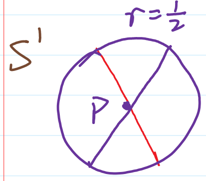
#2 $\gamma(t) = tp_1 + (1-t)p_2$ for some $\|p_1 - p_2\| = 1$, $p_1, p_2 \in \mathbb{R}^2$



center at $p$

$\mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times \mathbb{RP}^1 = \mathbb{R}^2 \times S^1$

$(p, \theta)$

$\theta \in S^1$

$M : \{\text{set of } \gamma(t)\}$

$M \sim \mathbb{R}^2 \times S^1 \sim \mathbb{R}^2 \times \mathbb{RP}^1$

Define $f : M \to \mathbb{R}^2 \times \mathbb{RP}^1$

$f(\gamma) = (p, [\gamma(0) - p])$ . $p = \frac{1}{2}(\gamma(1) + \gamma(0))$

$\gamma : [0,1] \to \mathbb{R}^2$

$\|\gamma\| = 1$

Injective: If $f(\gamma_1) = f(\gamma_2)$

$(p_1, [\gamma_1(0) - p_1]) = (p_2, [\gamma_2(0) - p_2])$

$p_1 = p_2$ and $\gamma_1(0) = \gamma_2(0)$ or $\gamma_1(0) = \gamma_2(1)$

$\gamma_1 = \gamma_2$

Surjective. $\forall (p, [x]) \in \mathbb{R}^2 \times \mathbb{RP}^1$,

let $\gamma(t) = t(p - x) + (1-t)(p + x)$

$f(\gamma) = (p . [x])$

$M \underset{f}{\cong} \mathbb{R}^2 \times \mathbb{RP}^1 \sim \underline{\mathbb{R}^2 \times S^1}$ , $\dim M = 3$

$M$ is oriented $\square$

#4. For any open covering $\bigcup_\beta A_\beta$ of $X$,

for $j$ fixed, $\bigcup_\beta A_\beta \supseteq \overline{U_j}$

Apply paracompactness of $\bar{U}_j$

$\exists$ locally finite subcover $\bigcup_{\beta \in I(j)} A_\beta \supseteq \bar{U}_j$

$X = \bigcup_{j \in \mathbb{N}} \bigcup_{\beta \in I(j)} A_\beta$, countably locally finite open covering

to $\bigcup_\beta A_\beta$.

$\bigcup_\beta A_\beta$ has count. locally. finite open subcover

$\underset{\text{lemma 41.3 on Munkres}}{\Longrightarrow}$ $\bigcup_\beta A_\beta$ has a locally finite subcover

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#6 $\bigoplus_{k=1}^{n+1} (S^n \times \mathbb{R})^k = S^n \times \mathbb{R} \times \mathbb{Z}_{n+1} = S^n \times (\mathbb{R} \times \mathbb{Z}_{n+1})$

$= S^n \times (\bigoplus^{n+1} \mathbb{R}) \simeq S^n \times \mathbb{R}^n$

$TS^n \sim (p, v, 0)$ $p \in S^n$, $v \in T_pS^n$, $0 \in \mathbb{R}$

$S^n \times \mathbb{R} \sim (p, 0, r)$ $p \in S^n$, $0 \in T_pS^n$, $r \in \mathbb{R}$

$TS^n$ vector bundle of $S^n$

$S^n \times \mathbb{R}$ 1-dim vector bundle of $S^n$

$S^n \times \mathbb{R} \oplus TS^n = \bigsqcup_{p \in S^n} \{(p, v, r),\ v \in T_pS^n,\ r \in \mathbb{R}\}$

$= TS^n \times \mathbb{R}$

Proving $TS^n \times \mathbb{R} = S^n \times \mathbb{R}^{n+1}$

Define $f : S^n \times \mathbb{R}^{n+1} \to TS^n \times \mathbb{R}$

Denote $\langle \, , \, \rangle$ inner product on $\mathbb{R}^{n+1}$

$$f(p,v) = (p, v - \langle p, v \rangle \cdot p, \langle p, v \rangle)$$

$\forall p \in S^n, v \in T_pS^n \subseteq \mathbb{R}^{n+1}, \quad S^n \hookrightarrow \mathbb{R}^{n+1}$

$f$ is surjective : trivial.

$f$ is injective : If $f(p_1, v_1) = f(p_2, v_2) \Rightarrow p_1 = p_2$

$v_1 - \langle p_1, v_1 \rangle p_1 = v_2 - \langle p_2, v_2 \rangle p_2, \quad \langle p_1, v_1 \rangle = \langle p_2, v_2 \rangle$

$\Rightarrow \quad v_1 = v_2 \quad \square$

$f$ is bijective. $\square$.

