Complex Network Analysis
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Problem 7-4 Random Failures in Uncorrelated Networks

1. Poisson d.: $p_k = e^{-\mu} \frac{\mu^k}{k!}$

From literature: $\langle k \rangle = \mu$ (1)

$\langle k^2 \rangle = \mu(1+\mu)$ (2)

Critical threshold:

$$f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1} \overset{(1),(2)}{=} 1 - \frac{1}{\frac{\cancel{\mu}(1+\mu)}{\cancel{\mu}} - 1} = 1 - \frac{1}{1+\mu-1}$$

$$= \underline{\underline{1 - \frac{1}{\mu}}} \left(= 1 - \frac{1}{\langle k \rangle} = f_c^{ER} \right)$$

2. Discrete exponential d.: $p_k = (1-e^{-\lambda})\, e^{-\lambda k}$

From literature: $\langle k \rangle = (e^{\lambda} - 1)^{-1}$ (3)

$\langle k^2 \rangle = (e^{\lambda}+1)(e^{\lambda}-1)^{-2}$ (4)

Critical threshold:

$$f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1} \overset{(3),(4)}{=} 1 - \left((e^{\lambda}+1)(e^{\lambda}-1)^{-2}(e^{\lambda}-1)^{+1} - 1 \right)^{-1}$$

$$= 1 - \left((e^{\lambda}+1)(e^{\lambda}-1)^{-1} - 1 \right)^{-1} = 1 - \left(\frac{e^{\lambda}+1-(e^{\lambda}-1)}{e^{\lambda}-1} \right)^{-1}$$

$$= 1 - \left(\frac{2}{e^{\lambda}-1} \right)^{-1} = \frac{2-(e^{\lambda}-1)}{2} = \frac{1}{2}(-e^{\lambda}+3)$$

$$= \underline{\underline{\frac{1}{2}(3-e^{\lambda})}}$$

3. Dirac delta d.: $p_k = \delta_{k,k_0} = \begin{cases} 1 & k=k_0 \\ 0 & \text{otherwise} \end{cases}$

From literature: $\langle k \rangle = k_0$

$\langle k^2 \rangle = k_0^2$

Critical threshold:

$$f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1} = 1 - \frac{1}{\frac{k_0^{\cancel{2}}}{\cancel{k_0}} - 1} = \frac{(k_0-1)-1}{k_0-1} = \underline{\underline{\frac{k_0-2}{k_0-1}}}$$

Consequences for the network robustness (towards random failures):

1. The network becomes more robust if $\mu \to \infty$, i.e. if $\langle k \rangle \to \infty$, which is the same result as for random networks. A larger average degree $\langle k \rangle$ improves the robustness, a small $\langle k \rangle$ worsens it.

2. The network becomes more robust towards random failures if $\lambda \to 0$ ($\lim_{\lambda \to 0} f_c = \lim_{\lambda \to 0} \frac{1}{2}(3-e^{\lambda}) = 1$), which leads to $\langle k \rangle \to \infty$ ($\lim_{\lambda \to 0} \frac{1}{\underbrace{e^{\lambda}}_{\to 1}-1} = \infty$), i.e. a large average degree. The network becomes less robust if $\lambda \to \ln(3)$, or if $\langle k \rangle \to \frac{1}{2}$.

3. The network becomes more robust towards random failures for $k_0 = \langle k \rangle \to \infty$ and less robust if $k_0 \to 2$.