July 19

Monday, July 19, 2021 2:07 PM

$$\begin{array}{ccc} M^m & \xrightarrow{f} & N^n \\ \psi \uparrow & & \uparrow \psi' \\ \mathbb{R}^m & \xrightarrow{\pi} & \mathbb{R}^n \end{array} \qquad m \geq n$$

$\pi : (x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n)$

let $K = (x_1, \dots, x_n, 0, \dots, 0)$ I want $\dim K = n$.

$K \subseteq \mathbb{R}^m$

$W := \psi(K)$ is a submanifold of M^m

$W \subseteq M^n$, $T_pM = T_pW \oplus \ker df_{(p)}N$

$f|_W$ is a diffeo. by IFT. on a nghd $U \subseteq M$ of p.

since $df_{(p)}|_W$ has rank $= n$

Let $V := f|_W(U)$, and V is the one we want

enlarge N by doing $N \times \mathbb{R}^{m-n}$

#6. General Idea:

Construct map $F : M(n) \to N$

want $F^{-1}(I_n) = Sp_{2n}(\mathbb{R})$

$Q^* := -JQ^TJ$

$N := \{Q \mid Q = Q^*\}$

$F(A) := A \cdot A^* = -AJA^TJ$

$F^{-1}(I_{2n}) = Sp(2n)$
$= \{Q \mid -QJQ^TJ = I_{2n}\}$

What's dF_Q

$Q^{*\prime} = (Q')^*$

Say $Q(t)$ a curve in $M(n)$

$$\frac{dF(Q(t))}{dt}\Big|_{t=0} = (Q \cdot Q^*)'(0) = Q'(0) \cdot Q^*(0) + Q(0) Q^{*\prime}(0)$$

$$\frac{dF(Q(t))}{dt}\Big|_{t=0} = (Q\cdot Q^T)'(0) = Q'(0)\cdot Q^T(0) + Q(0)Q^{T\prime}(0)$$

let $B \in \underbrace{T_{Q(0)}M(n)}_{\simeq M(n)}$, s.t. $B = Q'(0)$

If $A \in F^{-1}(I_{2n})$ $AA^* = I_{2n}$

$$d_A F(B) = A^*B + B^*A$$

Need to show d_AF is surjective. $T_AM \to T_{F(A)}N$.

$N = \{Q \mid Q = Q^*\}$. $\forall C \in N$. $C = C^*$.

$$d_AF(\tfrac{1}{2}AC) = \tfrac{1}{2}A^*AC + \tfrac{1}{2}(AC)^*A$$

$$= \tfrac{1}{2}I_{2n}\cdot C + \tfrac{1}{2}C^*\cdot \underbrace{A^*\cdot A}_{I_{2n}}$$

$$= \tfrac{1}{2}C + \tfrac{1}{2}C^* = C$$

$\Rightarrow$ d_AF is surjective. $\underset{I_{2n}}{F(A)}$ is a regular value

$F^{-1}(I_{2n}) = Sp_{2n}(\mathbb{R})$ is a submanifold.

$Q^TJQ = J \Rightarrow Q^TJQJ^{-1} = I_{2n}$

$O(n)$ where $A\cdot A^T = I_n$ $\quad F(Q)$ $\quad F^{-1}(I_{2n}) = Sp_{2n}(\mathbb{R})$

Ex. Define $F: M(n) \to N = F(M) = \{A \mid A = A^T\}$.

$F(A) = A\cdot A^T$.

Ex. $X = \{A \mid F'(A, \xi) = \xi\}$.

$$F(A) := F'(A,\xi)\xi^{-1} = I_n$$

$$X = F^{-1}(I_n)$$

$\dim Sp(2n)(\mathbb{R}) = \dim Lie(Sp(2n)(\mathbb{R}))$

$\dim M = \dim Sp_{2n}(\mathbb{R})$

$Lie(Sp(2n)) \cong T_{I_{2n}} Sp_{2n}(\mathbb{R}) = \ker d_{I_{2n}} F = \{A \mid A + A^* = 0\}$

Since $Sp_{2n} = F^{-1}(I_{2n})$, $\quad T_{I_{2n}} Sp_{2n}(\mathbb{R}) = \ker d_{I_{2n}} F$

$\gamma(t) \in Sp_{2n}$, $F(\gamma(t)) = I_{2n}$

$F'(\gamma(0)) = 0$

Define $A := \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$, a, b, c, d $n \times n$ matrices

If $A \in \ker d_{I_{2n}} F$. $\quad A + A^* = 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -A^* = J \begin{pmatrix} a & c \\ b & d \end{pmatrix} J$$

$$\begin{cases} a + d^T = 0 \\ b = b^T \quad c = c^T \end{cases}$$

$\dim Lie(Sp\, 2n)$

$\dim a = n^2$, $\quad d^T = a$

$\dim b = \frac{1}{2} n(n+1)$ $\quad \dim d = 0$

$\dim c = \frac{1}{2} n(n+1)$

$\dim Lie(Sp(2n)) = n^2 + \frac{1}{2} n(n+1) + \frac{1}{2} n(n+1)$ $\square$

$Lie = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a = -d^T \\ b = b^T \end{array}, \ c = c^T \right\}$

#7. Involutive $\Rightarrow$ $\alpha_j = 0$ $= \underbrace{dy_j}_{=0} + \sum_p y_p \pi^* w_{jp}$

$d\alpha_j = 0 = \sum_p dy_p \wedge \pi^* w_{jp} + \sum_p y_p \pi^*(dw_{jp})$

$dy_j = -\sum y_p \pi^* w_{jp}$

$= \sum_p \sum_k (-y_k \pi^* w_{pk}) \wedge \pi^*(w_{jp})$

$+ \sum_k y_k \pi^* dw_{jk}$

Collect ... [illegible]

$$+\sum_k y_k \pi^* dw_{jk}$$

Collect y_k

$$= \sum_k y_k \left(\pi^* dw_{jk} - \sum_p \pi^* (w_{pk} \wedge w_{jp})\right)$$

$$= \sum_k y_k \, \pi^* \left(dw_{jk} - \sum_p w_{pk} \wedge w_{jp}\right)$$

$$= 0 \quad = d\alpha_j$$

Each coeff. of y_k's is 0. $\Rightarrow \square$

$\Leftarrow$ Same, reverse the process in $\Rightarrow$.

$E = M \times \mathbb{R}^k$, $p \in M$

$e = (p, 0, 0, 0, \underset{j}{1}, 0, 0) \in E$.

$d\alpha_j(e) = \pi^* \big(\underbrace{dw_{jk} - \sum w_{pk} \wedge w_{jp}}_{=0} \big) = 0$ on H.

$\forall p \in N$. $\exists\, U_p \subseteq M$, $U_p \ni p$. on U_p, $p = (x_1, \dots, x_n, 0, \dots, 0)$

$N = (x_1, \dots, x_n, 0, \dots, 0)$

$f: N \to \mathbb{R}$.

$\exists\, f'_p : \mathbb{R}^m \to \mathbb{R}$ s.t. $\pi(f'_p) = f$

$\mathbb{R}^m \supseteq U$

$$\begin{array}{c} \mathbb{R}^m \\ \downarrow \pi \\ \mathbb{R}^n \end{array}$$

$f'_p : U_p \to \mathbb{R}$, $U_p \subseteq M$

$\bigcup_{p \in N} U_p \subseteq \bigcup_{j=1}^{T} U_p$, glue f'_p along U_p with P. U.:

$\{\beta_j\} \sim U_j$ $\quad F := \sum f'_p \cdot \beta_j$, $\quad F: M \to \mathbb{R}$