令和 6 年度
修士学位論文

論文用テンプレート

○○所属

○○課程 ○○専攻

○○分野

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1. Abstract

We study the family of elliptic curves $y^2 = x(x - a^2)(x + b^2)$, where (a, b, c) are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \tag{1.1}$$

It is known that the generic rank of the Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ is 0. We found an infinite subfamily of $E_{1,s}$ whose Mordell-Weil group has positive rank over $\overline{\mathbb{Q}}(s)$, which means that there are infinitely many Pythagorean triples (a, b, c) such that the Frey curve $y^2 = x(x - a^2)(x + b^2)$ has positive rank. TODO: $\mathbb{Q}(s)$ 上でランク正の無限族じゃないと，Frey curve が無限個とは言い難い Each elliptic curve over a function field corresponds to an elliptic surface. We prove that the Mordell-Weil group of the subfamily has exactly rank 1 over $\overline{\mathbb{Q}}(s)$ using the theory of elliptic surfaces.

2. Introduction

2.1 セクション

Theorem 2.1.1.

$$E_{1,s} : y^2 = x(x-4s^2)(x+(s^2-1)^2) \tag{2.1}$$

の $\overline{\mathbb{Q}}(s)$ 上のランクは 0 である．

証明

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \tag{2.2}$$

$$e(\mathcal{E}_{1,s}) = 24 \tag{2.3}$$

したがって $\mathcal{E}_{1,s}$ は K3 曲面であり． $\rho(\mathcal{E}_{1,s}) \leq 20$ である．Theorem 3.0.1 より

$$\mathrm{rank}(E_{1,s}) = 0 \tag{2.4}$$

□

Theorem 2.1.2.

$$E_{4,t} : y^2 = x(x-4s^2)(x+(s^2-1)^2), s = \frac{2t}{t^2-3} \tag{2.5}$$

は

$$\left(s^2-1, \sqrt{-1}s(s^2-1)\frac{t^2+3}{t^2-3}\right) \tag{2.6}$$

を通る．

$$1 \leq \mathrm{rank}\, E_{4,t}(\overline{\mathbb{Q}}(t)) \leq 2 \tag{2.7}$$

証明

$$\Delta_{E_{4,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \tag{2.8}$$

$$e(\mathcal{E}_{4,t}) = 48 \tag{2.9}$$

TODO: $\rho(\mathcal{E}_{4,t}) \leq 40$ である．Theorem 3.0.1 より

$$\operatorname{rank} E_{4,t}(\overline{\mathbb{Q}}(t)) \leq 2 \tag{2.10}$$

□

上の評価は不十分．生成元は 1 つしか見つかっていないので，ランクの上界が 1 であることを示したい．

Theorem 2.1.3.

$$E_{4,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \tag{2.11}$$

$$E_{1,s}^{(1+3s^2)} : (1+3s^2)y^2 = x(x-4s^2)(x+(s^2-1)^2) \tag{2.12}$$

$$\operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \operatorname{rank} E_{4,t}(\overline{\mathbb{Q}}(t)) \tag{2.13}$$

さらに

$$E_{0,s} : y^2 = x(x-4s)(x+(s-1)^2) \tag{2.14}$$

$$E_{0,s}^{(1+3s)} : (1+3s)y^2 = x(x-4s)(x+(s-1)^2) \tag{2.15}$$

$$E_{0,s}^{(s(1+3s))} : s(1+3s)y^2 = x(x-4s)(x+(s-1)^2) \tag{2.16}$$

$$\operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = \operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \tag{2.17}$$

証明

$$s = \frac{2t}{t^2-3} \tag{2.18}$$

を t について解くと

$$t = \frac{1 \pm \sqrt{1+3s^2}}{s} \tag{2.19}$$

したがって

$$E_{4,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \tag{2.20}$$

□

Theorem 2.1.4. TODO

$$\operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = ? \tag{2.21}$$

証明

$$\Delta(E_{1,s}^{(1+3s^2)}) = (1+3s^2)^6 \Delta(E_{1,s}) \tag{2.22}$$

$$e(\mathcal{E}_{1,s}^{(1+3s^2)}) = 36 \tag{2.23}$$

Theorem 3.0.1 からは

$$\operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) \leq 2 \tag{2.24}$$

しか分からない．K3 ですらないので，H^2 の次元が分からず，reduction を取る方法でも計算が進められない．

$$\operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) =?(1 or 2) \tag{2.25}$$

□

Theorem 2.1.5. TODO

$$\operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \tag{2.26}$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s-1)^4(s+1)^4(3s+1)^6 \tag{2.27}$$

