DATA PROC AND REP

HW 3

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1.a.

For general $k \in \mathbb{N}$, the j_{th} column of J^k is:

$$J_{i,j}^{k} = \begin{cases} 1 & ,i = (j+k)mod(n) \\ 0 & ,else \end{cases}$$

Specifically for $k = n$, the j_{th} column of J^n is:

$$J_{i,j}^{n} = \begin{cases} 1 & ,i = (j+n)mod(n) = j \\ 0 & ,else \end{cases}$$

Meaning $J^n = I_{nxn}$

1.b.

Recall the definition from class:

$$W^{kl} = e^{\frac{2\pi ikl}{n}}$$

As we saw in the previous course numeric algorithms:

Calculating the eigen values of a general circulant matrix C is given by calculating DFT on it's first row (transposed).

Therefore we get:

$$\Rightarrow \quad \lambda_k = \sum_{l=0}^{n-1} W_{k,l} J_{0,l} = W^{k(n-1)} = W^{k(-1)} = e^{\frac{-2\pi ik}{n}}$$

1.c.

As we saw in tutorial 7:

All the circulant matrices have the same eigenvectors - the DFT^* matrix diagonalizes any circulant matrix.

In our case, J is circulant then it is diagonalizable, with the eigenvalues as shown above, and eigenvectors as follows:

The eigenvector corresponds to the k_{th} eigenvalue is:

$$\begin{bmatrix} W^{0k} \\ \vdots \\ W^{lk} \\ \vdots \\ W^{(n-1)k} \end{bmatrix}$$

The decomposition of J is:

$$\Lambda = DFT \cdot J \cdot DFT^*$$

Where Λ is a diagonal matrix with the eigen values as it's main diagonal elements, and DFT is the corresponding eigenvectors matrix.

J can be diagonalized with a unitary basis since we saw that the DFT is a unitary matrix.

1.d.

Since H and J are both circulant matrices:

$$H = \sum_{i=0}^{n-1} h_i J^i = P(J)$$

- H is a circulant matrix, then it has at most n different values - h_i $(0 \le i \le n-1)$
- For all i, J^i is one shift of J^{i-1}

1.e.

As we mentioned before:

All the circulant matrices have the same eigenvectors - the DFT^* matrix diagonalizes any circulant matrix.

H is a circulant matrix, and therefore is it unitarily diagonalizable by the DFT matrix.

Since we saw that $H = P(J)$, and according to Cayley-Hamilton:

$$\lambda_k(H) = P\big(\lambda_k(J)\big) = P\big(W^{-k}\big)$$

- The decomposition of H is: $\Lambda_H = P(\Lambda) = DFT \cdot H \cdot DFT^*$

1.f.

we showed the requested property in the previous answer.

1.g.

As shown above:

$$\lambda_k(H) = P\big(W^{-k}\big) = \sum_{i=0}^{n-1} h_i W^{-ik} = \sum_{j=0}^{n-1} h_j e^{\frac{-2\pi ijk}{n}} = \sqrt{n} DTF_k^* \begin{bmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{bmatrix} = \sqrt{n} B_k \begin{bmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} = \sqrt{n} B \begin{bmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{bmatrix}$$

1.h.

Using the theorem above, we define:

$$H_1 = W^* \cdot \Lambda_1 \cdot W$$

$$H_2 = W^* \cdot \Lambda_2 \cdot W$$

Proof:

$$H_1 H_2 =$$

$$= W^* \cdot \Lambda_1 \cdot \underbrace{W \cdot W^*}_{I} \cdot \Lambda_2 \cdot W = W^* \cdot \Lambda_1 \cdot \Lambda_2 \cdot W$$

$$\underbrace{=}_{\substack{diagonal\\matrices\\commute}} W^* \cdot \Lambda_2 \cdot \Lambda_1 \cdot W = W^* \cdot \Lambda_2 \cdot I \cdot \Lambda_1 \cdot W = W^* \cdot \Lambda_2 \cdot W \cdot W^* \cdot \Lambda_1 \cdot W$$

$$= H_2 H_1 \quad \blacksquare$$

Compute $H_1 H_2$:

$$H_1 H_2 \underbrace{=}_{\substack{we\ saw\\above}} W^* \cdot \Lambda_2 \cdot \Lambda_1 \cdot W \underbrace{=}_{\substack{product\ of\\diagonal\ is\\diagonal}} W^* \cdot \Lambda_{1,2} \cdot W$$

Now since $H_1 H_2$ diagonalized by DFT, it is circulant.

1.i.

