

# On the Mordell-Weil groups of elliptic surfaces associated with Frey curves of degree two

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## 論文要旨

楕円曲線とは、種数 1 の滑らかな射影曲線のことである。楕円曲線上の点に対して加法を定義することができ、これにより楕円曲線上の点全体は、無限遠点を単位元とするアーベル群となる。体  $K$  上に定義された楕円曲線  $E$  に対して、Mordell-Weil 群  $E(K)$  は  $E$  上の  $K$ -有理点全体からなる群である。Mordell-Weil の定理によると、 $K$  に関する条件の下で Mordell-Weil 群は有限生成アーベル群である。Mordell-Weil 群は楕円曲線の研究において重要な対象であり、特にそのランクは一般的に決定が難しい。

本研究では、ピタゴラス数  $(a, b, c) \in \mathbb{Z}^3$ 、すなわち  $a^2 + b^2 = c^2$  を満たす整数  $a, b, c$  に対して、 $y^2 = x(x - a^2)(x + b^2)$  で与えられる楕円曲線の族について考察する。これは Frey 曲線の  $n = 2$  の場合である。ピタゴラス数  $(a, b, c)$  は、整数  $m, n \in \mathbb{Z}$  で  $(m, n) = 1$  を満たすものにより、 $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$  とパラメータ表示される。 $s = m/n$  とおけば、2 次 Frey 曲線の族は

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2)$$

という関数体  $\mathbb{Q}(s)$  上の楕円曲線として扱うことができる。この楕円曲線の  $\mathbb{Q}(s)$  上の Mordell-Weil 群のランクは 0 であることが知られている。本研究ではまずこの Mordell-Weil 群の捩れ部分群を完全に決定する。またこの楕円曲線の無限部分族で、その Mordell-Weil 群のランクが 1 であるものを構成する。

証明には楕円曲面の理論を用いる。一般に関数体上の楕円曲線は楕円曲面に対応し、Shioda-Tate の公式は楕円曲面の Néron-Severi 群のランクと Mordell-Weil 群のランクの関係を与える。

Tate のアルゴリズムにより楕円曲面の特異ファイバーの型を決定し Néron-Severi 群のランクの上界を見積もることで、無限部分族のランクが 2 以下であることを示すことができる。しかしながら、この場合得られる上界は最適ではない。Lefschetz の不動点定理を用いて  $l$ -進エタールコホモロジー に作用する Frobenius 自己同型の特性多項式を計算し、Néron-Severi 群のランクのより良い上界を得ることで、無限部分族のランクがちょうど 1 であることを示す。

## Thesis Abstract

An elliptic curve is a smooth projective curve of genus 1. On points on an elliptic curve we can define an addition law, which makes the set of points on an elliptic curve into an abelian group with the identity element being the point at infinity. For an elliptic curve  $E$  defined over a field  $K$ , the Mordell-Weil group  $E(K)$  is a group consisting of all  $K$ -rational points on  $E$ . The Mordell-Weil theorem states that the Mordell-Weil group is a finitely generated abelian group if  $K$  satisfies some finite conditions. The Mordell-Weil group is an important object in the study of arithmetic of elliptic curves. Especially, the rank of the Mordell-Weil group is important and difficult to determine, in general.

For Pythagorean triples  $(a, b, c) \in \mathbb{Z}^3$ , namely integers satisfies  $a^2 + b^2 = c^2$ , we study a family of elliptic curves  $y^2 = x(x - a^2)(x + b^2)$ . This is the case  $n = 2$  of the Frey curve. Using the parameterization of Pythagorean triples  $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$  by  $m, n \in \mathbb{Z}$  with  $(m, n) = 1$  and putting  $s = m/n$ , we can one-parameterize Frey curves of degree two, and regard the family as an elliptic curve

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2)$$

over a function field  $\mathbb{Q}(s)$ . It is known that the generic rank of the Mordell-Weil group of  $E_{1,s}$  over  $\overline{\mathbb{Q}}(s)$  is 0. We first determine the torsion subgroup of the Mordell-Weil group of  $E_{1,s}$ . Also, we construct an infinite subfamily of  $E_{1,s}$  whose Mordell-Weil rank over  $\overline{\mathbb{Q}}(s)$  is 1.

We use the theory of elliptic surfaces to prove it. Each elliptic curve over a function field corresponds to an elliptic surface. The Shioda-Tate formula gives the relation between the Mordell-Weil rank and the Néron-Severi rank of the corresponding elliptic surface.

We determine the types of special fibers of the elliptic surfaces by Tate's algorithm and estimate an upper bound of the rank of the Néron-Severi group. Using this we can prove that the generic rank of the subfamily is no more than 2. However, the upper bound is not sharp in this case. In order to prove that the generic rank is exactly 1, we calculate the characteristic polynomial of the action of the Frobenius automorphism on the second  $l$ -adic étale cohomology group using Lefschetz fixed point theorem, and get the sharp upper bound of the rank of the Néron-Severi group.

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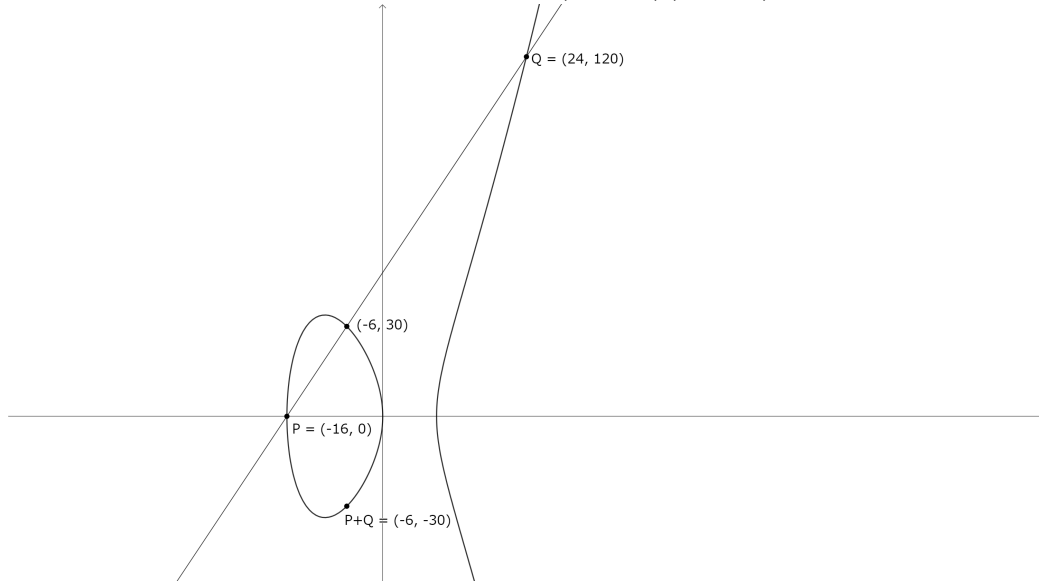
## 1 Introduction

