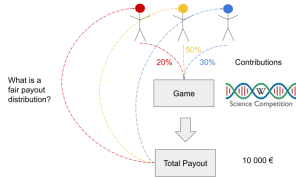


Interpretable Machine Learning

Shapley Values



Learning goals

- Learn what game theory is
- Understand the concept behind cooperative games
- Understand the Shapley value in game theory

COOPERATIVE GAMES IN GAME THEORY

► Shapley (1951)

- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors / agents with gains and losses of quantifiable utility value

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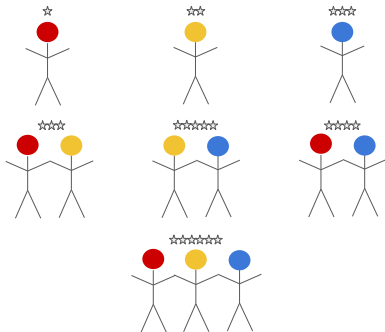
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- We call the individual payout per player $\phi_j, j \in P$ (later: Shapley value)

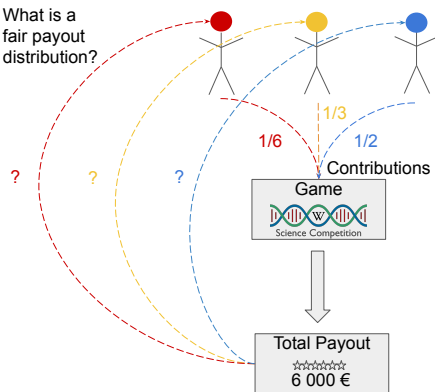
COOPERATIVE GAMES WITHOUT INTERACTIONS

Players do not interact
(payouts ☆ add up in each coalition)

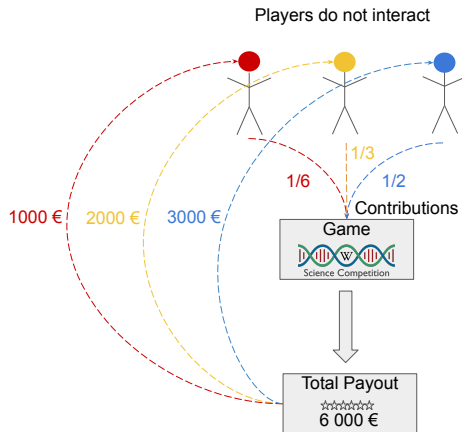
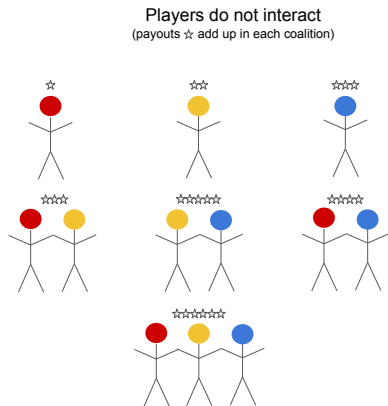


What is a fair payout distribution?

Players do not interact

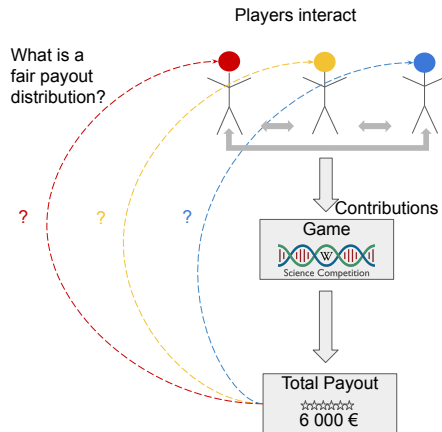
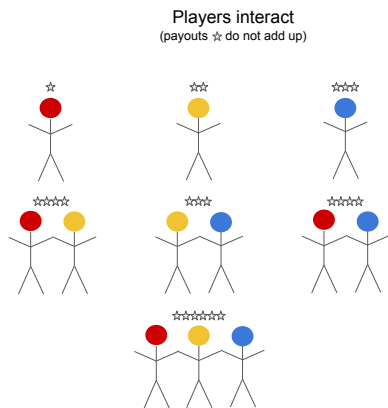


