July 7

Wednesday, July 7, 2021 12:20 PM

1 Class Build-up

- Complete 2 past quals as weekly Homework, and present your solution to the class during meetings.
- Each problem covered by a different person.
- Sign-up sheet for each problem on google drive. Record for your attendance.
- Everyone is encouraged to sign up.
- Suggestion: present a problem you had trouble with, or a problem that took a lot of time. So we may give feedback.
- Grading based only on attendance, but not important after all.

1.1 Goal: PASS QUALIFIER

1.2 PASS QUALIFIER

1.3 PASS QUALIFIER

1.4 Study

1. We plan to work on past quals from Aug 2016 to Jan 2021. This week HW: Aug16, Jan17.
2. Collect all failed problems. Redo the problems every week, until completely solved.
3. Mimic the testing environment while doing past quals.
   - Reserve 1pm - 5pm: no break, no internet, no books. Only papers and pen. Stop immediately at 5 pm.
   - Exam strategy on each HW practice: mark the problem taking you longer than 25-30 minutes, and look back when there is time left.
4. Finish remaining problems after a break.

1.5 On Aug 12

1. Arrive on time at Blocker 166.
2. Bring the best pen/pencils, erasers. Papers will be provided.
3. Find a good seat: dark or light, front or back, far or closed to AC.
4. Caffeine and sugar.
5. Bring extra clothes in case.

1.6 Email form Dr. Xu

Thanks. You and the students have probably noticed the recent update of the syllabus. The old ones are also available on the website. The first message is that the coverage of the qualify exam will be contained in the "Intersection" of the new and old syllabi.

I feel the following topics deserves special attention in this Prep Session (for the differential geometry part): 1. The definition of Gauss curvature 2. The computation of Gauss curvature if one only has the first fundamental form (there are a few problems in previous exams) 3. The Gauss-Bonnet theorem for surfaces (some concrete examples) 4. The definition of integration of differential forms and the Stokes theorem. 5. the implicit function theorem (both theory and application). There are other important theoretical materials in the related courses but these topics are associated with interesting exam problems.

Let me only say this much at this point. I will write a more detailed message later. Also please don't hesitate to ask me any questions, either general or specific.

Best,

Guangbo Xu

2 Topology Checklist

Munkres' book

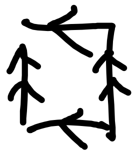
1. Topological spaces; closure and interior of sets. Generating topologies: subspaces, the order topology, the product topology, the quotient topology, the basis for a topology.
2. Continuous functions and homeomorphisms;
3. The gluing lemma(Pasting lemma).
   - Statement and proof
4. Connectedness and path-connectedness
   - Counterexamples.

   $y = \sin\frac{1}{x}$ at 0
5. Compactness.
6. Separation: Regular spaces, Normal spaces, Urysohn's Metrization Theorem; Tietze's extension theorem.
   - local finiteness.
   - Statement of theorems.
   - Direct application of theorems.
7. The Tychonoff Theorem.
   - Statement of theorems.
   - Direct application of theorems.
8. Homotopy of paths and the fundamental group of the circle.
9. Retraction.

- Brouwer fixed-point theorem.
  - Statement and proof

10. Fundamental group Examples:
    (a) $\Pi_1(Torus) = \mathbb{Z} \times \mathbb{Z}$
    (b) $\Pi_1(\mathbb{P}^n) = \mathbb{Z}_2$
    (c) $\Pi_1(Klein\ bottle) = \{a, b | a^2 = b^2\}$
    (d) $\Pi_1(Mobius) = \mathbb{Z}$

11. Van-Kampen theorem.
    - Know how to draw a diagram.
    - Surgery on 2-dim surfaces.



12. Covering space.
    - Definition.
    - Examples: Circle, Torus, Projective Space,...
    - Path lifting lemma.
      - Statement and proof.

3 Geometry Checklist

1. Differentiable manifolds; examples.
   - Change of coordinates.
2. Smooth maps, tangent spaces/bundle, differentials of smooth maps.

Bredon's Top. & Geo.

3. Implicit and Inverse Function Theorems; regular/ critical values/ points. , embeddings, immersions, immersed and embedded submanifolds.
   - Given a certain coordinate, how to find critical values/points.
   - Precise statement of theorems and definitions.
4. Vector fields and flows. Lie brackets of vector fields.
   - Compute a Lie bracket.
5. Tensor fields. The notions of pull-back and push-forward. Lie derivatives of tensor fields.
   - Given any map, compute the pullback of a form.
   - Given an embedding,
6. Differential forms α and exterior differential $d\alpha$ of them.
7. Basics on Lie groups: their Lie algebras, General Linear group, Special Linear group, orthogonal group, symplectic group, and their Lie algebras.

