July 14
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Find Gauss Curvature via local coframes.

$M \subseteq \mathbb{R}^3$, $\dim M = 2$, v_1, v_2 are orthonormal basis of T_pM.
Denote $\langle \, , \rangle_p$ as the metric of M at $p \in M$.

$\langle v_i, v_j \rangle = \delta_{ij}$ for $i,j = 1,2$

Let θ_1, θ_2 be a pair of coframe of v_1, v_2. i.e.
$\theta_i(v_j) = \delta_{ij}$ for $i,j = 1,2$. $\theta_i \in T_p^*M$.

~~Thm~~ Prop. $\exists$ 1-forms $w^1{}_2$ and $w^2{}_1$ s.t.

$d\theta_1 = \theta_2 \wedge w^1{}_2$ and $d\theta_2 = \theta_1 \wedge w^2{}_1$, $w^1{}_2 = -w^2{}_1$

For notation, let $w^i{}_i = 0$, $w = \begin{pmatrix} 0 & w^1{}_2 \\ w^2{}_1 & 0 \end{pmatrix}$

Ref. Spivak Vol 2.

Def: w is connection form of $\langle \, , \rangle_p$.

For 2-dim case

$K(p) = \langle R(v_1, v_2)v_2, v_1 \rangle$

$\langle v_1, v_2 \rangle = 0$ $v_1 \perp v_2$

$= \langle \sum_{i=1}^{2} \Omega^i{}_2(v_1, v_2) v_i, v_1 \rangle$ $i=1$

v_i are orthonormal

$= \langle \Omega^1{}_2(v_1, v_2) v_1, v_1 \rangle$

$\langle g \cdot v_1, v_1 \rangle = g \cdot \langle v_1, v_1 \rangle$
$= g \cdot 1 = g$

$= \langle (dw^1{}_2 + \sum_{k=1}^{2} w^1{}_k \wedge w^k{}_2)(v_1, v_2) v_1, v_1 \rangle$

$= dw^1{}_2(v_1, v_2) + \underbrace{\sum_{k=1}^{2} w^1{}_k \wedge w^k{}_2}_{=0}$, $w^1{}_1 = 0$, $w^2{}_2 = 0$

$= dw^1{}_2(v_1, v_2)(p)$

$= K(p)$

$K \cdot \theta^1 \wedge \theta^2 = dw^1{}_2$

① Find θ^1, θ^2 local coframes. $d\theta^1, d\theta^2$.

② Write $d\theta^1$ as $d\theta^1 = \theta_2 \wedge w^1{}_2$ under the basis of $\theta^2 \wedge w^1{}_2$

$d\theta^1$ is a 2-form. $\theta_1 \wedge \theta_2$ is the std. basis of $\Lambda^2 T_p^* M$.

Find $\omega^1{}_2$ by doing a change of basis on $\Lambda^2 T_p^* M$.

$\dim(\Lambda^2 T_p^* M) = 1$

(3) If $\omega^1{}_2$ is found, write $d\omega^1{}_2 = K \cdot \theta^1 \wedge \theta^2$

K is the Gauss curvature.

Critical and Regular pts / values Ref. Bredon's Top. & Geo.

Def: $f: M \to N$, $m = \dim M$, $n = \dim N$

A critical point of f is a point $p \in M$, where

df_p does not have a full rank, $df_p : T_pM \to T_pN$.

rank $(df)_p < n$, df_p is not surjective.

$f(p)$ is a critical value

If $m < n$ no regular pts on M.

A point p is regular if it's not critical

$f(p)$ is a regular value

Thm: $f: M \to N$, if $y \in N$ is a regular value, then

$f^{-1}(y)$ is a submanifold of M.

$\dim f^{-1}(y) = m - n$, $(m \geq n)$

Def: $f: M \to N$. If df_p is injective $\forall p \in M$, f immersion

If df_p is surjective for $\forall p \in M$, f submersion.

If f is inject, df_p is bijective, f embedding

Remark: $f: M \to N$, $y \in N$.

$y \notin f(M)$, y is also regular in N.

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$f^{-1}(y) = \emptyset$.

Ex. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (f_1(\cdots), f_2(\cdots))$.

Picks $p \in \mathbb{R}^2$. Ask whether $f^{-1}(p)$ is a submanifold.

Find $\{f^{-1}(p)\}$. $\forall q \in f^{-1}(p)$, $q \in M$.

Check $(df)_q$ for every $q \in f^{-1}(p)$

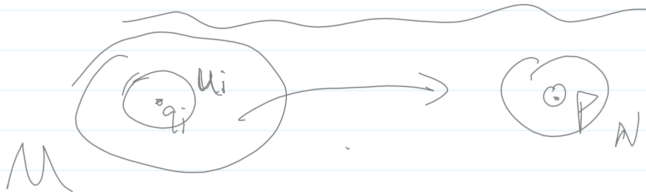
If $\operatorname{rank}(df)_q = 2$, then T

If $\operatorname{rank}(df)_q < 2$, then F.

