

令和6年度
修士学位論文

論文用テンプレート

〇〇所属

〇〇課程 〇〇専攻

〇〇分野

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1. Abstract

We study the family of elliptic curves $y^2 = x(x - a^2)(x + b^2)$, where (a, b, c) are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1.1)$$

It is known that the generic rank of the Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ is 0. We found an infinite subfamily of $E_{1,s}$ whose Mordell-Weil group has positive rank over $\overline{\mathbb{Q}}(s)$, which means that there are infinitely many Pythagorean triples (a, b, c) such that the Frey curve $y^2 = x(x - a^2)(x + b^2)$ has positive rank. **TODO: $\mathbb{Q}(s)$ 上でランク正の無限族じゃないと, Frey curve が無限個とは言い難い** Each elliptic curve over a function field corresponds to an elliptic surface. We prove that the Mordell-Weil group of the subfamily has exactly rank 1 over $\overline{\mathbb{Q}}(s)$ using the theory of elliptic surfaces.

2. Introduction

2.1 セクション

Theorem 2.1.1.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (2.1)$$

の $\overline{\mathbb{Q}}(s)$ 上のランクは 0 である.

証明

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \quad (2.2)$$

$$e(\mathcal{E}_{1,s}) = 24 \quad (2.3)$$

したがって $\mathcal{E}_{1,s}$ は K3 曲面であり. $\rho(\mathcal{E}_{1,s}) \leq 20$ である. Theorem 3.0.1 より

$$\text{rank}(E_{1,s}) = 0 \quad (2.4)$$

□

Theorem 2.1.2.

$$E_{4,t} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2), s = \frac{2t}{t^2 - 3} \quad (2.5)$$

は

$$\left(s^2 - 1, \sqrt{-1}s(s^2 - 1)\frac{t^2 + 3}{t^2 - 3} \right) \quad (2.6)$$

を通る.

$$1 \leq \text{rank } E_{4,t}(\overline{\mathbb{Q}}(t)) \leq 2 \quad (2.7)$$

証明

$$\Delta_{E_{4,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \quad (2.8)$$

$$e(\mathcal{E}_{4,t}) = 48 \quad (2.9)$$

TODO: $\rho(\mathcal{E}_{4,t}) \leq 40$ である. Theorem 3.0.1 より

$$\text{rank } E_{4,t}(\overline{\mathbb{Q}}(t)) \leq 2 \quad (2.10)$$

□

上の評価は不十分. 生成元は 1 つしか見つからないので, ランクの上界が 1 であることを示したい.

Theorem 2.1.3.

$$E_{4,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \quad (2.11)$$

$$E_{1,s}^{(1+3s^2)} : (1+3s^2)y^2 = x(x-4s^2)(x+(s^2-1)^2) \quad (2.12)$$

$$\text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{4,t}(\overline{\mathbb{Q}}(t)) \quad (2.13)$$

さらに

$$E_{0,s} : y^2 = x(x-4s)(x+(s-1)^2) \quad (2.14)$$

$$E_{0,s}^{(1+3s)} : (1+3s)y^2 = x(x-4s)(x+(s-1)^2) \quad (2.15)$$

$$E_{0,s}^{(s(1+3s))} : s(1+3s)y^2 = x(x-4s)(x+(s-1)^2) \quad (2.16)$$

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \quad (2.17)$$

証明

$$s = \frac{2t}{t^2 - 3} \quad (2.18)$$

を t について解くと

$$t = \frac{1 \pm \sqrt{1+3s^2}}{s} \quad (2.19)$$

したがって

$$E_{4,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \quad (2.20)$$

□

Theorem 2.1.4. TODO

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \quad (2.21)$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1+3s^2)^6 \Delta(E_{1,s}) \quad (2.22)$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \quad (2.23)$$

Theorem 3.0.1 からは

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \quad (2.24)$$

しか分からない。K3 ですらないので、 H^2 の次元が分からず、reduction を取る方法でも計算が進められない。

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ?(1 \text{ or } 2) \quad (2.25)$$

□

Theorem 2.1.5. TODO

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (2.26)$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \quad (2.27)$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \quad (2.28)$$

Theorem 3.0.1 からは

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (2.29)$$

□

Theorem 2.1.6.

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \quad (2.30)$$

証明

$$(s-1, \sqrt{-1}(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \quad (2.31)$$

より rank は正である。

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \quad (2.32)$$

上と同様に

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (2.33)$$

□

3. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, we can just find points of infinite order. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

Theorem 3.0.1. (Shioda-Tate formula, [1] Theorem 3.4)

$$\rho(S) = 2 + \sum_{v \in R} (m_v - 1) + \text{rank}(E(K)) \quad (3.1)$$

である.

Theorem 3.0.2. rational や K3 のときの ρ はわかっている.

In order to make it easier to calculate the rank, we can use the following theorem.

