DATA PROC AND REP

HW 2

IDO AMIT 315804831
IDO.AMIT@CAMPUS.TECHNION.AC.IL

HAY ELMALIAH 315777433
HAY.E@CAMPUS.TECHNION.AC.IL

1.a.a.

The best approximation of f using $\left\{\beta_{i_j}(x)\right\}_{j=1}^{k}$ is:

$$\tilde{f}^{\beta}(x) = \sum_{j=1}^{k} \langle f(x) \cdot \beta_{i_j}(x) \rangle \cdot \beta_{i_j}(x)$$

As we learned in class, the SE is:

$$\tilde{f}_{SE}\left(f - \tilde{f}^{\beta}|_k\right) = \int_0^1 f^2(x)dx - \sum_{j=1}^{k} \left\langle f(x) \cdot \beta_{i_j}(x) \right\rangle^2$$

1.a.b.

We reorder $\left\{\beta_{i_j}(x)\right\}_{j=1}^{n} \in F$ s.t $\underbrace{\langle f(x) \cdot \beta_1(x) \rangle^2}_{\beta_1^f} \geq \underbrace{\langle f(x) \cdot \beta_2(x) \rangle^2}_{\beta_2^f} \geq \cdots \geq \underbrace{\langle f(x) \cdot \beta_k(x) \rangle^2}_{\beta_k^f}$

Now as we learned, in order to minimize the error – we choose the k functions with the biggest squared inner product: $\beta_1, \beta_2, \dots, \beta_k$

And the SE would be:

$$\tilde{f}_{SE}\left(f - \tilde{f}^{\beta}|_k\right) = \int_0^1 f^2(x)dx - \sum_{i=1}^{k} \langle f(x) \cdot \beta_i(x) \rangle^2$$

The above k-approximation of f in F is not necessarily unique, in case that $\beta_k^f = \beta_{k+1}^f$ and than we could replace β_k^f with β_{k+1}^f in the SE expression and the error would be the same.

1.b.a.

As seen in 1.a.a.:

The SE of f n-approximation using $\{\beta_i(x)\}_{i=1}^{n}$ family in F:

$$\tilde{f}_{SE}\left(f - \tilde{f}^{\beta}|_n\right) = \int_0^1 f^2(x)dx - \sum_{i=1}^{n} \langle f(x) \cdot \beta_i(x) \rangle^2$$

The SE of f n-approximation using $\left\{\tilde{\beta}_i(x)\right\}_{i=1}^{n}$ family in F:

$$\tilde{f}_{SE}\left(f - \tilde{f}^{\tilde{\beta}}|_n\right) = \int_0^1 f^2(x)dx - \sum_{i=1}^{n} \left\langle f(x) \cdot \tilde{\beta}_i(x) \right\rangle^2$$

When comparing the two errors above:

$$\tilde{f}_{SE}\left(f - \tilde{f}^{\beta}|_n\right) - \tilde{f}_{SE}\left(f - \tilde{f}^{\tilde{\beta}}|_n\right) =$$

$$\int_0^1 f^2(x)dx - \sum_{i=1}^{n} \langle f(x) \cdot \beta_i(x) \rangle^2 - \left(\int_0^1 f^2(x)dx - \sum_{i=1}^{n} \left\langle f(x) \cdot \tilde{\beta}_i(x) \right\rangle^2\right) =$$

$$\sum_{i=1}^{n} \left\langle f(x)\cdot\tilde{\beta}_i(x)\right\rangle^2 - \langle f(x)\cdot\beta_i(x)\rangle^2$$

1.b.b.

First, we have no prior knowledge about the order of the functions' families in F. but even if the functions were ordered as shown in 1.a.b. (best k-approximations due to each family), we still won't be able to determine which family minimizes the SE for different values of k.

Even if the n-term SE is the same for both, we can't necessarily say the same for the k-term.

For example if the j_{th} coefficient of β may be much bigger than the j_{th} coefficient of $\tilde{\beta}$, in a way that for $k < j$ (not including the j_{th} projection) we get smaller SE using β family, and for $k \geq j$ (including the j_{th} projection) we get the opposite.

2.a.i.

We will show that: $H_4H_4^t = H_4^tH_4 = I_4$:

$$H_4H_4^t = \frac{1}{4}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I_4$$

$$H_4^tH_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I_4$$

2.a.ii.

as seen in class, the standard basis over $t \in [0,1]$ is given by:

$\left(\text{define: } \Delta_1 = \left[0,\frac{1}{4}\right)\ ,\ \Delta_2 = \left[\frac{1}{4},\frac{1}{2}\right)\ ,\ \Delta_3 = \left[\frac{1}{2},\frac{3}{4}\right)\ , \Delta_4 = \left[\frac{3}{4}\ ,1\right]\right)$

$$\begin{bmatrix} \sqrt{N}\cdot 1_{\Delta_1}(t) \\ \sqrt{N}\cdot 1_{\Delta_2}(t) \\ \sqrt{N}\cdot 1_{\Delta_3}(t) \\ \sqrt{N}\cdot 1_{\Delta_4}(t) \end{bmatrix}$$

For $N = 4$ we get:

$$\begin{bmatrix} 2\cdot 1_{\Delta_1}(t) \\ 2\cdot 1_{\Delta_2}(t) \\ 2\cdot 1_{\Delta_3}(t) \\ 2\cdot 1_{\Delta_4}(t) \end{bmatrix}$$

Now in order to get the Haar functions, we will apply the Haar matrix upon the standard basis:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 2\cdot 1_{\Delta_1}(t) \\ 2\cdot 1_{\Delta_2}(t) \\ 2\cdot 1_{\Delta_3}(t) \\ 2\cdot 1_{\Delta_4}(t) \end{bmatrix} = \begin{bmatrix} \psi_1^H(t) \\ \psi_2^H(t) \\ \psi_3^H(t) \\ \psi_4^H(t) \end{bmatrix}$$

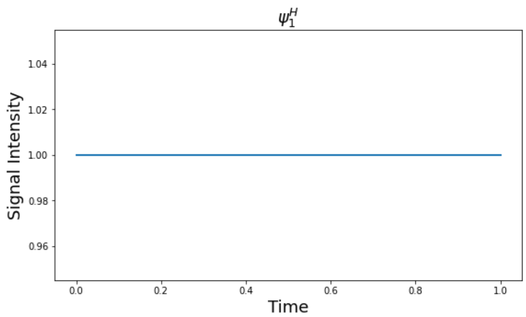
$$= \begin{bmatrix} 1_{\Delta_1}(t) + 1_{\Delta_2}(t) + \sqrt{2}\cdot 1_{\Delta_3}(t) + 0\cdot 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) + 1_{\Delta_2}(t) - \sqrt{2}\cdot 1_{\Delta_3}(t) + 0\cdot 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) - 1_{\Delta_2}(t) + 0\cdot 1_{\Delta_3}(t) + \sqrt{2}\cdot 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) - 1_{\Delta_2}(t) + 0\cdot 1_{\Delta_3}(t) - \sqrt{2}\cdot 1_{\Delta_4}(t) \end{bmatrix}$$

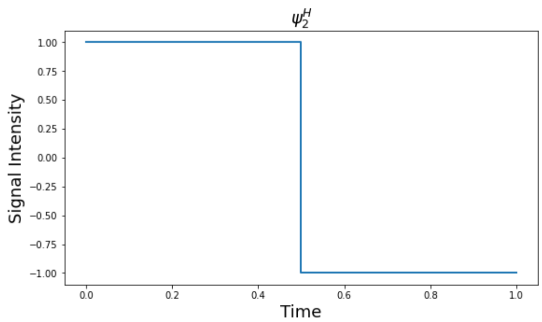
Therefore we get the following functions:

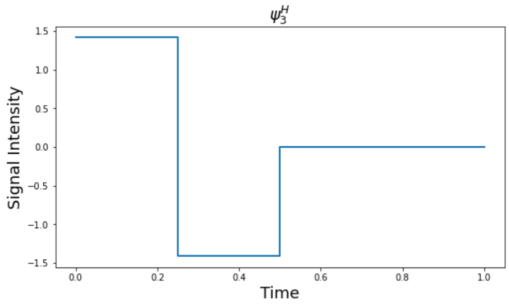
$$\psi_1^W(t) = 1 \qquad \psi_2^W(t) = \begin{cases} 1, t \in \left[0, \frac{1}{2}\right) \\ -1, t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

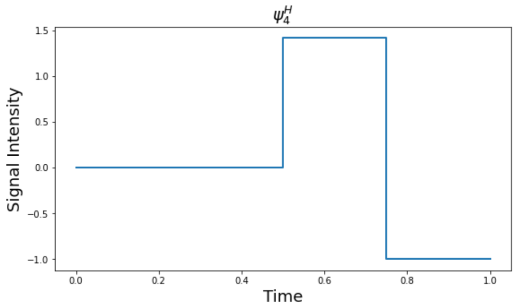
$$\psi_3^W(t) = \begin{cases} \sqrt{2}, t \in \left[0, \frac{1}{4}\right) \\ -\sqrt{2}, t \in \left[\frac{1}{4}, \frac{1}{2}\right) \\ 0, t \in \left[\frac{1}{2}, 1\right] \end{cases} \qquad \psi_4^W(t) = \begin{cases} 0\ , t \in \left[0, \frac{1}{2}\right) \\ \sqrt{2}\ , t \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ -\sqrt{2}\ , t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

Now we can plot the Walsh-Hadamard functions:









2.a.iii.

Define the i_{th} coefficient of the approximation of $\phi(t)$ using Haar basis:

$$\psi_i^H = \langle \phi(t), \psi_i^H(t) \rangle = \int_0^1 \phi(t) \cdot \psi_i^H(t)\, dt$$

As we learned in class, the best approximation of $\phi(t)$ using this Haar basis is:

$$\hat{\phi}(t) = \sum_{i=1}^{4} \psi_i^H \cdot \psi_i^H(t)$$

And the associated MSE is:

$$\hat{\phi}_{MSE}(\phi - \psi^H) = \int_0^1 \phi^2(t)\, dt - \sum_{i=1}^{4} \left(\psi_i^H\right)^2$$

2.a.iv.

First, the signal is:

$$\phi(t) = a + b \cdot cos(2\pi t) + c \cdot \cos^2(\pi t)$$

Let us remind:

$$\int_0^1 \cos^2(\pi t) dt = \int_0^1 \frac{\cos(2\pi t) + 1}{2} dt = \frac{1}{2}\int_0^1 \cos(2\pi t)\, dt + \int_0^1 \frac{1}{2} dt = \left[\frac{\sin(2\pi t)}{4\pi} + \frac{1}{2} t\right]_0^1$$

Now we can calculate the coefficients:

$$\boldsymbol{\psi_1^H} = \langle \phi(t), \psi_1^H(t) \rangle = \int_0^1 \phi(t) dt = \int_0^1 (a + b \cdot cos(2\pi t) + c \cdot \cos^2(\pi t)) dt =$$

$$\left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c \frac{\sin(2\pi t)}{4\pi} + \frac{c}{2} t\right]_0^1 = \boldsymbol{a} + \boldsymbol{\frac{c}{2}}$$

$$\boldsymbol{\psi_2^H} = \langle \phi(t), \psi_2^H(t) \rangle = \int_0^{\frac{1}{2}} \phi(t)dt - \int_{\frac{1}{2}}^{1} \phi(t)dt =$$

$$\left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_0^{\frac{1}{2}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{2}}^{1} =$$

$$= \left(\frac{a}{2} + \frac{c}{4}\right) - \left(\left(a + \frac{c}{2}\right) - \left(\frac{a}{2} + \frac{c}{4}\right)\right) = 2\left(\frac{a}{2} + \frac{c}{4}\right) - \left(a + \frac{c}{2}\right) = \mathbf{0}$$

$$\boldsymbol{\psi_3^H} = \langle \phi(t), \psi_3^H(t) \rangle = \sqrt{2}\left(\int_0^{\frac{1}{4}} \phi(t)dt - \int_{\frac{1}{4}}^{\frac{1}{2}} \phi(t)dt\right) =$$

$$\sqrt{2}\left(\left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_0^{\frac{1}{4}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{4}}^{\frac{1}{2}}\right)$$

$$= \sqrt{2}\left(\left(\frac{a}{4} + \frac{b}{2\pi} + \frac{c}{4\pi} + \frac{c}{8}\right) - \left(\left(\frac{a}{2} + \frac{c}{4}\right) - \left(\frac{a}{4} + \frac{b}{2\pi} + \frac{c}{4\pi} + \frac{c}{8}\right)\right)\right) =$$

$$= \sqrt{2}\left(\frac{a}{2} + \frac{b}{\pi} + \frac{c}{2\pi} + \frac{c}{4} - \left(\frac{a}{2} + \frac{c}{4}\right)\right) = \boldsymbol{\sqrt{2}\left(\frac{b}{\pi} + \frac{c}{2\pi}\right)}$$

$$\boldsymbol{\psi_4^H} = \langle \phi(t), \psi_4^H(t) \rangle = \sqrt{2}\left(\int_{\frac{1}{2}}^{\frac{3}{4}} \phi(t)dt - \int_{\frac{3}{4}}^{1} \phi(t)dt\right) =$$

$$\sqrt{2}\left(\left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{2}}^{\frac{3}{4}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{3}{4}}^{1}\right)$$

$$= \sqrt{2}\left(\left(\left(\frac{3a}{4} - \frac{b}{2\pi} - \frac{c}{4\pi} + \frac{3c}{8}\right) - \left(\frac{a}{2} + \frac{c}{4}\right)\right) - \left(\left(a + \frac{c}{2}\right) - \left(\frac{3a}{4} - \frac{b}{2\pi} - \frac{c}{4\pi} + \frac{3c}{8}\right)\right)\right) =$$

$$= \sqrt{2}\left(\frac{3a}{2} - \frac{b}{\pi} - \frac{c}{2\pi} + \frac{3c}{4} - \left(\frac{a}{2} + \frac{c}{4} + a + \frac{c}{2}\right)\right) = \boldsymbol{-\sqrt{2}\left(\frac{b}{\pi} + \frac{c}{2\pi}\right)}$$

The smallest absolute coefficient is $\psi_2^H = 0$

Since we have two coefficients with equal absolute values: $\left(|\psi_3^H| = |\psi_4^H| = \sqrt{2}\left(\frac{b}{\pi}+\frac{c}{2\pi}\right)\right)$,

We can compare only between the positive values of ψ_1^H and ψ_3^H:

- $\psi_1^H = a + \frac{c}{2}$
- $\psi_3^H = \sqrt{2}\left(\frac{b}{\pi}+\frac{c}{2\pi}\right)$

We notice that:

- $a \geq b \underbrace{\geq}_{\frac{\sqrt{2}}{\pi}<1\ ,b\geq 0} \frac{\sqrt{2}\cdot b}{\pi}$
- $\frac{c}{2} \underbrace{\geq}_{\frac{\sqrt{2}}{\pi}<1\ ,c\geq 0} \frac{\sqrt{2}c}{2\pi}$

Then we get:

$$\psi_1^H = a + \frac{c}{2} > \sqrt{2}\left(\frac{b}{\pi}+\frac{c}{2\pi}\right) = \psi_3^H$$

Overall we got: $\psi_1^H \geq \psi_3^H = |\psi_4^H| \geq \psi_2^H$

Therefore the best 1-term approximation of ϕ is: $\psi_1^H \cdot \psi_1^H(t)$

The best 2-term approximation of ϕ is: $\psi_1^H \cdot \psi_1^H(t) + \psi_3^H \cdot \psi_3^H(t)$

The best 3-term approximation of ϕ is: $\psi_1^H \cdot \psi_1^H(t) + \psi_3^H \cdot \psi_3^H(t) + \psi_4^H \cdot \psi_4^H(t)$

The best 4-term approximation of ϕ is: $\psi_1^H \cdot \psi_1^H(t) + \psi_3^H \cdot \psi_3^H(t) + \psi_4^H \cdot \psi_4^H(t) + \psi_2^H \cdot \psi_2^H(t)$

We notice that the best 3-term approximation of ϕ is the same as the 4-term, since $\psi_2^H \cdot \psi_2^H(t) = 0$.

2.a.v.

Now given the parameters we can plug them in the expressions above:

$$a = \frac{1}{\pi} \quad b = 1 \quad c = \frac{3}{2}$$

- $\psi_1^H = a + \frac{c}{2} = \frac{1}{\pi} + \frac{3}{4} \approx 1.06$
- $\psi_2^H = 0$
- $\psi_3^H = \sqrt{2}\left(\frac{b}{\pi}+\frac{c}{2\pi}\right) = \sqrt{2}\left(\frac{1}{\pi}+\frac{3}{4\pi}\right) \approx 0.78$
- $\psi_4^H = -\sqrt{2}\left(\frac{b}{\pi}+\frac{c}{2\pi}\right) = -\sqrt{2}\left(\frac{1}{\pi}+\frac{3}{4\pi}\right) \approx -0.78$

Even though now $a < b$ the order of the coefficients remained the same, thus the best k-approximation is the same as above.

2.b.i.

We will show that: $W_4 W_4^t = W_4^t W_4 = I_4$:

$$W_4 W_4^t = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I_4$$

$$W_4^t W_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I_4$$

2.b.ii.

as seen in class, the standard basis over $t \in [0,1]$ is given by:

$\left(\text{define: } \Delta_1 = \left[0, \frac{1}{4}\right) \text{ , } \Delta_2 = \left[\frac{1}{4}, \frac{1}{2}\right) \text{ , } \Delta_3 = \left[\frac{1}{2}, \frac{3}{4}\right) \text{ ,} \Delta_4 = \left[\frac{3}{4}\text{ ,} 1\right]\right)$

$$\begin{bmatrix} \sqrt{N} \cdot 1_{\Delta_1}(t) \\ \sqrt{N} \cdot 1_{\Delta_2}(t) \\ \sqrt{N} \cdot 1_{\Delta_3}(t) \\ \sqrt{N} \cdot 1_{\Delta_4}(t) \end{bmatrix}$$

For $N = 4$ we get:

$$\begin{bmatrix} 2 \cdot 1_{\Delta_1}(t) \\ 2 \cdot 1_{\Delta_2}(t) \\ 2 \cdot 1_{\Delta_3}(t) \\ 2 \cdot 1_{\Delta_4}(t) \end{bmatrix}$$

Now in order to get the Walsh-Hadamard functions, we will apply the Walsh-Hadamard matrix upon the standard basis:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \cdot 1_{\Delta_1}(t) \\ 2 \cdot 1_{\Delta_2}(t) \\ 2 \cdot 1_{\Delta_3}(t) \\ 2 \cdot 1_{\Delta_4}(t) \end{bmatrix} = \begin{bmatrix} \psi_1^W(t) \\ \psi_2^W(t) \\ \psi_3^W(t) \\ \psi_4^W(t) \end{bmatrix}$$

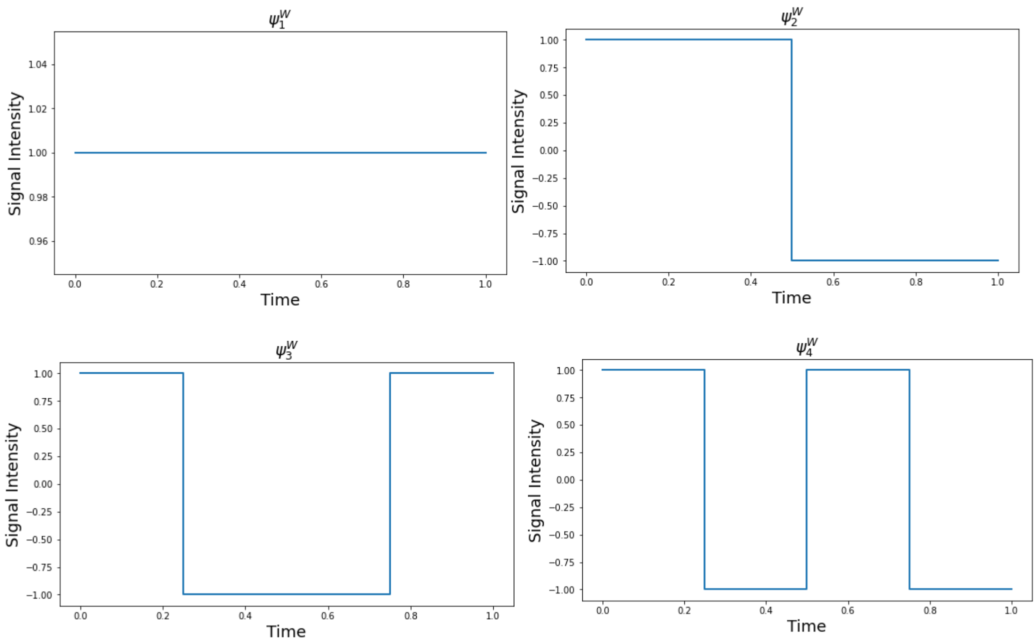
$$= \begin{bmatrix} 1_{\Delta_1}(t) + 1_{\Delta_2}(t) + 1_{\Delta_3}(t) + 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) + 1_{\Delta_2}(t) - 1_{\Delta_3}(t) - 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) - 1_{\Delta_2}(t) - 1_{\Delta_3}(t) + 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) - 1_{\Delta_2}(t) + 1_{\Delta_3}(t) - 1_{\Delta_4}(t) \end{bmatrix}$$

Therefore we get the following functions:

$$\psi_1^W(t) = 1 \qquad \psi_2^W(t) = \begin{cases} 1, t \in \left[0, \frac{1}{2}\right) \\ -1, t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\psi_3^W(t) = \begin{cases} 1, t \in \left[0, \frac{1}{4}\right) \\ -1, t \in \left[\frac{1}{4}, \frac{3}{4}\right) \\ 1, t \in \left[\frac{3}{4}, 1\right] \end{cases} \qquad \psi_4^W(t) = \begin{cases} 1\ , t \in \left[0, \frac{1}{4}\right) \\ -1, t \in \left[\frac{1}{4}, \frac{1}{2}\right) \\ 1\ , t \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ -1\ , t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

Now we can plot the Walsh-Hadamard functions:



2.b.iii.