- Lie groups of certain manifolds.

8. The Gauss map; The first and second fundamental forms; The Gaussian and mean curvatures. The Gaussian curvature of two-dimensional Riemannian manifold.
9. The Gauss Egregium theorem. The Gauss-Codazzi equations. The fundamental theorem of surface theory (the Bonnet theorem).
   - Connection is not tested in previous exams.
10. The Gauss-Bonnet theorem.
11. Stokes' Theorem
12. Distributions, foliations, integral submanifolds. Frobenius theorem
    - Determine whether a distribution is integrable/involutive or not.
13. Orientation.
    - Relation between top forms.

Gauss map & Gauss Curvature. Spivak's Vol 2. chap 3

M is a 2-dim sub-mnfd of $\mathbb{R}^3$, with $i: M \to \mathbb{R}^3$

Assume M is oriented.

$\forall p \in M$, assign a unique unit vector $\nu(p) \in T_p\mathbb{R}^3$

s.t. ① $\nu(p) \in T_p\mathbb{R}^3$ is perpendicular to T_pM

② $\nu(p), X_1, X_2$ is positively oriented in $\mathbb{R}^3$

$X_1, X_2 \in T_pM$ is an positively oriented basis

$\langle \nu(p), \nu(p) \rangle = 1$, $\nu: M \to S^2 \subset \mathbb{R}^3$

ν is called Gauss map

let $X: \underset{\subset \mathbb{R}^2}{U} \to M \subset \mathbb{R}^3$ be a localization of M,

$\forall (p_1, p_2) \in U \subset \mathbb{R}^2$. $X(p_1, p_2) = (f_1(p_1,p_2), f_2(p_1,p_2), f_3(p_1,p_2))$

Let $X_i = \dfrac{\partial X}{\partial p_i} = \left(\dfrac{\partial f_1}{\partial p_i}, \dfrac{\partial f_2}{\partial p_i}, \dfrac{\partial f_3}{\partial p_i} \right)$

X_1, X_2

Define $\vec{n} = \dfrac{X_1 \times X_2}{|X_1 \times X_2|}$, $\langle \vec{n}, \vec{n} \rangle = 1$

$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$X_1 \times X_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

then $\vec{n}(p_1, p_2) = V(p)$

Define The first fundamental form as

$$g_{ij} = \langle X_i, X_j \rangle \quad , \quad I(X_i, X_j)$$

g_{ij} depends on coordinate

Define The Weigarten map

$$L : T_pM \to \mathbb{R}^3 \quad \text{by} \quad L(X) = -X(\vec{n})$$

namely $\vec{n} = (n_1(p), n_2(p), n_3(p))$

$$X(\vec{n}) = (X(n_1), X(n_2), X(n_3))$$

X_1, X_2

$$L(X_i) = -X_i(V)(p) = -\frac{\partial \vec{n}}{\partial p_i}(p)$$

$$= \left(\frac{\partial n_1}{\partial p_i}, \frac{\partial n_2}{\partial p_i}, \frac{\partial n_3}{\partial p_i} \right)$$

$$\langle \vec{n}, \vec{n} \rangle = 1 \quad , \quad \frac{\partial}{\partial p_i} \langle \vec{n}, \vec{n} \rangle = \frac{\partial}{\partial p_i} 1 = 0$$

$$LHS = \langle \frac{\partial \vec{n}}{\partial p_i}, \vec{n} \rangle + \langle \vec{n}, \frac{\partial \vec{n}}{\partial p_i} \rangle = 0$$

$$\Rightarrow \quad \langle \vec{n}, \frac{\partial \vec{n}}{\partial p_i} \rangle = 0$$

$L: T_pM \to T_pM$ since $L(x_i) \perp \vec{n}$

Def : $K = \det(L)$. K is the Gauss Curvature

Def : Principal curvatures are eigenvalues of L.

k_1, k_2 be the eigenvalues. $\det(L) = k_1 k_2 = K$

Mean Curvature of M is H,

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2} \operatorname{trace}(L)$$

Defn $\mathrm{II}(p)(X, Y) = \langle L(X), Y \rangle$, $X, Y \in T_pM$

II is the second fundamental form of M.

