

令和 6 年度  
修士学位論文

論文用テンプレート

○○所属

○○課程 ○○専攻

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## 概要

We study the family of elliptic curves  $y^2 = x(x - a^2)(x + b^2)$ , where  $(a, b, c)$  are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1)$$

It is known that the generic rank of the Mordell-Weil group of  $E_{1,s}$  over  $\overline{\mathbb{Q}}(s)$  is 0. We found an infinite subfamily of  $E_{1,s}$  whose Mordell-Weil rank over  $\overline{\mathbb{Q}}(s)$  is 1, which means that there are infinitely many  $s \in \overline{\mathbb{Q}}$  such that the Mordell-Weil group of  $E_{1,s}$  has positive rank over  $\overline{\mathbb{Q}}$ .

We use the theory of elliptic surfaces to prove it. Each elliptic curve over a function field corresponds to an elliptic surface. The Shioda-Tate formula gives the relation between the Mordell-Weil rank and the Néron-Severi rank of elliptic surfaces. We compute the types of special fibers of the elliptic surfaces and consider the upper bound of the rank of the Néron-Severi group.

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# 1. Introduction

**Theorem 1.0.1.** Let

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1.1)$$

be an elliptic curve over  $\overline{\mathbb{Q}}(s)$ . Then, the Mordell-Weil group

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad (1.2)$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s+1)^2, 2s(s+1)^2(s^2+1)), \quad (1.3)$$

$$T_2 := (2is(s^2-1), 2is(s+i)^2(s^2-1)). \quad (1.4)$$

**Corollary 1.0.2.**

$$E_{1,s}(\mathbb{Q}(s)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \quad (1.5)$$

is generated by  $T_1$  and  $2T_2 = (0, 0)$ .

**Theorem 1.0.3.** Let

$$E_{2,t} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2), \quad s = \frac{2t}{t^2 - 3} \quad (1.6)$$

be an elliptic curve over  $\overline{\mathbb{Q}}(t)$ . Then, the Mordell-Weil group

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad (1.7)$$

especially the rank is 1. The torsion subgroup is generated by  $T_1$  and  $T_2$  in Theorem 1.0.1 and the free part is generated by

$$\left( s^2 - 1, is(s^2 - 1) \frac{t^2 + 3}{t^2 - 3} \right). \quad (1.8)$$

## 2. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, finding generators is enough. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

**Theorem 2.0.1.** (Shioda-Tate formula, [1] Theorem 3.4) Let  $C$  be a smooth irreducible projective curve over an algebraically closed field  $k$  and  $E$  an elliptic curve over a function field  $k(C)$ . Let  $\mathcal{E} \rightarrow C$  be the Néron model of  $E$ . Let  $R \subset C$  be the set of points where the special fiber of  $\mathcal{E}$  is singular. For each  $v \in R$ , let  $m_v$  be the number of components of the special fiber of  $\mathcal{E}$  at  $v$ . Let  $\rho(\mathcal{E})$  denote the rank of the Néron-Severi group of  $\mathcal{E}$ . Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \text{rank}(E(k(C))). \quad (2.1)$$

We can calculate  $R$  and  $m_v$  by Tate's algorithm, but it is still difficult to determine  $\rho(\mathcal{E})$ . We have the following theorem to get the upper bound of  $\rho(\mathcal{E})$ .

**Theorem 2.0.2.**

$$\rho(\mathcal{E}) \leq \frac{5}{6}e(\tilde{S}) + 2, \quad (2.2)$$

$$e(\tilde{S}) := \sum_{v \in R} e(F_v). \quad (2.3)$$

where  $e(\tilde{S})$  is the Euler number,  $e(F_v)$  is the local Euler number of the special fiber of  $\mathcal{E}$  at  $v$  for each  $v \in R$  and

$$e(F_v) = \begin{cases} m_v & \text{if the fiber has multiplicative reduction,} \\ m_v + 1 & \text{if the fiber has additive reduction.} \end{cases} \quad (2.4)$$

However, Theorem 2.0.2 is still not enough to get the upper bound of the ranks of the Mordell-weil groups in our case.

