Solution 1:

a) Calculation of Pearson correlation coefficient of x_1 and x_2

$$\rho(x_1, x_2) = \frac{\sum_{i=1}^{n}(x_1^{(i)} - \overline{x}_1)(x_2^{(i)} - \overline{x}_2)}{\sqrt{\sum_{i=1}^{n}(x_1^{(i)} - \overline{x}_1)^2}\sqrt{\sum_{i=1}^{n}(x_2^{(i)} - \overline{x}_2)^2}}$$

given the dataset

	1	2	3	4	5	6	7	8	9	$\sum_{i=1}^{n}$
y	-7.79	-5.37	-4.08	-1.97	0.02	2.05	1.93	2.16	2.13	-10.92
x_1	-1.00	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75	1.00	0
x_2	0.95	0.57	0.29	-0.03	0.02	0.08	0.23	0.54	0.98	3.63

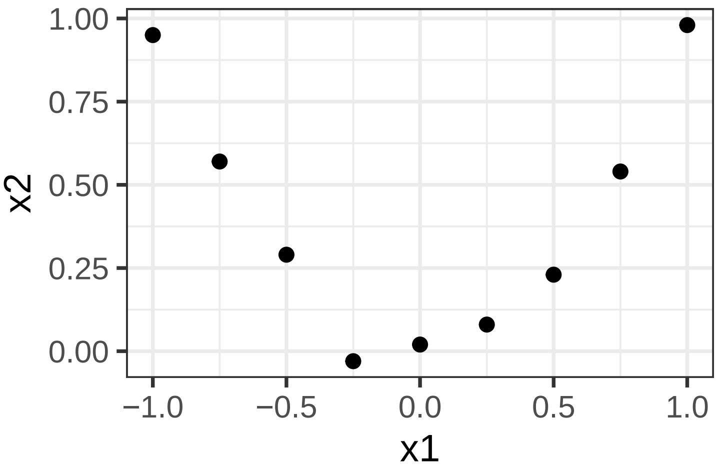
The individual differences to the means are

	1	2	3	4	5	6	7	8	9
$x_1^{(i)} - \overline{x}_1$	-1.00	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75	1.00
$x_2^{(i)} - \overline{x}_2$	0.55	0.17	-0.11	-0.43	-0.38	-0.32	-0.17	0.14	0.58

$$\begin{aligned}\rho(x_1, x_2) &= \frac{\sum_{i=1}^{n}(x_1^{(i)} - \overline{x}_1)(x_2^{(i)} - \overline{x}_2)}{\sqrt{\sum_{i=1}^{n}(x_1^{(i)} - \overline{x}_1)^2}\sqrt{\sum_{i=1}^{n}(x_2^{(i)} - \overline{x}_2)^2}} \\ &= \frac{-0.574 + -0.125 + 0.057 + 0.108 + 0 + -0.081 + -0.087 + 0.103 + 0.577}{2.086} = \frac{0.05}{2.086} = 0.002\end{aligned}$$

The Pearson correlation coefficient is close to 0 $\Rightarrow$ there is **no linear** relationship between x_1 and x_2.

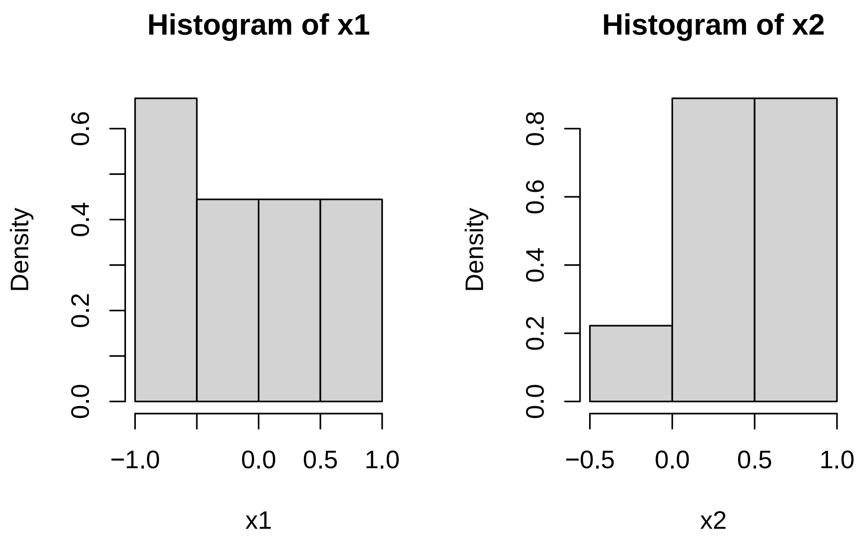
b) The scatter plot reveals that there is a strong non-linear/quadratic relationship between x_1 and x_2. The Pearson correlation coefficients is not suitable for detecting non-linear relationships.



