

令和 6 年度  
修士学位論文

論文用テンプレート

〇〇所属

〇〇課程 〇〇専攻

〇〇分野

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## 概要

We study a family of elliptic curves  $y^2 = x(x - a^2)(x + b^2)$ , where  $(a, b, c)$  are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1)$$

It is known that the generic rank of the Mordell-Weil group of  $E_{1,s}$  over  $\overline{\mathbb{Q}}(s)$  is 0. We found an infinite subfamily of  $E_{1,s}$  whose Mordell-Weil rank over  $\overline{\mathbb{Q}}(s)$  is 1, which means that there are infinitely many  $s \in \overline{\mathbb{Q}}$  such that the Mordell-Weil group of  $E_{1,s}$  has positive rank over  $\overline{\mathbb{Q}}$ .

We use the theory of elliptic surfaces to prove it. Each elliptic curve over a function field corresponds to an elliptic surface. The Shioda-Tate formula gives the relation between the Mordell-Weil rank and the Néron-Severi rank of elliptic surfaces. We compute the types of special fibers of the elliptic surfaces and consider the upper bound of the rank of the Néron-Severi group.

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# 1. Introduction

**Theorem 1.0.1.** Let

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \quad (1.1)$$

be an elliptic curve over  $\overline{\mathbb{Q}}(s)$ . Then, the Mordell-Weil group

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad (1.2)$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s+1)^2, 2s(s+1)^2(s^2+1)), \quad (1.3)$$

$$T_2 := (2\sqrt{-1}s(s^2-1), 2\sqrt{-1}s(s+\sqrt{-1})^2(s^2-1)). \quad (1.4)$$

**Corollary 1.0.2.**

$$E_{1,s}(\mathbb{Q}(s)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \quad (1.5)$$

is generated by  $T_1$  and  $2T_2 = (0, 0)$ .

*Remark 1.0.3.* 課題研究では、多項式の解の非存在から背理法を用いて 8-torsion point が存在しないことを示した。TODO: specialization theorem は injectivity しか言ってないので、各 special fiber について 8-torsion point が存在しないことを示した課題研究は上の系より真に強い？

**Theorem 1.0.4.** Let

$$E_{2,t} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2), \quad s = \frac{2t}{t^2 - 3} \quad (1.6)$$

be an elliptic curve over  $\overline{\mathbb{Q}}(t)$ . Then, the Mordell-Weil group

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad (1.7)$$

especially the rank is 1. The torsion subgroup is generated by  $T_1$  and  $T_2$  in Theorem 1.0.1 and the free part is generated by

$$\left( s^2 - 1, \sqrt{-1}s(s^2 - 1) \frac{t^2 + 3}{t^2 - 3} \right). \quad (1.8)$$

## 2. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, finding generators is enough. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

**Theorem 2.0.1.** (Shioda-Tate formula, [1, Corollary 5.3]) Let  $C$  be a smooth irreducible projective curve over an algebraically closed field  $k$  and  $E$  an elliptic curve over a function field  $k(C)$ . Let  $\mathcal{E} \rightarrow C$  be the Néron model of  $E$ . Let  $R \subset C$  be the set of points where the special fiber of  $\mathcal{E}$  is singular. For each  $v \in R$ , let  $m_v$  be the number of components of the special fiber of  $\mathcal{E}$  at  $v$ . Let  $\rho(\mathcal{E})$  denote the rank of the Néron-Severi group of  $\mathcal{E}$ . Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \text{rank}(E(k(C))). \quad (2.1)$$

We can calculate  $R$  and  $m_v$  by Tate's algorithm, but it is still difficult to determine  $\rho(\mathcal{E})$ . We have the following theorem to get the upper bound of  $\rho(\mathcal{E})$ .

**Theorem 2.0.2.**

$$\rho(\mathcal{E}) \leq \frac{5}{6}e(\tilde{S}) + 2, \quad (2.2)$$

$$e(\tilde{S}) := \sum_{v \in R} e(F_v). \quad (2.3)$$

where  $e(\tilde{S})$  is the Euler number,  $e(F_v)$  is the local Euler number of the special fiber of  $\mathcal{E}$  at  $v$  for each  $v \in R$  and

$$e(F_v) = \begin{cases} m_v & \text{if the fiber has multiplicative reduction,} \\ m_v + 1 & \text{if the fiber has additive reduction.} \end{cases} \quad (2.4)$$

証明 TODO: Naskrencki の PhD の 2.2.19(ii), 2.2.9, 2.2.10 (やその引用元) を引用するオイラー数については [2, pp. 136-137 付録 2], [3, p.14 Table II] □

**Theorem 2.0.3.** ([4, Lem.3.5])

$$E(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow \prod_{v \in R} G(F_v) \quad (2.5)$$

where  $G(F_v)$  is the group generated by all simple components of the fiber at  $v$ . If  $F_v$  is of type  $I_n$  in Kodaira's symbol, then  $G(F_v) \cong \mathbb{Z}/n\mathbb{Z}$ .

