DATA PROC AND REP

HW 4

IDO AMIT 315804831
IDO.AMIT@CAMPUS.TECHNION.AC.IL

HAY ELMALIAH 315777433
HAY.E@CAMPUS.TECHNION.AC.IL

1.a.

Given the following property of the operator:

$$\varphi_{data,j} = -\frac{1}{12}\varphi_{j-2[M]} + \frac{4}{3}\varphi_{j-1[M]} - \frac{5}{2}\varphi_{j[M]} + \frac{4}{3}\varphi_{j+1[M]} - \frac{1}{12}\varphi_{j+2[M]}$$

We look at the first few entries in the output:

$$\varphi_{data,0} = -\frac{1}{12}\varphi_{M-2} + \frac{4}{3}\varphi_{M-1} - \frac{5}{2}\varphi_0 + \frac{4}{3}\varphi_1 - \frac{1}{12}\varphi_2$$

$$= \begin{bmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \vdots \\ \vdots \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix}$$

Therefore $\begin{bmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} \end{bmatrix}$ would be the first row of H.

Similarly, the second row of H would be: $\begin{bmatrix} \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} \end{bmatrix}$

And so on.

We can see that each row is a one-entry shift of the previous one, thus we get:

$$H = \begin{bmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} \\ \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 \\ 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \\ -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} \end{bmatrix}$$

And we notice that it is a circulant matrix, which may be helpful ahead.

1.b.

In HW3 Q1.e. we calculated the decomposition of a general circulant matrix H:

$$\Lambda_H = DFT \cdot H \cdot DFT^*$$

As we mentioned before, our H is indeed circulant, then the equation above holds, then:

$$H = DFT^* \cdot \Lambda_H \cdot DFT$$

Where Λ_H is a diagonal matrix with the eigen values of H: $\begin{bmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{M-1} \end{bmatrix}$

Also, In HW3 Q1.g. we saw the following property:

$$\lambda_k(H) = \sum_{i=0}^{M-1} h_i W^{-ik}$$

- h_i – the i_{th} index the <u>first</u> column of H

Then:

$$\Lambda_H = \begin{bmatrix} \sum_{i=0}^{M-1} h_i W^{-i} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{i=0}^{M-1} h_i W^{-i(M-1)} \end{bmatrix}$$

The pseudo-inverse filter $H^\dagger$ is defined:

$$H^\dagger = DFT \cdot \begin{bmatrix} \sigma_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{M-1} \end{bmatrix} \cdot DFT^*$$

- $\sigma_j = \begin{cases} \frac{1}{\sum_{i=0}^{M-1} h_i W^{-ij}}, & \lambda_j \neq 0 \\ 0, & \lambda_j = 0 \end{cases}$

Now that we now the formula required in order to calculate $H^\dagger$, we are ready:

- $h = \begin{bmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} \end{bmatrix}$ (first column of H given as row)
- $W^{-kl} = e^{\frac{-2\pi ikl}{M}} \underbrace{=}_{\substack{euler's \\ formula}} \cos\left(\frac{-2\pi kl}{M}\right) + isin\left(\frac{-2\pi kl}{M}\right) \underbrace{=}_{\substack{\cos(x)=\cos(-x) \\ -\sin(x)=\sin(-x)}} \cos\left(\frac{2\pi kl}{M}\right) - isin\left(\frac{2\pi kl}{M}\right)$

$$\Rightarrow \lambda_k = \sum_{l=0}^{M-1} h_l W^{-kl} = -\frac{5}{2} W^{-k\cdot 0} + \frac{4}{3} W^{-k\cdot 1} - \frac{1}{12} W^{-k\cdot 2} + \frac{4}{3} W^{-k\cdot(M-1)} - \frac{1}{12} W^{-k\cdot(M-2)}$$

$$= -\frac{5}{2}\left(\cos\left(\frac{2\pi k0}{M}\right) - isin\left(\frac{2\pi k0}{M}\right)\right) + \frac{4}{3}\left(\cos\left(\frac{2\pi k1}{M}\right) - isin\left(\frac{2\pi k1}{M}\right)\right)$$

$$-\frac{1}{12}\left(\cos\left(\frac{2\pi k2}{M}\right) - isin\left(\frac{2\pi k2}{M}\right)\right) + \frac{4}{3}\left(\cos\left(\frac{2\pi k(M-1)}{M}\right) - isin\left(\frac{2\pi k(M-1)}{M}\right)\right)$$

$$-\frac{1}{12}\left(\cos\left(\frac{2\pi k(M-2)}{M}\right) - isin\left(\frac{2\pi k(M-2)}{M}\right)\right)$$

$$= -\frac{5}{2} + \frac{4}{3}\left(\cos\left(\frac{2\pi k}{M}\right) - isin\left(\frac{2\pi k}{M}\right)\right)$$

$$-\frac{1}{12}\left(\cos\left(\frac{4\pi k}{M}\right) - isin\left(\frac{4\pi k}{M}\right)\right) + \frac{4}{3}\left(\cos\left(\frac{2\pi k(M-1)}{M}\right) - isin\left(\frac{2\pi k(M-1)}{M}\right)\right)$$

$$-\frac{1}{12}\left(\cos\left(\frac{2\pi k(M-2)}{M}\right) - isin\left(\frac{2\pi k(M-2)}{M}\right)\right)$$

we notice the following:

- $\sin\left(\frac{2\pi k(M-1)}{M}\right) = \sin\left(2\pi k - \frac{2\pi k}{M}\right) \underbrace{=}_{\substack{\sin is \\ periodic \\ in\ 2\pi k}} sin\left(-\frac{2\pi k}{M}\right) = -sin\left(\frac{2\pi k}{M}\right)$
- $\sin\left(\frac{2\pi k(M-2)}{M}\right) = \sin\left(2\pi k - \frac{4\pi k}{M}\right) \underbrace{=}_{\substack{\sin is \\ periodic \\ in\ 2\pi k}} sin\left(-\frac{4\pi k}{M}\right) = -sin\left(\frac{4\pi k}{M}\right)$
- $\cos\left(\frac{2\pi k(M-1)}{M}\right) = \cos\left(2\pi k - \frac{2\pi k}{M}\right) \underbrace{=}_{\substack{\cos is \\ periodic \\ in\ 2\pi k}} \cos\left(-\frac{2\pi k}{M}\right) = \cos\left(\frac{2\pi k}{M}\right)$
- $\cos\left(\frac{2\pi k(M-2)}{M}\right) = \cos\left(2\pi k - \frac{4\pi k}{M}\right) \underbrace{=}_{\substack{\cos is \\ periodic \\ in\ 2\pi k}} \cos\left(-\frac{4\pi k}{M}\right) = \cos\left(\frac{4\pi k}{M}\right)$

We get:

$$= -\frac{5}{2} + \frac{8}{3}\cos\left(\frac{2\pi k}{M}\right) - \frac{1}{6}\cos\left(\frac{4\pi k}{M}\right)$$

Also, since:

$$\cos\left(\frac{4\pi k}{M}\right) = \cos\left(2 \cdot \frac{2\pi k}{M}\right) = 2\cos^2\left(\frac{2\pi k}{M}\right) - 1$$

We get:

$$= -\frac{5}{2} + \frac{8}{3}\cos\left(\frac{2\pi k}{M}\right) - \frac{1}{6}\left(2\cos^2\left(\frac{2\pi k}{M}\right) - 1\right) = -\frac{14}{6} + \frac{8}{3}\cos\left(\frac{2\pi k}{M}\right) - \frac{1}{3}\cos^2\left(\frac{2\pi k}{M}\right)$$

To sum up, we calculated: $\lambda_k = -\frac{14}{6} + \frac{8}{3}\cos\left(\frac{2\pi k}{M}\right) - \frac{1}{3}\cos^2\left(\frac{2\pi k}{M}\right)$

Since we wish to calculate $\sigma_i{}'s$ and we saw that the values depend on equality of $\lambda_i{}'s$ to zero, we will check which of the eigen values λ_i equal to zero.

