

令和6年度
修士学位論文

論文用テンプレート

〇〇所属

〇〇課程 〇〇専攻

〇〇分野

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概要

We study a family of elliptic curves $y^2 = x(x - a^2)(x + b^2)$, where (a, b, c) are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1)$$

It is known that the generic rank of the Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ is 0. We found an infinite subfamily of $E_{1,s}$ whose Mordell-Weil rank over $\overline{\mathbb{Q}}(s)$ is 1, which means that there are infinitely many $s \in \overline{\mathbb{Q}}$ such that the Mordell-Weil group of $E_{1,s}$ has positive rank over $\overline{\mathbb{Q}}$.

We use the theory of elliptic surfaces to prove it. Each elliptic curve over a function field corresponds to an elliptic surface. The Shioda-Tate formula gives the relation between the Mordell-Weil rank and the Néron-Severi rank of elliptic surfaces. We compute the types of special fibers of the elliptic surfaces and consider the upper bound of the rank of the Néron-Severi group.

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1. Introduction

Theorem 1.0.1. Let

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1.1)$$

be an elliptic curve over $\overline{\mathbb{Q}}(s)$. Then, the Mordell-Weil group

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad (1.2)$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s+1)^2, 2s(s+1)^2(s^2+1)), \quad (1.3)$$

$$T_2 := (2is(s^2-1), 2is(s+i)^2(s^2-1)). \quad (1.4)$$

Corollary 1.0.2.

$$E_{1,s}(\mathbb{Q}(s)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \quad (1.5)$$

is generated by T_1 and $2T_2 = (0, 0)$.

Theorem 1.0.3. Let

$$E_{2,t} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2), \quad s = \frac{2t}{t^2 - 3} \quad (1.6)$$

be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Then, the Mordell-Weil group

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad (1.7)$$

especially the rank is 1. The torsion subgroup is generated by T_1 and T_2 in Theorem 1.0.1 and the free part is generated by

$$\left(s^2 - 1, is(s^2 - 1) \frac{t^2 + 3}{t^2 - 3} \right). \quad (1.8)$$

2. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, finding generators is enough. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

Theorem 2.0.1. (Shioda-Tate formula, [1, Corollary 5.3]) Let C be a smooth irreducible projective curve over an algebraically closed field k and E an elliptic curve over a function field $k(C)$. Let $\mathcal{E} \rightarrow C$ be the Néron model of E . Let $R \subset C$ be the set of points where the special fiber of \mathcal{E} is singular. For each $v \in R$, let m_v be the number of components of the special fiber of \mathcal{E} at v . Let $\rho(\mathcal{E})$ denote the rank of the Néron-Severi group of \mathcal{E} . Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \text{rank}(E(k(C))). \quad (2.1)$$

We can calculate R and m_v by Tate's algorithm, but it is still difficult to determine $\rho(\mathcal{E})$. We have the following theorem to get the upper bound of $\rho(\mathcal{E})$.

Theorem 2.0.2.

$$\rho(\mathcal{E}) \leq \frac{5}{6}e(\tilde{S}) + 2, \quad (2.2)$$

$$e(\tilde{S}) := \sum_{v \in R} e(F_v). \quad (2.3)$$

where $e(\tilde{S})$ is the Euler number, $e(F_v)$ is the local Euler number of the special fiber of \mathcal{E} at v for each $v \in R$ and

$$e(F_v) = \begin{cases} m_v & \text{if the fiber has multiplicative reduction,} \\ m_v + 1 & \text{if the fiber has additive reduction.} \end{cases} \quad (2.4)$$

証明 TODO: Naskrencki の PhD の 2.2.19(ii), 2.2.9, 2.2.10 (やその引用元) を引用する □

However, Theorem 2.0.2 is still not enough to get the upper bound of the ranks of the Mordell-weil groups in our case.

TODO: étale cohomology を使うことを説明する.

