令和 6 年度

修士学位論文

論文用テンプレート

○○所属

○○課程 ○○専攻

○○分野

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概要

We study a family of elliptic curves $y^2 = x(x - a^2)(x + b^2)$, where (a, b, c) are Pythagorean triples. This is the family of the Frey curves of degree 2. We can 1-parameterize Pythagorean triples by rational numbers and consider the family as an elliptic curve over a function field.

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \tag{1}$$

It is known that the generic rank of the Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ is 0. We found an infinite subfamily of $E_{1,s}$ whose Mordell-Weil rank over $\overline{\mathbb{Q}}(s)$ is 1, which means that there are infinitely many $s \in \overline{\mathbb{Q}}$ such that the Mordell-Weil group of $E_{1,s}$ has positive rank over $\overline{\mathbb{Q}}$.

We use the theory of elliptic surfaces to prove it. Each elliptic curve over a function field corresponds to an elliptic surface. The Shioda-Tate formula gives the relation between the Mordell-Weil rank and the Néron-Severi rank of elliptic surfaces. We compute the types of special fibers of the elliptic surfaces and consider the upper bound of the rank of the Néron-Severi group.

目次

1. Introduction

Theorem 1.0.1. Let

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2) \tag{1.1}$$

be an elliptic curve over $\overline{\mathbb{Q}}(s)$. Then, the Mordell-Weil group

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \tag{1.2}$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s+1)^2, 2s(s+1)^2(s^2+1)), \tag{1.3}$$

$$T_2 := (2is(s^2-1), 2is(s+i)^2(s^2-1)). \tag{1.4}$$

Corollary 1.0.2.

$$E_{1,s}(\mathbb{Q}(s)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \tag{1.5}$$

is generated by T_1 and $2T_2 = (0, 0)$.

Remark 1.0.3. 課題研究では，多項式の解の非存在から背理法を用いて 8-torsion point が存在しないことを示した．

Theorem 1.0.4. Let

$$E_{2,t} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2), \quad s = \frac{2t}{t^2 - 3} \tag{1.6}$$

be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Then, the Mordell-Weil group

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \tag{1.7}$$

especially the rank is 1. The torsion subgroup is generated by T_1 and T_2 in Theorem 1.0.1 and the free part is generated by

$$\left(s^2 - 1, is(s^2 - 1)\frac{t^2 + 3}{t^2 - 3}\right). \tag{1.8}$$

2. Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, finding generators is enough. It is more difficult to get the upper bound of the rank. The following theorem behaves a key role in the proof of the main theorem.

Theorem 2.0.1. (Shioda-Tate formula, [1, Corollary 5.3]) Let C be a smooth irreducible projective curve over an algebraically closed field k and E an elliptic curve over a function field $k(C)$. Let $\mathcal{E} \to C$ be the Néron model of E. Let $R \subset C$ be the set of points where the special fiber of $\mathcal{E}$ is singular. For each $v \in R$, let m_v be the number of components of the special fiber of $\mathcal{E}$ at v. Let $\rho(\mathcal{E})$ denote the rank of the Néron-Severi group of $\mathcal{E}$. Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \operatorname{rank}(E(k(C))). \tag{2.1}$$

We can calculate R and m_v by Tate's algorithm, but it is still difficult to determine $\rho(\mathcal{E})$. We have the following theorem to get the upper bound of $\rho(\mathcal{E})$.

Theorem 2.0.2.

$$\rho(\mathcal{E}) \leq \frac{5}{6} e(\tilde{S}) + 2, \tag{2.2}$$

$$e(\tilde{S}) := \sum_{v \in R} e(F_v). \tag{2.3}$$

where $e(\tilde{S})$ is the Euler number, $e(F_v)$ is the local Euler number of the special fiber of $\mathcal{E}$ at v for each $v \in R$ and

$$e(F_v) = \begin{cases} m_v & \text{if the fiber has multiplicative reduction,} \\ m_v + 1 & \text{if the fiber has additive reduction.} \end{cases} \tag{2.4}$$

証明　TODO: Naskrencki の PhD の 2.2.19(ii), 2.2.9, 2.2.10（やその引用元）を引用する □

Theorem 2.0.3. ([2, Lem.3.5])

$$E(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow \prod_{v \in R} G(F_v) \tag{2.5}$$

where $G(F_v)$ is the group generated by all simple components of the fiber at v. If F_v is of type I_n in Kodaira notation, then $G(F_v) \cong \mathbb{Z}/n\mathbb{Z}$.

証明　TODO ◻

3.　Proof of Theorem 1.0.1

証明　of Theorem 1.0.1

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \tag{3.1}$$

Table 3.1　Singular fibers of $E_{1,s}$

Place	Type	m_v
$s=0$	I_4	4
$s=\pm 1$	I_4	4
$s=\pm i$	I_4	4
$s=\infty$	I_4	4

$$e(\mathcal{E}_{1,s}) = 24 \tag{3.2}$$

したがって $\mathcal{E}_{1,s}$ は K3 曲面であり．$\rho(\mathcal{E}_{1,s}) \leq 20$ である．Theorem 2.0.1 より

$$\mathrm{rank}(E_{1,s}) = 0 \tag{3.3}$$

As for the torsion subgroup, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{O, (0,0), (4s^2, 0), (-(s^2-1)^2, 0)\} \tag{3.4}$$

$$2T_1 = (4s^2, 0) \tag{3.5}$$

$$2T_2 = (0,0) \tag{3.6}$$

Theorem 2.0.3 より

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\mathrm{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12} \tag{3.7}$$

なので位数 8 の点は存在しない. ◻

4. The Generic Rank of $E_{2,t}$

In order to prove Theorem 1.0.4, Theorem 2.0.2 is not enough to get the exact upper bound of the ranks of the Néron-Severi group. Actually, we get rank $E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2$ from Theorem 2.0.1 and Theorem 2.0.2.

