Ordinary Differential Equation

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First Order Differential Equations

1.1 Existence and Uniqueness

A first order differential equation is usually written in one of the two equivalent standard forms depending on the method of solving it

$$\frac{dy}{dx} = F(x,y)^1 \quad \text{or} \quad P(x,y)dx + Q(x,y)dy = 0.^2 \tag{1.1}$$

The first from above allows us to essentially solve the differential equation without knowing the relation between x and y. Since the slope dy/dx is given as a function in terms of the position (x,y), we can simply draw a small line on every point whose the slope is $F(x,y)$. What we have created is called the direction field of the ODE and the solutions $y(x)$ are called the integral curves that are at each point tangent to the direction field. As one can see from fig. 1.1 that there are infinite of them, and we can specify any one of them by providing the coordinates of any point. Initial conditions specify the value of the independent variable and its derivatives $(y, y', \ldots, y^{(n-1)})$ at a single $x = x_0$ while boundary conditions specify them at two or more x values on the curve.

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[1]If a differential equation has more than one solution of the first form (perhaps the equation is quadratic in y', so there are two roots for y'), then the differential equation generally has more than one solution. Given the initial condition $(x,y) = (x_0, y_0)$, we also have to specify which integral curve we are following by specifying the slope at that point.

[2]The first form can always be recast into the second form (the symmetric form). However, $Q(x,y)$ may vanishes in some places, thus preventing one from dividing it and converting the symmetric form to the the first form. We shall assume that $|P(x,y)| + |Q(x,y)| \geq 0$, *i.e.,* $P(x,y)$ and $Q(x,y)$ do not vanish simultaneously, such that the symmetric form can be always recast into $dy/dx = F_y(x,y)$, or $dx/dy = F_x(x,y)$.

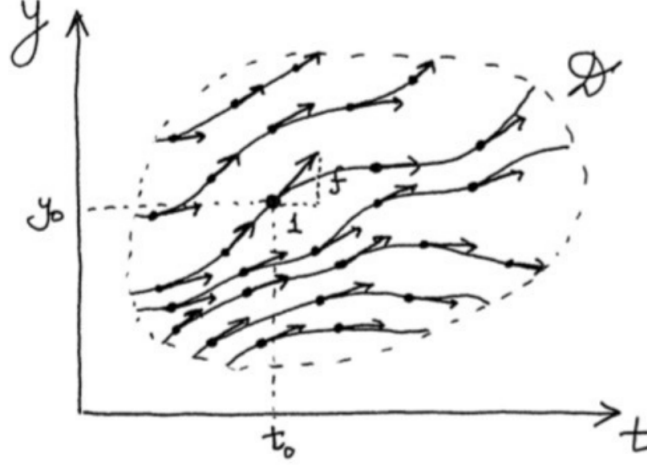singular solution *e.g.,* y'= cos(y+x) imp y= pi/2 - x or sth sim

Figure 1.1

The second form in eq. (1.1), on the other hand, treat the variables x and y on equal foots, as it should, since the assignation of y and x as the dependent and the independent variable respectively is entirely arbitrary (if x and y do not carry any physical meaning). It is crucial to understand that x and y are simply two variables that are related to each other, if $y(x)$ is a function of x then $x(y)$ is a function of y. The realm of possibilites of the integral curves is also slightly richer because now the integral curves can be vertical (see fig. 1.2).[4][5]

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[3]This is only true if both $F(x, y)$ and $\partial F(x, y)/\partial y$ exist at that point. Otherwise the integral curves may intersect at the point and we would not get an unique solution. For example, the differential equation $dy/dx = y^{2/3}$, accompanied with the initial condition $y(0) = 0$ is not unique, as $y(t) = 0$ and $y(t) = (t/3)^3$ are both valid solutions. And it does not guarantees that the integral curve found would not blows up in finite x. For example, the differential equation $dy/dx = ay^2$, accompanied by the initial condition $y(x_0) = x_0$ has the solution $y = 1/a(x_0 + 1/ay_0 - x)$, which blows up at a finite interval at $x = x_0 + 1/ay_0$. Beyond that point, the solution would still gives a solution of y for every x, but it does not make sense anymore. For linear differential equation, however, the existence and uniqueness is global, and the solution is valid at all x.

[4]In fact, higher order differential equations can also be inverted such that x becomes the dependent variable instead of y. However this cannot be done as easily as in first order differential equation (by simply rearranging the terms), and is rarely useful.

[5]A general theorem states that the general solution of all n^{th} order differential equation contains n integration constants. For example, the function $y = x^3/6 + C_1 x + C_2$ which is the solution to the differential equation $y'' = x$ has two integration constants C_1 and C_2 stemming from integration. Another example is the function $y = Ae^t + Be^{-t}$ which is the solution to the differential equation $y'' = y$ has two integration constants A and B although the equation is not necessarily solved by direct integration (but by guessing the solution $y = e^{\alpha t}$) and leveraging the homogenity of the equation. Consequently, there must be n boundary conditions or initial conditions to fix the values of these integration constants.
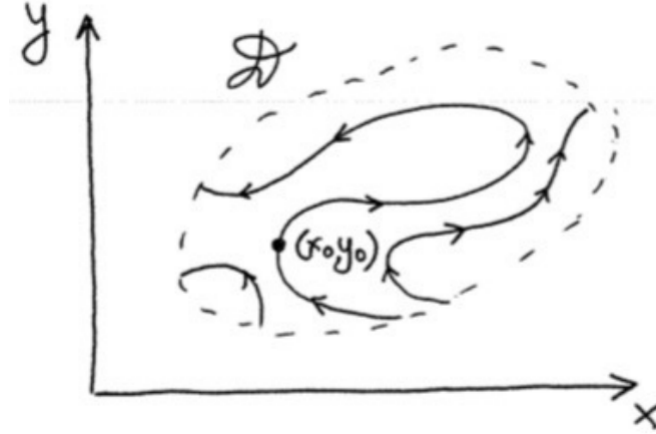
Figure 1.2

Sometimes to be rigourous we say that a differential equation are continuous on a rectangle $[a, b] \times [\alpha, \beta]$, which simply means that x is constrained by $a \le x \le b$ and y by $\alpha \le y \le \beta$.

1.2   First Order Separable

A first order differential equation is separable if it is in the form

$$\frac{dy}{dx} = f(x)g(y), \tag{1.2}$$

which can be solved by direct integration. Note that if $f(x) \to \infty$ or $g(y) \to 0$ for any $x = x_0$ or $y = y_0$ respectively then the constant functions $x = x_0$ or $y = y_0$ might also be solutions to the equation. These are called singular solutions and they can be hard to spot.

ODE in the form of $dy/dx = f(ax + by + c)$ is also separable simply by a change of the variable $z(x) = ax + by + c$.

Example: Singular Solutions.

Question: Solve

$$xy' + y = y^2. \tag{1.3}$$

Solution: Rewriting the equation as

$$\frac{dy}{dx} = \frac{y(y-1)}{x} \implies xdy = y(y-1)dx, \tag{1.4}$$

we can solve by direct integration that $y = 1/(1 + Cx)$.
However, we have to consider the cases when $x = 0, y = 0$ and $y = 1$. For $x = 0$, this is a solution to the symmetric form but not the original equation. On the other

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hand, $y = 0$ and $y = 1$ are both solutions to the differential equation but $y = 1$ is already covered by the genearl solution with $C = 0$. So the only singular solution we get is $y = 0$.

1.3   First Order Homogeneous

A first order differential equation is homogeneous $P(x, y)$ and $Q(x, y)$ in eq. (1.1) are homogenous functions of the same degree of x and y, i.e., the rescaling $x \to \lambda x$ and $y \to \lambda y$ does not affect the solution of the equation for all λ.[6][7]

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{1.5}$$

This reduces to a separable differential equation by introducing a new variable $z = y/x$, so

$$\frac{dz}{dx} = x\frac{dy}{dx} - \frac{y(x)}{x^2} = \frac{f(z) - z}{x} \tag{1.6}$$

which can be solved by direction integration,. An additional solution $z(x) = z_0$ for all x where z_0 satisfies $z_0 = f(z_0)$ is also possible.

The following equation

$$\frac{dy}{dx} = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right) \tag{1.7}$$

is not a homogeneous equation but it can be turned into one by a change of variables $X = x + u$ and $Y = y + v$ with appropriately chosen u and v so that it eliminates the constants c_1 and c_2. This happens if they satisfy

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{1.8}$$

which upon substitution leads to a homogeneous equation

$$\frac{dY}{dX} = f\left(\frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}\right), \tag{1.9}$$

Note that if the square matrix in eq. (1.8) has zero determinant meaning that there are no solutions for u and v, we have $a_2 x + b_2 y = k(a_1 x + b_1 y)$ for all x and y thus $k = a_2/a_1 = b_2/b_1$. In this case the equation is in the form

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[6]A function is said to be homogeneous if when each of the function's arguments is multiplied by the same scalar, then the function's value is multiplied by some power (called the degree of homogeneity) of this scalar. That is, if $f(\lambda x_1, \cdots, \lambda x_n) = \lambda^k f(x_1, \cdots, x_n)$.

[7]In genearl, a differential equation is homogeneous if the rescaling $y \to \lambda y$ does not affect the solutions of the equation for all λ.

$$\frac{dy}{dx} = f\left(\frac{a_1 x + b_1 y + c_1}{k(a_1 x + b_1 y) + c_2}\right) = g(a_1 x + b_1 y), \tag{1.10}$$

which is separable.

1.4 First Order Exact

An arbitrary differential

$$A(x, y)dx + B(x, y)dy \tag{1.11}$$

is exact if it is the differential of a function

$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{1.12}$$

Therefore, we have

$$A(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad B(x, y) = \frac{\partial f}{\partial y} \tag{1.13}$$

Since $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$, we obtain a necessary (and sufficient) condition for the differential to be exact, which is

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.^{8} \tag{1.14}$$

Finding exact differentials in a differential equation is crucial since it combines two differentials into one. The differential equation is then more likely to be integratable.