COOPERATIVE GAMES WITHOUT INTERACTIONS



⇒ Fair Payouts are Trivial Without Interactions

COOPERATIVE GAMES WITH INTERACTIONS

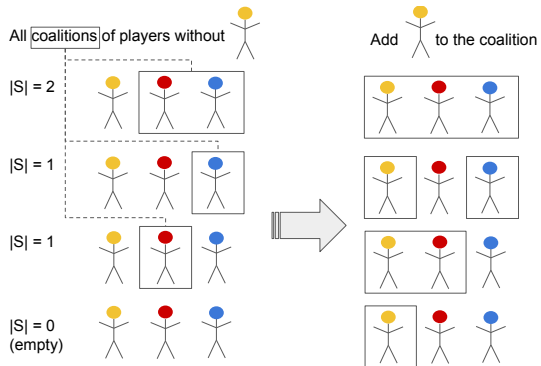


⇒ Unclear how to fairly distribute payouts when players interact

COOPERATIVE GAMES WITH INTERACTIONS

Question: What is a fair payout for player “yellow”?

Idea: Compute marginal contribution of the player of interest across different coalitions

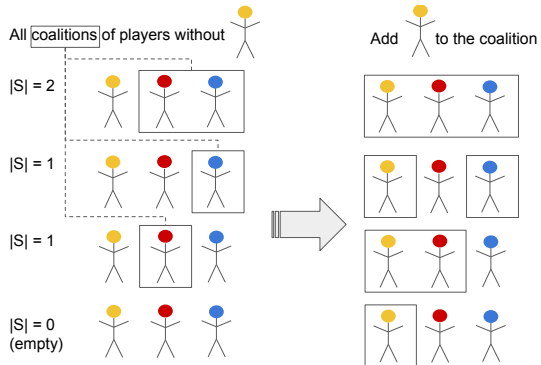


- Compute the total payout of each coalition
- Compute difference in payouts for each coalition with and without player “yellow” (= marginal contribution)
- Average marginal contributions using appropriate weights

COOPERATIVE GAMES WITH INTERACTIONS

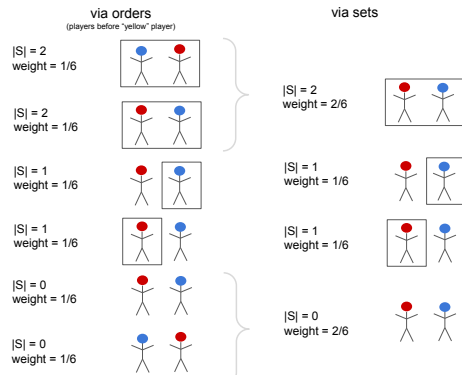
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Note: Each marginal contribution is weighted w.r.t. number of possible orders of its coalition

\leadsto More players in $S \Rightarrow$ more orderings of S



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SHAPLEY VALUE - SET DEFINITION

This idea refers to the **Shapley value** which assigns a payout value to each player according to its marginal contribution in all possible coalitions.

- Let $v(S \cup \{j\}) - v(S)$ be the marginal contribution of player j to coalition S
 \rightsquigarrow measures how much a player j increases the value of a coalition S

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- Average marginal contributions for all possible coalitions $S \subseteq P \setminus \{j\}$
 \rightsquigarrow order of how players join the coalition matters \Rightarrow different weights depending on size of S
- Shapley value via **set definition** (weighting via multinomial coefficient):

$$\phi_j = \sum_{S \subseteq P \setminus \{j\}} \frac{|S|!(|P| - |S| - 1)!}{|P|!} (v(S \cup \{j\}) - v(S))$$

SHAPLEY VALUE - ORDER DEFINITION

The Shapley value was introduced as summation over sets $S \subseteq P \setminus \{j\}$, but it can be equivalently defined as a summation of all orders of players:

$$\phi_j = \frac{1}{|P|!} \sum_{\tau \in \Pi} (v(S_j^\tau \cup \{j\}) - v(S_j^\tau))$$

- Π : All possible orders of players (we have $|P|!$ in total)

