King Saud University

College of Sciences

Department of Mathematics

106 Math Exercises

(3.2)

FUNDAMENTAL THEOREM OF CALCULUS

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The Fundamental Theorem of Calaulus :

If $f(x)$ is continuous on $[a, b]$, then

(i)

$$g(x) = \int_{a}^{x} f(t)dt \quad , \quad then\ g'(x) = f(x)$$

(ii)

$$\int_{a}^{b} f(x)dx = F(b) - F(a)$$

Where F is any antiderivative of f , that is , a function such that $F' = f$

Properties of Definite Integral :

$$\int_{b}^{a} f(x)dx = -\int_{a}^{b} f(x)dx$$

$$\int_{a}^{a} f(x)dx = 0$$

$$\int_{a}^{b} c\ dx = c\,(b - a)$$

$$\int_{a}^{b} [f(x) \pm g(x)]dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad : \quad c = constant$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \quad : \quad \forall c \in [a,b]$$

If $f(x) \geq 0 \;\; for \; a \leq x \leq b \;, \; then$

$$\int_a^b f(x)dx \geq 0$$

If $f(x) \geq g(x) \; for \; a \leq x \leq b \;, then$

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

If $m \leq f(x) \leq M \; for \; a \leq x \leq b \;, then$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$\frac{d}{dx}\int_a^b f(x)dx = 0$$

$$\int_a^b \frac{d}{dx}f(x)dx = f(b) - f(a)$$

$$\frac{d}{dx}\int_{a}^{x} f(t)dt \;=\; f(x)$$

$$\frac{d}{dx}\int_{a}^{v(x)} f(t)dt \;= f(v(x)\,)\,v'(x)$$

$$\frac{d}{dx}\int_{v(x)}^{a} f(t)dt \;= -\,f(v(x)\,)\,v'(x)$$

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt \;=\; f[v(x)\,]\,v'(x) - \; f[u(x)\,]\,u'(x)$$

$$\frac{d}{dx}\int f(x)dx = \; f(x)$$

$$\int \frac{d}{dx} f(x)dx = \; f(x) + \; c$$

<u>Average Value :</u>

$f(x) is\ continuous\ on\ [a,b] \quad \Rightarrow$

$$f_{av} = \frac{1}{b-a}\int_{a}^{b} f(x)dx$$

Mean Value Theorem:

If $f(x)$ *is continuous on* $[a,b]$, *then* $\exists z \in (a,b)$:

$$\int_a^b f(x)\ dx = (b-a)f(z)$$

$$\sqrt{x^2} \ = \ |x|$$

$$|x| = \begin{cases} x : x \geq 0 \\ -x : x < 0 \end{cases} \quad , \quad |x-3| = \begin{cases} x-3 : x \geq 3 \\ -(x-3) : x < 3 \end{cases}$$

$$-1 \leq \sin x \ \leq 1 \quad , \quad -1 \leq \cos x \ \leq 1$$

$$0 \leq \ sin^2 x \ \leq 1 \quad , \quad 0 \leq \ cos^2 x \ \leq 1$$

$$0 \ \leq |\sin x| \ \leq 1 \quad , \quad 0 \ \leq |\cos x| \ \leq 1$$

Exercieses

1) If $f(x) = \int \tan x \, dx$, find the value of

$1 + (\ tanx\ )f'(x) =$

2) Find $f(x) \quad if$

$$\int_0^{x^2} f(\sqrt{t})\ dt\ = x\ \ for\ x > 0$$

3) Find the following derivatives without evaluating the integrals :

i) $\frac{d}{dx}\left[\int_1^{x^2} \sin(2t)\ dt\right]$

ii) $\frac{d}{dx}\left[x\int_2^{x^3+x}\cos(\sqrt{t})\ dt\right]$

iii) Find $G'(x)$ if $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2}\ dt$

iv) Find $F'(\pi\ )\ if \quad F(x) = \int_1^x \ t\ \cos t \ \ dt$

v) $\frac{d}{dx}\left[\int_{\frac{1}{x}}^{x^2} \sin(2x) \ \ \frac{t}{\sqrt{t^2+3}} \ dt\right]$

vi) $\frac{d}{dx}\left[\int_{\sqrt{x}}^{x^4} \tan(x) \ \sqrt{t^2+2} \ dt\right]$

vii) If $F(x) = \int_{4x}^{x^3} f'(t)\,dt$, find $F'(x) = ?$

viii) If $F(x) = \int_{1}^{x^2} x^3\ \sqrt{5+t^2}\,dt$, find $F^{'}(1)$.