In order to do so, we will use the above equation's quadratic feature:

$$\lambda_k = -\frac{14}{6} + \frac{8}{3}\cos\left(\frac{2\pi k}{M}\right) - \frac{1}{3}\cos^2\left(\frac{2\pi k}{M}\right) = 0$$

$$\Rightarrow \cos\left(\frac{2\pi k}{M}\right)_{1,2} = \frac{-\frac{8}{3} \pm \sqrt{\left(\frac{8}{3}\right)^2 - 4\cdot\left(-\frac{14}{6}\right)\cdot(-\frac{1}{3})}}{2\cdot\left(-\frac{14}{6}\right)} \Rightarrow \cos\left(\frac{2\pi k}{M}\right)_1 = 1\,, \cos\left(\frac{2\pi k}{M}\right)_2 = 7$$

Since $0 \leq \cos(\alpha) \leq 1$ we get that $\cos\left(\frac{2\pi k}{M}\right)_2 = 7$ is the not a valid solution,

And that $\cos\left(\frac{2\pi k}{M}\right)_1 = 1$ is the only valid solution.

$$\cos\left(\frac{2\pi k}{M}\right) = 1 \Leftrightarrow \frac{2\pi k}{M} = 0 + 2\pi n \ (n \in \mathbb{N}) \Leftrightarrow k = nM \ (n \in \mathbb{N})$$

Since $0 \leq k \leq M-1$, we only get $\cos\left(\frac{2\pi k}{M}\right) = 1$ when $k = 0$.

Recall:

$$H^\dagger = DFT \cdot \begin{bmatrix} \sigma_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{M-1} \end{bmatrix} \cdot DFT^* \quad when \quad \sigma_j = \begin{cases} \frac{1}{\sum_{i=0}^{M-1} h_i W^{-ij}}, & \lambda_j \neq 0 \\ 0, & \lambda_j = 0 \end{cases}$$

$$\Rightarrow \sigma_0 = 0 \ (since\ we\ just\ showed\ that\ \lambda_0 = 0)$$

For $k \neq 0$: $\sigma_k = \frac{1}{-\frac{14}{6}+\frac{8}{3}\cos\left(\frac{2\pi k}{M}\right)-\frac{1}{3}\cos^2\left(\frac{2\pi k}{M}\right)}$

$$H^\dagger =$$

$$DFT^* \begin{bmatrix} \sigma_0 = 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_1 = \frac{1}{-\frac{14}{6}+\frac{8}{3}\cos\left(\frac{2\pi}{M}\right)-\frac{1}{3}\cos^2\left(\frac{2\pi}{M}\right)} & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 = \frac{1}{-\frac{14}{6}+\frac{8}{3}\cos\left(\frac{4\pi}{M}\right)-\frac{1}{3}\cos^2\left(\frac{4\pi}{M}\right)} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \sigma_{M-1} = \frac{1}{-\frac{14}{6}+\frac{8}{3}\cos\left(\frac{2\pi(M-1)}{M}\right)-\frac{1}{3}\cos^2\left(\frac{2\pi(M-1)}{M}\right)} \end{bmatrix} DFT$$

1.c.

In the previous question we saw that $\lambda_0(H) = 0$, which corresponds to the first column of DFT^* as it's eigen vector: $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

For a signal to be recoverable by the pseudo-inverse filter, all it's components must be in the directions of non-zero eigen-value's corresponding eigen-vectors.

Then the set of signals that are recoverable by the pseudo-inverse filter is:

$$\left\{\varphi \middle| \left\langle \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \varphi \right\rangle = 0 \right\} = \{\varphi | \Sigma_{i=0}^{M-1} \varphi_i = 0\}$$

2.a.

$$\varphi = \begin{bmatrix} M \\ \vdots \\ M \\ M+L \\ M \\ \vdots \\ M \end{bmatrix}$$

Define: φ_i – the i_{th} entry of vector φ

We will prove that the mean of φ is zero:

$$E(\varphi_i) \underbrace{=}_{\substack{law\ of \\ total \\ expectation}} E\big(E(\varphi_i|K)\big) \underbrace{=}_{\substack{expectation \\ definition}} \Sigma_{k=1}^{N} E(\varphi_i|K=k)p(K=k) \underbrace{=}_{\substack{K\sim U[1,N] \\ p(K=k)=\frac{1}{N}}}$$

$$\frac{1}{N}\Sigma_{k=1}^{N} E(\varphi_i|K=k) = \frac{1}{N}\big(\Sigma_{k\neq i} E(\varphi_i|K=k) + E(\varphi_i|K=i)\big)$$

$$\underbrace{=}_{\varphi_j=\begin{cases} M+L, j=k \\ M, else\end{cases}} \frac{1}{N}\big(\Sigma_{k\neq i} E(M) + E(M+L)\big) = \frac{N-1}{N}E(M) + \frac{1}{N}E(M+L)$$

$$\underbrace{=}_{\substack{expectation \\ linearity}} \frac{N-1}{N}E(M) + \frac{1}{N}E(M) + \frac{1}{N}E(L) = E(M) + \frac{1}{N}E(L)$$

- $E(L_1) = E(L_2) = 0$, either way $E(L) = 0$
- $E(M) = 0$

$$\Longrightarrow E(\varphi_i) = 0, \ \ \forall i \in [M]$$

$$\Longrightarrow E(\varphi) = \bar{0} \ \ \blacksquare$$

2.b.

We would like to calculate each of the entries $R_\varphi(i,j)$ of the autocorrelation matrix.

Separate to cases:

$i = j$:

$$R_\varphi(i,j) = R_\varphi(i,i) = E(\varphi_i \cdot \varphi_i) \underbrace{=}_{\substack{law\ of \\ total \\ expectation}} E\big(E(\varphi_i \cdot \varphi_i|K)\big) \underbrace{=}_{\substack{expectation \\ definition}}$$

$$\Sigma_{k=1}^{N} E(\varphi_i \cdot \varphi_i|K=k)p(K=k) =$$

$$= \Sigma_{k\neq i} E(\varphi_i \cdot \varphi_i|K=k)p(K=k) + E(\varphi_i \cdot \varphi_i|K=i)p(K=i) \underbrace{=}_{\substack{K\sim U[1,N] \\ p(K=k)=\frac{1}{N} \\ p(K=i)=\frac{1}{N}}}$$

$$\frac{1}{N}\Sigma_{k\neq i}E(\varphi_i\cdot\varphi_i|K=k)+\frac{1}{N}E(\varphi_i\cdot\varphi_i|K=i)\underbrace{=}_{\varphi_s=\begin{cases}M+L,s=k\\ M,else\end{cases}}$$

$$\frac{1}{N}\Sigma_{k\neq i}E(M^2)+\frac{1}{N}E((M+L)^2)=\frac{N-1}{N}E(M^2)+\frac{1}{N}E(M^2+2ML+L^2)\underbrace{=}_{\substack{expectation\\ linearity}}$$

$$E(M^2)+\frac{2}{N}E(ML)+\frac{1}{N}E(L^2)\underbrace{=}_{M,L\ independant}E(M^2)+\frac{2}{N}E(M)E(L)+\frac{1}{N}E(L^2)\underbrace{=}_{\substack{E(M^2)=c\\ E(M)=0}}$$

$$=c+\frac{1}{N}E(L^2)$$

- If $i\leq\frac{N}{2}$: $c+\frac{1}{N}E(L_1^2)=c+\frac{1}{N}Na=c+a$
- If $i\geq\frac{N}{2}$: $c+\frac{1}{N}E(L_2^2)=c+\frac{1}{N}Nb=c+b$

