Names: Klaus Kades (Uni ID: fw448, Matr.Nr: 3408463)
Lucas-Raphael Müller (Uni ID: al413, Matr.Nr: 3205638)
Melanie Schellenberg (Uni ID: gk400, Matr.Nr: 3146590)
Shuhan Xiao (Uni ID: kg410, Matr.Nr: 3160697)

08.11.2021
Complex Network Analysis

Assignment 2: "Graph Partitions and Random Graphs"

Problem 2-1 Erdős-Rényi Network

1. (i) expected number of links:

$$\langle L \rangle = \sum_{L=0}^{\frac{N(N-1)}{2}} L \underset{\substack{\uparrow \\ \text{(binomial distribution)}}}{p_L} = p \cdot \frac{N(N-1)}{2} = 0.05 \cdot \frac{80(80-1)}{2}$$

$$= \underline{\underline{158}}$$

(ii) expected degree:

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1) = 0.05 \cdot 79$$

$$= \underline{\underline{3.95}}$$

2. Since $\langle k \rangle = 3.95 > 1$ and $\langle k \rangle < \ln N = \ln(80) \approx 4.38$ the network is in the $\underline{\underline{\text{supercritical regime}}}$.

Also: $p = 0.05 > \frac{1}{N} = 0.0125$ and $p < \frac{\ln N}{N} = \frac{\ln(80)}{80} \approx 0.055$.

3. Probability to find $L = 200$ links:

using (binomial) probability distribution:

$$p_{L=200} = \binom{\frac{N(N-1)}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2} - L}$$

$$\overset{N=80,\ L=200}{\underset{p=0.05}{=}} \binom{\frac{80(80-1)}{2}}{200} 0.05^{200} (1-0.05)^{\frac{80(80-1)}{2} - 200}$$

$$= \binom{3160}{200} \cdot 0.05^{200} \cdot 0.95^{2960}$$

$$\approx \underline{\underline{0.000126 \quad \approx 10^{-4}}}$$

4. Probability that node i has degree $k_i = 5$:
using (binomial) degree distribution:

$$p_{k=5} = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

$$= \binom{80-1}{5} 0.05^5 (1-0.05)^{80-1-5}$$

$$= \binom{79}{5} 0.05^5 \cdot 0.95^{74}$$

$$\approx \underline{\underline{0.158}}$$

5. Maximum likelihood estimation for model parameters (N, p):

likelihood function, assuming that we are observing (i.i.d.) degrees $k_1, \dots, k_{12}$ of the nodes: for binomial probability density function $f(k; N, p) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$:

$$L(N,p) = \prod_{i=1}^{n=12} f(k_i; N, p)$$

$$= \prod_{i=1}^{n=12} \binom{N-1}{k_i} p^{k_i} (1-p)^{N-1-k_i}$$

maximum likelihood - argmax of $L(N,p)$ is the same as the argmax of the log-likelihood function:

$$\ln L(N,p) = \sum_{i=1}^{n=12} \ln f(k_i; N, p)$$

$$= \sum_{i=1}^{n=12} \left(\ln \binom{N-1}{k_i} + k_i \ln(p) + (N-1-k_i) \ln(1-p) \right)$$

Find maximum by setting $\frac{\partial \ln L(N)}{\partial N} = 0$, $\frac{\partial \ln L(p)}{\partial p} = 0$:

(i) $$\frac{\partial \ln L(N,p)}{\partial N} = \sum_{i=1}^{n=12} \left(\frac{\partial}{\partial N} \ln \binom{N-1}{k_i} + \ln(1-p) \right) \stackrel{!}{=} 0$$

(ii) $$\frac{\partial \ln L(N,p)}{\partial p} = \sum_{i=1}^{n=12} \left(\frac{k_i}{p} - \frac{(N-1-k_i)}{1-p} \right) \stackrel{!}{=} 0$$

(ii) $\frac{1}{p}\sum_{i=1}^{n} k_i - \frac{1}{1-p}\left(n\cdot(N-1) - \sum_{i=1}^{n} k_i\right) \overset{!}{=} 0$

$\Leftrightarrow \quad (1-p)\sum_{i=1}^{n} k_i - p\left(n(N-1) - \sum_{i=1}^{n} k_i\right) \overset{!}{=} 0$

$\Leftrightarrow \quad \sum_{i=1}^{n} k_i - p\cdot n\cdot(N-1) = 0$

$\Rightarrow \quad \hat{p} = \frac{\sum_{i=1}^{n} k_i}{n\cdot(\hat{N}-1)}$

assuming that $\hat{N} = 12$ nodes we get:

$$\hat{p} = \frac{3+2+2+1+5+1+1+3+1+1+1+1}{12\cdot(12-1)}$$

$$= \frac{22}{12\cdot 11} = \frac{1}{6}$$

$$\approx \underline{\underline{0.167}}$$

(i) since $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ we get:

$$\frac{\partial \ln L(N,p)}{\partial N} = \sum_{i=1}^{n}\left(\frac{\partial}{\partial N}\left[\ln\left(\frac{(N-1)!}{k_i!\,(N-1-k_i)!}\right)\right] + \ln(1-p)\right)$$

$$= \sum_{i=1}^{n}\left(\frac{\partial}{\partial N}\left[\ln(N-1)! - \cancel{\ln k_i!} - \ln(N-1-k_i)!\right] + \ln(1-p)\right)$$

$$= \left\{\sum_{i=1}^{n}\left(\frac{\partial}{\partial N}\left[\ln(N-1)! - \ln(N-1-k_i)!\right]\right\} + n\cdot\ln(1-p)$$

$$\overset{!}{=} 0$$