On the Mordell-Weil groups of elliptic surfaces associated with Frey curves of degree two

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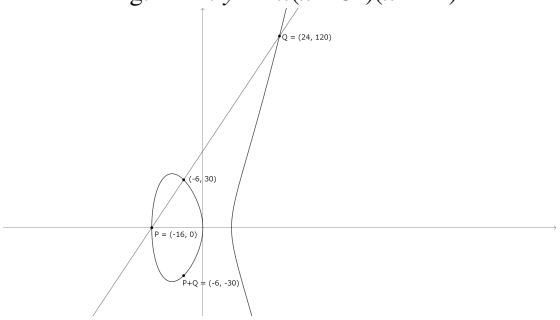
1 Introduction

An elliptic curve defined over a field K is a curve defined by a Weierstrass equation

$$E : y^2 = x^3 + Ax^2 + Bx + C$$

where $A, B, C \in K$ and the discriminant $\Delta = -4A^3C + A^2B^2 + 18ABC - 4B^3 - 27C^2$ is nonzero. On points on an elliptic curve defined over $\mathbb{Q}$, we can define an addition law geometrically. For two points P, Q on E, the point $-(P + Q)$ is defined as the third point of intersection of the line passing through P and Q with the curve. The sum $P + Q$ is the point symmetric to $-(P + Q)$ with respect to the x-axis.

Fig. 1 $E : y^2 = x(x - 3^2)(x + 4^2)$



The definition can be extended to any field K. The set of points on an elliptic curve forms an abelian group with the identity element being the point at infinity. The Mordell-Weil group $E(K)$ is a group consisting of all K-rational points on E. The Mordell-Weil theorem states that the Mordell-Weil group is a finitely generated abelian group. The Mordell-Weil group is an important object in the study of the arithmetic of elliptic curves. Especially, the rank of the Mordell-Weil group is important and difficult to determine in general.

Let $(a, b, c) \in \mathbb{Z}^3$ be a Pythagorean triple, namely integers satisfies $a^2 + b^2 = c^2$, and consider the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x - a^2)(x + b^2). \tag{1}$$

This is the $n = 2$ case of the Frey curve.

We can parameterize Pythagorean triples (a, b, c) by $m, n \in \mathbb{Z}$ with $(m, n) = 1$ as $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$. Then the equation (1) can be written as $y^2 = x(x-4m^2n^2)(x+(m^2 - n^2)^2)$. We replace x, y by n^2x, n^3y and put $s = m/n$. Then we get an elliptic curve

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2).$$

We consider $E_{1,s}$ as an elliptic curve over a function field $\overline{\mathbb{Q}}(s)$. We associate an elliptic surface $\mathcal{E}_{1,s} \to \mathbb{P}^1$ to $E_{1,s}$.

2 Main Theorem

Theorem 2.1. The Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ satisfies

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

By substituting $s = \frac{2t}{t^2-3}$ into $E_{1,s}$, we get a new family of elliptic curves

$$E_{2,t} : y^2 = x\left(x - 4\left(\frac{2t}{t^2-3}\right)^2\right)\left(x + \left(\left(\frac{2t}{t^2-3}\right)^2 - 1\right)^2\right),$$

which is a subfamily of $E_{1,s}$. The following is our main result.

Theorem 2.2. The Mordell-Weil group of $E_{2,t}$ over $\overline{\mathbb{Q}}(t)$ satisfies

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

The important point is that we prove that the generic rank of $E_{2,t}$ is exactly 1, not only the existence of a point of infinite order. Our proof is based on the method of Naskręcki in [1].

3 Sketch of proof

First, we compute types of singular fibers of elliptic surfaces by Tate's algorithm. The number of components m_v and the local Euler number $e(\mathcal{E}_v)$ of each special fiber at v are computed. The torsion subgroups are determined using these data. We know that there is a point of infinite order $\left(s^2 - 1, \sqrt{-1}s(s^2 - 1)\frac{t^2+3}{t^2-3}\right)$ on $E_{2,t}(\overline{\mathbb{Q}}(t))$. This implies that the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ is at least 1. The following theorem plays a key role in the proof of the main theorem.