An elliptic curve defined over a field  $K$  of characteristic  $\neq 2$  is a curve defined by a Weierstrass equation

$$E : y^2 = x^3 + Ax^2 + Bx + C$$

where  $A, B, C \in K$  and the discriminant  $\Delta = -4A^3C + A^2B^2 + 18ABC - 4B^3 - 27C^2$  is nonzero. When  $K = \mathbb{Q}$ , we can define an addition law geometrically on points on an elliptic curve defined over  $\mathbb{Q}$  as follows.

Figure 1:  $E : y^2 = x(x - 3^2)(x + 4^2)$



For two points  $P, Q$  on  $E$ , the point  $-(P + Q)$  is defined as the third point of intersection of the line passing through  $P$  and  $Q$  with the curve. The sum  $P + Q$  is the point symmetric to  $-(P + Q)$  with respect to the  $x$ -axis. The definition can be extended to any field  $K$  of characteristic  $\neq 2$ . The set of points on an elliptic curve forms an abelian group with the identity element being the point at infinity  $\mathcal{O}$ . The Mordell-Weil group  $E(K)$  is a group consisting of all  $K$ -rational points on  $E$ .

**Proposition 1.1.** The kernel of multiplication by 2 map  $[2] : E(K) \rightarrow E(K)$  is

$$E(K)[2] = \{\mathcal{O}\} \cup \{(x, y) \in E(K) \mid y = 0\}.$$

**Theorem 1.2.** (Mordell-Weil's Theorem) Let  $E$  be an elliptic curve defined over a number field  $K$ . Then the Mordell-Weil group  $E(K)$  is a finitely generated abelian group.

By the structure theorem of finite abelian groups, the Mordell-Weil group can be decomposed into a free part and a torsion part:

$$E(K) \cong \mathbb{Z}^{\oplus r} \oplus E(K)_{\text{tors}}$$

where  $r$  is the rank of the Mordell-Weil group and  $E(K)_{\text{tors}}$  is the torsion subgroup of  $E(K)$ . The Mordell-Weil group is an important object in the arithmetic of elliptic curves. Especially, the rank of the Mordell-Weil group is important and difficult to determine, in general.

Let  $a, b, c$  be positive integers which satisfy the Fermat's equation

$$a^n + b^n = c^n.$$

for any integer  $n \geq 2$  and consider the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x - a^n)(x + b^n),$$

which is called the Frey curve. The Frey curve played an important role in the proof of Fermat's Last Theorem by Wiles. Wiles proved that the Frey curves do not exist for  $n \geq 3$ , which implies that the Fermat's equation has no nontrivial solution.

In this paper, we consider the case  $n = 2$  of the Frey curves. In other words, let  $(a, b, c) \in \mathbb{Z}_{>0}^3$  be a Pythagorean triple and consider the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x - a^2)(x + b^2), \tag{1.1}$$

which we call the Frey curve of degree 2. The Frey curves of degree 2 do exist infinitely.

We can parameterize Pythagorean triples  $(a, b, c)$  by  $m, n \in \mathbb{Z}$  with  $(m, n) = 1$  as  $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$ . Then the equation (1.1) can be written as  $y^2 = x(x - 4m^2n^2)(x + (m^2 - n^2)^2)$ . We replace  $x, y$  by  $n^2x, n^3y$  and put  $s = m/n$ . Then we get an elliptic curve

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2). \quad (1.2)$$

We regard  $E_{1,s}$  as an elliptic curve over a function field  $\mathbb{Q}(s)$  instead of evaluating coefficients at particular values of  $s$ .

For any elliptic curves  $E$  over a function field  $k(C)$  of a smooth irreducible projective curve  $C$  over an algebraically closed field  $k$ , we can associate an elliptic surface  $\pi : \mathcal{E} \rightarrow C$ . We can use tools in the theory of surfaces by associating an elliptic surface  $\mathcal{E}_{1,s} \rightarrow \mathbb{P}^1$  to  $E_{1,s}$ .

**Definition 1.3.** ([8, III§3 Definition]) Let  $C$  be a non-singular projective curve. An elliptic surface over  $C$  consists of the following data:

- (i) a surface  $\mathcal{E}$ , by which we mean a two dimensional projective variety.
- (ii) a morphism  $\pi : \mathcal{E} \rightarrow C$  such that for all but finitely many points  $s \in C(k)$ , the special fiber  $\mathcal{E}_s := \pi^{-1}(s)$  at  $s$  is a non-singular curve of genus 1.

**Proposition 1.4.** ([8, Proposition III.3.8.]) Let  $E$  be an elliptic curve over a function field  $k(C)$  with the Weierstrass equation

$$E : y^2 = x^3 + a_2(s)x^2 + a_4(s)x + a_6(s), \quad a_2(s), a_4(s), a_6(s) \in k(C).$$

We associate an elliptic surface

$$\mathcal{E} = \{([X, Y, Z], s_0) \in \mathbb{P}^2 \times C \mid Y^2Z = X^3 + a_2(s_0)X^2Z + a_4(s_0)XZ^2 + a_6(s_0)Z^3\}$$

to  $E/k(C)$ , which is a subvariety of  $\mathbb{P}^2 \times C$  of dimension 2.

- (a) For two Weierstrass equations for  $E$ , the corresponding elliptic surfaces are  $k$ -birationally equivalent over  $C$ .
- (b) For an elliptic surface  $\mathcal{E}$  over  $C$  defined over  $k$ , the corresponding elliptic curve  $E$  over  $k(C)$  is uniquely determined up to  $k(C)$ -isomorphism. We say that  $E/k(C)$  is the generic fiber of  $\mathcal{E} \rightarrow C$ .
- (c) There is a natural isomorphism

$$k(\mathcal{E}) \cong k(C)(E)$$

as  $k(C)$ -algebras.

