

King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(4.2)

Equivalence Relations

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2018

Equivalence Relations

DEFINITION 1 A relation on a set A is called an *equivalence relation* if it is *reflexive*, *symmetric*, and *transitive*.

DEFINITION 2 Two elements a and b that are related by an equivalence relation are called *equivalent*. the notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation

Equivalence Classes

DEFINITION 3 Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class of a* . The equivalence class of a with respect to R is denoted by $[a]$ for this equivalence class . $[a] = \{b \in A : a R b\}$

Equivalence Classes and Partitions

THEOREM 1 Let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

$$(i) a R b \quad (ii) [a] = [b] \quad (iii) [a] \cap [b] \neq \emptyset$$

THEOREM 2 Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\mathfrak{S} = \{\{A_i : \emptyset \neq A_i \subseteq S, i \in I\}\}$ of the set S , there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes .

$$\forall (A_i, A_j \in \mathfrak{S}), (i \neq j \rightarrow A_i \cap A_j = \emptyset)$$

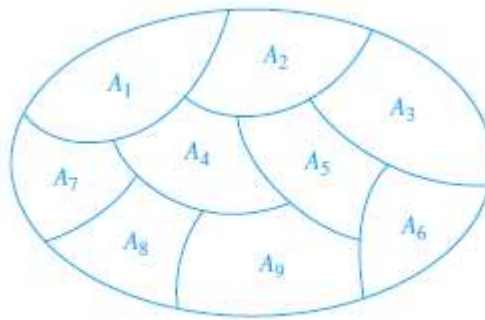


FIGURE 1 A Partition of a Set.

and

$$\bigcup_{i \in I} A_i = S$$

(Here the notation $\bigcup_{i \in I} A_i$ represents the union of the sets A_i for all $i \in I$.) Figure 1 illustrates the concept of a partition of a set.

EXAMPLE 1 Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S , because these sets are disjoint and their union is S .

$$\mathfrak{A} = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\} \text{ where } \forall (A_i, A_j \in \mathfrak{A}), (i \neq j \rightarrow A_i \cap A_j = \emptyset) \text{ and } \bigcup_{i=1,2,3} A_i = S$$

EXAMPLE 2 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 1.

Solution: The subsets in the partition are the equivalence classes of R . The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition. The pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, and $(3, 3)$ belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs $(4, 4)$, $(4, 5)$, $(5, 4)$, and $(5, 5)$ belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair $(6, 6)$ belongs to R because $\{6\}$ is an equivalence class. No pair other than those listed belongs to R .

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$$

EXAMPLE 3 Let \sim be a relation defined on $\mathbb{N} \times \mathbb{N}$, such that:

$$(m, n), (p, q) \in \mathbb{N} \times \mathbb{N} \quad (m, n) \sim (p, q) \Leftrightarrow m + q = p + n$$

- (i) Show that \sim is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.
- (ii) Find the *equivalence classes* $[(3,4)]$, $[(1,1)]$

Solution:

(i)

$$(a) \quad \forall (m, n) \in \mathbb{N} \times \mathbb{N} \Rightarrow \because m + n = m + n \Rightarrow \because (m, n) \sim (m, n) \Rightarrow \because \sim \text{ is reflexive}$$

$$(b) \quad (m, n), (p, q) \in \mathbb{N} \times \mathbb{N} : (m, n) \sim (p, q) \Rightarrow m + q = p + n$$

$$\Rightarrow p + n = m + q$$

$$\Rightarrow (p, q) \sim (m, n) \Rightarrow \because \sim \text{ is symmetric}$$

$$(c) \quad (m, n), (p, q), (r, s) \in \mathbb{N} \times \mathbb{N} : (m, n) \sim (p, q) \Rightarrow m + q = p + n$$

$$(p, q) \sim (r, s) \Rightarrow p + s = r + q$$

$$\Rightarrow m + q + p + s = p + n + r + q$$

$$\Rightarrow m + s = r + n \Rightarrow (m, n) \sim (r, s)$$

$$\Rightarrow \because \sim \text{ is transitive}$$

$$\because \sim \text{ is reflexive, symmetric and transitive } \Rightarrow \because \sim \text{ is an equivalence relation on } \mathbb{N} \times \mathbb{N}.$$