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \tag{2.28}$$

Theorem 3.0.1 からは

$$\operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \tag{2.29}$$

□

Theorem 2.1.6.

$$\operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \tag{2.30}$$

証明

$$(s-1, \sqrt{-1}(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \tag{2.31}$$

より rank は正である．

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \tag{2.32}$$

上と同様に

$$\operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \tag{2.33}$$

□

3. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, we can just find points of infinite order. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

Theorem 3.0.1. (Shioda-Tate formula, [1] Theorem 3.4)

$$\rho(S) = 2 + \sum_{v \in R}(m_v - 1) + \mathrm{rank}(E(K)) \tag{3.1}$$

である.

Theorem 3.0.2. rational や K3 のときの ρ はわかっている.

In order to make it easier to calculate the rank, we can use the following theorem.

Theorem 3.0.3. ([1] Proposition 4.1.)

$$\mathrm{rank}\, E(k(C)) + \mathrm{rank}\, E^{(u)}(k(C)) = \mathrm{rank}\, E(k(C)(\sqrt{u})) \tag{3.2}$$

However, Theorem 3.0.2 is still not enough to get the upper bound of the rank in our case.

TODO: etale cohomology を使う

4. Types of Special Fibers

Table 4.1 Singular fibers of $E_{1,s}$

Place	Type	m_v
$s = 0$	I_4	4
$s = \pm 1$	I_4	4
$s = \pm i$	I_4	4
$s = \infty$	I_4	4

Table 4.2 Singular fibers of $E_{4,t}$

Place	Type	m_v
$t = 0$	I_4	4
$t = \pm 1$	I_4	4
$t = \pm 3$	I_4	4
$t = \pm\sqrt{3}$	I_4	4
$t^4 - 2t^2 + 9 = 0$	I_4	4
$t = \infty$	I_4	4

Table 4.3 Singular fibers of $E_{1,s}^{(1+3s^2)}$

Place	Type	m_v
$s = 0$	I_4	4
$s = \pm 1$	I_4	4
$s = \pm i$	I_4	4
$s = \pm\frac{1}{\sqrt{-3}}$	I_0^*	5
$s = \infty$	I_4	4

Table 4.4 Singular fibers of $E_{0,s}^{(1+3s)}$

Place	Type	m_v
$s = 0$	I_2	2
$s = \pm 1$	I_4	4
$s = -\frac{1}{3}$	I_0^*	5
$s = \infty$	I_2^*	7

Table 4.5 Singular fibers of $E_{0,s}^{(s(1+3s))}$

Place	Type	m_v
$s = 0$	I_2^*	7
$s = \pm 1$	I_4	4
$s = -\frac{1}{3}$	I_0^*	5
$s = \infty$	I_2	2

5. Torsions

5.1 セクション

Theorem 5.1.1.

$$E_{4,t}(\overline{\mathbb{Q}}(t))_{\mathrm{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \tag{5.1}$$

$$T_1 = (2s(s+1)^2, 2s(s+1)^2(s^2+1)) \tag{5.2}$$

$$T_2 = (2is(s^2-1), 2is(s+i)^2(s^2-1)) \tag{5.3}$$

で生成される.

証明

$$E_{4,t}(\overline{\mathbb{Q}}(t))[2] = E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0,0), (4s^2, 0), (-(s^2-1)^2, 0)\} \tag{5.4}$$

$$2T_1 = (4s^2, 0) \tag{5.5}$$

$$2T_2 = (0, 0) \tag{5.6}$$

[1] の Lem.3.5 より

$$E_{4,t}(\overline{\mathbb{Q}}(t))_{\mathrm{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12} \tag{5.7}$$

なので位数 8 の点は存在しない. □

Remark 5.1.2. これは

$$E_{1,s}(\mathbb{Q}(s))_{\mathrm{tors}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \tag{5.8}$$

の別証明になっている.

6. Reductions

6.1 $E_{0,s}^{(1+3s)}$

K3 なので

$$\dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l) = 22 \tag{6.1}$$

である．Let V be the subspace of $\mathrm{NS}(\tilde{S})$ generated by the singular fibers and the zero section. Then V is of rank 19, on which the Frobenius automorphism acts by multiplication by p.

参考文献

[1] B. Naskręcki. Mordell-Weil ranks of families of elliptic curves associated to Pythagorean triples. eng. Acta Arithmetica 160.2, pp. 159–183, (2013). URL: http://eudml.org/doc/279803.