Theorem 3.0.3. ([1] Proposition 4.1.)

$$\text{rank } E(k(C)) + \text{rank } E^{(u)}(k(C)) = \text{rank } E(k(C)(\sqrt{u})) \quad (3.2)$$

However, Theorem 3.0.2 is still not enough to get the upper bound of the rank in our case.

TODO: etale cohomology を使う

4. Types of Special Fibers

Table 4.1 Singular fibers of $E_{1,s}$

| Place | Type | m_v |
|--------------|-------|-------|
| $s = 0$ | I_4 | 4 |
| $s = \pm 1$ | I_4 | 4 |
| $s = \pm i$ | I_4 | 4 |
| $s = \infty$ | I_4 | 4 |

Table 4.2 Singular fibers of $E_{4,t}$

| Place | Type | m_v |
|----------------------|-------|-------|
| $t = 0$ | I_4 | 4 |
| $t = \pm 1$ | I_4 | 4 |
| $t = \pm 3$ | I_4 | 4 |
| $t = \pm\sqrt{3}$ | I_4 | 4 |
| $t^4 - 2t^2 + 9 = 0$ | I_4 | 4 |
| $t = \infty$ | I_4 | 4 |

Table 4.3 Singular fibers of $E_{1,s}^{(1+3s^2)}$

| Place | Type | m_v |
|-------------------------------|---------|-------|
| $s = 0$ | I_4 | 4 |
| $s = \pm 1$ | I_4 | 4 |
| $s = \pm i$ | I_4 | 4 |
| $s = \pm \frac{1}{\sqrt{-3}}$ | I_0^* | 5 |
| $s = \infty$ | I_4 | 4 |

Table 4.4 Singular fibers of $E_{0,s}^{(1+3s)}$

| Place | Type | m_v |
|--------------------|---------|-------|
| $s = 0$ | I_2 | 2 |
| $s = \pm 1$ | I_4 | 4 |
| $s = -\frac{1}{3}$ | I_0^* | 5 |
| $s = \infty$ | I_2^* | 7 |

Table 4.5 Singular fibers of $E_{0,s}^{(s(1+3s))}$

| Place | Type | m_v |
|--------------------|---------|-------|
| $s = 0$ | I_2^* | 7 |
| $s = \pm 1$ | I_4 | 4 |
| $s = -\frac{1}{3}$ | I_0^* | 5 |
| $s = \infty$ | I_2 | 2 |

5. Torsions

5.1 セクション

Theorem 5.1.1.

$$E_{4,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad (5.1)$$

$$T_1 = (2s(s+1)^2, 2s(s+1)^2(s^2+1)) \quad (5.2)$$

$$T_2 = (2is(s^2-1), 2is(s+i)^2(s^2-1)) \quad (5.3)$$

で生成される.

証明

$$E_{4,t}(\overline{\mathbb{Q}}(t))[2] = E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0,0), (4s^2,0), (-(s^2-1)^2,0)\} \quad (5.4)$$

$$2T_1 = (4s^2, 0) \quad (5.5)$$

$$2T_2 = (0, 0) \quad (5.6)$$

[1] の Lem.3.5 より

$$E_{4,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12} \quad (5.7)$$

なので位数 8 の点は存在しない.

□

Remark 5.1.2. これは

$$E_{1,s}(\mathbb{Q}(s))_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad (5.8)$$

の別証明になっている.

6. Ranks

Theorem 6.0.1. TODO

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \quad (6.1)$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1 + 3s^2)^6 \Delta(E_{1,s}) \quad (6.2)$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \quad (6.3)$$

Theorem 3.0.1 からは

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \quad (6.4)$$

しか分からない。K3 ですらないので、 H^2 の次元が分からず、reduction を取る方法でも計算が進められない。

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? (1 \text{ or } 2) \quad (6.5)$$

□

Theorem 6.0.2. TODO

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (6.6)$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \quad (6.7)$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \quad (6.8)$$

Theorem 3.0.1 からは

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (6.9)$$

□

Theorem 6.0.3.

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \quad (6.10)$$

証明

$$(s-1, \sqrt{-1}(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \quad (6.11)$$

より rank は正である.

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \quad (6.12)$$

上と同様に

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (6.13)$$

□

7. Reductions

$$7.1 \quad E_{0,s}^{(1+3s)}$$

K3 なので

$$\dim_{\mathbb{Q}_l} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l) = 22 \tag{7.1}$$

である. Let V be the subspace of $\text{NS}(\tilde{S})$ generated by the singular fibers and the zero section. Then V is of rank 19, on which the Frobenius automorphism acts by multiplication by p .

参考文献

- [1] B. Naskręcki. Mordell-Weil ranks of families of elliptic curves associated to Pythagorean triples.
eng. Acta Arithmetica 160.2, pp. 159–183, (2013). URL: <http://eudml.org/doc/279803>.