_n$—

it follows that the universal quantification $\exists x\ P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \dots \vee P(x_n),$$

because this disjunction is true if and only if at least one of

$P(x_1), P(x_2), \dots, P(x_n)$ is true.

Table (1) Quantifiers

Statement	*When True?*	*When False?*
$\forall x\, P(x)$	$P(x)$ *is true for every* x	*There is an* x *for which* $P(x)$ *is false*
$\exists x\, P(x)$	*There is an* x *for which* $P(x)$ *is true*	$P(x)$ *is false for every* x

Negating Quantified Expressions

Table (2) De Morgan's Laws for Quantifiers

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	*For every* x $P(x)$ *is false*	*There is an* x *for which* $P(x)$ *is true*
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	*There is an* x *for which* $P(x)$ *is false*	$P(x)$ *is true for every* x

EXAMPLE 1 Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ "$3 \nless 2$" is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$.

Thus $\forall x\, Q(x)$ is false.

EXAMPLE 2 Suppose that $P(x)$ is "$x^2 > 0$" .To show that the statement $\forall x\, P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$ ($x^2 = 0 \ngtr 0$)

EXAMPLE 3 What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement " $x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x\, P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ because the domain consists of the integers 1, 2, 3, and 4.
Because $P(4)$, which is the statement " $4^2 < 10$," is false, ($4^2 = 16 \nless 10$)
it follows that $\forall x\, P(x)$ is false.

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EXAMPLE 4 What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x\, (x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 = \frac{1}{4} \ngeq \frac{1}{2}$.

Note that $x^2 \geq x \Leftrightarrow x^2 - x = x(x-1) \geq 0 \Leftrightarrow x \leq 0$ or $x \geq 1$

It follows that $\forall x\, (x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x\, (x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

EXAMPLE 5 Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x\, P(x)$, is true.

EXAMPLE 6 Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false.

EXAMPLE 7 What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement " $x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is {1, 2, 3, 4}, the proposition $\exists x\, P(x)$ is the same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4)$.

Because $P(4)$, which is the statement " $4^2 > 10$," is true, it follows that $\exists x\, P(x)$ is true.

EXAMPLE 8 What are the negations of the statements $\forall x\,(x^2 > x)$ and $\exists x\,(x^2 = 2)$?

Solution: (i) The negation of $\forall x\,(x^2 > x)$ is the statement $\neg\forall x\,(x^2 > x)$), which is equivalent to $\exists x \neg (x^2 > x)$. This can be rewritten as $\exists x\,(x^2 \leq x)$.

(ii) The negation of $\exists x\,(x^2 = 2)$ is the statement $\neg\exists x\,(x^2 = 2)$, which is equivalent to $\forall x \neg (x^2 = 2)$. This can be rewritten as $\forall x\,(x^2 \neq 2)$. The truth values of these statements depend on the domain.

EXAMPLE 9 Suppose that $Q(x)$ is " $x^2 \geq 2x$ ", where x is an integer.

(i) What is the negation of $\exists x\, Q(x)$?

<u>Solution:</u>

$\forall x \in \mathbb{Z}\ , x^2 < 2x$

(ii) What is the truth value of $\forall\, x\, Q(x)$? Justify your answer.

<u>Solution:</u>

False, take $x = 1$

(iii) What is the truth value of $\exists x\, Q(x)$? Justify your answer.

<u>Solution:</u>

True, take $x = 1$

EXAMPLE 10

Determine the truth value of each of the statements below given that the domain of each variable is the set of real numbers.

$(i)\ \exists x,\ \ (x^2 = 2)$

$(ii)\ \forall x,\ \ (x^2 \neq x).$

Solution:

(i) true , because $(\sqrt{2})^2 = 2$

(ii) false , because $(1)^2 = 1$.

Exercises

Q_1. Let *P(x)* be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$

d) $P(-1)$ **e)** $\exists x\ P(x)$ **f)** $\forall x\ P(x)$

Q_2. Let *Q(x)* be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$

d) $\exists x\ Q(x)$ **e)** $\forall x\ Q(x)$ **f)** $\exists x\ \neg Q(x)$

g) $\forall x\ \neg Q(x)$

Q3. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n\ (n + 1 > n)$

b) $\exists n\ (2n = 3n)$

c) $\exists n\ (n = -n)$

d) $\forall n\ (3n \leq 4n)$

Q4. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x\ (x^3 = -1)$

b) $\exists x\ (x^4 < x^2)$

c) $\forall x\ ((-x)^2 = x^2)$

d) $\forall x\ (2x > x)$

Q5. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n\ (n^2 \geq 0)$

b) $\exists n\ (n^2 = 2)$

c) $\forall n\ (n^2 \geq n)$

d) $\exists n\ (n^2 < 0)$

Q_6. Determine the truth value of each of these statements:

(1) $\forall x \in \mathbb{R}\,,\; x^2 - 4x + 4 \geq 0$

(2) $\forall x > 0\,,\; x \geq \frac{1}{x}$

(3) $\forall x \in \mathbb{Z}\,,\big((x \geq 2) \vee (x^2 \leq 2)\big)$

(4) $\exists x \in \{1,2,3,4\}\,,\; 2^x < x!$

(5) $\exists x \in \mathbb{Z}^* = \mathbb{Z} - \{0\}, \ \frac{x-1}{x} \in \mathbb{Z}$

(6) $\exists x \in \mathbb{R}, \quad x^2 = 5$

Q$_7$ Write *the negation* of the below statements:

(*i*) Some students did not listen to the instructions.

(*ii*) $\exists x \in D, x^2 > 3$

(*iii*) If you collect enough points, you will win the game.