We can define an addition law on  $\mathcal{E}(C/k)$ , which is induced by the addition law on each non-singular fiber, and make it into an abelian group.

**Proposition 1.5.** ([8, Proposition III.3.10. (c)]) Let  $E$  be an elliptic curve over a function field  $k(C)$  and  $\mathcal{E} \rightarrow C$  be the corresponding elliptic surface. Then there is a natural group isomorphism

$$E(k(C)) \cong \mathcal{E}(C/k).$$

We have to associate a Néron model to an elliptic curve over a function field to use the Tate's algorithm mentioned later.

**Definition 1.6.** ([8, IV§5 Definition]) Let  $A$  be a Dedekind domain with fraction field  $K$ , and let  $E/K$  be an elliptic curve. A Néron model for  $E/K$  is a smooth group scheme  $\mathcal{E}/A$  whose generic fiber is  $E/K$  and which satisfies the following universal property: Let  $\mathcal{X}/A$  be a smooth  $A$ -scheme with the generic fiber  $X/K$ , and let  $\phi_K : X \rightarrow E$  be a rational map defined over  $K$ . Then there exists a unique  $A$ -morphism  $\phi_A : \mathcal{X} \rightarrow \mathcal{E}$  extending  $\phi_K$ .

The universal property in Definition 1.6 is called the Néron mapping property. The following two propositions state the existence and the uniqueness of the Néron model.

**Proposition 1.7.** ([8, Proposition IV.5.2. (a)]) Let  $A$  be a Dedekind domain with fraction field  $K$ , and let  $E/K$  be an elliptic curve. Suppose that  $\mathcal{E}_1/A$  and  $\mathcal{E}_2/A$  are Néron models for  $E/K$ . Then there exists a unique  $A$ -isomorphism  $\mathcal{E}_1 \rightarrow \mathcal{E}_2$  whose restriction to the generic fiber is the identity. In other words, the Néron model is unique up to unique isomorphism.

**Proposition 1.8.** ([8, Theorem IV.6.1.]) Let  $A$  be a Dedekind domain with fraction field  $K$ , and let  $E/K$  be an elliptic curve. Let  $\mathcal{C}/A$  be a minimal proper regular model for  $E/K$ , and let  $\mathcal{E}/A$  be the largest subscheme of  $\mathcal{C}/A$  which is smooth over  $A$ . Then  $\mathcal{E}/A$  is a Néron model for  $E/K$ .

The Mordell-Weil's Theorem also holds for elliptic curves over a function field.

**Theorem 1.9.** ([8, Theorem III.6.1.]) Let  $\mathcal{E} \rightarrow C$  be an elliptic surface defined over a field  $k$  and  $E$  be the corresponding elliptic curve over the function field  $k(C)$ . If  $\mathcal{E} \rightarrow C$  does not split, then the Mordell-Weil group  $E(k(C))$  is a finitely generated abelian group.

On the relation between the Mordell-Weil group of an elliptic curve over a function field and its special fibers, the following theorem is known. We can reduce results obtained by treating a family of elliptic curves uniformly as an elliptic surface to results for each elliptic curve.

**Theorem 1.10.** (Specialization Theorem, [8, Theorem III.11.4.]) Let  $\mathcal{E} \rightarrow C$  be a non-split elliptic surface defined over a number field  $k$  and  $E$  be the corresponding elliptic curve over the function field  $k(C)$ . Let  $k' = k$  or  $\bar{k}$  where  $\bar{k}$  is the algebraic closure of  $k$ . Then for all but finitely many  $s \in C(k')$ , a homomorphism

$$E(k(C)) \hookrightarrow \mathcal{E}_s(k'),$$

called the specialization homomorphism at  $s$ , is injective.

**Lemma 1.11.** ([8, Exercise 3.9.(a)]) Let  $\mathcal{E}$  be an elliptic surface over  $k$ , and let  $j_{\mathcal{E}} : C \rightarrow \mathbb{P}^1$  be a morphism such that  $j_{\mathcal{E}}(v)$  gives the  $j$ -invariant for any non-singular fibers  $\mathcal{E}_v$ . If  $\mathcal{E}$  splits over  $k$ , then  $j_{\mathcal{E}}$  is a constant map.

*Remark 1.12.* All elliptic surfaces appearing in this paper have non-constant  $j$ -invariants, therefore, they do not split. Thus we can apply the Specialization Theorem (Theorem 1.10).

## 2 Main theorem

First, for the elliptic curve  $E_{1,s}$  defined in (1.2), we determine the Mordell-Weil group over  $\overline{\mathbb{Q}}(s)$  where  $\overline{\mathbb{Q}}$  is the algebraic closure of  $\mathbb{Q}$ .

**Theorem 2.1.** The Mordell-Weil group of  $E_{1,s}$  over  $\overline{\mathbb{Q}}(s)$  satisfies

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the rank is 0. The torsion subgroup is generated by

$$\begin{aligned} T_1 &:= (2s(s+1)^2, 2s(s+1)^2(s^2+1)), \\ T_2 &:= (2\sqrt{-1}s(s^2-1), 2\sqrt{-1}s(s+\sqrt{-1})^2(s^2-1)). \end{aligned}$$

**Corollary 2.2.** We have an isomorphism

$$E_{1,s}(\mathbb{Q}(s)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

and  $E_{1,s}(\mathbb{Q}(s))$  is generated by  $T_1$  and  $2T_2 = (0, 0)$ .

By Corollary 2.2 and the specialization theorem (Theorem 1.10), we have

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \hookrightarrow E_{1,s_0}(\mathbb{Q})$$

for all but finitely many  $s_0 \in \mathbb{Q}$ . Actually, we can prove that the image of the specialization homomorphism is  $E_{1,s_0}(\mathbb{Q})_{\text{tors}}$  for all  $s_0 \in \mathbb{Q}$  unless  $E_{1,s_0}$  is singular.



