جامعة الملك سعود
King Saud University

Theory of Computation
CSC 339 – Spring 2021

Chapter-7: part5
NP-Complete Problems

King Saud University

Department of Computer Science

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Recap

- **Problem A is NP-complete if:**

  1)A is in NP (it can be verified in polynomial time), and

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  2) **Every problem in NP reduces to A.**

- **In this part, we will look at more NP-complete problems and how we use reduction to prove they are NP-complete.**

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- **Again, why do we focus on NP-complete?**

  - **Assume we encountered a new NP-problem. Instead of spending lots of time seeking a polynomial time algorithm, we might want to try to prove that it is NP-complete.**

  - **Proving that a problem is NP-complete saves us time (instead of wasting too much time seeking a polynomial time solution that doesn’t exist).**

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- **e.g.,**

3SAT	$\xrightarrow{\text{Simulated by}}$ CLIQUE
variables	nodes
clauses	triples
a true variable	node part of clique
each clause must contain at least one true literal	each triple must contain one node in clique

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VERTEX-COVER = {⟨*G*, *k*⟩| *G is an undirected graph that has a k-node vertex cover*}**.**

Theorem 7.44

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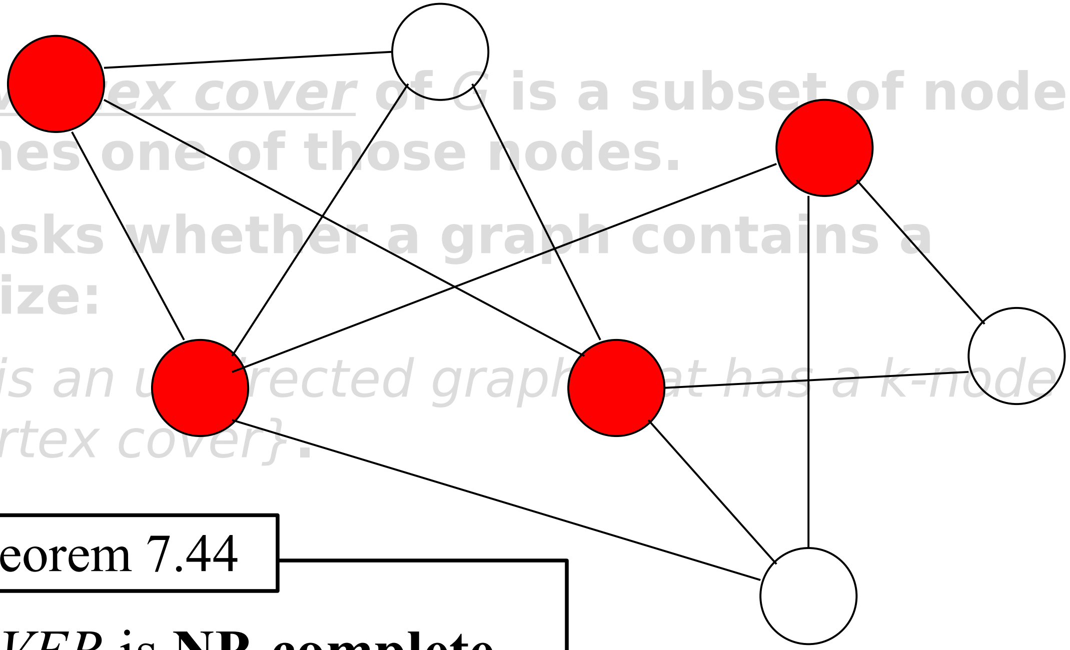