$i\neq j$:

$$R_\varphi(i,j)=E(\varphi_i\cdot\varphi_j)\underbrace{=}_{\substack{law\ of\\ total\\ expectation}}E\left(E(\varphi_i\cdot\varphi_j|K)\right)\underbrace{=}_{\substack{expectation\\ definition}}$$

$$\Sigma_{k=1}^{N}E(\varphi_i\cdot\varphi_j|K=k)p(K=k)=$$

$$\Sigma_{k\neq i,j}E(\varphi_i\cdot\varphi_i|K=k)p(K=k)+E(\varphi_i\cdot\varphi_i|K=i)p(K=i)$$

$$+E(\varphi_i\cdot\varphi_i|K=j)p(K=j)\underbrace{=}_{K\sim U[1,N]}$$

$$\frac{1}{N}\Sigma_{k\neq i,j}E(\varphi_i\cdot\varphi_i|K=k)+\frac{1}{N}E(\varphi_i\cdot\varphi_i|K=i)+\frac{1}{N}E(\varphi_i\cdot\varphi_i|K=j)=$$

$$\frac{N-2}{N}E(M^2)+\frac{1}{N}E\big(M(M+L)\big)+\frac{1}{N}E\big(M(M+L)\big)=$$

$$\frac{N-2}{N}E(M^2)+\frac{2}{N}E(M^2+ML)\underbrace{=}_{\substack{expectation\\ linearity}}$$

$$\frac{N-2}{N}E(M^2)+\frac{2}{N}E(M^2)+\frac{2}{N}E(ML)\underbrace{=}_{M,L\ independant}$$

$$E(M^2)+\frac{2}{N}E(M)E(L)\underbrace{=}_{E(M)=0}E(M^2)=c$$

In conclusion, we got:

$$R_\varphi = \begin{bmatrix} c+a & c & c & c & c & c \\ c & \ddots & c & c & c & c \\ c & c & c+a & c & c & c \\ c & c & c & c+b & c & c \\ c & c & c & c & \ddots & c \\ c & c & c & c & c & c+b \end{bmatrix}$$

- The first $\frac{N}{2}$ diagonal elements are $c + a$ the other $\frac{N}{2}$ are $c + b$

2.c.

The PCA matrix that corresponds to R_φ is the matrix of it's eigenvectors.

If the eigenvectors are DFT^*'s columns (so R_φ is diagonalized by the DFT^* matrix), then R_φ is circulant.

If we demand that R_φ is circulant, we will get that DFT^* matrix is the PCA matrix corresponding to the autocorrelation matrix R_φ.

In order for R_φ to be circulant, we demand that $a = b$, and that eigenvalues are in descending order (valid PCA).

We will assume that $a = b$, and validate the required descending order:

As we saw in HW3:

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \vdots \\ \lambda_{n-1} \end{bmatrix} = \sqrt{N} DFT^* \underbrace{\begin{bmatrix} r_0 \\ r_{n-1} \\ \vdots \\ \vdots \\ r_1 \end{bmatrix}}_{\substack{first\ row\ of \\ circulant\ R_\varphi}} = \sqrt{N} DFT^* \begin{bmatrix} a+c \\ c \\ \vdots \\ \vdots \\ c \end{bmatrix}$$

We will calculate general k_{th} eigen value:

$$\lambda_k = W^{-k0}(a+c) + c\left(\Sigma_{l=1}^{N-1} W^{-kl}\right) = W^0(a+c) + c\left(\Sigma_{l=1}^{N-1} W^{-kl}\right) =$$

$$(a+c) + c\left(\Sigma_{l=0}^{N-1} W^{-kl} - W^0\right) = (a+c) + c\left(\Sigma_{l=0}^{N-1} W^{-kl} - 1\right) =$$

- $k = 0$:

$$(a+c) + c\left(\Sigma_{l=0}^{N-1} 1 - 1\right) = (a+c) + c(N-1) = Nc + a$$

- $k \neq 0$:

$$(a+c) + c\left(W^k \underbrace{\Sigma_{l=0}^{N-1} W^l}_{=0} - 1\right) = a + c - c = a$$

Since $Nc > 0$ $\left(c \in (0,1)\right)$, we get that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$, as required.

$\Longrightarrow$ demanding $a = b$ is enough in order to guarantee that DFT^* matrix is the PCA matrix corresponding to the autocorrelation matrix R_φ.

Q.3.a.

We would like to calculate each of the entries $R_\varphi(i,j)$ of the autocorrelation matrix.

Separate to cases:

$i = j$:

$$\boldsymbol{E}(\boldsymbol{\varphi^i} \cdot \boldsymbol{\varphi^j}) = E\left(\left(\varphi^i\right)^2\right) =$$

$$P\left(i = k\left(mod\left(\frac{N}{2}\right)\right)\right)E((M+L)^2) + P\left(\neg\left(i = k\left(mod\left(\frac{N}{2}\right)\right)\right)\right)E(M^2) \underbrace{=}_{\substack{K\ is\ uniform\\+\\linearity\ of\\expectation}}$$

$$\frac{2}{N}\left(E(M^2) + 2E(ML) + E(L^2)\right) + \frac{N-2}{N}E(M^2) \underbrace{=}_{\substack{M\ and\ L\\independant}}$$

$$E(M^2) + \frac{4E(M)E(L)}{N} + \frac{2}{N}E(L^2) \underbrace{=}_{\substack{E(M^2)=c\\E(M)=0\\E(L^2)=\frac{N}{2}(1-c)}}$$

$$c + 0 + \frac{2}{N}\cdot\frac{N}{2}(1-c) = \mathbf{1}$$

$i \neq j$:

In that case, if $|i-j| = \frac{N}{2}$ then there is a probability that $\varphi^i = \varphi^j = M + L$

So we will separate to sub-cases:

- $|i-j| = \frac{N}{2}$:
  Given the index i, we know the values of both φ^i, φ^j which equal (together) M or $M+L$.

$$\boldsymbol{E}(\boldsymbol{\varphi^i} \cdot \boldsymbol{\varphi^j}) =$$

$$P\left(i = k\left(mod\left(\frac{N}{2}\right)\right)\right)E((M+L)^2) + P\left(\neg\left(i = k\left(mod\left(\frac{N}{2}\right)\right)\right)\right)E(M^2) =$$

$$\underbrace{=}_{\substack{as\ we\ have\\seen\ before}} \mathbf{1}$$

- $|i-j| \neq \frac{N}{2}$:
  In that case, only one of φ^i, φ^j could be $M+L$ simultaneously (or neither of them is).

$$\boldsymbol{E\left(\varphi^{i}\cdot\varphi^{j}\right)} =$$

$$P\left(i=k\left(mod\left(\frac{N}{2}\right)\right)\right)E\big(M(M+L)\big)+P\left(J=k\left(mod\left(\frac{N}{2}\right)\right)\right)E\big(M(M+L)\big)$$

$$+P\left(i\neq k\left(mod\left(\frac{N}{2}\right)\right)\ and\ j\neq k\left(mod\left(\frac{N}{2}\right)\right)\right)E(M^2)=$$

$$\underbrace{=}_{\substack{K\ is\\ uniform\\ +\\ linearity\ of\\ expectation}}\ \frac{4}{N}\big(E(M^2)+E(ML)\big)+\frac{N-4}{N}E(M^2)\ \underbrace{=}_{\substack{M\ and\ L\\ independant}}\ E(M^2)+\frac{4}{N}E(M)E(L)$$

$$\underbrace{=}_{\substack{E(M^2)=c\\ E(M)=0}}\ \boldsymbol{c}$$

Overall we got:

$$\boldsymbol{R_{\varphi}(i,j)}=\begin{cases}\boldsymbol{1}, & \boldsymbol{i-j\left(mod\left(\frac{N}{2}\right)\right)=0}\\ \boldsymbol{c}, & \boldsymbol{otherwise}\end{cases}$$

The autocorrelation matrix is indeed **circulant**, since as calculated above – the main diagonal $(where\ i=j)$ is all 1, as well as diagonal that satisfies $i-j\left(mod\left(\frac{N}{2}\right)\right)=0$, and all the other diagonals' values are c.

For example (N=4):

$$\begin{bmatrix}1 & c & 1 & c\\ c & 1 & c & 1\\ 1 & c & 1 & c\\ c & 1 & c & 1\end{bmatrix}$$

Q.3.b.

As we mentioned in previous HW, with the fact that R_{φ} is circulant, we can calculate the eigenvalues using:

$$\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_N\end{bmatrix}=\sqrt{N}\cdot DFT^{*}\cdot\big((first\ row\ of\ R_{\varphi}\big)^{T}$$

Where $\lambda_i{}'s$ are the eigenvalues of R_{φ}.

Q.4.a.

we denote φ^* as φ^{out} (so we can use $*$ for conjugate sign as taught in class).

$$\varphi^{out} = \mathcal{H}\varphi + n$$

Also, define $\mathcal{H}_i$ $-$ the i_{th} row of $\mathcal{H}$.