**Theorem 2.3.** For any  $s_0 \in \mathbb{Q} \setminus \{0, \pm 1\}$ ,  $E_{1,s_0}$  is non-singular and the torsion subgroup satisfies

$$E_{1,s_0}(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Inspite of the fact that the generic rank of  $E_{1,s}$  is 0, Yoshida ([10, Corollary 4.7.]) showed that there are infinitely many  $s \in \mathbb{Q}$  such that the Mordell-Weil group of  $E_{1,s}$  has positive rank over  $\mathbb{Q}$ . However, the infinite family is not explicitly parameterized. We find the subset of  $s \in \mathbb{Q}$  with positive rank of the Mordell-Weil group of  $E_{1,s}$  parameterized by a rational function of one variable  $t \in \overline{\mathbb{Q}}$ . In order to prove it, by the specialization theorem (Theorem 1.10), it is enough to find a subfamily of  $E_{1,s}$  whose generic rank is 1.

By substituting  $s = \frac{2t}{t^2-3}$  into  $E_{1,s}$ , we get a new family of elliptic curves

$$E_{2,t} : y^2 = x \left( x - 4 \left( \frac{2t}{t^2-3} \right)^2 \right) \left( x + \left( \left( \frac{2t}{t^2-3} \right)^2 - 1 \right)^2 \right),$$

which is a subfamily of  $E_{1,s}$ .

The following is our main result.

**Theorem 2.4.** The Mordell-Weil group of  $E_{2,t}$  over  $\overline{\mathbb{Q}}(t)$  satisfies

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the rank is 1. The torsion subgroup is generated by  $T_1$  and  $T_2$  in Theorem 2.1 with  $s = \frac{2t}{t^2-3}$ .

The important point is that we prove that the generic rank of  $E_{2,t}$  is exactly 1, not only the existence of a point of infinite order. We use Tate's algorithm, the Shioda-Tate formula, and the Lefschetz fixed point theorem, which we will explain in the following sections, to prove Theorem 2.4. Our proof is based on the method of Naskręcki in [4].

*Remark 2.5.* All points in  $E_{2,t}(\overline{\mathbb{Q}}(t))$  are defined over  $\mathbb{Q}(\sqrt{-1})(t)$ . Thus it follows that

$$E_{2,t}(\mathbb{Q}(\sqrt{-1})(t)) = E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the generic rank of  $E_{2,t}$  over  $\mathbb{Q}(\sqrt{-1})(t)$  is 1. However, the points of infinite order are not defined over  $\mathbb{Q}(t)$ . Thus the generic rank of  $E_{2,t}$  over  $\mathbb{Q}(t)$  is 0. If we consider the elliptic curve with Weierstrass equation

$$E_{2,t}^{(-1)} : -y^2 = x \left( x - 4 \left( \frac{2t}{t^2-3} \right)^2 \right) \left( x + \left( \left( \frac{2t}{t^2-3} \right)^2 - 1 \right)^2 \right),$$

then the generic rank over  $\mathbb{Q}(t)$  is 1.

### 3 Preliminaries

In order to get the lower bound of the rank of the Mordell-Weil group, finding points of infinite order is enough. It is quite difficult to get a sharp upper bound of the rank. The proof of the main theorem is divided into two major steps. First, we relate the Mordell-Weil rank to the Picard number, and then the Picard number is evaluated. The following theorem plays a key role in the first step.

**Theorem 3.1.** (Shioda-Tate formula, [7, Corollary 5.3]) Let  $\mathcal{E} \rightarrow C$  be an elliptic surface over a smooth projective curve  $C$  over an algebraically closed field  $k$ . Let  $R \subset C$  be the set of points where the special fiber of  $\mathcal{E}$  is singular. For each  $v \in R$ , let  $m_v$  be the number of components of the special fiber of  $\mathcal{E}$  at  $v$ . Let  $\rho(\mathcal{E})$  denote the rank of the Néron-Severi group of  $\mathcal{E}$ , and call it the Picard number. Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \text{rank}(E(k(C))).$$

We have the following theorem giving an upper bound of the Picard number  $\rho(\mathcal{E})$ . We use this estimate for some elliptic surfaces, but note that this value is not sharp for some surfaces that appear in this paper.

**Theorem 3.2.** ([5, Twierdzenie 2.2.9, 2.2.10, 2.2.19]) Let  $\mathcal{E}$  and  $R$  be as in Theorem 3.1. Let  $\chi(\mathcal{E})$  be the arithmetic genus of  $\mathcal{E}$ ,  $e(\mathcal{E}_v)$  be the local Euler number of the special fiber at  $v$ , and  $g(C)$  be the genus of  $C$ . Assume  $\text{ch } k = 0$ . Then

$$12\chi(\mathcal{E}) = e(\mathcal{E}) := \sum_{v \in R} e(\mathcal{E}_v),$$

$$\rho(\mathcal{E}) \leq 10\chi(\mathcal{E}) + 2g(C).$$

If  $\chi(\mathcal{E}) = 1$ , the elliptic surface  $\mathcal{E}$  is called rational. If  $\chi(\mathcal{E}) = 2$ , the elliptic surface  $\mathcal{E}$  is called an elliptic K3 surface.

We can compute types of each special fibers denoted by Kodaira symbols by Tate's algorithm. The following table shows the correspondence between Kodaira symbols and the values of  $m_v$  and  $e(\mathcal{E}_v)$  appearing in the two theorems above.

Table 1: Kodaira symbols ([6, pp.136-137 付録 2])

Kodaira symbol	$m_v$	$e(\mathcal{E}_v)$
$I_n$	$n$	$n$
II	1	2
III	2	3
IV	3	4
$I_n^*$	$n + 5$	$n + 6$
$II^*$	9	10
$III^*$	8	9
$IV^*$	7	8

**Algorithm 3.3.** (Tate's algorithm, [8, IV §9]) Let  $A$  be a discrete valuation ring with maximal ideal  $\mathfrak{m}$ , uniformizing element  $\pi$ , and fraction field  $K$ . Assume that the residue field  $k = A/\mathfrak{m}$  is perfect and of characteristic  $p$ . Let  $v$  be the normalized valuation on  $K$ . Let  $E/K$  be an elliptic curve given by the Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let  $\mathcal{E}/R$  be a Néron model of  $E/K$ . Let  $\tilde{\mathcal{E}} = \mathcal{E} \times_A k$  be the special fiber of  $\mathcal{E}$ .