> **Example: First Order Exact Differential Equation.**
>
> **Question:** Solve
>
> $$y' = \frac{(3x^2 + 2xy + y^2)\sin x - 2(3x + y)\cos x}{2(x + y)\cos x}. \tag{1.15}$$
>
> **Solution:** Rewriting the differential equation as
>
> $$2(x + y)\cos x\, dy + (2(3x + y)\cos x - (3x^2 + 2xy + y^2)\sin x)dx = 0, \tag{1.16}$$
>
> Since

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[8]Determining whether a differential containing many variables $x_1, x_2, \ldots, x_n$ is exact is a simple extension of the above: a differential $df = \sum_{i=1}^{n} g_i(x_1, x_2, \ldots, x_n)dx_i$ is exact if $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$ for all pairs i, j.

$$\frac{\partial}{\partial x}(2(x+y)\cos x) = \frac{\partial}{\partial y}(2(3x+y)\cos x - (3x^2 + 2xy + y^2)\sin x), \qquad (1.17)$$

the two terms is thus an exact differential, where

$$\frac{\partial f}{\partial y} = (2(x+y)\cos x) \quad \text{and} \quad \frac{\partial f}{\partial x} = (2(3x+y)\cos x - (3x^2 + 2xy + y^2)\sin x). \quad (1.18)$$

Integrating the first relation from the above equation,

$$f(x,y) = 2xy\cos x + y^2\cos x + C(x) \implies \frac{\partial f}{\partial x} = 2y(\cos x - x\sin x) - y^2\sin x + \frac{dC}{dx}. \tag{1.19}$$

Equating the the forms of $\partial f/\partial x$ found,

$$6x\cos x - 3x^2\sin x = \frac{\partial C}{\partial x} \implies C(x) = 3x^2\cos x + C'. \qquad (1.20)$$

Since $df(x,y) = 0$, we can integrate and get $f(x,y) = A$, where A is simply a constant (independent of x and y). Therefore,

$$f(x,y) = A = 2xy\cos x + y^2\cos x + 3x^2\cos x + C' \implies y = -x \pm \sqrt{B\sec x - 2x^2}. \tag{1.21}$$

Example: Finding Exact Differentials (1).

Question: Solve

$$y^2 dx - (xy + x^3)dx = 0. \qquad (1.22)$$

Solution: Rewriting the differential equation as

$$y(ydx - xdy) - x^3 dy = 0, \qquad (1.23)$$

we spot that $ydx - xdy$ is an exact differential of $f = y/x$ (multiplied by x^2). Thus

$$-yx^2 d\left(\frac{y}{x}\right) - x^3 dy = 0. \qquad (1.24)$$

The variable y can be eliminated, and instead be replaced by the new variable y/x. The differential equation then reduces to a homogeneous equation

$$yd\left(\frac{y}{x}\right) + xdy = 0 \implies y = -x(x \pm \sqrt{C + x^2}). \qquad (1.25)$$

Example: Finding Exact Differentials (2).

Question: Solve

$$2xydx + (x^2 - y^2)dy = 0. \tag{1.26}$$

Solution:

$$yd(x^2) + x^2dy - y^2dy = d(x^2y) - y^2dy = 0 \implies y^3 - 3x^2y + C = 0. \tag{1.27}$$

Example: Finding Exact Differentials (3).

Question: Solve

$$y(2ydx - xdy) + x^2(ydx + 2xdy) = 0. \tag{1.28}$$

Solution:

$$2\frac{dx}{x} - \frac{dy}{y} + \frac{x^2}{y}\left(\frac{dx}{x} + 2\frac{dy}{y}\right) = 0$$

$$2d(\ln x) - d(\ln y) + \frac{x^2}{y}(d(\ln x) + 2d(\ln y)) = 0$$

$$d\left(\ln\left(\frac{x^2}{y}\right)\right) + \frac{x^2}{y}d(\ln(xy^2)) = 0 \tag{1.29}$$

$$\ln(xy^2) = C - \frac{x^2}{y}.$$

Example: Finding Exact Differentials (4).

Question: Solve

$$(xy + y^4)dx + (x^2 - xy^3)dy = 0. \tag{1.30}$$

Solution:

$$x(ydx + xdy) + y^3(ydx - xdy) = 0$$

$$x(d(xy)) + y^5\left(d\left(\frac{x}{y}\right)\right) = 0 \tag{1.31}$$

Let $u \equiv xy$ and $v \equiv x/y$, then $y^5/x = u^2/v^3$, and the differential equation becomes separarble as

$$v^3 du + u^2 dv = 0. \tag{1.32}$$

Example: Finding Exact Differentials (5).

Question: Solve

$$1 + yy'' + y'^2 = 0. \tag{1.33}$$

Solution:
$$dx + d(yy') = 0 \implies y^2 + (x + C)^2 = C'. \tag{1.34}$$

1.5   Integrating factor

If the differential equation is not exact, we can always make it exact by multiplying the differential by some function $\mu(x, y)$, known as the integrating factor, so that

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}. \tag{1.35}$$

There is no general method for finding the integrating factor if μ is a function of both x and y. However, if μ only depend one of them (*i.e.*, $\mu(x, y) = \mu(x)$ or $\mu(y)$), then it can be found analytically. For example, if $\mu(x, y) = \mu(x)$, then the above equation becomes

$$\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{\partial \mu}{\partial x}, \tag{1.36}$$

which we can solve for $\mu(x)$ as

$$\mu(x) = \exp\left(\frac{1}{B} \int \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx\right). \tag{1.37}$$

Whether the integrating factor $\mu(x, y)$ is dependent of x only can be determined by whether the proposed $\mu(x)$ in the above equation is dependent of x only.

A first order differential equation is linear if it is in the form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1.38}$$

It can be shown using eq. (1.37) that the integrating factor μ always depends on x alone and has the general form

$$\mu(x) = \exp\left(\int P(x)dx\right), \tag{1.39}$$

without needing to add an integration constant.

An alternative way to solve a first order linear differential equation is to treat it as a general linear equation and solve it using the methods detailed in chapter 3.

Example: Exact Differential and Integrating Factor.

Question: Solve

$$(x^2 + y^2 + y)dx - xdy = 0. \tag{1.40}$$

Solution: Rewrite the equation as

$$(x^2 + y^2)dx - x^2 d\left(\frac{y}{x}\right) = 0. \tag{1.41}$$

Spot that the integrating factor is $\mu(x, y) = 1/(x^2 + y^2)$, as if we multiply the equation by $\mu(x, y)$, we get

$$dx - \frac{d\left(\frac{y}{x}\right)}{1 + (\frac{y}{x})^2} = 0 \implies x - \arctan\frac{y}{x} = C. \tag{1.42}$$

1.6 First Order Isobaric

A first order differential equation is isobaric if it is in the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)}, \tag{1.43}$$

where the equation is dimensionally consistent if $y = zx^m$.

Example: Isobaric Differential Equation

Question: Solve

$$\left(y^2 + \frac{2}{x}\right)dx + 2xydy = 0. \tag{1.44}$$

Solution: Giving y and dy weight m and x and dx the weight 1, the sums of powers in each term on the LHS are $2m + 1, 0$ and $2m + 1$ respectively. These are equal if $m = -1/2$. Therefore we substitute $y = zx^{-\frac{1}{2}}$ and get

$$zdz + \frac{dx}{x} = 0 \implies \frac{1}{2}y^2x + \ln x = C. \tag{1.45}$$

1.7 Bernoulli's Equation

A first order differential equation is a Bernoulli's equation if it has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad y \neq 1. \tag{1.46}$$

This equation can be solved by the substitution

$$z(x) = y^{1-n} \implies \frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dz}{dx}, \tag{1.47}$$

which makes the equation linear, as

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).^9 \tag{1.48}$$

1.8 Riccati's Equation

A first order differential is a Riccati's equation if it has the form

$$\frac{dy}{dx} + P(x)y = Q(x) + R(x)y^2. \tag{1.49}$$

If one can guess one of the solutions $y_0(x)$, then the substitution

$$z(x) = y(x) - y_0(x) \tag{1.50}$$

reduces the equation to a Bernoulli's equation, as

$$\frac{dz}{dx} = (2R(x)y_0(x) + P(x))z + R(x)z^2. \tag{1.51}$$

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[9]Alternatively, one can first find the homogenous solution $y_H(x)$ by solving $y' + P(x)y = 0$ and use the method of variations of parameters to guess the particular solution as $y_P(x) = \psi(x)y_H(x)$ and solve for $\psi(x)$. This only works for the Bernoulli's equation where the inhomogenous term is in the form $Q(x)y^n$ but does not work for a general function $F(x, y)$ (except, of course if $F(x, y) = F(x)$ does not depend on y). This method has no rationale other than mere coincidence, so it is put in the footnotes.

1.9 Introduction of Parameter

If we cannot (and do not want to) change dy/dx to be the subject of the equation, but can change y to be the subject, then the substitution $p = y'$ implies that

$$p\,dx = dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial p}dp \implies \left(\frac{\partial f}{\partial x} - p\right)dx + \frac{\partial f}{\partial p}dp = 0. \tag{1.52}$$

Integrating gives the solution in the form $p = p(x)$, which can be directly substituted into the equation where y is the subject to get the solution.

Example: Introduction of Parameter (1).

Question: Solve

$$y = xy' - y'^2. \tag{1.53}$$

Solution: Introducing the parameter $p = y'$, we have

$$p\,dx = dy = x\,dp + p\,dx - 2p\,dp \implies (x - 2p)dp = 0. \tag{1.54}$$

The solutions are

$$p = \frac{x}{2} \implies y = \frac{x^2}{4} \quad \text{or} \quad dp = 0 \implies p = C \implies y = C(x - C). \tag{1.55}$$

Example: Introduction of Parameter (2).

Question: Solve

$$xy'^2 - 2yy' - x = 0. \tag{1.56}$$

Solution: Introducing the parameter $p = y'$, we have

$$y = \frac{xp^2 - x}{2p} \implies \frac{dy}{dx} = p = \frac{(2pxp' + p^2 - 1)p - p'(xp^2 - x)}{2p^2} \implies p' = \frac{p}{x}. \tag{1.57}$$

Integrating we get

$$p = y' = Cx \implies y = Cx^2 + D. \tag{1.58}$$

Subsitute this back into the original equation we get $D = -C/4$, since there should only be 1 integration constant from a first order differential equation. Two integral curves pass through any given point (x, y), since the differential equation is quadratic in y', so at every point there are two possible slopes.