(ii)

$$\begin{aligned} [(1,1)] &= \{(a,b) \in \mathbb{N} \times \mathbb{N} : (a,b) \sim (1,1) \Rightarrow a+1 = b+1 \Rightarrow a=b\} \\ &= \{(a,a) : a \in \mathbb{N}\} = \{(1,1), (2,2), (3,3), \dots\} \end{aligned}$$

$$\begin{aligned} [(3,4)] &= \{(a,b) \in \mathbb{N} \times \mathbb{N} : (a,b) \sim (3,4) \Rightarrow a+4 = b+3 \Rightarrow b = a+1\} \\ &= \{(a, a+1) : a \in \mathbb{N}\} = \{(1,2), (2,3), (3,4), \dots\} \end{aligned}$$

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EXAMPLE 4 Let S be a relation defined on \mathbb{R} such that:

$$x, y \in \mathbb{R}, x S y \Leftrightarrow x^2 - y^2 = 2(y - x) \Leftrightarrow x^2 + 2x = y^2 + 2y$$

(i) Show that S is an equivalence relation on \mathbb{R} .(ii) Find the *equivalence classes* $[1], [0]$ *Solution:*

(i)

$$(a) \forall x \in \mathbb{R}, x^2 + 2x = x^2 + 2x \Rightarrow \therefore x S x \Rightarrow \therefore S \text{ is } \textit{reflexive}$$

$$(b) x, y \in \mathbb{R}, x S y \Rightarrow x^2 + 2x = y^2 + 2y \Rightarrow y^2 + 2y = x^2 + 2x \Rightarrow y S x \\ \Rightarrow \therefore S \text{ is } \textit{symmetric}$$

$$(c) x, y, z \in \mathbb{R} : x S y \Rightarrow x^2 + 2x = y^2 + 2y$$

$$y S z \Rightarrow y^2 + 2y = z^2 + 2z$$

$$\Rightarrow x^2 + 2x = y^2 + 2y = z^2 + 2z$$

$$\Rightarrow x^2 + 2x = z^2 + 2z$$

$$\Rightarrow x S z \Rightarrow \therefore S \text{ is } \textit{transitive}$$

$\therefore S$ is *reflexive*, *symmetric* and *transitive* $\Rightarrow \therefore S$ is an *equivalence relation* on \mathbb{R} .

(ii)

$$[0] = \{x \in \mathbb{R} : x S 0 \Rightarrow x^2 + 2x = 0^2 + 2(0)\}$$

$$= \{x \in \mathbb{R} : x(x+2) = 0\} = \{-2, 0\}$$

$$[1] = \{x \in \mathbb{R} : x S 1 \Rightarrow x^2 + 2x = 1^2 + 2(1) = 3\}$$

$$= \{x \in \mathbb{R} : x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0\} = \{-3, 1\}.$$

#

EXAMPLE 5 Let T be a relation defined on \mathbb{Z} such that:

$$a, b \in \mathbb{Z}, \quad a T b \Leftrightarrow |a| = |b|$$

(i) Show that T is an equivalence relation .

(ii) Find $\mathfrak{I}(T)$.

Solution:

(i)

$$1- \forall a \in \mathbb{Z}, \because |a| = |a| \Rightarrow \therefore a T a \Rightarrow \therefore T \text{ is } \textcolor{green}{\text{reflexive}}$$

$$2- a, b \in \mathbb{Z}, \quad a T b \Leftrightarrow |a| = |b| \Rightarrow |b| = |a| \Rightarrow b T a \Rightarrow \therefore T \text{ is } \textcolor{red}{\text{symmetric}}$$

$$3- a, b, c \in \mathbb{Z}, \quad a T b \Leftrightarrow |a| = |b|$$

&

$$b T c \Leftrightarrow |b| = |c|$$

$$|a| = |b| = |c| \Rightarrow |a| = |c| \Rightarrow \therefore a T c \Rightarrow \therefore T \text{ is } \textcolor{violet}{\text{transitive}}$$

$\therefore T$ is $\textcolor{green}{\text{reflexive}}$, $\textcolor{red}{\text{symmetric}}$ and $\textcolor{violet}{\text{transitive}} \Rightarrow \therefore T$ is an $\textcolor{blue}{\text{equivalence relation}}$ on \mathbb{Z} .