First we compute DFT^2:

$$DFT^2 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & W^{-1} & \dots & W^{-(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W^{-(n-1)} & \dots & W^{-(n-1)^2} \end{pmatrix} \cdot \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & W^{-1} & \dots & W^{-(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W^{-(n-1)} & \dots & W^{-(n-1)^2} \end{pmatrix}$$

$$= \frac{1}{n} \begin{pmatrix} \Sigma_{j=0}^{n-1} W^{-j \cdot 0 - 0 \cdot j} & \Sigma_{j=0}^{n-1} W^{-j \cdot 0 - 1 \cdot j} & \dots & \Sigma_{j=0}^{n-1} W^{-0 \cdot j - j(n-1)} \\ \Sigma_{j=0}^{n-1} W^{-j \cdot 1 - 0 \cdot j} & \Sigma_{j=0}^{n-1} W^{-j \cdot 1 - 1 \cdot j} & \dots & \Sigma_{j=0}^{n-1} W^{-1 \cdot j - j(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \Sigma_{j=0}^{n-1} W^{-j \cdot (n-1) - 0 \cdot j} & \Sigma_{j=0}^{n-1} W^{-j \cdot (n-1) - 1 \cdot j} & \dots & \Sigma_{j=0}^{n-1} W^{-(n-1) \cdot j - j(n-1)} \end{pmatrix}$$

$$\Rightarrow \forall 0 \le k, l \le n-1: \quad DFT^2_{k,l} =$$

$$= \frac{1}{n} \Sigma_{j=0}^{n-1} W^{-j \cdot (k+l)} = \frac{1}{n} \Sigma_{j=0}^{n-1} W^{-jk} \cdot W^{-jl} =$$

$$\underbrace{=}_{W\ property} \frac{1}{n} \Sigma_{j=0}^{n-1} W^{-jk} \cdot W^{-j(n-l)} = \frac{1}{n} \langle DFT_k, DFT_{n-l} \rangle$$

Therefore, the entries in each anti-diagonal are equal, and:

- If $k(mod(n)) = (n-l)(mod(n))$, then $DFT^2_{k,l} = 1$
- Else, $DFT^2_{k,l} = 0$

For example:

$DFT^2_{0,0} = 1$ since $0 = (n-0)mod(n)$

$DFT^2_{n-1,1} = 1$ since $n-1 = n-1$

We get the following matrix:

$$DFT^2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

We can see that DFT^2 is a circulant matrix, and is also a premutation matrix.

A look at a general $k(mod(4))$ power of DFT matrix:

- $k\big(mod(4)\big) = 0 \implies DFT^k = (DFT^4)^{\frac{k}{4}} = (DFT^2 \cdot DFT^2)^{\frac{k}{4}} = I^{\frac{k}{4}} = I$
  according to theory in the previous course numerical algorithms, square of permutation matrix is the identity matrix.

- $k\big(mod(4)\big) = 1 \implies DFT^k = DFT^{4i+1} = DFT^{4i} \cdot DFT \underbrace{=}_{\substack{as\ seen\ above \\ DFT^4=I}} DFT$
- $k\big(mod(4)\big) = 2 \implies DFT^k = DFT^{4i+2} = DFT^{4i} \cdot DFT^2 = \underbrace{DFT^2}_{\substack{as\ calculated \\ above}}$
- $k\big(mod(4)\big) = 3 \implies DFT^k = DFT^{4i-1} = DFT^{4i} \cdot DFT^{-1} \underbrace{=}_{\substack{DFT\ is \\ unitary}} DFT^*$

$$\implies \forall k \in \mathbb{N}:\ DFT^k = \begin{cases} I & ,k(mod4) = 0 \\ DFT & ,k(mod4) = 1 \\ DFT^2 & ,k(mod4) = 2 \\ DFT^* & ,k(mod4) = 3 \end{cases}$$

1.j.

In numerical algorithms we saw that a convolution can be expressed as a matrix operation:

$$z = x \otimes y = \underbrace{\begin{bmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{bmatrix}}_{A} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Since A is a circulant matrix (as a convolution matrix), it can be unitarily diagonalized by DFT:

$$A = DFT^* \cdot \Lambda_A \cdot DFT$$

Where Λ_A is the diagonal matrix with $A's$ eigenvalues.

Therefore,

$$z = DFT^* \cdot \Lambda_A \cdot DFT \cdot y$$

$$\Longrightarrow DFT \cdot z = \Lambda_A \cdot DFT \cdot y \underbrace{=}_{\substack{Hadamard\\product}} \begin{bmatrix} \lambda_{A_0} \\ \lambda_{A_1} \\ \vdots \\ \lambda_{A_{n-1}} \end{bmatrix} \odot DFT \cdot y$$

As seen in q.1.g.

$$\begin{bmatrix} \lambda_{A_0} \\ \lambda_{A_1} \\ \vdots \\ \lambda_{A_{n-1}} \end{bmatrix} = \sqrt{n} \cdot DFT \cdot x$$

Let us plug in the expression above, and get:

$$DFT \cdot z = \sqrt{n} \cdot DFT \cdot x \odot DFT \cdot y$$

2.a.