Define the i_{th} coefficient of the approximation of $\phi(t)$ using Walsh-Hadamard basis:

$$\psi_i^W = \langle \phi(t), \psi_i^W(t) \rangle = \int_0^1 \phi(t) \cdot \psi_i^W(t)\, dt$$

As we learned in class, the best approximation of $\phi(t)$ using this Walsh-Hadamard basis is:

$$\hat{\phi}(t) = \sum_{i=1}^{4} \psi_i^W \cdot \psi_i^W(t)$$

And the associated MSE is:

$$\hat{\phi}_{MSE}(\phi - \psi^W) = \int_0^1 \phi^2(t)\,dt - \sum_{i=1}^{4} \left(\psi_i^W\right)^2$$

2.b.iv.

First, the signal is:

$$\phi(t) = a + b \cdot cos(2\pi t) + c \cdot \cos^2(\pi t)$$

Let us remind:

$$\int_0^1 \cos^2(\pi t) dt = \int_0^1 \frac{\cos(2\pi t) + 1}{2} dt = \frac{1}{2}\int_0^1 \cos(2\pi t)\,dt + \int_0^1 \frac{1}{2} dt = \left[\frac{\sin(2\pi t)}{4\pi} + \frac{1}{2}t\right]_0^1$$

Now we can calculate the coefficients:

$$\boldsymbol{\psi_1^W} = \left\langle \phi(t), \psi_1^W(t) \right\rangle = \int_0^1 \phi(t) dt = \int_0^1 (a + b \cdot cos(2\pi t) + c \cdot \cos^2(\pi t)) dt =$$

$$\left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_0^1 = \boldsymbol{a + \frac{c}{2}}$$

$$\boldsymbol{\psi_2^W} = \left\langle \phi(t), \psi_2^W(t) \right\rangle = \int_0^{\frac{1}{2}} \phi(t) dt - \int_{\frac{1}{2}}^1 \phi(t) dt =$$

$$\left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_0^{\frac{1}{2}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{2}}^1 =$$

$$= \left(\frac{a}{2} + \frac{c}{4}\right) - \left(\left(a + \frac{c}{2}\right) - \left(\frac{a}{2} + \frac{c}{4}\right)\right) = 2\left(\frac{a}{2} + \frac{c}{4}\right) - \left(a + \frac{c}{2}\right) = \mathbf{0}$$

$$\boldsymbol{\psi_3^W} = \langle \phi(t), \psi_3^W(t) \rangle = \int_0^{\frac{1}{4}} \phi(t)dt - \int_{\frac{1}{4}}^{\frac{3}{4}} \phi(t)dt + \int_{\frac{3}{4}}^{1} \phi(t)dt =$$

$$= \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_0^{\frac{1}{4}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{4}}^{\frac{3}{4}}$$

$$+ \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{3}{4}}^{1} =$$

$$= 2\left(\frac{a}{4} + \frac{b}{2\pi} + \frac{c}{4\pi} + \frac{c}{8}\right) - 2\left(\frac{3a}{4} - \frac{b}{2\pi} - \frac{c}{4\pi} + \frac{3c}{8}\right) + \left(a + \frac{c}{2}\right) = \boldsymbol{\frac{2b}{\pi} + \frac{c}{\pi}}$$

$$\boldsymbol{\psi_4^W} = \langle \phi(t), \psi_4^W(t) \rangle = \int_0^{\frac{1}{4}} \phi(t)dt - \int_{\frac{1}{4}}^{\frac{1}{2}} \phi(t)dt + \int_{\frac{1}{2}}^{\frac{3}{4}} \phi(t)dt - \int_{\frac{3}{4}}^{1} \phi(t)dt =$$

$$= \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_0^{\frac{1}{4}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{4}}^{\frac{1}{2}} +$$

$$+ \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{1}{2}}^{\frac{3}{4}} - \left[at + \frac{b}{2\pi} \cdot \sin(2\pi t) + c\frac{\sin(2\pi t)}{4\pi} + \frac{c}{2}t\right]_{\frac{3}{4}}^{1} =$$

$$2\left(\frac{a}{4} + \frac{b}{2\pi} + \frac{c}{4\pi} + \frac{c}{8}\right) - 2\left(\frac{a}{2} + \frac{c}{4}\right) + 2\left(\frac{3a}{4} - \frac{b}{2\pi} - \frac{c}{4\pi} + \frac{3c}{8}\right) - \left(a + \frac{c}{2}\right) = \mathbf{0}$$

The smallest absolute coefficients are $\psi_2^W = \psi_4^W = 0$

We will compare between the positive values of ψ_1^W and ψ_3^W:

- $\psi_1^W = a + \frac{c}{2}$
- $\psi_3^W = \frac{2b}{\pi} + \frac{c}{\pi}$

We notice that:

- $a \geq b \underbrace{\geq}_{\frac{2}{\pi} < 1, b \geq 0} \frac{2b}{\pi}$
- $\frac{c}{2} \underbrace{\geq}_{\pi > 2, c \geq 0} \frac{c}{\pi}$

Then we get:

$$\psi_1^W = a + \frac{c}{2} > \frac{2b}{\pi} + \frac{c}{\pi} = \psi_3^W$$

Overall we got: $\psi_1^W \geq \psi_3^W \geq \psi_4^W = \psi_2^W$

Therefore the best 1-term approximation of ϕ is: $\psi_1^W \cdot \psi_1^W(t)$

The best 2-term approximation of ϕ is: $\psi_1^W \cdot \psi_1^W(t) + \psi_3^W \cdot \psi_3^W(t)$

The best 3-term approximation of ϕ is: $\psi_1^W \cdot \psi_1^W(t) + \psi_3^W \cdot \psi_3^W(t) + \psi_4^W \cdot \psi_4^W(t)$

The best 4-term approximation of ϕ is: $\psi_1^W \cdot \psi_1^W(t) + \psi_3^W \cdot \psi_3^W(t) + \psi_4^W \cdot \psi_4^W(t) + \psi_2^W \cdot \psi_2^W(t)$

We notice that the best 2-term approximation of ϕ is the same as the 3-term and 4-term, since $\psi_2^W \cdot \psi_2^W(t) = \psi_4^W \cdot \psi_4^W(t) = 0$.

2.b.v.

Now given the parameters we can plug them in the expressions above:

$$a = \frac{1}{\pi} \quad b = 1 \quad c = \frac{3}{2}$$

- $\psi_1^W = a + \frac{c}{2} = \frac{1}{\pi} + \frac{3}{4} \approx 1.06$
- $\psi_2^W = 0$
- $\psi_3^W = \frac{2b}{\pi} + \frac{c}{\pi} = \frac{2}{\pi} + \frac{3}{2\pi} \approx 1.11$
- $\psi_4^W = 0$

Now the coefficients order is: $\psi_3^W \geq \psi_1^W \geq \psi_4^W = \psi_2^W$

The only thing that changes is that the best 1-term approximation of ϕ is: $\psi_3^W \cdot \psi_3^W(t)$

The rest is the same as above.

3.a.

Given a $N = 2^n$ dimension Hadamard matrix, we wish to prove the following:

H_N is symmetric, real, unitary and can be written as $H_N = \lambda_N A$ - where $\lambda_N = \frac{1}{\sqrt{N}} \in \mathbb{R}$, A is a matrix with only ± 1 entries.

We will use induction to prove the above:

<u>Base case</u> $(n = 1)$:

$$N = 2^1 = 2$$

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

As we can see H_2 is symmetric, real, and if we define $\lambda_2 = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{N}} \in \mathbb{R}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ we can write: $H_2 = \lambda_2 A$.

H_2 is unitary: $H_2^T H_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I_2$

Inductive step:

assuming all the above hold for $1 \leq k < N$, we now prove for N:

H_N is symmetric:

$$H_N^T = H_{2^n}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}}^T & H_{2^{n-1}}^T \\ H_{2^{n-1}}^T & -H_{2^{n-1}}^T \end{bmatrix} =$$

According to the inductive assumption: $H_{2^{n-1}}^T = H_{2^{n-1}}$ since $H_{2^{n-1}}$ is symmetric, then:

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} = H_{2^n} = H_{\mathrm{N}}$$

H_N is unitary:

$$H_N^T H_N = H_{2^n}^T H_{2^n} = \frac{1}{2}\begin{bmatrix} H_{2^{n-1}}^T & H_{2^{n-1}}^T \\ H_{2^{n-1}}^T & -H_{2^{n-1}}^T \end{bmatrix}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} 2H_{2^{n-1}}^T H_{2^{n-1}} & 0 \\ 0 & 2H_{2^{n-1}}^T H_{2^{n-1}} \end{bmatrix} =$$

According to the inductive assumption: $H_{2^{n-1}}^T H_{2^{n-1}} = I_{2^{n-1}}\ since\ H_{2^{n-1}}$ is unitary, then:

$$= \frac{1}{2}\begin{bmatrix} 2I_{2^{n-1}} & 0 \\ 0 & 2I_{2^{n-1}} \end{bmatrix} = I_N$$

H_N is real:

$$H_{\mathrm{N}} = H_{2^n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}$$

According to the inductive assumption: $H_{2^{n-1}}$ is real, therefore H_{N} is real as well, as a real blocks matrix.

H_N can be written as $H_N = \lambda_N A$ where $\lambda_N \in \mathbb{R}$, A is a matrix with only ± 1 entries:

$$H_N = H_{2^n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} =$$

According to the inductive assumption: $H_{2^{n-1}}$ can be written as $H_{2^{n-1}} = \lambda_{2^{n-1}}\tilde{A}$ where $\lambda_{2^{n-1}} = \frac{1}{\sqrt{2^{n-1}}} \in \mathbb{R}$, and $\tilde{A}$ is a matrix with only ± 1 entries, then:

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \lambda_{2^{n-1}}\tilde{A} & \lambda_{2^{n-1}}\tilde{A} \\ \lambda_{2^{n-1}}\tilde{A} & -\lambda_{2^{n-1}}\tilde{A} \end{bmatrix} = \frac{\lambda_{2^{n-1}}}{\sqrt{2}}\begin{bmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & -\tilde{A} \end{bmatrix} = \frac{1}{\sqrt{2^{n-1}}\sqrt{2}}\begin{bmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & -\tilde{A} \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2^n}}\begin{bmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & -\tilde{A} \end{bmatrix} = \frac{1}{\sqrt{N}}\begin{bmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & -\tilde{A} \end{bmatrix}$$

Since each $\tilde{A}$ is a matrix with only ± 1 entries, we can define $A = \begin{bmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & -\tilde{A} \end{bmatrix}$ a matrix with only ± 1 entries, and $\boldsymbol{\lambda_N} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{N}}}$, then: $H_N = \lambda_N A$ ■

3.b.i.

For two sequences s_1, s_2 of digits ± 1, define: $I(s_1, s_2) = \begin{cases} 1, & s_1[Last] \neq s_2[First] \\ 0, & else \end{cases}$

Then:

$$S(s_1, s_2) = S(s_1) + S(s_2) + I(s_1, s_2)$$

3.b.ii.

Claim 1:

For two sequences s_1, s_2 of digits ± 1, **both start with 1**:

$$I(s_1 s_2) = \begin{cases} 1, & S(s_1) \ is \ odd \\ 0, & else \end{cases}$$

Proof:

- $S(s_1)$ is odd $\Longrightarrow s_1$ starts with 1 so it must end with (-1)

  Also we know that s_2 starts with 1,

  Therefore $s_1[Last] = -1 \neq 1 = s_2[First]$

  By definition of I in 3.b.i. - $I(s_1 s_2) = 1$

- $S(s_1)$ is even $\Longrightarrow s_1$ starts with 1 so it must end with 1
  Also we know that s_2 starts with 1,
  Therefore $s_1[Last] = 1 = s_2[First]$
  By definition of I in 3.b.i. - $I(s_1 s_2) = 0$

Claim 2:

For two sequences s_1, s_2 of digits ± 1, **both start with 1**:

$$I(s_1 \overline{s_2}) = \begin{cases} 1, & S(s_1) \ is \ even \\ 0, & else \end{cases}$$

When $\bar{s}[i] = -s[i]$

Proof:

- $S(s_1)$ is even $\Longrightarrow s_1$ starts with 1 so it must end with 1

  Also we know that s_2 starts with 1, then $\overline{s_2}$ starts with (-1)

  Therefore $s_1[Last] = 1 \neq -1 = \overline{s_2}[First]$

  By definition of I in 3.b.i. - $I(s_1 \overline{s_2}) = 1$

- $S(s_1)$ is odd $\Longrightarrow s_1$ starts with 1 so it must end with (-1)
  Also we know that s_2 starts with 1, then $\overline{s_2}$ starts with (-1)
  Therefore $s_1[Last] = -1 = \overline{s_2}[First]$
  By definition of I in 3.b.i. - $I(s_1 \overline{s_2}) = 0$

Claim 3:

In Hadamard matrix H_{2^n}, $\forall i$: r_i starts with 1, for all n.

Proof by induction:

Base case ($n = 1$):

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

we can see that every row starts with 1.

Inductive step:

Assume that for H_{2^n} all rows start with 1.

We will show that all rows of $H_{2^{n+1}}$ start with 1 as well:

$$H_{2^{n+1}} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

Every rows of $H_{2^{n+1}}$ start with the same digits H_{2^n} start with, therefore all rows of $H_{2^{n+1}}$ start with 1.

Claim 4:

We will show that $\{S(r_1), S(r_2), \dots, S(r_N)\} = \{0,1, \dots, N-1\}$.

Proof by induction:

Base case ($n = 1$):

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S(r_1) = 0$$

$$S(r_2) = 1$$

Therefore, $\{S(r_1), S(r_2)\} = \{0,1\}$

Inductive step:

Assume that $\{S(r_1), \dots, S(r_{2^n})\} = \{0, \dots, 2^n - 1\}$

We will show that $\{S(r_1), \dots, S(r_{2^{n+1}})\} = \{0, \dots, 2^{n+1} - 1\}$

Observing the first 2^n rows of $H_{2^{n+1}}$:

$$\begin{bmatrix} H_{2^n} & H_{2^n} \end{bmatrix}$$

The sequences will be (not necessarily by that order):

$$\begin{cases} s_0 & 0+0+I(s_{r_0}s_{r_0}) = 0+0+0 = 0 \\ s_1 & 1+1+I(s_{r_1}s_{r_1}) = 1+1+1 = 3 \\ s_2 & 2+2+I(s_{r_2}s_{r_2}) = 2+2+0 = 4 \\ \vdots & \vdots \\ \vdots & \vdots \\ s_{2^n-1} & (2^n-1)+(2^n-1)+I\left(s_{r_{2^n-1}}s_{r_{2^n-1}}\right) = (2^n-1)+(2^n-1)+1 = 2^{n+1}-1 \end{cases}$$

when s_{r_i} is the sequence in H_{2^n} rows, with i sign changes.

And observing the last 2^n rows of $H_{2^{n+1}}$:

$$[H_{2^n} \quad -H_{2^n}]$$

The sequences will be (not necessarily by that order):

$$\begin{cases} \widetilde{s_0} & 0+0+I(s_{r_0}\overline{s_{r_0}}) = 0+0+1 = 1 \\ \widetilde{s_1} & 1+1+I(s_{r_1}\overline{s_{r_1}}) = 1+1+0 = 2 \\ \widetilde{s_2} & 2+2+I(s_{r_2}\overline{s_{r_2}}) = 2+2+1 = 5 \\ \vdots & \vdots \\ \vdots & \vdots \\ \widetilde{s_{2^n-1}} & (2^n-1)+(2^n-1)+I\left(s_{r_{2^n-1}}\overline{s_{r_{2^n-1}}}\right) = (2^n-1)+(2^n-1)+0 = 2^{n+1}-2 \end{cases}$$

when s_{r_i} is the sequence in H_{2^n} rows, with i sign changes.

Overall we see that $\{s_i, \widetilde{s_i}\} = \{2p, 2p+1\}$ where p is running over the values $\{0,..,2^n-1\}$

$Hence, \{s_0, \dots, s_{2^n-1}, \widetilde{s_0}, \dots, \widetilde{s_{2^n-1}}\} = \{0,1,\dots,2^n-1,\dots,2^{n+1}-2,2^{n+1}-1\}$ ■

4.a.

For $N=1$, H_{2N} is symmetric since $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H_2^T$

For $N>1$, H_{2N} is not symmetric:

For $N=2$:

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = H_4^T$$

For general $N>1$:

If we look at the most lower right part of H_{2N}, we see that:

$$\begin{bmatrix} H_{2N_{2N-1,2N-1}} & H_{2N_{2N-1,2N}} \\ H_{2N_{2N,2N-1}} & H_{2N_{2N,2N}} \end{bmatrix} = \begin{bmatrix} I_{N_{(N-1,N)}} \otimes (1,-1) \\ I_{N_{(N,N)}} \otimes (1,-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

And it is independent of previous N, only in the unit matrix behavior.

Since $\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix} \neq \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}^{transpose} = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}$

We get that H_N is not symmetric (else every lower right square sub-matrix of it would be symmetric by itself).

4.b.

$\boldsymbol{H_{2N}}$ **is orthogonal for all $\boldsymbol{N \geq 1}$** (with rows/columns not normalized, it's not orthonormal – and since we deal with real numbers, it is also not unitary).

Meaning $H_{2N}H_{2N}^T = Diag_{2Nx2N}$

Proof using induction:

Base (N=1):

$$H_2 H_2^T = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

Inductive step:

Assume $H_N H_N^T = Diag_{N\times N}$, we will prove that $H_{2N}H_{2N}^T = Diag_{2Nx2N}$:

$$\boldsymbol{H_{2N}H_{2N}^T} =$$

$$= \begin{bmatrix} H_N \otimes (1,1) \\ I_N \otimes (1,-1)\end{bmatrix}\begin{bmatrix} H_N \otimes (1,1) \\ I_N \otimes (1,-1)\end{bmatrix}^T =$$

$$= \begin{bmatrix} H_N \otimes (1,1) \\ I_N \otimes (1,-1)\end{bmatrix}\begin{bmatrix} H_N^T \otimes \begin{pmatrix}1\\1\end{pmatrix} & I_N^T \otimes \begin{pmatrix}1\\-1\end{pmatrix}\end{bmatrix} =$$

$$= \begin{bmatrix} [H_N \otimes (1,1)]\left[H_N^T \otimes \begin{pmatrix}1\\1\end{pmatrix}\right] & [H_N \otimes (1,1)]\left[I_N^T \otimes \begin{pmatrix}1\\-1\end{pmatrix}\right] \\ [I_N \otimes (1,-1)]\left[H_N^T \otimes \begin{pmatrix}1\\1\end{pmatrix}\right] & [I_N \otimes (1,-1)]\left[I_N^T \otimes \begin{pmatrix}1\\-1\end{pmatrix}\right]\end{bmatrix} \underbrace{=}_{\substack{(A\otimes B)(C\otimes D)\\=(AC\otimes BD)}}$$

$$= \begin{bmatrix} [H_N H_N^T \otimes (2)] & \left[H_{2(N-1)} I_{2(N-1)}^T \otimes (0)\right] \\ [I_N H_N^T \otimes (0)] & \left[I_{2(N-1)} I_{2(N-1)}^T \otimes (2)\right]\end{bmatrix} \underbrace{=}_{\substack{inductive\\hypothesis}}$$

$$= \begin{bmatrix} Diag_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & 2I_{N\times N}\end{bmatrix} =$$

$$= \boldsymbol{Diag_{2N\times 2N}}$$

4.c.