Definition 2.0.3. Let C be a smooth curve over an algebraically closed field k . Let E be an elliptic curve over a function field $k(C)$ given by the Weierstrass equation

$$E : y^2 = x^3 + a_2x^2 + a_4x + a_6 \quad (2.5)$$

where $a_2, a_4, a_6 \in k(C)$. For a fixed $u \in k(C)^*$, we denote

$$E^{(u)} : uy^2 = x^3 + a_2x^2 + a_4x + a_6 \quad (2.6)$$

to be the quadratic twist of E by u .

The following theorem is used to reduce the order of coefficients in the Weierstrass equation and make the computation feasible.

Theorem 2.0.4. ([2, Exercise 10.16]) Let E be an elliptic curve over a function field $k(C)$ and $u \in k(C)^*$. Then, the following equation holds

$$\text{rank } E(k(C)) + \text{rank } E^{(u)}(k(C)) = \text{rank } E(k(C)(\sqrt{u})). \quad (2.7)$$

Theorem 2.0.5. TODO: どこに書くか検討

$$E(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow \prod_{v \in R} G(F_v) \quad (2.8)$$

where $G(F_v)$ is the group generated by all simple components of the fiber at v . If F_v is of type I_n in Kodaira notation, then $G(F_v) \cong \mathbb{Z}/n\mathbb{Z}$.

証明 TODO

□

Table 3.1 Singular fibers of $E_{1,s}$

Place	Type	m_v
$s = 0$	I_4	4
$s = \pm 1$	I_4	4
$s = \pm i$	I_4	4
$s = \infty$	I_4	4

Table 3.2 Singular fibers of $E_{2,t}$

Place	Type	m_v
$t = 0$	I_4	4
$t = \pm 1$	I_4	4
$t = \pm 3$	I_4	4
$t = \pm\sqrt{3}$	I_4	4
$t^4 - 2t^2 + 9 = 0$	I_4	4
$t = \infty$	I_4	4

3. Types of Special Fibers

4. Torsions

4.1 セクション

Theorem 4.1.1.

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad (4.1)$$

Table 3.3 Singular fibers of $E_{1,s}^{(1+3s^2)}$

Place	Type	m_v
$s = 0$	I_4	4
$s = \pm 1$	I_4	4
$s = \pm i$	I_4	4
$s = \pm \frac{1}{\sqrt{-3}}$	I_0^*	5
$s = \infty$	I_4	4

Table 3.4 Singular fibers of $E_{0,s}^{(1+3s)}$

Place	Type	m_v
$s = 0$	I_2	2
$s = \pm 1$	I_4	4
$s = -\frac{1}{3}$	I_0^*	5
$s = \infty$	I_2^*	7

$$T_1 = (2s(s+1)^2, 2s(s+1)^2(s^2+1)) \quad (4.2)$$

$$T_2 = (2is(s^2-1), 2is(s+i)^2(s^2-1)) \quad (4.3)$$

で生成される.

証明

$$E_{2,t}(\overline{\mathbb{Q}}(t))[2] = E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0,0), (4s^2,0), (-(s^2-1)^2,0)\} \quad (4.4)$$

$$2T_1 = (4s^2, 0) \quad (4.5)$$

$$2T_2 = (0, 0) \quad (4.6)$$

[3] の Lem.3.5 より

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12} \quad (4.7)$$

なので位数 8 の点は存在しない. □

Remark 4.1.2. これは

$$E_{1,s}(\mathbb{Q}(s))_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad (4.8)$$

の別証明になっている.

Table 3.5 Singular fibers of $E_{0,s}^{(s(1+3s))}$

Place	Type	m_v
$s = 0$	I_2^*	7
$s = \pm 1$	I_4	4
$s = -\frac{1}{3}$	I_0^*	5
$s = \infty$	I_2	2

5. Ranks

証明 of Theorem 1.0.1

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \quad (5.1)$$

$$e(\mathcal{E}_{1,s}) = 24 \quad (5.2)$$

したがって $\mathcal{E}_{1,s}$ は K3 曲面であり． $\rho(\mathcal{E}_{1,s}) \leq 20$ である． Theorem 2.0.1 より

$$\text{rank}(E_{1,s}) = 0 \quad (5.3)$$

□

証明 of Theorem 1.0.3

$$\Delta_{E_{2,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \quad (5.4)$$

$$e(\mathcal{E}_{4,t}) = 48 \quad (5.5)$$

TODO: $\rho(\mathcal{E}_{4,t}) \leq 40$ である． Theorem 2.0.1 より

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2 \quad (5.6)$$

□

上の評価は不十分．生成元は1つしか見つかっていないので，ランクの上界が1であることを示したい．

Theorem 5.0.1.

