令和 6 年度

修士学位論文

# 論文用テンプレート


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○○課程 ○○専攻

○○分野

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## 概要

We study a family of elliptic curves $y^2 = x(x - a^2)(x + b^2)$, where $(a, b, c)$ are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \tag{1}$$

It is known that the generic rank of the Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ is 0. We found an infinite subfamily of $E_{1,s}$ whose Mordell-Weil rank over $\overline{\mathbb{Q}}(s)$ is 1, which means that there are infinitely many $s \in \overline{\mathbb{Q}}$ such that the Mordell-Weil group of $E_{1,s}$ has positive rank over $\overline{\mathbb{Q}}$.

We use the theory of elliptic surfaces to prove it. Each elliptic curve over a function field corresponds to an elliptic surface. The Shioda-Tate formula gives the relation between the Mordell-Weil rank and the Néron-Severi rank of elliptic surfaces. We compute the types of special fibers of the elliptic surfaces and consider the upper bound of the rank of the Néron-Severi group.

# 目次

# 1. Introduction

**Theorem 1.0.1.** Let

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \tag{1.1}$$

be an elliptic curve over $\overline{\mathbb{Q}}(s)$. Then, the Mordell-Weil group

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \tag{1.2}$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s+1)^2, 2s(s+1)^2(s^2+1)), \tag{1.3}$$

$$T_2 := (2\sqrt{-1}s(s^2-1), 2\sqrt{-1}s(s+\sqrt{-1})^2(s^2-1)). \tag{1.4}$$

**Corollary 1.0.2.**

$$E_{1,s}(\mathbb{Q}(s)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \tag{1.5}$$

is generated by $T_1$ and $2T_2 = (0, 0)$.

*Remark* 1.0.3. 課題研究では，多項式の解の非存在から背理法を用いて 8-torsion point が存在しないことを示した．TODO: specialization theorem は injectivity しか言ってないので, 各 special fiber について 8-torsion point が存在しないことを示した課題研究は上の系より真に強い？

**Theorem 1.0.4.** Let

$$E_{2,t} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2), \quad s = \frac{2t}{t^2 - 3} \tag{1.6}$$

be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Then, the Mordell-Weil group

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \tag{1.7}$$

especially the rank is 1. The torsion subgroup is generated by $T_1$ and $T_2$ in Theorem 1.0.1 and the free part is generated by

$$\left(s^2 - 1, \sqrt{-1}s(s^2 - 1)\frac{t^2 + 3}{t^2 - 3}\right). \tag{1.8}$$

# 2. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, finding generators is enough. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

**Theorem 2.0.1.** (Shioda-Tate formula, [1, Corollary 5.3]) Let $C$ be a smooth irreducible projective curve over an algebraically closed field $k$ and $E$ an elliptic curve over a function field $k(C)$. Let $\mathcal{E} \to C$ be the Néron model of $E$. Let $R \subset C$ be the set of points where the special fiber of $\mathcal{E}$ is singular. For each $v \in R$, let $m_v$ be the number of components of the special fiber of $\mathcal{E}$ at $v$. Let $\rho(\mathcal{E})$ denote the rank of the Néron-Severi group of $\mathcal{E}$. Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \operatorname{rank}(E(k(C))). \tag{2.1}$$

We can calculate $R$ and $m_v$ by Tate's algorithm, but it is still difficult to determine $\rho(\mathcal{E})$. We have the following theorem to get the upper bound of $\rho(\mathcal{E})$.

**Theorem 2.0.2.**

$$\rho(\mathcal{E}) \leq \frac{5}{6} e(\tilde{S}) + 2, \tag{2.2}$$

$$e(\tilde{S}) := \sum_{v \in R} e(F_v). \tag{2.3}$$

where $e(\tilde{S})$ is the Euler number, $e(F_v)$ is the local Euler number of the special fiber of $\mathcal{E}$ at $v$ for each $v \in R$ and

$$e(F_v) = \begin{cases} m_v & \text{if the fiber has multiplicative reduction,} \\ m_v + 1 & \text{if the fiber has additive reduction.} \end{cases} \tag{2.4}$$

証明　TODO: Naskrencki の PhD の 2.2.19(ii), 2.2.9, 2.2.10 （やその引用元）を引用するオイラー数については [2, pp. 136-137 付録 2], [3, p.14 Table II] □

**Theorem 2.0.3.** ([4, Lem.3.5])

$$E(\overline{\mathbb{Q}}(s))_{\mathrm{tors}} \hookrightarrow \prod_{v \in R} G(F_v) \tag{2.5}$$

where $G(F_v)$ is the group generated by all simple components of the fiber at $v$. If $F_v$ is of type $I_n$ in Kodaira's symbol, then $G(F_v) \cong \mathbb{Z}/n\mathbb{Z}$.

