On the Mordell-Weil groups of elliptic surfaces associated with Frey curves of degree two

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1 Introduction

An elliptic curve defined over a field K of characteristic $\neq 2$ is a curve defined by a Weierstrass equation

$$E : y^2 = x^3 + Ax^2 + Bx + C$$

where $A, B, C \in K$ and the discriminant $\Delta = -4A^3C + A^2B^2 + 18ABC - 4B^3 - 27C^2$ is nonzero. When $K = \mathbb{Q}$, we can define an addition law geometrically on points on an elliptic curve defined over $\mathbb{Q}$. For two points P, Q on E, the point $-(P+Q)$ is defined as the third point of intersection of the line passing through P and Q with the curve. The sum $P+Q$ is the point symmetric to $-(P+Q)$ with respect to the x-axis. The definition can be extended to any field K of characteristic $\neq 2$. The set of points on an elliptic curve forms an abelian group with the identity element being the point at infinity. The Mordell-Weil group $E(K)$ is a group consisting of all K-rational points on E. The Mordell-Weil theorem states that the Mordell-Weil group is a finitely generated abelian group. The Mordell-Weil group is an important object in the study of the arithmetic of elliptic curves. Especially, the rank of the Mordell-Weil group is important and difficult to determine, in general.

Let $(a, b, c) \in \mathbb{Z}^3$ be a Pythagorean triple, namely integers satisfies $a^2 + b^2 = c^2$, and consider the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x - a^2)(x + b^2). \tag{1}$$

This is the case $n = 2$ of the Frey curve.

We can parameterize Pythagorean triples (a, b, c) by $m, n \in \mathbb{Z}$ with $(m, n) = 1$ as $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$. Then the equation (1) can be written as $y^2 = x(x-4m^2n^2)(x+(m^2 - n^2)^2)$. We replace x, y by n^2x, n^3y and put $s = m/n$. Then we get an elliptic curve

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2).$$

We consider $E_{1,s}$ as an elliptic curve over a function field $\overline{\mathbb{Q}}(s)$. We associate an elliptic surface $\mathcal{E}_{1,s} \to \mathbb{P}^1$ to $E_{1,s}$.

2 Main Theorem

Proposition 2.1. The Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ satisfies

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s+1)^2, 2s(s+1)^2(s^2+1)),$$
$$T_2 := (2\sqrt{-1}s(s^2-1), 2\sqrt{-1}s(s+\sqrt{-1})^2(s^2-1)).$$

By substituting $s = \frac{2t}{t^2-3}$ into $E_{1,s}$, we get a new family of elliptic curves

$$E_{2,t} : y^2 = x\left(x - 4\left(\frac{2t}{t^2-3}\right)^2\right)\left(x + \left(\left(\frac{2t}{t^2-3}\right)^2 - 1\right)^2\right),$$

which is a subfamily of $E_{1,s}$. The following is our main result.

Theorem 2.2. The Mordell-Weil group of $E_{2,t}$ over $\overline{\mathbb{Q}}(t)$ satisfies

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the rank is 1. The torsion subgroup is generated by T_1 and T_2 in Theorem 2.1 with $s = \frac{2t}{t^2-3}$.

The important point is that we prove that the generic rank of $E_{2,t}$ is exactly 1, not only the existence of a point of infinite order. Our proof is based on the method of Naskręcki in [2].

3 Sketch of the proof

The torsion subgroup is relatively simple, so only rank is discussed here. The proof is divided into two major steps. First, the Mordell-Weil rank is tied to the Picard number, and then the Picard number is evaluated. The following theorem plays a key role in the first step.

Theorem 3.1. (Shioda-Tate formula, [4, Corollary 5.3]) Let $\mathcal{E} \to C$ be an elliptic surface over a smooth projective curve C over an algebraically closed field k. Let $R \subset C$ be the set of points where the special fiber of $\mathcal{E}$ is singular. For each $v \in R$, let m_v be the number of components of the special fiber of $\mathcal{E}$ at v. Let $\rho(\mathcal{E})$ denote the rank of the Néron-Severi group of $\mathcal{E}$, and call it the Picard number. Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R}(m_v - 1) + \operatorname{rank}(E(k(C))).$$

The number of components m_v of each special fiber at v is computed by Tate's algorithm.