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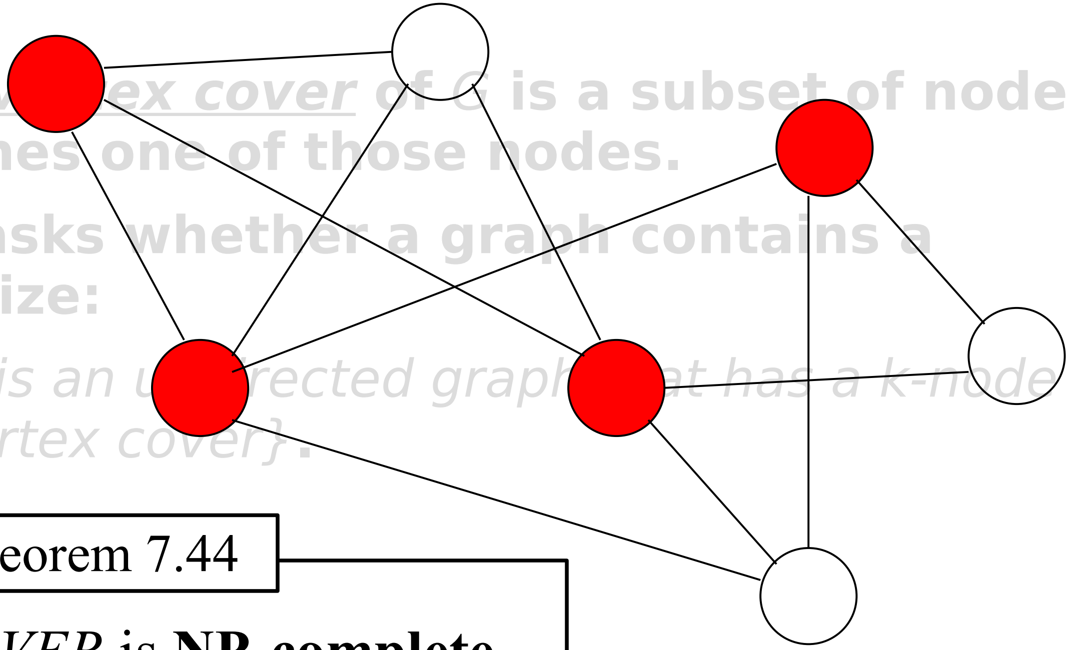
S **is a vertex cover for this graph**

$|S| = 4$



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1. *Show that a vertex-cover solution can be verified in polynomial time*
2. *Show that all NP problems are reducible to it*

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  - **A Vertex-cover verifier can clearly verify whether the certificate is valid or not in polynomial time.**
    1) **Check if cover size is** k**. If not, <u>reject</u>.**
    2) **Iterate over the edges of the graph and check whether one of the nodes (at the two ends of the edge) exists in the vertex cover presented in the certificate.**

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  - **A Vertex-cover verifier can clearly verify whether the certificate is valid or not in polynomial time.**

    1)Check if cover size is k. If not, <u>reject</u>.

    2)Iterate over the edges of the graph and check whether one of the nodes (at the two ends of the edge) exists in the vertex cover presented in the certificate.

    3)If it finds one edge whose both nodes are not in the certificate, then it <u>rejects</u> the solution. Otherwise, it <u>accepts</u>.

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  - **Show that *3SAT* is p-reducible to *VERTEX-COVER*.**
  - **This entails converting a 3cnf-formula into a graph G and a number k.**
  - **The formula φ is satisfiable whenever G has a vertex cover with k nodes.**
  - **Simply put, G should simulate φ.**

VERTEX-COVER (yet, another NP-complete problem)

- **2) Polynomial time reduction (p-reduction)**

Let φ be a 3CNF formula with $\boldsymbol{m}$ variables and $\boldsymbol{l}$ clauses

$$\varphi = \underbrace{(x_1 \vee x_2 \vee x_3)}_{\text{Clause 1}} \wedge \underbrace{(\overline{x_1} \vee \overline{x_2} \vee \overline{x_4})}_{\text{Clause 2}} \wedge \underbrace{(\overline{x_1} \vee x_3 \vee x_4)}_{\text{Clause 3}}$$

$\boldsymbol{m} = 4$
$\boldsymbol{l} = 3$

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- **2) Polynomial time reduction (p-reduction)**