(ii)

$$\textcolor{violet}{[a]} = \{b \in \mathbb{Z} : |a| = |b|\} = \{b \in \mathbb{Z} : a = \pm b\}$$

$$= \textcolor{violet}{\{a, -a\}}$$

$$\therefore \mathfrak{I}(T) = \{[a] : a \in \mathbb{Z}\} = \{\{0\}, \{-1, 1\}, \{-2, 2\}, \dots\} \quad .$$

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EXERCISES

1. Let R be the relation defined on the integers set \mathbb{Z} , such that:

$$a, b \in \mathbb{Z}, a R b \Leftrightarrow 6a \equiv b \pmod{5} \Leftrightarrow 5|(6a - b), \quad 5 \text{ divides } (6a - b)$$

(i) Show that R is an equivalence relation.

(ii) Find the equivalence class $[0]$.

(iii) Show that $9 \in [4]$.

Solution: (i) $a R b \Leftrightarrow 6a \equiv b \pmod{5} \Leftrightarrow 5|(6a - b) \Rightarrow 6a - b = 5h : h \in \mathbb{Z}$

$$1- \forall a \in \mathbb{Z}, 5|(6a - a) = 5a \Rightarrow 5|5a \Rightarrow \therefore a R a \Rightarrow \therefore R \text{ is reflexive}$$

$$2- a, b \in \mathbb{Z}, a R b \Leftrightarrow 6a \equiv b \pmod{5} \Leftrightarrow 5|(6a - b) \Rightarrow 6a - b = 5h : h \in \mathbb{Z}$$

$$\begin{aligned} b &= 6a - 5h \Rightarrow 6b - a = 6(6a - 5h) - a = 36a - 5h - a \\ &= 35a - 5h = 5(7a - h) \Rightarrow \therefore 5|(6b - a) \Rightarrow b R a \Rightarrow \therefore R \text{ is symmetric} \end{aligned}$$

$$3- a, b, c \in \mathbb{Z} \quad a R b \Leftrightarrow 6a \equiv b \pmod{5} \Leftrightarrow 5|(6a - b) \Rightarrow 6a - b = 5h : h \in \mathbb{Z}$$

&

$$b R c \Leftrightarrow 6b \equiv c \pmod{5} \Leftrightarrow 5|(6b - c) \Rightarrow 6b - c = 5k : k \in \mathbb{Z}$$

$$\Rightarrow (+) \underline{\hspace{2cm}}$$

$$6a + 5b - c = 5h + 5k$$

$$6a - c = 5(h + k - b) = 5l$$

$$\therefore (h + k - b) = l \in \mathbb{Z} \Rightarrow \therefore 5|6a - c \Rightarrow \therefore a R c \Rightarrow \therefore R \text{ is transitive}$$

$\therefore R$ is reflexive, symmetric and transitive $\Rightarrow \therefore R$ is an equivalence relation on \mathbb{Z} .

$$(ii) [0] = \{x \in \mathbb{Z} : 0 R x\} = \{x \in \mathbb{Z} : 6(0) - x = 5h : h \in \mathbb{Z}\}$$

$$= \{x \in \mathbb{Z} : x = 5(-h) = 5r : r = -5h\}$$

$$= \{\dots, -15, -10, -5, 0, 5, 10, 15, \dots\}$$

$$(iii) \therefore 5|6(9) - 4 = 50 \Rightarrow \therefore 9 R 4 \Rightarrow \therefore 9 \in [4]$$

2. Let S be the relation defined on the set $A = \{-2, -1, 0, 1, 2\}$, such that:

$$a, b \in A, \quad a S b \Leftrightarrow 3|(a + 2b), \quad 3 \text{ divides } (a + 2b)$$

(i) Show that S is an equivalence relation.

(ii) Find all *equivalence classes*.

Solution :

3. Let R be the relation defined on the integers set \mathbb{Z} , such that:

$$m, n \in \mathbb{Z}, \quad m R n \Leftrightarrow 4|(m - n + 8), \quad 4 \text{ divides } (m - n + 8)$$

(i) Show that R is an equivalence relation.

(ii) Show that $[10] = [-6]$.

Solution :

4. Assume T is an equivalence relation defined on the set $A = \{a, b, c, d, e\}$,

and the matrix of T given such that

$$M_T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the number of *equivalence classes* .

Solution :

5. Let $T = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$ be a relation defined on the set $A = \{a, b, c, d\}$

(i) Represent T by the directed graph (diagraph)

(ii) Show that R is an equivalence relation.

(iii) Find all *equivalence classes*.

Solution :

6. Let R be the relation defined on the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, such that:

$$a, b \in \mathbb{Z}^+ , \quad a R b \Leftrightarrow (\sqrt{a} - \sqrt{b}) \in \mathbb{Z} , \quad (\sqrt{a} - \sqrt{b}) \text{ is integer}$$

(i) Show that R is an equivalence relation.