We offer the following scaling method:

$$\widetilde{H}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\widetilde{H}_{2N} = \frac{1}{\sqrt{2}}\begin{bmatrix} \widetilde{H}_N \otimes (1,1) \\ I_N \otimes (1,-1) \end{bmatrix}$$

Using this scaling, we get the Haar matrix of general size $2N$ as we saw in class, and we proved it is unitary.

4.d.

$$[A \otimes B]^T \underbrace{=}_{\substack{Kronecker \\ product \\ definition}}$$

$$= \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,n}B \end{bmatrix}^T =$$

$$= \begin{bmatrix} a_{1,1}^T B^T & a_{2,1}^T B^T & \cdots & a_{n,1}^T B^T \\ a_{1,2}^T B^T & a_{2,2}^T B^T & \cdots & a_{n,2}^T B^T \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n}^T B^T & a_{2,n}^T B^T & \cdots & a_{n,n}^T B^T \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,1}B^T & a_{2,1}B^T & \cdots & a_{n,1}B^T \\ a_{1,2}B^T & a_{2,2}B^T & \cdots & a_{n,2}B^T \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n}B^T & a_2, nB^T & \cdots & a_{n,n}B^T \end{bmatrix} \underbrace{=}_{\substack{Kronecker \\ product \\ definition}}$$

$$= A^T \otimes B^T$$

4.e.

We offer the following recursive definition for $\widetilde{H}_{2N}^T$ Using the property of 4.d:

$$\widehat{H}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\widehat{H}_{2N} = \frac{1}{\sqrt{2}}\left[\widehat{H}_{2N} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad I_{2N} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

Implementation

1.a.

$$\phi(x,y) = A\,cos(2\pi\omega_x x)sin(2\pi\omega_y y) \qquad for\ (x,y) \in [0,1] \times [0,1]$$

$$A = 2500, \quad \omega_x = 2, \quad \omega_y = 7$$

value range:

$\forall x, y \in [0,1]X[0,1] : -1 \leq cos(2\pi w_x x) \leq 1, -1 \leq sin(2\pi w_y y) \leq 1$
Then: $-A \leq \phi(x,y) \leq A$

horizontal derivative energy:

$\frac{\partial\phi}{\partial x} = -2\pi w_x A sin(2\pi w_x x) sin(2\pi w_y y)$

$\int_0^1 \int_0^1 (\frac{\partial\phi}{\partial x})^2 dxdy =$

$= \int_0^1 \int_0^1 4\pi^2 w_x^2 A^2 sin^2(2\pi d w_x x) sin^2(2\pi w_y y) dxdy =$

$\int_0^1 4\pi^2 w_x^2 A^2 sin^2(2\pi w_y y) \int_0^1 \frac{1}{2}(1 - cos(4\pi w_x x))dxdx =$

$\int_0^1 2\pi^2 w_x^2 A^2 sin^2(2\pi w_y y)(x - \frac{sin(4\pi w_x x)}{4\pi w_x}|_0^1)dy =$

$\int_0^1 2\pi^2 w_x^2 A^2 sin^2(2\pi w_y y)(1 - \frac{sin(4\pi w_x)}{4\pi w_x})dy =$

$\pi^2 w_x^2 A^2 (1 - \frac{sin(4\pi w_x)}{4\pi w_x})(1 - \frac{sin(4\pi w_y)}{4\pi w_y}) = \pi^2 \cdot 4 \cdot 2500^2$

vertical derivative energy:

$\frac{\partial\phi}{\partial y} = 2\pi w_y A cos(2\pi w_x x) cos(2\pi w_y y)$

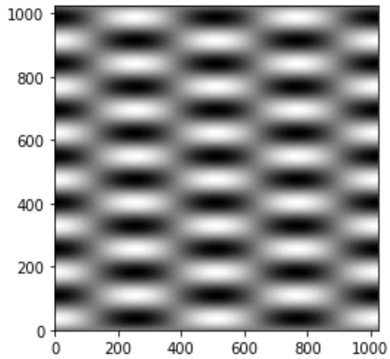
$\int_0^1 \int_0^1 (\frac{\partial\phi}{\partial y})^2 dydx =$

$4\pi^2 w_y^2 A^2 \int_0^1 \int_0^1 cos^2(2\pi d w_x x) \cdot \frac{1}{2}(cos(4\pi w_y y) + 1)dydx =$

$2\pi^2 w_y^2 A^2 \int_0^1 cos^2(2\pi d w_x x)(1 + \frac{sin(4\pi w_y)}{4\pi w_y})dx =$

$\pi^2 w_y^2 A^2 (1 + \frac{sin(4\pi w_y)}{4\pi w_y})(1 + \frac{sin(4\pi w_x)}{4\pi w_x}) = 49 \cdot 2500^2 \pi^2$

1.b.



1.c.

We got the following output in the notebook:

Phi value-range is: [-2490.2314393881684 , 2509.762666363762]

The horizontal-derivative energy of Phi is: 246261562.39850235

The vertical-derivative energy of Phi is: 3028459379.449531

First we see that the value-range is close to the range we stated before:

$[-A, A] = [-2500\,, 2500]$

The relative deviation of the numerical horizontal derivative energy compared to the analytical horizontal derivative energy we calculated before is:

$$\frac{(4\pi^2 2500^2 - 246261562.39850235)}{4\pi^2 2500^2} = 0.0019$$

$\Longrightarrow$ the numerical horizontal derivative energy is close to the analytical.

The relative deviation of the numerical vertical derivative energy compared to the analytical vertical derivative energy we calculated before is:

$$\frac{(49\pi^2 2500^2 - 3028459379.449531)}{49\pi^2 2500^2} = -0.0019$$

$\Longrightarrow$ the numerical vertical derivative energy is close to the analytical.

1.d.

As we saw in the tutorial, the MSE approximation for a high resolution digitization is given by:

$$MSE(N_x, N_y, b) \approx \frac{1}{12N_x^2}En_x + \frac{1}{12N_y^2}En_y + \frac{1}{12}\frac{R_\phi^2}{2^{2b}}$$

When:

$$En_x = \int_0^1 \int_0^1 \left(\frac{\partial \phi}{\partial x}\right)^2 dxdy$$

$$En_y = \int_0^1 \int_0^1 \left(\frac{\partial \phi}{\partial y}\right)^2 dxdy$$

$$R_\phi = \phi_H - \phi_L$$

We wish to quantize the optimal samples in order to get the best approximation.

We allocate c bits for sampling ($N_x \cdot N_y = c$) and b bits for quantizing, when $b = \frac{B}{c}$

Now, in order to distribute optimally c bits between N_x and N_y we will minimize the sampling MSE, without respect to b:

$$MSE(N_x, N_y) \approx \frac{1}{12N_x^2}En_x + \frac{1}{12N_y^2}En_y \underbrace{=}_{N_x=\frac{c}{N_y}} \frac{\mathrm{N_y^2}}{12\mathrm{c}^2}En_x + \frac{1}{12N_y^2}En_y$$

Find optimum by taking derivative:

$$\frac{\partial MSE}{\partial N_y} = \frac{2N_y}{12c^2}En_x - \frac{2}{12N_y^3}En_y = 0$$

$$\Rightarrow \frac{2N_y}{12c^2}En_x = \frac{2}{12N_y^3}En_y \Rightarrow N_y^4 = c^2\frac{En_y}{En_x} \underbrace{\Rightarrow}_{c=N_xN_y} N_y^4 = N_x^2N_y^2\frac{En_y}{En_x} \Rightarrow N_y = N_x\sqrt{\frac{En_y}{En_x}}$$

Now we will consider b again in order to find the optimal sampling:

$$MSE(N_x, N_y, b) \approx \frac{1}{12N_x^2}En_x + \frac{1}{12N_x^2}\frac{En_x}{En_y}En_y + \frac{1}{12}\frac{R_\phi^2}{2^{2b}} = \frac{1}{6N_x^2}En_x + \frac{1}{12}\frac{R_\phi^2}{2^{2b}}$$

As we mentioned before, we use the connection: $b = \frac{B}{c}$ which gives us: $B = N_xN_yb$, then:

$$B = N_xN_x\sqrt{\frac{En_y}{En_x}}b \;\Rightarrow N_x^2 = \frac{B}{b}\sqrt{\frac{En_x}{En_y}}$$

Finally we plug N_x^2 in the MSE expression above in order to get a function of b:

$$MSE(N_x, N_y, b) \approx \frac{\mathrm{b}}{6\mathrm{B}}\sqrt{En_x En_y} + \frac{1}{12}\frac{R_\phi^2}{2^{2b}}$$

Now we will find the optimal b by taking derivative:

$$\frac{\partial MSE}{\partial b} = \frac{1}{6B}\sqrt{En_x En_y} + \frac{R_\phi^2}{12}\left(-\frac{2}{2^{2b}}\ln(2)\right) = 0$$

$$\Rightarrow \frac{1}{6B}\sqrt{En_x En_y} = \frac{R_\phi^2}{6 \cdot 2^{2b}}\ln(2) \Rightarrow 2^{2b} = \frac{R_\phi^2 \ln(2) B}{\sqrt{En_x En_y}}$$

$$\Rightarrow \log_2(2^{2b}) = \log_2\left(\frac{R_\phi^2 \ln(2) B}{\sqrt{En_x En_y}}\right)$$

$$\Rightarrow b = \frac{\log_2\left(\frac{B R_\phi^2 \ln(2)}{\sqrt{En_x En_y}}\right)}{2}$$

1.e.

The optimal **low** budget $N_x = 21 \;\; N_y = 73 \;\; b = 3 \; (B_{low} = 5{,}000)$

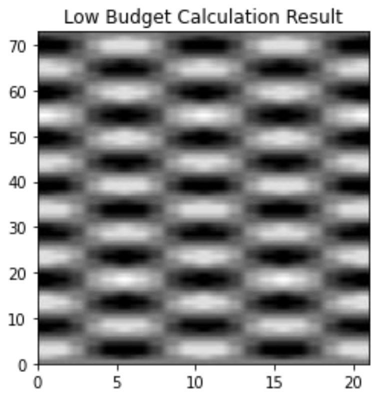
The optimal **high** budget $N_x = 53 \;\; N_y = 188 \;\; b = 5 \; (B_{high} = 50{,}000)$

1.g.

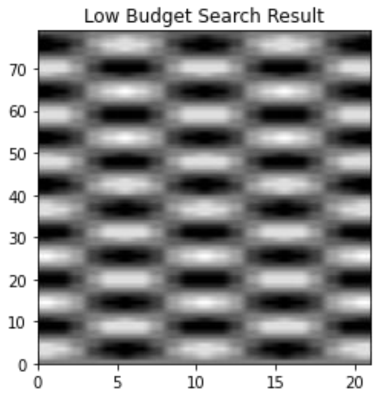
The optimal search low budget $N_x = 21 \;\; N_y = 79 \;\; b = 3$

The optimal search high budget $N_x = 54 \;\; N_y = 185 \;\; b = 5$

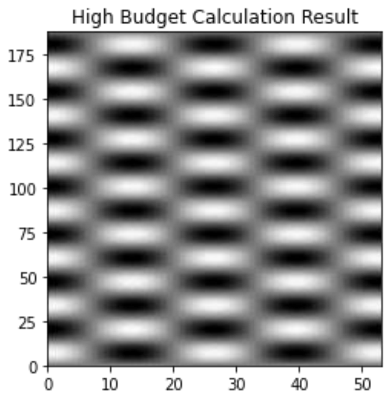
The image using the calculated low budget values:



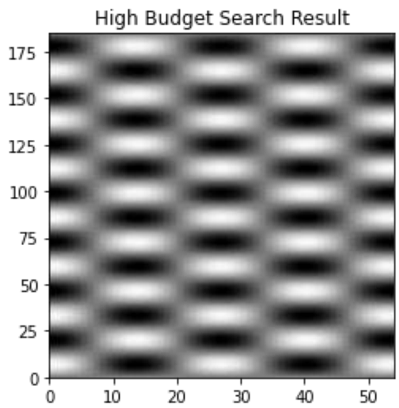
The image using the search procedure low budget values:



The image using the calculated high budget values:



The image using the search procedure high budget values:



Generally we see that the values we get from the numerical procedure and the values we get from the search, are pretty similar (as well as the images). We do notice a small minor difference between the two sets of values, we assume it could come as result of some cases:

- In the numerical calculation, we take into account the energy values which are much bigger compared to the other parameters in this calculation. Mixing big and small values and performing mathematical operations all together might lead to numerical errors.
- In the numerical calculation, in case of optimal result not being an integer we round it, whereas the rounded integer might not be the optimal integer. In the search algorithm it is not the case since it strictly searches for optimal integer.

1.h.

Repeat 1.a.

After the changes, the value-range remain the same, and the vertical derivative energy swaps values with the horizontal derivative energy:

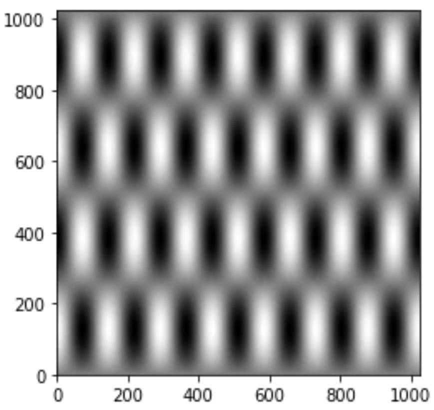
value range:

$\forall x, y \in [0,1]X[0,1] : -1 \leq cos(2\pi w_x x) \leq 1, -1 \leq sin(2\pi w_y y) \leq 1$
Then: $-A \leq \phi(x,y) \leq A$

horizontal derivative energy: $49\pi^2 2500^2$

vertical derivative energy: $4\pi^2 2500^2$

Repeat 1.b.



Repeat 1.c.

We got the following output in the notebook:

Phi value-range is: [-2490.2314393881684 , 2509.762666363762]

The horizontal-derivative energy of Phi is: 3016704139.3816543

The vertical-derivative energy of Phi is: 247221173.83261475

First we see that the value-range is close to the range we stated before:

$[-A, A] = [-2500\,, 2500]$

The relative deviation of the numerical horizontal derivative energy compared to the analytical horizontal derivative energy we calculated before is:

$$\frac{(49\pi^2 2500^2 - 3016704139.3816543)}{49\pi^2 2500^2} = 0.0019$$

$\Longrightarrow$ the numerical horizontal derivative energy is close to the analytical.

The relative deviation of the numerical vertical derivative energy compared to the analytical vertical derivative energy we calculated before is:

$$\frac{(4\pi^2 2500^2 - 247221173.83261475)}{4\pi^2 2500^2} = -0.0019$$

$\Longrightarrow$ the numerical vertical derivative energy is close to the analytical.

Repeat 1.d.

Implemented

Repeat 1.e.

The optimal **low** budget $N_x \;=\; 72 \;\; N_y \;=\; 21 \;\; b \;=\; 3 \; (B_{low} = 5{,}000)$

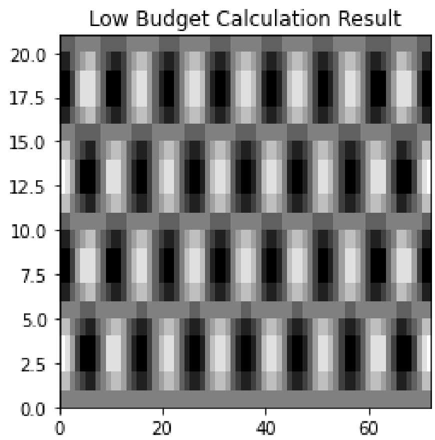
The optimal **high** budget $N_x \;=\; 187 \;\; N_y \;=\; 53 \;\; b \;=\; 4 \; (B_{high} = 50{,}000)$

Repeat 1.g.

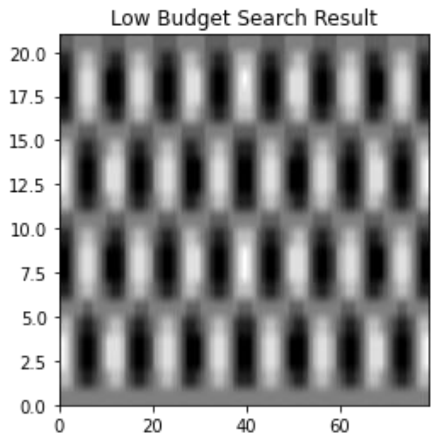
The optimal search low budget $N_x = 79$ $N_y = 21$ $b = 3$

The optimal search high budget $N_x = 185$ $N_y = 54$ $b = 5$

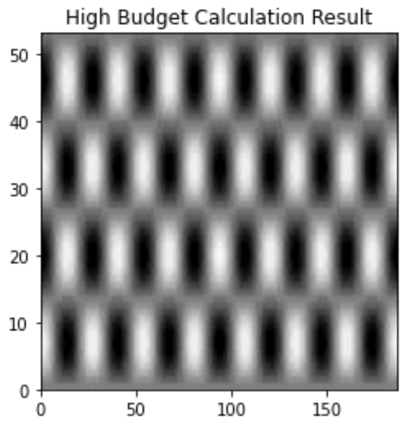
The image using the calculated low budget values:



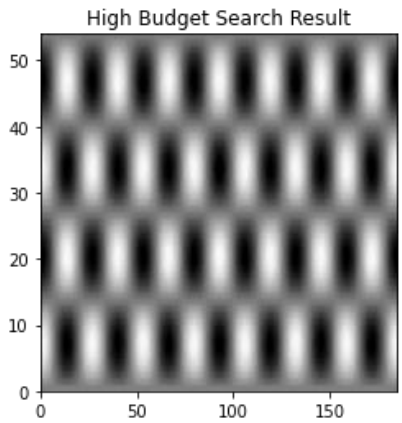
The image using the search low budget values:



The image using the calculated high budget values:



The image using the search high budget values:



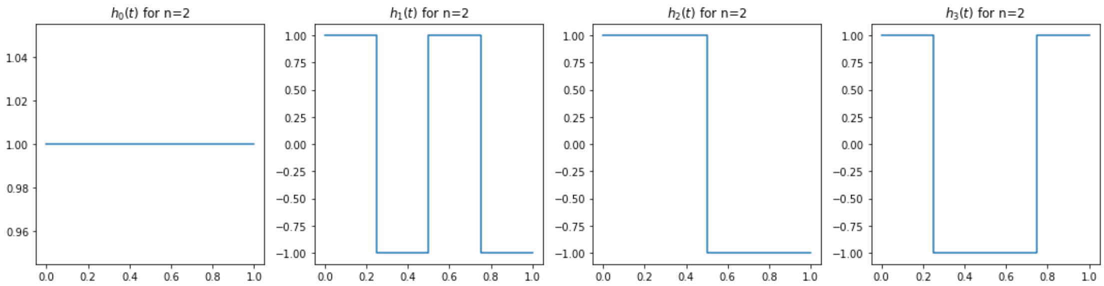
As we saw before, the values we calculated numerically and the values we found using the search algorithm, are pretty similar.
This time the difference between the images from the calculated values and the images from the searched values is bigger (the image formed from the values found using the search algorithm is smoother – which satisfies our previous claim that it may be more accurate and a better approximation).

2.a.

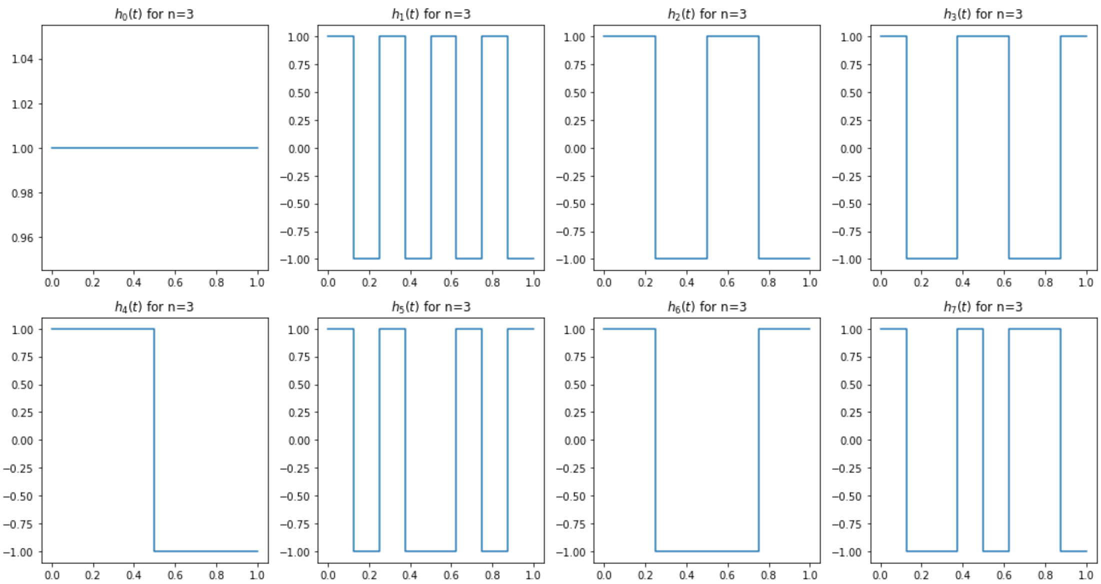
Implemented in the notebook.

