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Complex Network Analysis

Assignment 4 "Scale-Free Networks"

Problem 4-1 Power Laws

1. Network (b) is approximately scale-free.
In the log-log plot the data points of the degree distribution function p_k for network (b) follow a linear function, whereas the data points for network (a) have a plateau for small k at first ($k < 10^1$).
p_k of network (b) can therefore be better described by the relation $p_k \sim k^{-\gamma}$ than (a), p_k follows a power law, which ~~is~~ always apply for scale-free networks.
$\log p_k \sim -\gamma \log k$

2. Estimate γ using formula from slide 4-28:
$$\gamma = 1 + N\left[\sum_{i=1}^{N} \ln\left(\frac{k_i}{k_{min} - \frac{1}{2}}\right)\right]^{-1}$$
using $k_{min} = 10$, $N = 20$ (data from twenty nodes are available),
and $k_i \in \{16, 17, \dots, 22, 10\}$
$\Rightarrow \underline{\underline{\gamma \approx 2.53}}$
Calculate error of estimation σ:
$$\sigma = \frac{\gamma - 1}{\sqrt{N}} \approx \frac{2.53 - 1}{\sqrt{N}} \approx \underline{\underline{0.34}}$$

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Problem 4-2 Friendship Paradoxon

1. Probability that a node at one of a randomly chosen node's ends has degree k:

$q_k = \frac{1}{c} k \times p_k$

$q_k = \frac{k p_k}{\sum_{k'} k' p_{k'}} = \frac{k p_k}{\langle k \rangle}$ $= c$

$q_k = \frac{1}{c} \cdot k \cdot p_k$

(1) Assuming that k is discrete:

Normalisation condition for a probability density function:

$$\sum_{k=k_{min}}^{k_{max}} q_k = 1 \Rightarrow \frac{1}{c} \sum_{k=k_{min}}^{k_{max}} k p_k = 1$$

$$\Leftrightarrow \quad c = \sum_{k=k_{min}}^{k_{max}} k p_k$$

$$= \sum_{k=k_{min}}^{k_{max}} k \cdot c' \cdot k^{-\gamma}$$ (using power law: $p_k = c' k^{-\gamma}$ with $c' = [\sum_{k=k_{min}}^{k_{max}} k^{-\gamma}]^{-1}$)

$$\boxed{= \sum_{k=k_{min}}^{k_{max}} k^{1-\gamma} \cdot \left[\sum_{k'=k_{min}}^{k_{max}} (k')^{-\gamma}\right]^{-1}}$$

(2) Assuming that k is continuous:

Normalisation condition:

$$\int_{k_{min}}^{k_{max}} q(k)\, dk = 1 \Rightarrow \frac{1}{c} \int_{k_{min}}^{k_{max}} k\, p(k)\, dk = 1$$

$$\Leftrightarrow \quad c = \int_{k_{min}}^{k_{max}} k\, p(k)\, dk$$

$$= \int_{k_{min}}^{k_{max}} k \cdot c' \cdot k^{-\gamma} dk$$ (power law: $p(k) = c' k^{-\gamma}$ with $c' = [\int_{k_{min}}^{k_{max}} k^{-\gamma} dk]^{-1}$)

$$= c' \int_{k_{min}}^{k_{max}} k^{1-\gamma} dk$$

$$= c' \cdot \left[\frac{1}{2-\gamma} k^{2-\gamma}\right]_{k_{min}}^{k_{max}}$$

$$= c' \cdot \frac{1}{2-\gamma} [k_{max}^{2-\gamma} - k_{min}^{2-\gamma}]$$

$$= \left[\int_{k_{min}}^{k_{max}} k^{-\gamma} dk\right]^{-1} \cdot \frac{1}{2-\gamma} [k_{max}^{2-\gamma} - k_{min}^{2-\gamma}]$$

$$= \left[\left[\frac{1}{-\gamma+1} k^{-\gamma+1}\right]_{k_{min}}^{k_{max}}\right]^{-1} \frac{1}{2-\gamma} [k_{max}^{2-\gamma} - k_{min}^{2-\gamma}]$$

$$\boxed{= \frac{1-\gamma}{2-\gamma} \cdot \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{k_{max}^{1-\gamma} - k_{min}^{1-\gamma}}}$$

Here we also assume that $p_k = 0$ for $k > k_{max}$ and $k < k_{min}$.

$$\langle k \rangle = \sum_k k q_k = \sum_k k \cdot \frac{k p_k}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