(ii) Decide whether $4 \in [9]$.

Solution :

7. Let S be the relation defined on the set $\mathbb{N} = \{1, 2, 3, \dots\}$, such that:

$$a, b \in \mathbb{Z}, \quad a R b \Leftrightarrow 5a \equiv b \pmod{4} \Leftrightarrow 4 \mid (5a - b), \quad 4 \text{ divides } (5a - b)$$

(i) Show that S is an equivalence relation.

(ii) Find the *equivalence class* $[1]$.

Solution:

8. Let S be the relation defined on the Rational set \mathbb{Q} , such that:

$$x, y \in \mathbb{Q}, x S y \Leftrightarrow x - y \in \mathbb{Z}, \quad (x - y) \text{ is integer}$$

(i) Show that S is an equivalence relation.

(ii) Decide whether $\frac{9}{4} \in [\frac{1}{2}]$ or not?

Solution: (i)

$$1- \forall x \in \mathbb{Q}, x - x = 0 \in \mathbb{Z} \Rightarrow x S x \Rightarrow \therefore S \text{ is reflexive}$$

$$2- x, y \in \mathbb{Q}, x S y \Leftrightarrow x - y = m \in \mathbb{Z}$$

$$(\text{multiply by } -1) \Rightarrow y - x = -m \in \mathbb{Z} \Rightarrow \therefore S \text{ is symmetric}$$

$$3- x, y, z \in \mathbb{Z}, x S y \Rightarrow x - y = m_1 \in \mathbb{Z}$$

&

$$y S z \Rightarrow y - z = m_2 \in \mathbb{Z}$$

$$(+)\Rightarrow \underline{\hspace{2cm}}$$

$$x - z = m_1 + m_2 = m \in \mathbb{Z}$$

$$\Rightarrow \therefore x S z \Rightarrow \therefore S \text{ is transitive}$$

$\therefore S$ is reflexive, symmetric and transitive $\Rightarrow \therefore S$ is an equivalence relation on \mathbb{Q}

$$(ii) \quad \therefore \frac{9}{4} - \frac{1}{2} = \frac{9}{4} - \frac{2}{4} = \frac{7}{4} \notin \mathbb{Z} \Rightarrow \therefore \frac{9}{4} \notin [\frac{1}{2}]$$

$$[\frac{1}{2}] = \{x \in \mathbb{Q} : x S \frac{1}{2}\} = \{x \in \mathbb{Q} : x - \frac{1}{2} = m \in \mathbb{Z}\}$$

$$= \{x \in \mathbb{Q} : x = \frac{1}{2} + m : m \in \mathbb{Z}\}$$

$$= \{\dots, -\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, \dots\}$$

9. Let S be the relation defined on the set $E = \{2a \mid a \in \mathbb{Z}\}$ (even Integers set), such that:

$$m, n \in E, \quad m S n \Leftrightarrow 4 \mid (m + n), \quad 4 \text{ divides } (m + n)$$

(i) Show that S is an equivalence relation.

(ii) Find the *equivalence class* $[0]$.

Solution :

10. Let R be the relation defined on the Rational set $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, such that:

$$x, y \in \mathbb{Z}^*, \quad x R y \Leftrightarrow xy > 0$$

(i) Show that S is an equivalence relation.

(ii) Find the *equivalence classes* $[-1], [1]$

Solution: (i)

$$1- \forall x \in \mathbb{Z}^*, \quad xx = x^2 > 0 \Rightarrow \therefore x R x \Rightarrow \therefore R \text{ is } \textit{reflexive}$$

$$2- x, y \in \mathbb{Z}^*, \quad x R y \Leftrightarrow xy > 0 \Rightarrow yx > 0 \Rightarrow y R x \Rightarrow \therefore R \text{ is } \textit{symmetric}$$

$$3- x, y, z \in \mathbb{Z}^*, \quad x R y \Leftrightarrow xy > 0 \quad (1)$$

&

$$y R z \Leftrightarrow yz > 0 \quad (2)$$

$$(1) \times (2) \Rightarrow \underline{\hspace{2cm}}$$

$$\Rightarrow x y^2 z > 0$$

$$(\text{divide both sides on } y^2 > 0) \Rightarrow xz > 0 \Rightarrow x R z \Rightarrow \therefore R \text{ is } \textit{transitive}$$

$\therefore R$ is *reflexive*, *symmetric* and *transitive* $\Rightarrow \therefore R$ is an equivalence relation on \mathbb{Z}^* .