証明　TODO □

# 3.　Proof of Theorem 1.0.1

証明　of Theorem 1.0.1

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \tag{3.1}$$

Table 3.1　Singular fibers of $E_{1,s}$

| Place | Type | $m_v$ |
|---|---|---|
| $s=0$ | $I_4$ | 4 |
| $s=\pm 1$ | $I_4$ | 4 |
| $s=\pm\sqrt{-1}$ | $I_4$ | 4 |
| $s=\infty$ | $I_4$ | 4 |

$$e(\mathcal{E}_{1,s}) = 24 \tag{3.2}$$

したがって $\mathcal{E}_{1,s}$ は K3 曲面であり． $\rho(\mathcal{E}_{1,s}) \leq 20$ である．Theorem 2.0.1 より

$$\mathrm{rank}(E_{1,s}) = 0 \tag{3.3}$$

As for the torsion subgroup, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0,0), (4s^2,0), (-(s^2-1)^2, 0)\}, \tag{3.4}$$

and we can check by calculation that

$$2T_1 = (4s^2, 0), \tag{3.5}$$

$$2T_2 = (0,0). \tag{3.6}$$

By Theorem 2.0.3, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\mathrm{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^6. \tag{3.7}$$

Therefore, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\mathrm{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \tag{3.8}$$

□

# 4. The Generic Rank of $E_{2,t}$

In order to prove Theorem 1.0.4, Theorem 2.0.2 is not enough to get the exact upper bound of the ranks of the Néron-Severi group. Actually, we get $\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2$ from Theorem 2.0.1 and Theorem 2.0.2.

Table 4.1 Singular fibers of $E_{2,t}$

| Place | Type | $m_v$ |
|---|---|---|
| $t = 0$ | $I_4$ | 4 |
| $t = \pm 1$ | $I_4$ | 4 |
| $t = \pm 3$ | $I_4$ | 4 |
| $t = \pm\sqrt{3}$ | $I_4$ | 4 |
| $t^4 - 2t^2 + 9 = 0$ | $I_4$ | 4 |
| $t = \infty$ | $I_4$ | 4 |

証明

$$\Delta_{E_{2,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \tag{4.1}$$

$$e(\mathcal{E}_{2,t}) = 48 \tag{4.2}$$

TODO: $\rho(\mathcal{E}_{2,t}) \leq 40$ である．Theorem 2.0.1 より

$$\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2 \tag{4.3}$$

□

**Lemma 4.0.1.**

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\mathrm{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \tag{4.4}$$

証明　By Theorem 2.0.3, we have

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\mathrm{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12}. \tag{4.5}$$

Obviously, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\mathrm{tors}} \subset E_{2,t}(\overline{\mathbb{Q}}(t))_{\mathrm{tors}}. \tag{4.6}$$

□

On the other hand, we have only one point of infinite order in $E_{2,t}(\overline{\mathbb{Q}}(t))$.

**Lemma 4.0.2.**

$$\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) \geq 1 \tag{4.7}$$

Now, our goal is to show the upper bound of the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ is 1.

We use another method to estimate the upper bound of the rank of Néron-Severi group, which we will explain in Chapter 5. Beforehand, we express the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ in terms of ranks of elliptic curves with lower order coefficients in the Weierstrass equations to make the later computation feasible.

**Definition 4.0.3.** Let $C$ be a smooth curve over an algebraically closed field $k$. Let $E$ be an elliptic curve over a function field $k(C)$ given by the Weierstrass equation

$$E : y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{4.8}$$

where $a_2, a_4, a_6 \in k(C)$. For a fixed $u \in k(C)^*$, we denote

$$E^{(u)} : uy^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{4.9}$$

to be the quadratic twist of $E$ by $u$.

