

Special Relativity

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1.1 Lorentz Transformation

An event is something that an observer can specify with spatial and time coordinates. For example somebody clap their hands, two spaceships crashes onto each other, or more abstractly a specific point at a given time. Something that is not an event is when the observer cannot specify the point in space where or the time when it occurs, such as a person staring at a pebble for one minute (because the time coordinate is not specified), or a train moving by an observer (because the train has spatial extent).

Lorentz transformation attempts to relate the coordinates of two different events¹ observed in two different inertial frames (S and S') which have a relative speed v . That is, we want to find the constants A, B, C, D in the relations²³

$$\begin{aligned}\Delta x &= A\Delta x' + B\Delta t' \\ \Delta t &= C\Delta t' + D\Delta x'.\end{aligned}\tag{1.1}$$

The four independent equations depending on four independent facts, which are

1. S' moves with velocity v with respect to S .
2. S moves with velocity $-v$ with respect to S' .
3. Time dilation (or length contraction) looks the same from either frame.
4. A light pulse with speed c in S' also has speed c in S .

¹Since there is no preferred origin, an event can be described by arbitrary x and t values simply by shifting the origin. Therefore there is no physical significance to compare the coordinates between two frames describing one event (except if the origin is specified but this already constitutes as an event).

²We have assumed A, B, C and D are constants (does not depend on x, t, x' or t'), since all points in space are indistinguishable, the presence of dependency on absolute coordinates would mean that the absolute location in space (and not just the relative position) is important. In short, the absolute coordinate is arbitrarily defined based on the choice of the origin; only the change in coordinates is what we concern.

³We also assumed that Δx and Δt are linear functions of $\Delta x'$ and $\Delta t'$. This can be justified by noting that any interval can be built up from a series of many infinitesimal intervals. But for infinitesimal intervals $\Delta x'$ and $\Delta t'$, any nonlinear terms such as $(\Delta t')^2$ are negligible. So if we add up all the infinitesimal intervals to obtain the given interval, we will be left with only the linear terms.

The first two facts translate into

1. $\Delta x' = 0$ and $v = \Delta x / \Delta t \implies B/C = v$.
2. $\Delta x = 0$ and $-v = \Delta x' / \Delta t' \implies A = C$.

The third fact can be used by asking how fast does a person in S see a clock in S' ticks and the analogous question how fast does a person in S' see a clock in S ticks?

The answer to the first question can be solved by letting $\Delta x' = 0$, which gives $\Delta t = \Delta t'$. The second question can be solved by letting $\Delta x = 0$, which gives $\Delta t' = \Delta t / (A - Dv)$. Comparing the two equations, we have $D = (A - 1/A)/v$.

The fourth fact can be used to say that if $\Delta x' = c\Delta t'$, then $\Delta x = c\Delta t$. Solving for A gives $A = \gamma$.

Gathering all the constants found, we have the Lorentz transformation

$$\begin{aligned}\Delta x &= \gamma(\Delta x' + v\Delta t') \\ \Delta t &= \gamma(\Delta t' + v\Delta x').\end{aligned}\tag{1.2}$$

Solving for $\Delta x'$ and $\Delta t'$, we have the same form as above except v is replaced by $-v$ (as expected, as there is no superior frame).

If two events occur simultaneously in frame S , then $\Delta t = 0$, so $(\Delta x', \Delta t') = (\gamma\Delta x, -\gamma v\Delta x)$. Thus the two events does not occur simultaneously in frame S' and length is contracted.

1.1.1 Time Dilation

To prove that proper time (the time interval in which the events occur at the same place) is dilated we substitute $(\Delta x', \Delta t') = (0, \Delta t')$ into the Lorentz transformation which gives $(\Delta x, \Delta t) = (v\gamma\Delta t', \gamma\Delta t')$.

1.1.2 Length Contraction

To prove that proper length (the length of an object in its rest frame) is contracted we substitute $(\Delta x', \Delta t) = (\Delta x', 0)$ into the Lorentz transformation which gives $(\Delta x, \Delta t') = (\Delta x' / \gamma, -v\Delta x' / c^2)$.

The two relevant events here are that the measurement of the position of the head and the end of the object, which happens simultaneously in the unprimed frame S but not according to S' .

1.1.3 Velocity Addition

Both longitudinal and transverse velocity addition can be easily done with Lorentz transformation, as

$$\begin{aligned}v_x &\equiv \frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + v\Delta x'} = \frac{v'_x + v}{1 + vv'_x} \\ v_y &\equiv \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma(\Delta t' + v\Delta x')} = \frac{v'_y}{\gamma(1 + vv'_x)},\end{aligned}\tag{1.3}$$

where v is the relative speed of the two frames and v' is the velocity of the object observed in the primed frame.

1.1.4 Spacetime Interval

As will be proved in later chapters, the spacetime interval (or invariant interval)

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (1.4)$$

is an invariant (does not depend on which reference frame you are measuring it from).

There are 3 possible signs for $(\Delta s)^2$:

1. $(\Delta s)^2 > 0 \implies |x/t| < c$ (timelike separation): We can always find a frame S' in which the two events happen at the same place, since $\Delta x' = \gamma(\Delta x - v\Delta t)$ can be equals to zero. This implies that it is possible for an object to travel from one event to the other.

In this case, the invariant interval Δs is the proper time τ , since $\Delta x = \Delta y = \Delta z$ in the frame S' where the two events happen at the same place.

2. $(\Delta s)^2 < 0 \implies |x/t| > c$ (spacelike separation): We can always find a frame S' in which the two events happen at the same time, since $\Delta t' = \gamma(\Delta t - v\Delta x)$ can be equals to zero.

Now the invariant interval Δs is no longer proper time, since there is no frame in which the two events happen in the same place.

The invariant interval Δs is now the negative of proper length, since $\Delta t = 0$ in the frame S' where the two events happen at the same time.

3. $(\Delta s)^2 = 0 \implies |x/t| = c$ (lightlike separation): It is not possible to find a frame S' in which the two events happen at the same place or the same time. However, a photon emitted at one of the events will arrive at the other in every frame.

In a spacetime diagram, the third case corresponds to the events happen at the origin and on the light cone. Timelike events are the events at the origin and above or below the light cone. Spacelike separation are the events at the origin and at the left or right of the light cone.

Example: Exceeding the speed of light.

Question: A laser beam is swepted through the space as it shines on the moon, the speed of the spot on the moon can be made arbitrarily large as it is directly proportional to the distance from the moon to the Earth, why does it not violate special relativity?