TODO: étale cohomology を使う

**Definition 2.0.3.** Let  $C$  be a smooth curve over an algebraically closed field  $k$ . Let  $E$  be an elliptic curve

over a function field  $k(C)$  given by the Weierstrass equation

$$E : y^2 = x^3 + a_2x + a_4x + a_6 \quad (2.5)$$

where  $a_2, a_4, a_6 \in k(C)$ . For a fixed  $u \in k(C)^*$ , we denote

$$E^{(u)} : uy^2 = x^3 + a_2x + a_4x + a_6 \quad (2.6)$$

to be the quadratic twist of  $E$  by  $u$ .

The following theorem is used to reduce the order of coefficients in the Weierstrass equation and make the computation feasible.

**Theorem 2.0.4.** ([1] Proposition 4.1.)

$$\text{rank } E(k(C)) + \text{rank } E^{(u)}(k(C)) = \text{rank } E(k(C)(\sqrt{u})) \quad (2.7)$$

**Theorem 2.0.5.** TODO: どこに書くか検討

$$E(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow \prod_{v \in R} G(F_v) \quad (2.8)$$

where  $G(F_v)$  is the group generated by all simple components of the fiber at  $v$ . If  $F_v$  is of type  $I_n$  in Kodaira notation, then  $G(F_v) \cong \mathbb{Z}/n\mathbb{Z}$ .

### 3. Types of Special Fibers

Table 3.1 Singular fibers of  $E_{1,s}$

Place	Type	$m_v$
$s = 0$	$I_4$	4
$s = \pm 1$	$I_4$	4
$s = \pm i$	$I_4$	4
$s = \infty$	$I_4$	4

Table 3.2 Singular fibers of  $E_{2,t}$ 

Place	Type	$m_v$
$t = 0$	$I_4$	4
$t = \pm 1$	$I_4$	4
$t = \pm 3$	$I_4$	4
$t = \pm\sqrt{3}$	$I_4$	4
$t^4 - 2t^2 + 9 = 0$	$I_4$	4
$t = \infty$	$I_4$	4

Table 3.3 Singular fibers of  $E_{1,s}^{(1+3s^2)}$ 

Place	Type	$m_v$
$s = 0$	$I_4$	4
$s = \pm 1$	$I_4$	4
$s = \pm i$	$I_4$	4
$s = \pm \frac{1}{\sqrt{-3}}$	$I_0^*$	5
$s = \infty$	$I_4$	4

## 4. Torsions

### 4.1 セクション

**Theorem 4.1.1.**

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad (4.1)$$

$$T_1 = (2s(s+1)^2, 2s(s+1)^2(s^2+1)) \quad (4.2)$$

$$T_2 = (2is(s^2-1), 2is(s+i)^2(s^2-1)) \quad (4.3)$$

で生成される.

証明

$$E_{2,t}(\overline{\mathbb{Q}}(t))[2] = E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0,0), (4s^2,0), (-(s^2-1)^2,0)\} \quad (4.4)$$

Table 3.4 Singular fibers of  $E_{0,s}^{(1+3s)}$

Place	Type	$m_v$
$s = 0$	$I_2$	2
$s = \pm 1$	$I_4$	4
$s = -\frac{1}{3}$	$I_0^*$	5
$s = \infty$	$I_2^*$	7

Table 3.5 Singular fibers of  $E_{0,s}^{(s(1+3s))}$

Place	Type	$m_v$
$s = 0$	$I_2^*$	7
$s = \pm 1$	$I_4$	4
$s = -\frac{1}{3}$	$I_0^*$	5
$s = \infty$	$I_2$	2

$$2T_1 = (4s^2, 0) \quad (4.5)$$

$$2T_2 = (0, 0) \quad (4.6)$$

[1] の Lem.3.5 より

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12} \quad (4.7)$$

なので位数 8 の点は存在しない. □

*Remark 4.1.2.* これは

$$E_{1,s}(\mathbb{Q}(s))_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad (4.8)$$

の別証明になっている.