Making a change of variables, we may assume that the Weierstrass equation has coefficients  $a_1, a_2, a_3, a_4, a_6 \in A$ . Let

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= a_1a_3 + 2a_4, \\ b_6 &= a_3^2 + 4a_6, \\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \\ \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6. \end{aligned}$$

Step 1. If  $\pi \nmid \Delta$ , then the special fiber  $\tilde{\mathcal{E}}$  is non-singular, that is to say,  $\tilde{\mathcal{E}}$  is an elliptic curve and the type is  $I_0$ .

Step 2. Assume  $\pi \mid \Delta$ . Make a change of variables to move the singular point to the origin. Then  $\pi \mid a_3, a_4, a_6$ . If  $\pi \nmid b_2$ , then the type is  $I_n$  with  $n = v(\Delta)$ . Further, let  $k'$  be the splitting field over  $k$  of the polynomial  $T^2 + a_1T - a_2$ . If  $k' = k$ , then  $E$  has split multiplicative reduction, whereas if  $k' \neq k$ , then  $E$  has non-split multiplicative reduction.

Step 3. Assume now that  $\pi \mid b_2$ . If  $\pi^2 \nmid a_6$ , then the type is II.

Step 4. Assume that  $\pi^2 \mid a_6$ . If  $\pi^3 \nmid b_8$ , then the type is III.

Step 5. Assume that  $\pi^3 \mid b_8$ . If  $\pi^3 \nmid b_6$ , then the type is IV.

Step 6. Assume that  $\pi^3 \mid b_6$ . Then we can change coordinates to get

$$\pi \mid a_1, a_2, \quad \pi^2 \mid a_3, a_4, \quad \pi^3 \mid a_6.$$

We consider the factorization over the algebraically closed field  $\bar{k}$  of the polynomial

$$P(T) = T^3 + \pi^{-1}a_2T^2 + \pi^{-2}a_4T + \pi^{-3}a_6.$$

If  $P(T)$  has distinct roots in  $\bar{k}$ , then the type is  $I_0^*$ .

Step 7. If  $p \neq 2$  and  $P(T)$  has one simple root and one double root in  $\bar{k}$ , then the type is  $I_n^*$  with  $n = v(\Delta) - 6$ .

Although the step continues, we omit the rest of the steps since they are not used in this paper.

For the torsion subgroup of the Mordell-Weil group, we have the following theorems.

**Theorem 3.4.** ([4, Lem.3.5]) Let  $\mathcal{E}$  and  $R$  be as in Theorem 3.1. Let  $E$  be the generic fiber of  $\mathcal{E}$ . Then there is an injective homomorphism

$$E(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow \prod_{v \in R} G(\mathcal{E}_v),$$

where  $G(\mathcal{E}_v)$  is the group generated by all simple components of the fiber at  $v$ . If  $\mathcal{E}_v$  is of type  $I_n$  in Kodaira symbol, then  $G(\mathcal{E}_v) \cong \mathbb{Z}/n\mathbb{Z}$ .

**Theorem 3.5.** (Mazur's Theorem, [9, Theorem VIII.7.5.]) Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$ . Then the torsion subgroup  $E(\mathbb{Q})_{\text{tors}}$  is isomorphic to one of the following groups.

- |   |  |
|---|--|
| (1) $\mathbb{Z}/n\mathbb{Z}$                                | $(1 \leq n \leq 10 \text{ or } n = 12),$ |
| (2) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}$ | $(1 \leq n \leq 4).$                     |

## 4 Proof of Theorem 2.1 and Theorem 2.3

*Proof of Theorem 2.1.* Let  $\mathcal{E}_{1,s} \rightarrow \mathbb{P}^1$  be the elliptic surface with generic fiber  $E_{1,s}$ . The discriminant of  $E_{1,s}$  is

$$\Delta_{E_{1,s}} = 256s^4(s+1)^4(s-1)^4(s^2+1)^4 \quad (4.1)$$

and the  $j$ -invariant is

$$j_{\mathcal{E}_{1,s}} = \frac{16(s^4 - 2s^3 + 2s^2 + 2s + 1)^3(s^4 + 2s^3 + 2s^2 - 2s + 1)^3}{s^4(s+1)^4(s-1)^4(s^2+1)^4}.$$

By Tate's algorithm, we have the following table.

Table 2: Singular fibers of  $E_{1,s}$

Place	Type	$m_v$	$e$
$s = 0$	$I_4$	4	4
$s = \pm 1$	$I_4$	4	4
$s = \pm\sqrt{-1}$	$I_4$	4	4
$s = \infty$	$I_4$	4	4

Then  $e(\mathcal{E}_{1,s}) = 4 \times 6 = 24$  and we have  $\rho(\mathcal{E}_{1,s}) \leq 20$  by Theorem 3.2. By Shioda-Tate formula (Theorem 3.1), we have

$$\text{rank}(E_{1,s}) \leq 20 - (2 + (4 - 1) \times 6) = 0$$

As for the torsion subgroup, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))[2] = \{\mathcal{O}, (0, 0), (4s^2, 0), (-(s^2 - 1)^2, 0)\},$$

and we can check by calculation that

$$2T_1 = (4s^2, 0), \quad (4.2)$$

$$2T_2 = (0, 0). \quad (4.3)$$

By Theorem 3.4, we have  $E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^6$ . Therefore, we have

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

□

*Proof of Theorem 2.3.* The non-singularity of the special fibers of  $\mathcal{E}_{1,s}$  at  $s \in \mathbb{Q} \setminus \{0, \pm 1\}$  follows from the equation (4.1). By the equation Corollary 2.2 and Mazur's Theorem (Theorem 3.5), the only possibility of  $E_{1,s_0}(\mathbb{Q})_{\text{tors}}$  is  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$  or  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$  since  $T_2|_{s=s_0}$  is not defined over  $\mathbb{Q}$  for any  $s_0 \in \mathbb{Q} \setminus \{0, \pm 1\}$ . For a point  $P = (x, y) \in E_{1,s_0}(\mathbb{Q})$ , we can calculate the  $x$ -coordinate of  $2P$  as

$$x(2P) = \frac{16s^4(s^2 - 1)^4 + 8s^2(s^2 - 1)^2x^2 + x^4}{4x(x - 4s^2)(x + (s^2 - 1)^2)}.$$

