ST5209/X Assignment 4

Due 31 Mar, 11.59pm

Set up

1. Make sure you have the following installed on your system: $\LaTeX$, R4.2.2+, RStudio 2023.12+, and Quarto 1.3.450+.
2. Pull changes from the course repo.

Submission

1. Render the document to get a .pdf printout.
2. Submit both the .qmd and .pdf files to Canvas.

1. AR polynomial

Consider the AR(2) model

$$X_t = 4 + 0.5X_{t-1} - 0.25X_{t-2} + W_t. \tag{1}$$

a. What is the autoregressive polynomial?

b. What are its roots?

c. Is this model causal? Why?

d. Given the representation

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j W_{t-j},$$

solve for $\psi_0, \psi_1, \psi_2, \psi_3$. *Hint: Follow the strategy discussed in `linear_process_coefs.qmd`.*

2. Likelihood

Consider the AR(1) model

$$X_t = 1 - 0.6X_{t-1} + W_t,$$

where $W_t \sim WN(0, 0.25)$ is Gaussian white noise. We are given observations $x_1 = 0.2, x_2 = -0.3, x_3 = 0.4$.

a. What is the mean of X_t?

b. Write the conditional likelihood of this model, when conditioning on the value of x_1.

3. Reversibility

a. Using `set.seed(5209)` to fix a random seed, create a sample trajectory of length $n = 500$ from the AR(2) model from Problem 1 using `arima.sim()`.

b. Reverse the time index of the vector you obtain using `rev()`.

c. Fit an AR(2) model to the reversed time series using `fable`. Hint: You may use the code snippet `model(AR(X ~ order(2)))`.

d. Inspect the model parameters using `tidy()`. Why are they similar to the those in Equation 1?

e. Make a forecast with $h = 10$. What does this correspond to in terms of the original time series?

4. Yule-Walker

a. Write the Yule-Walker equations for the AR(2) model from Problem 1.

b. Arrange the equations in the following matrix form (i.e. fill in the missing entries):

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \end{bmatrix}.$$

c. Solve the system from part b) for $\gamma(0), \gamma(1), \gamma(2)$ numerically using `solve()`.

d. Given

$$\Gamma_2 = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix},$$

what is the top left entry of Γ_2^{-1} in terms of $\gamma(0)$ and $\gamma(1)$?

e. Write a 95% confidence interval for ϕ_1 using your answers for d) and 3d).

5. Recursive forecasting for ARMA(1, 1)

Consider the $ARMA(1,1)$ model

$$X_t - 0.5X_{t-1} = W_t + 0.5W_{t-1}.$$

In this question, we will investigate recursive forecasting. The following code snippet generates a sequence of length $n = 50$ drawn from the above model.

```
set.seed(5209)
n <- 50
wn <- rnorm(n)
xt <- arima.sim(model = list(ar = 0.5, ma = 0.5), innov = wn, n = n)
```

a. Fill in the following code snippet using equation (11.14) to generate a sequence `wn_hat`.

```
wn_hat <- rep(0, n)
wn_hat[[1]] <- xt[[1]]
for (i in 2:n) {
  # FILL IN
}
```

b. Make a time plot of the log absolute difference between `wn` and `wn_hat`.

c. What consequence does this have for truncated forecasts?

d. Compute the truncated forecast for X_{53}.

6. ACF, PACF, and BLPs

Let (X_t) be a mean zero stationary process with the following autocovariance values:

$$\gamma_X(0) = 2, \gamma_X(1) = 1.4, \gamma_X(2) = 0.6, \gamma_X(3) = 0.4, \gamma_X(4) = 0.2.$$

a. Can (X_t) be an MA(2) process? Explain why or why not.

b. Can (X_t) be an AR(1) process? Explain why or why not.

c. What is the best linear predictor $\hat{X}_4$ for X_4 given only $X_3 = 2$?

d. Using the notation in part c), what is the variance of $X_4 - \hat{X}_4$?

e. What is the best linear predictor $\hat{X}_4$ for X_4 given only $X_2 = 2$?

f. Using the notation in part e), what is the variance of $X_4 - \hat{X}_4$?

g. Let α_X denote the partial autocorrelation function of (X_t). What is $\alpha_X(1)$?

h. What is $\alpha_X(3)$?