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- S_j^τ : Set of players before player j in order $\tau = (\tau^{(1)}, \dots, \tau^{(p)})$ where $\tau^{(i)}$ is i -th element
 - \Rightarrow Example: Players 1, 2, 3 $\Rightarrow \Pi = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$
 - \rightsquigarrow For order $\tau = (2, 1, 3)$ and player of interest $j = 3 \Rightarrow S_j^\tau = \{2, 1\}$
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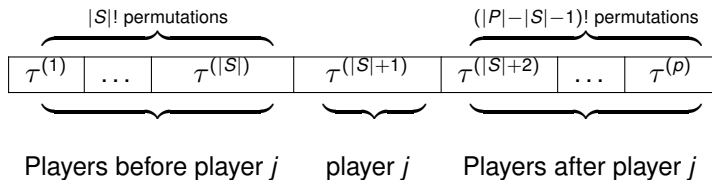
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 - \rightsquigarrow For order $\tau = (3, 1, 2)$ and player of interest $j = 3 \Rightarrow S_j^\tau = \emptyset$
- Order definition: Marginal contribution of orders that yield set $S = \{\textcolor{red}{1}, \textcolor{red}{2}\}$ is summed twice
 - \rightsquigarrow In set definition, it has the weight $\frac{2!(3-2-1)!}{3!} = \frac{2 \cdot 0!}{6} = \frac{2}{6}$

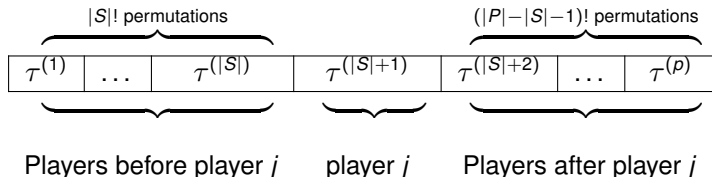
SHAPLEY VALUE - COMMENTS ON ORDER DEFINITION

- Order and set definition are equivalent
- Reason: The number of orders which yield the same coalition S is $|S|!(|P| - |S| - 1)!$
 - \Rightarrow There are $|S|!$ possible orders of players within coalition S
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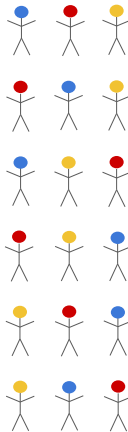
- Relevance of the order definition: Approximate Shapley values by sampling permutations
 \rightsquigarrow randomly sample a fixed number of M permutations and average them:

$$\phi_j = \frac{1}{M} \sum_{\tau \in \Pi_M} (v(S_j^\tau \cup \{j\}) - v(S_j^\tau))$$

where $\Pi_M \subset \Pi$ is a random subset of Π containing only M orders of players

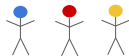
WEIGHTS FOR MARGINAL CONTRIBUTION - ILLUSTRATION

$|P|! = 6$ orders



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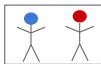
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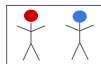
via orders

(players before "yellow" player)

$|S| = 2$
weight = $1/6$



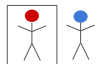
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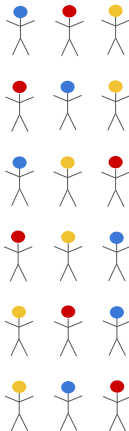


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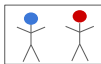
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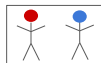
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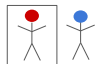
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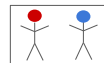


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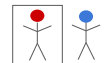
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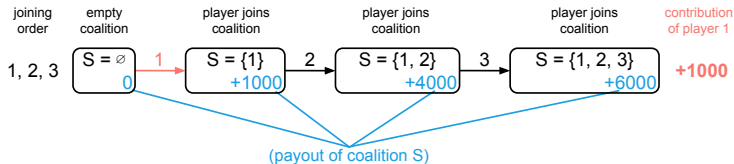


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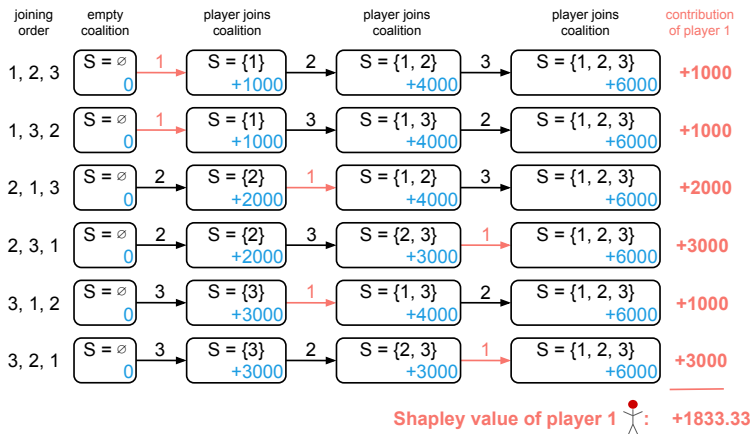
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- Shapley value of player j is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions



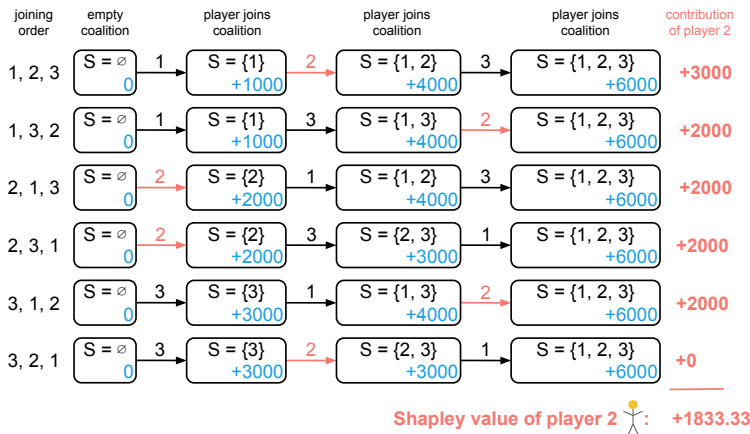
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- Measure and average the difference in payout after player 1 enters the coalition



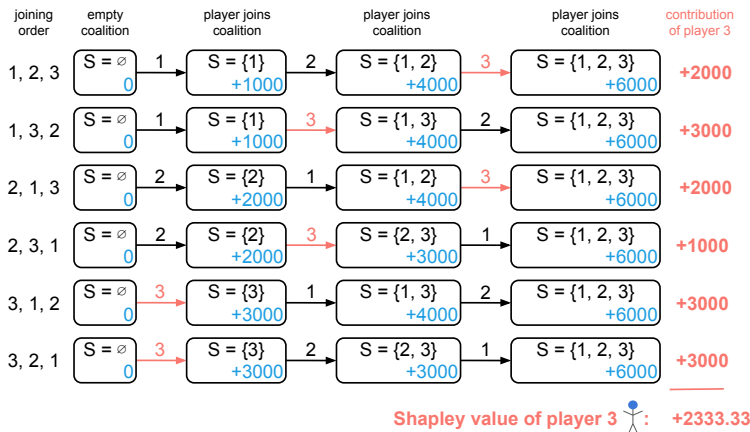
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- Measure and average the difference in payout after player 2 enters the coalition



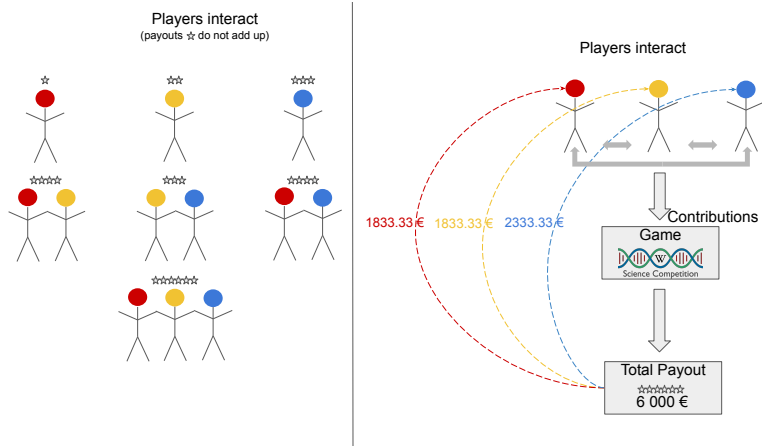
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AXIOMS OF FAIR PAYOUTS

Why is this a fair payout solution?

One possibility to define fair payouts are the following axioms for a given value function v :

- **Efficiency:** Player contributions add up to the total payout of the game: $\sum_{j=1}^p \phi_j = v(P)$

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- **Additivity:** For a game v with combined payouts $v(S) = v_1(S) + v_2(S)$, the payout is the sum of payouts: $\phi_{j,v} = \phi_{j,v_1} + \phi_{j,v_2}$