A Study of Driven Harmonic Motion

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Abstract

In this experiment, we investigated the dynamics of driven simple harmonic motion using a torsional pendulum. Firstly, we determined the effective torsional spring constant κ and the natural frequency ω_0 of the pendulum. Then, by adding damping of different degrees to the system we measure the damping constant γ, oscillating frequency ω_γ and the quality factor Q. Lastly, we . Most experimental results have shown good agreement with the theoretical prediction. We are also able to provide explanations when the experimental results have shown discrepancies.

1 Introduction

The motion of the torsional pendulum is governed by the (rotational) Newton's second law

$$I\frac{d^2\theta}{dt^2} = \tau_D(t) - \gamma I\frac{d\theta}{dt} - \kappa\theta, \tag{1}$$

where θ is the angular displacement of the disc, I is the rotaional inertia of the disc, γ is the damping constant and κ is the effective torsional spring constant of the springs. The three terms on the right-hand side of the equation represent the driving torque provided by the motor ($\tau_D(t)$), the damping torque due to the interaction with the magnet ($-\gamma I\dot{\theta}$), and the restoring torque provided by the springs ($-\kappa\theta$), respectively.

Dividing the whole equation by I and defining the natural frequency ω_0 via $\omega_0^2 \equiv \kappa/I$, we can rewrite the equation as

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2\theta = \frac{\tau_D(t)}{I}. \tag{2}$$

The homogeneous part of the differential equation can be solved by substituting the ansatz $\theta_H = e^{i\omega t}$ to find

$$\omega^2 + i\gamma\omega + \omega_0^2 = 0 \implies \omega = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \equiv \frac{i\gamma}{2} \pm \omega_\gamma, \tag{3}$$

where ω_γ is the underdamped freqency, *i.e.,* the frequency at which the system is oscillating if there is no driving force.

If the driving torque is sinusoidal, it can be written in the form of $\tau_D(t) = \tau_0 e^{i\omega t}$. Then the particular solution of $\theta(t)$ can be solved by guessing $\theta_P = Ae^{i\omega t}$, which upon substitution into eq. (2) gives

$$(-\omega^2 + i\gamma\omega + \omega_0^2)A = \frac{\tau_0}{I} \implies A = \frac{\tau_0 \omega_0^2}{\kappa\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} e^{-i\varphi}, \quad \tan\varphi = \frac{\omega\gamma}{\omega_0^2 - \omega^2}. \tag{4}$$

Due to the linearity of the differential equation, the genearl solution is given by the linear combination of all possible homogeneous solution plus the particular solution, which is

$$\theta(t) = e^{-\frac{\gamma}{2}} \left(A\cos(\omega t) + B\sin(\omega_\gamma t)\right) + \frac{\tau_0 \omega_0^2}{\kappa\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \cos(\omega t - \varphi). \tag{5}$$

Conventionally, the quality factor of an oscillating system is defined as the ratio of the natural frequecny to the damping constant, so

$$Q \equiv \frac{\omega_0}{\gamma} \approx \frac{\omega_\gamma}{\gamma}, \tag{6}$$

where the aproximation is based on the premise that the quality factor is sufficiently high, *i.e.,* $Q \gg 1$, since if it is true then from the definition of ω_γ from eq. (3) we can see that

$$\omega_\gamma = \omega_0\sqrt{1 - \frac{1}{4Q^2}} \approx \omega_0. \tag{7}$$

2 Method

3 Results

4 Interpretation

5 References

- Lab Script: GP04.