証明 TODO

□

### 3. Proof of Theorem 1.0.1

証明 of Theorem 1.0.1

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \quad (3.1)$$

Table 3.1 Singular fibers of  $E_{1,s}$

Place	Type	$m_v$
$s = 0$	$I_4$	4
$s = \pm 1$	$I_4$	4
$s = \pm\sqrt{-1}$	$I_4$	4
$s = \infty$	$I_4$	4

$$e(\mathcal{E}_{1,s}) = 24 \quad (3.2)$$

したがって  $\mathcal{E}_{1,s}$  は K3 曲面であり.  $\rho(\mathcal{E}_{1,s}) \leq 20$  である. Theorem 2.0.1 より

$$\text{rank}(E_{1,s}) = 0 \quad (3.3)$$

As for the torsion subgroup, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0, 0), (4s^2, 0), (-(s^2 - 1)^2, 0)\}, \quad (3.4)$$

and we can check by calculation that

$$2T_1 = (4s^2, 0), \quad (3.5)$$

$$2T_2 = (0, 0). \quad (3.6)$$

By Theorem 2.0.3, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^6. \quad (3.7)$$

Therefore, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \quad (3.8)$$

□

## 4. The Generic Rank of $E_{2,t}$

In order to prove Theorem 1.0.4, Theorem 2.0.2 is not enough to get the exact upper bound of the ranks of the Néron-Severi group. Actually, we get  $\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2$  from Theorem 2.0.1 and Theorem 2.0.2.

Table 4.1 Singular fibers of  $E_{2,t}$

Place	Type	$m_v$
$t = 0$	$I_4$	4
$t = \pm 1$	$I_4$	4
$t = \pm 3$	$I_4$	4
$t = \pm\sqrt{3}$	$I_4$	4
$t^4 - 2t^2 + 9 = 0$	$I_4$	4
$t = \infty$	$I_4$	4

証明

$$\Delta_{E_{2,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \quad (4.1)$$

$$e(\mathcal{E}_{2,t}) = 48 \quad (4.2)$$

TODO:  $\rho(\mathcal{E}_{2,t}) \leq 40$  である. Theorem 2.0.1 より

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2 \quad (4.3)$$

□

**Lemma 4.0.1.**

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \quad (4.4)$$

証明 By Theorem 2.0.3, we have

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12}. \quad (4.5)$$

Obviously, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \subset E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}}. \quad (4.6)$$

□

On the other hand, we have only one point of infinite order in  $E_{2,t}(\overline{\mathbb{Q}}(t))$ .

**Lemma 4.0.2.**

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \geq 1 \quad (4.7)$$

Now, our goal is to show the upper bound of the rank of  $E_{2,t}(\overline{\mathbb{Q}}(t))$  is 1.

We use another method to estimate the upper bound of the rank of Néron-Severi group, which we will explain in Chapter 5. Beforehand, we express the rank of  $E_{2,t}(\overline{\mathbb{Q}}(t))$  in terms of ranks of elliptic curves with lower order coefficients in the Weierstrass equations to make the later computation feasible.

**Definition 4.0.3.** Let  $C$  be a smooth curve over an algebraically closed field  $k$ . Let  $E$  be an elliptic curve over a function field  $k(C)$  given by the Weierstrass equation

$$E : y^2 = x^3 + a_2x^2 + a_4x + a_6 \quad (4.8)$$

where  $a_2, a_4, a_6 \in k(C)$ . For a fixed  $u \in k(C)^*$ , we denote

$$E^{(u)} : uy^2 = x^3 + a_2x^2 + a_4x + a_6 \quad (4.9)$$

to be the quadratic twist of  $E$  by  $u$ .

**Theorem 4.0.4.** ([5, Exercise 10.16]) Let  $E$  be an elliptic curve over a function field  $k(C)$  and  $u \in k(C)^*$ .

Then, the following equation holds

$$\text{rank } E(k(C)(\sqrt{u})) = \text{rank } E(k(C)) + \text{rank } E^{(u)}(k(C)). \quad (4.10)$$

**Theorem 4.0.5.** Let

$$E_{0,u} : y^2 = x(x - 4u)(x + (u - 1)^2) \quad (4.11)$$



be an elliptic curve over  $\overline{\mathbb{Q}}(u)$ . Then, we have

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) = \text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)), \quad (4.12)$$

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) + \text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \quad (4.13)$$

Therefore, we have

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) = \text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) + \text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \quad (4.14)$$

証明 Since solving  $s = \frac{2t}{t^2-3}$  for  $t$  yields  $t = \frac{1 \pm \sqrt{1+3s^2}}{s}$ , we have

