

On the Mordell-Weil groups of elliptic surfaces associated with Frey curves of degree two

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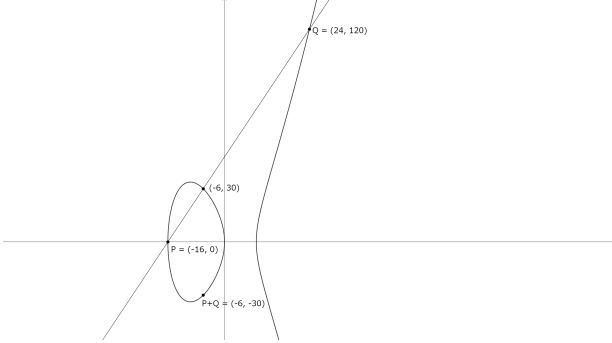
1 Introduction

An elliptic curve defined over a field K is a curve defined by a Weierstrass equation

$$E : y^2 = x^3 + Ax^2 + Bx + C$$

where $A, B, C \in K$ and the discriminant $\Delta = -4A^3C + A^2B^2 + 18ABC - 4B^3 - 27C^2$ is nonzero. On points on an elliptic curve defined over \mathbb{Q} , we can define an addition law geometrically. For two points P, Q on E , the point $-(P + Q)$ is defined as the third point of intersection of the line passing through P and Q with the curve. The sum $P + Q$ is the point symmetric to $-(P + Q)$ with respect to the x -axis.

Fig. 1 $E : y^2 = x(x - 3^2)(x + 4^2)$



The definition can be extended to any field K . The set of points on an elliptic curve forms an abelian group with the identity element being the point at infinity. The Mordell-Weil group $E(K)$ is a group consisting of all K -rational points on E . The Mordell-Weil theorem states that the Mordell-Weil group is a finitely generated abelian group. The Mordell-Weil group is an important object in the study of the arithmetic of elliptic curves. Especially, the rank of the Mordell-Weil group is important and difficult to determine.

In this paper, we consider elliptic curves in the form of the Frey curves for $n = 2$. In other words, let $(a, b, c) \in \mathbb{Z}^3$ be a Pythagorean triple, namely integers satisfies $a^2 +$

$b^2 = c^2$, and consider the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x - a^2)(x + b^2). \quad (1)$$

We can parameterize Pythagorean triples (a, b, c) by $m, n \in \mathbb{Z}$ with $(m, n) = 1$ as $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$. Then the equation (1) can be written as $y^2 = x(x - 4m^2n^2)(x + (m^2 - n^2)^2)$. We replace x, y by n^2x, n^3y and put $s = m/n$. Then we get an elliptic curve

$$E_{1,s} : y^2 = x(x - 4s^2)(x + (s^2 - 1)^2). \quad (2)$$

We consider $E_{1,s}$ as an elliptic curve over a function field $\overline{\mathbb{Q}}(s)$. We associate an elliptic surface $\mathcal{E}_{1,s} \rightarrow \mathbb{P}^1$ to $E_{1,s}$.

2 Main Theorem

Theorem 2.1. The Mordell-Weil group of $E_{1,s}$ over $\overline{\mathbb{Q}}(s)$ satisfies

$$E_{1,s}(\overline{\mathbb{Q}}(s)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the rank is 0. The torsion subgroup is generated by

$$T_1 := (2s(s + 1)^2, 2s(s + 1)^2(s^2 + 1)),$$

$$T_2 := (2\sqrt{-1}s(s^2 - 1), 2\sqrt{-1}s(s + \sqrt{-1})^2(s^2 - 1)).$$

By substituting $s = \frac{2t}{t^2 - 3}$ into $E_{1,s}$, we get a new family of elliptic curves

$$E_{2,t} : y^2 = x \left(x - 4 \left(\frac{2t}{t^2 - 3} \right)^2 \right) \left(x + \left(\left(\frac{2t}{t^2 - 3} \right)^2 - 1 \right)^2 \right),$$

which is a subfamily of $E_{1,s}$. The following is our main result.

Theorem 2.2. The Mordell-Weil group of $E_{2,t}$ over $\overline{\mathbb{Q}}(t)$ satisfies

$$E_{2,t}(\overline{\mathbb{Q}}(t)) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},$$

especially the rank is 1. The torsion subgroup is generated by T_1 and T_2 in Theorem 2.1 with $s = \frac{2t}{t^2 - 3}$.

The important point is that we prove that the generic rank of $E_{2,t}$ is exactly 1, not only the existence of a point of infinite order. Our proof is based on the method of Naskręcki in [1].

3 Proof of Main Theorem

Lemma 3.1.

$$\text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) \geq 1$$

Proof.

$$\left(s^2 - 1, \sqrt{-1}s(s^2 - 1)\frac{t^2 + 3}{t^2 - 3} \right) \in E_{2,t}(\overline{\mathbb{Q}}(t)) \setminus E_{2,t}(\overline{\mathbb{Q}}(t))_{\text{tors}}.$$

□

The following theorem behaves a key role in the proof of the main theorem.