⇒ More suitable: **Mutual Information (MI)**

$$MI(x_1;x_2) = \mathbb{E}_{p(x_1,x_2)}\left[log\left(\frac{p(x_1,x_2)}{p(x_1)p(x_2)}\right)\right] = \sum_{x_1}\sum_{x_2} p(x_1,x_2) log\left(\frac{p(x_1,x_2)}{p(x_1)p(x_2)}\right)$$

Problem: distribution needed.
Solution: e.g. histograms with Gaussian kernel:



Now taking the mean values as replacement for the values in x_1 and x_2:

	1	2	3	4	5	6	7	8	9
$x_1^\star$	-0.75	-0.75	-0.75	-0.25	-0.25	0.25	0.25	0.75	0.75
$x_2^\star$	0.75	0.75	0.25	-0.25	0.25	0.25	0.25	0.75	0.75

Table with joint and marginal distribution:

$x_1^\star$ / $x_2^\star$	-0.25	0.25	0.75	p_{x_1}
-0.75	0.00	0.11	0.22	0.33
-0.25	0.11	0.11	0.00	0.22
0.25	0.00	0.22	0.00	0.22
0.75	0.00	0.00	0.22	0.22
p_{x_2}	0.11	0.44	0.44	1.00

Now we can calculate the approximate MI:

$$\begin{aligned}
MI(x_1^\star;x_2^\star) = &\sum_{x_1^\star}\sum_{x_2^\star} p(x_1^\star,x_2^\star)\, log\left(\frac{p(x_1^\star,x_2^\star)}{p(x_1^\star)p(x_2^\star)}\right)\\
&= 0\, log\left(\frac{0}{0.33\cdot 0.11}\right) + 0.11\, log\left(\frac{0.11}{0.33\cdot 0.44}\right) + 0.22\, log\left(\frac{0.22}{0.33\cdot 0.44}\right)\\
&\quad + 0.11\, log\left(\frac{0.11}{0.22\cdot 0.11}\right) + 0.11\, log\left(\frac{0.11}{0.22\cdot 0.44}\right) + 0\, log\left(\frac{0}{0.22\cdot 0.44}\right)\\
&\quad + 0\, log\left(\frac{0}{0.22\cdot 0.11}\right) + 0.22\, log\left(\frac{0.22}{0.22\cdot 0.44}\right) + 0\, log\left(\frac{0}{0.22\cdot 0.44}\right)\\
&\quad + 0\, log\left(\frac{0}{0.22\cdot 0.11}\right) + 0\, log\left(\frac{0}{0.22\cdot 0.44}\right) + 0.22\, log\left(\frac{0.22}{0.22\cdot 0.44}\right)\\
&= 0.603
\end{aligned}$$