**Theorem 4.0.4.** ([5, Exercise 10.16]) Let $E$ be an elliptic curve over a function field $k(C)$ and $u \in k(C)^*$. Then, the following equation holds

$$\operatorname{rank} E(k(C)(\sqrt{u})) = \operatorname{rank} E(k(C)) + \operatorname{rank} E^{(u)}(k(C)). \tag{4.10}$$

**Theorem 4.0.5.** Let

$$E_{0,u} : y^2 = x(x-4u)(x+(u-1)^2) \tag{4.11}$$

be an elliptic curve over $\overline{\mathbb{Q}}(u)$. Then, we have

$$\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) = \operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)), \tag{4.12}$$

$$\operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) + \operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \tag{4.13}$$

Therefore, we have

$$\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) = \operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) + \operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \tag{4.14}$$

証明　Since solving $s = \frac{2t}{t^2-3}$ for $t$ yields $t = \frac{1\pm\sqrt{1+3s^2}}{s}$, we have

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \tag{4.15}$$

By Theorem 4.0.4, we get (4.12). Similarly, $E_{1,s}$ is obtained by substituting $u = s^2$ into $E_{0,u}$, so we have

$$E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)(\sqrt{u})), \tag{4.16}$$

then we get (4.13). □

**Theorem 4.0.6.** TODO

$$\operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 1 \tag{4.17}$$

証明

$$\Delta(E_{0,u}^{(1+3u)}) = 256u^2(u-1)^4(u+1)^4(3u+1)^6 \tag{4.18}$$

Table 4.2　Singular fibers of $E_{0,u}^{(1+3u)}$

| Place | Type | $m_v$ |
|---|---|---|
| $u = 0$ | $I_2$ | 2 |
| $u = \pm 1$ | $I_4$ | 4 |
| $u = -\frac{1}{3}$ | $I_0^*$ | 5 |
| $u = \infty$ | $I_2^*$ | 7 |

$$e(\mathcal{E}_{0,u}^{(1+3u)}) = 24 \tag{4.19}$$

Theorem 2.0.1 からは

$$\operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 1 \tag{4.20}$$

□

**Theorem 4.0.7.**

$$\operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) = 1 \tag{4.21}$$

証明

$$(u-1, \sqrt{-1}(u-1)) \in E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) \tag{4.22}$$

より rank は正である.

$$\Delta(E_{0,u}^{(u(1+3u))}) = 256u^8(u-1)^4(u+1)^4(3u+1)^6 \tag{4.23}$$

上と同様に

Table 4.3 Singular fibers of $E_{0,u}^{(u(1+3u))}$

| Place | Type | $m_v$ |
|---|---|---|
| $u=0$ | $I_2^*$ | 7 |
| $u=\pm 1$ | $I_4$ | 4 |
| $u=-\frac{1}{3}$ | $I_0^*$ | 5 |
| $u=\infty$ | $I_2$ | 2 |

$$\operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) \leq 1 \tag{4.24}$$

□

# 5. Reductions

Let $A$ be a discrete valuation ring of a number field $K$ with maximal ideal $\mathfrak{m}$, whose residue field $k$ has $q = p^r$ elements with $p$ prime. Let $S$ be an integral scheme with a morphism $S \longrightarrow \operatorname{Spec} A$ that is projective and smooth of relative dimension 2. Then the projective surface $\overline{S} = S_{\overline{\mathbb{Q}}}$ and $\tilde{S} = S_{\overline{k}}$ are smooth over the algebraically closed field $\overline{\mathbb{Q}}$ and $\overline{k}$, respectively. We will assume that $\overline{S}$ and $\tilde{S}$ are integrals, i.e., they are irreducible, nonsingular, projective surfaces.

Let $l \neq p$ be a prime number. For $X = \overline{S}$ or $\tilde{S}$, we denote by $H^2_{\text{ét}}(X, \mathbb{Q}_l)$ the $l$-adic étale cohomology group of $X$ and by $H^2_{\text{ét}}(X, \mathbb{Q}_l)(1)$ the Tate twist of it.

**Theorem 5.0.1.** ([6, Proposition 6.2.])

There are natural injective homomorphisms

$$\mathrm{NS}(\overline{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow \mathrm{NS}(\tilde{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)(1) \tag{5.1}$$

of finite-dimensional vector spaces over $\mathbb{Q}_l$.

