Problem 3-2 Parametrized Random Networks

$p = \frac{a}{N^z}$, $a > 0$, $z \geq 0$ (also assuming $N > 0$)

1.

$\langle k \rangle = p(N-1) = \frac{a}{N^z}(N-1)$

- $a = 0.5$, $z = 1$ $\to$ $\langle k \rangle = \frac{1}{2} \cdot \frac{1}{N}(N-1) = \frac{1}{2}(1 - \frac{1}{N})$

$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} \frac{1}{2}(1 - \underbrace{\frac{1}{N}}_{\to 0}) = \frac{1}{2}$

- $a = 2$, $z = 1$ $\to$ $\langle k \rangle = \frac{2}{N}(N-1) = 2(1 - \frac{1}{N})$

$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} 2(1 - \underbrace{\frac{1}{N}}_{\to 0}) = 2$

- $a > 0$, $z = 2$ $\to$ $\langle k \rangle = \frac{a}{N^2}(N-1) = a(\frac{1}{N} - \frac{1}{N^2})$

$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} a \underbrace{\frac{(N-1)}{N^2}}_{\to 0} = 0$

- $a > 0$, $z = 0.5$ $\to$ $\langle k \rangle = \frac{a}{N^{1/2}}(N-1) = a(N^{1/2} - N^{-1/2})$

$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} a(\underbrace{N^{1/2}}_{\to\infty} - \underbrace{N^{-1/2}}_{\to 0}) = \infty$

In a random network a giant component emerges if $\langle k \rangle \geq 1$, so in the second ($a=2, z=1$) and fourth ($a>0, z=0.5$) case does the random network contain a giant component for $N \to \infty$.

2. $\langle k \rangle = a(N^{1-z} - N^{-z}) = a\frac{(N-1)}{N^z}$

We can regard different cases for different values of $a > 0$, $z \geq 0$:

$\underline{z = 0}$: $\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} a(N-1) = \infty$

$\underline{0 < z < 1}$: $\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} a(\underbrace{N^{\overbrace{1-z}^{>0, <1}}}_{\to\infty} - \underbrace{N^{-z}}_{\to 0}) = \infty$

$\underline{z = 1}$: $\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} a\frac{(N-1)}{N} = \lim_{N\to\infty} a(1 - \underbrace{\frac{1}{N}}_{\to 0}) = a$

$\underline{z > 1}$: $\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} a(\underbrace{N^{\overbrace{1-z}^{<0}}}_{\to 0} - \underbrace{N^{\overbrace{-z}^{<-1}}}_{\to 0}) = 0$

3. Generally, a random network is critical if $\langle k \rangle = 1$.

Since we have already shown that

$\lim_{N \to \infty} \langle k \rangle = a$ for $z = 1$

it follows that $\lim_{N \to \infty} \langle k \rangle = 1$ for $a = 1$ and $z = 1$

$\left(\lim_{N \to \infty} \langle k \rangle = \lim_{N \to \infty} p(N-1) = \lim_{N \to \infty} \frac{1}{N^1}(N-1) = 1 \right)$

The network is subcritical for $a = 0.5, z = 1$, since $\lim_{N \to \infty} \langle k \rangle = \frac{1}{2} < 1$,

and for $a > 0, z = 0$, since $\lim_{N \to \infty} \langle k \rangle < 1$.

The network is supercritical for the second case $a = 2, z = 1$

since $\lim_{N \to \infty} \langle k \rangle = 2 > 1$ ~~and~~ $\cancel{\lim_{N \to \infty} \langle k \rangle < \ln N \to \infty}$

$\lim_{N \to \infty} \langle k \rangle < \lim_{N \to \infty} \ln N = \infty$.

The network is in the connected regime for the last case $a > 0, z = 0.5$.

$(\langle k \rangle > \ln N$ for $N \to \infty)$

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