We will calculate the autocorrelation matrix of φ^{out} $\left(R_{\varphi^{out}}\right)$:

$$\boldsymbol{R_{\varphi^{out}}(i,j)} =$$

$$E\left((\varphi^{out})^i \cdot \overline{(\varphi^{out})^j}\right) =$$

$$E\left((\mathcal{H}\varphi + n)^i \cdot \left((\mathcal{H}\varphi + n)^j\right)^*\right) =$$

$$E\left(\left(\mathcal{H}_i\varphi + n^i\right) \cdot \left(\mathcal{H}_j\varphi + n^j\right)^*\right) =$$

$$E\left(\left(\mathcal{H}_i\varphi + n^i\right) \cdot \left((\mathcal{H}_j\varphi)^* + n^{j^*}\right)\right) =$$

$$E\left(\mathcal{H}_i\varphi \cdot \varphi^*\mathcal{H}_j^*\right) + E(\mathcal{H}_i\varphi)E\left(n^{j^*}\right) + E\left(\mathcal{H}_j^*\varphi^*\right)E\left(n^i\right) + E\left(n^i n^{j^*}\right) \underbrace{=}_{\substack{R_n(i,j)=E\left(n^i\overline{n^j}\right) \\ E(n)=0}}$$

$$E\left(\mathcal{H}_i\varphi \cdot \varphi^*\mathcal{H}_j^*\right) + R_n(i,j) =$$

$$\mathcal{H}_i E(\varphi\varphi^*)\mathcal{H}_j^* + R_n(i,j) =$$

$$\boldsymbol{\mathcal{H}_i R_\varphi \mathcal{H}_j^* + R_n(i,j)}$$

To sum it up, using the original notation:

$$\varphi^* = \mathcal{H}\varphi + n$$

We got:

$$\boldsymbol{R_{\varphi^*} = \mathcal{H}R_\varphi\mathcal{H}^* + R_n}$$

Q.4.b.

As we have seen in class, the Weiner Filter is given by:

$$W = R_\varphi\mathcal{H}^*\left(\mathcal{H}R_\varphi\mathcal{H}^* + \sigma_n^2 I\right)^{-1}$$

Since in our case: $\sigma_n^2 I = R_n$, we can rewrite Weiner Filter as follows:

$$W = R_\varphi\mathcal{H}^*\left(\mathcal{H}R_\varphi\mathcal{H}^* + R_n\right)^{-1} \underbrace{=}_{\substack{as\ seen \\ in\ the \\ previous \\ section}} R_\varphi\mathcal{H}^*{R_{\varphi^*}}^{-1}$$

Q.4.c.

Let A be a NxN matrix diagonalized by DFT^*_{NxN}, we will prove that A is circulant:

First, since A is diagonalized by DFT^* we get:

$$A = DFT^* \cdot \Lambda_A \cdot DFT$$

As mentioned before, the eigenvalues of any circulant matrix C can be calculated by:

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \sqrt{N} \cdot DFT^* \cdot (first\,row\,of\,C)^T$$

Then, given the eigenvalues of A (from Λ_A above) as: $\begin{bmatrix} \lambda_1(A) \\ \vdots \\ \lambda_N(aA) \end{bmatrix}$

We can calculate:

$$\begin{bmatrix} \lambda_1(A) \\ \vdots \\ \lambda_N(aA) \end{bmatrix} = \sqrt{N} \cdot DFT^* \cdot b^T$$

For some horizontal vector b.

Then,

$$b^T = \frac{1}{\sqrt{N}} \cdot DFT \cdot \begin{bmatrix} \lambda_1(A) \\ \vdots \\ \lambda_N(aA) \end{bmatrix}$$

Satisfying the property mentioned above, b is the first row of some circulant matrix B,

With eigenvalues equal to those of matrix A.

Since B is circulant, it could be diagonalized by DFT^*:

$$B = DFT^* \cdot \Lambda_B \cdot DFT \underbrace{=}_{\substack{explained \\ right\,above}} DFT^* \cdot \Lambda_A \cdot DFT = A$$

Since B is circulant, and $A = B$, we proved that A is circulant.

Q.4.d.

The Weiner Filter is not a shift-invariant system for any arbitrary $R_\varphi, \mathcal{H}\ and\ \sigma_n$:

For example, let $R_\varphi = I, \sigma_n = 1\ and\ \mathcal{H} = \begin{bmatrix} 3 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ a diagonal matrix with 3 in the first entry of the diagonal, and the rest is 1.

$$W \underbrace{=}_{definition} R_\varphi \mathcal{H}^* \left(\mathcal{H} R_\varphi \mathcal{H}^* + \sigma_n^2 I\right)^{-1} = \mathcal{H}^*(\mathcal{H}^2 + I)^{-1} = \begin{bmatrix} \frac{3}{10} & & & \\ & 0.5 & & \\ & & \ddots & \\ & & & 0.5 \end{bmatrix}$$

We got a diagonal matrix with different diagonal elements (not all elements are equal).

As we have seen in the tutorial, a shift-invariant system is expressed by a circulant matrix.

Since the matrix above is not circulant – it is not a shift-invariant system.

Q.4.e.

We will show the general condition for the Weiner Filter to be a shift-invariant system is that both R_φ and $\mathcal{H}$ are shift-invariant systems.

Let R_φ and $\mathcal{H}$ be shift-invariant systems. According to the theorem mentioned in the previous section, both are circulant matrices.

Recall, any circulant matrix A is diagonalized by the DFT^* matrix, s.t:

$$A = DFT^* \cdot \Lambda_A \cdot DFT$$

Therefore,

$$R_\varphi = DFT^* \cdot \Lambda_{R_\varphi} \cdot DFT$$

$$\mathcal{H} = DFT^* \cdot \Lambda_{\mathcal{H}} \cdot DFT$$

Let us also note that $\mathcal{H}^* = (DFT^* \cdot \Lambda_{\mathcal{H}} \cdot DFT)^* = DFT^* \cdot (\Lambda_{\mathcal{H}})^* \cdot DFT = DFT^* \cdot \Lambda_{\mathcal{H}^*} \cdot DFT$

$$W \underbrace{=}_{definition} R_\varphi \mathcal{H}^* \left(\mathcal{H} R_\varphi \mathcal{H}^* + \sigma_n^2 I\right)^{-1}$$

$$= DFT^* \Lambda_{R_\varphi} \underbrace{DFT \cdot DFT^*}_{I} \Lambda_{\mathcal{H}^*} DFT \left(DFT^* \Lambda_{\mathcal{H}} \underbrace{DFT \cdot DFT^*}_{I} \Lambda_{R_\varphi} \underbrace{DFT \cdot DFT^*}_{I} \Lambda_{\mathcal{H}} DFT + \sigma_n^2 I \right)^{-1}$$

$$= DFT^* \Lambda_{R_\varphi} \Lambda_{\mathcal{H}^*} DFT \left(DFT^* \Lambda_{\mathcal{H}} \Lambda_{R_\varphi} \Lambda_{\mathcal{H}} DFT + \sigma_n^2 I \right)^{-1}$$

- The product of two diagonal matrices $\Lambda_{R_\varphi} \Lambda_{\mathcal{H}^*}$ is a diagonal matrix, mark as: D_1
- Similarly, we ,mark $\Lambda_{\mathcal{H}} \Lambda_{R_\varphi} \Lambda_{\mathcal{H}}$ as: D_2

Then,

$$= DFT^* D_1 DFT (DFT^* D_2 DFT + \sigma_n^2 I)^{-1}$$

Both matrices $M_1 = DFT^* D_1 DFT$ and $M_2 = DFT^* D_2 DFT$ are diagonalized by DFT^*.

The matrix $\sigma_n^2 I$ (diagonal with all equal elements in the diagonal), is diagonalized by any matrix and in particular by DFT^* matrix.

Summation, Inverse, and multiplication of matrices that are all diagonalized by DFT^* matrix, yield a matrix diagonalized by DFT^* as well.

Therefore, according to the equation above – the Weiner Filter in our case (of R_φ and $\mathcal{H}$ being shift-invariant systems) is diagonalized by the DFT^* matrix.

In previous section we proved that any matrix diagonalized by DFT^*, is circulant.

Then in that case, the Weiner Filter is circulant.

From the theorem we mentioned above, since the Weiner Filter is circulant – it is a shift-invariant system.