Assume that there is a rational point of order 8, then there is a point  $P = (x, y) \in E_{1,s_0}(\mathbb{Q})$  such that  $2P = T_1 = (2s(s+1)^2, 2s(s+1)^2(s^2+1))$ . Then we have

$$\frac{16s^4(s^2 - 1)^4 + 8s^2(s^2 - 1)^2x^2 + x^4}{4x(x - 4s^2)(x + (s^2 - 1)^2)} = 2s(s+1)^2.$$

Put  $x' = x - 2s(s+1)^2$ , then we have

$$\begin{aligned} x'^4 &= 8s(s^2 + 1)(s+1)^4(x' + 2s(s^2 + 1))^2, \\ x'^2 &= \pm \sqrt{8s(s^2 + 1)}(s+1)^2(x' + 2s(s^2 + 1)). \end{aligned}$$

Since  $x', s \in \mathbb{Q}$ , we have  $\sqrt{8s(s^2 + 1)} \in \mathbb{Q}$ . Then  $(2s, \sqrt{8s(s^2 + 1)})$  is a rational point on the elliptic curve  $y^2 = x^3 + 4x$ . However, the study of this elliptic curve goes back to Fetmat, and it is well-known that the Mordell-Weil group of  $y^2 = x^3 + 4x$  over  $\mathbb{Q}$  is

$$\{\mathcal{O}, (0, 0), (2, \pm 4)\}.$$

This contradicts the assumption that  $s \in \mathbb{Q} \setminus \{0, \pm 1\}$ . □

## 5 The Generic Rank of $E_{2,t}$

In order to prove Theorem 2.4, Theorem 3.2 is not enough to get the sharp upper bound of the ranks of the Néron-Severi group. Actually, the discriminant of  $E_{2,t}$  is

$$\Delta_{E_{2,t}} = 4096t^4(t-1)^4(t+1)^4(t-3)^4(t+3)^4(t^2-3)^4(t^4-2t^2+9)^4,$$

and the types of the singular fibers of  $E_{2,t}$  are calculated as in Table 3 by Tate's algorithm. Then in a similar way to the proof of Theorem 2.1, we get  $\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \leq 2$ . We will see later that this bound is not sharp.

Table 3: Singular fibers of  $E_{2,t}$

Place	Type	$m_v$	$e$
$t = 0$	$I_4$	4	4
$t = \pm 1$	$I_4$	4	4
$t = \pm 3$	$I_4$	4	4
$t = \pm\sqrt{3}$	$I_4$	4	4
$t^4 - 2t^2 + 9 = 0$	$I_4$	4	4
$t = \infty$	$I_4$	4	4

**Lemma 5.1.**

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

*Proof.* Since  $E_{2,t}$  is given by substituting  $s = \frac{2t}{t^2-3}$  into  $E_{1,s}$ ,

$$E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \subset E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}}.$$

By Theorem 3.4, we have

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})^{12}.$$

Since  $E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \cong (\mathbb{Z}/4\mathbb{Z})^2$  by Theorem 2.1, we have

$$(\mathbb{Z}/4\mathbb{Z})^2 \cong E_{1,s}(\overline{\mathbb{Q}}(s))_{\text{tors}} \subset E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \subset (\mathbb{Z}/4\mathbb{Z})^{12}.$$

There are no more points of order 2 in  $E_{2,t}(\overline{\mathbb{Q}}(t))$  than  $E_{1,s}(\overline{\mathbb{Q}}(s))$ . Therefore, we have

$$E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

□

**Lemma 5.2.**

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \geq 1$$

*Proof.* We have a point of infinite order

$$\left( s^2 - 1, \sqrt{-1}s(s^2 - 1) \frac{t^2 + 3}{t^2 - 3} \right) \in E_{2,t}(\overline{\mathbb{Q}}(t)) \setminus E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}}.$$

□

Now, our goal is to show the rank of  $E_{2,t}(\overline{\mathbb{Q}}(t))$  is  $\leq 1$ . We use another method to estimate an upper bound of the rank of Néron-Severi group, which we will explain in Section 6. Beforehand, we express the rank of  $E_{2,t}(\overline{\mathbb{Q}}(t))$  in terms of ranks of elliptic curves with lower order coefficients in the Weierstrass equations to make the later computation feasible.

**Definition 5.3.** Let  $C$  be a smooth curve over an algebraically closed field  $k$ . Let  $E$  be an elliptic curve over a function field  $k(C)$  given by the Weierstrass equation

$$E : y^2 = x^3 + a_2x^2 + a_4x + a_6$$

where  $a_2, a_4, a_6 \in k(C)$ . For a fixed  $u \in k(C)^*$ , we denote

$$E^{(u)} : uy^2 = x^3 + a_2x^2 + a_4x + a_6$$

to be the quadratic twist of  $E$  by  $u$ .

**Proposition 5.4.** ([9, Exercise 10.16]) Let  $E$  be an elliptic curve over a function field  $k(C)$  and  $u \in k(C)^*$ . Then, the following equation holds

$$\text{rank } E(k(C)(\sqrt{u})) = \text{rank } E(k(C)) + \text{rank } E^{(u)}(k(C)).$$

We omit the proof of Proposition 5.4.

**Theorem 5.5.** Let

$$E_{0,u} : y^2 = x(x - 4u)(x + (u - 1)^2)$$

be an elliptic curve over  $\overline{\mathbb{Q}}(u)$ . Then, we have

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) = \text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) + \text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)), \quad (5.1)$$

$$\text{rank } E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = \text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) + \text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \quad (5.2)$$

Therefore, we have

$$\begin{aligned} \text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) &= \text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) \\ &\quad + \text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \\ &\quad + \text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \end{aligned} \quad (5.3)$$

*Proof.* Since solving  $s = \frac{2t}{t^2-3}$  for  $t$  yields  $t = \frac{1 \pm \sqrt{1+3s^2}}{s}$ , we have

$$E_{2,t}(\overline{\mathbb{Q}}(t)) = E_{1,s}(\overline{\mathbb{Q}}(s)(\sqrt{1+3s^2})).$$

By Proposition 5.4, we get (5.1). Similarly,  $E_{1,s}$  is obtained by substituting  $u = s^2$  into  $E_{0,u}$ , so we have

$$E_{1,s}^{(1+3s^2)}(\overline{\mathbb{Q}}(s)) = E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)(\sqrt{u})),$$

then we get (5.2). □



We already know that the rank of  $E_{1,s}(\overline{\mathbb{Q}}(s))$  is 0. The rank of  $E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u))$  in the equation (5.3) can also be calculated easily as follows.