First we saw that $f(t) * g(t) = \int_{-\infty}^{\infty} f(\phi) g(t-\phi) d\phi$

Define:

$$\tilde{f}(t) = f(t-1)$$

$$\tilde{g}(t) = g(t+1)$$

$$\Longrightarrow \tilde{f}(t) * \tilde{g}(t) = \int_{-\infty}^{\infty} \tilde{f}(\phi)\tilde{g}(t-\phi) d\phi = \int_{-\infty}^{\infty} f(\phi-1) g(t-\phi+1) d\phi$$

Define $\omega = \phi - 1$, then:

$$\underbrace{=}_{\substack{d\phi = d\omega\\ and\\ limits\ are\\ the\ same}} \int_{-\infty}^{\infty} f(\omega) g(t-\omega) d\omega = f(t) * g(t) = h(t)$$

$$\Longrightarrow f(t-1) * g(t+1) = h(t)$$

2.b.

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xu} dx$$

$$\mathcal{G}(u) = \int_{-\infty}^{\infty} g(y) e^{-i2\pi yu} dy$$

$$\mathcal{F}(u)\mathcal{G}(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xu} dx \cdot \int_{-\infty}^{\infty} g(y) e^{-i2\pi yu} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, g(y) e^{-i2\pi u(x+y)} dxdy$$

Define $z = x + y$ (so $y = z - x \ and \ dy = dz$), then:

$$\boldsymbol{\mathcal{F}(u)\mathcal{G}(u)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,g(z-x)e^{-i2\pi uz}dxdz = \int_{-\infty}^{\infty} e^{-i2\pi uz}\underbrace{\int_{-\infty}^{\infty} f(x)\,g(z-x)dx}_{f(z)*g(z)=h(z)}\,dz$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi uz}h(z)dz \underbrace{=}_{\substack{definition\\ of\\ fourier\\ tranform}} \boldsymbol{\mathcal{H}(u)}$$

Where $\mathcal{H}$ is the Fourier transform of h, then we get the following transform as well:

$$h(u) = \int_{-\infty}^{\infty} e^{i2\pi uz}\mathcal{H}(z)dz$$

then:

$$h(0) = \int_{-\infty}^{\infty} e^{i2\pi 0z}\mathcal{H}(z)dz = \int_{-\infty}^{\infty}\mathcal{H}(z)dz = \underbrace{\int_{-\infty}^{\infty}\mathcal{F}(z)\mathcal{G}(z)dz}_{*}$$

Also, we know that $h(z) = \int_{-\infty}^{\infty} f(x)\,g(z-x)dx \implies h(0) = \underbrace{\int_{-\infty}^{\infty} f(x)\,g(-x)dx}_{**}$

From the two equations above, we get:

$$\int_{-\infty}^{\infty}\mathcal{F}(z)\mathcal{G}(z)dz = \int_{-\infty}^{\infty} f(x)\,g(-x)dx \ \blacksquare$$

3.a.

The DFT matrix of ϕ is given by: $W\phi$ (where each entrance in W is $w_{k,l} = \frac{1}{\sqrt{2N}}e^{-\frac{i2\pi kl}{2N}}$)

Let us denote the j_{th} column of W as w_j:

$$W\phi = \frac{1}{\sqrt{2N}}\begin{bmatrix} \vdots & \vdots & & \vdots \\ w_0 & w_1 & \cdots & w_{2N-1} \\ \vdots & \vdots & & \vdots \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ \frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{2N}}\left(1\cdot w_0 + \frac{1}{2}\cdot w_1 + \underbrace{\cdots}_{=0} + \frac{1}{2}\cdot w_{2N-1}\right)$$

3.b.

First, the DFT of ψ is: $W\psi = \psi^F$

Each entrance in ψ^F is given by: $\psi_k^F = \frac{1}{\sqrt{N}}\left(\sum_{j=0}^{N-1}\psi_j\cdot e^{-\frac{-i2\pi jk}{N}}\right)$ for $k \in \{0,..,N-1\}$

In the same manner, the DFT of γ is given by $W\gamma = \gamma^F$

Each entrance in γ^F:

$$\gamma_k^F = \frac{1}{\sqrt{2N}}\left(\sum_{j=0}^{2N-1} \gamma_j \cdot e^{-\frac{-i2\pi jk}{2N}}\right) \text{ for } k \in \{0,..,2N-1\}$$