ix) If $G(x) = \int_{2}^{x} \sqrt{2t+5}\ dt$, $then\ find$ $G''(2)$

x)

$$\frac{d}{dx} \int_{-x}^{3} f(t)\ dt\ =$$

xi)

$$\frac{d}{dx} \int_{2x-1}^{3} \sqrt{5t+4}\ dt =$$

xii)

$$\frac{d}{dx} \int_{0}^{3} \sqrt{x^2+1}\ dx\ =$$

xiii)

$$\frac{d}{dx} \int \sqrt{x^2+1}\ dx\ =$$

xiv)

$$\int \frac{d}{dx} \sqrt{x^2 + 1} \; dx \; =$$

xv)

$$\int_0^3 \frac{d}{dx} \; \sqrt{x^2 + 1} \; dx \; =$$

xvi)

$$\int_{-x}^{x} \frac{d}{dt} [f(t)] dt =$$

xvii)

$$\frac{d}{dx} \int_3^x (2 + \frac{d}{dt} \cos t \;) \; dt =$$

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4) Find the average value of the function on the indicated interval :

i) $f(x) = \sqrt{x + 1} \qquad on \quad [0,3]$

ii) $f(x) = x^2 - 2x \qquad on \quad [1,4]$

iii) $f(x) = (2x + 1)^2 \qquad on \quad [0,2]$

iv) $f(x) = \sin x \cos x \qquad on \quad \left[0, \frac{\pi}{4}\right]$

v) $f(x) = \sqrt[3]{x+1}$ $\quad$ *on* $[-2,0]$

vi) $f(x) = \frac{x}{\sqrt{x^2+9}}$ $\quad$ *on* $[0,4]$

v) $f(x) = \frac{1}{\sqrt{x+2}}$ $\quad$ *on* $[-1,2]$

5) Find the number z that satisfies the conclusion of the Mean Value Theorem for :

i) $f(x) = x^2 + 2x \qquad on \ [1,4]$

ii) $f(x) = x^2 + 1 \qquad on \ [-2,1]$

iii) $f(x) = 3 + x^2 \qquad on\ [0,1]$

iv) $f(x) = \cos x \qquad on\ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

v) $f(x) = \frac{8}{x^3}$ $\quad on\ [-2, -1]$

6) Verify the inequalities without evaluating the integrals :

i)

$$\int_0^{\pi} (1 + sin^2 x)\, dx \leq \int_0^{\pi} 2\, dx$$

ii)

$$\int_{0}^{\frac{\pi}{4}} \cos x \, dx \ \geq \ \int_{0}^{\frac{\pi}{4}} sin\ x \, dx$$

iii)

$$\int_{1}^{2} (\ 3x^2 + 4\ )\ dx \ \ \geq \ \ \int_{1}^{2} (\ 2x^2 + 5\ )\ dx$$

iv)

$$\int_{-1}^{1} \frac{x^2}{x^2 + 4}\, dx \le \int_{-1}^{1} x^2 \, dx$$

v)

$$\int_{2}^{4} (x^2 - 6x + 8)\, dx \le 0$$

vi)

$$\int_{2}^{4} (5x^2 - x + 1)\ dx \geq 0$$

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7) Evaluate the following integrals :

i)

$$\int_{-3}^{6} |x - 4| d$$

ii)

$$\int_0^4 x\sqrt{x^2+9}\,dx$$

ii)

$$\int_4^9 \frac{t-3}{\sqrt{t}}\,dt$$

iii)

$$\int_{1}^{0} x^2\left(\sqrt[3]{x} - \sqrt{x}\right)dx$$

iv)

$$\int_{1}^{4} \left(2 + 3\sqrt{x}\right)dx$$

v)

$$\int_{0}^{1} \frac{1}{\sqrt{t^2 + t}}(2t + 1)dt$$

vi)

$$\int_{-1}^{1} 2|x|^3 \ dx$$

vii)

$$\int_0^4 \frac{\sqrt{x}}{(x^{3/2}+1)^3}\,dx$$

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8) Find the definite integral representing $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{3+x}\,\Delta x$ using regular partition of the interval [1,2]

9)

$$\int_3^7 (x^2+1)dx \; - \int_3^5 (x^2+1)dx \; =$$