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})) \quad (4.15)$$

By Theorem 4.0.4, we get (4.12). Similarly,  $E_{1,s}$  is obtained by substituting  $u = s^2$  into  $E_{0,u}$ , so we have

$$E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)(\sqrt{u})), \quad (4.16)$$

then we get (4.13). □

**Theorem 4.0.6.** TODO

$$\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 1 \quad (4.17)$$

証明

$$\Delta(E_{0,u}^{(1+3u)}) = 256u^2(u-1)^4(u+1)^4(3u+1)^6 \quad (4.18)$$

Table 4.2 Singular fibers of  $E_{0,u}^{(1+3u)}$

Place	Type	$m_v$
$u = 0$	$I_2$	2
$u = \pm 1$	$I_4$	4
$u = -\frac{1}{3}$	$I_0^*$	5
$u = \infty$	$I_2^*$	7

$$e(\mathcal{E}_{0,u}^{(1+3u)}) = 24 \quad (4.19)$$

Theorem 2.0.1 からは

$$\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 1 \quad (4.20)$$

□

**Theorem 4.0.7.**

$$\text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) = 1 \quad (4.21)$$

証明

$$(u-1, \sqrt{-1}(u-1)) \in E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) \quad (4.22)$$

より rank は正である.

$$\Delta(E_{0,u}^{(u(1+3u))}) = 256u^8(u-1)^4(u+1)^4(3u+1)^6 \quad (4.23)$$

上と同様に

Table 4.3 Singular fibers of  $E_{0,u}^{(u(1+3u))}$

Place	Type	$m_v$
$u = 0$	$I_2^*$	7
$u = \pm 1$	$I_4$	4
$u = -\frac{1}{3}$	$I_0^*$	5
$u = \infty$	$I_2$	2

$$\text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) \leq 1 \quad (4.24)$$

□

## 5. Reductions

Let  $A$  be a discrete valuation ring of a number field  $K$  with maximal ideal  $\mathfrak{m}$ , whose residue field  $k$  has  $q = p^r$  elements with  $p$  prime. Let  $S$  be an integral scheme with a morphism  $S \rightarrow \text{Spec } A$  that is projective and smooth of relative dimension 2. Then the projective surface  $\overline{S} = S_{\overline{\mathbb{Q}}}$  and  $\tilde{S} = S_{\overline{k}}$  are smooth over the algebraically closed field  $\overline{\mathbb{Q}}$  and  $\overline{k}$ , respectively. We will assume that  $\overline{S}$  and  $\tilde{S}$  are integrals, i.e., they are irreducible, nonsingular, projective surfaces.

For  $l \neq p$  be a prime number, we denote by  $H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)$  the  $l$ -adic étale cohomology group of  $X$  and by  $H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)(1)$  the Tate twist of it.

**Theorem 5.0.1.** ([6, Proposition 6.2.])

There are natural injective homomorphisms

$$\mathrm{NS}(\bar{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow \mathrm{NS}(\tilde{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow H_{\mathrm{\acute{e}t}}^2(\tilde{S}, \mathbb{Q}_l)(1) \quad (5.1)$$

of finite-dimensional vector spaces over  $\mathbb{Q}_l$ .

Let  $F : S_k \rightarrow S_k$  denote the absolute Frobenius, which acts as the identity on the points and by  $f \mapsto f^p$  on the structure sheaf. Set  $\varphi := F^r$  and let  $\varphi^{(i)}$  denote the automorphism on  $H_{\mathrm{\acute{e}t}}^i(\tilde{S}, \mathbb{Q}_l)$  induced by  $\varphi \times 1$  acting on  $S_k \times_{\mathrm{Spec} k} \mathrm{Spec} \bar{k} \cong \tilde{S}$ .

**Corollary 5.0.2.** ([6, Corollary 6.4.]) The ranks of  $\mathrm{NS}(\bar{S})$  and  $\mathrm{NS}(\tilde{S})$  are bounded from above by the number of eigenvalues  $\lambda$  of  $\varphi^{(2)}$  for which  $\lambda/q$  is a root of unity, counted with multiplicity.

*Remark 5.0.3.* ([6, Remark 6.5.]) Tate's conjecture states that the upper bound mentioned in Corollary 5.0.2 is actually equal to the rank of  $\mathrm{NS}(\tilde{S})$ . Tate's conjecture has been proven for elliptic K3 surfaces.

Now we want to calculate the characteristic polynomial  $\mathrm{char}(\varphi^{(2)})$ . Beforehand, we recall the Lefschetz trace formula.

