Problem 5-2 Role of Preferential Attachment

Probability that a link of a new node connects to node i:

$$\Pi = \frac{1}{m_0+t-1} \qquad (1)$$

1. $\frac{dk_i}{dt} = 1 - \underbrace{(1-\Pi)^m}_{\text{probability not to get any of the } m \text{ new links when a node is added}}$ (as derived in the lecture)

Taylor expansion of $(1-\Pi)^m$ around $\Pi \approx 0$:

$$(1-\Pi)^m \approx (1-\Pi)^m\Big|_{\Pi=0} + \frac{d}{d\Pi}(1-\Pi)^m\Big|_{\Pi=0} \cdot (\Pi - 0)$$

$$= 1 - m(1-\Pi)^{m-1}\Big|_{\Pi=0} \cdot \Pi$$

$$= 1 - m\Pi \qquad (2)$$

$$\Rightarrow \frac{dk_i}{dt} \overset{(2)}{\approx} 1 - (1 - m\Pi)$$

$$= m\Pi$$

$$\overset{(1)}{=} m \cdot \frac{1}{m_0+t-1} = \frac{m}{m_0+t-1}$$

$\square$

2. $\frac{dk_i}{dt} = \frac{m}{m_0+t-1}$

$\Leftrightarrow dk_i = \frac{m}{m_0+t-1} dt$ (separation of variables)

$\Leftrightarrow \int_{m}^{k_i(t)} dk_i = m \int_{t_i}^{t} \frac{1}{m_0+t'-1} dt'$ (using initial conditions: $k_i(t_i) = m$) ($t_i \leftarrow$ node i is added at $t'=t_i$)

$\Leftrightarrow k_i'\Big|_{k_i'=m}^{k_i(t)} = m \cdot \log(m_0+t'-1)\Big|_{t'=t_i}^{t'=t}$

$\Leftrightarrow k_i(t) - m = m \cdot [\log(m_0+t-1) - \log(m_0+t_i-1)]$

$\Leftrightarrow k_i(t) = m \cdot \left[1 + \log\left(\frac{m_0+t-1}{m_0+t_i-1}\right)\right]$

$\square$

3. $k_i(t) < k$

$\Leftrightarrow m\left[1 + \log\left(\frac{m_0+t-1}{m_0+t_i-1}\right)\right] < k$ $\quad | \cdot \frac{1}{m}, -1 \quad (m \neq 0)$

$\Leftrightarrow \log(m_0+t-1) - \log(m_0+t_i-1) < \frac{k}{m} - 1$ $\quad | + \log(m_0+t_i-1)$

$\Leftrightarrow \log(m_0+t-1) < \frac{k}{m} + \log(m_0+t_i-1) - 1$ $\quad | -\frac{k}{m} + 1$

$\Leftrightarrow \log(m_0+t-1) - \frac{k}{m} + 1 < \log(m_0+t_i-1)$ $\quad | \exp(\ldots)$

$\Leftrightarrow (m_0+t-1) \cdot \exp(-\frac{k}{m}+1) < m_0+t_i-1$ $\quad | -m_0, +1$

$\Leftrightarrow 1 - m_0 + (m_0+t-1)\exp(1-\frac{k}{m}) < t_i$

$\Leftrightarrow t_i > 1 - m_0 + (m_0+t-1)\exp(1-\frac{k}{m})$

$\square$