Solution: Each photon travels independently at the speed of light. The speed of the spot does not bear any physical significance since it is defined based on uncorrelated events (each photon hitting the moon), and thus cannot be used to transfer any information.

1.2 Doppler Effect

1.2.1 Classical Doppler Effect

The classical limit is when the source and observer speeds, v_s and v_o respectively, are much smaller than the speed of light. Note, however, that the speed of the wave v can be close to the speed of light c , *i.e.*, $v_o \ll c$ and $v_s \ll c$.

One-Dimensional

In one-dimensional classical case, the doppler effect formula is trivially given by

$$f_o = \frac{v \pm v_o}{v \pm v_s} f_s, \quad (1.5)$$

where the \pm sign can be deduced from common sense.

Two-Dimensional

In two-dimensional classical case, the doppler effect formula is modified to only take into account only the relevant components of velocities

$$f_s = \frac{v \pm v_o \cos \theta_o}{v \pm v_s \cos \theta_s} f_o, \quad (1.6)$$

where θ_s is the angle between the angle between the sound and the source and θ_o is the angle between the sound and the observer.

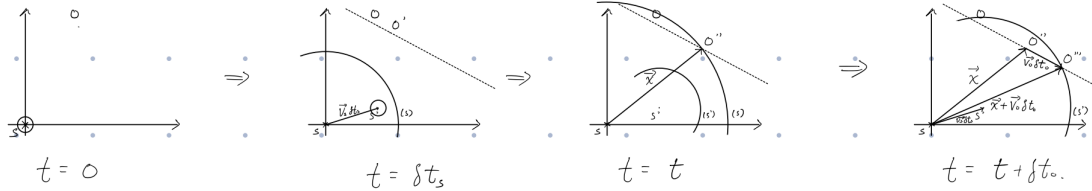


Figure 1.1

From fig. 1.1 we have

$$|\mathbf{x}| = ct \quad \text{and} \quad |\mathbf{x} + \mathbf{v}_o \delta t_o - \mathbf{v}_s \delta t_s|^2 = c^2(t + \delta t_o - \delta t_s)^2, \quad (1.7)$$

which implies

$$\mathbf{x} \cdot \mathbf{v}_o \delta t_o - \mathbf{x} \cdot \mathbf{v}_s \delta t_s = |\mathbf{x}|c(\delta t_o - \delta t_s), \quad (1.8)$$

so we have

$$f_o = \frac{\delta t_s}{\delta t_o} f_s = \frac{v - \mathbf{v}_o \cdot \hat{\mathbf{x}}}{v - \mathbf{v}_s \cdot \hat{\mathbf{x}}} f_s. \quad (1.9)$$

Note that for relativistic case this reduces to

$$f_o = \frac{\delta t_s}{\delta \tau_o} f_s = \frac{\gamma_s \delta t_s}{\gamma_o \delta t_o} f_s. \quad (1.10)$$

1.2.2 Relativistic Doppler Effect

The relativistic limit is that the relative speed of the source and the observer v is comparable to the speed of light c , *i.e.*, $v \sim c$.

One-Dimensional

In one-dimensional relativistic case, the doppler effect formula is given by

$$f_o = \sqrt{\frac{c \pm v}{c \mp v}} f_s, \quad (1.11)$$

where the \pm sign can be deduced from common sense.

The formula is easiest proven in the observer's frame (see fig. 1.2), where from the one-dimensional classical doppler effect formula and time dilation formula we have

$$f_o = \frac{c}{c+v} f'_s, \quad f'_s = \frac{f_s}{\gamma} \implies f_o = \sqrt{\frac{1-\beta}{1+\beta}} f_s. \quad (1.12)$$

where f'_s is the frequency of the source observed by the observer, while f_s is the frequency of the source in the source's rest frame.

One can also proof the formula in the source's frame (see fig. 1.3), using the Lorentz transformation

$$(\Delta x, \Delta t) = \left(\frac{vc}{c-v} \frac{1}{f_s}, \frac{c}{c-v} \frac{1}{f_s} \right) \implies (\Delta x', \Delta t') = \left(0, \sqrt{\frac{1+\beta}{1-\beta}} \frac{1}{f_s} \right). \quad (1.13)$$



Figure 1.2

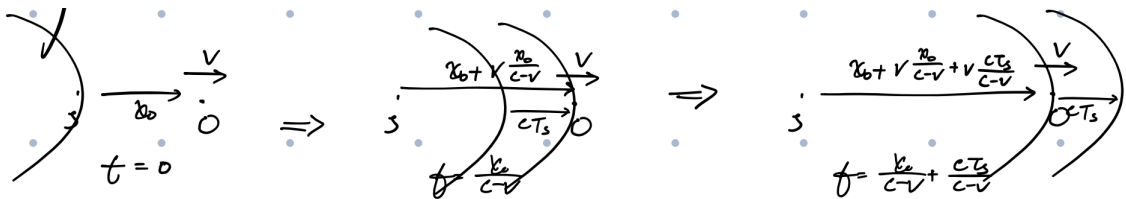


Figure 1.3

We also note that when $v \ll c$ the formula is reduced to

$$f_o \approx \frac{c+v}{c} f_s \approx \frac{v}{c+v} f_s, \quad (1.14)$$

as it should by the correspondence principle.

Also note that the frequencies and period here can be referring to both discrete pulses or that of a single light pulse, since the Doppler shift always compares the spacing of successive emission events to the spacing of successive reception events, whether you call those events “wave-crests” or “discrete flashes”, the kinematics is identical so you get the same formula.

Example: Terrell–Penrose Rotation.

Question: A square with side L flies past you at speed v , in a direction parallel to two of its sides. You stand in the plane of the square. When you see the square at its nearest point to you, show that it looks to you like it is simply rotated, instead of contracted. (Assume that L is small compared to the distance between you and the square.)

Solution: The top view of the square is shown in fig. 1.4a. This is the shape in my frame of reference (where is-ness is defined by where all the points of an object are at simultaneous times).

To find how the square look to me, we notice that the far side of the square have to travel an extra distance L to get to my eye, compared to ones from the near side, so they need an extra time L/c of flight. During this time L/c , the square moves a distance $Lv/c = L\beta$ sideways. Therefore, the trailing side of length L spans a distance $L\beta$ across my field of vision, while the near side spans a distance $L/\gamma = L\sqrt{1-\beta^2}$ across my field of vision. But this is exactly what a rotated square of side L looks like, as shown in fig. 1.4b.