$\mathbb{R}^2 \sim \mathbb{R} \oplus \mathbb{R}$

#9 : let $N \overset{i}{\hookrightarrow} M$. $\exists \, X' = (di)X$ and $Y' = (di)Y$ ($X'$ is $i$-ralated to $X$)

$X, Y \in TN \subseteq TM \quad X', Y' \in TN \subseteq TM$

If $X', Y' \in TN$, $[X', Y'] \in TN$, $[X', Y'] = (di)[X, Y]$

Lie bracket is closed under $i$-related relationship

$X'$ is $i$-related, $Y'$ is $i$-related

$\Rightarrow [X', Y']$ is $i$-related to $[X, Y]$

~~Ød~~ $[X, Y] \in TN$.

#10 $\exp_p(v) : T_pM \to M$.

$\exists!$ geodesic $\gamma(t) : \mathbb{R} \to M$.

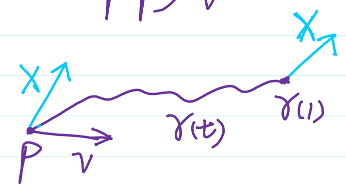
s.t. $\gamma(0) = p \quad \gamma'(0) = v$

$\exp_p(v) \longmapsto \gamma(1)$

$\tau(s) = \exp_p(s\cdot v)$ is defined on $\mathbb{R}$

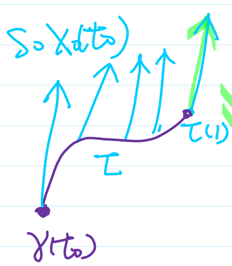
$d(\exp_p)_v : T(T_pM) \sim T_pM \to T_pM$

on $M$




$\alpha'_{t_0}(S_0) = \frac{d}{ds} \exp_{\gamma(t_0)}(S X(t_0))\Big|_{S=S_0}$ , Say $\tau(t)$ is the geodesic at $\gamma(t_0)$

$= \frac{d}{ds}\left(\tau(1, \gamma(t_0), S X(t_0))\right)\Big|_{S=S_0}$

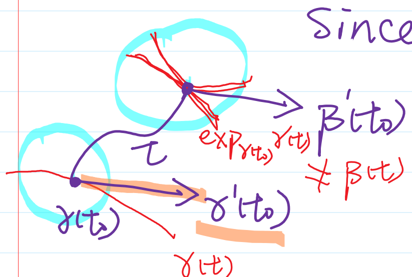



$= \frac{d}{ds}\left(\tau(S, \gamma(t_0), X(t_0))\right)\Big|_{S=S_0}$

$= d(\exp_{\gamma(t_0)})_{(S_0X(t_0))}(X(t_0))$

$\beta'(t_0) = \frac{d}{dt}\left(\exp_{\gamma(t)}(S_0X(t))\right)\Big|_{t=t_0}$ ,

Since $\exp_{\gamma(t_0)}$ is a global diffeo.




$\beta'(t_0) = d(\exp_{\gamma(t_0)})_{(S_0X(t_0))}(\gamma'(t_0))$

$\langle \gamma'(t_0), X(t_0)\rangle = 0$

Apply Gauss lemma

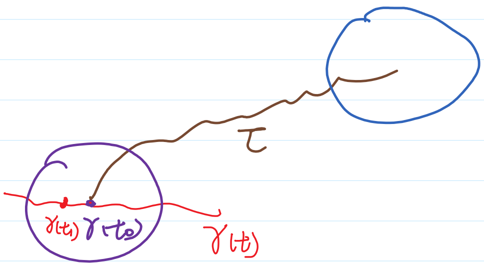
$\Rightarrow \langle d(\exp_{\gamma(t_0)})_{(S_0X(t_0))}\gamma'(t_0), d(\exp_{\gamma(t_0)})_{S_0X(t_0)}(X(t_0))\rangle$

$= \langle \gamma'(t_0), X(t_0)\rangle = 0$

$\Rightarrow \langle \alpha'_{t_0}(S_0), \beta'(t_0)\rangle = 0$ at $(S_0, t_0)$

$\square$

$\square$




$\exp_{\gamma(t_1)}$ & $\exp_{\gamma(t_0)}$ share the same diffeo.

i.e. $\exp_{\gamma(t_0)}$ is the diffeo for $\gamma(t_1)$.

$$\frac{d}{dt}\beta(t) = \frac{d}{dt}\underline{\exp_{\gamma(t)}}\, S_0 X(t)$$

fix $t_0$ a certain $t_0$.