Every odd-index element of γ is 0, so instead of summing over all j to $2N-1$, we can sum over (2j) to $N-1$,

and instead of using γ_j we can use ψ_j since for every even-index element it is the same value:

$$\gamma_k^F = \frac{1}{\sqrt{2N}}\left(\sum_{j=0}^{N-1} \gamma_j \cdot e^{-\frac{-i2\pi k(2j)}{2N}}\right) = \frac{1}{\sqrt{2N}}\left(\sum_{j=0}^{N-1} \psi_j \cdot e^{-\frac{-i2\pi kj}{N}}\right)$$

Since we saw that $\psi_k^F = \frac{1}{\sqrt{N}}\left(\sum_{j=0}^{N-1} \psi_j \cdot e^{-\frac{-i2\pi jk}{N}}\right)$, we can re-write the above:

$$\gamma_k^F \underbrace{=}_{\substack{W^k = W^{k(mod(N))} \\ and\ \gamma\ size\ is\ 2N}} \begin{cases} \frac{1}{\sqrt{2}}\psi_k^F & \text{for } k \in \{0,..,N-1\} \\ \frac{1}{\sqrt{2}}\psi_k^F & \text{for } k \in \{N,..,2N-1\} \end{cases}$$

$$\Rightarrow \gamma^F = \frac{1}{\sqrt{2}}\begin{bmatrix}\psi^F \\ \psi^F\end{bmatrix}$$

3.c.

Let us denote the k_{th} entrance of the convolution vector $\gamma * \phi$ as h_k:

$$h_k = \sum_{j=0}^{2N-1} \gamma_{(k-j)mod(2N)}\phi_j = 1 \cdot \gamma_k + \frac{1}{2}\cdot\gamma_{k-1} + \frac{1}{2}\cdot\gamma_{k+1} = \gamma_k + \frac{1}{2}(\gamma_{k-1} + \gamma_{k+1})$$

- For odd $k(mod(n))$, $\gamma_{k(mod(n))} = 0$, and both $(k-1), (k+1)$ are even, and:

$$\gamma_{k-1} = \psi_{\frac{k-1}{2}},\ \gamma_{k+1} = \psi_{\frac{k+1}{2}}$$

  Then: $h_k = 0 + \frac{1}{2}\left(\psi_{\frac{k-1}{2}} + \psi_{\frac{k+1}{2}}\right) = \frac{\psi_{\frac{k-1}{2}} + \psi_{\frac{k+1}{2}}}{2}$

- For even $k\big(mod(n)\big)$, both $\gamma_{k-1}, \gamma_{k+1}$ equal 0, and $\gamma_k = \psi_{\frac{k}{2}}$

  Then: $h_k = \psi_{\frac{k}{2}} + 0$

$$\Rightarrow h = \gamma * \phi = \left[\psi_0, \frac{\psi_0 + \psi_1}{2}, \psi_1, \frac{\psi_1 + \psi_2}{2}, \ldots, \psi_{N-1}, \frac{\psi_{N-1} + \psi_0}{2}\right]^T$$

3.d.

As we proved before in q.1.j.

Since $h = \gamma * \phi$, the DFT of h is given by: $DFT(\gamma * \phi) = \sqrt{2N} \cdot DFT(\gamma) \odot DFT(\phi)$ when $\odot$ is the Hadamard product.

Each of the two expressions above was calculated in previous sections:

$$DFT(\gamma) = \gamma^F = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi^F \\ \psi^F \end{bmatrix}$$

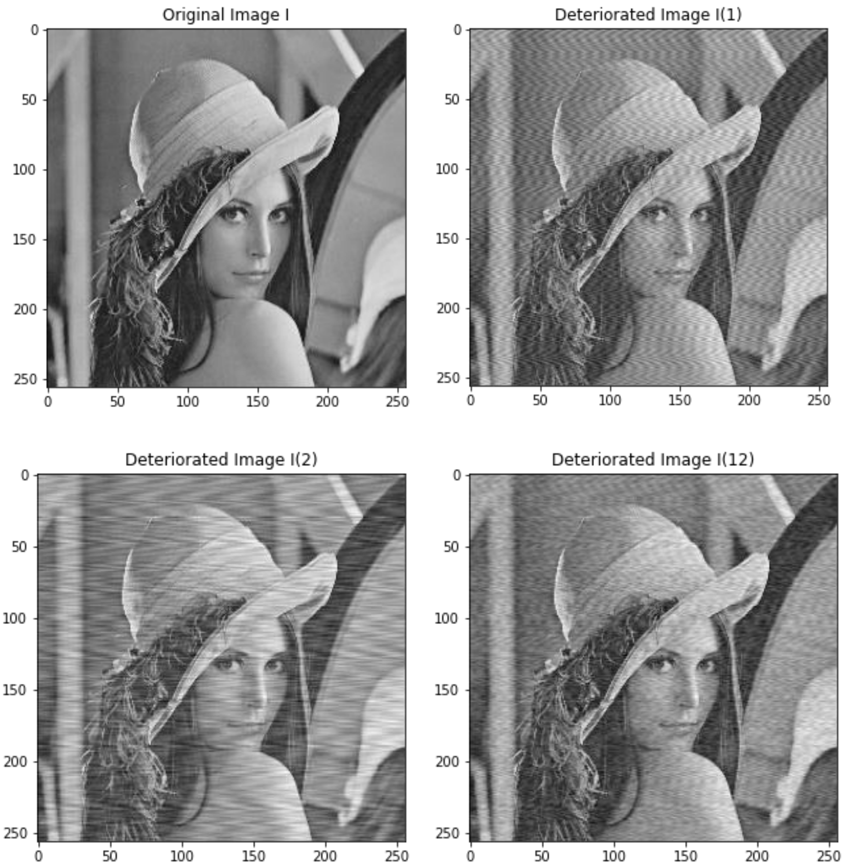
$$DFT(\phi) = \frac{1}{\sqrt{2N}} \left(\mathbf{1}_{2N} + \frac{1}{2} \cdot w_1 + \frac{1}{2} \cdot w_{2N-1} \right)$$