Formula φ can be converted to a graph G such that:

φ is satisfiable ***iff***

G has a vertex cover
of size $\mathbf{k} = \boldsymbol{m} + \boldsymbol{2l}$

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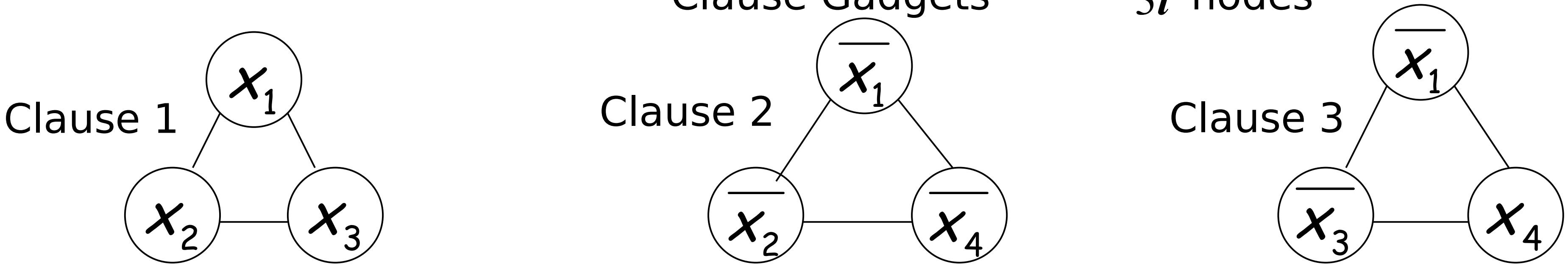
Variable Gadgets $\quad 2m$ nodes

Clause Gadgets $\quad 3l$ nodes



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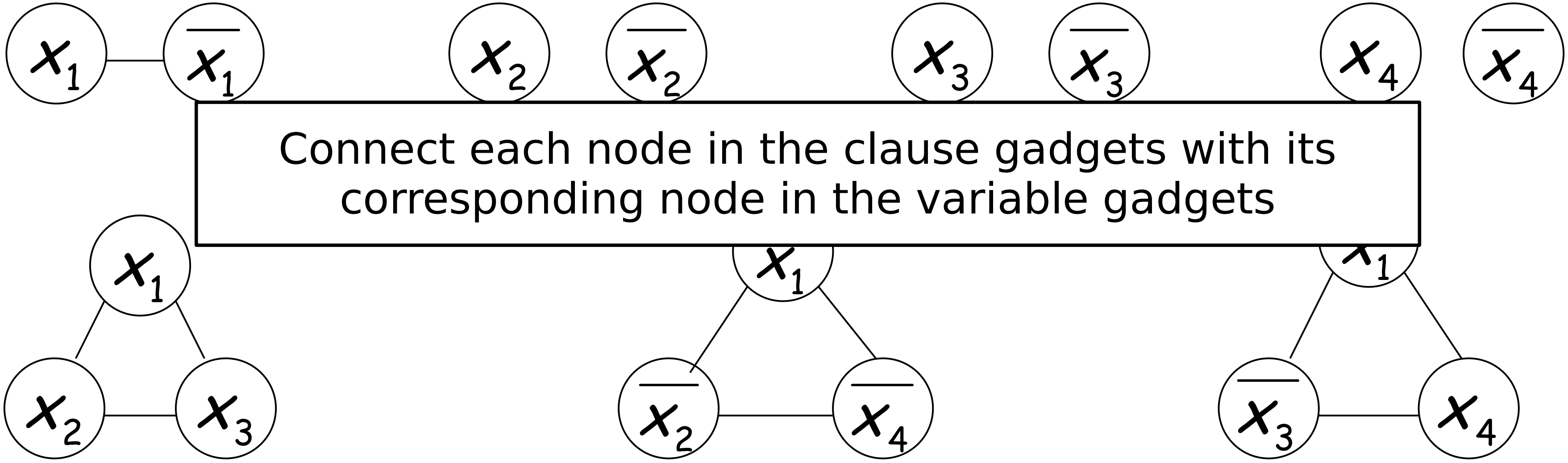

