Jan 17 #8

a) $w \in T^*M$, θ_1, θ_2 basis of T^*M,

Assume $w = u_1\theta_1 + u_2\theta_2$.
where u_1, u_2 are variables of each basis.

Show u_1, u_2 have a solution

$$d\theta^1 = -(u_1\theta_1 + u_2\theta_2) \wedge \theta^2$$

.....

write into basis

$$d\theta_1(X_1, X_2) \longrightarrow \theta_1 \wedge \theta_2 \in \Lambda^2 T^*M$$

b) $\tilde{X}_i = A \cdot X_i$, A is a transition of 2 basis

$$\tilde{\theta}_i(\tilde{X}_i) = 1 = \tilde{\theta}_i(AX_i) = A \cdot \tilde{\theta}_i(X_i)$$

$$\tilde{\Theta}_i = A^{-1}\Theta_i \quad , \quad d\theta_i = -\sum_j w_{ij} \wedge \theta_j$$

from a)

$$\boxed{\Theta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{matrix} \theta_1 \\ \theta_2 \end{matrix} \qquad \Theta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} \theta_1 \\ \theta_2 \end{matrix}}$$

$$(w_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\tilde{\Theta} = -w \wedge \Theta$$

$$d\tilde{\Theta}_i = dA^{-1} \wedge \Theta_i + A^{-1} d\Theta_i$$

write in component

$$d\tilde{\theta}_i = \sum_j da^{-1}_{ij} \wedge \theta_j - \sum_j a^{-1}_{ij} \sum_k w_{jk} \wedge \theta_k$$

Collect θ_k term

$$= \sum_k \left(da^{-1}_{ik} - \sum_j a^{-1}_{ij} w_{jk} \right) \wedge \theta_k$$

$$= \sum_k \Big(\cdots \Big)$$

write θ_k in terms of $\tilde{\theta}_k$, $A \cdot \tilde{\Theta}_k = \tilde{\Theta}_k$

$$= \sum_k \sum_p \Big(da^{-1}_{ik} a_{kp} - \sum_j a^{-1}_{ij} w_{jk} a_{kp} \Big) \wedge \tilde{\theta}_p$$

On the other hand

$$d\tilde{\theta}_k = - \sum_p \tilde{w}_{ip} \wedge \tilde{\theta}_p$$

$$\tilde{w}_{ip} = \sum_k \Big(- da^{-1}_{ik} a_{kp} + \sum_j a^{-1}_{ij} w_{jk} a_{kp} \Big)$$

Trick in forms:

$A^{-1} \cdot A = Id \Rightarrow d(A^{-1}A) = 0$

$\Rightarrow dA^{-1} \cdot A + A^{-1} \cdot dA = 0$

$\Rightarrow dA^{-1} A = \boxed{-A^{-1} dA}$

$$\tilde{w}_{ip} = \sum_k \Big(a^{-1}_{ik} da_{kp} + \sum_j a^{-1}_{ij} w_{jk} a_{kp} \Big)$$

$A^{-1} dA$ $\qquad$ $A^{-1} \cdot w \cdot A$

$$(\tilde{w})_{ip} = (A^{-1} \cdot dA)_{ip} + (A^{-1} w A)_{ip}$$

$$\tilde{w} = A^{-1} dA + A^{-1} w A$$

$$d\tilde{w} = dA^{-1} \wedge dA + d\big(A^{-1} w A\big)$$

Since $A \in SO(2)$ i.e. $A \cdot A^T = Id$

$\dim SO(2) = 1$, $dA^{-1} \wedge dA = 0$

$d\tilde{w} = d(A^{-1} w A)$ $\qquad$ $A^{-1} w A = w$ $\rightarrow$ leave to check

$d\widetilde{w} = dw$ $\square$