$$\Rightarrow DFT(h) = \sqrt{2N} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} \psi^F \\ \psi^F \end{bmatrix} \odot \frac{1}{\sqrt{2N}} \left(\mathbf{1}_{2N} + \frac{1}{2} \cdot w_1 + \frac{1}{2} \cdot w_{2N-1} \right)$$

$$\boldsymbol{DFT(h)} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \begin{bmatrix} \boldsymbol{\psi^F} \\ \boldsymbol{\psi^F} \end{bmatrix} \odot \left(\mathbf{1}_{\mathbf{2N}} + \frac{\mathbf{1}}{\mathbf{2}} \cdot \boldsymbol{w_1} + \frac{\mathbf{1}}{\mathbf{2}} \cdot \boldsymbol{w_{2N-1}} \right)$$

$\boldsymbol{Implementation}$

1.a.



1.b.

First, define:

$DFT(D_{k,l})$ – The DFT representation of the (k, l) entry of a degraded image D (originally I).

$$D_{i,j} = I_{i,j} + A_i \cos\left(2\pi f j + \varphi_i\right)$$

Calculate a general case $DFT(D_{k,l})$:

Let us remind that since W matrix (the DFT matrix) is symmetric, then $W^T = W$.

$$DFT\left(D_{k,l}\right) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, D_{k,l}$$

$$= \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\left(I_{k,l} + A_k \cos\left(2\pi f l + \varphi_k\right)\right)$$

$$= \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l} + A_k \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl} \cos\left(2\pi f l + \varphi_k\right)$$

$$e^{i2\pi kl} = \cos(2\pi kl) + i \cdot sin(2\pi kl)$$

$$\Longrightarrow\ e^{i2\pi kl} + e^{-i2\pi kl} = \cos(2\pi kl) + isin(2\pi kl) + \cos(2\pi kl) - isin(2\pi kl) = 2\cos(2\pi kl)$$

$$\Longrightarrow \underbrace{\cos(2\pi kl) = \frac{1}{2}\left(e^{i2\pi kl} + e^{-i2\pi kl}\right)}_{*}$$

Then:

$$\frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} I_{k,l} + A_k \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} \cos(2\pi fl + \varphi_k)$$

$$\underbrace{=}_{*} \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} I_{k,l} + A_k \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} \frac{1}{2}\left(e^{i(2\pi fl+\varphi_k)} + e^{-i(2\pi fl+\varphi_k)}\right)$$

$$\underbrace{=}_{\substack{DFT\\ entry\\ definiton}} \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} I_{k,l} + A_k \frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} \left(\omega^{fl} e^{i\varphi_k} + \omega^{-fl} e^{-i\varphi_k}\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} I_{k,l} + A_k e^{i\varphi_k} \frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{fl} + A_k e^{-i\varphi_k} \frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{-fl}$$

As mentioned in the question, $\frac{1}{f}$ divides n, then exists an integer $q \in \mathbb{N}$ such that:

$\frac{n}{\frac{1}{f}} = q \Longrightarrow f = \frac{q}{n}$.

