#5, $x\in f^{-1}(y_1)$, $f(x)=\{y_1\}$ closed. $\{y_2\}$ closed

X normal, $\exists\, U_1 \supseteq f^{-1}(y_1)$. $U_2\supseteq f^{-1}(y_2)$, $U_1\cap U_2=\phi$

$f(U_1^c)$ closed in Y $\quad y_1\notin f(U_1^c)$

$f(U_2^c)$ closed in Y $\quad y_2\notin f(U_2^c)$

$f(U_1^c)\cup f(U_2^c)=Y$
surjectivity of f.

$\Rightarrow [f(U_1^c)\cup f(U_2^c)]^c = Y^c$

$\Rightarrow f(U_1^c)^c\cap f(U_2^c)^c=\phi$

$y_1\in f(U_1^c)^c \quad y_2\in f(U_2^c)^c$

since $y_1\notin f(U_1^c)$

Done □

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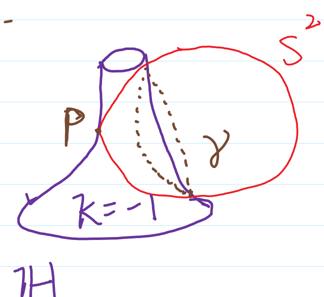
a) Skip.

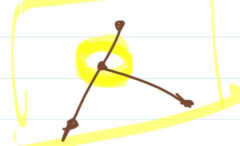
c)



$\mathbb{H}\cap S^2=\{p\}\cup\{\gamma\}$
0-mfd ∪ 1-mfd

2-dim

?



b) Fix $p\in S_1\cap S_2$, $S_1\subset M^n$ $\quad s_1=\dim S_1$
$s_2=\dim S_2$

By I.F.T. $\exists\,\varphi_1: U\subseteq M\to\mathbb{R}^{n-s_1}$ s.t.

0 is a regular value $\Rightarrow U\cap S_1=\varphi_1^{-1}(0)$

$n-s_2$

$\exists\ \varphi_2 : U \subseteq M \to \mathbb{R}^{n-s_2}$ s.t. $U \cap S_2 = \varphi_2^{-1}(0)$

Consider $\varphi_1 \times \varphi_2 : U \to \mathbb{R}^{n-s_1} \times \mathbb{R}^{n-s_2}$

$$(\varphi_1 \times \varphi_2)_* : T_pM \to \mathbb{R}^{n-s_1} \times \mathbb{R}^{n-s_2}$$

$$(\varphi_1 \times \varphi_2)(p) = (\varphi_1(p), \varphi_2(p))$$

Want to show $(\varphi_1 \times \varphi_2)$ is regular at $(0,0)$

$$\Rightarrow (\varphi_1 \times \varphi_2)^{-1}(0,0) = \varphi_1^{-1}(0) \cap \varphi_2^{-1}(0) = S_1 \cap S_2$$

To show $\dim \mathrm{Im}\,(\varphi_1 \times \varphi_2)_* = \dim(\mathbb{R}^{n-s_1} \times \mathbb{R}^{n-s_2})$

$$\ker(\varphi_{1*}) \cap \ker(\varphi_{2*}) = T_pS_1 \cap T_pS_2$$

$$= \ker(\varphi_1 \times \varphi_2)_*$$

$$\dim(T_pS_1 \cap T_pS_2) \overset{\text{tras.}}{=} \dim T_pS_1 + \dim T_pS_2 - \dim T_pM$$

$$= s_1 + s_2 - n$$

$$\dim(\mathrm{Im}) + \dim(\ker) = \dim M$$

$$\dim(\mathrm{Im}) = n - (s_1 + s_2 - n) = 2n - s_1 - s_2$$

$$\dim(\mathrm{Im}(\varphi_1 \times \varphi_2)_*) = \dim(\mathbb{R}^{n-s_1} \times \mathbb{R}^{n-s_2})$$

Surjective

0 is regular

$$S_1 \cap S_2 = (\varphi_1 \times \varphi_2)^{-1}(0) \qquad \square.$$

$$\dim(S_1 \cap S_2) = \dim T_p(S_1 \cap S_2)$$

$$= s_1 + s_2 - n$$

Suffices to show true for any $u \cdot dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \in \Omega^*(M)$

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$LHS = d\,F^*(u\,dx_{i_1} \cdots dx_{i_k}) = d\left[u \circ F\, d(x_{i_1} \circ F) \cdots dx_{i_k} \circ F\right]$

$= d(u \circ F) \wedge d(x_{i_1} \circ F) \cdots \wedge d(x_{i_k} \circ F)$

RHS $\quad F^*(d(u\,dx_k)) = F^*(du \wedge dx_{i_1} \cdots dx_{i_k})$

$= d(u \circ F) \wedge d(x_{i_1} \circ F) \cdots \wedge d(x_{i_k} \circ F)$

$L = R$

a) $\alpha \in \Omega^*(M_2) \qquad \varphi^*\alpha \in \Omega^*_k(M_1)$

$\varphi^*\alpha(v_1, \ldots, v_k)_p = \alpha(d\varphi_p v_1, \ldots, d\varphi_p v_k)$

#9 Aug 2016 a) b) c)

Jan 2021, Find K under g.

a) ① Find a orthonormal by Gram-Schmitz

b) ② Find the co-frame

c) ③ Find K. $\qquad dx \otimes dx = dx^2$

2. $dxdy := dx \otimes dy + dy \otimes dx$

$dx \otimes dy \curvearrowright dxdy \qquad$ If $x = y \quad 2dx^2 = 2dx^2$

#10 a) let $\omega = \omega / \int_N \omega$ WLoG

assume $\int_N \omega = 1$

R at dim $= n$

Assume $\int_N \omega = 1$

By Poincaré lemma: $H_c^*(M) = \begin{cases} \mathbb{R} & \text{at } \dim = n \\ 0 & \text{else} \end{cases}$

$H_c^*(M) = $ closed forms / exact forms

Say $\int_N \omega = \int_N \alpha = 1$,

$\exists\ \beta$ $(n-1)$ form s.t. $\omega = \alpha + d\beta$

$[\omega] = [\alpha] \in H_c^*(M)$

$$\int_M \varphi^*\omega = \int_M \varphi^*(\alpha + d\beta) = \int_M \varphi^*\alpha + \int_M d\varphi^*\beta$$

= Stokes

$$= \int_M \varphi^*\alpha + 0 = \int_M \varphi^*\alpha = \deg(\varphi)$$

$\square$

Remark $H_c^*(M) \underset{\int_M \alpha}{\overset{\int}{\sim}} \mathbb{R}$

$\forall \alpha \in H_c^*(M)$ $\int_M \alpha \mapsto x \in \mathbb{R}$

b) $H_c^n(S^n) = \mathbb{R}$ $\qquad \varphi: M \to S^n$

$\varphi^*: H_c^n(S^n) \to H_c^n(M)$

$\int_{S^n} \alpha = 1$, α is a generator of $H_c^n(S^n)$

$\varphi^*\alpha$ is a multiple of $\mathbb{1} \in \mathbb{R}$

Assume $\deg \varphi \neq 1$.

$\varphi^*\alpha$ is always multiple of $\mathbb{1}$.

? φ^* is not surjective.