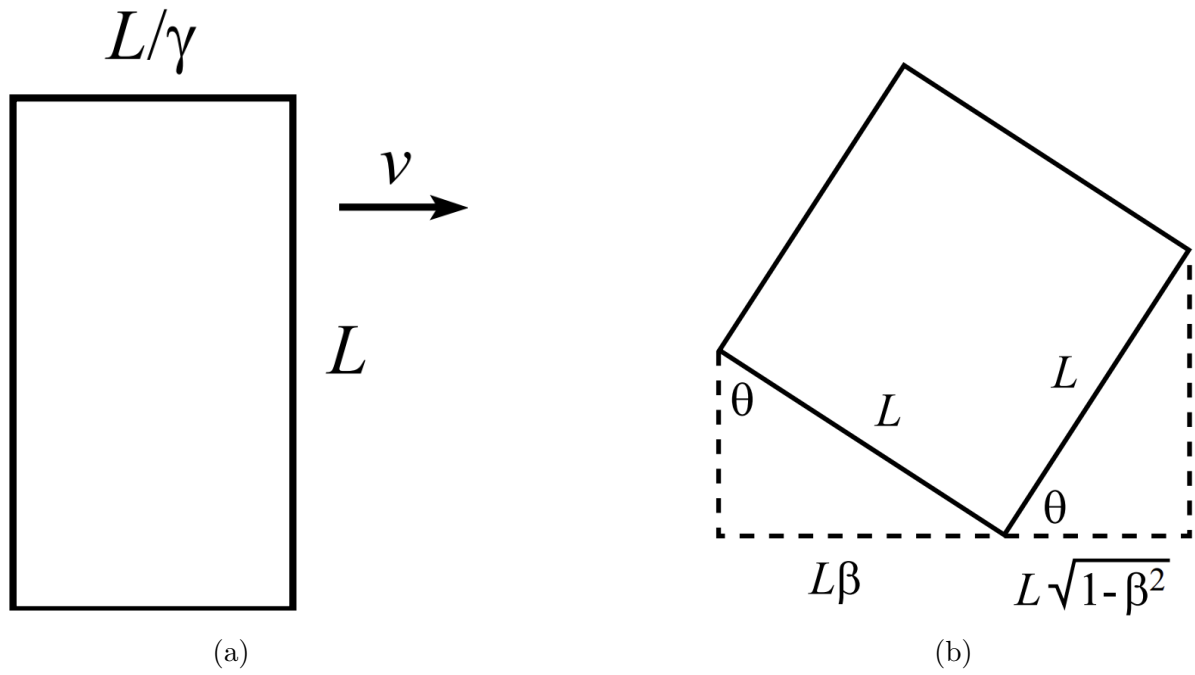


Figure 1.4

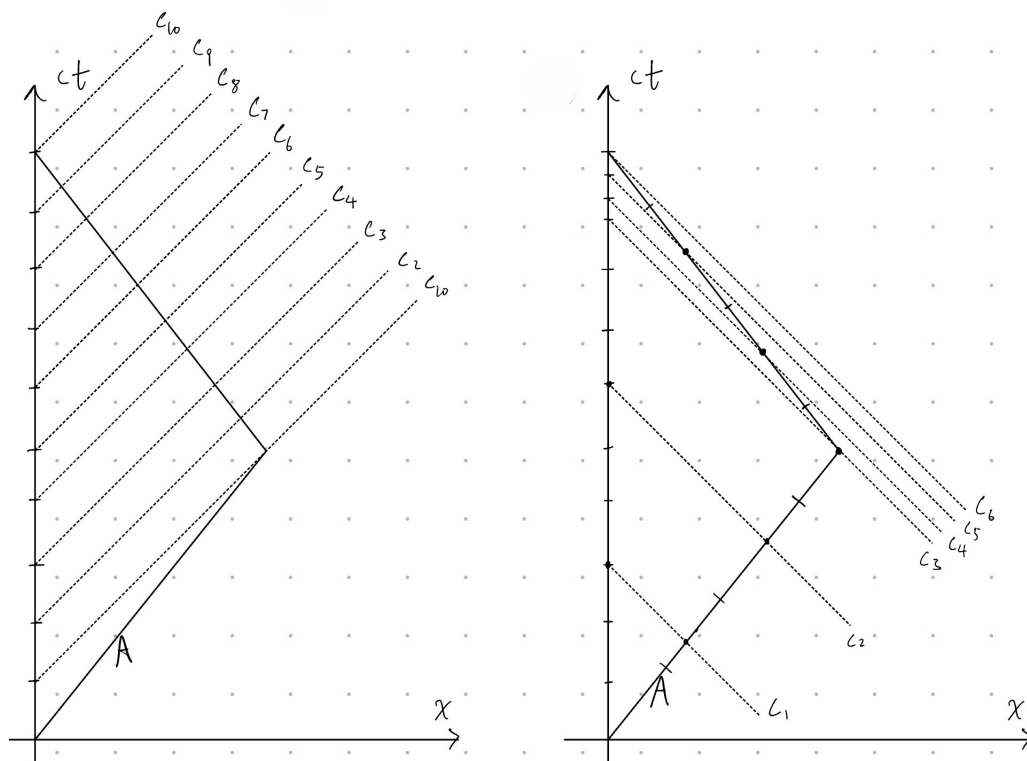


Figure 1.5

Example: Twin Paradox.

Question: Twin A stays on the Earth and twin B flies at speed v to a distant star and back at a distance L from the Earth. A sends out flashes at

intervals of t seconds in its own frame. Find the numbers of red-shifted flashes n_b and blue-shifted flashes n_r that B receives. Hence show that B is younger by a factor γ when B returns.

Solution: Consider the emission time $t_e = kt$ and the reception time t_r of the k^{th} flash. Equating the x -coordinates of the flash and B we have

$$c(t_r - t_e) = c(t_r - kt) = vt_r. \quad (1.15)$$

Requiring this to be less than L/v we get

$$K_{\max} = n_r = \frac{1}{t} \left(\frac{L}{v} - \frac{L}{c} \right). \quad (1.16)$$

A wrong approach is illustrated in fig. 1.6.

To find n_b we simply set

$$c(t_r - kt) = L - vt_r \leq \frac{L}{v} \implies n_b = \frac{1}{t} \left(\frac{L}{v} + \frac{L}{c} \right). \quad (1.17)$$

The time elapsed measured by A is

$$T_A = t(n_r + n_b) = \frac{2L}{v}, \quad (1.18)$$

while the time elapsed measured by B is

$$T_B = t \left(n_r \sqrt{\frac{1-\beta}{1+\beta}} + n_b \sqrt{\frac{1+\beta}{1-\beta}} \right). \quad (1.19)$$

Therefore we get $T_A = \gamma T_B$.