(ii)

$$[-1] = \{x \in \mathbb{Z}^* : x R (-1)\}$$

$$= \{x \in \mathbb{Z}^* : x(-1) = -x > 0\}$$

$$= \{x \in \mathbb{Z}^* : x < 0\} = \mathbb{Z}^-$$

$$[1] = \{x \in \mathbb{Z}^* : x R (1)\}$$

$$= \{x \in \mathbb{Z}^* : x(1) = x > 0\} = \mathbb{Z}^+$$

11. Let $S = \{(a, a), (a, c), (b, b), (b, e), (c, a), (c, c), (d, d), (e, b), (e, e)\}$ be a relation defined on the set $A = \{a, b, c, d, e\}$

(i) Show that S is an equivalence relation.

(ii) Find all *equivalence classes*.

Solution: (i)

1- $\because (a, a), (b, b), (c, c), (d, d), (e, e) \in S \Rightarrow \therefore S$ is *reflexive*

2- $\because (a, c), (c, a) \in S$ & $(b, e), (e, b) \in S \Rightarrow \therefore S$ is *symmetric*

3- $\because (a, c), (c, a), (a, a) \in S$

& $(b, e), (e, b), (b, b) \in S$

& $(e, b), (b, e), (e, e) \in S$

& $(c, a), (a, c), (c, c) \in S \Rightarrow \therefore S$ is *transitive*

$\because S$ is *reflexive*, *symmetric* and *transitive* $\Rightarrow \therefore S$ is an equivalence relation on A .

(ii) $[a] = \{a, c\}$

$[b] = \{b, e\}$

$[d] = \{d\}$

$\Rightarrow \mathfrak{I}(S) = \{\{a, c\}, \{b, e\}, \{d\}\}$

12. Let S be the relation defined on the set $A = \{0,1,2,3,4\}$, such that:

$$a, b \in A, \quad a S b \Leftrightarrow 3 \mid (2a + b), \quad 3 \text{ divides } (2a + b)$$

- (i) Show that S is an equivalence relation.
- (ii) Find the *equivalence classes* $[0], [1]$
- (iii) Find the number of *equivalence classes* of the relation S .

Solution :

13. Let R be the relation defined on the set $\mathbb{N} = \{1, 2, 3, \dots\}$, such that:

$$x, y \in \mathbb{N}, \quad x R y \Leftrightarrow x + y \text{ is even}.$$

(i) Show that R is an equivalence relation.

(ii) Find the **equivalence class** $[2]$.

(iii) Decide whether $4 \in [11]$ or not?

Solution: (i)

$$1- \quad \forall x \in \mathbb{N}, \quad x + x = 2x \text{ (is even)} \Rightarrow \therefore x R x \Rightarrow \therefore R \text{ is reflexive}$$

$$2- \quad x, y \in \mathbb{N}, \quad x R y \Leftrightarrow x + y = 2m \text{ (is even)} : m \in \mathbb{N} \\ \Rightarrow y + x = 2m \text{ (is even)} \Rightarrow y R x \therefore R \text{ is symmetric}$$

$$3- \quad x, y, z \in \mathbb{N}, \quad x R y \Leftrightarrow x + y = 2m_1 \text{ (is even)} : m_1 \in \mathbb{N}$$

&

$$y R z \Leftrightarrow y + z = 2m_2 \text{ (is even)} : m_2 \in \mathbb{N}$$

$$(+)\Rightarrow \frac{x + y + y + z}{x + 2y + z} = \frac{2m_1 + 2m_2}{x + 2y + z} \\ \Rightarrow x + z = 2(m_1 + m_2 - y) = 2m \text{ (is even)}$$

$$: m = (m_1 + m_2 - y) \in \mathbb{N} \Rightarrow \therefore x R z \Rightarrow \therefore R \text{ is transitive}$$

$\therefore R$ is reflexive, symmetric and transitive $\Rightarrow \therefore R$ is an equivalence relation on \mathbb{N} .

$$(ii) \quad [2] = \{x \in \mathbb{N} : x R 2\}$$

$$= \{x \in \mathbb{N} : x + 2 = 2m : m \in \mathbb{N}\}$$

$$= \{x \in \mathbb{N} : x = 2m - 2 = 2(m - 1) = 2k : k = m - 1 \in \mathbb{N}\}$$

$$= \{x \in \mathbb{N} : x = 2k, k \in \mathbb{N}\} = \{2, 4, 6, \dots\}$$

$$(iii) \quad \because 4 + 11 = 15 \text{ (is an odd)} \Rightarrow \therefore 4 \notin [11]$$

14. Let S be the relation defined on the Rational set \mathbb{Q} , such that:

$$a, b \in \mathbb{Q}, a S b \Leftrightarrow a - b = 2m : m \in \mathbb{Z}, (a - b) \text{ is even integer}.$$

(i) Show that S is an equivalence relation.

