#7 Jan 19

(b) Let M and N be 2-dimensional smooth manifolds. Let (α_1, α_2) be a coframe on M and (ω_1, ω_2) be a coframe on N such that there are constants k_1 and k_2 so that

$$d\alpha_1 = k_1\,\alpha_1 \wedge \alpha_2, \quad d\alpha_2 = k_2\,\alpha_1 \wedge \alpha_2, \quad d\omega_1 = k_1\,\omega_1 \wedge \omega_2, \quad d\omega_2 = k_2\,\omega_1 \wedge \omega_2.$$

Prove that for every $q \in M$ and $p \in N$ there exist a neighborhood U of q in M, a neighborhood V of p in N, and a diffeomorphism $F : U \to V$ such that $\alpha_1 = F^*\omega_1$, $\alpha_2 = F^*\omega_2$. in U.

Hint: Let $\pi : M \times N \to M$ and $p : M \times N \to N$ are canonical projections. Work with the distribution on $M \times N$ defined by 1-forms $\pi^*\alpha_1 - p^*\omega_1$ and $\pi^*\alpha_2 - p^*\omega_2$.

$\beta_1 = \pi^*\alpha_1 - p^*w_1$, $\beta_2 = \pi^*\alpha_2 - p^*w_2$. on $M \times N$

$$d\beta_1 = \pi^* d\alpha_1 - p^* dw_1 = \pi^*(k_1\alpha_1 \wedge \alpha_2) - p^*(k_1 w_1 \wedge w_2)$$
$$= k_1(\pi^*\alpha_1 \wedge \pi^*\alpha_2 - p^*w_1 \wedge p^*w_2)$$
$$= k_1(\pi^*\alpha_1 \wedge \pi^*\alpha_2 - \pi^*\alpha_1 \wedge p^*w_2 + \pi^*\alpha_1 \wedge p^*w_2 - p^*w_1 \wedge p^*w_2)$$
$$= k_1(\pi^*\alpha_1 \wedge \underbrace{(\pi^*\alpha_2 - p^*w_2)}_{\beta_2} - p^*w_2 \underbrace{(\pi^*\alpha_1 - p^*w_1)}_{\beta_1})$$
$$= k_1(\pi^*\alpha_1 \wedge \beta_2 - p^*w_2 \wedge \beta_1)$$

By the 1-form Criterion of Invol.

$d\beta_j = \sum_{i=1}^{2} w_{ij} \wedge \beta_i$. $\Rightarrow$ $\hat{D} = \ker\beta_1 \cap \ker\beta_2$ is invol.

$\hat{D}$ is involutive in $M \times N$.

$\Rightarrow$ $\exists$ W be the integral manifold $\hat{D}$. $T_pZ = \hat{D}_p$.
Fer. Thm. $p \in Z$.

$$\begin{tikzcd}
& M \times N \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
M \arrow[rr, bend right=60] & W \arrow[u, "i"] \arrow[l, "\pi \circ i"] \arrow[r, "p \circ i"'] & N
\end{tikzcd}$$

To show In fact
$F := (p \circ i) \circ (\pi \circ i)^{-1}$

to show $(\pi \circ i)$ and $(p \circ i)$ are diffeo.
and $(\pi \circ i)^*\alpha_j = w_j$.

Look at $d(\pi \circ i)$

$\forall$ $v \in \ker(d(\pi \circ i))$, $d\pi(v) = d(\pi \circ i)(v) = 0$

$$0 = \alpha_j(d(\pi \circ i)(v)) \underset{\text{pull back}}{\overset{\text{def}}{=}} (\pi \circ i)^*\alpha_j(v) = (p \circ i)^* w_j(v) \quad ①$$

$\hat{D}$ is involutive. on W, $i^*\beta_1 = i^*\beta_2 = 0$

back

$\hat{D}$ is involutive, on W, $i^*\beta_1 = i^*\beta_2 = 0$

$$\Rightarrow (\pi\circ i)^*\alpha_j = (p\circ i)^* w_j, \quad j=1,2$$

since $\beta_1 = \pi^*\alpha_1 - p^* w_1$

$\Rightarrow$ ① $\overset{def}{\underset{p.b.}{=}} w_j(d(p\circ i)(v)) = 0$ for $j = 1, 2$.