Now, we are interested in the upper bound of the Picard number to prove that the rank of $E_{2,t}(\overline{\mathbb{Q}}(t))$ is exactly 1, since we know that there is a point of infinite order $(s^2-1, \sqrt{-1}s(s^2-1)\frac{t^2+3}{t^2-3})$ on $E_{2,t}(\overline{\mathbb{Q}}(t))$. We calculate the characteristic polynomial of the action of the Frobenius automorphism on the second l-adic étale cohomology group using Lefschetz fixed point theorem to get the sharp upper bound of the Picard number.

Theorem 3.2. Let

$$E_{0,u} : y^2 = x(x-4u)(x+(u-1)^2)$$

be an elliptic curve over $\overline{\mathbb{Q}}(u)$. Then, we have

$$\begin{aligned}\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) &= \operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) \\ &\quad + \operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \\ &\quad + \operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)).\end{aligned}$$

We have an estimation that $\rho(\mathcal{E}) \leq \frac{5}{6}e(\mathcal{E})$ by [3] where the Euler number $e(\mathcal{E})$ is calculated by Tate's algorithm, which gives

$$\operatorname{rank} E_{1,s}(\overline{\mathbb{Q}}(s)) = 0,$$
$$\operatorname{rank} E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) = 1.$$

However, the upper bounds computed in the same way for $E_{0,u}^{(1+3u)}$ and $E_{2,t}$ are not sharp. $E_{0,u}^{(1+3u)}$ is a K3 surface, which is relatively well studied, while $E_{2,t}$ is not. Also, it has lower order coefficients in the Weierstrass equation. These properties make the following computation feasible.

Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Assume that the residue field $k = A/\mathfrak{m}$ has $q = p^r$ elements with p prime. Let S be an integral scheme with a morphism $S \to \operatorname{Spec} A$ that is projective and smooth of relative dimension 2. Then the projective surface $\overline{S} = S_{\overline{\mathbb{Q}}}$ and $\tilde{S} = S_{\overline{k}}$ are smooth over the algebraically closed field $\overline{\mathbb{Q}}$ and $\overline{k}$, respectively. For a prime number $l \neq p$, we denote by $H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)$ the l-adic étale cohomology group of X and by $H^2_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)(1)$ its Tate twist. Let $\varphi^{(i)}$ be the Frobenius automorphism acting on $H^i_{\text{ét}}(\tilde{S}, \mathbb{Q}_l)$. Under some assumptions, we have the following theorem.

Theorem 3.3. ([1, Corollary 6.4.]) The ranks of $\operatorname{NS}(\overline{S})$ and $\operatorname{NS}(\tilde{S})$ are bounded from above by the number of eigenvalues λ of $\varphi^{(2)}$ for which λ/q is a root of unity, counted with multiplicity.

We apply Theorem 3.3 with $A = \mathbb{Z}_{(5)}$ and $S = \mathcal{E}_{0,u}^{(1+3u)} \to \mathbb{P}^1$. The characteristic polynomial of $\varphi^{(2)}$ can be calculated if we know the traces of $(\varphi^{(2)})^m$ for $m = 1, 2, 3$. The traces can be calculated by the Lefschetz fixed point theorem:

$$\#\tilde{S}(\mathbb{F}_{q^m}) = \sum_{i=0}^{n}(-1)^i \operatorname{Tr}((\varphi^{(i)})^m).$$

Tate's algorithm gives the types of singular fibers of $\mathcal{E}_{0,u}^{(1+3u)}$ as shown in Table 1, and the number of points on $\tilde{S}(\mathbb{F}_{5^m})$ are calculated as shown in Table 2.

Tab. 1 Singular fibers of $E_{0,u}^{(1+3u)}$

Place	Type	m_v	e
$u=0$	I_2	2	2
$u=\pm 1$	I_4	4	4
$u=-\frac{1}{3}$	I_0^*	5	6
$u=\infty$	I_2^*	7	8

Tab. 2 $\#\tilde{S}(\mathbb{F}_{5^m})$

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264

Using the computation above, we obtain

$$\operatorname{char}(\varphi^{(2)}) = (x-5)^{19}(x^3+x^2+11x-77).$$

By Theorem 3.3, $\rho(\mathcal{E}_{0,u}^{(1+3u)}) \leq 19$. Then by Theorem 3.1, we have $\operatorname{rank} E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 0$, and get $\operatorname{rank} E_{2,t}(\overline{\mathbb{Q}}(t)) = 1$.

References

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[3] B. Naskręcki, *Rangi w rodzinach krzywych eliptycznych i formy modularne*, PhD thesis, Adam Mickiewicz University, 2014.

[4] T. Shioda, On the Mordell-Weil Lattices, Commentarii mathematici universitatis sancti pauli 39.2 (1990), pp. 211–240.