$p \in N$, $f^{-1}(p) = \{q_1, \cdots, q_n, \cdots\}$

p is criticial, $\underbrace{\operatorname{rank}(df q_i) = n \text{ for some}}$ q_i's

q_i's



$f^{-1}(p)$

$\operatorname{rank}(df q_0) < n$.

$\odot$ q_0

Distribution, Frobenius Thm.

Def. M manifold, TM tagent bundle.

$TM = \bigsqcup_{p \in M} T_pM$ (disjoint union)

A distribution D is a subbundle of TM. $D \subseteq TM$

The rank of D is $\operatorname{rank} D = \dim(D_q)$ $q \in M$

$\{ TM = \bigsqcup T_pM$ Assign a k-dim subspace ... $\subset T_qM$ ...

$TM = \bigsqcup_p T_pM$. Assign a k-dim subspace $V_p \subseteq T_pM$ to each $p \in M$.

$D = \bigsqcup_p V_p \subseteq \bigsqcup_p T_pM$, D is a subbundle

Def: $N \subseteq M$, N is called integral manifold of D, if $T_pN = D_p$ for any $p \in M$.

$D_p := V_p$.

D is defined on the entire M

Def: D is involutive if $\forall X, Y \in D$, then $[X, Y] \in D$. i.e. is closed under Lie bracket

1-form Criterion for smooth Distr.

D is a smooth distri. iff $\forall p \in U \subset M$, $n = \dim M$

$\exists$ 1-forms $\alpha_1, \dots, \alpha_{n-k}$ s.t.

$$D_q = (\ker \alpha_1 \cap \cdots \cap \ker \alpha_{n-k})_q \quad \forall q \in U.$$

If $\alpha_1, \dots, \alpha_{n-k}$ linearly independent, then $\dim D_q = k$.

Def: $\alpha_1, \dots, \alpha_{n-k}$ are the defining forms for D.

Eqv. $D = \ker \alpha_1 \cap \cdots \cap \ker \alpha_{n-k}$

If $\beta \in T^*M$, and $\beta(D) = 0$, then β annihilates D

Def: A distribution D on M is integral if $\forall p \in M$ is contained in an integral manifold of D.

Prop: Integral distribution $\Rightarrow$ involutive distri.

1-Form Criterion for involutivity. $D \subset TM$.

1-Form Criterion for involutivity. $D \subset TM$.

D is involutive iff the following is true:

If β is a 1-form annihilating D ($\beta(D)=0$) on $U \subset M$,

then $d\beta(D)=0$ on U.

Local Coframe Criterion for Involutivity.

Let D be a rank-k distr. with $\alpha_1, \dots, \alpha_{n-k}$ as the defining 1-forms for M.

The followings are equivalent.

① D is involutive

② $d\alpha_1, \dots, d\alpha_{n-k}$ annihilate D

③ $\exists$ 1-forms $\{\beta^i{}_j\}$ $i,j=1,\dots,n-k$. $(n-k)^2$ many 1-forms

s.t. $d\alpha_i = \sum_{j=1}^{n-k} \alpha_i \wedge \beta^i{}_j$

Sketch of ① ⇔ ② (Aug 2016 # 8)

① ⇒ ② Say $\eta(D)=0$ on D, where $\eta \in T^*M$.

☆ $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$

Since $[X,Y] \in D$, $\eta(D)$, so $\eta([X,Y])=0$

$d\eta(X,Y) = X(0) - Y(0) - 0 = 0$ □

$d\eta(D)=0$

② ⇒ ① $\alpha_1, \dots, \alpha_{n-k}$ define D.

$\alpha_i([X,Y]) = X(\underbrace{\alpha_i(Y)}_{=0}) - Y(\underbrace{\alpha_i(X)}_{=0}) - \underbrace{d\alpha_i(X,Y)}_{=0 \text{ by ②}}$

$=0$

n-k

$$[X,Y] \in \overset{=0}{\bigcap_{i=1}^{n-k}} \ker \alpha_i = D$$

$\square$

Def: $D \subseteq TM$ is called <u>completely integrable</u>, $n = \dim M$

if $\exists$ a chart $(U, x_1, \ldots, x_n)$ s.t.

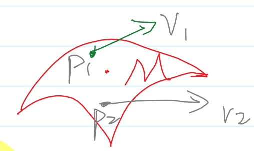
$$D = \operatorname{span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}\right), \quad k = \dim D, \quad k \leq n$$

Rem. If $M = \mathbb{R}^n$, then D is already com. int.

Completely integrable $\Rightarrow$ integrable $\Rightarrow$ involutive

Frobenius Thm

Frobenius Thm: Every involutive distri. is completely integrable.



$D = \operatorname{span}(X_1, \ldots, X_k)$, $X_i \in TM$

If there is another coordinates $(u_1, \ldots, u_n)$ s.t. $X_1 = \frac{\partial}{\partial u_1}$, $\vdots$, $X_k = \frac{\partial}{\partial u_k}$

It depends on M.