(ii) Show that $[m] = [0]$ for every even integer m , and $[n] = [1]$ for every odd integer n .

Solution: (i)

$$\begin{aligned} \text{(ii)} \quad [0] &= \{x \in \mathbb{Q} : x S 0\} \\ &= \{x \in \mathbb{Q} : x - 0 = x = m \text{ is an even integer}\} \\ &= [m] : m \text{ is an even integer} \end{aligned}$$

$$\begin{aligned} [1] &= \{x \in \mathbb{Q} : x S 1\} \\ &= \{x \in \mathbb{Q} : x - 1 = k \text{ is an even integer}\} \\ &= \{x \in \mathbb{Q} : x = k + 1 = n \text{ is an odd integer}\} \\ &= [n] : n \text{ is an odd integer} \end{aligned}$$

15. Let R be the relation defined on the integers set \mathbb{Z} , such that:

$$a, b \in \mathbb{Z}, a R b \Leftrightarrow a \equiv b \pmod{7} \Leftrightarrow 7|(a-b), \quad 7 \text{ divides } (a-b)$$

(i) Show that R is an equivalence relation.

(ii) Decide whether $9 \in [2]$.

(iii) If R is defined on the set $= \{1,5,12,22,35,41,55\}$, find all *equivalence classes*.

Solution: (i) $a, b \in \mathbb{Z}, a R b \Leftrightarrow a \equiv b \pmod{7} \Leftrightarrow 7|(a-b) \Rightarrow a-b = 7m : m \in \mathbb{Z}$

$$1- \quad \forall a \in \mathbb{Z}, 7|(a-a) = 0 \Rightarrow \therefore a R a \Rightarrow \therefore R \text{ is reflexive}$$

$$2- \quad a, b \in \mathbb{Z}, a R b \Leftrightarrow a \equiv b \pmod{7} \Leftrightarrow 7|(a-b) \Rightarrow a-b = 7m : m \in \mathbb{Z}$$

$$(\text{multiply both sides by } -1) \Rightarrow b-a = 7(-m) \Rightarrow 7|(b-a) \Rightarrow b R a \Rightarrow \therefore R \text{ is symmetric}$$

$$3- \quad a, b, c \in \mathbb{Z}, a R b \Leftrightarrow a \equiv b \pmod{7} \Leftrightarrow 7|(a-b) \Rightarrow a-b = 7m_1 : m_1 \in \mathbb{Z}$$

&

$$b R c \Leftrightarrow b \equiv c \pmod{7} \Leftrightarrow 7|(b-c) \Rightarrow b-c = 7m_2 : m_2 \in \mathbb{Z}$$

(+) \Rightarrow

$$a-c = 7(m_1 + m_2) = 7m$$

$$: m = (m_1 + m_2) \in \mathbb{Z} \Rightarrow 7|(a-c) \Rightarrow a R c \Rightarrow \therefore R \text{ is transitive}$$

$\therefore R$ is reflexive, symmetric and transitive $\Rightarrow \therefore R$ is an equivalence relation on \mathbb{Z} .

$$(ii) \quad \therefore 9-2 = 7 \Rightarrow 7|(9-2) \Rightarrow \therefore 9 R 2 \quad \therefore 9 \in [2]$$

$$(iii) \quad [1] = \{1,22\}$$

$$[5] = \{5,12\}$$

$$[35] = \{35\}$$

$$[41] = \{41,55\}$$

$$\mathfrak{I}(R) = \{\{1,22\}, \{5,12\}, \{35\}, \{41,55\}\}$$

16. Let R be the relation defined on the integers set \mathbb{Z} , such that:

$$a, b \in \mathbb{Z}, a R b \Leftrightarrow a^2 \equiv b^2 \pmod{7} \Leftrightarrow 7 | (a^2 - b^2), \quad , \quad 7 \text{ divides } (a^2 - b^2)$$

(i) Show that R is an equivalence relation.