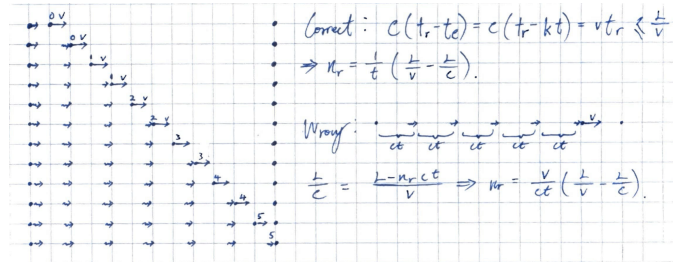


Figure 1.6

Example: Superluminal Motion.

Question: In the accompanying figure, a blob of material is ejected from the centre of a distant galaxy, at speed $v = 0.9c$, at 10° to the line of sight of the observer on Earth. Analyse the situation to show that the projected position of the blob on the plane of the sky appears to move faster than the speed of light.

Solution: The apparent velocity is

$$v_{\text{app}} = \frac{\text{transverse distance}}{\text{arrival-time interval}} = \frac{v \sin \theta t}{(1 - \beta \cos \theta)t} = 1.375c. \quad (1.20)$$

4-Vectors and Dynamics

2.1 Definition and Examples of 4-Vectors

Definition 2.1.1. A 4-tuple, $A = (A_0, A_1, A_2, A_3)$ is called a “4-vector” if the A_i transform between frames in the same way that $(\Delta t, \Delta x, \Delta y, \Delta z)$ do. In other words, A is a 4-vector if it transforms according to the Lorentz transformation

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \gamma \begin{pmatrix} 1 & v & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix}. \quad (2.1)$$

The displacement 4-vector between two events that are infinitesimally close together is

$$dS \equiv (dt, dx, dy, dz) \quad (2.2)$$

Dividing the displacement 4-vector with the proper time $d\tau = \frac{dt}{\gamma}$ (which is an invariant since it is independent of the frame in which it is measured), we obtain

$$V \equiv \frac{dS}{d\tau} = \gamma \left(1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \gamma(1, \mathbf{v}). \quad (2.3)$$

We can then take the derivative of the velocity 4-vector with respect to τ . The result is a four vector because the numerator is just an infinitesimal change of the displacement 4-vector (which is also a 4-vector due to linearity) and the denominator is an invariant. So

$$A \equiv \frac{dV}{d\tau} = \gamma \left(\frac{d\gamma}{dt}, \frac{d(\gamma\mathbf{v})}{dt} \right) = (\gamma^4 v \dot{v}, \gamma^4 v \dot{v} \mathbf{v} + \gamma^2 \mathbf{a}) \quad (2.4)$$

Multiply the velocity 4-vector by another invariant, mass m , we get the energy-momentum 4-vector

$$P \equiv mV = \gamma m(1, v) = (E, \mathbf{p}). \quad (2.5)$$

The force 4-vector is then the derivative of the momentum 4-vector with respect to τ . This gives

$$F \equiv \frac{dP}{d\tau} = \gamma \left(\frac{dE}{dt}, \mathbf{f} \right), \quad (2.6)$$

where $\mathbf{f} \equiv d(\gamma m \mathbf{v})/dt$ is the usual 3-force defined as the rate of change of momentum.

In terms of acceleration 4-vector, we can write

$$F = mA = m(\gamma^4 v \dot{v}, \gamma^4 v \dot{\mathbf{v}} + \gamma^2 \mathbf{a}) = m(\gamma^4 v_x a_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z), \quad (2.7)$$

if $\mathbf{v} = v_x \hat{\mathbf{x}}$. Combining the two forms of F , we see that the 3-force is

$$\mathbf{f} = m(\gamma^3 a_x, \gamma a_y, \gamma a_z). \quad (2.8)$$

2.2 Properties of 4-Vectors

If A and B are 4-vectors, then $C \equiv aA + bB$ is also a 4-vector, this is due to the linearity of Lorentz transformation. For example, the first component of C is

$$C_0 \equiv aA_0 + bB_0 = a(A'_0 + vA'_1) + b(B'_0 + vB'_1) = (aA'_0 + bB'_0) + v(aA'_1 + bB'_1) = C'_0 + vC'_1 \quad (2.9)$$

which is the correct transformation for the first component of C under Lorentz transformation.

The inner product of two 4-vectors A and B are defined as

$$\sum_{\mu} A^{\mu} B_{\mu} \equiv A^{\mu} B_{\mu} \equiv \mathbf{A} \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3, \quad (2.10)$$

where $\mathbf{P}_{\mu} = (\text{diag}(1, -1, -1, -1) \mathbf{p}^{\mu})^T$.

It can be shown (by direct substitution) that the inner product is an invariant. This invariance under Lorentz Transformation (by multiplying the Lorentz matrix) is analogous to the invariance of the scalar (dot) product $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta$ under rotations (by multiplying the rotational matrix) in 3-space (since neither $|\mathbf{A}|, |\mathbf{B}|$ nor θ is affected by rotation).

The norm of a 4-vector is then defined as the square root of the inner product of A with itself:

$$|A|^2 \equiv A \cdot A \equiv A_0 A_0 - A_1 A_1 - A_2 A_2 - A_3 A_3 = A_0^2 - |\mathbf{A}|^2. \quad (2.11)$$

The invariance of the norm $\sqrt{A \cdot A}$ is analogous to the invariance of the norm $\sqrt{\mathbf{A} \cdot \mathbf{A}} \equiv |\mathbf{A}|$ for rotations in 3-space.

If a certain one of the components of a 4-vector is zero in every frame, then all four components are zero in every frame. This is analgous to how if someone comes along and says the she has a vector in 3-space that has no x component, no matter how you rotate the axes, then you would certainly say that the vector must be the zero vector. The situation in Lorentzian 4-sapce is the same, because all of the coordinates get intertwined with each other in the Lorentz (and rotation) transformations.