## 5. Ranks

証明 of Theorem 1.0.1

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \quad (5.1)$$



$$e(\mathcal{E}_{1,s}) = 24 \quad (5.2)$$

したがって  $\mathcal{E}_{1,s}$  は K3 曲面であり．  $\rho(\mathcal{E}_{1,s}) \leq 20$  である． Theorem 2.0.1 より

$$\text{rank}(E_{1,s}) = 0 \quad (5.3)$$

□

証明 of Theorem 1.0.3

$$\Delta_{E_{2,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \quad (5.4)$$

$$e(\mathcal{E}_{4,t}) = 48 \quad (5.5)$$

TODO:  $\rho(\mathcal{E}_{4,t}) \leq 40$  である． Theorem 2.0.1 より

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2 \quad (5.6)$$

□

上の評価は不十分．生成元は 1 つしか見つかっていないので，ランクの上界が 1 であることを示したい．

**Theorem 5.0.1.**

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \quad (5.7)$$

$$E_{1,s}^{(1+3s^2)} : (1+3s^2)y^2 = x(x-4s^2)(x+(s^2-1)^2) \quad (5.8)$$

$$\text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \quad (5.9)$$

さらに

$$E_{0,s} : y^2 = x(x-4s)(x+(s-1)^2) \quad (5.10)$$

$$E_{0,s}^{(1+3s)} : (1+3s)y^2 = x(x-4s)(x+(s-1)^2) \quad (5.11)$$

$$E_{0,s}^{(s(1+3s))} : s(1+3s)y^2 = x(x-4s)(x+(s-1)^2) \quad (5.12)$$

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \quad (5.13)$$

証明

$$s = \frac{2t}{t^2-3} \quad (5.14)$$

を  $t$  について解くと

$$t = \frac{1 \pm \sqrt{1 + 3s^2}}{s} \quad (5.15)$$

したがって

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1 + 3s^2})) \quad (5.16)$$

□

**Theorem 5.0.2.** TODO

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \quad (5.17)$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1 + 3s^2)^6 \Delta(E_{1,s}) \quad (5.18)$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \quad (5.19)$$

Theorem 2.0.1 からは

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \quad (5.20)$$

しか分からない。K3 ですらないので、 $H^2$  の次元が分からず、reduction を取る方法でも計算が進められない。

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? (1 \text{ or } 2) \quad (5.21)$$

□

**Theorem 5.0.3.** TODO

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.22)$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \quad (5.23)$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \quad (5.24)$$

Theorem 2.0.1 からは

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.25)$$

□

**Theorem 5.0.4.**

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \quad (5.26)$$

証明

$$(s-1, i(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \quad (5.27)$$

より rank は正である.

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \quad (5.28)$$

上と同様に

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.29)$$

□

**Theorem 5.0.5.** TODO

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \quad (5.30)$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1+3s^2)^6 \Delta(E_{1,s}) \quad (5.31)$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \quad (5.32)$$

Theorem 2.0.1 からは

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \quad (5.33)$$

しか分からない. K3 ですらないので,  $H^2$  の次元が分からず, reduction を取る方法でも計算が進められない.

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? (1 \text{ or } 2) \quad (5.34)$$

□

**Theorem 5.0.6.** TODO

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.35)$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \quad (5.36)$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \quad (5.37)$$

Theorem 2.0.1 からは

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.38)$$

□

**Theorem 5.0.7.**

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \quad (5.39)$$

証明

$$(s-1, i(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \quad (5.40)$$

より rank は正である.

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \quad (5.41)$$

上と同様に

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.42)$$

□

## 6. Reductions

### 6.1 $E_{0,s}^{(1+3s)}$

K3 なので

$$\dim_{\mathbb{Q}_l} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l) = 22 \quad (6.1)$$

である. Let  $V$  be the subspace of  $\text{NS}(\tilde{S})$  generated by the singular fibers and the zero section. Then  $V$  is of rank 19, on which the Frobenius automorphism acts by multiplication by  $p$ .

$$\text{char}(\Phi_{\tilde{S}}^*||V) = (x-5)^{19} \quad (6.2)$$

Note that all the multiplicative fibers are split.

$$t_m := \text{Tr}((\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)^m) = \#\tilde{S}(\mathbb{F}_{5^m}) - 1 - 5^{2m} - 19 \cdot 5^m \quad (6.3)$$

$$\text{char}(\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*) = x^3 + x^2 + 11x - 77 \quad (6.4)$$

Table 6.1 Sample Table

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264
$t_m$	-1	-21	263

## 参考文献

- [1] B. Naskręcki. Mordell-Weil ranks of families of elliptic curves associated to Pythagorean triples.  
eng. Acta Arithmetica 160.2, pp. 159–183, (2013). URL: <http://eudml.org/doc/279803>.