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Problem 8_3

$$M(G,C) = \sum_{C=1}^{n_C} \left(\frac{L_C}{L} - \left(\frac{k_C}{2L}\right)^2 \right)$$

1.) $L_{C=A} = 5$ $\quad k_{C=A} = 4+3+3+3 = 13$ $\quad L = 15$

$L_{C=B} = 2$ $\quad k_{C=B} = 2+4+3 = 9$

$L_{C=C} = 3$ $\quad k_{C=C} = 3+2+3 = 8$

$$M(G,C) = \frac{L_{C=A}}{L} - \left(\frac{k_{C=A}}{2L}\right)^2 + \frac{L_{C=B}}{L} - \left(\frac{k_{C=B}}{2L}\right)^2$$

$$+ \frac{L_{C=C}}{L} - \left(\frac{k_{C=C}}{2L}\right)^2$$

$$= \frac{5}{15} - \left(\frac{13}{30}\right)^2 + \frac{2}{15} - \left(\frac{9}{30}\right)^2 + \frac{3}{15} - \left(\frac{8}{30}\right)^2$$

$$= \frac{10}{15} - \frac{169+81+64}{900} = \frac{10}{15} - \frac{314}{900}$$

$$= \frac{300}{450} - \frac{157}{450} = \frac{143}{450} \sim 0{,}317$$

2.) $$M(G,C) = \underbrace{\frac{L_{C_1}}{L} + \frac{L_{C_2}}{L} + \frac{L_{C_3}}{L} + \dots + \frac{L_{C_n}}{L}} - \left(\frac{k_{C_1}}{2L}\right)^2 - \left(\frac{k_{C_2}}{2L}\right)^2 - \dots \left(\frac{k_{C_n}}{2L}\right)^2$$

$$= \frac{L_{C_1} + L_{C_2} + \dots + L_{C_n}}{L} - \frac{k_{C_1}^2 + k_{C_2}^2 + \dots + k_{C_n}^2}{4L^2}$$

$$= \frac{\sum L_C}{L} - \underbrace{\frac{\sum k_C^2}{(2L)^2}}_{>0 \text{ due to square}}$$

for n clusters.

This means the modularity is a sum of all links within the clusters divided by the total amount of links minus a positive number (positive due to the square). As the sum of all links within the clusters is maximally equal to the total amount of links in the cluster in simple graphs:

$$\sum L_c \leq L,$$

the quotient is maximally 1:

$$\frac{\sum L_c}{L} \leq 1$$

Therefore, $M(G,C) = \underbrace{\frac{\sum L_c}{L}}_{\leq 1} - \underbrace{\frac{\sum k_c^2}{(2L)^2}}_{\geq 0} \leq 1$ //