(ii) Find $\mathfrak{I}(R)$.

Solution:

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17. Let S be a relation defined on $\mathbb{R}^* \times \mathbb{R}^*$, such that:

$$(x, y), (a, b) \in \mathbb{R}^* \times \mathbb{R}^*, (x, y) S (a, b) \Leftrightarrow xy(a^2 + b^2) = ab(x^2 + y^2)$$

$$\Leftrightarrow \frac{xy}{x^2 + y^2} = \frac{ab}{a^2 + b^2}$$

(i) Show that S is an equivalence relation.

(ii) Find the *equivalence classes* $[(3,4)]$, $[(2,1)]$

Solution:

18. Let T be the relation defined on the integers set \mathbb{Z} , such that:

$$x, y \in \mathbb{Z}, \quad x T y \Leftrightarrow |x - 3| = |y - 3|$$

(i) Show that T is an equivalence relation.

(ii) Find $[0], [3], [-2]$

(ii) Find $\mathfrak{I}(T)$.

Solution:

19. Let \sim be a relation defined on $\mathbb{Z} \times \mathbb{Z}^+$, such that:

$$(m, n), (p, q) \in \mathbb{Z} \times \mathbb{Z}^+ \quad (m, n) \sim (p, q) \Leftrightarrow mq = pn \Leftrightarrow \frac{m}{n} = \frac{p}{q}$$

(i) *Show that \sim is an equivalence relation.*

(ii) *Find the equivalence classes $[(3,4)]$, $[(1,2)]$*

Solution:

20. Let T be the relation defined on the Rational set \mathbb{R} , such that:

$$x, y \in \mathbb{Q}, x S y \Leftrightarrow x - y \in \mathbb{Z}, \quad (x - y) \text{ is integer}$$

(i) Show that S is an equivalence relation.

(ii) Find $[0]$ and $[\frac{1}{2}]$.

Solution:

21. Let S be a relation defined on \mathbb{R} such that:

$$x, y \in \mathbb{R}, x S y \Leftrightarrow x - y \in \mathbb{Q}, \quad (x - y) \text{ is rational}.$$

(i) *Show that S is an equivalence relation.*

(ii) *Find [0]*

Solution:

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22. Let T be a relation defined on $\mathbb{R} \times \mathbb{R}$, such that:

$$(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}, (a, b) T (c, d) \Leftrightarrow b - a^2 = d - c^2$$

(i) Show that T is an equivalence relation.

(ii) Find the equivalence classes $[(0,0)]$, $[(1,2)]$

Solution: (i)

1- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}, b - a^2 = b - a^2 \Rightarrow (a, b) T (a, b) \Rightarrow \therefore T$ is reflexive

2- $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}, (a, b) T (c, d) \Leftrightarrow b - a^2 = d - c^2$
 $\Rightarrow d - c^2 = b - a^2$
 $\Rightarrow (c, d) T (a, b) \Rightarrow \therefore T$ is symmetric

3-

$$(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}, (a, b) T (c, d) \Leftrightarrow b - a^2 = d - c^2$$

&

$$(c, d) T (e, f) \Leftrightarrow d - c^2 = f - e^2$$

$$b - a^2 = d - c^2 = f - e^2 \Rightarrow b - a^2 = f - e^2 \Rightarrow (a, b) T (e, f) \Rightarrow \therefore T$$
 is transitive

$\therefore T$ is reflexive, symmetric and transitive $\Rightarrow \therefore T$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

(ii)

$$\begin{aligned} [(0,0)] &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x, y) T (0,0)\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y - x^2 = 0 - 0^2 = 0\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \end{aligned}$$

$$\begin{aligned} [(1,2)] &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x, y) T (1,2)\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y - x^2 = 2 - 1^2 = 1\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2 + 1\} \end{aligned}$$

23. Let R be the relation defined on the integers set \mathbb{Z} , such that:

$$x, y \in \mathbb{Z}, \quad x R y \Leftrightarrow 4 \mid (3x + y), \quad 4 \text{ divides } (3x + y)$$

- (i) Show that R is an equivalence relation.
- (ii) Find $[0], [1]$.

Solution :

24. Let R be the relation defined on the integers set \mathbb{Z} , such that:

$$a, b \in \mathbb{Z}, a R b \Leftrightarrow a \equiv 4b \pmod{3} \Leftrightarrow 3|(a - 4b), \quad 3 \text{ divides } (a - 4b)$$

- (i) Show that R is an equivalence relation.
- (ii) Find the *equivalence class* $[0]$.
- (iii) Decide whether $2 \in [5]$?