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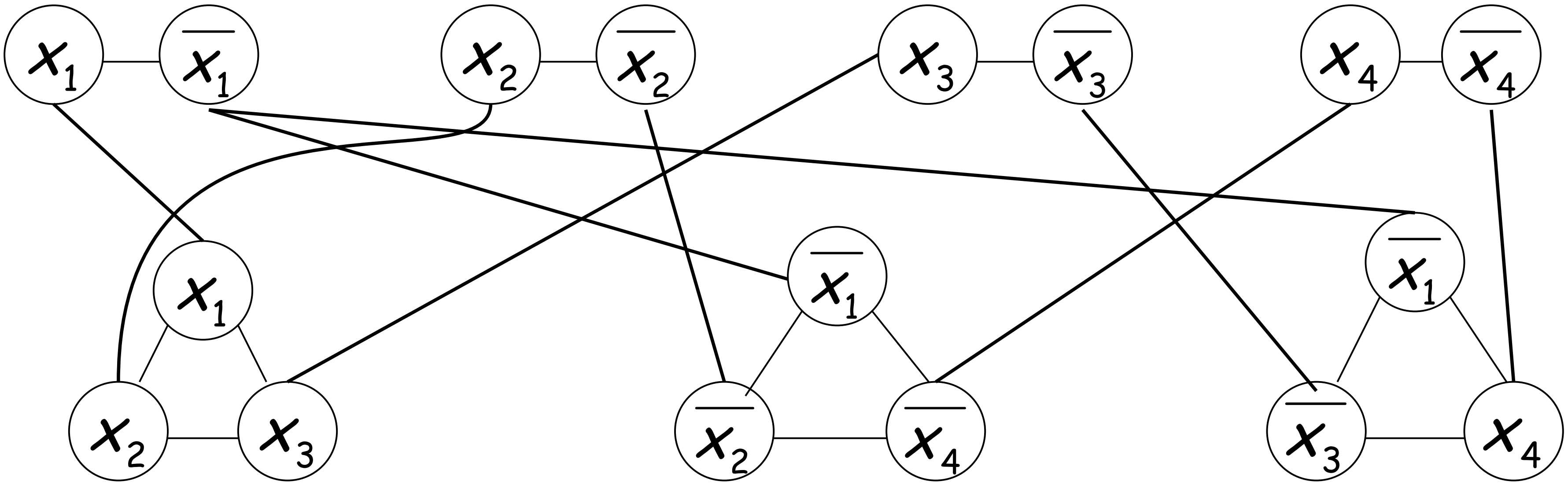