2.3 Force and Acceleration

In frame S' of a given particle travelling at speed v in the x direction relative to the lab frame S , we have the force 4-vector

$$F' = \gamma\left(\frac{dE'}{dt}, \mathbf{f}'\right) = \gamma\left(\frac{dm}{dt}, \mathbf{f}'\right) = (0, \mathbf{f}') \quad (2.12)$$

Transfomring F' into F , we get

$$\begin{aligned} F_0 &= \gamma \frac{dE}{dt} = \gamma(F'_0 + vF'_1) = \gamma v f'_x, \\ F_1 &= \gamma f_x = \gamma(F'_1 + vF'_0) = \gamma f'_x, \\ F_2 &= \gamma f_y = F'_2 = f'_y, \\ F_3 &= \gamma f_z = F'_3 = f'_z. \end{aligned} \quad (2.13)$$

Thus we get

$$\frac{dE}{dt} = v f'_x, \quad f_x = f'_x, \quad f_y = \frac{f'_y}{\gamma}, \quad f_z = \frac{f'_z}{\gamma}. \quad (2.14)$$

The acceleration 4-vector, on the other hand, is

$$A' = (\gamma^4 v' \dot{v}', \gamma^4 v' \dot{v}' \mathbf{v}' + \gamma^2 \mathbf{a}') = (0, \mathbf{a}'), \quad (2.15)$$

Transforming A' into A , we get

$$\begin{aligned} A_0 &= \gamma^4 v a_x = \gamma(A'_0 + v A'_1) = \gamma v a'_x, \\ A_1 &= \gamma^4 a_x = \gamma(A'_0 + v A'_1) = \gamma a'_x, \\ A_2 &= \gamma^2 a_y = A'_2 = a'_y, \\ A_3 &= \gamma^2 a_z = A'_3 = a'_z. \end{aligned} \quad (2.16)$$

Thus we get

$$a_x = \frac{a'_x}{\gamma^3}, \quad a_y = \frac{a'_y}{\gamma^2}, \quad a_z = \frac{a'_z}{\gamma^2}. \quad (2.17)$$

Example: Acceleration for Circular Motion

Question: A particle moves with constant speed v around the circle $x^2 + y^2 = r^2$, $z = 0$, in the lab frame. At the instant the particle crosses the negative y axis, find the 3-acceleration and 4-acceleration in both the lab frame and the instantaneous inertial frame of the particle.

Solution: The 3-acceleration in S is simply

$$\mathbf{a} = \left(0, \frac{v^2}{r}, 0\right). \quad (2.18)$$

There is nothing fancy going on here; the standard nonrelativistic proof of the centripetal acceleration works just fine again in the relativistic case.

The 4-acceleration in S is then

$$A = \left(0, 0, \gamma^2 \frac{v^2}{r}, 0\right). \quad (2.19)$$

Transforming to S' , we have

$$A' = \left(0, 0, \gamma^2 \frac{v^2}{r}, 0\right). \quad (2.20)$$

So the 3-acceleration in S' is then the space part of A'

$$\mathbf{a}' = \left(0, \gamma^2 \frac{v^2}{r}, 0\right). \quad (2.21)$$

Of course, we can arrive at this result with a simple time-dilation argument, since

$$a'_y = \frac{d^2 y'}{dt'^2} = \frac{d^2 y}{(dt/\gamma)^2} = \gamma^2 a_y = \gamma^2 \frac{v^2}{r}. \quad (2.22)$$

Note that in special relativity, if a statement has any chance of being true in all frames, it must involve only 4-vectors. Consider a 4-vector equation $A = B$ that is true in frame S . Applying Lorentz transformation on both sides, we get $A' = B'$ so the law is therefore also true in frame S' . However, $\mathbf{f} = m\mathbf{a}$ cannot be a physical law, since the two sides of this equation transform differently when going from one frame to another, so the statement cannot be true in all frames.

All of this is exactly analogous to the situation in 3-dimensional space. In Newtonian mechanics, $\mathbf{f} = m\mathbf{a}$ is a possible law (and indeed a physical law), because both sides are 3-vectors. But $\mathbf{f} = m(2a_x, a_y, a_z)$ has no chance of being a physical law, because the right hand side is not a 3-vector; it depends on which axis you label as the x axis.

Physical laws may also take the form of scalar equations, such as $P \cdot P = m^2$. A scalar is by definition a quantity that is frame independent. So if a scalar statement is true in one inertial frame, then it is true in all inertial frames.

2.4 Collisions

From the energy-momentum 4-vector we see that the momentum and energy of a particle governed by special relativity are

$$\mathbf{p} = \gamma m \mathbf{v} \text{ and } E = \gamma m. \quad (2.23)$$

Taylor expanding γ , we get

$$\begin{aligned} \mathbf{p} &= (1 + \mathcal{O}(v^2))m\mathbf{v} = m\mathbf{v} + \mathcal{O}(v^3) \\ E &= (1 + \frac{1}{2}v^2 + \mathcal{O}(v^4))m = m + \frac{1}{2}mv^2 + \mathcal{O}(v^4), \end{aligned} \quad (2.24)$$

which, according to correspondence principle, must give the same answer in Newtonian physics.

Note that the energy of a particle even in Newtonian physics consists of the rest mass energy term m .

In an elastic collision, no kinetic energy is transformed into other forms of energy, so the mass is conserved (To not introduce another degree of freedom, we assume that individual masses are conserved, not just the sum of them, so particles cannot transfer their internal energy to one another), so we can neglect writing down the rest mass energies when applying the law of conservation of energy.

An inelastic collision, on the other hand, converts kinetic energy to internal energy (e.g., thermal energy) which increases the masses of the particles¹, so the mass is not conserved and thus the Newtonian kinetic energy $mv^2/2$ is not conserved.

Similar to the relation $E = p^2/2m$ or $p = \sqrt{2mE}$ in Newtonian physics, we can relate the energy and the momentum of a particle by this “very-important-relationship”

$$E^2 = p^2 + m^2. \quad (2.25)$$

Whenever we know two of the three quantities E, p and m , this equation gives you the third.

There is no need to use 4-vectors to solve collision problems, they can be always be solved solely by writing down the energy and momentum conservation laws, as well as the “very-important-relationship” but they often prove to be useful bookkeeping tools, since the norm of the energy-momentum 4-vector of a particle (note that this does not work for a collection of particles) is simply the mass of the particle:

$$P \cdot P = E^2 - \mathbf{p}^2 = m^2. \quad (2.26)$$

The power of writing energy and momentum of a particle in a single energy-momentum 4-vector can be illustrated by the example below:

¹All internal energy such as thermal, rotational and potential energy contributes to the rest mass of an object. In fact, the vast majority of the mass of an atom is due to the internal energy between quarks that make up the nucleus rather than the rest mass of the quarks themselves.

Example: Decay of a Particle.

Question: A particle with mass M and energy E decays into two identical particles. In the lab frame, one of them is emitted at a 90° angle. What are the energy of the created particles?

Solution: Let P, P_1 and P_2 be the energy-momentum 4-vectors of the decayed particle, the particle emitted at 90° angle, and the remaining particle respectively.