To sum up, we showed that a general condition for the Weiner Filter to be a shift-invariant system is **$\boldsymbol{R_\varphi}$ and $\boldsymbol{\mathcal{H}}$ being shift-invariant systems**.

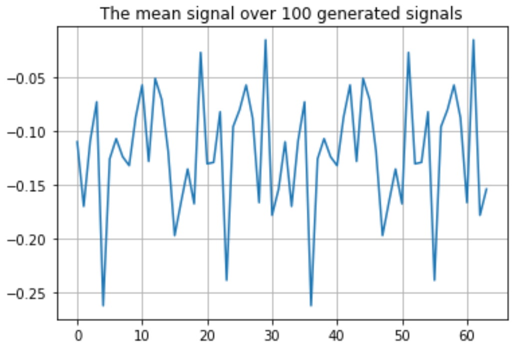
Implementation

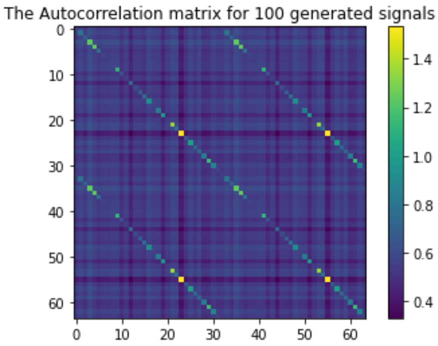
Q.a.

We wish to find the number of realizations needed for obtaining a good empirical approximation of the second-order statistics s.t the empirical results approximate the analytical results obtained in the third question in the Theory part.

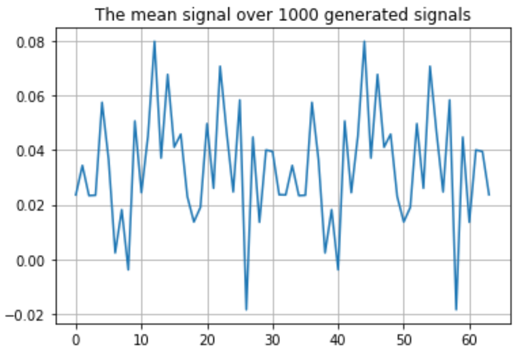
In order to do so, we tried generating signals in a few rows, each time different number of signals. We reached the following results:

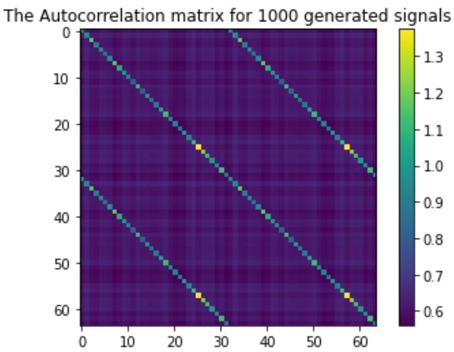
Generating 100 signals:



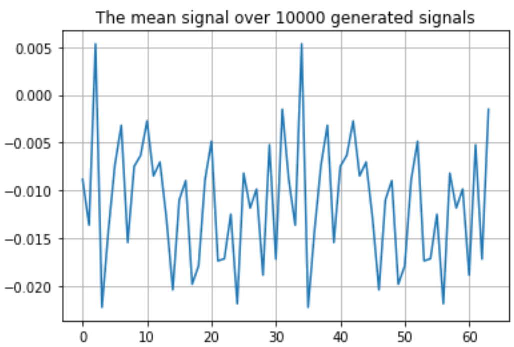


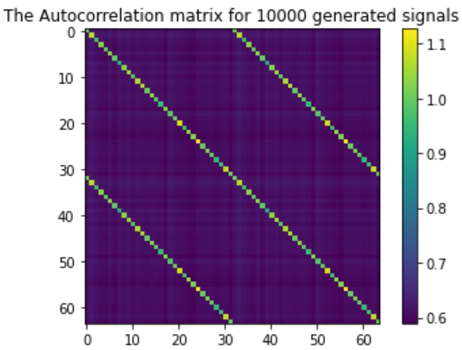
Generating 1,000 signals:





Generating 10,000 signals:





Given the plots above, we can see that for 1,000 signals and more we reach quite good approximation of the autocorrelation matrix.

Recall Q.3. autocorrelation matrix calculation:

$$R_\varphi(i,j) = \begin{cases} 1, & i - j \left(mod\left(\frac{N}{2}\right)\right) = 0 \\ c, & otherwise \end{cases}$$

Indeed, we can see a circulant pattern with values near 1, and the rest of the values are near c (=0.6).

Therefore, a good empirical approximation is obtained by sampling 1,000 signals and above.

Q.b.

The Weiner Filter is given by:

$$W = R_\varphi \mathcal{H}^* \left(\mathcal{H} R_\varphi \mathcal{H}^* + \sigma_n^2 I\right)^{-1}$$

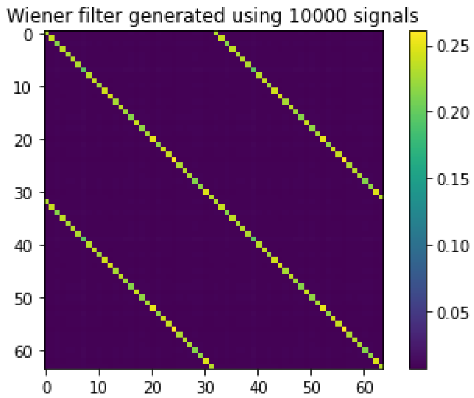
In our case, we don't use operator $\mathcal{H}$, and $\sigma_n^2 = 1$, then:

$$W = R_\varphi \left(R_\varphi + I\right)^{-1}$$

Where R_φ is the autocorrelation matrix of φ.

In Q.4.e. we claimed that a general condition for W to be circulant is that both R_φ and $\mathcal{H}$ are circulant. In our case, $\mathcal{H} = I$ and in Q.3.a. we showed that R_φ (under similar constraints) is circulant. Then indeed we expect our W to be circulant as well in the empirical case.

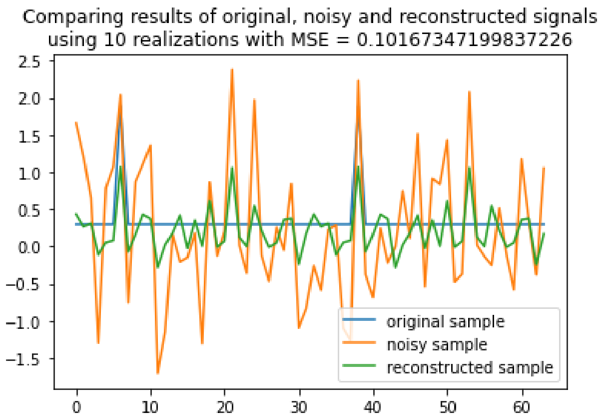
We generated the Wiener Filter matrix and got the result as follows:

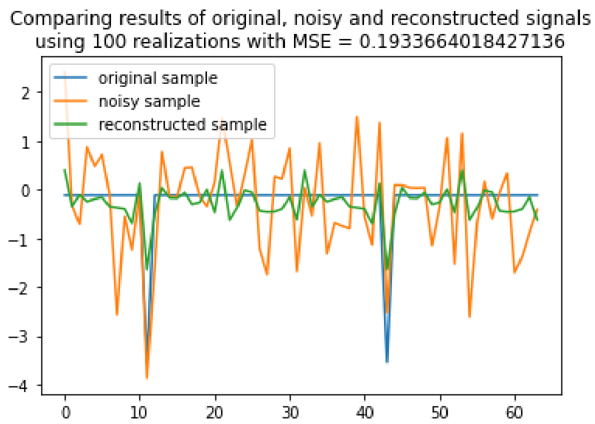


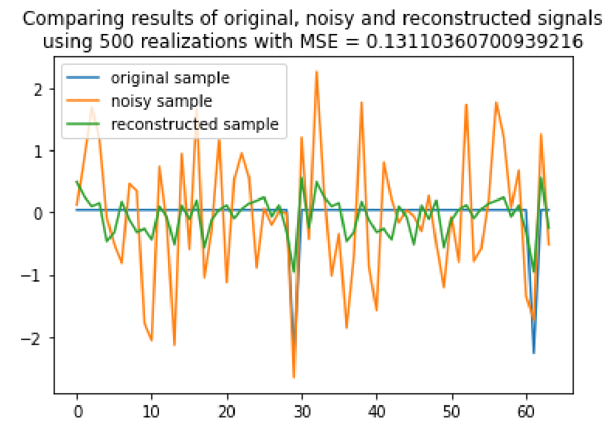
It is indeed (approximately) a circulant matrix, and is pretty similar to the autocorrelation matrix.