Table 4.1 Singular fibers of $E_{2,t}$

Place	Type	m_v
$t = 0$	I_4	4
$t = \pm 1$	I_4	4
$t = \pm 3$	I_4	4
$t = \pm\sqrt{3}$	I_4	4
$t^4 - 2t^2 + 9 = 0$	I_4	4
$t = \infty$	I_4	4

証明

$$\Delta_{E_{2,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4 \tag{4.1}$$

$$e(\mathcal{E}_{4,t}) = 48 \tag{4.2}$$

TODO: $\rho(\mathcal{E}_{4,t}) \leq 40$ である．Theorem 2.0.1 より

$$\text{rank}\, E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2 \tag{4.3}$$

□

On the other hand, we have only one point of infinite order in $E_{2,t}(\overline{\mathbb{Q}}(t))$. Now, our goal is to show the upper bound of the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ is 1.

We use another method to estimate the upper bound of the rank of Néron-Severi group, which we will explain in Chapter 5. Beforehand, we express the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ in terms of ranks of elliptic curves with lower order coefficients in the Weierstrass equations to make the later computation feasible.

Definition 4.0.1. Let C be a smooth curve over an algebraically closed field k. Let E be an elliptic curve over a function field $k(C)$ given by the Weierstrass equation

$$E : y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{4.4}$$

where $a_2, a_4, a_6 \in k(C)$. For a fixed $u \in k(C)^*$, we denote

$$E^{(u)} : uy^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{4.5}$$

to be the quadratic twist of E by u.

Theorem 4.0.2. ([3, Exercise 10.16]) Let E be an elliptic curve over a function field $k(C)$ and $u \in k(C)^*$. Then, the following equation holds

$$\operatorname{rank} E(k(C)(\sqrt{u})) = \operatorname{rank} E(k(C)) + \operatorname{rank} E^{(u)}(k(C)). \tag{4.6}$$

Theorem 4.0.3. Let

$$E_{0,s} : y^2 = x(x - 4s)(x + (s - 1)^2) \tag{4.7}$$

bet an elliptic curve over $\overline{\mathbb{Q}}(s)$. Then, we have

$$\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) = \operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)), \tag{4.8}$$

$$\operatorname{rank} E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)). \tag{4.9}$$

Therefore, we have

$$\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) = \operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) + \operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)). \tag{4.10}$$

証明　Since solving $s = \frac{2t}{t^2-3}$ for t yields $t = \frac{1\pm\sqrt{1+3s^2}}{s}$, we have

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1 + 3s^2})) \tag{4.11}$$

By Theorem 4.0.2, we get (4.8). Similarly, we have

$$E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)(\sqrt{s})), \tag{4.12}$$

then we get (4.9). □

Theorem 4.0.4. TODO

$$\operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \tag{4.13}$$

証明

$$\Delta(E_{0,s}^{(1+3s)}) = 256s^2(s - 1)^4(s + 1)^4(3s + 1)^6 \tag{4.14}$$

Table 4.2　Singular fibers of $E_{0,s}^{(1+3s)}$

Place	Type	m_v
$s=0$	I_2	2
$s=\pm 1$	I_4	4
$s=-\frac{1}{3}$	I_0^*	5
$s=\infty$	I_2^*	7

$$e(\mathcal{E}_{0,s}^{(1+3s)}) = 24 \tag{4.15}$$

Theorem 2.0.1 からは

$$\operatorname{rank} E_{0,s}^{(1+3s)}(\overline{\mathbb{Q}}(s)) \leq 1 \tag{4.16}$$

□

Theorem 4.0.5.

$$\operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) = 1 \tag{4.17}$$

証明

$$(s-1, i(s-1)) \in E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \tag{4.18}$$

より rank は正である.

$$\Delta(E_{0,s}^{(s(1+3s))}) = 256s^8(s-1)^4(s+1)^4(3s+1)^6 \tag{4.19}$$

上と同様に

Table 4.3　Singular fibers of $E_{0,s}^{(s(1+3s))}$

Place	Type	m_v
$s=0$	I_2^*	7
$s=\pm 1$	I_4	4
$s=-\frac{1}{3}$	I_0^*	5
$s=\infty$	I_2	2

$$\operatorname{rank} E_{0,s}^{(s(1+3s))}(\overline{\mathbb{Q}}(s)) \leq 1 \tag{4.20}$$

□

5. Reductions

5.1 $E_{0,s}^{(1+3s)}$

We denote by $\mathcal{E}_{0,s}^{(1+3s)} \to \mathbb{P}^1$ the elliptic surface with the generic fiber $E_{0,s}^{(1+3s)}$.

Table 4.2 より K3 なので

$$\dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l) = 22 \tag{5.1}$$

である．Let V be the subspace of $\mathrm{NS}(\tilde{S})$ generated by the singular fibers and the zero section. Then V is of rank 19, on which the Frobenius automorphism acts by multiplication by p.

$$\mathrm{char}(\Phi^*_{\tilde{S}}|V) = (x-5)^{19} \tag{5.2}$$

Note that all the multiplicative fibers are split.

$$t_m := \mathrm{Tr}((\Phi^*_{\tilde{S},H^2_{\text{ét}}/V})^m) = \#\tilde{S}(\mathbb{F}_{5^m}) - 1 - 5^{2m} - 19 \cdot 5^m \tag{5.3}$$

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264
t_m	-1	-21	263

$$\mathrm{char}(\Phi^*_{\tilde{S},H^2_{\text{ét}}/V}) = x^3 + x^2 + 11x - 77 \tag{5.4}$$

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