Theorem 3.2. (Shioda-Tate formula, [2, Corollary 5.3]) Let $\mathcal{E} \rightarrow C$ be an elliptic surface over a smooth projective curve C over an algebraically closed field k . Let $R \subset C$ be the set of points where the special fiber of \mathcal{E} is singular. For each $v \in R$, let m_v be the number of components of the special fiber of \mathcal{E} at v . Let $\rho(\mathcal{E})$ denote the rank of the Néron-Severi group of \mathcal{E} . Then, we have

$$\rho(\mathcal{E}) = 2 + \sum_{v \in R} (m_v - 1) + \text{rank}(E(k(C))).$$

Theorem 3.3. Let

$$E_{0,u} : y^2 = x(x - 4u)(x + (u - 1)^2)$$

be an elliptic curve over $\overline{\mathbb{Q}}(u)$. Then, we have

$$\begin{aligned} \text{rank } E_{2,t}(\overline{\mathbb{Q}}(t)) &= \text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) \\ &+ \text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \\ &+ \text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)). \end{aligned} \quad (3)$$

Lemma 3.4.

$$\text{rank } E_{1,s}(\overline{\mathbb{Q}}(s)) = 0,$$

$$\text{rank } E_{0,u}^{(u(1+3u))}(\overline{\mathbb{Q}}(u)) = 1.$$

It is relatively easy to prove Lemma 3.4. We can compute m_v and upper bounds of $\rho(\mathcal{E})$ in Theorem 3.2 by Tate's algorithm. However, the upper bound computed in the same way for $E_{0,u}^{(1+3u)}$ is not sharp. We need a more sophisticated method to estimate the rank of $E_{0,u}^{(1+3u)}$, which we will explain below.

Let A be a discrete valuation ring with maximal ideal \mathfrak{m} and fraction field K . Assume that the residue field $k = A/\mathfrak{m}$ has $q = p^r$ elements with p prime. Let S be an integral scheme with a morphism $S \rightarrow \text{Spec } A$ that is projective and smooth of relative dimension 2. Then the projective surface $\overline{S} = S_{\overline{\mathbb{Q}}}$ and $\tilde{S} = S_{\bar{k}}$ are smooth over the algebraically closed field $\overline{\mathbb{Q}}$ and \bar{k} , respectively. For a prime number $l \neq p$, we denote by $H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)$ the l -adic

étale cohomology group of X and by $H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)(1)$ its Tate twist. Let $\varphi^{(i)}$ be the Frobenius automorphism acting on $H_{\text{ét}}^i(\tilde{S}, \mathbb{Q}_l)$. Under some assumptions, we have the following theorem.

Corollary 3.5. ([3, Corollary 6.4.]) The ranks of $\text{NS}(\overline{S})$ and $\text{NS}(\tilde{S})$ are bounded from above by the number of eigenvalues λ of $\varphi^{(2)}$ for which λ/q is a root of unity, counted with multiplicity.

We apply Corollary 3.5 with $A = \mathbb{Z}_{(5)}$ and $S = \mathcal{E}_{0,u}^{(1+3u)} \rightarrow \mathbb{P}^1$. The characteristic polynomial of $\varphi^{(2)}$ can be calculated if we know the traces of $(\varphi^{(2)})^m$ for $m = 1, 2, 3$. The traces can be calculated by the Lefschetz fixed point theorem:

$$\#\tilde{S}(\mathbb{F}_{q^m}) = \sum_{i=0}^n (-1)^i \text{Tr}((\varphi^{(i)})^m).$$

Tate's algorithm gives the types of singular fibers of $\mathcal{E}_{0,u}^{(1+3u)}$ as shown in Table 1, and the number of points on $\tilde{S}(\mathbb{F}_{5^m})$ are calculated as shown in Table 2.

Tab. 1 Singular fibers of $E_{0,u}^{(1+3u)}$

Place	Type	m_v	e
$u = 0$	I_2	2	2
$u = \pm 1$	I_4	4	4
$u = -\frac{1}{3}$	I_0^*	5	6
$u = \infty$	I_2^*	7	8

Tab. 2 $\#\tilde{S}(\mathbb{F}_{5^m})$ and t_m

m	1	2	3
$\#\tilde{S}(\mathbb{F}_{5^m})$	120	1080	18264

Then, we have

$$\text{char}(\varphi^{(2)}) = (x - 5)^{19}(x^3 + x^2 + 11x - 77).$$

By Corollary 3.5, $\rho(\mathcal{E}_{0,u}^{(1+3u)}) \leq 19$. Then by Theorem 3.2, we have $\text{rank } E_{0,u}^{(1+3u)}(\overline{\mathbb{Q}}(u)) \leq 0$.

References

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- [3] R. van Luijk. "An elliptic K3 surface associated to Heron triangles". In: *Journal of Number Theory* 123.1 (2007), pp. 92–119.