25. Let S be a relation defined on $A \times A$, where $A = \{1, 2, 3, 4, 5\}$ such that:

$$(a, b), (c, d) \in A \times A, (a, b) S (c, d) \Leftrightarrow a + b = c + d$$

(i) Show that S is an equivalence relation.

(ii) Find the equivalence classes $[(3,3)]$, $[(5,5)]$, $[(2,3)]$

Solution: (i)

1- $\forall (a, b) \in A \times A, a + b = a + b \Rightarrow (a, b) S (a, b) \Rightarrow \therefore S$ is reflexive

2- $(a, b), (c, d) \in A \times A, (a, b) S (c, d) \Leftrightarrow a + b = c + d$
 $\Rightarrow c + d = a + b \Rightarrow (c, d) S (a, b) \Rightarrow \therefore S$ is symmetric

3- $(a, b), (c, d), (e, f) \in A \times A, (a, b) S (c, d) \Leftrightarrow a + b = c + d$
 $\&$
 $(c, d) S (e, f) \Leftrightarrow c + d = e + f$

$a + b = c + d = e + f \Rightarrow a + b = e + f \Rightarrow (a, b) S (e, f) \Rightarrow \therefore S$ is transitive

$\therefore S$ is reflexive, symmetric and transitive $\Rightarrow \therefore S$ is an equivalence relation on $A \times A$.

(ii)

$$[(3,3)] = \{(3,3), (1,5), (2,4), (5,1), (4,2)\}$$

$$[(5,5)] = \{(5,5)\}$$

$$[(2,3)] = \{(1,4), (2,3), (4,1), (3,2)\}$$

26. Let R be the relation defined on the integers set $\mathbb{N} = \{1, 2, 3, \dots\}$, such that:

$$a, b \in \mathbb{N}, a R b \Leftrightarrow ab = k^2 \quad : k \in \{1, 2, 3, \dots\}$$

(i) Show that R is an equivalence relation.

(ii) Find the *equivalence class* $[1]$.

Solution: (i)

1- $\forall a \in \mathbb{N}, aa = a^2 \Rightarrow \therefore a R a \Rightarrow \therefore R$ is *reflexive*

2- $a, b \in \mathbb{N}, a R b \Leftrightarrow ab = k^2 \quad : k \in \{1, 2, 3, \dots\}$
 $\Rightarrow ba = k^2 \Rightarrow b R a \Rightarrow \therefore R$ is *symmetric*

3- $a, b, c \in \mathbb{N}, a R b \Leftrightarrow ab = k_1^2 \quad : k_1 \in \{1, 2, 3, \dots\} \quad (1)$

&

$b R c \Leftrightarrow bc = k_2^2 \quad : k_2 \in \{1, 2, 3, \dots\} \quad (2)$

$(1) \times (2) \Rightarrow \underline{\hspace{2cm}}$

$$ab^2c = k_1^2 k_2^2$$

$$\Rightarrow ac = \frac{k_1^2 k_2^2}{b^2} = \left(\frac{k_1 k_2}{b}\right)^2 = k^2$$

$\therefore \frac{k_1 k_2}{b} = k \quad : (k \text{ is a positive integer, cause } b \text{ divides both } k_1 \text{ and } k_2)$

$\Rightarrow ac = k^2 \Rightarrow a R c \Rightarrow \therefore$ *متعدية* R

$\therefore R$ is *reflexive*, *symmetric* and *transitive* $\Rightarrow \therefore R$ is an equivalence relation on

(ii)

$$[1] = \{a \in \mathbb{N} : a R 1\}$$

$$= \{a \in \mathbb{N} : a(1) = a = k^2 : k \in \mathbb{N}\}$$

$$= \{1, 4, 9, 16, 25, \dots\}$$

- 27.** Let T be the equivalence relation defined on the set $A = \{1,2,3,4\}$, where $\{1,3\}, \{2\}, \{4\}$ are equivalence classes. Represent T in ordered pairs .

- 28.** Let T be the equivalence relation defined on the set $A = \{1,2,3,4,5,6,7,8\}$, where $\mathfrak{S}(T) = \{\{1\}, \{2,3\}, \{4,5,6\}, \{7,8\}\}$. Represent T in ordered pairs .