By conservation of energy and momentum,

$$\begin{aligned}
P &= P_1 + P_2 \\
P - P_1 &= P_2 \\
(P - P_1)(P - P_1) &= P_2 \cdot P_2 \\
P^2 - 2P \cdot P_1 + P_1^2 &= P_2^2 \\
M^2 - 2EE_1 + m^2 &= m^2 \\
E_1 &= \frac{M^2}{2E}.
\end{aligned} \tag{2.27}$$

Again, the energy-momentum vector introduce no new physics, it simply write the conservation laws in a way that help simplify the algebra using the “norm squared equals mass” property.

The only non-trivial property of the energy-momentum vector is that since the sum of the energy-momentum 4-vectors of a collection of particles $\sum_{i=1}^n P$ is also a 4-vector due to linearity, the norm squared of this 4-vector

$$\left(\sum P \right)^2 = \left(\sum E \right)^2 - \left(\sum \mathbf{p} \right)^2 \tag{2.28}$$

is also an invariant. Note that the sums are taken before squaring in the above equation. If we square before the addition it would simply give the sum of the squares of the masses (which of course is also an invariant but is trivial why).

Example: Threshold Kinetic Energy.

Question: What is the threshold kinetic energy in lab frame for production of an electron positron pair by collision of an incident electron with mass m_e with a stationary electron?

Solution: Since $\sum \mathbf{p}_{\text{c.m.}} = 0$ and $\sum E_{\text{c.m.}} = \sum \gamma_{\text{c.m.}} m$, the kinetic energy in the center of mass frame is minimized if $\gamma_i = 1$ for all i .

Using the subscript “lab” and “c.m.” to distinguish the lab frame and the center of mass frame; primed and unprimed to distinguish the quantity before or after collision, we have 4 equal quantities:

The norm squared of the sum of the energy-momentum 4-vectors of all the particles

in the center of mass frame after the collision as

$$\left(\sum P'_{\text{c.m.}}\right)^2 = \left(\sum E'_{\text{c.m.}}\right)^2 - \left(\sum \mathbf{p}'_{\text{c.m.}}\right)^2 = \left(\sum m\right)^2 = 4m_e^2, \quad (2.29)$$

which must be equals to the normed squared of the sum of the energy-momentum 4-vectors of all the particles in the lab frame after the collision due to 4-vector property

$$\left(\sum P'_{\text{lab}}\right)^2 = \left(\sum E'_{\text{lab}}\right)^2 - \left(\sum \mathbf{p}'_{\text{lab}}\right)^2 \text{ (not important)}, \quad (2.30)$$

which must be equals to the normed squared of the sum of the energy-momentum 4-vectors of all the particles in the lab frame before the collision due to energy and momentum conservation

$$\left(\sum P_{\text{lab}}\right)^2 = \left(\sum E_{\text{lab}}\right)^2 - \left(\sum \mathbf{p}_{\text{lab}}\right)^2 = (E + m_e)^2 - p^2 = (E + m_e)^2 - E^2 + m_e^2. \quad (2.31)$$

which must be equals to the normed squared of the sum of the energy-momentum 4-vectors of all the particles in the center of mass frame before the collision due to 4-vector property

$$\left(\sum P_{\text{c.m.}}\right)^2 = \left(\sum E_{\text{c.m.}}\right)^2 - \left(\sum \mathbf{p}_{\text{c.m.}}\right)^2 \text{ (not important)}. \quad (2.32)$$

Alternatively, we can use the lorentz transformation of the energy-momentum 4-vectors. Here there is only one γ (or β) is important, since the γ of the particle in the center of mass frame is equals to the γ of the center of mass frame relative to the lab frame, as one of the electron is originally at rest and the masses of the two electrons are equal.

In the center of mass frame before the collision, we have

$$\sum P_{\text{c.m.}} = \begin{pmatrix} 2\gamma m_e \\ 0 \end{pmatrix}. \quad (2.33)$$

Boosting into the lab frame before the collision, we have

$$\sum P_{\text{lab}} = \begin{pmatrix} 2m_e\gamma^2 \\ -2m_e\gamma\beta \end{pmatrix}. \quad (2.34)$$

By conservation of energy and momentum, we have

$$\sum P'_{\text{lab}} = \begin{pmatrix} 2m_e\gamma^2 \\ -2m_e\gamma\beta \end{pmatrix}. \quad (2.35)$$

Boosting into the center of mass frame after the collision, we have

$$\sum P'_{\text{c.m.}} = \begin{pmatrix} 2m_e\gamma \\ 0 \end{pmatrix}. \quad (2.36)$$

However, we also know that in the center of mass frame all the particles are at rest, so

$$\sum P'_{\text{c.m.}} = \begin{pmatrix} 4m_e \\ 0 \end{pmatrix}. \quad (2.37)$$

Therefore we have $\gamma = 2$.

Both methods yields $E = 7m_e \implies K.E. = 6m_e$.

Example: Photon Emission.

Question: Show that a free electron moving in a vacuum cannot emit a single photon.

Solution: An incomplete approach would be to write down the energy and momentum conservation laws in the lab frame, since the the direction of photon emission need not be the same with the direction in which the electron is originally moving. Working in the center of mass frame, we have

$$\begin{pmatrix} m_e \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma m_e \\ \gamma \beta m_e \end{pmatrix} + \begin{pmatrix} E_p \\ E_p \end{pmatrix}. \quad (2.38)$$

The only solution is that $\gamma = 1, E_p = 0, \beta = 0$, which means that there is no emission of photon.

Example: Decay of an Atom.

Question: An atom in an excited state of energy Q_0 (as measured in its rest frame) above the ground state is moving with velocity v as observed from the lab frame. The atom decays to its ground state by emitting a photon of energy Q and comes completely at rest. Find Q if the mass of the atom is M .

Solution: Comparing the norm squared of the energy-momentum 4-vector between the center of mass frame before the decay and the lab frame after the decay, we have

$$\left(\sum P\right)^2 = (Q_0 + M)^2 = (M + Q)^2 - Q^2 \implies Q = Q_0 \left(1 + \frac{Q_0}{2Mc^2}\right). \quad (2.39)$$

Alternatively, we have

$$\begin{pmatrix} Q_0 + M \\ 0 \end{pmatrix}_{\text{c.m.}} \rightarrow \begin{pmatrix} \gamma(Q_0 + M) \\ v\gamma(Q_0 + M) \end{pmatrix}_{\text{lab}} = \begin{pmatrix} Q + M \\ Q \end{pmatrix}'_{\text{lab}}. \quad (2.40)$$

Therefore, we have $\gamma = (Q + M)/(Q_0 + M)$ and $v\gamma = Q/(Q_0 + M)$. Using the identity $\gamma^2 - \beta^2\gamma^2 = 1$, we get the same result $Q = Q_0 (1 + Q_0/2Mc^2)$.