$$I(p)(X,Y) = \langle X, Y \rangle$$

Prop If $v_1, v_2 \in T_pM$ orthonormal. then

$$k(p) = \det\left(\mathrm{II}(v_i, v_j)\right)$$

$$= \det\begin{pmatrix} \mathrm{II}(v_1, v_1) & \mathrm{II}(v_1, v_2) \\ \mathrm{II}(v_2, v_1) & \mathrm{II}(v_2, v_2) \end{pmatrix}$$

Prop If $v_1, v_2 \in T_pM$ independent

$$k(p) = \frac{\det\left(\mathrm{II}(v_i, v_j)\right)}{\det\left(I(v_i, v_j)\right)}$$

Gauss Bonnet's Thm

$$\iint_M k\, d_A = 2\pi \chi \quad , \qquad \chi : \text{Euler characteris}$$

$$V - E + F$$

vertices edges faces

$$d_A = \sqrt{\det(\langle V_i, V_j \rangle)}\, dp_1 dp_2$$

$$\iint_M K \sqrt{\det(I(v_i, v_j))}\, dp_1 dp_2 = 2\pi \cdot (V - E + F)$$

v_i, v_j orthonormal $\quad \sqrt{(\cdots)} = 1$

$$\iint_M k \, dp_1 dp_2 = 2\pi \cdot \chi$$

Alternative:

$$k(p) = \lim_{A \to 0} \underbrace{\frac{\text{Area}(V(A))}{\text{Area}(A)}}$$

$p \quad A$

$V(A)$ $\quad p \in A \subset M$

$V(A) \subseteq S^2 \subset \mathbb{R}^3$

$V(A) = \text{constant}$

Ex. For planes $\quad k \equiv 0$

For cylinders $\quad k \equiv 0$, $\quad V(A) = \text{arc in } S^2$

For a sphere with radius r. $\quad S^2(r)$

$K \equiv \frac{1}{r^2}$, $\quad k = \lim \frac{V(A)}{A}$ $\quad V(A) \in S^2$

$A \subset S^2(r)$

Spivak Vol. 2. chap 3

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad , P(a,0,0)$$

$$x=a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$$

Lie bracket : $X=\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$, $Y=\sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}$ $\frac{\partial}{\partial y_i}$

$$[X,Y]= X(Y)-Y(X)=\sum_{i=1}^{n} X(Y_i)\frac{\partial}{\partial x_i}-\sum_{i=1}^{n} Y(X_i)\frac{\partial}{\partial x_i}$$

$Y_i, X_i : M \to \mathbb{R}$, $X(Y_i)$, $Y(X_i)$ are directional directive on $U \subset M$

$(x_1, \cdots, x_n)$ be a coordinate of M.

$(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ be a basis of T_pM.

$X, Y \in TU \subseteq TM$, $X=\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$, $Y=\sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}$

$(u_1, \cdots, u_n)$ on M

$(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n})$ on T_pM , $X=\sum_{i=1}^{n} X_i \frac{\partial}{\partial u_i}$, $Y=\sum_{i=1}^{n} Y_i \frac{\partial}{\partial u_i}$

Jacobi Identity $X, Y, Z \in TM$ (tangent bundle)

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Rm: Sometimes one of the bracket $= 0$.

Lie derivative: A global (local) flow is map

$$\theta : \mathbb{R}\,(U) \times M \to M \qquad (t,p) \mapsto \theta(t,p)$$

s.t. $\theta(t, \theta(s,p)) = \theta(t+s, p)$, $\theta(0,p) = p := Id_M$

For each t, $\theta_t(p) : M \to M$,

$$\theta_t \circ \theta_s = \theta_{t+s}$$

In DG1 Fundamental Thm of Flows

$\forall X \in TM$, $\exists$ unique local flow $\theta(t,p)$ on $U \times M$

s.t.: $X(p) = \dfrac{\partial \theta}{\partial t}(0,p)$ for any $p \in M$.

Lie derivative: $X, Y \in TM$

$$(L_X Y)_p = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(Y_{\theta_t(p)}\right)$$

$$= \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(Y_{\theta_t(p)}\right) - Y_p}{t}$$

Prop $(L_X Y)_p = [X, Y]_{(p)}$ for any p

$$L_X Y = [X, Y] \in TM$$

Ref: Lee, John's Intro to sth mnfd.