Then:

$$\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{fl} = \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{\frac{q}{n}l} = \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{\frac{ql}{n}(mod(n))} = \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{ql(mod(n))} = \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{ql}$$

We plug in the last expression:

$$= \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl} I_{k,l} + A_k e^{i\varphi_k} \frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{ql} + A_k e^{-i\varphi_k} \frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{-ql}$$

When $k = q$:

$$\sum_{l=1}^{n-1} \omega^{-kl}\,\omega^{ql} = \sum_{l=1}^{n-1} \omega^{(q-k)l} = \sum_{l=1}^{n-1} \omega^{0} = \sum_{l=1}^{n-1} 1 = n$$

When $k \neq q$:

$$\sum_{l=1}^{n-1} \omega^{-kl}\,\omega^{ql} = \sum_{l=1}^{n-1} \omega^{(q-k)l} \underbrace{=}_{\substack{geometric\\ sum\\ formula}} \frac{\omega^{n(q-k)l} - 1}{\omega^{(q-k)l} - 1} = \frac{\omega^{n(q-k)l(mod(n))} - 1}{\omega^{(q-k)l} - 1}$$

$$= \frac{\omega^{0(q-k)l(mod(n))} - 1}{\omega^{(q-k)l} - 1} = \frac{\omega^{0(q-k)l} - 1}{\omega^{(q-k)l} - 1} = \frac{\omega^{0} - 1}{\omega^{(q-k)l} - 1} = 0$$

As we saw in class, we can define:

$$\sum_{l=1}^{n-1} \omega^{-kl}\,\omega^{ql} = n \cdot\ \delta_{(k,q)}$$

$$\sum_{l=1}^{n-1} \omega^{-kl}\,\omega^{-ql} = n \cdot\ \delta_{(k,n-q)}$$

We plug in the last expression:

$$= \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l} + A_k e^{i\varphi_k} \frac{1}{2\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\,\omega^{ql} + A_k e^{-i\varphi_k} \frac{1}{2\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\,\omega^{-ql}$$

$$\underbrace{=}_{\substack{above\\ explanation}} \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l} + A_k e^{i\varphi_k} \frac{1}{2\sqrt{n}} n\delta_{(k,q)} + A_k e^{-i\varphi_k} \frac{1}{2\sqrt{n}} n\delta_{(k,n-q)}$$

$$= \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l} + \frac{\sqrt{n}}{2} A_k \left(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}\right)$$

By definition, $\frac{1}{\sqrt{n}}\sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l}$ is the DFT representation of entry (k,l) of image I

$$\Rightarrow DFT(D_{k,l}) = DFT(I_{k,l}) + \frac{\sqrt{n}}{2} A_k \left(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}\right)$$

In our case, $n = 256$, then:

$$DFT(D_{k,l}) = DFT(I_{k,l}) + \frac{\sqrt{256}}{2} A_k \left(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}\right)$$

$$= DFT(I_{k,l}) + 8A_k \left(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}\right)$$

We define:

$DFT(D_k)$ – The DFT representation of the k_{th} row of a degraded image D (originally I).

According to the results above:

$$DFT(D_k) = \begin{bmatrix} DFT(D_{k,0}) \\ DFT(D_{k,1}) \\ \vdots \\ \vdots \\ DFT(D_{k,n-2}) \end{bmatrix}^T = \begin{bmatrix} DFT(I_{k,0}) + 8A_k(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}) \\ DFT(I_{k,1}) + 8A_k(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}) \\ \vdots \\ \vdots \\ DFT(I_{k,n-1}) + 8A_k(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}) \end{bmatrix}^T$$

1.c.

First, define:

$DFT(\widehat{D}_{k,l})$ – The DFT representation of the (k,l) entry of a degraded image $\widehat{D}$ (originally I).

$$\widehat{D}_{k,l} = I_{k,l} + \beta_1 A_{1_k}\cos(2\pi f_1 l + \varphi_{1_k}) + \beta_2 A_{2_k}\cos(2\pi f_2 l + \varphi_{2_k})$$

* β_1 and β_2 are normalized weights as requested in the question.

Calculate a general case $DFT(\widehat{D}_{k,l})$:

$$DFT(\widehat{D}_{k,l}) = \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,\widehat{D}_{k,l}$$

$$= \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\left(I_{k,l} + \beta_1 A_{1_k}\cos(2\pi f_1 l + \varphi_{1_k}) + \beta_2 A_{2_k}\cos(2\pi f_2 l + \varphi_{2_k})\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,I_{k,l} + \beta_1 A_{1_k}\frac{1}{\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\cos(2\pi f_1 l + \varphi_{1_k}) + \beta_2 A_{2_k}\frac{1}{\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\cos(2\pi f_2 l + \varphi_{2_k})$$

$$\underbrace{=}_{\substack{as\\ seen\\ above}}$$

$$\frac{1}{\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,I_{k,l}$$

$$+ \beta_1 A_{1_k}e^{i\varphi_{1k}}\frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,\omega^{f_1 l} + \beta_1 A_{1_k}e^{-i\varphi_{1k}}\frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,\omega^{-f_1 l}$$

$$+ \beta_2 A_{2_k}e^{i\varphi_{2k}}\frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,\omega^{f_2 l} + \beta_2 A_{2_k}e^{-i\varphi_{2k}}\frac{1}{2\sqrt{n}}\sum_{l=1}^{n-1}\omega^{-kl}\,\omega^{-f_2 l}$$