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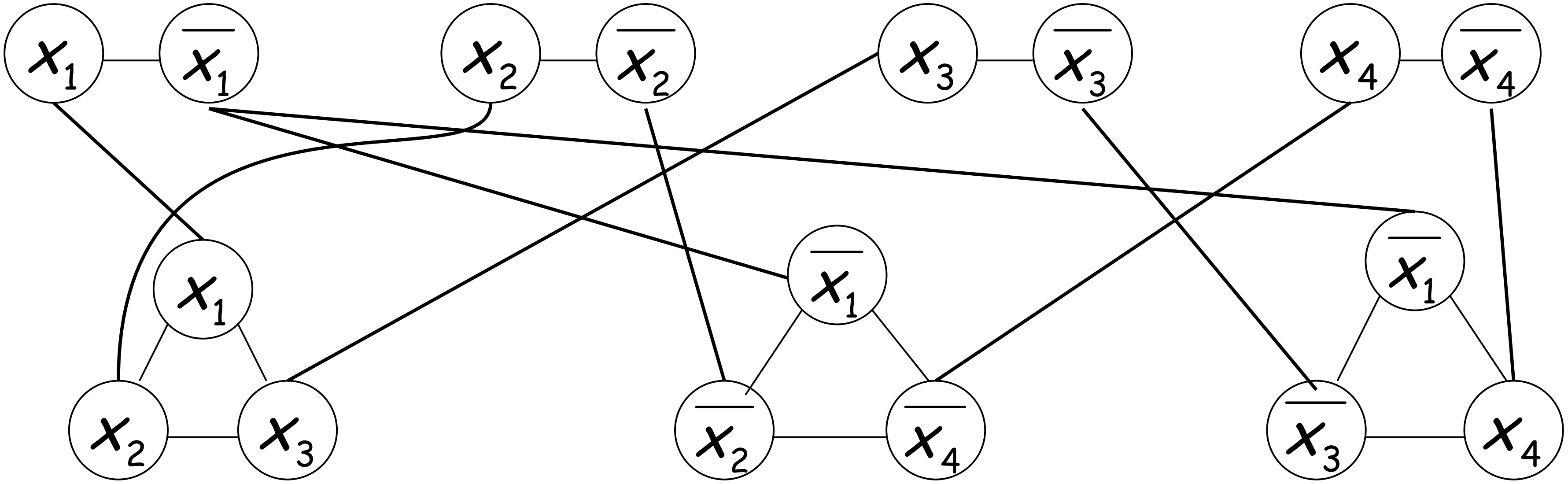
Start with a satisfying assignment $x_1 = 1$ $x_2 = 0$ $x_3 = 0$ $x_4 = 1$



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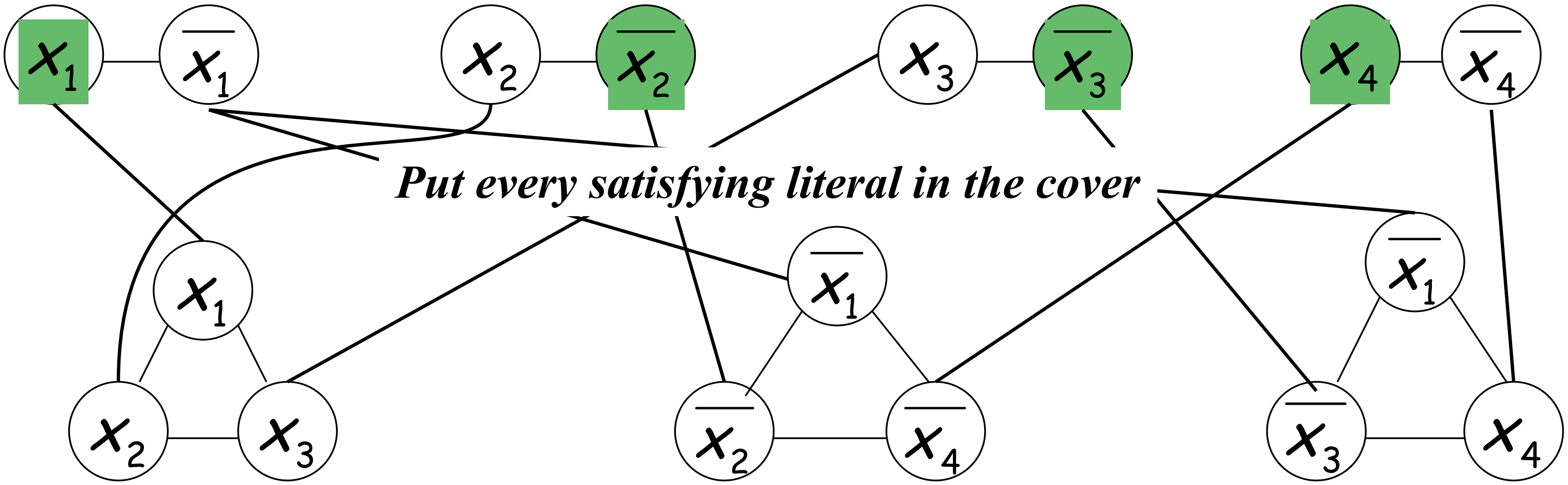