$d(p\circ i)(v)$ is 0 under w_1, w_2 which are basis of T^*N. $\dim N = 2$

$\Rightarrow d(p\circ i)(v) = 0$

$\Rightarrow v \in \ker(d(p\circ i))$

$$\underset{v}{T_{p,q}(M\times N)} = T_pM \oplus T_qN$$

$d(\pi\circ i)(v) \rightsquigarrow 0$ $\quad d(p\circ i)(v) = 0$

$\Rightarrow v = 0 \in T_{p,q}(M\times N) \Rightarrow \ker(d(\pi\circ i)) = 0$

$d(\pi\circ i)$ injective $\quad U \xrightarrow{\pi\circ i} \pi\circ i(U)$ (Inverse F. Thm)

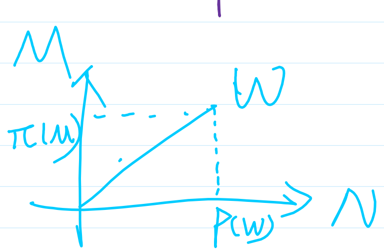
$\exists\, U$ open in W, diffeomorphic to $\pi\circ i(U)$ via $\pi\circ i$.

Similarly, $\exists\, V$ on W s.t. $V \underset{p\circ i}{\sim} p\circ i(V)$

$U' = U \cap V$

F' define on U', s.t. $F' = p\circ i \circ (\pi\circ i)^{-1}$

$F : U' \underset{F'}{\rightsquigarrow} F'(U')$



Remark: Locally connectedness $\not\Leftrightarrow$ connectedness

Sketch #1. To show $\forall x \in A$, $\forall$ open set U of x in A

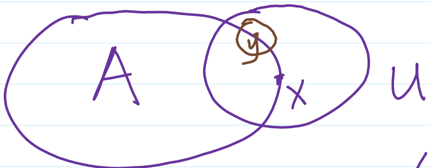
$\exists\, V$ open & connected in A s.t. $x \in V \subseteq U$

Case 1: if $x \in Int(A)$ $\quad Int(A) \subseteq Y$ open in Y

Trivial.

Case 2: if $x \in \partial A$. $\exists\, U$ connected in Y s.t.

$U \cap \partial A$ is connected in ∂A, thus $U \cap \partial A$ in connected $U \cap A$



$U \cap \partial A$ connected in $\bar{A}$

$\forall y \in U \cap \partial A$, $\exists\, W_y \subseteq U$ s.t.

$y \in W_y$, W_y is connected in ∂A

connected in $U \cap \partial A$.

let $W = \left(\bigcup_{y \in U \cap \partial A} W_y \right) \cap \bar{A}$

Claim W is connected in $\bar{A}$.

Skip the proof

If W is connected in $\bar{A}$. $x \in W \subseteq U$ $\square$

#4: $\mathcal{P}(J) \sim \{ f \mid f : J \to \mathbb{Z}_2 = \{0, 1\} \}$

$\sim \{ char._A \mid A \in \mathcal{P}(J) \}$

$\sim \prod_{j \in J} (\mathbb{Z}_2)_j$

$N_{A,B}$ is the basis of product topo. of $\prod_{j \in J} \mathbb{Z}_{2\,j}$

a) $\square$

b) T. Thm

c) $K \sim \{ f : J \to \mathbb{Z}_2$ with a countable set $\}$

Hint

Idea : Any open set of $P(J)$, namely

To show $\prod_{\beta \in J} V_\beta \cap K \neq \phi$

d) Let $f_n \in H$, $f_n \to f$. to show $f \in K$

to show $\text{supp}(f) \subseteq \bigcup_{n=1}^{\infty} \text{supp}(f_n)$.

$\Rightarrow f \in K. \Rightarrow \overline{H} \subseteq K$

#7, No, Assume $w = \sum f_{ij} \, dx_i \, dx_j$

$v_1 = \sum a_i \frac{\partial}{\partial x_i}$, $v_2 = \sum b_i \frac{\partial}{\partial x_i}$

$w(v_1, v_2) = 0$ has non-0 solns for a_i, b_i

Actually dim of solns is 2

#9 2017 $dw = d\tilde{w}$.

$A \cdot A^{-1} = Id$

trick $(dA \cdot A^{-1} + A \cdot dA^{-1} = 0)$

#10: On $\mathbb{R}^3$, $F(x,y,z)=0$ defines a surface

$$K = -\frac{1}{\|DF\|^4} \det \begin{pmatrix} D^2F & (DF)^T \\ DF & 0 \end{pmatrix}_{4\times 4}$$

$$DF = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$$

$$D^2F = \begin{pmatrix} \frac{2}{a^2}, & 0, & 0 \\ 0, & \frac{2}{b^2}, & 0 \\ 0, & 0, & \frac{2}{c^2} \end{pmatrix}$$