2. Average degree of the neighbors of a randomly chosen node $\langle k \rangle_{q_k}$ Complex Network Analysis

(1) Assuming that k is discrete:

$$\langle k \rangle_{q_k} = \sum_{k=k_{min}}^{k_{max}} k\, q_k = \sum_{k=k_{min}}^{k_{max}} k \cdot \underbrace{\frac{1}{c} k\, p_k}_{q_k} = \sum_{k=k_{min}}^{k_{max}} \frac{1}{c} k^2 \underbrace{c' k^{-\gamma}}_{p_k}$$

$$\underset{\text{Part 1.(1)}}{=} \underbrace{\left[\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma} \cdot \cancel{c'}\right]^{-1}}_{\frac{1}{c}} \cdot \cancel{c'} \cdot \sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}$$

$$\boxed{= \frac{\sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}}{\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma}}} \rightarrow \frac{\langle k^2 \rangle}{\langle k \rangle}$$

$$\langle k_c \rangle = \langle k \rangle + \frac{\sigma_k^2}{\langle k \rangle}$$

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2$$

(2) Assuming that k is continuous

$$\langle k \rangle_{q_k} = \int_{k_{min}}^{k_{max}} k\, q(k)\, dk = \int_{k_{min}}^{k_{max}} k \frac{1}{c} k\, p(k)\, dk = \int_{k_{min}}^{k_{max}} \frac{1}{c} k^2 c' k^{-\gamma} dk$$

$$\underset{\text{Part 1.(2)}}{=} \underbrace{\left[\cancel{c'} \cdot \frac{1}{2-\gamma} \left[k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right]\right]^{-1}}_{\frac{1}{c}} \int_{k_{min}}^{k_{max}} k^{2-\gamma} dk \cdot \cancel{c'}$$

$$= \frac{2-\gamma}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}} \left[\frac{1}{3-\gamma} k^{3-\gamma}\right]_{k_{min}}^{k_{max}}$$

$$\boxed{= \frac{2-\gamma}{3-\gamma} \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}}$$

3 power-law degree distr. with $N = 10^4$, $\gamma = 2.3$, $k_{min} = 1$, $k_{max} = 1000$

average degree of neighbors:

use $\langle k^n \rangle$, $c = (\gamma-1) k_{min}^{\gamma-1}$

$\langle k^2 \rangle \approx 231.94$

$\langle k \rangle = 3.78$

(1) discrete:

$$\langle k \rangle_{q_k} = \frac{\sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}}{\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma}} \quad (\approx 50.96) \approx \underline{\underline{51.0}}$$

(with Wolfram Alpha)

$$\langle k_c \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} = 61.234$$

(2) continuous

$$\langle k \rangle_{q_k} = \frac{2-\gamma}{3-\gamma} \cdot \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}} \approx \underline{\underline{61.2}}$$

average degree

(1) discrete:

$$\langle k \rangle_{p_k} = \sum_{k=k_{min}}^{k_{max}} k\, p_k = c' \sum_{k=k_{min}}^{k_{max}} k^{1-\gamma} = \left[\sum_{k'=k_{min}}^{k_{max}} k'^{1-\gamma}\right]^{-1} \sum_{k=k_{min}}^{k_{max}} k^{1-\gamma}$$

$\langle k \rangle_{pk} = \ldots \approx 2.5 = \langle k \rangle$

(2) continuous

$$\langle k \rangle_{pk} = \int_{k_{min}}^{k_{max}} k\, p(k)\, dk = \int_{k_{min}}^{k_{max}} k \cdot C' \cdot k^{-\gamma}\, dk$$

$$= \underbrace{\left[\int_{k_{min}}^{k_{max}} k^{-\gamma} dk\right]^{-1}}_{C'} \int_{k_{min}}^{k_{max}} k^{1-\gamma} dk$$

$$= \left[\frac{1}{1-\gamma}\left(k_{max}^{1-\gamma} - k_{min}^{1-\gamma}\right)\right]^{-1} \cdot \left[\frac{1}{2-\gamma}\left(k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right)\right]$$

$$= \frac{1-\gamma}{2-\gamma} \cdot \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{k_{max}^{1-\gamma} - k_{min}^{1-\gamma}}$$

≈ 3.8

4. Nodes with lower degrees are on average more likely to be connected to high degree nodes than other low degree nodes, which is also explained by the Friendship paradoxon.

4-3

$\gamma \approx 3.54$ ($\gamma = 1.0$) → highly depends on kmin ~~the~~

use powerlaw package

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use np.random choice
always normalize weights
→ sum up to 1