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onwards

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Autonomous Second Order Differential Equations

An autonomous system is described by a differential equation that does not explicitly depend on the independent variable. Newton's second law is an example if the force depends on time only through instantenous position and velocity. The general form of a differential equation describing such system is in the form of

$$\ddot{r} = f(r, \dot{r}), \tag{2.1}$$

which can be solved by noting $\ddot{r} = \dot{r}(d\dot{r}/dr)$, which would give a first order differential equation upon substituion:

$$\frac{d\dot{r}}{dr} = \frac{f(r, \dot{r})}{\dot{r}}. \tag{2.2}$$

The $(r, \dot{r})$ space is referred to as the phase space, whose evolution is characterised by a one-parameter family of curves. For a given initial condition, the evolution is restricted to a single curve in phase space. Note that a second order differential equation would generally give a two-paramter family of curves, so the phase space of (r,t) would be very messy as the curves inevitably intersect.

2.1 Conservative Systems

Conservative systems have a even more strict requirement on $f(r, v)$, where $v \equiv \dot{r}$: It requires that $f(r, v)$ depends explicitly only on r but not on v, so $f(r, v) = f(r)$. In this case the differential equation is separable, as

$$v\frac{dv}{dr} = f(r) \implies \frac{1}{2}v^2 + V(r) = E \tag{2.3}$$

Here the integration constant E corresponds to the total energy of the system and $V(r) = -\int f(r)dr$ is the potential of the force.

Solving further for $v = dr/dt$,

$$\frac{dr}{dt} = \pm\sqrt{2(E - V(r))} \implies \int \frac{dr}{\sqrt{2(E - V(r))}} = \pm \int dt = \pm t + C. \qquad (2.4)$$

Example: Phase Space of a Simple Harmonic Oscillator

Question: Plot the integral curves in phase space and explain the motion in phase space for a simple harmonic oscillator.

Solution: For a simple harmonic oscillator,

$$\ddot{r} = f(r) = -kr \implies \frac{1}{2}v^2 + \frac{1}{2}kr^2 = E. \qquad (2.5)$$

This equation describes an ellipse in phase space centered on $(r, v) = (0, 0)$, with axes of length $\sqrt{2E}$ and $\sqrt{\frac{2E}{k}}$ as shown in fig. 2.1:
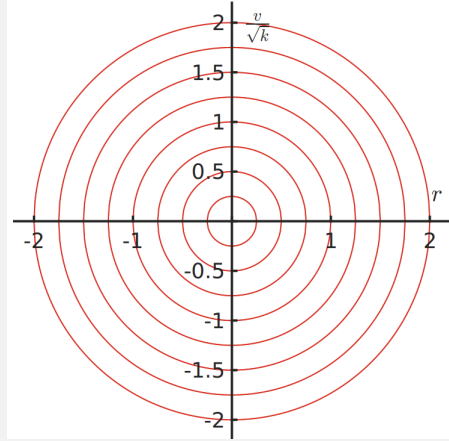
Figure 2.1

The potential $V(r) = \frac{1}{2}kr^2$ has a minimum at $r = 0$ and all curves in phase space oscillate around this point perpetually.

Example: Phase Space of a Pendulum

Question: Plot the integral curves in phase space and explain the motion in phase space for a pendulum.

Solution: For a pendulum we have

$$\ddot{\theta} = -k\sin\theta \implies \frac{1}{2}\omega^2 - k\cos\theta = E \implies \omega(\theta) = \pm\sqrt{2(E + k\cos\theta)}. \qquad (2.6)$$

The integral curves are shown in fig. 2.2 below.

Figure 2.2

The potential $V(\theta) = -k\cos\theta$ has a minimum at $\theta = 2n\pi$ corresponding to the lowest point and a maximum at $\theta = (2n+1)\pi$ corresponding to the highest point, where n is an integer.

At $E = -k$, the term inside the square root is only real when $\cos\theta = 1 \implies \theta = 0$, so the pendulum is at rest at the potential minimum, a stable fixed point.

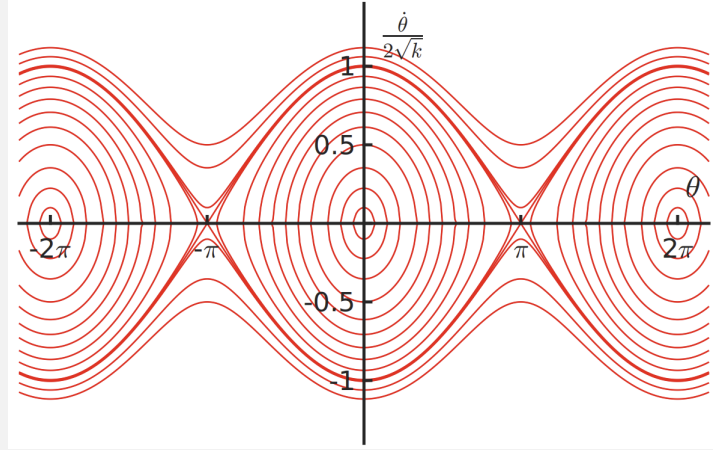For $E \gtrsim -k$, we have $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ and we would get $\frac{1}{2}\omega^2 + \frac{1}{2}k\theta^2 = E$ which is a simple harmonic oscillator as described above.

If $-k < E < k$, the pendulum goes back and forth around $(\theta, \omega) = (0,0)$ but the shape of the curve is not an ellipse.

When $E = k$, the pendulum is fixed at $(\theta, \omega) = (\pi, 0)$ whihc is an unstable fixed point.

For $E > k$, the pendulum goes around perpetually respectively either with $\omega > 0$ or $\omega < 0$ throughout the evolution, since the pendulum has enough energy to overcome the potential barrier at the highest point.

2.2 Nonconservative Systems

In this section we lessen the restriction on $f(r,v)$ and consider the case where $f(r,v)$ depends on both position and velocity.

In some cases, the second order differential equation can still be reduced to an analytically solvable first order differential equation (though it may not be separable), such as the example below:

Example: Phase Space of a Damped Harmonic Oscillator (1)

Question: Plot the integral curves in phase space and explain the motion in phase space for the damped harmonic oscillator.

Solution: For a damped simple harmonic oscillator,

$$\ddot{r} = f(r, v) = -kr - \gamma v \implies \frac{dv}{dr} = -\gamma - k\frac{r}{v}. \tag{2.7}$$

This is no longer a separable differential equation but a homogeneous one, which can be solved by substituting $z = \frac{v}{r}$ (not $\frac{r}{v}$). Solving, we get the implicit solution

$$\frac{1}{2}\ln\left|u^2 + D\right| - \frac{\gamma}{2\sqrt{D}}\arctan\frac{u}{\sqrt{D}} = -\ln|r| + C, \tag{2.8}$$

where $u = z + \frac{1}{2}\gamma$ and $D = k - \frac{1}{4}\gamma^2$. A parametric solution $(v(u), r(u))$ may be derived directly from this equation by solving for r, which gives

$$r(u) = \pm\frac{C_2}{\sqrt{r^2 + D}}e^{\frac{\gamma}{2\sqrt{D}}\arctan\frac{u}{\sqrt{D}}} \text{ and } v(u) = z(u)r(u) = (u - \frac{1}{2}\gamma)r(u). \tag{2.9}$$

which is plotted in fig. 2.3.
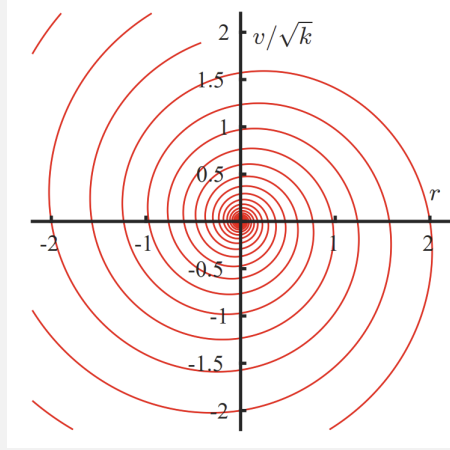
Figure 2.3

For a more general case, where the first order differential equation is not analytically solvable, we employ the pertubation method to obtain qualitative understanding of the system.

We start with the more general equation

$$\ddot{r} = f(r) - \gamma v \implies \frac{dv}{dr} = \frac{f(r)}{v} - \gamma \implies \frac{d}{dr}(\frac{1}{2}v^2 + V(r)) = -\gamma v. \tag{2.10}$$

If γ is sufficiently small, we may obtain the solution perturbatively around a known solution as

$$v(r) = v_0(r) + \delta v(r), \tag{2.11}$$

where v_0 is defined to be the solution to the same equation without the damping term, so

$$\frac{dv}{dr} = \frac{f(r)}{v} \implies \frac{1}{2}v_0 + V(r) = E \implies v_0(r) = \pm\sqrt{2(E - V(r))}. \tag{2.12}$$

Substituting $v = v_0 + \delta v$, we have

$$\frac{d}{dr}(v_0 \delta v) = -\gamma(v_0 + \delta v), \tag{2.13}$$

which is a first order linear differential equation which can be solved with the techniques introduced in the last chapter.

$$v_0' \delta v + v_0 \delta v' = -\gamma v_0 - \gamma \delta v \implies \delta v' + \left(\frac{v_0'}{v_0} + \gamma\right)\delta v = -\gamma \implies \delta v' + (f(r) + \gamma)\delta v = -\gamma, \tag{2.14}$$

where in the last equality we used the definition of $v_0(r)$.

Example: Phase Space of a Damped Harmonic Oscillator (2)

Question:

Solution: We start with the same equation $\ddot{r} = -kr - \gamma v$ and rearrange it to be

$$\frac{d}{dr}\underbrace{\frac{1}{2}v^2 + V(r)}_{E(r,v)} = -\gamma v. \tag{2.15}$$

If γ is sufficiently small, we may obtain the solution perturbatively around a known solution, assuming that it may be written as

$$v(r) = v_0(r) + \delta v(r), \tag{2.16}$$

where $v_0(r)$ is the known solution for the case of $\gamma = 0$.

3

Linear Differential Equations

In this chapter we examine the nth order linear differential equation with the general form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x). \tag{3.1}$$

We start by noticing that if y_1 and y_2 are two solutions then their difference $y_1 - y_2$ solves the corresponding homogeneous equation which can be easily verified by direct substitution.