The important point here to note is the the internal energy Q_0 is added to M are transformed together, thus internal energy is equivalent to mass.

Example: Meson-Photoproduction.

Question:

1. Consider a group of N particle, where the i^{th} particle has energy E_i in the laboratory frame. Starting from the Lorentz transformations of energy and momentum show that the Lorentz factor of the center of mass frame, $\gamma_{\text{c.m.}}$, and the speed of the center of mass $v_{\text{c.m.}}$, can be written as

$$\gamma_{\text{c.m.}} = \frac{E_{\text{lab}}}{E_{\text{c.m.}}} \quad \text{and} \quad v_{\text{c.m.}} = \frac{p_{\text{lab}}}{E_{\text{lab}}}. \quad (2.41)$$

where E_{lab} , $E_{\text{c.m.}}$, p_{lab} and $p_{\text{c.m.}}$ are the total energy and momentum observed from the lab frame and the center of mass frame, respectively, respectively.

2. Consider a photon which collides with a proton at rest in the laboratory frame. The collision produces a neutral π meson and a proton via

$$\gamma + p \rightarrow \pi + p. \quad (2.42)$$

Find the threshold photon energy for this reaction to occur.

3. A π meson is produced at threshold via the interaction given in (b), and subsequently decays to two photons. One photon travels in a direction parallel, and the other anti-parallel, to the initial line of light of the π meson in the laboratory frame. Calculate the energy of each photon in the laboratory frame.

Solution:

1. The Lorentz transformation of energy and momentum of the i^{th} particle are

$$\begin{cases} E'_i &= \gamma(E_i - vp_i), \\ p'_i &= \gamma(p_i - vE_i). \end{cases} \quad (2.43)$$

Setting the primed frame S' to be the center of mass frame we get, upon summation

$$\begin{cases} \sum_{i=1}^N E'_i &= E_{\text{c.m.}} = \gamma_{\text{c.m.}} \left(\sum_{i=1}^N E_i - v_{\text{c.m.}} \sum_{i=1}^N p_i \right), \\ \sum_{i=1}^N p'_i &= p_{\text{c.m.}} = 0 = \gamma_{\text{c.m.}} \left(\sum_{i=1}^N p_i - v_{\text{c.m.}} \sum_{i=1}^N E_i \right). \end{cases} \quad (2.44)$$

Eliminating $\sum_{i=1}^N p_i$ we get

$$\gamma_{\text{c.m.}} = \frac{\sum_{i=1}^N E_i}{\sum_{i=1}^N E'_i} = \frac{E_{\text{lab}}}{E_{\text{c.m.}}}. \quad (2.45)$$

the center of mass speed can then be found as

$$v_{\text{c.m.}} = \frac{1}{\sqrt{1 - \gamma_{\text{c.m.}}^{-2}}} = \frac{\sqrt{E_{\text{lab}}^2 - E_{\text{c.m.}}^2}}{E_{\text{lab}}} \frac{p_{\text{lab}}}{E_{\text{lab}}}. \quad (2.46)$$

These two formulae are incidentally very useful results. Note that $v_{\text{c.m.}} = p_{\text{lab}}/E_{\text{lab}}$ is by no means trivial because the standard $v = p/E$ only holds for a

single particle with a well-defined v , but now this formula states that we can regard the collection of particle as a single particle moving at velocity $v_{\text{c.m.}}$.

2. From the energy-momentum invariant we get

$$(m_\pi + m_p)^2 = (E_\gamma + m_p)^2 - E_\gamma^2 \implies E_\gamma = m_\pi \left(1 + \frac{m_\pi}{2m_p}\right). \quad (2.47)$$

3. We first notice that the Lorentz factor of the π meson is the same as the Lorentz factor of the center of mass, since the π meson is stationary in the center of mass frame after the threshold collision, so

$$\gamma_\pi = \gamma_{\text{c.m.}} = \frac{E_{\text{lab}}}{E_{\text{c.m.}}} = \frac{E_\gamma + m_p}{m_\pi + m_p}. \quad (2.48)$$

In the π meson's frame, the momentum and energy of the photons are

$$p_\gamma = \pm \frac{m_\pi}{2} \quad \text{and} \quad E_\gamma = \frac{m_\pi}{2}. \quad (2.49)$$

We can now simply boost to lab frame to obtain the final answer

$$E_{\text{lab}} = \gamma_\pi(E_\gamma + v_\pi p_\gamma) = 77.2 \text{ MeV} \quad \text{or} \quad 59.0 \text{ MeV}. \quad (2.50)$$

Example: Collision (1).

Question: A ball of rest mass M and energy E collides elastically with a stationary ball of rest mass m , such that the direction of motion is along the line joining the centers of the two balls. Find the final energy of M .

Solution: From the conservation of the energy-momentum 4-vectors, we have

$$\left(\frac{E}{\sqrt{E^2 - M^2}}\right) + \left(\frac{m}{0}\right) = P_m + \left(\frac{E'}{\sqrt{E'^2 - M^2}}\right). \quad (2.51)$$

Isolating P_m on one side and squaring both sides,

$$\begin{aligned} 2M^2 + m^2 + 2(mE) - 2(E E' - \sqrt{E^2 - M^2} \sqrt{E'^2 - M^2}) - 2(mE') &= m^2 \\ (M^2 + mE - mE' - EE')^2 + (E^2 - M^2)(E'^2 - M^2) &= 0 \\ m^2(E - E')^2 + 2M^2m(E - E') - 2M^2EE' - 2m(E - E')EE' + M^2E^2 + M^2E'^2 &= 0 \\ m^2(E - E')^2 + 2M^2m(E - E') - 2m(E - E')EE' + M^2(E - E')^2 &= 0 \\ E' &= \frac{2mM^2 + E(m^2 + M^2)}{2Em + m^2 + M^2}, \end{aligned} \quad (2.52)$$

where we have divided the whole equation by $E - E'$, indicating that $E = E'$ is a trivial solution.

Example: Collision (2).

Question: In the jet of a quasar, an ultra-relativistic electron with energy E_1 and p_1 collides head-on with a very low energy photon with energy E_2 of the cosmic microwave background. No new particles are created. Find the energy of the photon after the collision.