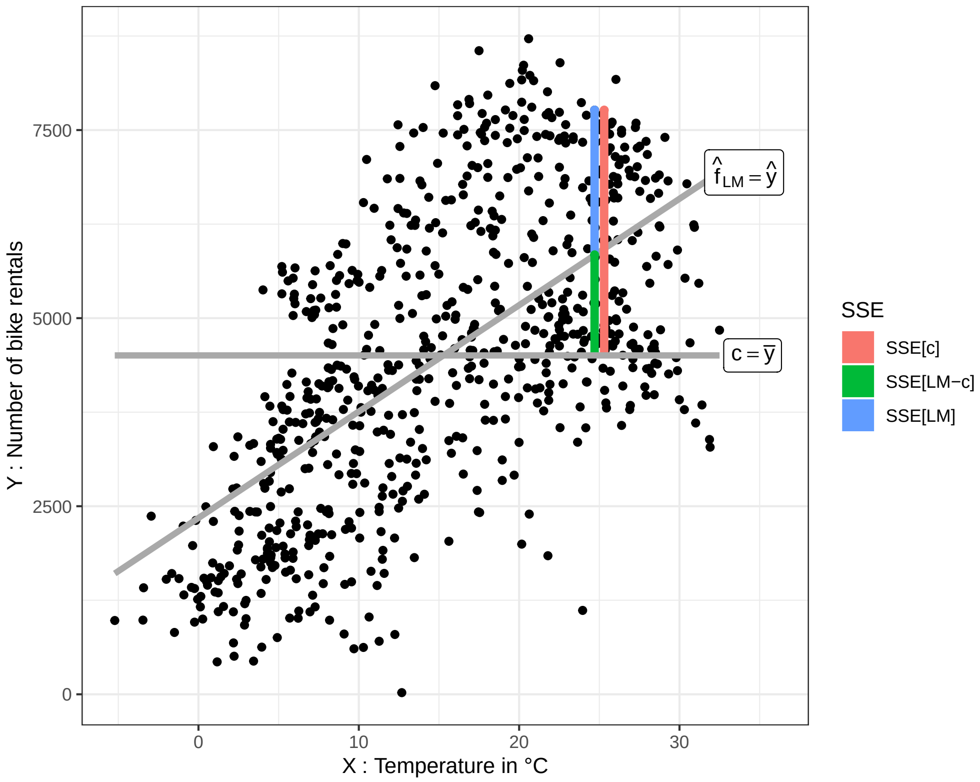
⇒ MI shows that there is a dependency.

Solution 2:

Recall that the formula for the coefficient of determination R^2 is:

$$R^2 = 1 - \frac{SSE_{LM}}{SSE_c} = 1 - \frac{\sum_{i=1}^n (y^{(i)} - \hat{f}_{LM}(x^{(i)}))^2}{\sum_{i=1}^n (y^{(i)} - \bar{y})^2} = 1 - \frac{\sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)})^2}{\sum_{i=1}^n (y^{(i)} - \bar{y})^2}$$

where $SSE_{LM} = \sum_{i=1}^n (y^{(i)} - \hat{f}_{LM}(x^{(i)}))^2$ is the sum of squares due to regression (error) and $SSE_c = \sum_{i=1}^n (y^{(i)} - \bar{y})^2$ is the total sum of squares.



First it is shown that

$$R^2 = 1 - \frac{SSE_{LM}}{SSE_c} = 1 - \frac{\sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)})^2}{\sum_{i=1}^n (y^{(i)} - \bar{y})^2} = \frac{\sum_{i=1}^n (\hat{y}^{(i)} - \bar{y})^2}{\sum_{i=1}^n (y^{(i)} - \bar{y})^2} = \frac{SSE_{LM-c}}{SSE_c} \tag{1}$$

Note that

$$\sum_{i=1}^n (y^{(i)} - \bar{y})^2 = \sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)})^2 + \sum_{i=1}^n (\hat{y}^{(i)} - \bar{y})^2. \tag{2}$$

Proof:

$$\begin{aligned}
\sum_{i=1}^n (y^{(i)} - \bar{y})^2 &= \sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)} + \hat{y}^{(i)} - \bar{y})^2 \\
&= \sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)})^2 + (\hat{y}^{(i)} - \bar{y})^2 + 2(y^{(i)} - \hat{y}^{(i)})(\hat{y}^{(i)} - \bar{y}) \\
&= \sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)})^2 + \sum_{i=1}^n (\hat{y}^{(i)} - \bar{y})^2 + 2\sum_{i=1}^n (y^{(i)} - \hat{y}^{(i)})(\hat{y}^{(i)} - \bar{y})
\end{aligned}$$