As we saw above, each of the f_i's satisfies the following with corresponding q_i:

$$\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{f_1 l} = \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{q_1 l}$$

$$\sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{f_2 l} = \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{q_2 l}$$

We plug in the expression above:

$$= \frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l}$$

$$+ \beta_1 A_{1k} e^{i\varphi_{1k}} \frac{1}{2\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{q_1 l} + \beta_1 A_{1k} e^{-i\varphi_{1k}} \frac{1}{2\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{-q_1 l}$$

$$+ \beta_2 A_{2k} e^{i\varphi_{2k}} \frac{1}{2\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{q_2 l} + \beta_2 A_{2k} e^{-i\varphi_{2k}} \frac{1}{2\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, \omega^{-q_2 l}$$

$$\underbrace{=}_{\substack{as\\ seen\\ above}}$$

$$\frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l}$$

$$+ \beta_1 A_{1k} e^{i\varphi_{1k}} \frac{1}{2\sqrt{n}} n\delta_{(k,q_1)} + \beta_1 A_{1k} e^{-i\varphi_{1k}} \frac{1}{2\sqrt{n}} n\delta_{(k,n-q_1)}$$

$$+ \beta_2 A_{2k} e^{i\varphi_{2k}} \frac{1}{2\sqrt{n}} n\delta_{(k,q_2)} + \beta_2 A_{2k} e^{-i\varphi_{2k}} \frac{1}{2\sqrt{n}} n\delta_{(k,n-q_2)}$$

$$=$$

$$\frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l}$$

$$+\beta_1 A_{1k} \frac{\sqrt{n}}{2} \left(e^{i\varphi_{1k}} \delta_{(k,q_1)} + e^{-i\varphi_{1k}} \delta_{(k,n-q_1)} \right)$$

$$+\beta_2 A_{2k} \frac{\sqrt{n}}{2} \left(e^{i\varphi_{2k}} \delta_{(k,q_2)} + e^{-i\varphi_{2k}} \delta_{(k,n-q_2)} \right)$$

By definition, $\frac{1}{\sqrt{n}} \sum_{l=1}^{n-1} \omega^{-kl}\, I_{k,l}$ is the DFT representation of entry (k, l) of image I

$$\Longrightarrow DFT(\widehat{D}_{k,l}) = DFT(I_{k,l}) + \beta_1 A_{1k} \frac{\sqrt{n}}{2}\left(e^{i\varphi_{1k}}\delta_{(k,q_1)} + e^{-i\varphi_{1k}}\delta_{(k,n-q_1)}\right)$$

$$+\beta_2 A_{2k} \frac{\sqrt{n}}{2}\left(e^{i\varphi_{2k}}\delta_{(k,q_2)} + e^{-i\varphi_{2k}}\delta_{(k,n-q_2)}\right)$$

In our case, $n = 256$, then:

$$DFT(\widehat{D}_{k,l}) = DFT(I_{k,l}) + \beta_1 A_{1k} 8\left(e^{i\varphi_{1k}}\delta_{(k,q_1)} + e^{-i\varphi_{1k}}\delta_{(k,n-q_1)}\right)$$

$$+\beta_2 A_{2k} 8\left(e^{i\varphi_{2k}}\delta_{(k,q_2)} + e^{-i\varphi_{2k}}\delta_{(k,n-q_2)}\right)$$

We define:

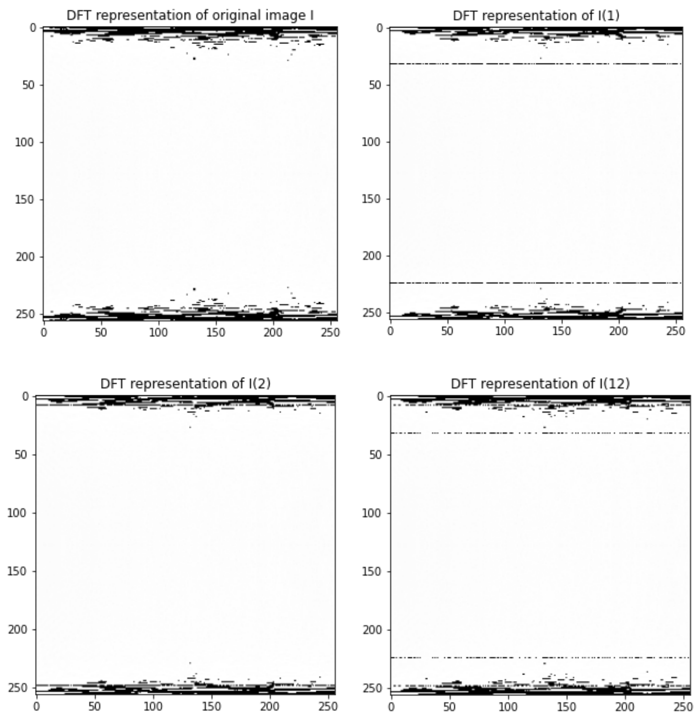
$DFT(\widehat{D}_k)$ – The DFT representation of the k_{th} row of a degraded image $\widehat{D}$ (originally I).