2.b.

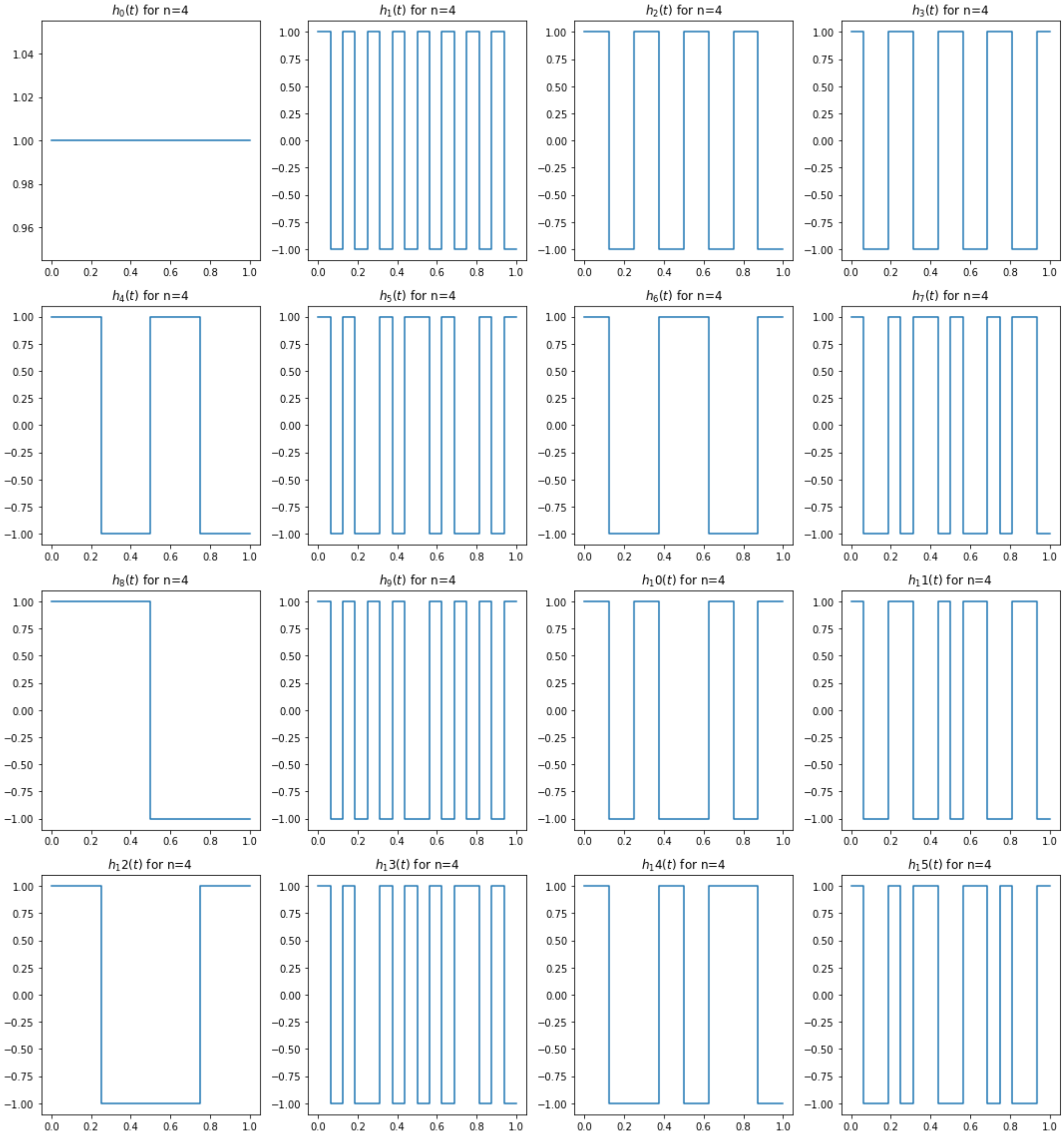
<u>N=2:</u>



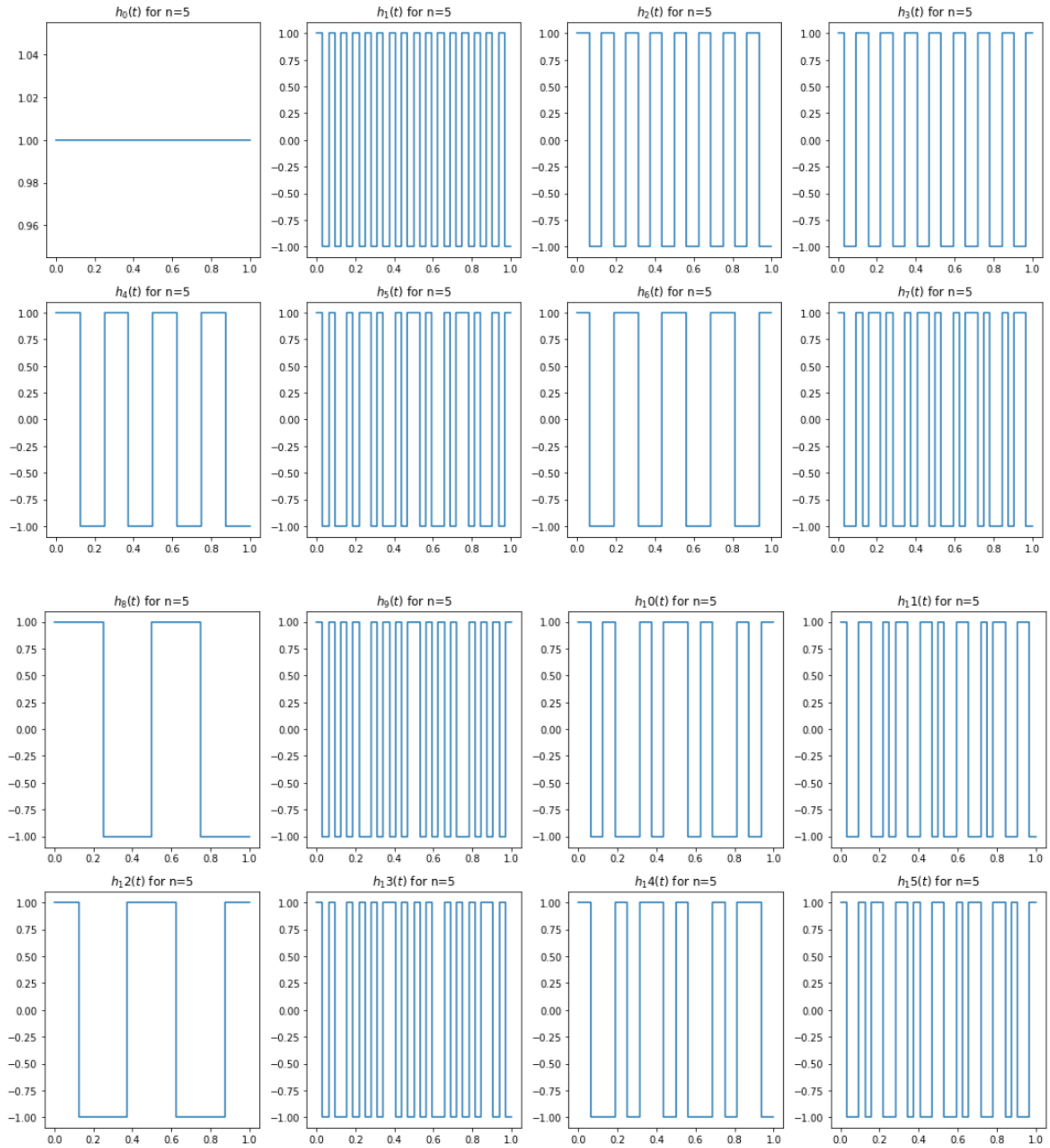
<u>N=3:</u>



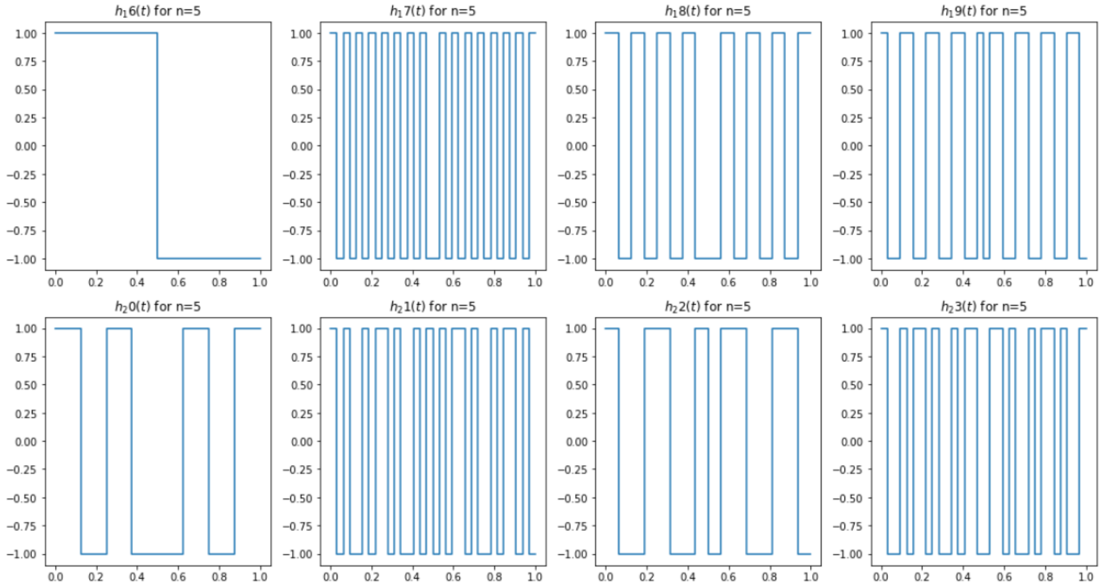
<u>N=4:</u>

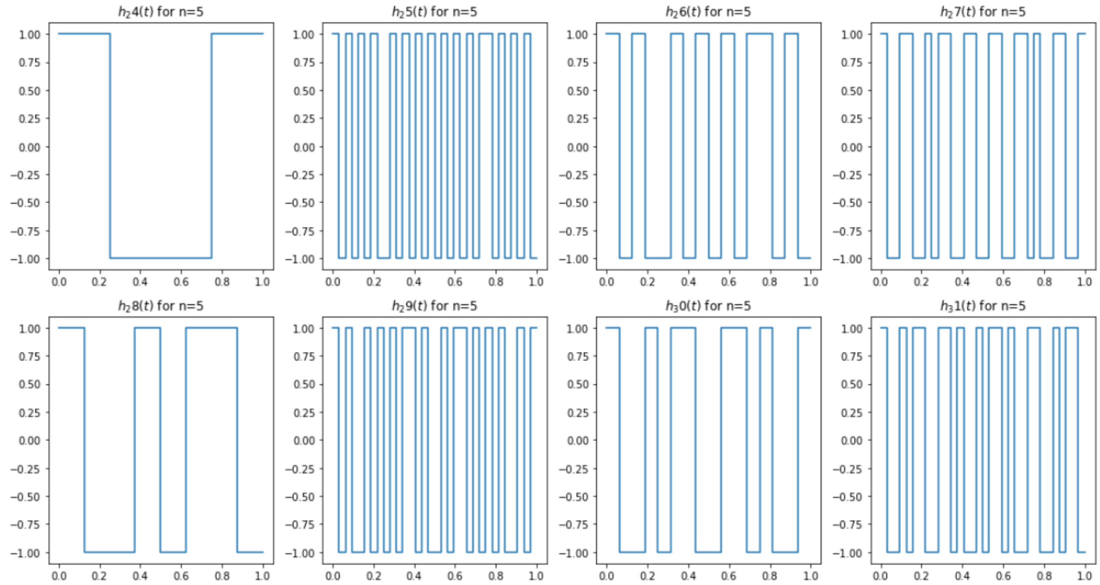

$h_0(t)$ for n=4
$h_1(t)$ for n=4
$h_2(t)$ for n=4
$h_3(t)$ for n=4
$h_4(t)$ for n=4
$h_5(t)$ for n=4
$h_6(t)$ for n=4
$h_7(t)$ for n=4
$h_8(t)$ for n=4
$h_9(t)$ for n=4
$h_1 0(t)$ for n=4
$h_1 1(t)$ for n=4
$h_1 2(t)$ for n=4
$h_1 3(t)$ for n=4
$h_1 4(t)$ for n=4
$h_1 5(t)$ for n=4

N=5:

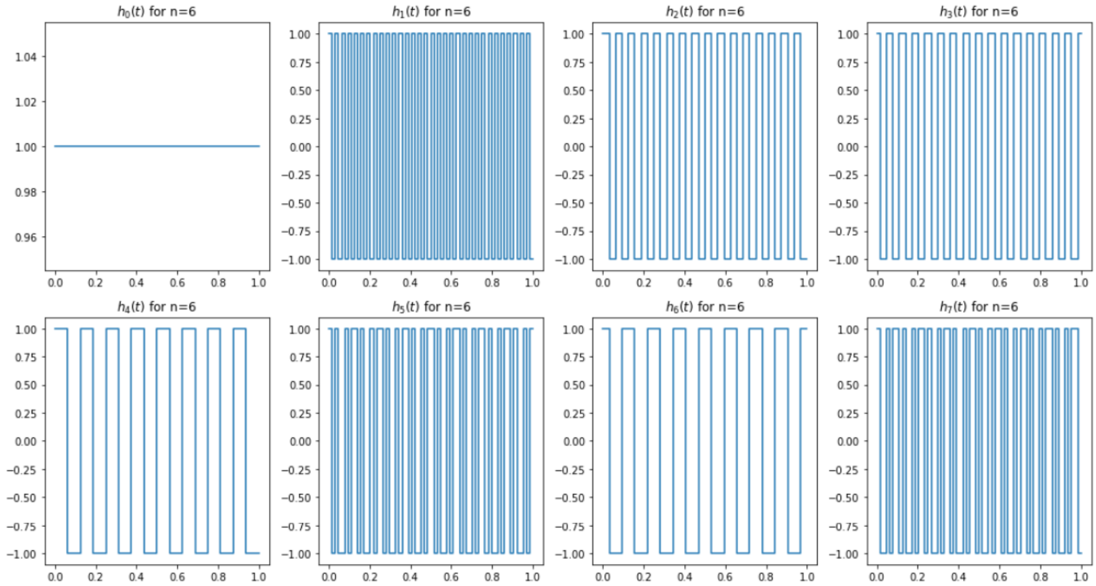


h0(t) for n=5
h1(t) for n=5
h2(t) for n=5
h3(t) for n=5
h4(t) for n=5
h5(t) for n=5
h6(t) for n=5
h7(t) for n=5
h8(t) for n=5
h9(t) for n=5
h10(t) for n=5
h11(t) for n=5
h12(t) for n=5
h13(t) for n=5
h14(t) for n=5
h15(t) for n=5

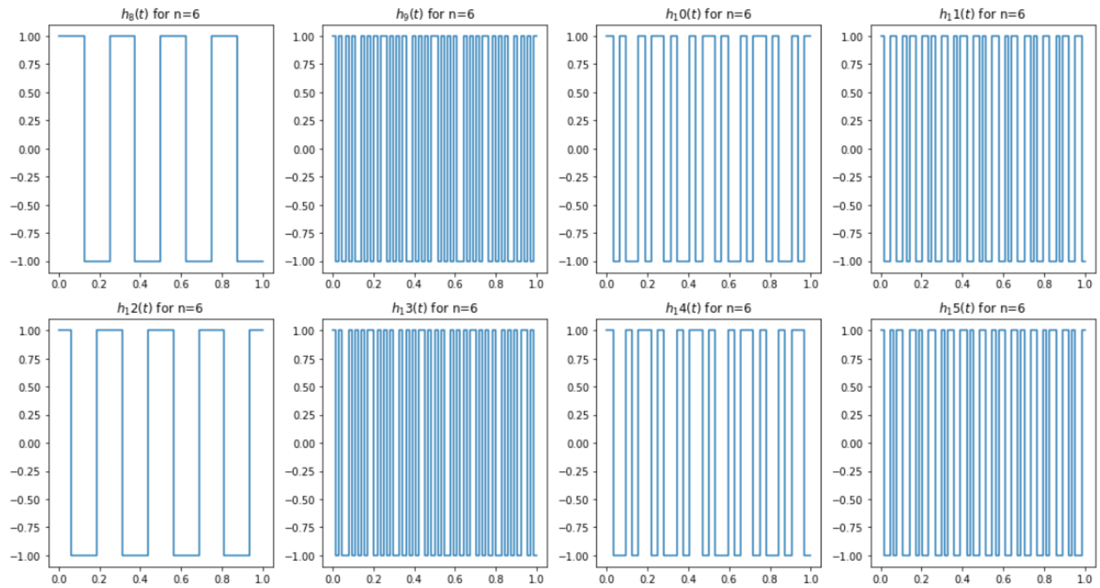

$h_1 6(t)$ for n=5
$h_1 7(t)$ for n=5
$h_1 8(t)$ for n=5
$h_1 9(t)$ for n=5
$h_2 0(t)$ for n=5
$h_2 1(t)$ for n=5
$h_2 2(t)$ for n=5
$h_2 3(t)$ for n=5


$h_2 4(t)$ for n=5
$h_2 5(t)$ for n=5
$h_2 6(t)$ for n=5
$h_2 7(t)$ for n=5
$h_2 8(t)$ for n=5
$h_2 9(t)$ for n=5
$h_3 0(t)$ for n=5
$h_3 1(t)$ for n=5

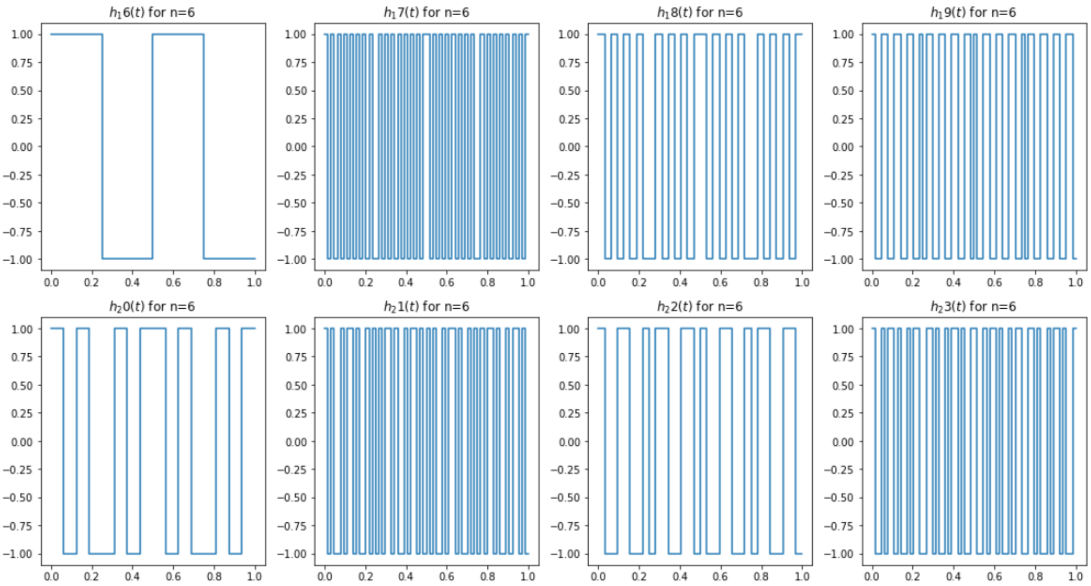
N=6:

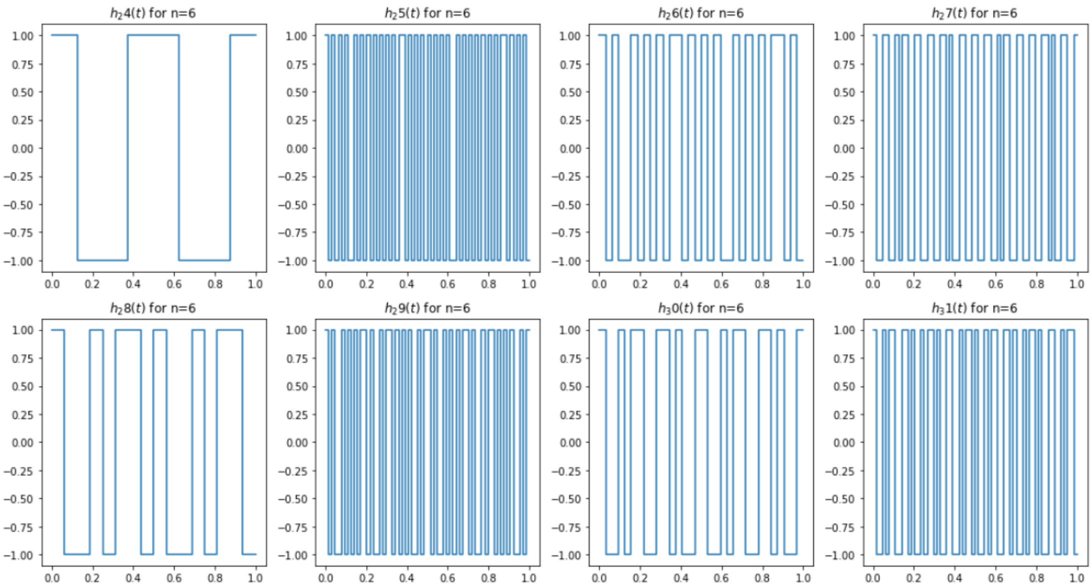


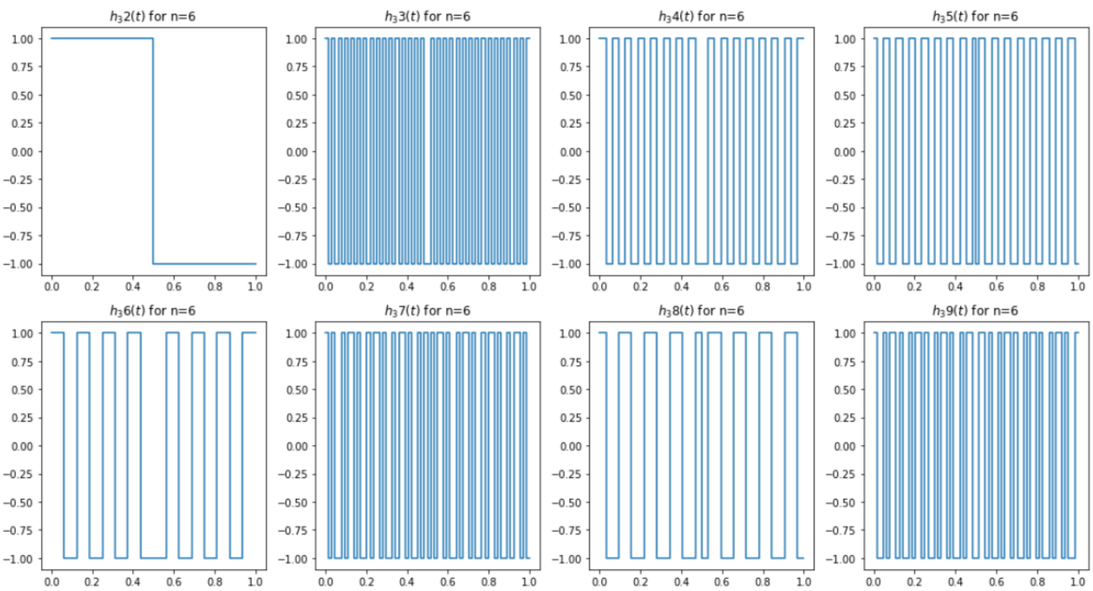
$h_0(t)$ for n=6
$h_1(t)$ for n=6
$h_2(t)$ for n=6
$h_3(t)$ for n=6
$h_4(t)$ for n=6
$h_5(t)$ for n=6
$h_6(t)$ for n=6
$h_7(t)$ for n=6

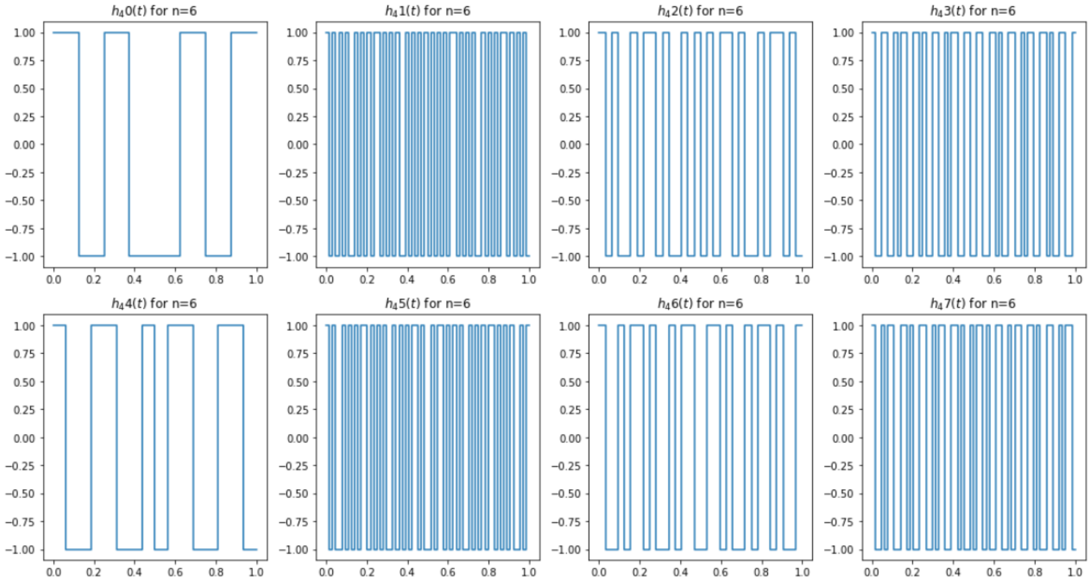


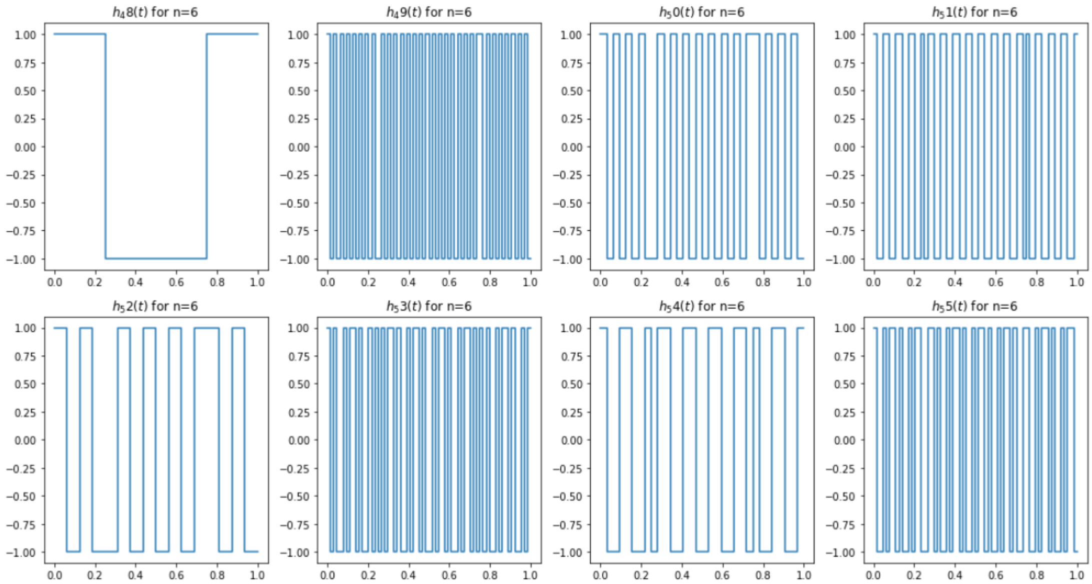
$h_8(t)$ for n=6
$h_9(t)$ for n=6
$h_10(t)$ for n=6
$h_11(t)$ for n=6
$h_12(t)$ for n=6
$h_13(t)$ for n=6
$h_14(t)$ for n=6
$h_15(t)$ for n=6

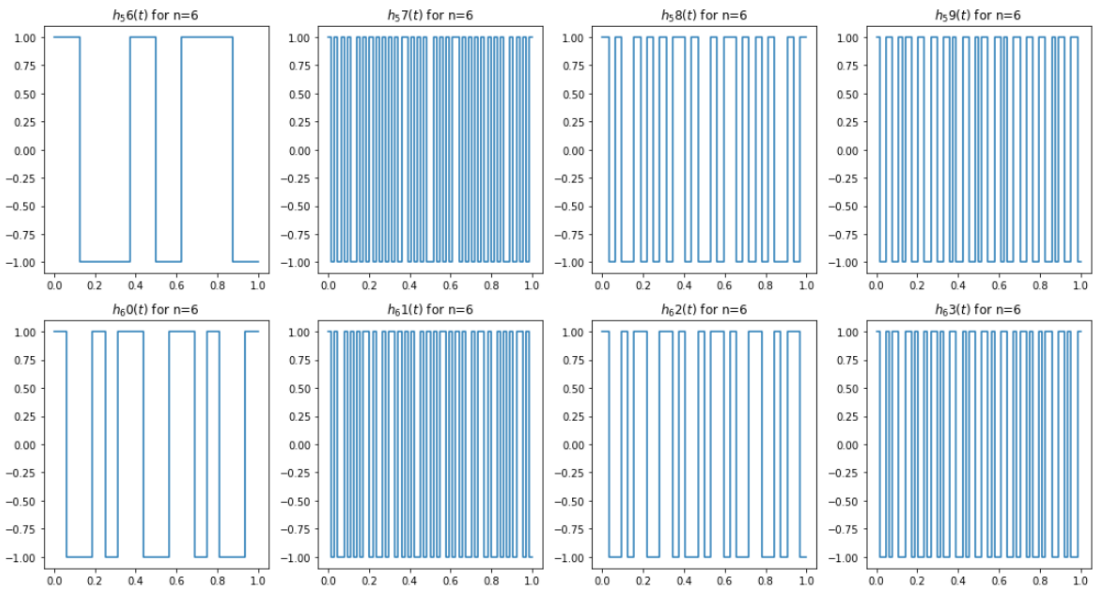

$h_16(t)$ for n=6
$h_17(t)$ for n=6
$h_18(t)$ for n=6
$h_19(t)$ for n=6
$h_20(t)$ for n=6
$h_21(t)$ for n=6
$h_22(t)$ for n=6
$h_23(t)$ for n=6


$h_24(t)$ for n=6
$h_25(t)$ for n=6
$h_26(t)$ for n=6
$h_27(t)$ for n=6
$h_28(t)$ for n=6
$h_29(t)$ for n=6
$h_30(t)$ for n=6
$h_31(t)$ for n=6


$h_32(t)$ for n=6
$h_33(t)$ for n=6
$h_34(t)$ for n=6
$h_35(t)$ for n=6
$h_36(t)$ for n=6
$h_37(t)$ for n=6
$h_38(t)$ for n=6
$h_39(t)$ for n=6


$h_40(t)$ for n=6
$h_41(t)$ for n=6
$h_42(t)$ for n=6
$h_43(t)$ for n=6
$h_44(t)$ for n=6
$h_45(t)$ for n=6
$h_46(t)$ for n=6
$h_47(t)$ for n=6


$h_48(t)$ for n=6
$h_49(t)$ for n=6
$h_50(t)$ for n=6
$h_51(t)$ for n=6
$h_52(t)$ for n=6
$h_53(t)$ for n=6
$h_54(t)$ for n=6
$h_55(t)$ for n=6

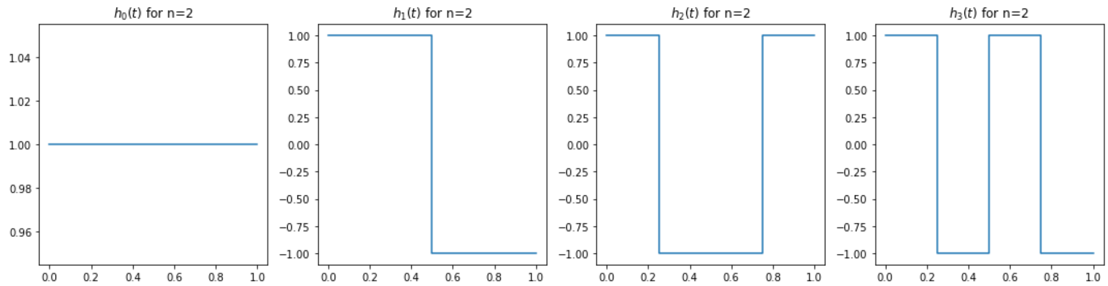

$h_56(t)$ for n=6
$h_57(t)$ for n=6
$h_58(t)$ for n=6
$h_59(t)$ for n=6
$h_60(t)$ for n=6
$h_61(t)$ for n=6
$h_62(t)$ for n=6
$h_63(t)$ for n=6

2.c.

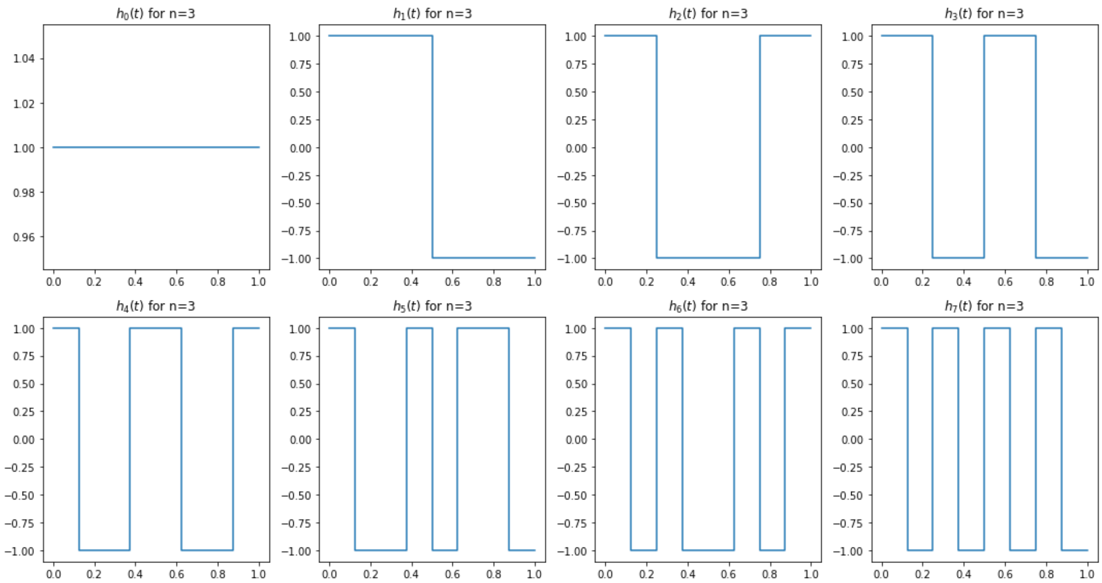
Implemented in the notebook.

2.d.

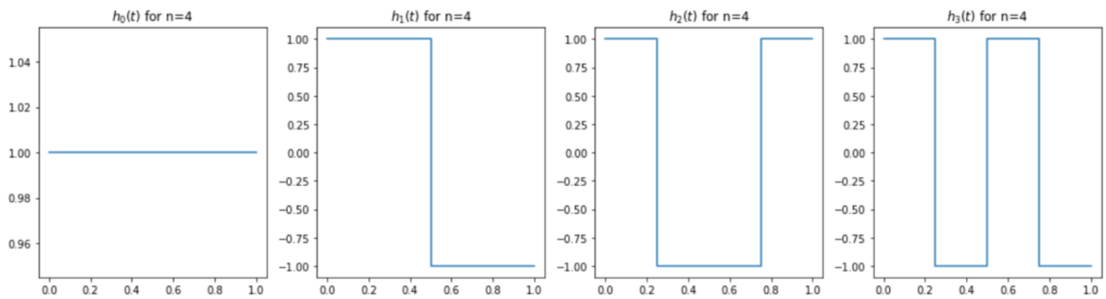
<u>N=2:</u>

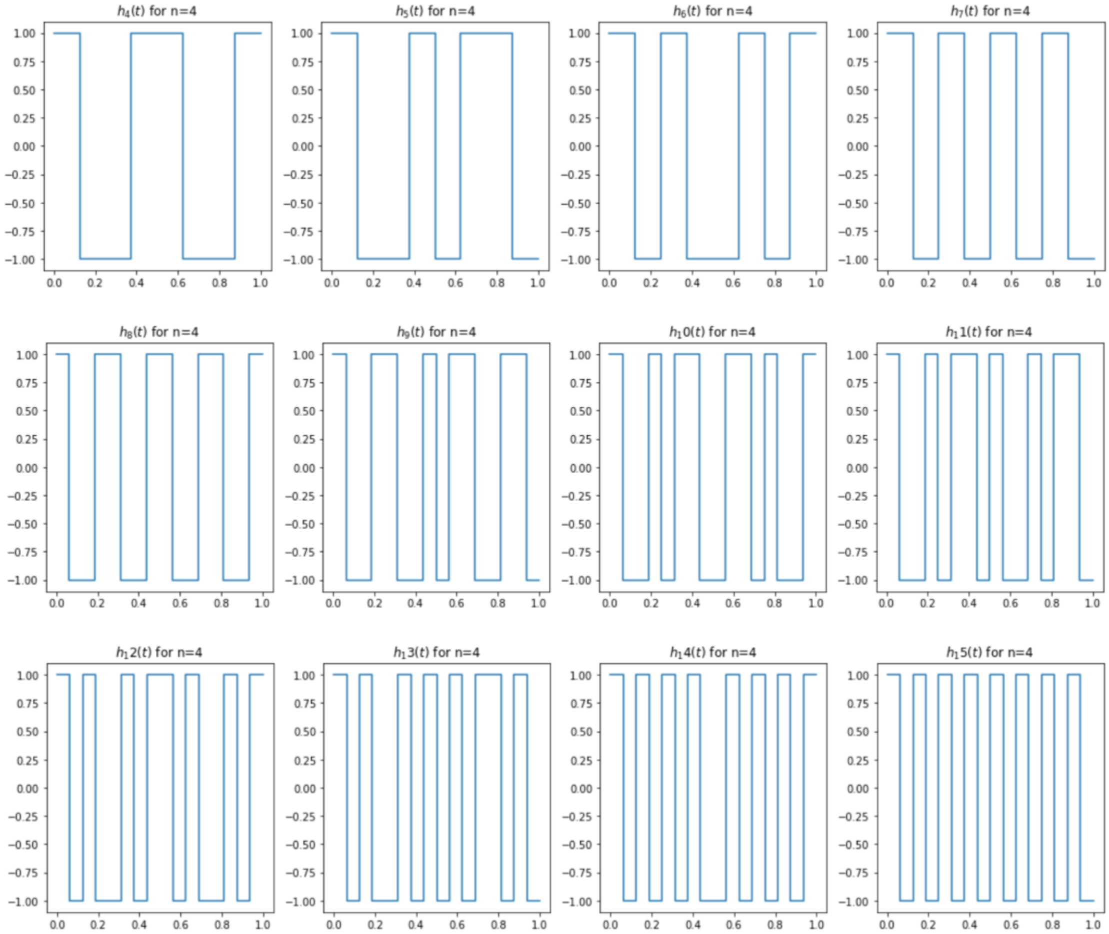


<u>N=3:</u>

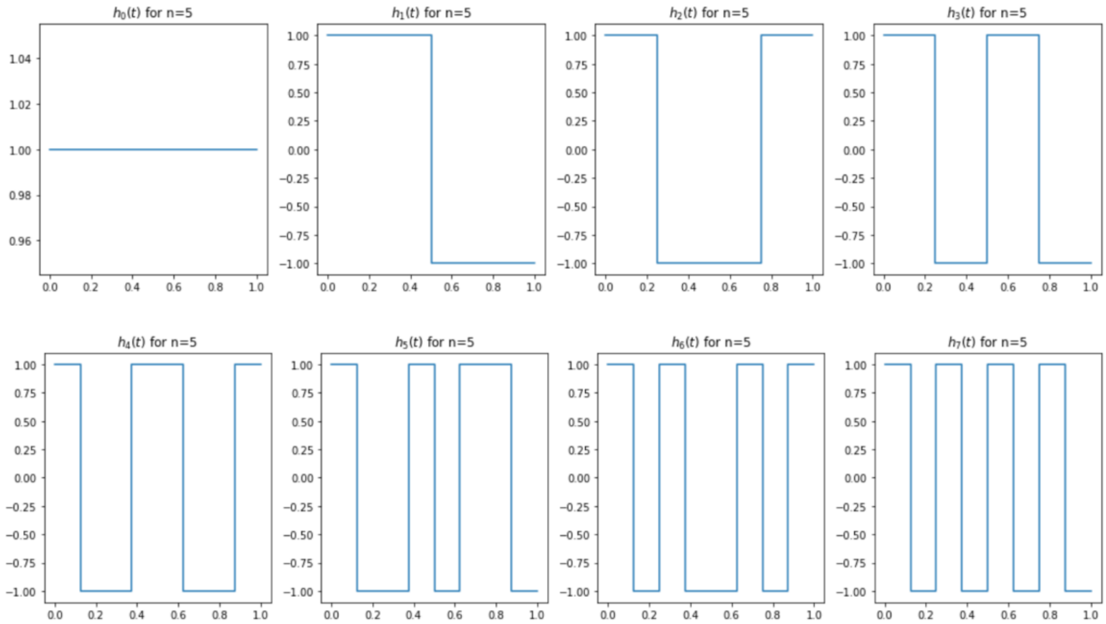


<u>N=4:</u>

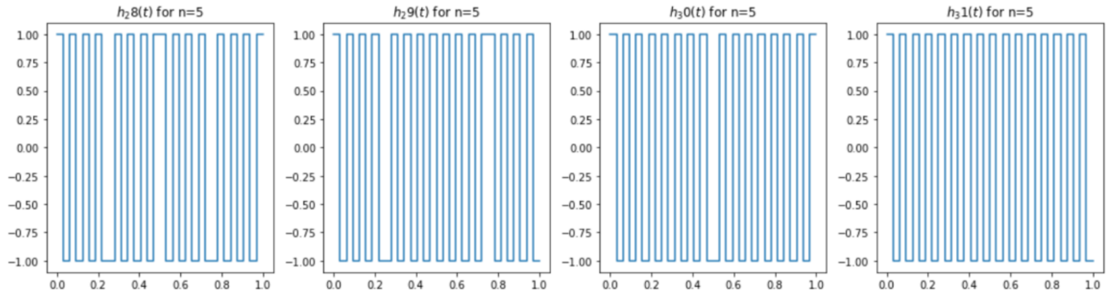



$h_4(t)$ for n=4
$h_5(t)$ for n=4
$h_6(t)$ for n=4
$h_7(t)$ for n=4
$h_8(t)$ for n=4
$h_9(t)$ for n=4
$h_10(t)$ for n=4
$h_11(t)$ for n=4
$h_12(t)$ for n=4
$h_13(t)$ for n=4
$h_14(t)$ for n=4
$h_15(t)$ for n=4