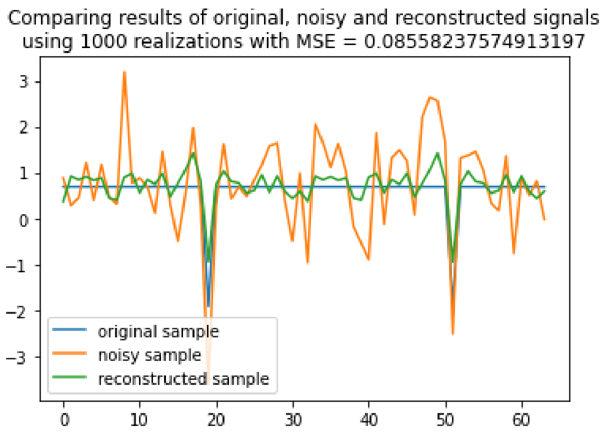
We wish to compare the results of using Wiener Filter for reconstructing signals, of different batches, with noise constructed as requested in the exercise.

Our results in the next page:







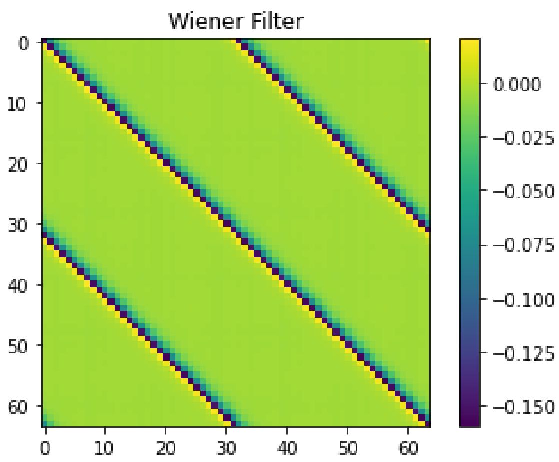


Finally, the average MSE for all signals is: 0.2314941228671783

Q.c.

We wish to repeat the previous section, now considering the operator $\mathcal{H}$ as well (as given in the question).

First, we plotted the Wiener Filter:

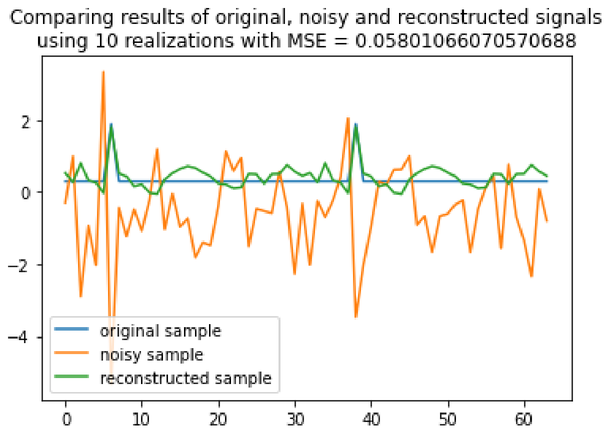


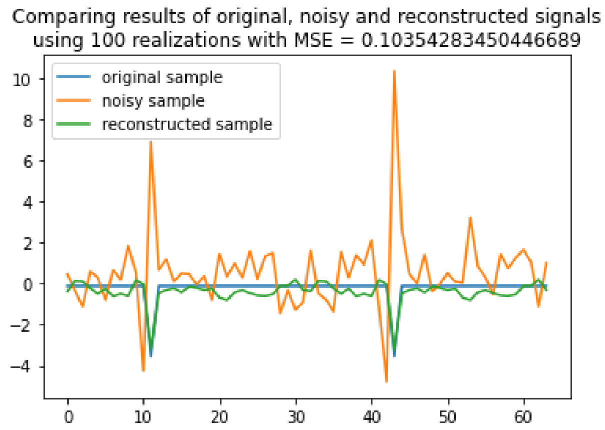
As we mentioned before, a general condition for W to be circulant is that both R_φ and $\mathcal{H}$ are circulant. We saw that R_φ is circulant and now having $\mathcal{H}$ as circulant matrix as well, leads to the Wiener Filter matrix to be circulant.

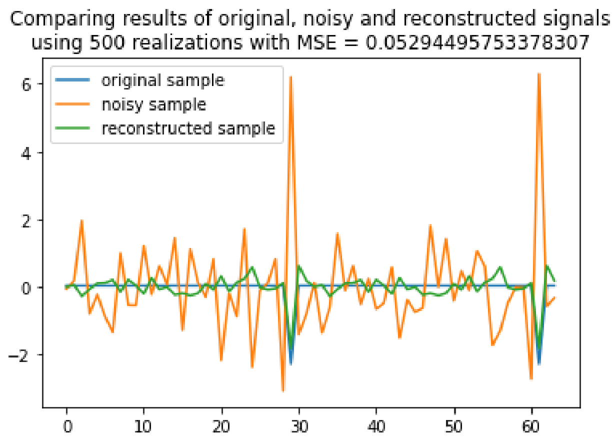
In the plot above, we indeed received a (approximately) circulant matrix.

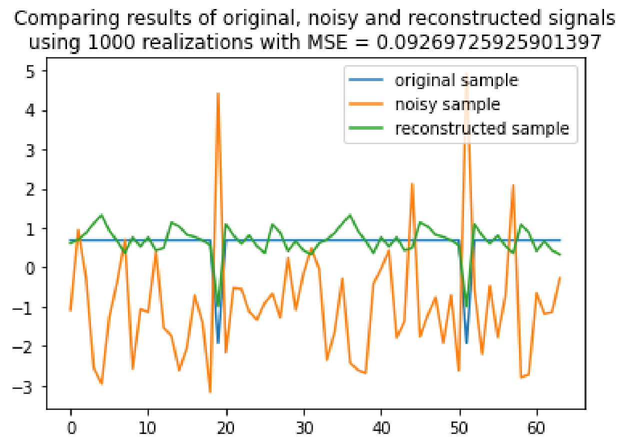
Again, We wish to compare the results of using Wiener Filter for reconstructing signals, of different batches, with noise constructed as requested in the exercise, now using the operator $\mathcal{H}$ in order to produce noise, as well.

The results as follows:









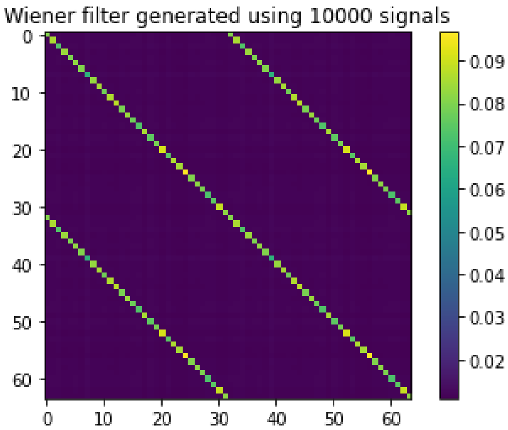
Finally, the average MSE for all signals is: 0.07755749243852317.

We can see that the average MSE is lower than that of the previous section.

Q.d.