This suggest that the general solution must be written in the form of

$$y(x) = y_H(x) + y_P(x). \tag{3.2}$$

Here $y_H(x)$ is the general solution of the corresponding homogeneous equation which comprise of an n-parameter family of curves and $y_P(x)$ is one particular solution of the inhomogeneous equation so that for any two solutions $y_1(x)$ and $y_2(x)$ having the same $y_P(x)$ but different $y_{H_1}(x)$ and $y_{H_2}(x)$, their difference $y_{H_1}(x) - y_{H_2}(x)$ will always satisfy the homogeneous equation.

3.1 Homogeneous Solutions

If y_1 and y_2 are solutions to a linear homogeneous differential equation, then the linear combination of them $C_1y_1 + C_2y_2$ (where C_1 and C_2 may be complex) will also be a solution the the equation, which can be easily verified by direct substitution.

Combining with the fact that there are n integration constants, this suggest that the general solution to a linear homogeneous differential equation can be written in the form of

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x), \tag{3.3}$$

where $y_1(x), y_2(x), \ldots, y_n(x)$ are each solutions to the equation and linearly independent of each other (or else at least one of the terms in the above equation is redundant and

there will be less than n integration constants). In the language of linear algebra, the solutions form a basis for the vector space of all the possible solutions.

To ensure linear independency, we must have

$$
\begin{aligned}
C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) &= 0 \\
C_1 y_1'(x) + C_2 y_2'(x) + \cdots + C_n y_n'(x) &= 0 \\
&\vdots \\
C_1 y_1^{(n-1)} + C_2 y_2^{(n-1)}(x) + \cdots + C_n y_n^{(n-1)}(x) &= 0,
\end{aligned}
\tag{3.4}
$$

or equivalently

$$
\begin{pmatrix}
y_1(x) & y_2(x) & \cdots & y_n(x) \\
y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\
\vdots & \vdots & & \vdots \\
y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x)
\end{pmatrix}
\begin{pmatrix}
C_1 \\ C_2 \\ \vdots \\ C_n
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ \vdots \\ 0
\end{pmatrix}
\tag{3.5}
$$

for all x only if $C_1 = C_2 = \cdots = C_n = 0$.

Therefore, the determinant of the leftmost matrix $W(x)$, known as the Wronskian must not vanish. So

$$
W(x) \equiv
\begin{vmatrix}
y_1(x) & y_2(x) & \cdots & y_n(x) \\
y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\
\vdots & \vdots & & \vdots \\
y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x)
\end{vmatrix}
\neq 0.
\tag{3.6}
$$

It must be noted, however, that the vanishing of the Wronskian does not imply that the solutions are linearly dependent.

3.2 Linear Second Order

3.2.1 Variation of Parameters

The general form of a second order linear differential equation is

$$
y'' + a(x)y' + b(x)y = f(x).
\tag{3.7}
$$

There is no analytic solution for a general second order differential equations, even if it is linear. However, if we know one of the solution we can use the method of variation of parameters to find the general solution. Specifically, if $y_1(x)$ is a solution, we look for another independent solution in the form of

$$
y_2(x) = \psi(x)y_1(x).
\tag{3.8}
$$

Substituting into the equation, we have

$$(\psi y_1'' + 2\psi'y_1' + \psi''y_1) + a(x)(\psi y_1' + \psi'y_1) + b(x)\psi y_1 = f(x)$$
$$\implies (y_1'' + a(x)y_1' + b(x)y_1)\psi + (2y_1' + a(x)y_1)\psi' + y_1\psi'' = f(x). \tag{3.9}$$

But the term proportional to ψ is zero since y_1 solve the original differential equation so we have the first order differential equation (in ψ')

$$\psi'' + \left(2\frac{y_1'}{y_1} + a(x)\right)\psi' = \frac{f(x)}{y_1} \tag{3.10}$$

The integrating factor of this equation is

$$\mu(x) = \exp\left(\int\left(2\frac{y_1'}{y_1} + a(x)\right)dx\right) = \exp\left(2\int\frac{d(\ln(y_1))}{dx}dx + \int a(x)dx\right) = y_1^2\exp\left(\int a(x)dx\right) \tag{3.11}$$

so the solution is

$$\psi'(x) = \frac{1}{\mu(x)}\left(\int\frac{\mu(x)f(x)dx}{y_1(x)} + C\right),^1 \tag{3.12}$$

and thus

$$\psi(x) = C\int^x\frac{dx}{\mu(x)} + \int^x\frac{1}{\mu(x)}\left(\int^v\frac{\mu(v)f(v)dv}{y_1(v)}dv\right)dx. \tag{3.13}$$

where the integration constant is omitted since it would just give the original solution $y_1(x)$. The notation $\int^x$ is simply to change back the dummy variable u or v to x but is not the upper bound of integration because we are doing indefinite integral.

The first term is the second homogeneous solution and the second term is the particular solution.

<div style="border:1px solid orange">order of integration simplify</div>

3.2.2 Finding the First Solution

There is no better method than guessing for finding the first solution. If the coefficient funcions of a nth order differential equation are $a(x)$ and $b(x)$ are polynomials, the following order of guessing is recommended: $e^{\lambda x} \to x^n \to c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} + x^n \to (c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} + x^n)e^{\lambda x}$. For example, the Euler's equation $x^2y'' + axy' + by = 0$ can be solved by guessing $y(x) = x^n$.

---

[1]Here the integration constant C is introduced from the integration $\int d(\psi'(x)\mu(x))$. The integral $\int\mu(x)f(x)dx$ would give another integration constant but it will simply combine with C which is consistent with the fact that integrating a differential equation once gives one integration constant. Normally, C can be omitted and be introduced only when the second integral is carried out. But since the second integral is not carried out in this case we add the integration constant earlier. If it is not added here, we would still get the correct result, as the first term in eq. (3.13) would vanish but is encoded in the inner integral of the second term. Now we do not have to add an integration constant when evaluating this integral.

> **Example: Finding the First Solution.**
>
> **Question:** Solve
>
> $$x^2 y'' + xy' + y = x. \tag{3.14}$$
>
> **Solution:** Guess $y = x^n$, then we would get $n = \pm i$, so the homogenous solution is
>
> $$\begin{aligned}
y_H &= Ax^i + Bx^{-i} = Ae^{i\ln x} + Be^{-i\ln x} \\
&= A(\cos(\ln x) + i\sin(\ln x)) + B(\cos(\ln x) + i\sin(\ln x)) \\
&= A'\cos(\ln x) + B'\sin(\ln x).
\end{aligned} \tag{3.15}$$
>
> The particular solution is $y_P = \frac{x}{2}$, so the complete solution is $y = y_H + y_P = A'\cos(\ln x) + B'\sin(\ln x) + \frac{x}{2}$.

3.2.3 Linear Constant Coefficients

An important special case where the solution can be obtained analytically is when the coefficients are constant, where we guess the solution $y(x) = e^{\lambda x}$ to get a nth degree polynomial with n (generally complex) roots, corresponding to the n linearly independent solutions.[2]

In the case where there are repeated roots, we have to look for additional solutions which is not in the form of $y(x) = e^{\lambda x}$. Instead we use the method of variation of parameters outlined above to get extra indepedent solutions. For example, for a second order homogenous linear differential equation with constant coefficients,

$$a_2 y'' + a_1 y' + a_0 y = 0, \tag{3.16}$$

we get (see eq. (3.10))

$$\psi'' + 2\left(\frac{y_1'}{y_1} + a(x)\right)\psi' = \psi'' + \left(2\lambda + \frac{a_1}{a_2}\right)\psi' = \psi'' = \frac{f(x)}{y_1} = 0, \tag{3.17}$$

where $2\lambda + a_1/a_2$ is set to zero, since $\lambda = -a_1/2a_2$ is a repeated root. Thus, we get

$$\psi'' = 0 \implies \psi = (C_1 + C_2 x), \tag{3.18}$$

where the term containing C_1 is the original solution, and the term with $C_2 x$ is the extra independent solution.

---

[2]If all the coefficents $(a_1, \ldots, a_n)$ are real, then the complex solutions come in pairs, *i.e.,* if λ is a root, then its complex conjugate λ^* must also be a root. This can be shown by taking the complex conjugte on both sides of the equation after substituting one of the solutions.

This can be generalized to higher dimensions easily. For example, if the characteristic polynomial of some 7th order linear homogeneous differential equation is $(\lambda - a)^3(\lambda - b)(\lambda - c)(\lambda - d)$, then the general solution of the ODE is $y(x) = (C_1 + C_2 x + C_3^2)e^{ax} + (C_4 + C_5 x)e^{bx} + C_6 e^{cx} + C_7 e^{dx}$.

The table below shows some common solutions to a second order linear homogenous equation

λ_1	λ_2	$y(x)$	$y(x)$ (alternative form)	$y(x)$ (alternative form 2)
0	0	$C_1 + C_2 x$	—	—
κ	κ	$(C_1 + C_2 x)e^{\kappa x}$	—	—
κ_1	κ_2	$C_1 e^{\kappa_1 x} + C_2 e^{\kappa_2 x}$	$C_1 \sinh(\kappa_1 x) + C_2 \cosh(\kappa_1 x)$	—
$i\omega$	$-i\omega$	$C_1 \sin(\omega x) + C_2 \cos(\omega x)$	$C\cos(\omega x + \delta)$	$\mathfrak{Re}(Ce^{i\delta}e^{i\omega x})$
$\kappa + i\omega$	$\kappa - i\omega$	$[C_1 \sin(\omega x) + C_2 \cos(\omega x)]e^{\kappa x}$	$C\cos(\omega x + \delta)e^{\kappa x}$	$\mathfrak{Re}(Ce^{i\delta}e^{\kappa x + i\omega x})$

To find the particular solution, we again use the method of variation of parameters and substitute $y_p = \psi(x)e^{\lambda_1 x}$ (or $\psi(x)e^{\lambda_2 x}$) into eq. (3.16), but this time the RHS of the equation is $f(x)$ instead of 0. From eq. (3.10), we get

$$\psi'' + \left(2\lambda_1 + \frac{a_1}{a_2}\right)\psi' = \frac{1}{a_2}e^{-\lambda_1 x}f(x) \implies \psi'' + (\lambda_1 - \lambda_2)\psi' = \frac{1}{a_2}e^{-\lambda_1 x}f(x), \qquad (3.19)$$

where we have used the sum of root $\lambda_1 + \lambda_2 = -a_1/a_2$ and the factor $\frac{1}{a_2}$ is introduced since in section 3.2.1 we assumed the coefficient for y'' in the original differential equation is 0.