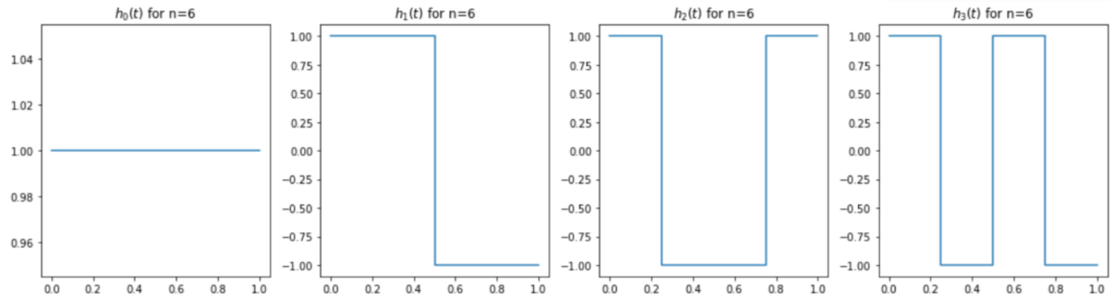
<u>N=5:</u>

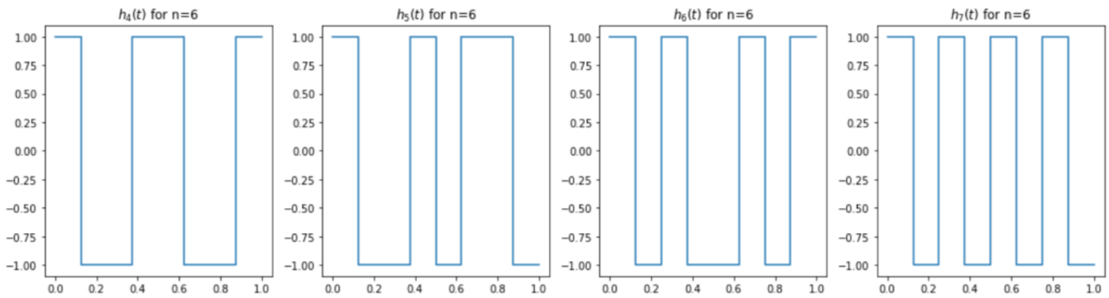

$h_0(t)$ for n=5
$h_1(t)$ for n=5
$h_2(t)$ for n=5
$h_3(t)$ for n=5
$h_4(t)$ for n=5
$h_5(t)$ for n=5
$h_6(t)$ for n=5
$h_7(t)$ for n=5

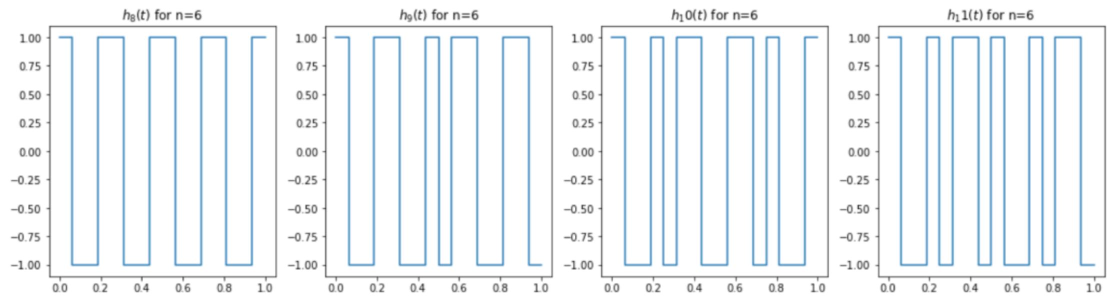
$h_8(t)$ for n=5
$h_9(t)$ for n=5
$h_10(t)$ for n=5
$h_11(t)$ for n=5
$h_12(t)$ for n=5
$h_13(t)$ for n=5
$h_14(t)$ for n=5
$h_15(t)$ for n=5
$h_16(t)$ for n=5
$h_17(t)$ for n=5
$h_18(t)$ for n=5
$h_19(t)$ for n=5
$h_20(t)$ for n=5
$h_21(t)$ for n=5
$h_22(t)$ for n=5
$h_23(t)$ for n=5
$h_24(t)$ for n=5
$h_25(t)$ for n=5
$h_26(t)$ for n=5
$h_27(t)$ for n=5

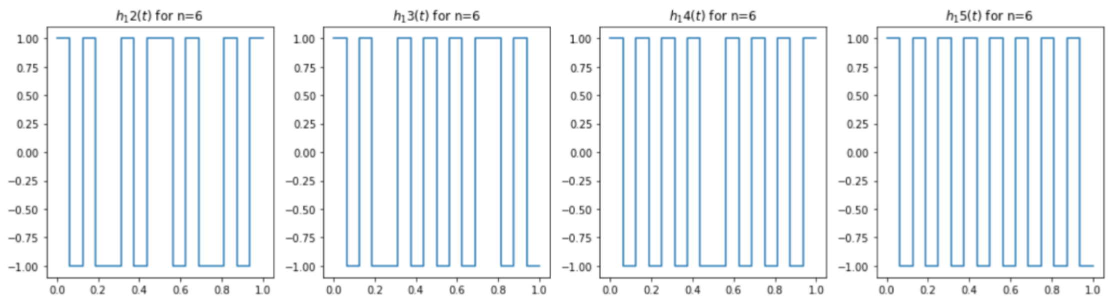

$h_2 8(t)$ for n=5
$h_2 9(t)$ for n=5
$h_3 0(t)$ for n=5
$h_3 1(t)$ for n=5

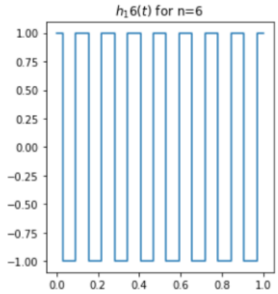
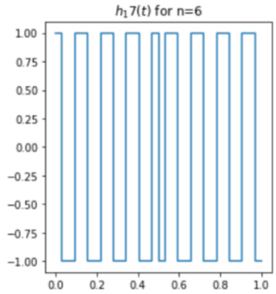
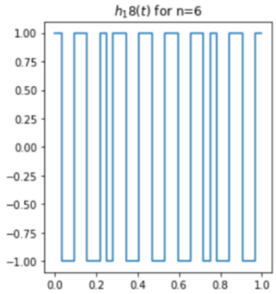
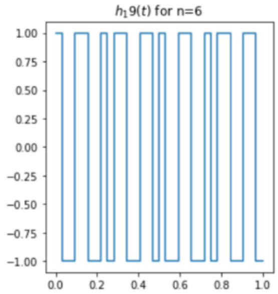
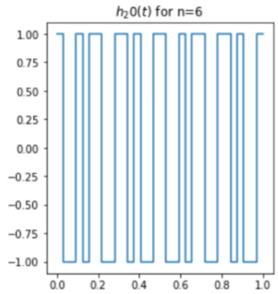
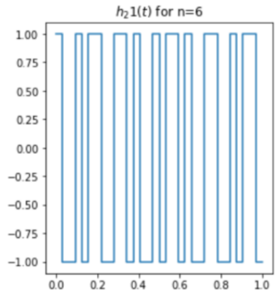
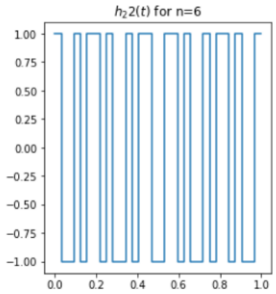
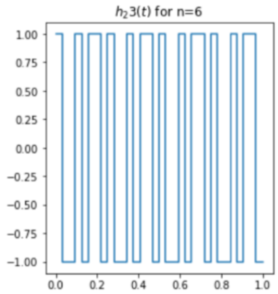
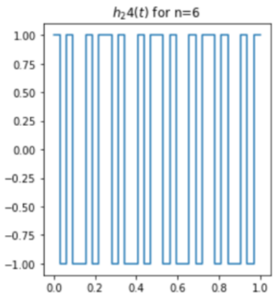
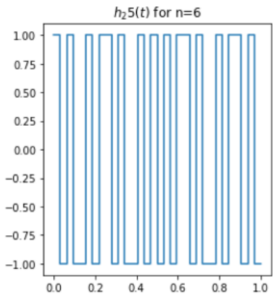
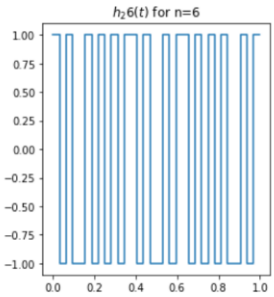
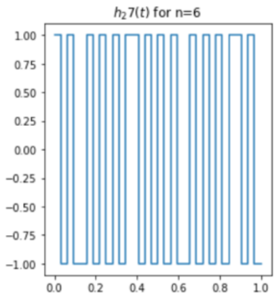
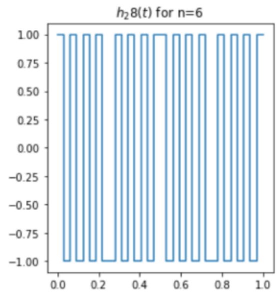
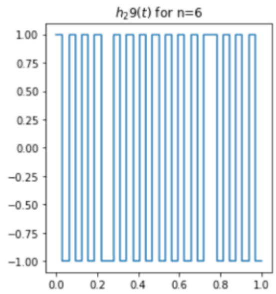
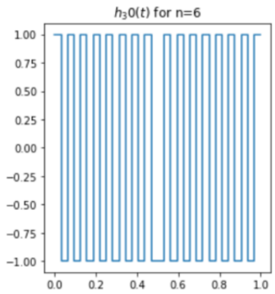
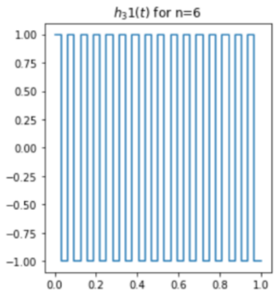
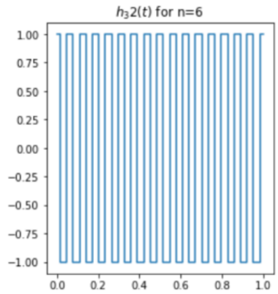
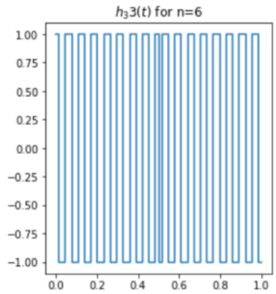
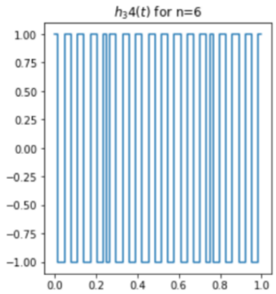
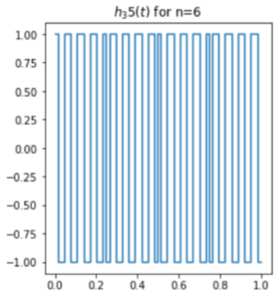
N=6:

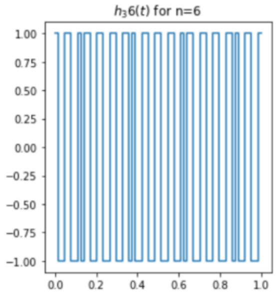
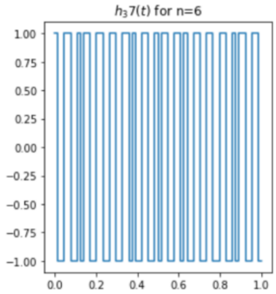
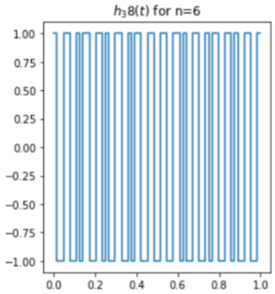
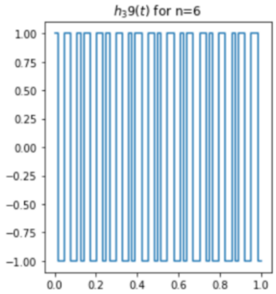
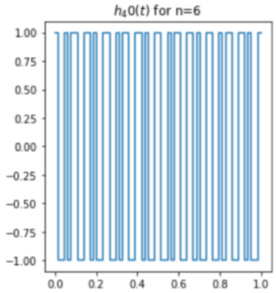
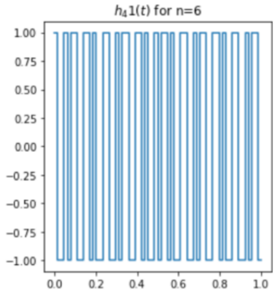
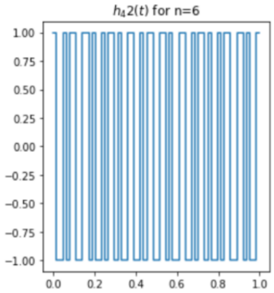
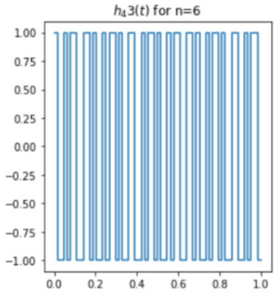
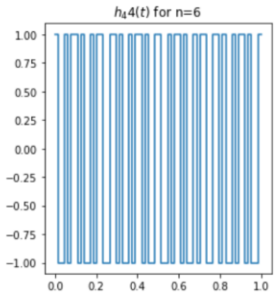
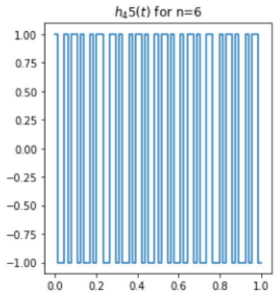
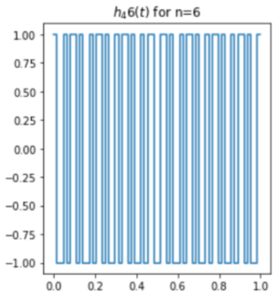
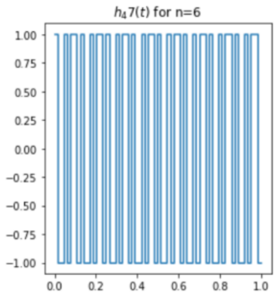
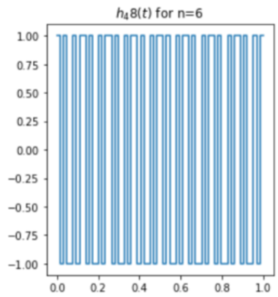
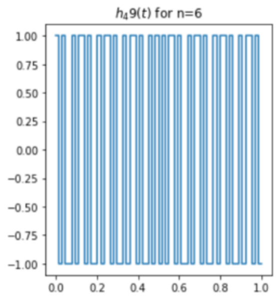
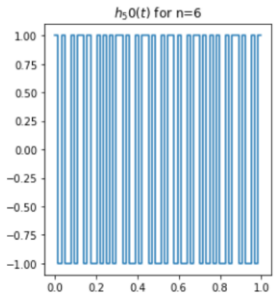
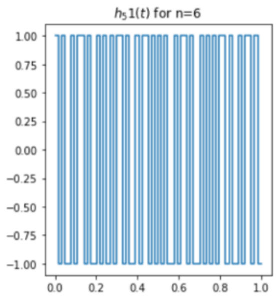
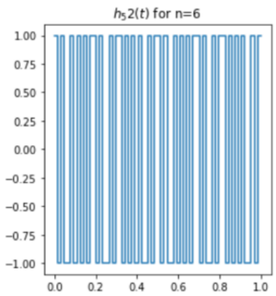
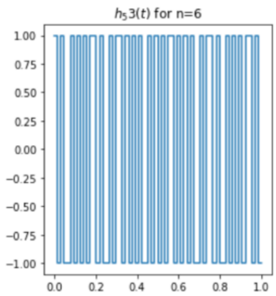
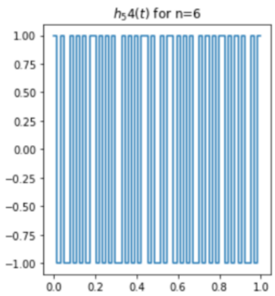
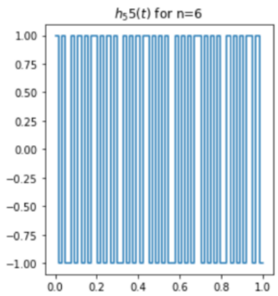

$h_0(t)$ for n=6
$h_1(t)$ for n=6
$h_2(t)$ for n=6
$h_3(t)$ for n=6

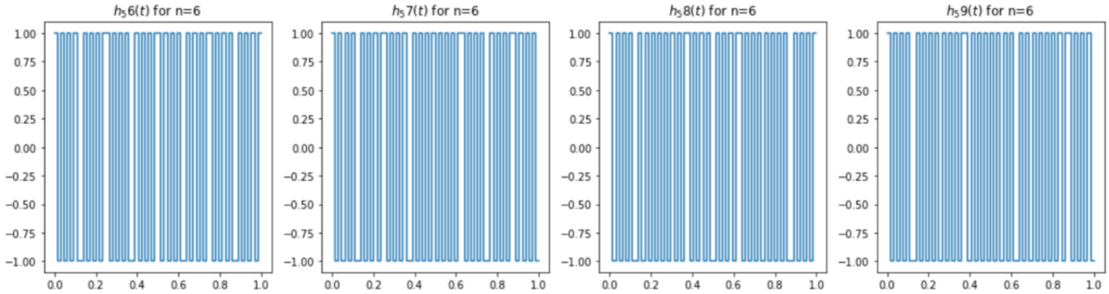

$h_4(t)$ for n=6
$h_5(t)$ for n=6
$h_6(t)$ for n=6
$h_7(t)$ for n=6

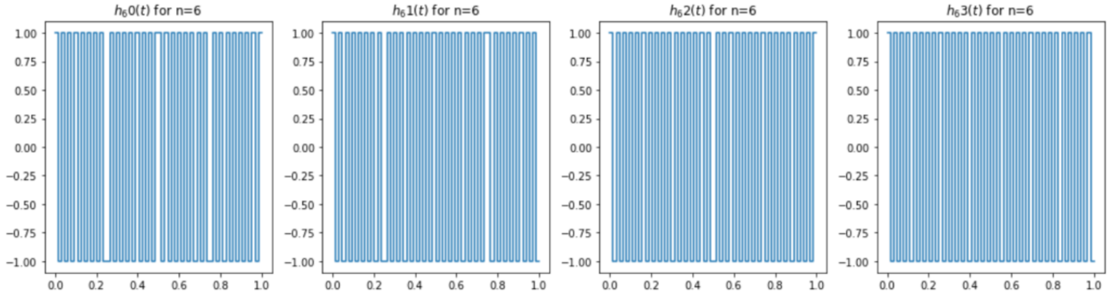

$h_8(t)$ for n=6
$h_9(t)$ for n=6
$h_1 0(t)$ for n=6
$h_1 1(t)$ for n=6


$h_1 2(t)$ for n=6
$h_1 3(t)$ for n=6
$h_1 4(t)$ for n=6
$h_1 5(t)$ for n=6


$h_1 6(t)$ for n=6

$h_1 7(t)$ for n=6

$h_1 8(t)$ for n=6

$h_1 9(t)$ for n=6

$h_2 0(t)$ for n=6

$h_2 1(t)$ for n=6

$h_2 2(t)$ for n=6

$h_2 3(t)$ for n=6

$h_2 4(t)$ for n=6

$h_2 5(t)$ for n=6

$h_2 6(t)$ for n=6

$h_2 7(t)$ for n=6

$h_2 8(t)$ for n=6

$h_2 9(t)$ for n=6

$h_3 0(t)$ for n=6

$h_3 1(t)$ for n=6

$h_3 2(t)$ for n=6

$h_3 3(t)$ for n=6

$h_3 4(t)$ for n=6

$h_3 5(t)$ for n=6


$h_{3}6(t)$ for n=6

$h_{3}7(t)$ for n=6

$h_{3}8(t)$ for n=6

$h_{3}9(t)$ for n=6

$h_{4}0(t)$ for n=6

$h_{4}1(t)$ for n=6

$h_{4}2(t)$ for n=6

$h_{4}3(t)$ for n=6

$h_{4}4(t)$ for n=6

$h_{4}5(t)$ for n=6

$h_{4}6(t)$ for n=6

$h_{4}7(t)$ for n=6

$h_{4}8(t)$ for n=6

$h_{4}9(t)$ for n=6

$h_{5}0(t)$ for n=6

$h_{5}1(t)$ for n=6

$h_{5}2(t)$ for n=6

$h_{5}3(t)$ for n=6

$h_{5}4(t)$ for n=6

$h_{5}5(t)$ for n=6



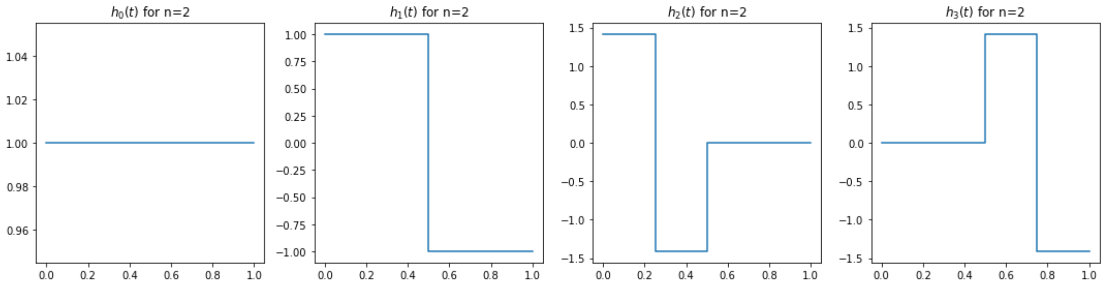


2.e.