**Theorem 5.0.4.**

$$\#\tilde{S}(\mathbb{F}_{q^m}) = \sum_{i=0}^n (-1)^i \mathrm{Tr}((\varphi^{(i)})^m) \quad (5.2)$$

**Corollary 5.0.5.**

$$\mathrm{Tr}((\varphi^{(2)})^m) = \#\tilde{S}(\mathbb{F}_{q^m}) - 1 - q^{2m} \quad (5.3)$$

証明

$$\dim H_{\mathrm{\acute{e}t}}^1(\tilde{S}, \mathbb{Q}_l) = \dim H_{\mathrm{\acute{e}t}}^3(\tilde{S}, \mathbb{Q}_l) = 0 \quad (5.4)$$

and  $\varphi^{(4)}$  acts on  $H_{\mathrm{\acute{e}t}}^4(\tilde{S}, \mathbb{Q}_l) \cong \mathbb{Q}_l$  by multiplication by  $q^2$ .  $\square$

Let  $V$  be the linear subspace of  $H_{\mathrm{\acute{e}t}}^2(\tilde{S}, \mathbb{Q}_l)$  generated by the components of the singular fibers and by the zero section and  $W = H_{\mathrm{\acute{e}t}}^2(\tilde{S}, \mathbb{Q}_l)/V$ , then

$$\dim V = \sum_{v \in R} (m_v - 1) + 2. \quad (5.5)$$

By the multiplicativity of the characteristic polynomial, we have

$$\mathrm{char}(\varphi^{(2)}) = \mathrm{char}(\varphi^{(2)}|V) \cdot \mathrm{char}(\varphi_W^{(2)}) \quad (5.6)$$

and

$$\mathrm{Tr}((\varphi^{(2)})^m) = \mathrm{Tr}((\varphi^{(2)}|V)^m) + \mathrm{Tr}((\varphi_W^{(2)})^m) \quad (5.7)$$

for any  $m \in \mathbb{Z}$ , where  $\varphi_W^{(2)} : W \rightarrow W$  is induced by  $\varphi^{(2)}$ . Since  $\varphi^{(2)}$  acts on  $V$  by multiplication by  $q$ , we have

$$\text{char}(\varphi^{(2)}|V) = (x - q)^{\dim V}. \quad (5.8)$$

As for the characteristic polynomial of  $\varphi_W^{(2)}$ , let  $t_m := \text{Tr}((\varphi_W^{(2)})^m)$ , then  $\text{char}(\varphi_W^{(2)})$  is the polynomial part of

$$\frac{x^{\dim W}}{\exp\left(\sum_{m=1}^{\infty} \frac{t_m}{m} x^{-m}\right)} = x^{\dim W} \left(1 + t_1 x^{-1} + \frac{t_1^2 - t_2}{2} x^{-2} + \frac{-t_1^3 + 3t_1 t_2 - 2t_3}{6} x^{-3} + \dots\right). \quad (5.9)$$

Here, by (5.7) and Corollary 5.0.5, we have

$$t_m = \#\tilde{S}(\mathbb{F}_{q^m}) - 1 - q^{2m} - \dim V \cdot q^m. \quad (5.10)$$

**Lemma 5.0.6.** ([7, Theorem 4, Part III]) If  $\tilde{S}$  is a K3 surface, then the second Betti number of  $\tilde{S}$  is 22.

**Theorem 5.0.7.**

$$\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) = 0 \quad (5.11)$$

**証明** We denote by  $S = \mathcal{E}_{0,u}^{(1+3u)} \rightarrow \mathbb{P}^1$  the elliptic surface with the generic fiber  $E_{0,u}^{(1+3u)}$ .

Table 4.2 より K3 なので Lemma 5.0.6 より

$$\dim_{\mathbb{Q}_l} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l) = 22 \quad (5.12)$$

である.  $V$  is of rank 19, on which the Frobenius automorphism acts by multiplication by  $p$ .

$$\text{char}(\varphi^{(2)}|V) = (x - 5)^{19} \quad (5.13)$$

Note that all the multiplicative fibers are split in  $\mathbb{F}_{5^m}$  for  $m = 1, 2, 3$ .

$$t_m = \#\tilde{S}(\mathbb{F}_{5^m}) - 1 - 5^{2m} - 19 \cdot 5^m \quad (5.14)$$

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264
$t_m$	-1	-21	263

$$\text{char}(\varphi_W^{(2)}) = x^3 + x^2 + 11x - 77 \quad (5.15)$$

If  $\text{char}(\varphi_W^{(2)})$  has a root of the form  $x = 5\zeta$  for some root of unity  $\zeta$ , then  $\zeta$  is a root of the polynomial

$$125x^3 + 25x^2 + 55x - 77, \quad (5.16)$$

which is irreducible over  $\mathbb{Q}$ . It contradicts the fact that  $\zeta$  is an algebraic integer. By Corollary 5.0.2,  $\rho(\mathcal{E}_{0,u}^{(1+3u)}) \leq 19$ . Then by Theorem 2.0.1, we have

$$\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) = 0. \quad (5.17)$$

□

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