We will repeat the previous sections with $\sigma_n^2 = 5$:

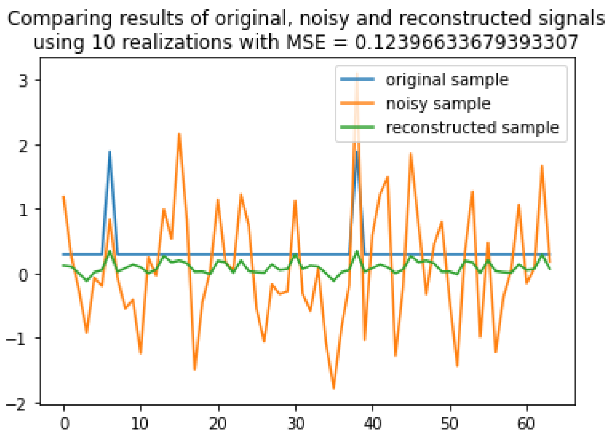
First, examine the case when $\mathcal{H} = I$:

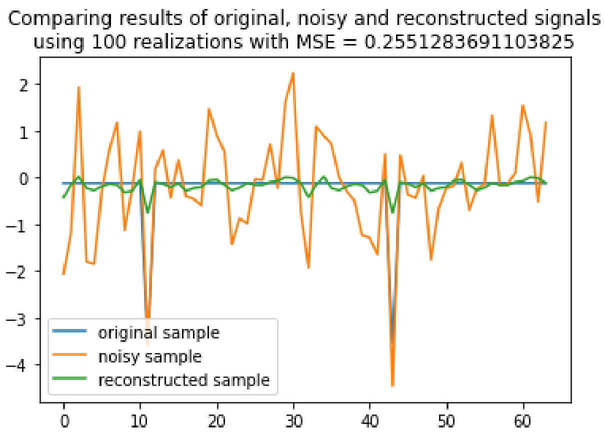


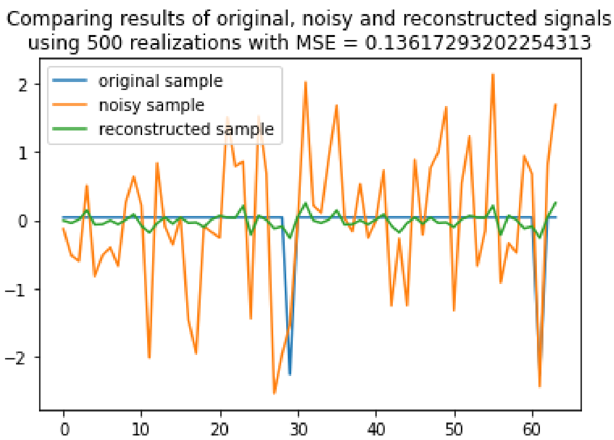
As we explained before, it seems to be circulant and satisfies the analytic expression we obtained.

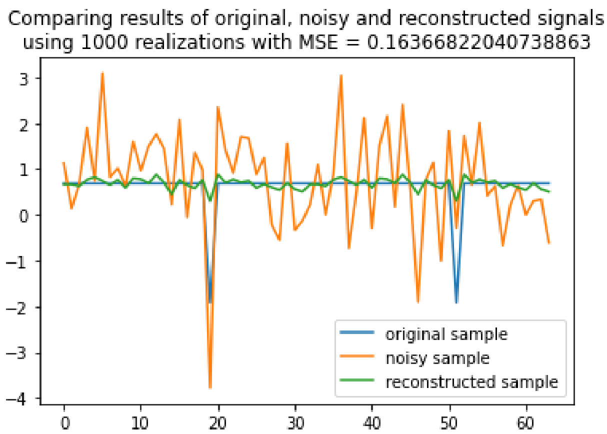
Again, comparing the results of using Wiener Filter for reconstructing signals, of different batches, with noise constructed as requested in the exercise.

The results as follows:





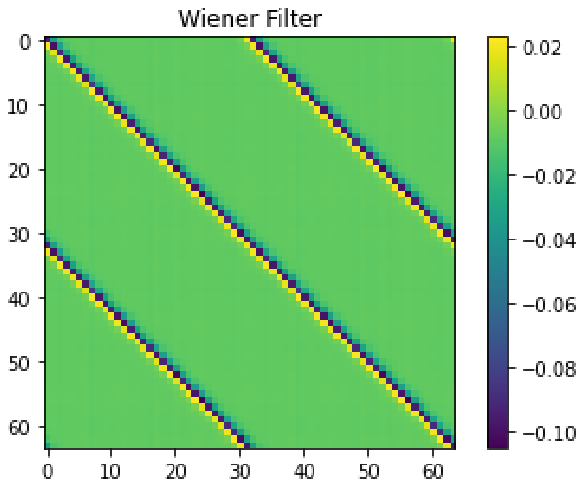




Finally, the average MSE for all signals is: 0.3197041069074448.

Now to the case using $\mathcal{H}$ as described in the exercise:

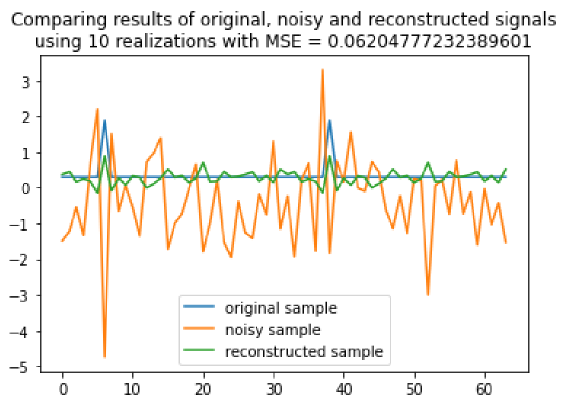
The Wiener Filter we plotted:

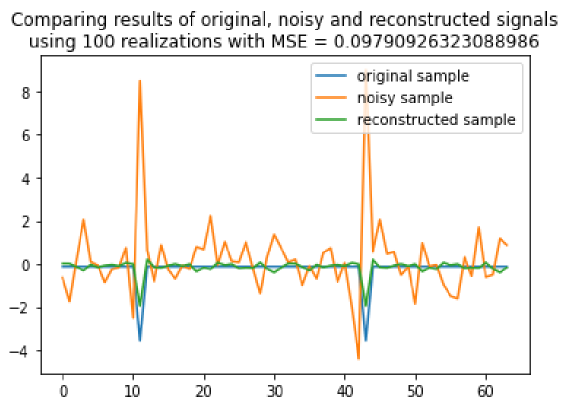


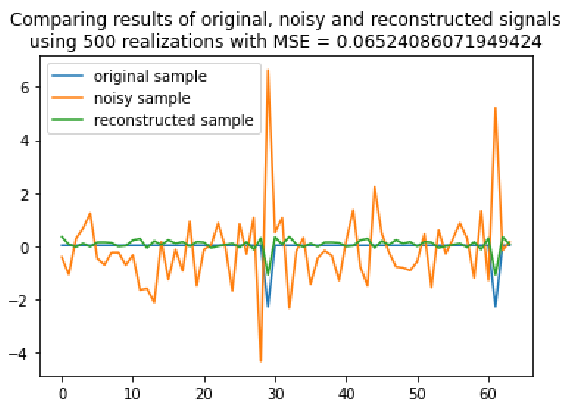
As we explained before, it seems to be circulant and satisfies the analytic expression we obtained.

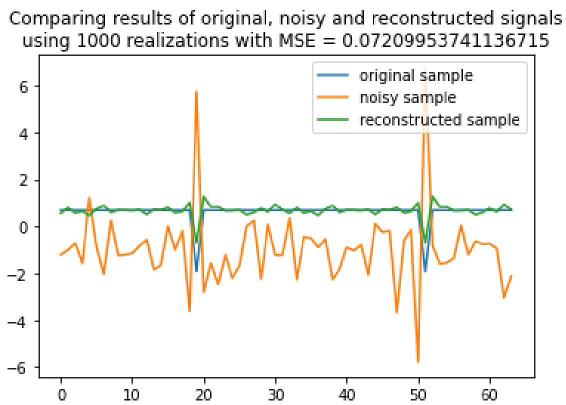
Again, comparing the results of using Wiener Filter for reconstructing signals, of different batches, with noise constructed as requested in the exercise, now using the operator $\mathcal{H}$ in order to produce noise, as well.