The integrating factor is $\mu(x) = e^{\int(\lambda_1 - \lambda_2)dx} = e^{(\lambda_1 - \lambda_2)x}$, so

$$\psi'(x) = \frac{1}{\mu(x)}\int^x \mu(u)\left[\frac{1}{a_2}e^{-\lambda_1 u}f(u)\right]du = \frac{e^{(\lambda_2 - \lambda_1)x}}{a_2}\int^x e^{-\lambda_2 u}f(u)du, \qquad (3.20)$$

which upon integration gives

$$\begin{aligned}
\psi(x) &= \int^x \psi'(v)dv = \int^x \frac{e^{-(\lambda_2 - \lambda_1)v}}{a_2}\left(\int^v e^{-\lambda_2 u}f(u)du\right)dv \\
&= \int^x \left(\int_u^x \frac{e^{(\lambda_2 - \lambda_1)v}}{a_2}dv\right)e^{-\lambda_2 u}f(u)du = \int^x \frac{\left[e^{(\lambda_2 - \lambda_1)x} - e^{(\lambda_2 - \lambda_1)u}\right]e^{-\lambda_2 u}f(u)}{a_2(\lambda_2 - \lambda_1)}du \\
&= \frac{e^{(\lambda_2 - \lambda_1)x}\int^x e^{-\lambda_2 u}f(u)du - \int^x e^{-\lambda_1 u}f(u)du}{a_2(\lambda_2 - \lambda_1)}.
\end{aligned}$$

$$(3.21)$$

Here we omit the integration constant since it would give back the homogeneous solution $e^{\lambda_1 x}$.

In many cases, the form of the particular solution can be guessed easily. For example,

$$f(x) = x \sin x, \quad \lambda = 1 \text{ and } 2 \qquad \Longrightarrow \; y_P = (Ax + B)\sin x + (Cx + D)\cos x$$
$$f(x) = e^{2x} + \cos^2 x, \quad \lambda = 1 \text{ and } 2 \qquad \Longrightarrow \; y_P = Axe^{2x} + B + C\cos(2x) + D\sin(2x)$$
$$f(x) = 8e^{\frac{x}{2}}, \quad \lambda = \frac{1}{2} \text{ and } \frac{1}{2} \qquad \Longrightarrow \; y_P = Ax^2 e^{\frac{x}{2}}$$
$$f(x) = xe^{-x}, \quad \lambda = -1 \text{ and } -1 \qquad \Longrightarrow \; y_P = x(Ax + B)e^{-x}$$
$$f(x) = x + \cos(2x), \quad \lambda = \pm 2i \qquad \Longrightarrow \; y_P = Ax + B + x(C\cos(2x) + D\sin(2x))$$
$$f(x) = e^x(1 + \sin x), \quad \lambda = 1 \pm i \qquad \Longrightarrow \; y_P = e^x(A + x(C\cos x + D\sin(x)))$$
$$f(x) = (\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m)e^{\mu x} \qquad \Longrightarrow \; y_P = x^k(\beta_0 + \beta_1 x + \cdots + \beta_m x^m)e^{\mu x}$$

$$\tag{3.22}$$

In the last example, k is the degree of degeneracy of μ if it is a root of the charateristic polynomial, *i.e.*, the number of times that μ occurs as a root.

3.3 Harmonic Oscillator

3.3.1 The General Solution

The appearance of harmonic oscillator is too often in physics that it deserves its own section. The most general equation describing a forced and dapmed harmonic oscillator is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 \cos(\omega t + \delta). \tag{3.23}$$

where $2\gamma, \omega_0$ and ω are the damping constant, natural frequency and driving frequency respectively.

The fastest way to solve this equation is to first consider the associated complex differential equation

$$\ddot{\tilde{x}} + 2\gamma\dot{\tilde{x}} + \omega_0^2 \tilde{x} = F_0 e^{i(\omega t + \delta)}. \tag{3.24}$$

Taking the real part of this complex differential equation,

$$\mathfrak{Re}\left(\ddot{\tilde{x}} + 2\gamma\dot{\tilde{x}} + \omega_0^2 \tilde{x}\right) = \frac{d^2}{dt^2}\left(\mathfrak{Re}(\tilde{x})\right) + 2\gamma\frac{d}{dt}\left(\mathfrak{Re}(\dot{\tilde{x}})\right) + \omega_0^2 \mathfrak{Re}(\tilde{x}) = \mathfrak{Re}\left(F_0 e^{i(\omega t + \delta)}\right). \tag{3.25}$$

Comparing the two associating equations above we can indeed say that x is the real part of $\tilde{x}$.[3] Note that this methods hinges on the fact that the real function $\mathfrak{Re}$ is linear, so that we can distribute $\mathfrak{Re}$ separately to the three terms in the above equation, and also the derivative function (since differentiation is also a linear process).

Also, the differential equation itself has to be linear. For example, if the equation contains a term $\dot{x}^2$, then when we take the real part of the complex associated equation, we would

---

[3] We usually use the tilde symbol to denote the complex numbers we want to take the real part of.

get $\mathfrak{Re}(\dot{\tilde{x}}^2) \neq d/dt\mathfrak{Re}^2(\tilde{x})$, so we cannot follow the argument above and claim that x is the real part of $\tilde{x}$.

The homogeneous solution is

$$x_H = Ae^{\lambda_+ t} + Be^{\lambda_- t}, \quad \lambda_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{3.26}$$

Depending on sign of the discriminant, there are 3 cases: overdamping when the determinant is positive where there is no oscillation; underdamping when there is osillation; and critical damping in the critical case (where the general homogeneous solution is $x_H = (A + Bt)e^{\lambda t}$), as there is only one repeated root λ (we will ignore this special case in the discussion below).

For the particular solution, we guess

$$
\begin{aligned}
\tilde{x_P} = Ce^{i(\omega t + \delta)} &\implies \dot{\tilde{x}}_P = i\omega Ce^{i(\omega t + \delta)} \implies \ddot{\tilde{x}}_P = -\omega^2 Ce^{i(\omega t + \delta)} \\
x_P = \mathfrak{Re}(Ce^{i(\omega t + \delta)}) &\implies \dot{x}_P = \mathfrak{Re}(i\omega Ce^{i(\omega t + \delta)}) \implies \ddot{x}_P = \mathfrak{Re}(-\omega^2 Ce^{i(\omega t + \delta)}).^4
\end{aligned}
\tag{3.27}
$$

This explicit calculation confirms that the real function $\mathfrak{Re}$ is indeed linear, as $\mathfrak{Re}(\ddot{\tilde{x}}_P) = d^2/dt^2 (\mathfrak{Re}(\tilde{x}_P))$. In the event where the driving function is not a pure sinusodial function, we can always use Fourier transform to write it as sum of sinusodial functions, so the general scheme of the above analysis still works. For example, the driving force is modulated as $F(t) = mA\cos(\sigma t)\sin(\omega t)$, then we cannot guess the particular solution as $x_P = C\mathfrak{Re}(e^{i\sigma t})\mathfrak{Re}(-ie^{i\omega t})$ or something like that, but instead we should use the idea of a Fourier transform and rewrite the driving function as sum of trigonometric functions $f(t) = mA(\sin(\sigma + \omega) + \sin(\sigma - \omega))/2$, and proceeds as above.

Upon substitution, we get

$$C = \frac{F_0}{\omega_0^2 - \omega^2 + 2i\gamma\omega} = \frac{F_0}{\sqrt{4\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}e^{-i\varphi}, \quad \varphi = \arctan\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right). \tag{3.28}$$

Taking the real part and combining the homogeneous and inhomogeneous solution, we have

$$x = x_H + x_P = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2 - \omega_0^2}t} + Be^{\sqrt{\gamma^2 - \omega_0^2}t}\right) + \frac{F_0\cos(\omega t + \delta + \varphi)}{\sqrt{4\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}. \tag{3.29}$$

3.4 Special Cases of ω

3.4.1 Undamped Resonance

When $\gamma = 0$ and as $\omega \to \pm\omega_0$, we have $y_P \to \infty$, so the guess in eq. (3.27) does not work anymore. The mathematical reason is that since $i\omega \to i\omega_0 = \lambda_\pm$, the particular solution coincides with the homogenous solutions, so we would have to find another independent solution.

The way we would find the extra solution is by considering a more cumbersome but a more general particular solution

$$x_P = \mathfrak{Re}(C(e^{i(\omega t + \delta)} - e^{\pm i\omega_0 t})). \tag{3.30}$$

What we have done basically, is to call some part of the homogenous solution as part of the particular solution. More concretely, we split the coefficient A (or B) into two parts of C and $A - C$ (or $B - C$), and we add the part of coefficient C to the particular solution. Of course, the method of variation of parameters also works by guessing $x_P = \psi(t)e^{\lambda t}$, where λ can be λ_+ or λ_-.

In some sense, it is a more general particular solution because it not only works for most cases where $\omega \neq \omega_0$, we can now also take the limit as $\omega \to \omega_0$ to get a new independent solution as

$$x_P = F_0\mathfrak{Re}\left(\lim_{\omega \to \pm\omega_0}\left(\frac{e^{i(\omega t + \delta)} - e^{\pm i\omega_0 t}}{\omega_0^2 - \omega^2}\right)\right) = F_0\mathfrak{Re}\left(\lim_{\omega \to \pm\omega_0}\left(\frac{ite^{i(\omega t + \delta)}}{-2\omega}\right)\right) = \frac{F_0 t \sin(\omega_0 t + \delta)}{2\omega_0}. \tag{3.31}$$

Therefore, if $\omega = \omega_0$, then $\varphi = -\pi/2$ or $\pi/2$ and $A \approx F_0/\gamma\omega_0$. Mathematically, the acceleration cancels with the restoring force, while the driving force cancels the damping force.