**Theorem 5.6.**

$$\text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) = 1$$

*Proof.* We have a point of infinite order

$$(u-1, \sqrt{-1}(u-1)) \in E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u))$$

and thus the rank is positive. The discriminant of  $E_{0,u}^{(u(1+3u))}$  is

$$\Delta_{E_{0,u}^{(u(1+3u))}} = 256u^8(u-1)^4(u+1)^4(3u+1)^6,$$

and the types of the singular fibers of  $E_{0,u}^{(u(1+3u))}$  are calculated as in Table 4 by Tate's algorithm.

Table 4: Singular fibers of  $E_{0,u}^{(u(1+3u))}$

Place	Type	$m_v$	$e$
$u = 0$	$I_2^*$	7	8
$u = \pm 1$	$I_4$	4	4
$u = -\frac{1}{3}$	$I_0^*$	5	6
$u = \infty$	$I_2$	2	2

Using Theorem 3.1 and Theorem 3.2, we have

$$e(\mathcal{E}_{0,u}^{(u(1+3u))}) = 8 + 4 \times 2 + 6 + 2 = 24$$

and

$$\begin{aligned} \text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) &\leq 20 - (2 + (7-1) + (4-1) \times 2 + (5-1) + (2-1)) \\ &= 1. \end{aligned}$$

□

The remaining task is to calculate the rank of  $E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u))$ . Theorem 3.2 gives the rank is  $\leq 1$ , which is not sharp. We will show the rank is 0 in the next section.

## 6 Reductions

Let  $A$  be a discrete valuation ring with maximal ideal  $\mathfrak{m}$  and fraction field  $K$ . Assume that the residue field  $k = A/\mathfrak{m}$  has  $q = p^r$  elements with  $p$  prime. Let  $S$  be an integral scheme with a morphism  $S \rightarrow \operatorname{Spec} A$  that is projective and smooth of relative dimension 2. Then the projective surface  $\bar{S} = S_{\bar{\mathbb{Q}}}$  and  $\tilde{S} = S_{\bar{k}}$  are smooth over the algebraically closed field  $\bar{\mathbb{Q}}$  and  $\bar{k}$ , respectively. We will assume that  $\bar{S}$  and  $\tilde{S}$  are integrals, i.e., they are irreducible, nonsingular, projective surfaces.

For a prime number  $l \neq p$ , we denote by  $H_{\text{ét}}^i(\tilde{S}, \mathbb{Q}_l)$  the  $l$ -adic étale cohomology group of  $X$  and by  $H_{\text{ét}}^i(\tilde{S}, \mathbb{Q}_l)(1)$  its Tate twist.

**Theorem 6.1.** ([2, Proposition 6.2.]) There are natural injective homomorphisms

$$\operatorname{NS}(\bar{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow \operatorname{NS}(\tilde{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l(1)) = H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)(1)$$

of finite-dimensional vector spaces over  $\mathbb{Q}_l$ .

Let  $F : S_k \rightarrow S_k$  denote the absolute Frobenius, which acts as the identity on points and by  $f \mapsto f^p$  on the structure sheaf. Set  $\varphi := F^r$  where  $r$  is the integer such that  $q = p^r$ , and let  $\varphi^{(i)}$  denote the automorphism on  $H_{\text{ét}}^i(\tilde{S}, \mathbb{Q}_l)$  induced by  $\varphi \times 1$  acting on  $S_k \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \cong \tilde{S}$ .

**Corollary 6.2.** ([2, Corollary 6.4.]) The ranks of  $\operatorname{NS}(\bar{S})$  and  $\operatorname{NS}(\tilde{S})$  are bounded from above by the number of eigenvalues  $\lambda$  of  $\varphi^{(2)}$  for which  $\lambda/q$  is a root of unity, counted with multiplicity.

*Remark 6.3.* ([2, Remark 6.5.]) Tate's conjecture states that the upper bound mentioned in Corollary 6.2 is actually equal to the rank of  $\operatorname{NS}(\tilde{S})$ . Tate's conjecture was proven for elliptic K3 surfaces by Artin and Swinnerton-Dyer [1].

Now we want to calculate the characteristic polynomial  $\operatorname{char}(\varphi^{(2)})$ . Beforehand, we recall the Lefschetz fixed point theorem.

**Theorem 6.4.** (Lefschetz fixed point theorem)

$$\#\tilde{S}(\mathbb{F}_{q^m}) = \sum_{i=0}^n (-1)^i \operatorname{Tr}((\varphi^{(i)})^m)$$

**Corollary 6.5.**

$$\operatorname{Tr}((\varphi^{(2)})^m) = \#\tilde{S}(\mathbb{F}_{q^m}) - 1 - q^{2m}$$

*Proof.* Since

$$\dim H_{\text{ét}}^1(\tilde{S}, \mathbb{Q}_l) = \dim H_{\text{ét}}^3(\tilde{S}, \mathbb{Q}_l) = 0$$

and  $\varphi^{(4)}$  acts on  $H_{\text{ét}}^4(\tilde{S}, \mathbb{Q}_l) \cong \mathbb{Q}_l(-2)$  by multiplication by  $q^2$ , we get the conclusion from Theorem 6.4.  $\square$