It remains to show that

$$\begin{aligned} 2\sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})(\hat{y}^{(i)}-\bar{y}) &= 0 \\ \sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})\hat{y}^{(i)} - \sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})\bar{y} &= 0 \\ \bar{y}\sum_{i=1}^{n} y^{(i)}-\hat{y}^{(i)} &= 0 \\ \sum_{i=1}^{n} y^{(i)}-\hat{y}^{(i)} &= 0 \end{aligned}$$

where we have used the fact that $\sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})\hat{y}^{(i)} = 0$ as the residuals $(y^{(i)}-\hat{y}^{(i)})$ and $\hat{y}^{(i)}$ are not correlated.

(proof of (2)) □

It follows:

$$\begin{aligned} R^2 = 1 - \frac{\sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} &= \frac{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2 - \sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} \\ = \frac{\sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})^2 + \sum_{i=1}^{n}(\hat{y}^{(i)}-\bar{y})^2 - \sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} &= \frac{\sum_{i=1}^{n}(\hat{y}^{(i)}-\bar{y})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} \end{aligned}$$

(proof of (1)) □

And further:

$$R^2 = \frac{\sum_{i=1}^{n}(\hat{y}^{(i)}-\bar{y})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} = \frac{\sum_{i=1}^{n}(\hat{\beta}_0+\hat{\beta}_1 x^{(i)} - (\hat{\beta}_0+\hat{\beta}_1\bar{x}))^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} = \frac{\hat{\beta}_1^2\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2}$$

Now, starting with ρ^2, we can write:

$$\begin{aligned} \rho^2 &= \left(\frac{\sum_{i=1}^{n}(x^{(i)}-\bar{x})(y^{(i)}-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2}\sqrt{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2}}\right)^2 \\ &= \frac{\left(\sum_{i=1}^{n}(x^{(i)}-\bar{x})(y^{(i)}-\bar{y})\right)^2}{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} \\ &= \frac{\left(\sum_{i=1}^{n}(x^{(i)}-\bar{x})(y^{(i)}-\bar{y})\right)^2}{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2}\frac{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2}{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2} \\ &= \left(\frac{\sum_{i=1}^{n}(x^{(i)}-\bar{x})(y^{(i)}-\bar{y})}{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2}\right)^2\frac{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} \\ &= \hat{\beta}_1^2\frac{\sum_{i=1}^{n}(x^{(i)}-\bar{x})^2}{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2} = R^2 \end{aligned}$$

Hence, we have shown that $R^2 = \rho^2$, which completes the proof. Note that this result is valid only for simple linear regression, where there is only one independent variable. For multiple regression, the coefficient of determination is defined differently and does not necessarily equal the square of the Pearson correlation coefficient.

□

Solution 3:

Problem: The function $f(\mathbf{x}) = 2x_1 + 3x_2 - x_1|x_2|$ is not differentiable for $x_2 = 0$. Hence, different cases need to be considered:

$$\text{Case 1: } x_2 > 0 \quad ; \quad \text{Case 2: } x_2 < 0 \quad ; \quad \text{Case 3: } x_2 = 0$$

Case 1: $x_2 > 0$

$$\left(\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2}\right)^2 = \left(\frac{\partial^2}{\partial x_1 \partial x_2}\,(2x_1 + 3x_2 - x_1 x_2)\right)^2 = \left(\frac{\partial}{\partial x_2}\,(2 - x_2)\right)^2 = (-1)^2 = 1 > 0$$

Case 2: $x_2 < 0$

$$\left(\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2}\right)^2 = \left(\frac{\partial^2}{\partial x_1 \partial x_2}\,(2x_1 + 3x_2 - x_1(-x_2))\right)^2 = \left(\frac{\partial}{\partial x_2}\,(2 + x_2)\right)^2 = 1^2 = 1 > 0$$

Case 3: $x_2 = 0$
Not considered, as analysis of interactions via definition requires the consideration of intervals. The examination of single points does not make sense.

$\Rightarrow$ x_1 and x_2 interact with each other.