In hindsight, one can simply guess $x_P = \mathfrak{Re}(Cte^{i(\omega t + \delta)})$, since it is the simpliest mean to obtain another independent solution, and it works also.

If $\gamma = \omega_0$ and for some reason $\omega \to i\omega_0$, then the roots $\lambda_\pm = \lambda$ are repeated and the particular solution coincides with the homogenous solution, and the particular solution can be found by similar means, either by the method of variation of parameters and guess $x_P = \psi(x)e^{\lambda t}$, or by taking the limit of the generalized particular solution

$$x_P = \mathfrak{Re}(C(e^{i(\omega t + \delta)} - e^{-\omega_0 t})) = F_0\mathfrak{Re}\left(\lim_{\omega \to i\omega_0}\left(\frac{e^{i(\omega t + \delta)} - e^{-\omega_0 t}}{\omega_0^2 - \omega^2 + 2i\omega_0\omega}\right)\right)$$
$$= \mathfrak{Re}\left(\lim_{\omega \to i\omega_0}\left(\frac{ite^{i(\omega t + \delta)}}{-2\omega + 2i\omega_0}\right)\right) = \mathfrak{Re}\left(\frac{t^2 e^{i(\omega t + \delta)}}{2}\right) = \frac{1}{2}t^2 \cos(\omega t + \delta). \tag{3.32}$$

> **Example: Resonance.**
>
> **Question:** Find the solution to the solution to the equation
>
> $$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + (\beta^2 - 1)y = e^x(1 + \cos x). \tag{3.33}$$
>
> **Solution:** The roots of the charateristic equation is $\lambda = 1 \pm i\beta$, so the homogenous solution for $\beta \neq 0$ is
>
> $$y_H(x) = e^x(A\cos\beta x + B\sin\beta x). \tag{3.34}$$

For $\beta^2 \neq 1$, the particular solution is

$$y_P(x) = Ce^x + e^x(D\cos x + E\sin x). \tag{3.35}$$

Solving gives $C = \beta^{-2}, D = (\beta^2 - 1)^{-1}$ and $E = 0$.

If $\beta^2 = 1$, however, then since one of the particular solution, namely $e^x\cos x$, coincide with one of the homogenous solution, we must modify our guess to the particular solution as

$$y_P(x) = Ce^x + xe^x(D\cos x + E\sin x). \tag{3.36}$$

Solving gives $C = \beta^{-2}, D = 0$ and $E = 1/2$.

Alternatively we can take the limit of the generalized particular solution as $\beta \to \pm 1$

$$y_P(x) = \frac{e^x}{\beta^2} + \lim_{\beta \to \pm 1} \frac{\cos x e^x - \cos(\beta x)e^x}{\beta^2 - 1} = e^x + \frac{1}{2}x\sin x e^x. \tag{3.37}$$

If $\beta = 0$, then the roots of the charateristic equation are both 1 so the homogenous solution is

$$y_H(x) = (A + Bx)e^x. \tag{3.38}$$

Now for the particular solution we have to guess

$$y_P(x) = Cx^2 e^x + De^x\cos x \tag{3.39}$$

to get $C = 1/2$ and $D = -1$.

Alternatively we can take the limit of the generalized particular solution as $\beta \to 0$

$$\begin{aligned}
y_P(x) &= (\beta^2 - 1)^{-1}e^x\cos x + \lim_{\beta \to 0} \frac{e^x - e^x\cos(\beta x)}{\beta^2} \\
&= -e^x\cos x + e^x x \lim_{\beta \to 0} \frac{\sin(\beta x)}{2\beta} = -e^x\cos x + \frac{1}{2}e^x x^2.
\end{aligned} \tag{3.40}$$

3.4.2   Extreme Cases

If $\omega \ll \omega_0$, then from the general solution we have $\varphi \approx 0$ and $A \approx F_0/\omega_0^2$. Mathematically, the acceleration and the damping term is negligible, so the mass simply follows the driving force, which is cancelled by the restoring force.

If $\omega \gg \omega_0$ (and $\gamma \ll \omega$), then from the general solution we have $\varphi \approx -\pi$ and $A \approx F_0/\omega^2$. Mathematically, the damping term and the restoring term is negligible, so the driving force is soley responsible for the acceleration.

3.4.3   Damped Resonance

Definition of Quality Factor

For an underdamped unforced oscillator, the general solution can be seen from eq. (3.29) as

$$x = Ae^{-\gamma t}\cos(\omega_u t + \varphi)^5 \implies \dot{x} = -A\left(e^{-\gamma t}\omega_u\sin(\omega_u t + \varphi) + \gamma\cos(\omega_u t + \varphi)\right), \quad (3.41)$$

where $\omega_u \equiv \sqrt{\omega_0^2 - \gamma^2}$ is the underdamped frequency.

The quality factor Q of an oscillator is defined as[6]

$$Q \equiv 2\pi\frac{\text{Average energy stored in a cycle}}{\text{Total energy lost per period}}. \quad (3.42)$$

Sometimes the word "average" is omitted, since as we will see both definitions give the same result.

The average energy stored in a cycle E can be calculated directly via, assuming γ small (and thus $\omega_u \approx \omega_0$),

$$\begin{aligned}
E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \\
&= \frac{1}{2}m\left(A^2 e^{-2\gamma t}\left(\omega_u\sin(\omega_u t + \varphi) + \gamma\cos(\omega_u t + \varphi)\right)^2\right) + \frac{1}{2}\left(\omega_0^2(Ae^{-\gamma}\cos(\omega_u t + \varphi))^2\right) \\
&= \frac{1}{2}mA^2 e^{-2\gamma t}\left((\omega_0^2 - \gamma^2)\sin^2(\omega_u t + \varphi) + (\gamma^2 + \omega_0^2)\cos^2(\omega_u t + \varphi) + \frac{\gamma}{2}\omega_u\sin(2(\omega_u t + \varphi))\right) \\
&= \frac{1}{2}mA^2 e^{-2\gamma t}\left(\omega_0^2 + \frac{\gamma}{2}\omega_u\sin(2(\omega_u t + \varphi)) + \gamma^2(\cos^2(\omega_u t + \varphi) - \sin^2(\omega_u t + \varphi))\right) \\
&= \frac{1}{2}mA^2 e^{-2\gamma t}\left(\omega_0^2 + \frac{\gamma}{2}\omega_u\sin(2(\omega_u t + \varphi)) + \gamma^2\cos(2(\omega_u t + \varphi))\right) \\
\langle E\rangle &= \frac{1}{2}mA^2\omega_0^2 e^{-2\gamma t} \approx \frac{1}{2}mA^2\omega_0^2.
\end{aligned}$$
$$(3.43)$$

since when averaging over a long period of time $T \to \infty$, simple trigonometric functions integrate to be zero (while their squared to $\frac{1}{2}$).

Therfore the quality factor Q is

$$\begin{aligned}
Q &= 2\pi\frac{\langle E\rangle}{|\Delta\langle E\rangle|} = 2\pi|\Delta\left(\ln(\langle E\rangle)\right)|^{-1} = 2\pi\left|\Delta\left(-2\gamma t + \ln\left(\frac{1}{2}mA^2\omega_0^2\right)\right)\right|^{-1} \\
&= 2\pi|(-2\gamma\Delta t)|^{-1} = \frac{2\pi}{\gamma\Delta t} = \frac{\omega_0}{\gamma}.
\end{aligned}$$
$$(3.44)$$

Another way to calculate the quality factor is by considering the power of the resistive force. For simplicity, we simply calculate the fractional change of energy of the first cycle. This does not lose generality as every cycle can be regarded as the first cycle by redefining the amplitude and the time. Therefore we have

---

[5]A and φ are real if the initial conditions are real, but in general A and φ are complex.

[6]The oscillator can be forced or unforced. Unforced oscillator simply means that the energy lost from damping is not replenished by the driving force.

$$|\Delta\langle E\rangle| = \int_0^{\Delta t} m\gamma\dot{x}^2 dt = m\gamma \int_0^{\Delta t} A^2 e^{-2\gamma t}\left(\omega_u \sin(\omega_u t + \varphi) + m\gamma \cos(\omega_u t + \varphi)\right)^2 dt$$

$$\approx m\gamma A^2 \int_0^{\Delta t} e^{-2\gamma t}\left(\omega_0^2 \sin^2(\omega_0 t + \varphi) + \omega_0\gamma \sin(2(\omega_0 t + \varphi))\right) dt$$

$$= m\gamma A^2 \omega_0^2 \int_0^{\frac{2\pi}{\omega_0}} e^{-2\gamma t}\left(\frac{1 - \sin(\omega_0 t + \varphi)}{2}\right) dt$$

$$= \frac{1}{2} m\gamma A^2 \omega_0^2 \left(\frac{1}{-2\gamma}\right)\left(1 - e^{-2\gamma\left(\frac{2\pi}{\omega_0}\right)}\right) \approx \frac{1}{4} m A^2 \omega_0^2 \left(2\gamma\left(\frac{2\pi}{\omega_0}\right)\right) = m\pi\gamma\omega_0 A^2.$$

$$(3.45)$$

Therefore the quality factor Q is

$$Q = 2\pi \frac{\langle E\rangle}{|\langle E\rangle|} = 2\pi \frac{\frac{1}{2} m A^2 \omega_0^2}{m\pi\gamma\omega_0 A^2} \approx \frac{\omega_0}{\gamma}. \tag{3.46}$$

Note that the quality factor is an inherent property of the circuit itself, so the same quality factor can be derived in the steady state solution as well, not just the trasient solution, which we have demonstrated on, given that the driving frequency is also equals to the natural freqeucy, *i.e.*, $\omega = \omega_0$.