Theorem 3.1. (Shioda-Tate formula, [3, Corollary 5.3]) Let $\mathcal{E} \to C$ be an elliptic surface over a smooth projective curve C over an algebraically closed field k. Let $R \subset C$ be the set of points where the special fiber of $\mathcal{E}$ is singular. For each $v \in R$, let m_v be the number of components of the special fiber of $\mathcal{E}$ at v. Let $\rho(\mathcal{E})$ denote the rank of the Néron-Severi group of $\mathcal{E}$. Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R}(m_v - 1) + \operatorname{rank}(E(k(C))).$$

Now, we are interested in the upper bounds of the rank of the Néron-Severi groups. To make the later computation feasible, we decompose the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ into the ranks of elliptic curves with lower order coefficients in the Weierstrass equations.

Theorem 3.2. Let

$$E_{0,u} : y^2 = x(x - 4u)(x + (u - 1)^2)$$

be an elliptic curve over $\overline{\mathbb{Q}}(u)$. Then, we have

$$\begin{aligned}\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) = &\operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) \\ &+ \operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \\ &+ \operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)).\end{aligned}$$

We have an estimation that $\rho(\mathcal{E}) \leq \frac{5}{6}\sum_{v \in R} e(\mathcal{E}_v)$ by [2], which gives

$$\operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) = 0,$$

$$\operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) = 1.$$

However, the upper bound computed in the same way for $E_{0,u}^{(1+3u)}$ is not sharp. We need a more sophisticated method to determine the rank of $E_{0,u}^{(1+3u)}$, which we will explain below.

Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Assume that the residue field $k = A/\mathfrak{m}$ has $q = p^r$ elements with p prime. Let S be an integral scheme with a morphism $S \to \operatorname{Spec} A$ that is projective and smooth of relative dimension 2. Then the projective surface $\overline{S} = S_{\overline{\mathbb{Q}}}$ and $\tilde{S} = S_{\overline{k}}$ are smooth over the algebraically closed field $\overline{\mathbb{Q}}$ and $\overline{k}$, respectively. For a prime number $l \neq p$, we denote by $H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)$ the l-adic étale cohomology group of X and by $H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)(1)$ its Tate twist. Let $\varphi^{(i)}$ be the Frobenius automorphism acting on $H^i_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)$. Under some assumptions, we have the following theorem.

Theorem 3.3. ([4, Corollary 6.4.]) The ranks of $\mathrm{NS}(\overline{S})$ and $\mathrm{NS}(\tilde{S})$ are bounded from above by the number of eigenvalues λ of $\varphi^{(2)}$ for which λ/q is a root of unity, counted with multiplicity.

We apply Theorem 3.3 with $A = \mathbb{Z}_{(5)}$ and $S = \mathcal{E}_{0,u}^{(1+3u)} \to \mathbb{P}^1$. The characteristic polynomial of $\varphi^{(2)}$ can be calculated if we know the traces of $(\varphi^{(2)})^m$ for $m = 1, 2, 3$. The traces can be calculated by the Lefschetz fixed point theorem:

$$\#\tilde{S}(\mathbb{F}_{q^m}) = \sum_{i=0}^{n}(-1)^i \operatorname{Tr}((\varphi^{(i)})^m).$$

Tate's algorithm gives the types of singular fibers of $\mathcal{E}_{0,u}^{(1+3u)}$ as shown in Table 1, and the number of points on $\tilde{S}(\mathbb{F}_{5^m})$ are calculated as shown in Table 2.

Tab. 1 Singular fibers of $E_{0,u}^{(1+3u)}$

Place	Type	m_v	e
$u = 0$	I_2	2	2
$u = \pm 1$	I_4	4	4
$u = -\frac{1}{3}$	I_0^*	5	6
$u = \infty$	I_2^*	7	8

Tab. 2 $\#\tilde{S}(\mathbb{F}_{5^m})$

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264

Using the computation above, we obtain

$$\operatorname{char}(\varphi^{(2)}) = (x - 5)^{19}(x^3 + x^2 + 11x - 77).$$

By Corollary 3.3, $\rho(\mathcal{E}_{0,u}^{(1+3u)}) \leq 19$. Then by Theorem 3.1, we have $\operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 0$.

References

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