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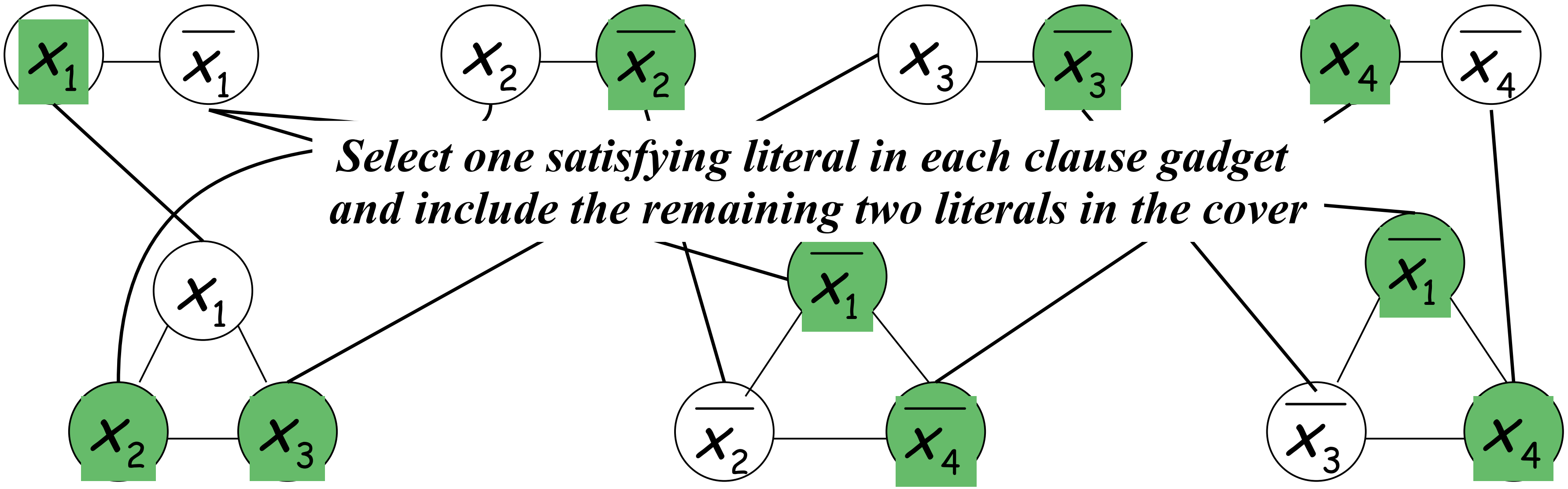

