

# ST5209/X Assignment 4

Due 31 Mar, 11.59pm

## Set up

1. Make sure you have the following installed on your system:  $\text{\LaTeX}$ , R4.2.2+, RStudio 2023.12+, and Quarto 1.3.450+.
2. Pull changes from the course [repo](#).

## Submission

1. Render the document to get a .pdf printout.
2. Submit both the .qmd and .pdf files to Canvas.

## 1. AR polynomial

Consider the AR(2) model

$$X_t = 4 + 0.5X_{t-1} - 0.25X_{t-2} + W_t. \quad (1)$$

- a. What is the autoregressive polynomial?
- b. What are its roots?
- c. Is this model causal? Why?
- d. Given the representation

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j W_{t-j},$$

solve for  $\psi_0, \psi_1, \psi_2, \psi_3$ . *Hint: Follow the strategy discussed in `linear_process_coefs.qmd`.*

## 2. Likelihood

Consider the AR(1) model

$$X_t = 1 - 0.6X_{t-1} + W_t,$$

where  $W_t \sim WN(0, 0.25)$  is Gaussian white noise. We are given observations  $x_1 = 0.2, x_2 = -0.3, x_3 = 0.4$ .

- What is the mean of  $X_t$ ?
- Write the conditional likelihood of this model, when conditioning on the value of  $x_1$ .

## 3. Reversibility

- Using `set.seed(5209)` to fix a random seed, create a sample trajectory of length  $n = 500$  from the AR(2) model from Problem 1 using `arma.sim()`.
- Reverse the time index of the vector you obtain using `rev()`.
- Fit an AR(2) model to the reversed time series using `fable`. Hint: You may use the code snippet `model(AR(X ~ order(2)))`.
- Inspect the model parameters using `tidy()`. Why are they similar to the those in Equation 1?
- Make a forecast with  $h = 10$ . What does this correspond to in terms of the original time series?

## 4. Yule-Walker

- Write the Yule-Walker equations for the AR(2) model from Problem 1.
- Arrange the equations in the following matrix form (i.e. fill in the missing entries):

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \end{bmatrix}.$$

- Solve the system from part b) for  $\gamma(0), \gamma(1), \gamma(2)$  numerically using `solve()`.
- Given

$$\Gamma_2 = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix},$$

what is the top left entry of  $\Gamma_2^{-1}$  in terms of  $\gamma(0)$  and  $\gamma(1)$ ?

- Write a 95% confidence interval for  $\phi_1$  using your answers for d) and 3d).

## 5. Recursive forecasting for ARMA(1, 1)

Consider the  $ARMA(1, 1)$  model

$$X_t - 0.5X_{t-1} = W_t + 0.5W_{t-1}.$$

In this question, we will investigate recursive forecasting. The following code snippet generates a sequence of length  $n = 50$  drawn from the above model.

```
set.seed(5209)
n <- 50
wn <- rnorm(n)
xt <- arima.sim(model = list(ar = 0.5, ma = 0.5), innov = wn, n = n)
```

a. Fill in the following code snippet using equation (12.14) to generate a sequence `wn_hat`.

```
wn_hat <- rep(0, n)
wn_hat[[1]] <- xt[[1]]
for (i in 2:n) {
  # FILL IN
}
```

- b. Make a time plot of the log absolute difference between `wn` and `wn_hat`.
- c. What consequence does this have for truncated forecasts?
- d. Compute the truncated forecast for  $X_{53}$ .

## 6. ACF, PACF, and BLPs

Let  $(X_t)$  be a mean zero stationary process with the following autocovariance values:

$$\gamma_X(0) = 2, \gamma_X(1) = 1.4, \gamma_X(2) = 0.6, \gamma_X(3) = 0.4, \gamma_X(4) = 0.2.$$

- a. Can  $(X_t)$  be an MA(2) process? Explain why or why not.
- b. Can  $(X_t)$  be an AR(1) process? Explain why or why not.
- c. What is the best linear predictor  $\hat{X}_4$  for  $X_4$  given only  $X_3 = 2$ ?
- d. Using the notation in part c), what is the variance of  $X_4 - \hat{X}_4$ ?
- e. What is the best linear predictor  $\hat{X}_4$  for  $X_4$  given only  $X_2 = 2$ ?
- f. Using the notation in part e), what is the variance of  $X_4 - \hat{X}_4$ ?

- g. Let  $\alpha_X$  denote the partial autocorrelation function of  $(X_t)$ . What is  $\alpha_X(1)$ ?
- h. What is  $\alpha_X(3)$ ?