According to the results above:

$$DFT(\widehat{D}_k) = \begin{bmatrix} DFT(\widehat{D}_{k,0}) \\ DFT(\widehat{D}_{k,1}) \\ \vdots \\ \vdots \\ DFT(\widehat{D}_{k,n-2}) \end{bmatrix}^T$$

1.d.

Below are the empirical calculations of DFT representation of I, I^1, I^2, I^{12} :



By projecting the noised images to the Fourier domain (using DFT), we see in the plots above:

For frequency=1/8: the 32_{th} row and the $(256-32)_{th}$ row of the projected noised image are black (values near 1), while the rest of the rows are white (values near 0).

For example, the calculated median of the 32_{th} row and the $(256-32)_{th}$ row is 1 (most values are 1).

For frequency=1/32: the 8_{th} row and the $(256-8)_{th}$ row of the projected noised image are black (value 1), while the rest of the rows are white (value 0).

For example, the calculated median of the 8_{th} row and the $(256-8)_{th}$ row is 1 (most values are 1).

Let us remind that:

$$DFT(D_k) = \begin{bmatrix} DFT(I_{k,0}) + 8A_k(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}) \\ DFT(I_{k,1}) + 8A_k(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}) \\ \vdots \\ \vdots \\ DFT(I_{k,n-1}) + 8A_k(e^{i\varphi_k}\delta_{(k,q)} + e^{-i\varphi_k}\delta_{(k,n-q)}) \end{bmatrix}^T$$

For frequency=1/8, we get: $q = 32$.

The empirical values fit our theory, since:

$$\delta_{(k,32)} = \begin{cases} 1, & k = 32 \\ 0, & else \end{cases}$$

$$\delta_{(k,256-32)} = \begin{cases} 1, & k = 256-32 \\ 0, & else \end{cases}$$

So only for rows 32 and 256-32 we get values 1 (black rows), and the rest are 0 (white).

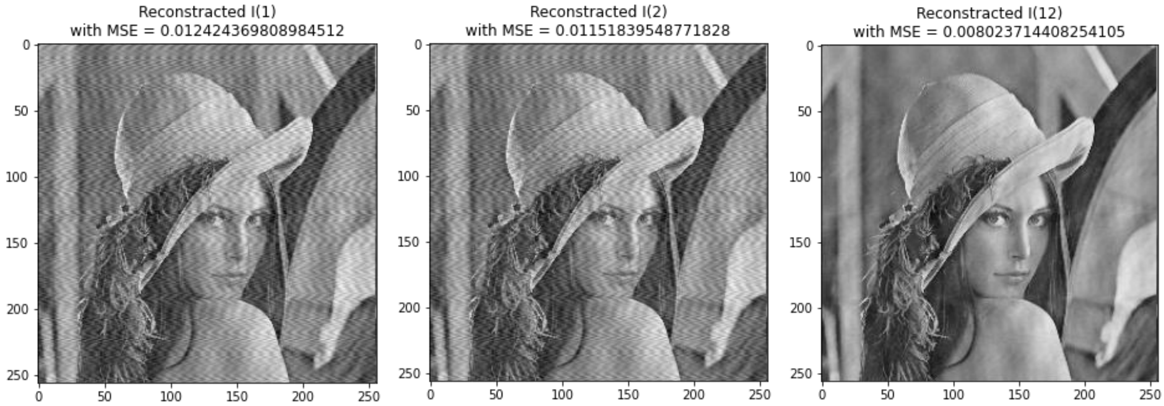
Same happens for frequency=1/32.

Analyzing noised image with frequency r_{12}, we get 4 rows with average around 0.5 (all the black rows of the two cases above: 32, 256-32, 8, 256-8) and for all other rows we get average 0 (white).

It also fits the theory, since in the expression for $DFT\left(\widehat{D}_{k,l}\right)$ above, we got 4 δ −expressions corresponding to 4 rows.

1.e.

By applying the filter mentioned in the question, and reconstructing the images using the invers DFT matrix, we got the following images with the corresponding MSEs:



As we can see above, the reconstructions are not perfect, but we obtained a high similarity to the original images (low MSE values).

The reconstruction of $I(12)$ seems the best (smooth and similar to the original image, with the minimal MSE among the 3 images).