FWHM of the Resonance Peak

In steady state (*i.e.*, $t \to \infty$), the general solution (and for simplicity we assume $\delta = 0$) can be seen from eq. (3.29) as

$$x = \frac{F_0 \cos(\omega t + \varphi)}{\sqrt{4\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} = A\cos(\omega t + \varphi). \tag{3.47}$$

The resonance frequency ω_{res} is defined to be the driving frequency at which the response[7] has maximum amplitiude, which can be found by differentiating A to get

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \implies A_{\max} = \frac{F}{\gamma\omega_0}. \tag{3.48}$$

It turns out that the FWHM (full width at half maximum) of the resonance peak (*i.e.*, the peak of the A v.s. ω graph), which is the quantity $|\omega_2 - \omega_1|$, is (to the first order) inversely proportional to the quality factor Q. We would like to find the $\omega = \omega_1$ or ω_2 for which

$$A(\omega) = \frac{F_0}{\sqrt{4\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} = \frac{A}{2} = \frac{F}{2\gamma\omega_0}$$
$$\omega^4 + (\gamma^2 - 2\omega_0^2)\omega^2 + \omega_0^4 - 4\gamma^2\omega_0^2 = 0 \tag{3.49}$$

---

[7]In this case, the response we are interested in is the displacement x. However, if the response we are interested is velocity or (current in electricity analogy), then the resonance frequency will be the natural frequency ω_0.

One can obviously find the roots expcitly and subtract them to get the answer. The faster way to do this would be to use the sum and the product of roots. We have

$$\omega_2 - \omega_1 = \sqrt{\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2} = \sqrt{-(\gamma^2 - 2\omega_0^2) - 2\sqrt{(\omega_0^2 - 4\gamma^2)\omega_0^2}}$$

$$\approx \sqrt{2\omega_0^2 - \gamma^2 - 2\omega_0^2 \left(1 - 2\frac{\gamma^2}{\omega_0^2}\right)} = \sqrt{3}\gamma \tag{3.50}$$

3.4.4 Rapidly Oscillating Force

As can be seen from the genearl solution of the forced oscillator in eq. (3.29), the amplitude of oscillation C goes to zero as the frequency of the driving force ω goes to infinity. This is mass does not have time to respond to the rapidly changing direction of the force.

However, if the driving force depends also on the displacement, for example, when the pivot of a pendulum is oscillating vertically with vertical displacement $a\cos(\omega t)$, then in the non-inertial frame the pendulum experiences a vertical force

3.5 Non-Linear Second Order

Example: The Equation Does Not Contain x.

Question: Solve

$$y''(y - 1) + y'(y - 1)^2 = y'^2. \tag{3.51}$$

Solution: Let $p = y'$, then $y'' = dp/dy = dy/dx = pp'$. Substituting into the equation we get

$$pp'(y - 1) + p(y - 1)^2 = p^2 \implies x = \frac{1}{C}\ln\left|\frac{y - 1}{C + 1 - y}\right| + C'. \tag{3.52}$$

However, the solution blows up at $C = 0$, so we have to evaluate the integral again to get $y = \frac{1}{x+B}$, as another answer.
Finally, $y = A$ is trivally true, since every term in the equation contains y'.

PS2 2.5
onwards

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Coupled Linear Differential Equations

4.1 1ˢᵗ Order Vector Equation

4.1.1 Single Scalar Equation

For a general n^{th} order linear differential equation

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x), \tag{4.1}$$

it can be treated as a system of n first order linear differential equations of n unknown functions $y(x), y'(x), \ldots, y^{(n-1)}(x)$,

$$
\begin{aligned}
y'(x) &= \frac{d}{dx}y(x), \\
y''(x) &= \frac{d}{dx}y'(x), \\
&\vdots, \\
y^{(n-1)}(x) &= \frac{d}{dx}y^{(n-2)}(x),
\end{aligned}
\tag{4.2}
$$

$$\frac{d}{dx}y^{(n-1)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x).$$

Introducing the natural vector

$$\mathbf{y} = (y, y', \ldots, y^{(n-2)}, y^{(n-1)}), \tag{4.3}$$

we can rewrite the system of equation into a more compact form as

$$\mathbf{y}' = A(x)\mathbf{y} + \mathbf{f}(x), \quad \mathbf{f}(x) = (0, \ldots 0, \ldots, f(x)), \tag{4.4}$$

and

$$
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-a_0(x) & -a_1(x) & -a_2(x) & \cdots & -a_{n-2}(x) & -a_{n-1}(x)
\end{pmatrix}. \tag{4.5}
$$

Therefore, we have sucessfully simplified the problem of solving a nth order linear differential equation into solving a single first order vector differential eqution.

However, we would almost directly solve the nth order differential equation instead of writing it in this form.

4.1.2 Coupled Scalar Equation

For a system of k n^{th} order differential equations, they can be written as a system of $n \times k$ first order differential equations, or one first order vector differential equation, like the above equation, but the vector has $n \times k$ entries (not a $n \times k$ matrix).

For example, the equations of motion of two coupled harmonic oscillators read

$$
\begin{cases}
\ddot{y}_1 & = -\gamma_1 \dot{y}_1 - k_1 y_1 - k_2(y_1 - y_2), \\
\ddot{y}_2 & = -\gamma_2 \dot{y}_2 - k_2(y_2 - y_1).
\end{cases} \tag{4.6}
$$

Introducing the natural vector

$$
\mathbf{y}(t) = (y_1(t)\dot{y}_1(t), y_2(t), \dot{y}_2(t)), \tag{4.7}
$$

we have

$$
\dot{\mathbf{y}}(t) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-k_1 - k_2 & -\gamma_1 & k_2 & 0 \\
0 & 0 & 0 & 1 \\
k_2 & 0 & -k_2 & -\gamma_2
\end{pmatrix} \mathbf{y}(t). \tag{4.8}
$$

Note that we could have labeled the variables in a different order which does not affect the solution.

4.2 Solving 1st Order Vector Equation

4.2.1 Homogeneous

Diagonal

We start with the case where $A(x) = A$ is independent of x and $\mathbf{f}(x) = \mathbf{0}$, so

$$\mathbf{y}'(x) = A\mathbf{y}(x). \tag{4.9}$$

If A is diagonal with elements $A_{ii} = a_i$, then the equations are independent, as the ith component of the equation simply reads

$$y_i' = a_i y_i(x) \implies y_i(x) = C_i e^{a_i x}. \tag{4.10}$$

Explicitly, we have

$$\mathbf{y}(x) = C_1 e^{a_i x} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + C_2 e^{a_2 x} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + C_n e^{a_n x} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{4.11}$$

Diagonalizable

If A is not diagonal, but still diagonalizable, we can diagonalize A by introducing the change of basis matrix P containing the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$

$$P = (\mathbf{v}_1, \dots, \mathbf{v}_n), \quad \mathbf{y}(x) = P\mathbf{z}(x). \tag{4.12}$$

The equation now becomes

$$P\mathbf{z}'(x) = AP\mathbf{z}(x) \implies \mathbf{z}'(x) = P^{-1}AP\mathbf{z}(x) = \text{diag}(\lambda_1, \dots, \lambda_n)\mathbf{z}(x). \tag{4.13}$$

The solution is therefore

$$z_i(x) = C_i e^{\lambda_i x} \implies \mathbf{y} = P\mathbf{z} = (\mathbf{v}_1, \dots, \mathbf{v}_n)(\text{diag}(e^{\lambda_1 x}, \dots, e^{\lambda_n x}))\mathbf{C} \equiv PE(x)\mathbf{C} \tag{4.14}$$

where $\mathbf{C} = (C_1, C_2, \dots, C_n)$ is a constant vector.

Explicity, we have

$$\mathbf{y}(x) = C_1 e^{\lambda_1 x} \mathbf{v}_1 + \cdots + C_n e^{\lambda_n x} \mathbf{v}_n. \tag{4.15}$$

The usual solution $y(x)$ of the nth order linear differential solution is simply the first row of this vector equation. However, this solution also provides the solutions for $y'(x), y''(x), \ldots y^{(n-1)}(x)$.

4.2.2 Inhomogeneous

Now suppose $\mathbf{f}(x) \neq \mathbf{0}$ and we would like to find the particular solution $\mathbf{y}_P(x)$ given the homogenous solution $\mathbf{y}_H(x)$. We use the method of variation of parameters and guess

$$\mathbf{y}_P(x) = \boldsymbol{\psi}(x)\mathbf{y}_H(x).^1 \tag{4.16}$$

Substituting into the differential equation we get

$$\boldsymbol{\psi}'(x)\mathbf{y}_H(x) + \boldsymbol{\psi}(x)\mathbf{y}'_H(x) = A(x)\boldsymbol{\psi}(x)\mathbf{y}_H(x) + \mathbf{f}(x). \tag{4.17}$$

Since $\mathbf{y}_H(x)$ solves the homogeneous equation, two terms vanish and we have

$$\boldsymbol{\psi}'(x)\mathbf{y}_H(x) = \mathbf{f}(x) \implies \boldsymbol{\psi}(x) = \int \mathbf{y}_H(x)^{-1}\mathbf{f}(x)dx,^2 \tag{4.18}$$

and the particular solution is

$$\mathbf{y}_P(x) = \mathbf{y}_H(x) \int \mathbf{y}_H(x)^{-1}\mathbf{f}(x)dx. \tag{4.19}$$

Usually, however, the particular solution is find by simple guessing. For example, if $\mathbf{f}(x) = e^{5t}(4, 0)$ then we would guess $e^{5t}(A, B)$.

4.2.3 Complete Solution with Initial Conditions

The complete solution is therefore

$$\mathbf{y}(x) = \mathbf{y}_H(x) + \mathbf{y}_P(x) = PE(x)\mathbf{C} + PE(x) \int E^{-1}(x)P^{-1}\mathbf{f}(x)dx. \tag{4.20}$$

Given initial conditions $\mathbf{y}(x) = \mathbf{y}(x_0)$ at $x = x_0$, we have $\mathbf{C} = E(x)^{-1}P^{-1}\mathbf{y}(x_0)$ if we choose the lower limit of the integral to be x_0.[3] and the general solution is given by

$$\mathbf{y}(x) = \mathbf{y}_H(x) + \mathbf{y}_P(x) = PE(x - x_0)P^{-1}\mathbf{y}(x_0) + PE(x) \int_{x_0}^{x} E^{-1}(x')P^{-1}\mathbf{f}(x')dx'. \tag{4.21}$$

---

[1]There is no need to use the complete general solution of the homogeneous solution, we can just choose one of the independent solutions (as long as it solves the homogeneous equation so that cancellation can be acheived), but there is no harm either.

[2]The constant vector of intgration gives back the homogenous solution, as usual.