Let  $V$  be the linear subspace of  $H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)$  generated by the components of the singular fibers and by the zero section and  $W = H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)/V$ , then

$$\dim V = \sum_{v \in R} (m_v - 1) + 2. \quad (6.1)$$

By the multiplicativity of the characteristic polynomial, we have

$$\text{char}(\varphi^{(2)}) = \text{char}(\varphi^{(2)}|V) \cdot \text{char}(\varphi_W^{(2)})$$

and

$$\text{Tr}((\varphi^{(2)})^m) = \text{Tr}((\varphi^{(2)}|V)^m) + \text{Tr}((\varphi_W^{(2)})^m) \quad (6.2)$$

for any  $m \in \mathbb{Z}$ , where  $\varphi_W^{(2)} : W \rightarrow W$  is induced by  $\varphi^{(2)}$ . Since  $\varphi^{(2)}$  acts on  $V$  by multiplication by  $q$ , we have

$$\text{char}(\varphi^{(2)}|V) = (x - q)^{\dim V}.$$

As for the characteristic polynomial of  $\varphi_W^{(2)}$ , let  $t_m := \text{Tr}((\varphi_W^{(2)})^m)$ , then  $\text{char}(\varphi_W^{(2)})$  is the polynomial part of

$$\frac{x^{\dim W}}{\exp\left(\sum_{m=1}^{\infty} \frac{t_m}{m} x^{-m}\right)} = x^{\dim W} \left(1 + t_1 x^{-1} + \frac{t_1^2 - t_2^2}{2} x^{-2} + \frac{-t_1^3 + 3t_1 t_2 - 2t_3}{6} x^{-3} + \dots\right). \quad (6.3)$$

Here, by (6.2) and Corollary 6.5, we have

$$t_m = \#\tilde{S}(\mathbb{F}_{q^m}) - 1 - q^{2m} - \dim V \cdot q^m. \quad (6.4)$$

**Lemma 6.6.** ([3, Theorem 4, Part III]) If  $\tilde{S}$  is a elliptic K3 surface, then the second Betti number of  $\tilde{S}$  is 22.

**Theorem 6.7.**

$$\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) = 0$$

*Proof.* We denote by  $S = \mathcal{E}_{0,u}^{(1+3u)} \rightarrow \mathbb{P}^1$  the elliptic surface with generic fiber  $E_{0,u}^{(1+3u)}$ . The discriminant of  $E_{0,u}^{(1+3u)}$  is

$$\Delta_{E_{0,u}^{(1+3u)}} = 256u^2(u-1)^4(u+1)^4(3u+1)^6$$

and the types of the singular fibers of  $E_{0,u}^{(1+3u)}$  are calculated as in Table 5 by Tate's algorithm. Put  $\mathfrak{p} = (5) \in \text{Spec } \mathbb{Z}$  and  $A = \mathbb{Z}_{\mathfrak{p}}$ . The residue field is  $k = A/\mathfrak{p} \cong \mathbb{F}_5$ . Then the elliptic surface  $S$  defines an elliptic surface  $\tilde{S} = S_{\overline{\mathbb{F}_5}} \rightarrow \mathbb{P}^1$ . We can check that  $\tilde{S}$  has exactly the same types of singular fibers as  $S$ .

Table 5: Singular fibers of  $E_{0,u}^{(1+3u)}$

Place	Type	$m_v$	$e$
$u = 0$	$I_2$	2	2
$u = \pm 1$	$I_4$	4	4
$u = -\frac{1}{3}$	$I_0^*$	5	6
$u = \infty$	$I_2^*$	7	8

Then since  $e(\tilde{S}) = 2 + 4 \times 2 + 6 + 8 = 24$ ,  $\tilde{S}$  is a elliptic K3 surface. By Lemma 6.6, we have

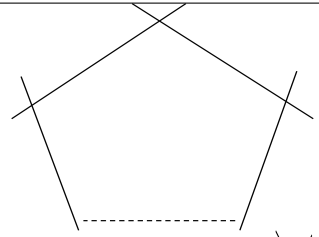
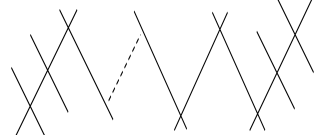
$$\dim_{\mathbb{Q}_l} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l) = 22.$$

The subspace  $V$  is of rank 19 by (6.1), on which the Frobenius automorphism acts by multiplication by 5. Thus

$$\text{char}(\varphi^{(2)}|V) = (x - 5)^{19}.$$

In order to calculate the characteristic polynomial of  $\varphi_W^{(2)}$ , we need to compute the number of points on  $\tilde{S}$  over finite fields  $\mathbb{F}_{5^m}$ . Note that all the multiplicative fibers are split in  $\mathbb{F}_{5^m}$  for  $m = 1, 2, 3$ . We count the number of points on each special fibers. For non-singular fibers, the Schoof-Elkies-Atkin (SEA) algorithm, which calculates the number of points on an elliptic curve over a finite field, is known and can be used. For each singular fiber, the number of points is determined by the type of the fiber. Singular fibers' configurations are as shown in Table 6 where each component is  $\mathbb{P}^1$ . We get the number of points by subtracting from  $m_v \times \#\mathbb{P}^1(\mathbb{F}_{q^m}) = m_v(q^m + 1)$  the number of points that are counted twice or more.

Table 6: Number of points on each singular fiber over a finite field

Type	Configuration	$m_v$	$\mathcal{E}_v(\mathbb{F}_{q^m})$
$I_n$		$n$	$nq^m$
$I_n^*$		$n + 5$	$(n + 5)q^m + 1$

By the equation (6.4), we have

$$t_m = \#\tilde{S}(\mathbb{F}_{5^m}) - 1 - 5^{2m} - 19 \cdot 5^m.$$

These calculations give the number of points on  $\tilde{S}$  over  $\mathbb{F}_{5^m}$  and  $t_m$  for  $m = 1, 2, 3$  as shown in Table 7.

Table 7:  $\#\tilde{S}(\mathbb{F}_{5^m})$  and  $t_m$

$m$	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264
$t_m$	-1	-21	263

By the equation (6.3), we have

$$\text{char}(\varphi_W^{(2)}) = x^3 + x^2 + 11x - 77.$$

If  $\text{char}(\varphi_W^{(2)})$  has a root of the form  $x = 5\zeta$  for some root of unity  $\zeta$ , then  $\zeta$  is a root of the polynomial

$$125x^3 + 25x^2 + 55x - 77,$$

which is irreducible over  $\mathbb{Q}$ . It contradicts the fact that  $\zeta$  is an algebraic integer. By Corollary 6.2,  $\rho(\mathcal{E}_{0,u}^{(1+3u)}) \leq 19$ . Then by Theorem 3.1, we have

$$\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 19 - (2 + (2 - 1) + (4 - 1) \times 2 + (5 - 1) + (7 - 1)) = 0.$$

□

This concludes the proof of the main theorem.

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