Let $F : S_k \to S_k$ denote the absolute Frobenius, which acts as the identity on the points and by $f \mapsto f^p$ on the structure sheaf. Set $\varphi := F^r$ and let $\varphi^{(i)}$ denote the automorphism on $H^i_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)$ induced by $\varphi \times 1$ acting on $S_k \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \cong \tilde{S}$.

**Corollary 5.0.2.** ([6, Corollary 6.4.]) The ranks of $\mathrm{NS}(\overline{S})$ and $\mathrm{NS}(\tilde{S})$ are bounded from above by the number of eigenvalues $\lambda$ of $\varphi^{(2)}$ for which $\lambda/q$ is a root of unity, counted with multiplicity.

*Remark* 5.0.3. ([6, Remark 6.5.]) Tate's conjecture states that the upper bound mentioned in Corollary 5.0.2 is actually equal to the rank of $\mathrm{NS}(\tilde{S})$. Tate's conjecture has been proven for elliptic K3 surfaces.

**Lemma 5.0.4.**

$$\#\tilde{S}(\mathbb{F}_{q^m}) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}((\varphi^{(i)})^m) \tag{5.2}$$

**Corollary 5.0.5.**

$$\operatorname{Tr}((\varphi^{(2)})^m) = \#\tilde{S}(\mathbb{F}_{q^m}) - 1 - q^{2m} \tag{5.3}$$

**Theorem 5.0.6.** Let $V$ be the linear subspace of $H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)$ generated by the components of the singular fibers and by the zero section. Let $W = H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)/V$.

$$\operatorname{char}(\varphi^{(2)}) = \operatorname{char}(\varphi^{(2)}|V) \cdot \operatorname{char}(\varphi^{(2)}_W) \tag{5.4}$$

where $\varphi^{(2)}_W : W \to W$ is induced by $\varphi^{(2)}$.

$$\dim V = \sum_{v \in R} (m_v - 1) + 2 \tag{5.5}$$

$$\operatorname{char}(\varphi^{(2)}|V) = (x-q)^{\dim V} \tag{5.6}$$

$$t_m := \operatorname{Tr}((\varphi^{(2)}_W)^m) \tag{5.7}$$

$$\operatorname{char}(\varphi^{(2)}_W) = \frac{x^{\dim W}}{\exp\left(\sum_{m=1}^{\infty} \frac{t_m}{m} x^{-m}\right)} \text{のテイラー展開の多項式部分} \tag{5.8}$$

$$\frac{1}{\exp\left(\sum_{m=1}^{\infty} \frac{t_m}{m} x^{-m}\right)} = 1 + t_1 x^{-1} + \frac{t_1^2 - t^2}{2} x^{-2} + \frac{-t_1^3 + 3t_1 t_2 - 2t_3}{6} x^{-3} + \cdots \tag{5.9}$$

$$\operatorname{Tr}((\varphi^{(2)})^m) = \operatorname{Tr}(\varphi^{(2)}|V)^m + \operatorname{Tr}((\varphi^{(2)}_W)^m) \tag{5.10}$$

$$t_m = \#\tilde{S}(\mathbb{F}_{q^m}) - 1 - q^{2m} - \dim V \cdot q^m \tag{5.11}$$

We denote by $\mathcal{E}_{0,u}^{(1+3u)} \to \mathbb{P}^1$ the elliptic surface with the generic fiber $E_{0,u}^{(1+3u)}$.
Table 4.2 より K3 なので

$$\dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l) = 22 \tag{5.12}$$

である． $V$ is of rank 19, on which the Frobenius automorphism acts by multiplication by $p$.

$$\mathrm{char}(\varphi^{(2)}|V) = (x-5)^{19} \tag{5.13}$$

Note that all the multiplicative fibers are split in $\mathbb{F}_{5^m}$ for $m = 1, 2, 3$.

$$t_m = \#\tilde{S}(\mathbb{F}_{5^m}) - 1 - 5^{2m} - 19 \cdot 5^m \tag{5.14}$$

| m | 1 | 2 | 3 |
|---|---|---|---|
| $\#\tilde{S}(\mathbb{F}_{5^m})$ | 120 | 1080 | 18264 |
| $t_m$ | -1 | -21 | 263 |

$$\mathrm{char}(\varphi_W^{(2)}) = x^3 + x^2 + 11x - 77 \tag{5.15}$$

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