The results as follows:









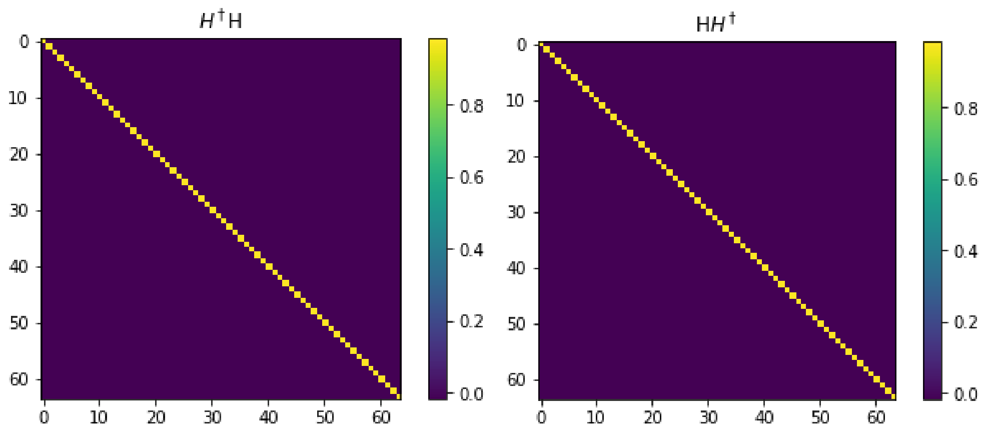
Finally, the average MSE for all signals is: 0.12699693431585643.

The difference in the obtained results is that now the noise variates more, which lead to a more noisy signal post degradation, as we can see in the plots above.
As result, the MSE of the reconstruction (comparing the original signal to the reconstructed signal) is bigger.

$\mathcal{H}$	σ_n^2	$reconstraction\ MSE$
I	1	0.2314941228671783
	5	0.07755749243852317
$given\ circulant\ matrix \neq I$	1	0.3197041069074448
	5	0.12699693431585643

Q.e.

First, we plot the matrices: $\mathcal{H}^{\dagger}\mathcal{H}$ and $\mathcal{H}\mathcal{H}^{\dagger}$:



The minimal value of the 'purple' area is 0.015624999999969894

The maximal value of the diagonal is 0.9843750000000288

This means that even though the matrices above look much identical to the unit matrix, they are in fact not equal.

We examined the eigenvalues of $\mathcal{H}$ and found out that one is very small (1.345e-15) which is practically zero, and the next smallest eigenvalue is a small but not zero (0.0096382).

Meaning there is a null space in both $\mathcal{H}$ and $\mathcal{H}^{\dagger}$.

We use this property of $\mathcal{H}^{\dagger}$ in order to find the signals ϕ_1 and $\phi 2$:

We note that $\lambda_N(\mathcal{H}^\dagger) = 0$ is corresponding to the eigenvector of $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Since $\mathcal{H}^\dagger$ is a circulant matrix and so diagonalized by DFT^*.

Therefore choosing $\phi_1 = \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ will result in $\mathcal{H}^\dagger \phi_1 \underbrace{=}_{\substack{eigenvalue \\ and\ co. \\ eigenvector}} \lambda_N \cdot \phi_1 = \bar{0}$.

By choosing $\phi_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ would lead to $\mathcal{H}^\dagger \phi_2 = \bar{0} = \mathcal{H}^\dagger \phi_1$.

This way, choosing α that will satisfy $\|\phi_1 - \phi_2\|_2 \geq 256$, we would still get:

$$\left\|\mathcal{H}^\dagger \phi_1 - \mathcal{H}^\dagger \phi_2\right\|_2 = 0$$

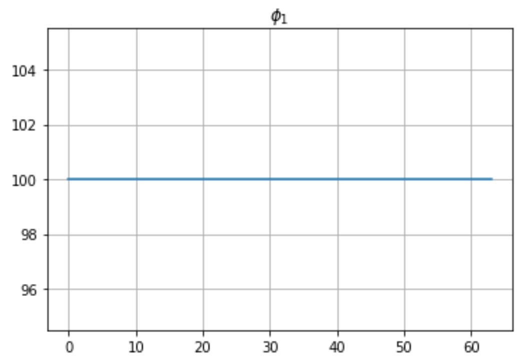
as requested.

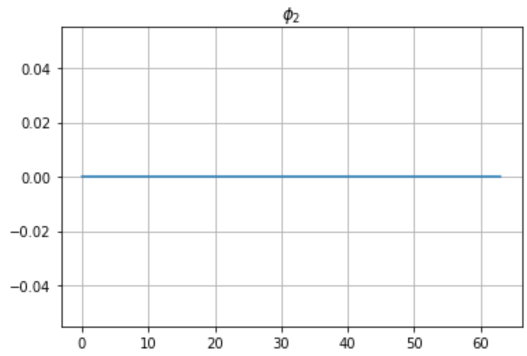
Finally, we choose $\phi_1 = 100 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $\phi_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

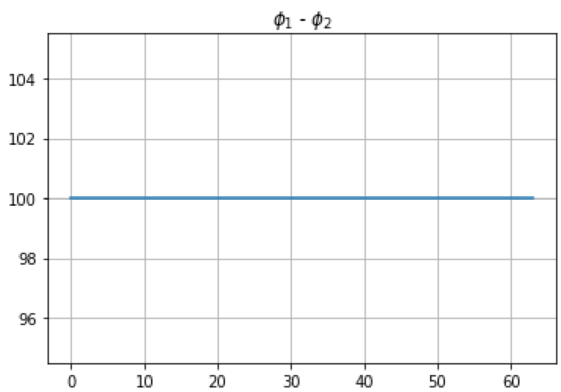
It satisfies the requirements: $\|\phi_1 - \phi_2\|_2 = 800 > 256$ and:

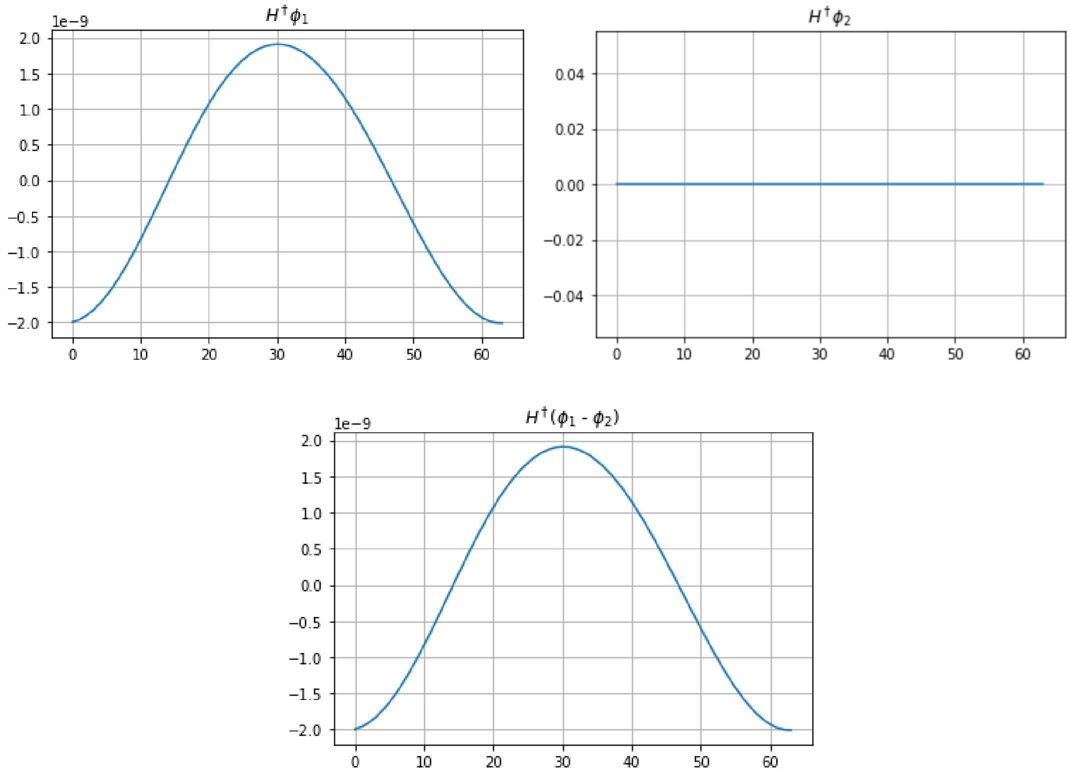
$$\left\|\mathcal{H}^\dagger \phi_1 - \mathcal{H}^\dagger \phi_2\right\|_2 = 1.108844\text{e-}08 \underbrace{\approx}_{\substack{numerical \\ error}} 0$$

The requested plots as follows:









As we expected and explained above, the maximal value in the last plot - which describes $\left\|\mathcal{H}^\dagger\phi_1 - \mathcal{H}^\dagger\phi_2\right\|_2$ is around 2e-9 which is practically zero.