[3]We have this freedom since making any other choice only differs from this choice by an integration vector constant, which can be absorbed by $\mathbf{C}$. In other words, $\mathbf{C}$ and x_0 can be chosen arbitrarily as long as they together satisfies the initial conditions, and we choose x_0 conviently so that $\mathbf{C}$ is simple.

4.3 Non-diagonlizable

4.3.1 Triangulation

For any matrix A, there exists an unitary matrix U such that the transformed matrix $T = U^{-1}AU$ is upper triangular, where the diagonal carries the eigenvalues of A, since

$$\det(A - \lambda\mathbb{I}) = \det(UTU^{-1} - \lambda UU^{-1}) = \det(U(T - \lambda\mathbb{I})U^{-1}) = \det(T - \lambda\mathbb{I}). \quad (4.22)$$

Following the footsteps in section 4.2.1, substituting $A = UTU^{-1}$, we have

$$\mathbf{z}' = T\mathbf{z}, \quad \mathbf{z} = T^{-1}\mathbf{y}. \quad (4.23)$$

The system of linear differential equations encoded in the above equation can then be solved one by one, where each one requires the results of all the subsequent solutions. After finding $\mathbf{z}$, we can invert the equation to get $\mathbf{y} = T\mathbf{z}$.

4.3.2 Jordan Form

Consider an eigenvalue λ_1 with degeneracy m, with a single eigenvector $\mathbf{v}_1$. The set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called a Jordan chain of λ_1, or the generalized eigenvectors of A, if they satisfy

$$
\begin{cases}
(A - \lambda_1\mathbb{I})\mathbf{v}_1 = \mathbf{0}, \\
(A - \lambda_1\mathbb{I})\mathbf{v}_2 = \mathbf{v}_1, \\
\quad\vdots \\
(A - \lambda_1\mathbb{I})\mathbf{v}_k = \mathbf{v}_{k-1}.
\end{cases}
\text{ or }
\begin{cases}
(A - \lambda_1\mathbb{I})\mathbf{v}_1 = \mathbf{0}, \\
(A - \lambda_1\mathbb{I})^2\mathbf{v}_2 = \mathbf{0}, \\
\quad\vdots \\
(A - \lambda_1\mathbb{I})^k\mathbf{v}_k = \mathbf{0}.
\end{cases}
\text{ or }
\begin{cases}
A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \\
A\mathbf{v}_2 = \lambda_1\mathbf{v}_2 + \mathbf{v}_1, \\
\quad\vdots \\
A\mathbf{v}_k = \lambda_1\mathbf{v}_k + \mathbf{v}_{k-1}.
\end{cases}
$$
$$(4.24)$$

As usual, we introduce the change of basis matrix P containing all the n generalized eigenvectors $\mathbf{v}_1, \dots, v_n$

$$P = (\mathbf{v}_1, \dots, \mathbf{v}_n), \quad \mathbf{y}(x) = P\mathbf{z}(x). \quad (4.25)$$

The equation now becomes

$$P\mathbf{z}'(x) = AP\mathbf{z}(x) \implies \mathbf{z}'(x) = P^{-1}AP\mathbf{z}(x) \equiv J\mathbf{z}(x), \quad (4.26)$$

where J is the transformed matrix, called the jordan form

$$J \equiv P^{-1}AP = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda_i \end{pmatrix}, \quad (4.27)$$

where s is the number of distinct eigenvalues.

For example,

$$J = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 3 \end{pmatrix}. \quad (4.28)$$

For simplicity, we assume that $\lambda_1 = \lambda_2 = \cdots = \lambda_k \equiv \lambda \neq \lambda_{k+1} \neq \lambda_{k+1} \neq \cdots \neq \lambda_n$, *i.e.*, there is only 1 degenerate eigenvalue $\lambda_1 = \lambda_2 = \cdots = \lambda_k \equiv \lambda$ with degeneracy k and all other eigenvalues are distinct.

Then expanding the vector equation we have

$$\begin{cases} z_1' = \lambda z_i + z_2, \\ \quad\vdots \\ z_{k-1}' = \lambda z_{k-1} + z_k, \\ \quad z_k' = \lambda z_k, \\ z_{k+1}' = \lambda_{k+1} z_{k+1}, \\ z_{k+2}' = \lambda_{k+2} z_{k+2}, \\ \quad\vdots \\ \quad z_n' = \lambda_n z_n. \end{cases} \quad (4.29)$$

Withing the first jordan block, the last equation can solved direcctly and then substituted upwards consecutively and the solutions become

$$\begin{cases} z_k = C_k e^{\lambda x}, \\ z_{k-1} = (C_{k-1} + C_k x)e^{\lambda x}, \\ \quad\vdots \\ z_1 = \left(C_1 + C_2 x + \cdots + C_k \dfrac{x^{k-1}}{(k-1)!} \right) e^{\lambda x}, \\ z_{k+1} = C_{k+1} e^{\lambda_{k+1} x}, \\ z_{k+2} = C_{k+2} e^{\lambda_{k+2} x}, \\ \quad\vdots \\ z_n = C_n e^{\lambda_n x}. \end{cases} \tag{4.30}$$

The general solution is therefore

$$\mathbf{y}(x) = P\mathbf{z}(x) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)E(x)\mathbf{C}, \tag{4.31}$$

which has the same form as eq. (4.14), but $E(x)$ is more complex now. In the above example, $E(x)$ is given by

$$E(x) = \begin{pmatrix} e^{\lambda x} & xe^{\lambda x} & & \frac{x^{k-1}}{(k-1)!}e^{\lambda x} & 0 & \cdots & & 0 \\ 0 & e^{\lambda x} & xe^{\lambda x} & & & & & \\ & 0 & \ddots & \ddots & & \vdots & & \iddots \\ & & 0 & e^{\lambda x} & xe^{\lambda x} & & & \\ \vdots & & & 0 & e^{\lambda x} & 0 & & \vdots \\ & & & & 0 & e^{\lambda_{k+1}x} & 0 & \\ & & \iddots & & & 0 & e^{\lambda_{k+2}x} & 0 \\ & & & & & & 0 & \ddots & 0 \\ 0 & & & \cdots & & & & 0 & e^{\lambda_n x} \end{pmatrix}. \tag{4.32}$$

Explicitly, we have

$$\mathbf{y}(x) = C_1 e^{\lambda x}\mathbf{v}_1 + C_2 e^{\lambda x}(x\mathbf{v}_1 + \mathbf{v}_2) + \cdots + C_k e^{\lambda x}\left(\frac{x^{k-1}}{(k-1)!}\mathbf{v}_1 + \cdots + x\mathbf{v}_{k-1} + \mathbf{v}_k \right)$$
$$+ C_{k+1}e^{\lambda_{k+1}x}\mathbf{v}_{k+1} + \cdots + C_n e^{\lambda_n x}\mathbf{v}_n. \tag{4.33}$$

This solution has a similar form as if A is not defective, with only the eigenvectors replaced by the generalized eigenvectors.

Example: Non-Diagonalizable System.

Question: Solve the equation $\mathbf{y}'(x) = A\mathbf{y}(x)$, where

$$A = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}. \tag{4.34}$$

Solution: A is defective with eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 1$. The eigevectors are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}. \tag{4.35}$$

We find a generalized eigenvector

$$(A - (-1)\mathbb{I})^2 \mathbf{v}_2 = \mathbf{0} \implies \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{4.36}$$

Given the eigenvalues and the generalized eigenvectors we give the solution directly as

$$\mathbf{y}(x) = e^{-t}\left(C_1 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + C_2 \left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \right) + C_3 e^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}. \tag{4.37}$$

4.4   Solving Higher Order Vector Equation

Note that for the coupled differential equation in eq. (4.6)

$$\begin{cases} \ddot{y}_1 &= -\gamma_1 \dot{y}_1 - k_1 y_1 - k_2(y_1 - y_2), \\ \ddot{y}_2 &= -\gamma_2 \dot{y}_2 - k_2(y_2 - y_1), \end{cases} \tag{4.38}$$

instead of introducing the state vector $\mathbf{y} = (y_1, \dot{y}_1, y_2, \dot{y}_2)$, we can simply introduce the vector $\mathbf{y} = (x, y)$,

and rewrite the coupled second ordre scalar differential equation in a single second order vector differential equation

$$\ddot{\mathbf{y}} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \dot{\mathbf{y}} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \mathbf{y} = 0. \tag{4.39}$$

We can then solve this equation as usual, by guessing

$$\mathbf{y} = \mathbf{u}e^{\lambda t} \tag{4.40}$$

to obatin

$$\left(\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \right) \mathbf{u} = 0. \tag{4.41}$$

Reqruiring the determinant equals to zero we get four roots for λ. The general solution is then given by

$$\sum_{i=1}^{4} C_i \mathbf{u}_i e^{\lambda_i t}. \tag{4.42}$$

Example: Second Order Vector Differential Equation.

Question: Solve the coupled differential equation

$$\begin{cases} 2y'' - 3y' + 2z' + 3y + z &= e^{2x}, \\ y'' - 3y' + z' + 2y - z &= 0. \end{cases} \tag{4.43}$$

Solution: Introducing the vector $\mathbf{v} = (y, z)$ we rewrite the coupled scalar equation in a single vector equation

$$\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{v}'' + \begin{pmatrix} -3 & 2 \\ -3 & 1 \end{pmatrix} z' + \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} e^{2x} \\ 0 \end{pmatrix}. \tag{4.44}$$

We substitute the homogenous equation

$$\mathbf{v}_H(x) = \mathbf{u}_H e^{\lambda x} \tag{4.45}$$

and obtain

$$\left(\lambda^2 \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 & 2 \\ -3 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \right) \mathbf{u}_H = 0. \tag{4.46}$$

Requiring that the determinant of the matrix in fron to be zero we get $\lambda = 1$ and the corresponding eigenvector $\mathbf{u} = (3, -2)$.
Guess the particular solution

$$\mathbf{v}_P = \mathbf{u}_P e^{2x}, \tag{4.47}$$

we obtain $\mathbf{u}_P = (1/5, 0)$. Therefore the general solution is

$$\begin{pmatrix} y \\ z \end{pmatrix} = Ce^x \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2x}. \tag{4.48}$$

If a first order differential equation has the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{4.49}$$

then one can also write the solution as

$$\mathbf{x} = \mathbf{C}e^{At}, \quad e^{At} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n. \tag{4.50}$$