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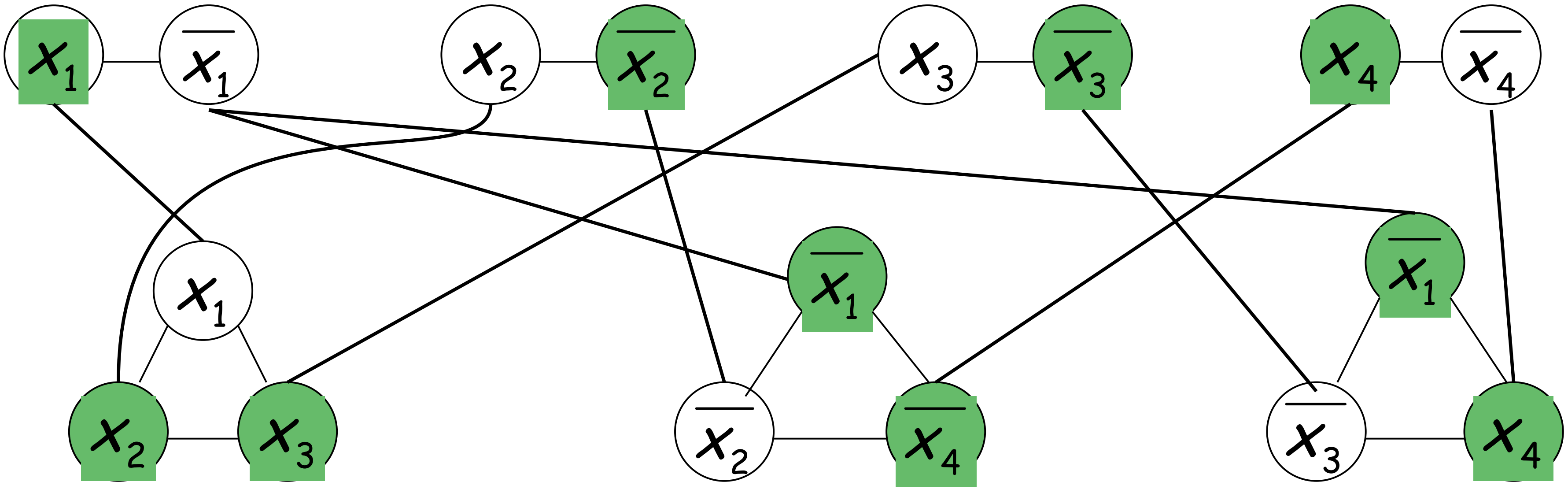

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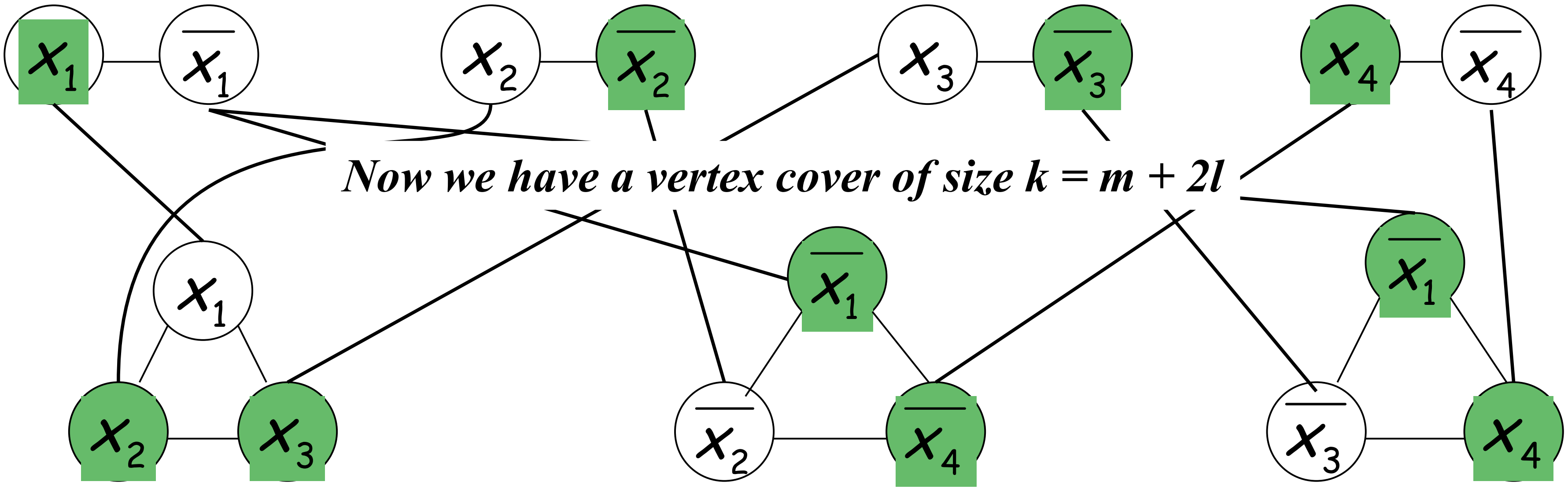

Now we have a vertex cover of size k = m + 2l

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This concludes the proof for Theorem 7.44