Solution: Using the conservation of the energy-momentum 4-vectors, we have

$$\begin{pmatrix} E_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ -E_2 \end{pmatrix} = \begin{pmatrix} E_e \\ p_e \end{pmatrix} + \begin{pmatrix} E \\ E \end{pmatrix}, \quad (2.53)$$

where it is crucial to add the negative sign to the initial photon's energy-momentum 4-vector, otherwise the electron would not collide with the photon. Isolating the electron's energy-momentum 4-vector after the collision and taking the norm squared we get

$$E = \frac{(E_1 + p_1)E_2}{(E_1 - p_1) + 2E_2}. \quad (2.54)$$

Example: Photon Rocket.

Question: Derive the photon rocket equation

$$\frac{m}{M} = \sqrt{\frac{1-v}{1+v}}, \quad (2.55)$$

where M is the initial mass of the rocket (when it is at rest).

Solution: Consider the energy and momentum conservation at time t and $t + dt$

$$\begin{cases} \gamma m &= (\gamma + d\gamma)(m + dm) + dE, \\ \gamma m v &= (\gamma + d\gamma)(m + dm)(v + dv) - dp. \end{cases} \quad (2.56)$$

Dividing one by another gives

$$v + dv = \frac{\gamma m v + dE}{\gamma m - dE} = \frac{\gamma m(1+v) - (\gamma + d\gamma)(m + dm)}{(\gamma + d\gamma)(m + dm)}. \quad (2.57)$$

Expanding the brackets we have

$$v\gamma dm + vmd\gamma + vm\gamma + m\gamma dv + \gamma dm + md\gamma + m\gamma - m\gamma - \gamma m v = (1+v)d(\gamma m) + \gamma m(dv) = 0, \quad (2.58)$$

which can be integrated as

$$\ln\left(\frac{\gamma m}{M}\right) = -\ln(1+v) \implies \frac{m}{M} = \sqrt{\frac{1-v}{1+v}}. \quad (2.59)$$

Alternatively, from the energy and momentum conservations we simply have

$$d(\gamma mv) = -dp = -dE = -d(\gamma m) \implies \gamma mv = M - \gamma m \implies \frac{m}{M} = \sqrt{\frac{1-v}{1+v}}. \quad (2.60)$$

2.5 Force

The Newton's law in special relativity is

$$F = \frac{dp}{dt} = \frac{d(\gamma mv)}{dt} = m(\dot{\gamma}v + \gamma\dot{v}) = m(\gamma^3 v a v + \gamma a) = m a \gamma (\gamma^2 v^2 + 1) = \gamma^3 m a, \quad (2.61)$$

for one-dimensional case while the work energy theorem is

$$\int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} (\gamma^3 m a) dx = m \int_{v_1}^{v_2} \gamma^3 v dv = m \int_{\gamma_1}^{\gamma_2} d\gamma = E_2 - E_1 = \Delta E. \quad (2.62)$$

which is the same as in nonrelativistic physics.

The Newton's law can be easily generalized into two dimensions by

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(\gamma m \mathbf{v})}{dt} = \frac{d}{dt} \left(\frac{m(v_x, v_y)}{\sqrt{1 - v_x^2 - v_y^2}} \right) = m(\gamma^3 a_x, \gamma a_y). \quad (2.63)$$

The asymmetric arises from the fact that $v_x = v$ and $v_y = 0$ at the start by default.

Example: Bead on a Rod

Question: A spring with a tension has one end attached to the end of a rod, and the other end attached to a bead that is constrained to move along the rod. The rod makes an angle θ' with respect to the x' axis and is fixed at rest in the s' frame (see fig. 2.1a). Right after the bead is released, find the directions of the rod, the acceleration of the bead and the force on the bead in frame S .

Solution: The horizontal span of the rod is decreased by a factor γ due to length contraction while the vertical span is unchanged. So we have $\tan \theta = \gamma \tan \theta'$.

The acceleration, on the other hand, must point along the rod, simply because the bead always lies on the rod (and the rod itself does not accelerate).

The force, however, does not point along the rod. This is because the y component of the force on the bead is decreased by a factor of γ while the x component is unchanged. So F_y/F_x is smaller than F'_y/F'_x by a factor γ . So we have $\tan \phi = \tan \theta'/\gamma$.

To double check the direction of acceleration,

$$\frac{a_y}{a_x} = \gamma^2 \frac{F_y}{F_x} = \gamma^2 \frac{F'_y/\gamma}{F'_x} = \gamma \tan \theta' = \tan \theta. \quad (2.64)$$

Note that the rod does not exert a force of constraint. The bead does not need to touch the rod in S' so it does not need to touch it in S . \mathbf{F} simply does not have to be collinear with \mathbf{a} in relativistic physics.

The situation in frame S is shown in fig. 2.1b.

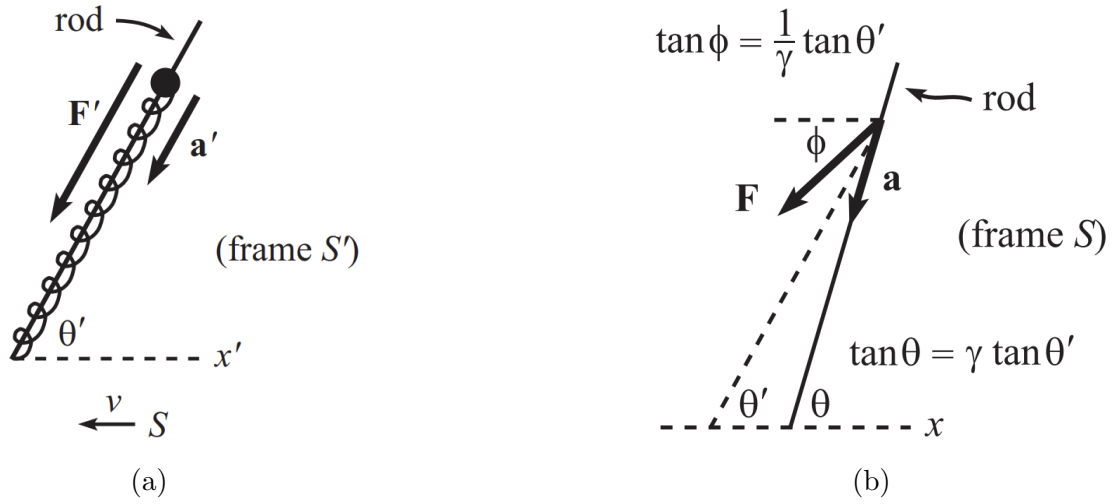


Figure 2.1