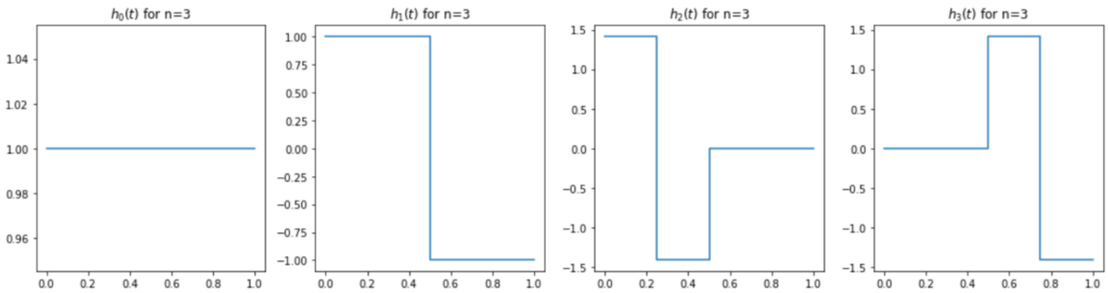
Implemented in the notebook.

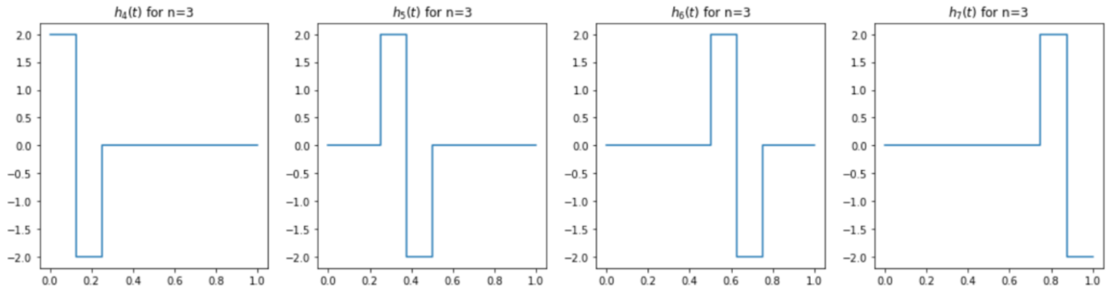
2.f.

<u>N=2:</u>

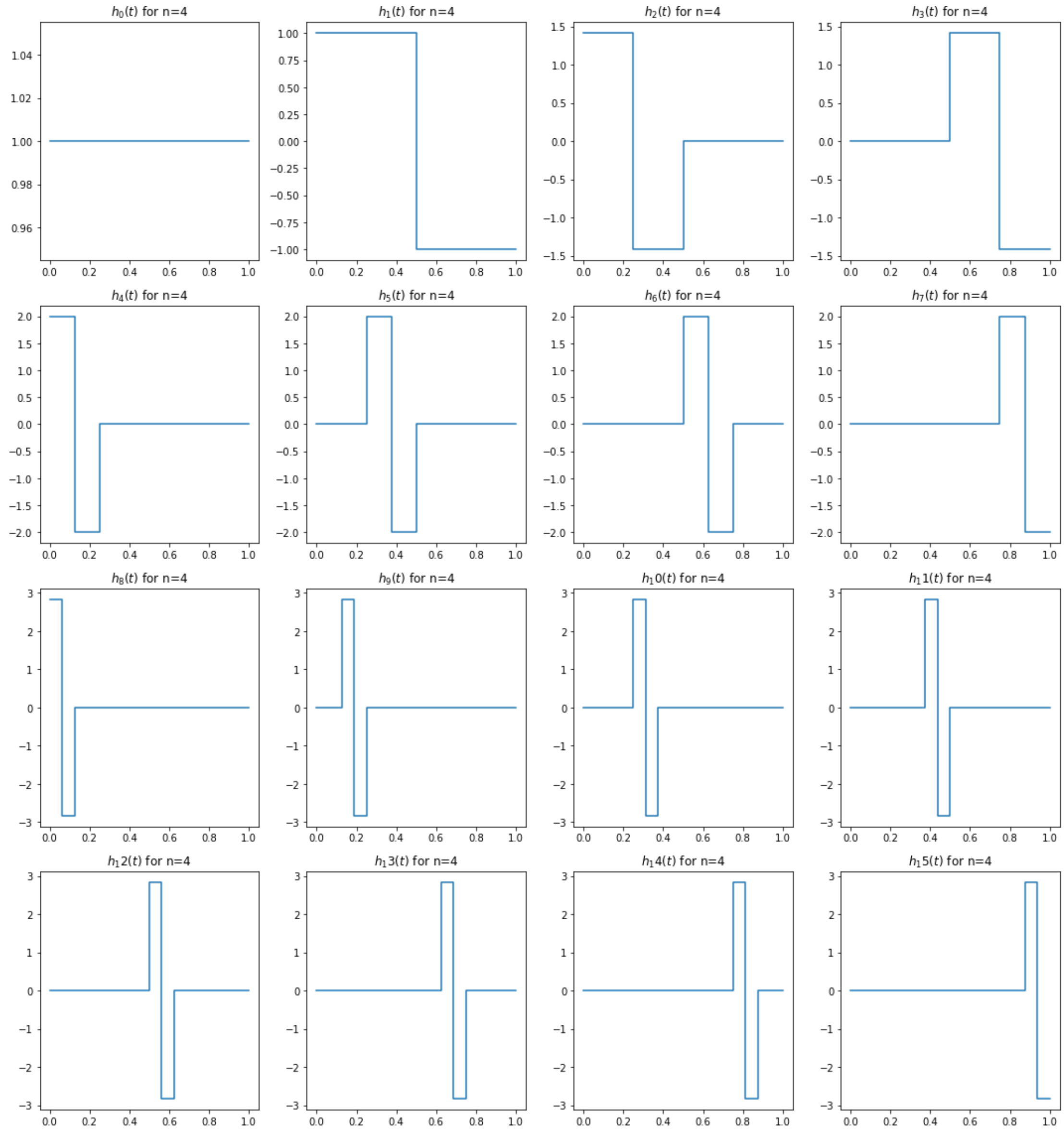


<u>N=3:</u>

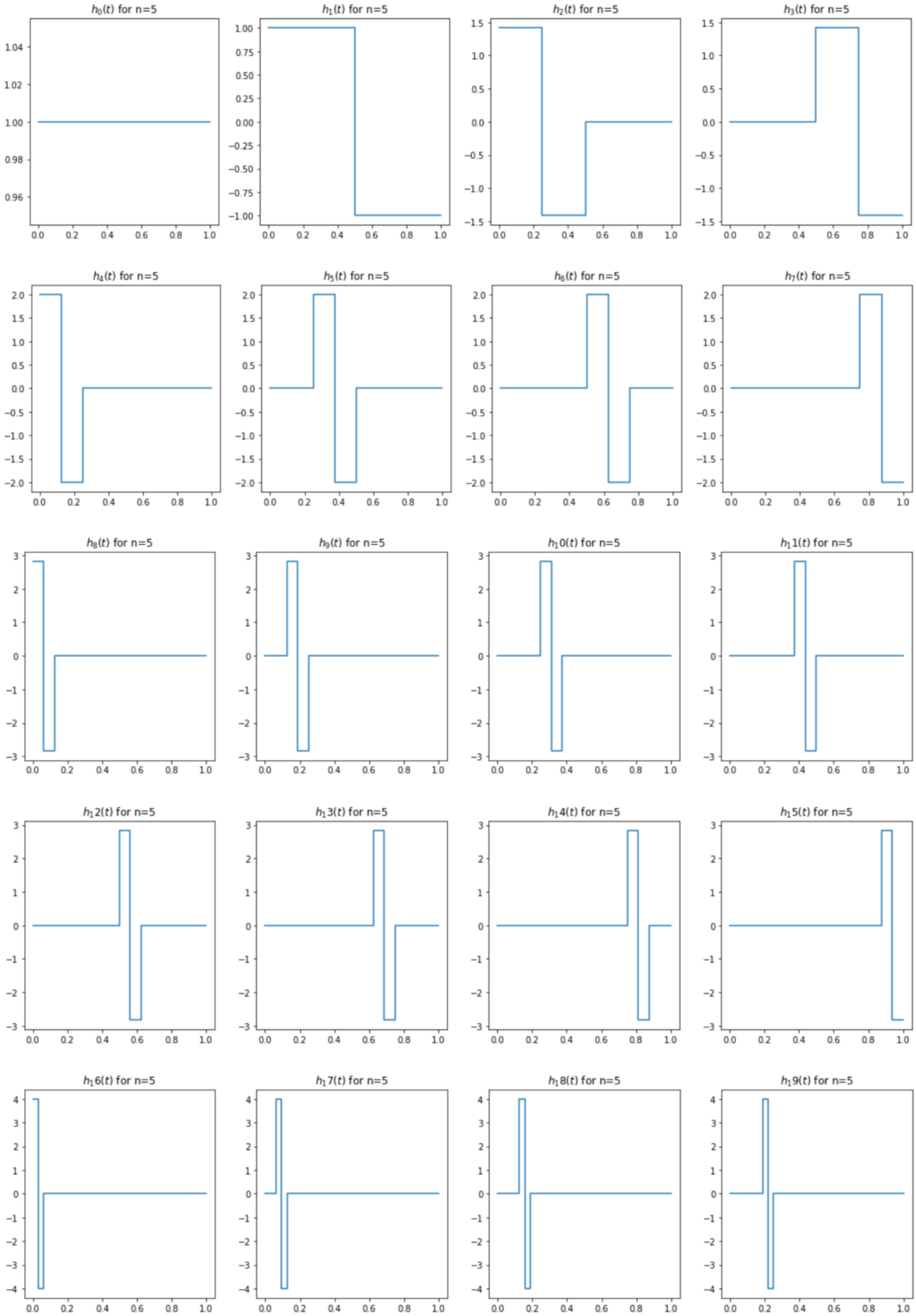


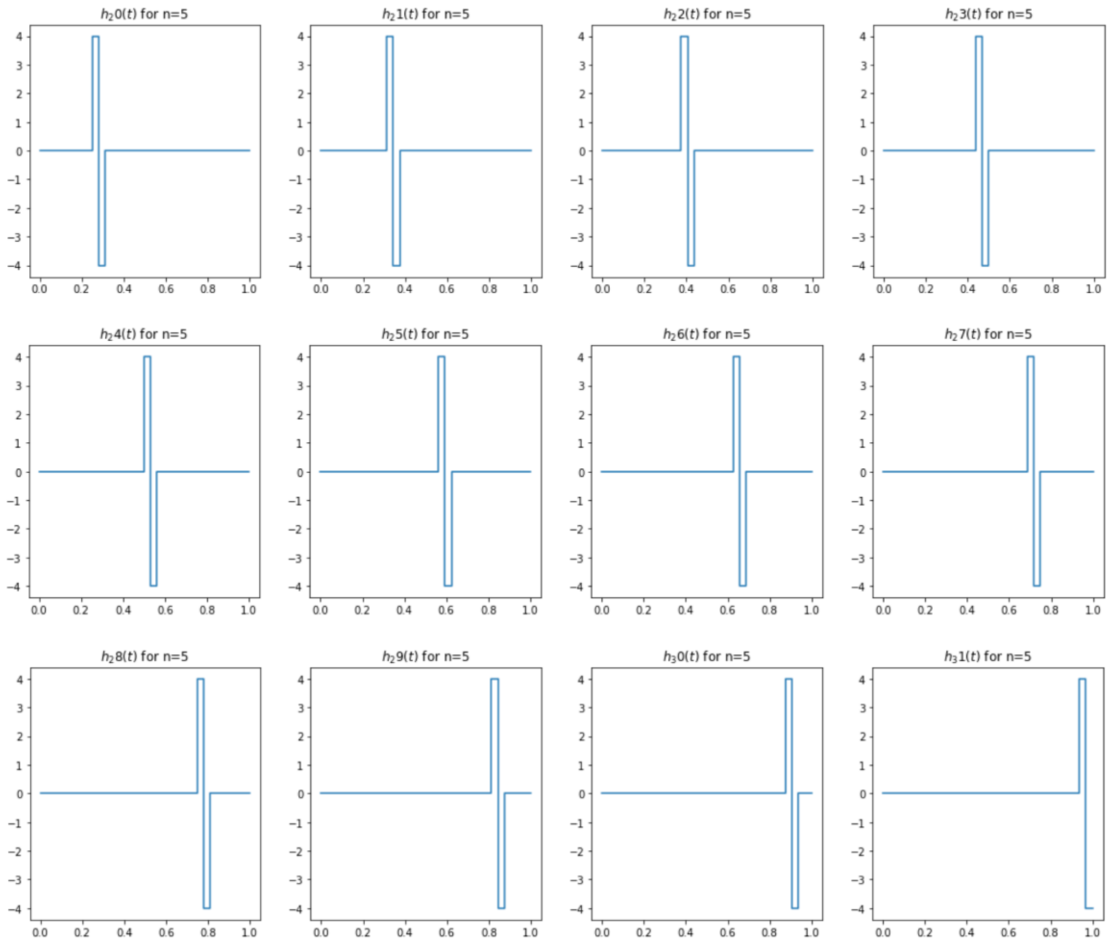

$h_4(t)$ for n=3
$h_5(t)$ for n=3
$h_6(t)$ for n=3
$h_7(t)$ for n=3

N=4:

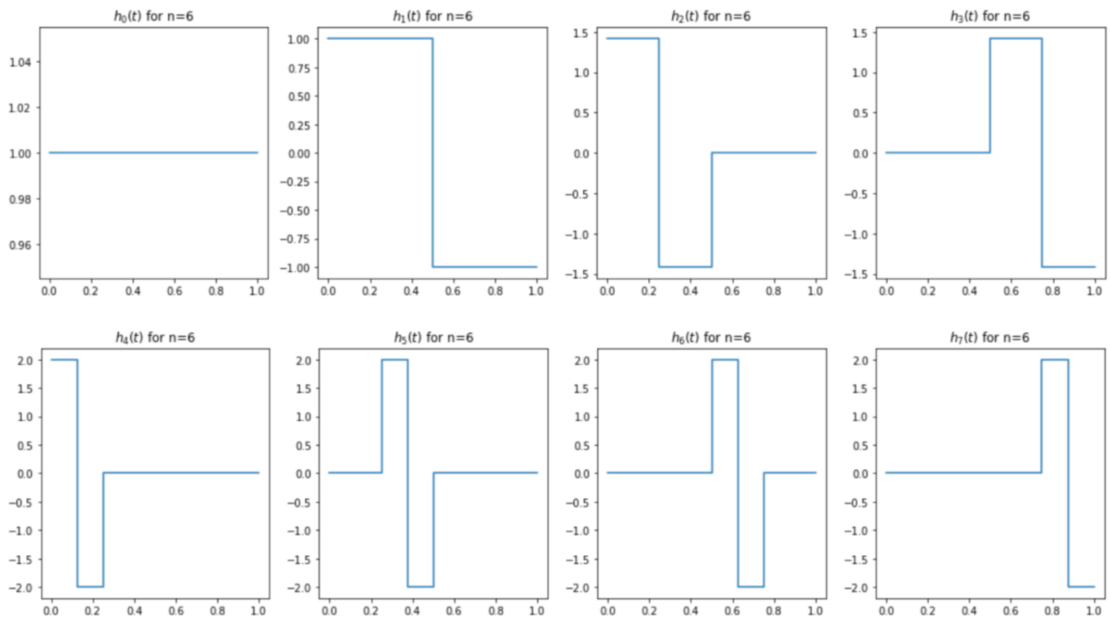

$h_0(t)$ for n=4
$h_1(t)$ for n=4
$h_2(t)$ for n=4
$h_3(t)$ for n=4
$h_4(t)$ for n=4
$h_5(t)$ for n=4
$h_6(t)$ for n=4
$h_7(t)$ for n=4
$h_8(t)$ for n=4
$h_9(t)$ for n=4
$h_1 0(t)$ for n=4
$h_1 1(t)$ for n=4
$h_1 2(t)$ for n=4
$h_1 3(t)$ for n=4
$h_1 4(t)$ for n=4
$h_1 5(t)$ for n=4

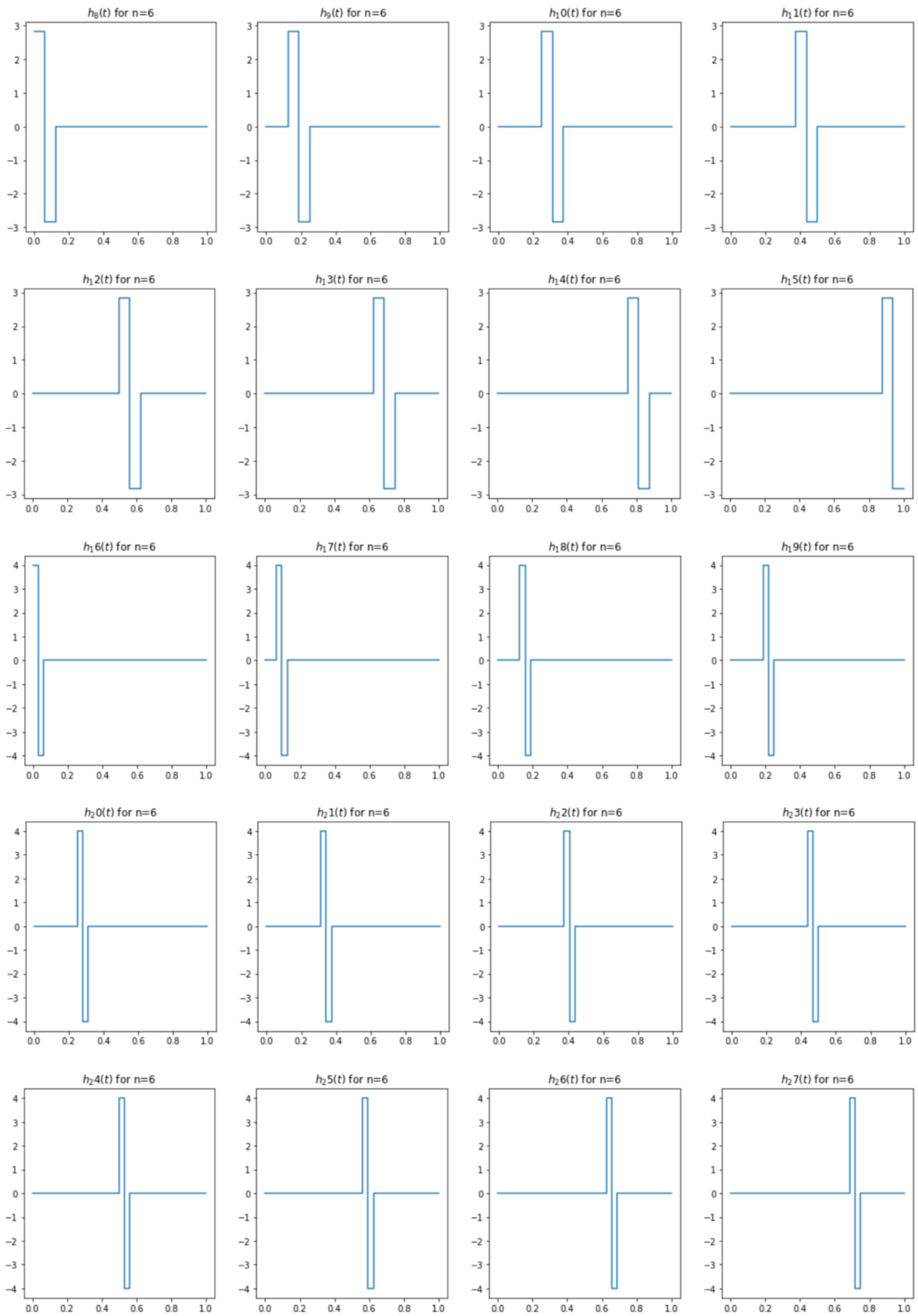
N=5:


$h_0(t)$ for n=5
$h_1(t)$ for n=5
$h_2(t)$ for n=5
$h_3(t)$ for n=5
$h_4(t)$ for n=5
$h_5(t)$ for n=5
$h_6(t)$ for n=5
$h_7(t)$ for n=5
$h_8(t)$ for n=5
$h_9(t)$ for n=5
$h_1 0(t)$ for n=5
$h_1 1(t)$ for n=5
$h_1 2(t)$ for n=5
$h_1 3(t)$ for n=5
$h_1 4(t)$ for n=5
$h_1 5(t)$ for n=5
$h_1 6(t)$ for n=5
$h_1 7(t)$ for n=5
$h_1 8(t)$ for n=5
$h_1 9(t)$ for n=5

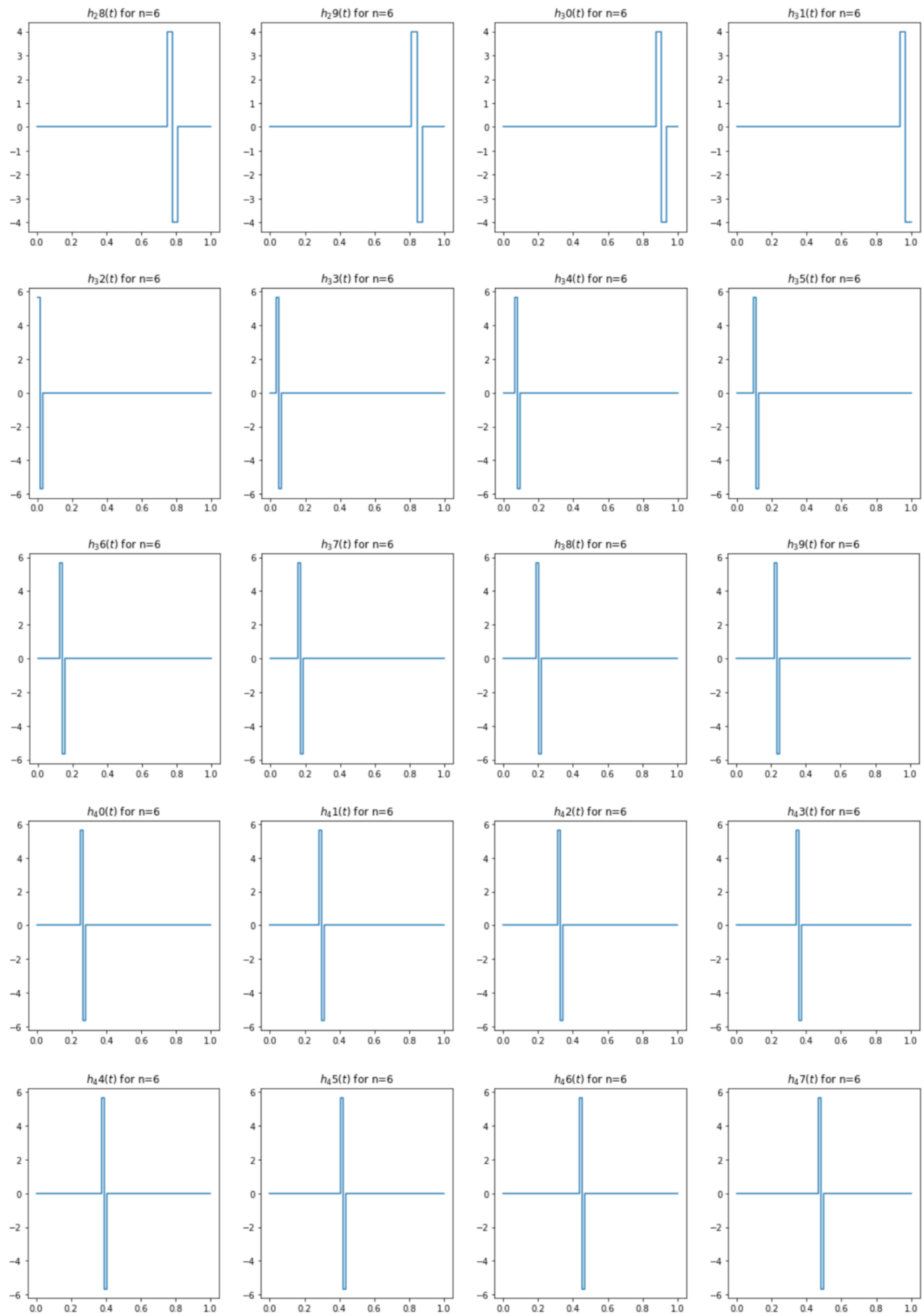

$h_20(t)$ for n=5
$h_21(t)$ for n=5
$h_22(t)$ for n=5
$h_23(t)$ for n=5
$h_24(t)$ for n=5
$h_25(t)$ for n=5
$h_26(t)$ for n=5
$h_27(t)$ for n=5
$h_28(t)$ for n=5
$h_29(t)$ for n=5
$h_30(t)$ for n=5
$h_31(t)$ for n=5

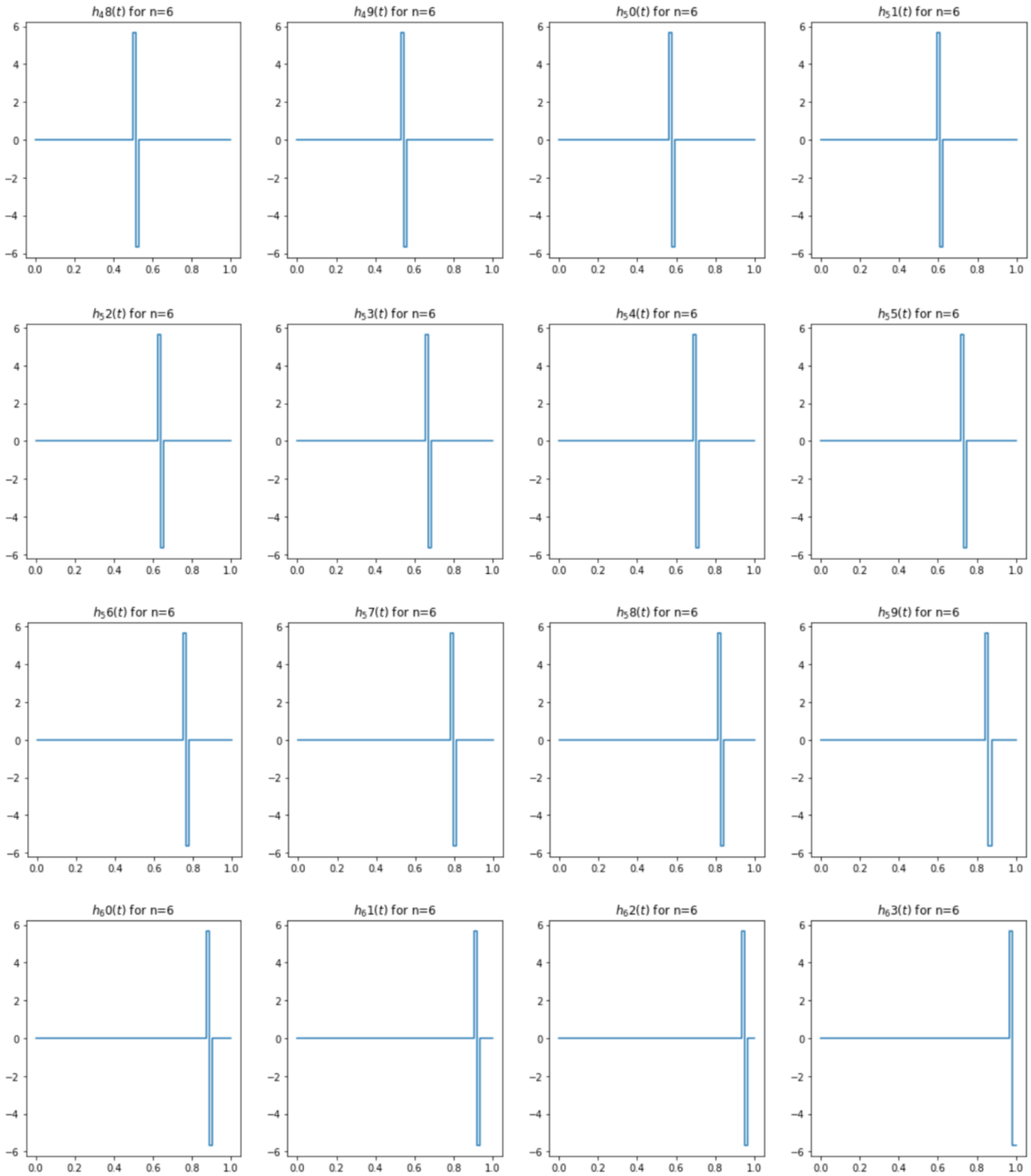
N=6:


$h_0(t)$ for n=6
$h_1(t)$ for n=6
$h_2(t)$ for n=6
$h_3(t)$ for n=6
$h_4(t)$ for n=6
$h_5(t)$ for n=6
$h_6(t)$ for n=6
$h_7(t)$ for n=6



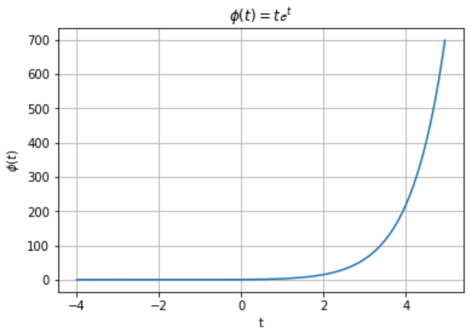
$h_8(t)$ for n=6
$h_9(t)$ for n=6
$h_10(t)$ for n=6
$h_11(t)$ for n=6
$h_12(t)$ for n=6
$h_13(t)$ for n=6
$h_14(t)$ for n=6
$h_15(t)$ for n=6
$h_16(t)$ for n=6
$h_17(t)$ for n=6
$h_18(t)$ for n=6
$h_19(t)$ for n=6
$h_20(t)$ for n=6
$h_21(t)$ for n=6
$h_22(t)$ for n=6
$h_23(t)$ for n=6
$h_24(t)$ for n=6
$h_25(t)$ for n=6
$h_26(t)$ for n=6
$h_27(t)$ for n=6


$h_2 8(t)$ for n=6
$h_2 9(t)$ for n=6
$h_3 0(t)$ for n=6
$h_3 1(t)$ for n=6
$h_3 2(t)$ for n=6
$h_3 3(t)$ for n=6
$h_3 4(t)$ for n=6
$h_3 5(t)$ for n=6
$h_3 6(t)$ for n=6
$h_3 7(t)$ for n=6
$h_3 8(t)$ for n=6
$h_3 9(t)$ for n=6
$h_4 0(t)$ for n=6
$h_4 1(t)$ for n=6
$h_4 2(t)$ for n=6
$h_4 3(t)$ for n=6
$h_4 4(t)$ for n=6
$h_4 5(t)$ for n=6
$h_4 6(t)$ for n=6
$h_4 7(t)$ for n=6

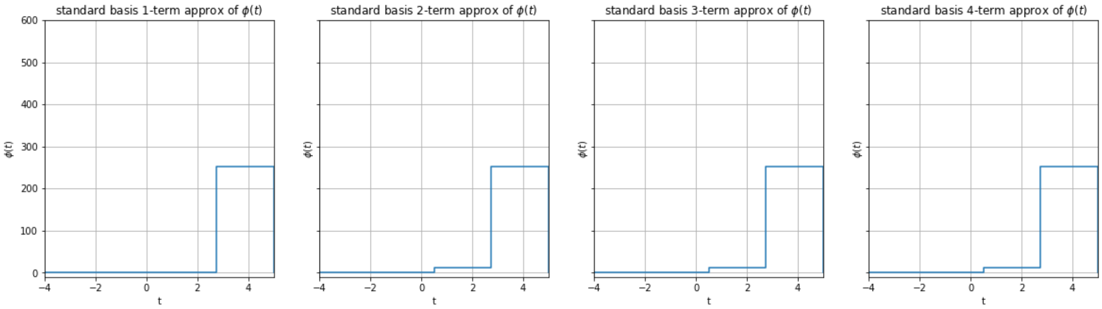


2.g.

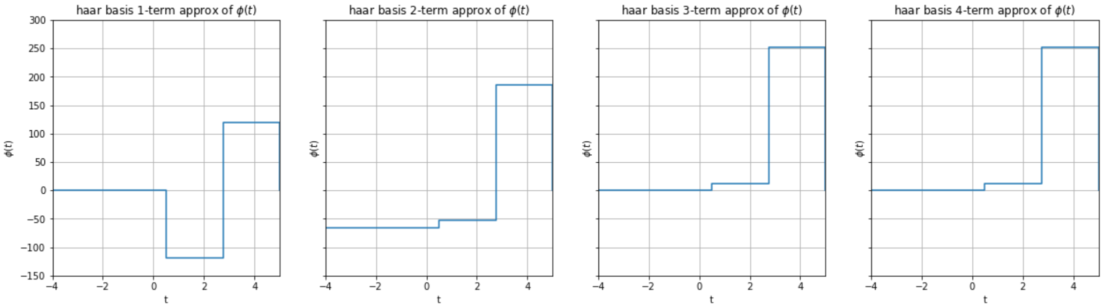
Consider $\phi(t) = te^t$ for $t \in [-4,5]$:



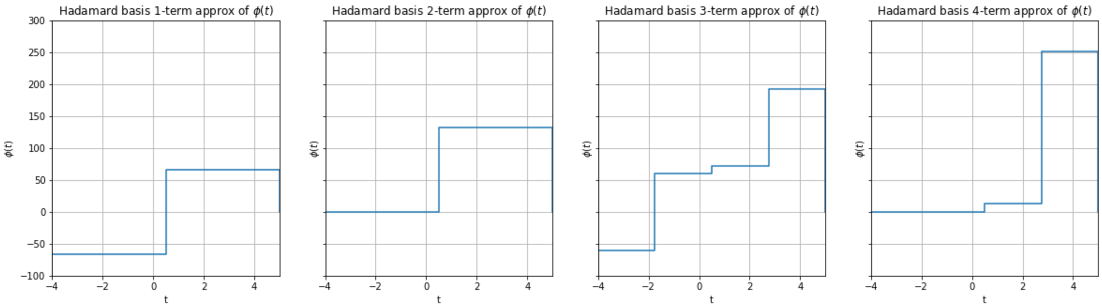
Best k-term approximation of $\phi(t)$ with standard basis:



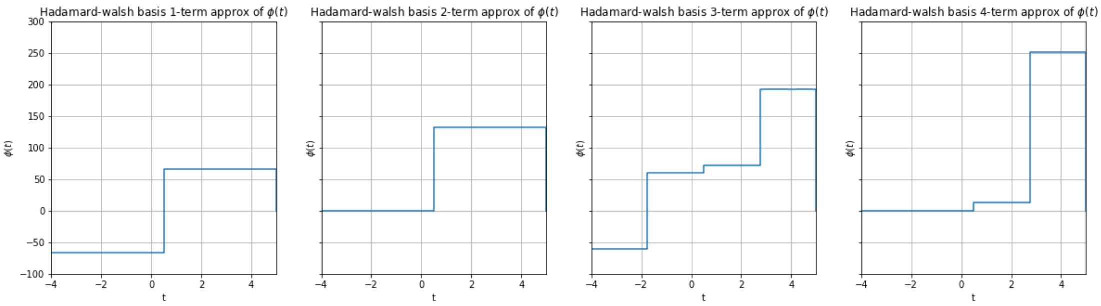
Best k-term approximation of $\phi(t)$ with Haar basis:



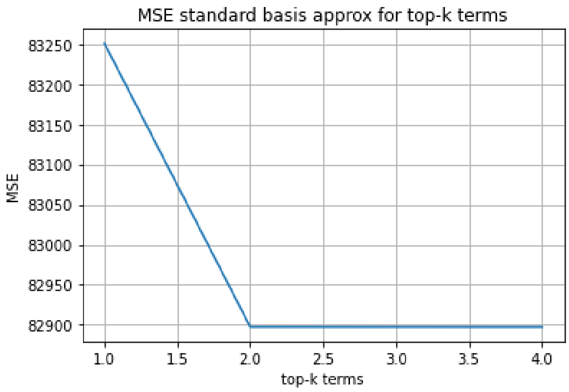
Best k-term approximation of $\phi(t)$ with Hadamard basis:

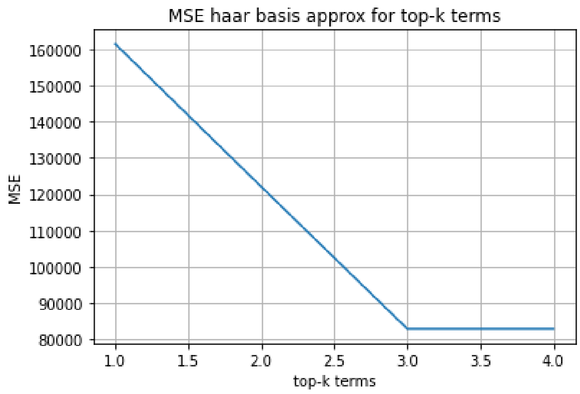


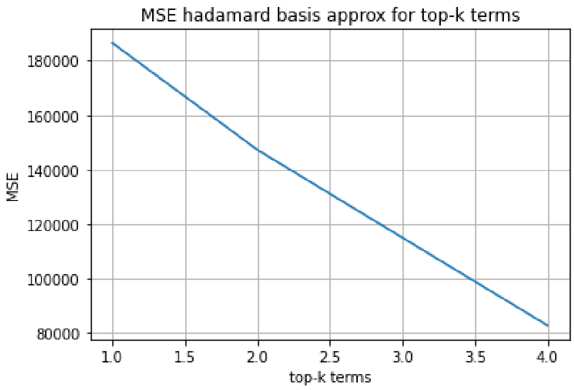
Best k-term approximation of $\phi(t)$ with Walsh-Hadamard basis:

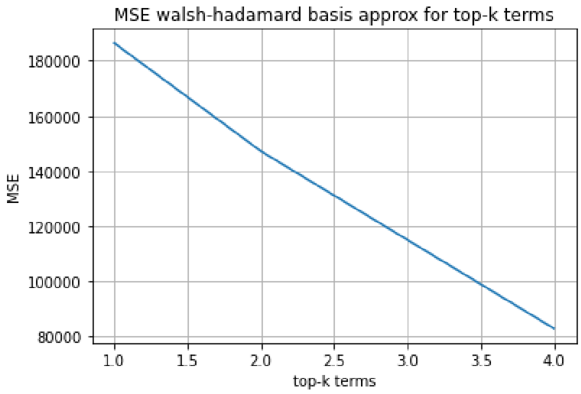


The MSE analysis in the nest page


MSE standard basis approx for top-k terms
MSE
top-k terms


MSE haar basis approx for top-k terms
MSE
top-k terms


MSE hadamard basis approx for top-k terms
MSE
top-k terms


MSE walsh-hadamard basis approx for top-k terms
MSE
top-k terms