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \quad (5.7)$$

$$E_{1,s}^{(1+3s^2)} : (1+3s^2)y^2 = x(x-4s^2)(x+(s^2-1)^2) \quad (5.8)$$

$$\text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \quad (5.9)$$

さらに

$$E_{0,s} : y^2 = x(x-4s)(x+(s-1)^2) \quad (5.10)$$

$$E_{0,s}^{(1+3s)} : (1+3s)y^2 = x(x-4s)(x+(s-1)^2) \quad (5.11)$$

$$E_{0,s}^{(s(1+3s))} : s(1+3s)y^2 = x(x-4s)(x+(s-1)^2) \quad (5.12)$$

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \quad (5.13)$$

証明

$$s = \frac{2t}{t^2 - 3} \quad (5.14)$$

を t について解くと

$$t = \frac{1 \pm \sqrt{1+3s^2}}{s} \quad (5.15)$$

したがって

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \quad (5.16)$$

□

Theorem 5.0.2. TODO

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \quad (5.17)$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1+3s^2)^6 \Delta(E_{1,s}) \quad (5.18)$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \quad (5.19)$$

Theorem 2.0.1 からは

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \quad (5.20)$$

しか分からない. K3 ですらないので, H^2 の次元が分からず, reduction を取る方法でも計算が進められない.

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? (1 \text{ or } 2) \quad (5.21)$$

□

Theorem 5.0.3. TODO

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.22)$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \quad (5.23)$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \quad (5.24)$$

Theorem 2.0.1 から

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.25)$$

□

Theorem 5.0.4.

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \quad (5.26)$$

証明

$$(s-1, i(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \quad (5.27)$$

より rank は正である.

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \quad (5.28)$$

上と同様に

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.29)$$

□

Theorem 5.0.5. TODO

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \quad (5.30)$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1+3s^2)^6 \Delta(E_{1,s}) \quad (5.31)$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \quad (5.32)$$

Theorem 2.0.1 からは

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \quad (5.33)$$

しか分からない。K3 ですらないので、 H^2 の次元が分からず、reduction を取る方法でも計算が進められない。

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ?(1 \text{ or } 2) \quad (5.34)$$

□

Theorem 5.0.6. TODO

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.35)$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \quad (5.36)$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \quad (5.37)$$

Theorem 2.0.1 からは

$$\text{rank } E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.38)$$

□

Theorem 5.0.7.

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \quad (5.39)$$

証明

$$(s-1, i(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \quad (5.40)$$

より rank は正である。

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \quad (5.41)$$

上と同様に

$$\text{rank } E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \quad (5.42)$$

□

6. Reductions

6.1 $E_{0,s}^{(1+3s)}$

K3 なので

$$\dim_{\mathbb{Q}_l} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l) = 22 \quad (6.1)$$

である. Let V be the subspace of $\text{NS}(\tilde{S})$ generated by the singular fibers and the zero section. Then V is of rank 19, on which the Frobenius automorphism acts by multiplication by p .

$$\text{char}(\Phi_{\tilde{S}}^*|V) = (x - 5)^{19} \quad (6.2)$$

Note that all the multiplicative fibers are split.

$$t_m := \text{Tr}((\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)^m) = \#\tilde{S}(\mathbb{F}_{5^m}) - 1 - 5^{2m} - 19 \cdot 5^m \quad (6.3)$$

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264
t_m	-1	-21	263

$$\text{char}(\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*) = x^3 + x^2 + 11x - 77 \quad (6.4)$$

参考文献

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