In #8 Jan 17, Given $A \in SO(2)$, $W \in Sp(2n)$

$A^{-1}WA = W$ is due to W being skew-symmetric

and also $A^{-1} = A^T$ $\qquad W^T = -W$

$\qquad\qquad\qquad\qquad \mathrm{diag}(W) = 0$

$$A^{-1}WA = WA^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A, \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= W\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = W$$

Aug 18 #4

a) To show, there is no nonvanishing vector field on S^2 (S^n)

By contradiction, $\exists\, V \in TS^n$, $V := \frac{V}{\|V\|}$.

WLOG, assume $\|V\| = 1$, i.e.

$\|V(p)\| = 1$, $p \in S^n$

$\|\cdot\|$ norm induced by $\mathbb{R}^{n+1}$

$\langle\ ,\ \rangle$ inner product of $\mathbb{R}^{n+1}$

$\langle p, V(p)\rangle = 0$

Define: $F : S^n \times [0, \pi] \to S^n$

$F(x, \theta) = x\cos\theta + V(x)\sin\theta \in S^n$

F is well-defined on S^n

$F(x, 0) = x = \mathrm{Id}_x$, $F(x, \pi) = -x$ (antipodal map)

$Id_X \underset{F}{\simeq}$ Antipodal map   (assume M compact)

Define : Degree of a map $f: M \to N$ at $y \in N$

$$\deg f = \sum_{x \in f^{-1}(y)} \mathrm{sign}(df_x)$$

If $f \underset{F}{\simeq} g$ then $\deg f = \deg g$.

$\deg(Id_X) = \mathrm{sign}(df_x) = 1$ $\mathbb{R}^{n+1} \simeq \mathbb{C}^{\frac{n+1}{2}}$

$$\deg(-X) = \begin{cases} 1 & n+1 \text{ even} \\ -1 & n+1 \text{ odd} \end{cases} \Rightarrow \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$$

$\deg(Id_X) = \deg(-X) \Rightarrow$ n is odd

S^2 is even. $\square$

b) $S^3 \subseteq \mathbb{R}^4 \simeq \mathbb{C}^2$, $\mathbb{C}^2 = (z_1, z_2)$

$x_1 = \mathrm{Re}\, z_1$, $x_2 = \mathrm{Im}\, z_1$, $x_3 = \mathrm{Re}\, z_2$, $x_4 = \mathrm{Im}\, z_2$

$(x_1, x_2, x_3, x_4) \sim (z_1, z_2)$

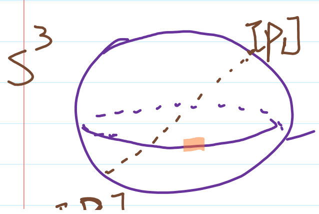
$p \in S^3 \in \mathbb{R}^4$, $V(p) := i \cdot p = i(z_1, z_2)$

$V(p)$ is non-vanishing

and $\langle V(p), p \rangle = 0$

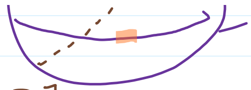
$V(p)$ is what we want. $\square$

c) $S^3 \xrightarrow{\pi} \mathbb{R}P^3$, $V(p)$ is defined above



$\pi_* V(p) \in T\mathbb{R}P^3$

$\pi_*(V)(p)$ is well-defined on S^3.



$\pi_*(V)(p)$ is well-defined on S^3.

$[P_2]$

$\pi(p_1)=\pi(p_2)=[p_1]$

$V(p_1)$

$\pi_* V(p_1) \overset{?}{=} \pi_* V(p_2)$

p_1

If yes,

$\pi_* V(p)$ is well-defined

p_2

$V(p_2)$

$F: M \to N$. $V \in TM$

$F_* V(q) := dF_{F^{-1}(q)} V(F^{-1}(q))$, $dF_p V(p)$

$q \in N$

$= \boxed{(F_* \circ V)} \circ F^{-1}$

$[2xy\,\partial x] \circ F^{-1}$

$(dF)_p$ = Jacobi of F at p.

Jacobi is of F's $\quad (dF)_{F^{-1}(q)}$

domain $(x,y) \in \mathbb{R}^2$

$(u,t) \in \mathbb{R}^2$ image

$q \in N \quad q=(x,y) \quad ((dF)_{F^{-1}(x,y)})(V(F^{-1}(x,y)))$

$F^{-1}(u,t) = (\sqrt{u/t}, \sqrt{ut})$, $F(x,y) = (xy, \frac{y}{x})$

$F_* X(u,t) = dF_{F^{-1}(u,t)} \quad F^{-1}: \; x=\sqrt{u/t}, \; y=\sqrt{ut}$

$$dF|_p = J|_p = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix} = \begin{pmatrix} \sqrt{ut} & \sqrt{u/t} \\ -\frac{\sqrt{ut}}{u/t} & \sqrt{\frac{t}{u}} \end{pmatrix}$$

T_pM

$(\sqrt{\frac{u}{t}}$

T_pM

$$X(F^{-1}(u,t)) = \sqrt{u/t}\,\frac{\partial}{\partial x} + \sqrt{ut}\,\frac{\partial}{\partial y} = \begin{pmatrix} \sqrt{\frac{u}{t}} \\ \sqrt{ut} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{ut} & \sqrt{u/t} \\ -\frac{\sqrt{ut}\cdot t}{u} & \sqrt{\frac{t}{u}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{u}{t}} \\ \sqrt{ut} \end{pmatrix} = \begin{pmatrix} 2u \\ t-t \end{pmatrix} = \begin{pmatrix} 2u \\ 0 \end{pmatrix}$$

$$= 2u\frac{\partial}{\partial u}$$

$$= 2x\frac{\partial}{\partial x}$$

change of basis