Names: Klaus Kades, Lucas-Raphael Müller, Melanie Schellenberg, Shuhan Xiao 13.11.2021

Assignment 6: "Degree Correlations and Assortativity" — Complex Network Analysis

Problem 6-1 The t-Party Evolving Network Model

1. time evolution of the node degrees:

$$\frac{dk_i}{dt} = 1-(1-\Pi_i)^m$$

$$= \Pi_i \qquad (m=1 : \text{new node only establishes one link})$$

$$= \frac{\eta_i}{t\langle\eta\rangle}$$

$$\left[\Rightarrow dk_i = \frac{\eta_i}{t\langle\eta\rangle} dt \qquad (\text{separation of variables})\right]$$

$$\Rightarrow \int_{m=1}^{k_i(t)} dk_i = \frac{\eta_i}{\langle\eta\rangle} \int_{t_i}^{t} \frac{1}{t'} dt'$$

$t$ ← current time; $t_i$ ← time node $i$ joins

$$\Leftrightarrow k_i(t) - 1 = \frac{\eta_i}{\langle\eta\rangle} [\log(t) - \log(t_i)] \qquad [\text{here: } \log = \ln]$$

$$\Leftrightarrow k_i(t) = \boxed{\frac{\eta_i}{\langle\eta\rangle} \log\left(\frac{t}{t_i}\right) + 1 = k_i(t, \eta_i)} = \text{number of dances a node } i \text{ had at time } t > t_i$$

2. degree distribution $p_k(\eta)$:

In order to derive the cumulative distribution $P(k_i \leq k) = 1 - P(k_i > k)$, first we derive the number of nodes that have a degree $> k$ : $N_{>k}$.

From the condition that a node has a degree $k_i > k$ it follows:

$$k_i(t, \eta_i) = \frac{\eta_i}{\langle\eta\rangle} \log\left(\frac{t}{t_i}\right) + 1 \overset{!}{>} k$$

$$\Leftrightarrow \log\left(\frac{t}{t_i}\right) > \frac{\langle\eta\rangle}{\eta_i}(k-1) \qquad \Big| -1 ; \cdot \frac{\eta_i}{\langle\eta\rangle}$$

$$\Leftrightarrow \frac{t}{t_i} > \exp\left(\frac{\langle\eta\rangle}{\eta_i}(k-1)\right) \qquad \Big| \exp(\ldots)$$

$$\Leftrightarrow t_i < t \cdot \exp\left(-\frac{\langle\eta\rangle}{\eta_i}(k-1)\right) \qquad (\text{note that here } t = N = \text{number of nodes at party})$$

Number of nodes this condition is satisfied for (on average, for all $\eta$)

$$N_{>k} = \int_{\eta_{min}}^{\eta_{max}} \rho(\eta) \cdot t \cdot \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) d\eta \qquad (*)$$

↑ attractiveness distribution

$$\Rightarrow P(k_i \leq k) = 1 - P(k_i > k)$$
$$= 1 - \frac{N_{>k}}{N}$$
$$\overset{(*)}{=} 1 - \frac{1}{N} \cdot \int_{\eta_{min}}^{\eta_{max}} \rho(\eta)\, t \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) d\eta$$
$$\overset{N=t}{=} 1 - \cancel{\tfrac{1}{N}}\cancel{t} \cdot \int_{\eta_{min}}^{\eta_{max}} \rho(\eta) \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) d\eta$$

derivative with respect to $k$:

$$p_k = \frac{dP(k_i \leq k)}{dk}$$
$$= -\int_{\eta_{min}}^{\eta_{max}} \left(-\frac{\langle\eta\rangle}{\eta}\right) \cdot \rho(\eta) \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) d\eta$$
$$\Rightarrow \quad p_k = \int_{\eta_{min}}^{\eta_{max}} \frac{\langle\eta\rangle}{\eta} \rho(\eta) \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) d\eta$$

---

3. $$\rho(\eta) = \begin{cases} \frac{1}{2} & \text{for } \eta = 1 \\ \frac{1}{2} & \text{for } \eta = 2 \end{cases} \quad \Rightarrow \langle\eta\rangle = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 2 = \frac{3}{2}$$

$$\Rightarrow p_k \overset{(2.)}{=} \int_{\eta_{min}}^{\eta_{max}} \frac{\langle\eta\rangle}{\eta} \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) \rho(\eta)\, d\eta$$
$$= \int_{\eta_{min}}^{\eta_{max}} \frac{\langle\eta\rangle}{\eta} \exp\left(-\frac{\langle\eta\rangle}{\eta}(k-1)\right) \left[\frac{1}{2} \cdot (\delta(\eta-1) + \delta(\eta-2))\right] d\eta$$

Dirac Delta function: $\int f(x)\delta(x)dx = f(0)$

$$= \frac{3}{2} \cdot \frac{1}{2} \left[\frac{1}{1} \exp\left(-\frac{\frac{3}{2}}{1}(k-1)\right) + \frac{1}{2} \exp\left(-\frac{\frac{3}{2}}{2}(k-1)\right)\right]$$
$$= \frac{1}{2} \cdot \frac{3}{4} \left[2 \cdot \exp\left(-\frac{3}{2}(k-1)\right) + \exp\left(-\frac{3}{4}(k-1)\right)\right]$$
$$\Rightarrow \boxed{p_k = \frac{3}{4}\left[\exp\left(-\frac{3}{2}(k-1)\right) + \frac{1}{2}\exp\left(-\frac{3}{4}(k-1)\right)\right]}$$