Problem 4-2 Friendship Paradoxon

1. Probability that a node at one of a randomly chosen node's ends has degree k:

$$q_k = \frac{1}{c} \cdot k \cdot p_k$$

(1) Assuming that k is discrete:

Normalisation condition for a probability density function:

$$\sum_{k=k_{min}}^{k_{max}} q_k = 1 \Rightarrow \frac{1}{c} \sum_{k=k_{min}}^{k_{max}} k\, p_k = 1$$

$$\Leftrightarrow \quad c = \sum_{k=k_{min}}^{k_{max}} k\, p_k$$

$$= \sum_{k=k_{min}}^{k_{max}} k \cdot c' \cdot k^{-\gamma} \quad \left(\text{using power law: } p_k = c' k^{-\gamma} \text{ with } c' = \left[\sum_{k=k_{min}}^{k_{max}} k^{-\gamma}\right]^{-1}\right)$$

$$\boxed{= \sum_{k=k_{min}}^{k_{max}} k^{1-\gamma} \cdot \left[\sum_{k'=k_{min}}^{k_{max}} (k')^{-\gamma}\right]^{-1}}$$

(2) Assuming that k is continuous:

Normalisation condition:

$$\int_{k_{min}}^{k_{max}} q(k)\, dk = 1 \Rightarrow \frac{1}{c} \int_{k_{min}}^{k_{max}} k\, p(k)\, dk = 1$$

$$\Leftrightarrow \quad c = \int_{k_{min}}^{k_{max}} k\, p(k)\, dk$$

$$= \int_{k_{min}}^{k_{max}} k \cdot c' \cdot k^{-\gamma} dk \quad \left(\text{power law: } p(k) = c' k^{-\gamma} \text{ with } c' = \left[\int_{k_{min}}^{k_{max}} k^{-\gamma} dk\right]^{-1}\right)$$

$$= c' \int_{k_{min}}^{k_{max}} k^{1-\gamma} dk$$

$$= c' \cdot \left[\frac{1}{2-\gamma} k^{2-\gamma}\right]_{k_{min}}^{k_{max}}$$

$$= c' \cdot \frac{1}{2-\gamma} \left[k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right]$$

$$= \left[\int_{k_{min}}^{k_{max}} k^{-\gamma} dk\right]^{-1} \cdot \frac{1}{2-\gamma} \left[k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right]$$

$$= \left[\left[\frac{1}{-\gamma+1} k^{-\gamma+1}\right]_{k_{min}}^{k_{max}}\right]^{-1} \frac{1}{2-\gamma} \left[k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right]$$

$$\boxed{= \frac{1-\gamma}{2-\gamma} \cdot \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{k_{max}^{1-\gamma} - k_{min}^{1-\gamma}}}$$

Here we also assume that $p_k = 0$ for $k > k_{max}$ and $k < k_{min}$.

2. Average degree of the neighbors of a randomly chosen node $\langle k\rangle_{q_k}$ — Complex Network Analysis

(1) Assuming that k is discrete:

$$\langle k\rangle_{q_k} = \sum_{k=k_{min}}^{k_{max}} k\, q_k = \sum_{k=k_{min}}^{k_{max}} k \cdot \underbrace{\frac{1}{c} k p_k}_{q_k} = \sum_{k=k_{min}}^{k_{max}} \frac{1}{c} k^2 \underbrace{c' k^{-\gamma}}_{p_k}$$

$$\underset{\text{Part 1.(1)}}{=} \underbrace{\left[\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma} \cdot \cancel{c'}\right]^{-1}}_{\frac{1}{c}} \cdot \cancel{c'} \cdot \sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}$$

$$\boxed{= \frac{\sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}}{\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma}}}$$

(2) Assuming that k is continuous

$$\langle k\rangle_{q_k} = \int_{k_{min}}^{k_{max}} k\, q(k)\, dk = \int_{k_{min}}^{k_{max}} k \frac{1}{c} k p(k)\, dk = \int_{k_{min}}^{k_{max}} \frac{1}{c} k^2 c' k^{-\gamma} dk$$

$$\underset{\text{Part 1.(2)}}{=} \underbrace{\left[\cancel{c'} \cdot \frac{1}{2-\gamma}\left[k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right]\right]^{-1}}_{\frac{1}{c}} \int_{k_{min}}^{k_{max}} k^{2-\gamma} dk \cdot \cancel{c'}$$

$$= \frac{2-\gamma}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}} \left[\frac{1}{3-\gamma} k^{3-\gamma}\right]_{k_{min}}^{k_{max}}$$

$$\boxed{= \frac{2-\gamma}{3-\gamma} \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}}$$

3 power-law degree distr. with $N = 10^4$, $\gamma = 2.3$, $k_{min} = 1$, $k_{max} = 1000$

average degree of neighbors:

(1) discrete:

$$\langle k\rangle_{q_k} = \frac{\sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}}{\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma}} \quad (\approx 50.96) \approx \underline{\underline{51.0}}$$

(with Wolfram Alpha)

(2) continuous

$$\langle k\rangle_{q_k} = \frac{2-\gamma}{3-\gamma} \cdot \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}} \approx \underline{\underline{61.2}}$$

average degree

(1) discrete:

$$\langle k\rangle_{p_k} = \sum_{k=k_{min}}^{k_{max}} k\, p_k = c' \sum_{k=k_{min}}^{k_{max}} k^{1-\gamma} = \left[\sum_{k'=k_{min}}^{k_{max}} k'^{-\gamma}\right]^{-1} \sum_{k=k_{min}}^{k_{max}} k^{1-\gamma}$$

$\langle k \rangle_{p_k} = \ldots \approx \underline{\underline{2.5}} = \langle k \rangle$

(2) continuous

$$\langle k \rangle_{p_k} = \int_{k_{min}}^{k_{max}} k\, p(k)\, dk = \int_{k_{min}}^{k_{max}} k \cdot C' \cdot k^{-\gamma}\, dk$$

$$= \underbrace{\left[\int_{k_{min}}^{k_{max}} k^{-\gamma} dk \right]^{-1}}_{C'} \int_{k_{min}}^{k_{max}} k^{1-\gamma} dk$$

$$= \left[\frac{1}{1-\gamma} \left(k_{max}^{1-\gamma} - k_{min}^{1-\gamma} \right) \right]^{-1} \cdot \left[\frac{1}{2-\gamma} \left(k_{max}^{2-\gamma} - k_{min}^{2-\gamma} \right) \right]$$

$$= \frac{1-\gamma}{2-\gamma} \cdot \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{k_{max}^{1-\gamma} - k_{min}^{1-\gamma}}$$

$$\approx \underline{\underline{3.8}}$$

4. Nodes with lower degrees are on average more likely to be connected to high degree nodes than other low degree nodes, which is also explained by the Friendship paradoxon.