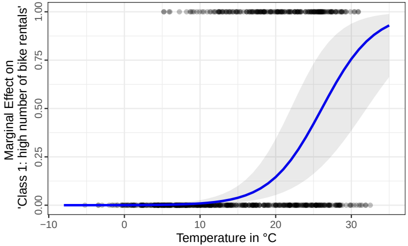
# Interpretable Machine Learning

# Generalized Linear Models



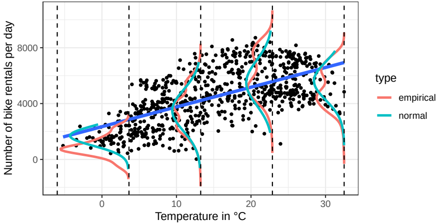

### Learning goals

- Definition of GLMs
- Logistic regression as example
- Interpretation in logistic regression

# GENERALIZED LINEAR MODEL (GLM) ▸ Nelder and Wedderburn 1972

**Problem**: Target variable given the features not always normally distributed $\rightsquigarrow$ LM not suitable

- Target is binary (e.g., disease classification)
  $\rightsquigarrow$ Bernoulli / Binomial distribution
- Target is count variable (e.g., number of sold products)
  $\rightsquigarrow$ Poisson distribution
- Time until an event occurs (e.g., time until death)
  $\rightsquigarrow$ Gamma distribution



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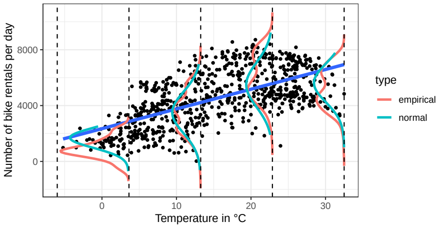


**Solution**: GLMs - extend LMs by allowing other distributions from exponential family

$$g(\mathbb{E}(y \mid \mathbf{x})) = \mathbf{x}^\top \theta \;\Leftrightarrow\; \mathbb{E}(y \mid \mathbf{x}) = g^{-1}(\mathbf{x}^\top \theta)$$

- Link function $g$ links linear predictor $\mathbf{x}^\top \theta$ to expectation $\mathbb{E}$ of specified distribution of $y \mid \mathbf{x}$
  $\rightsquigarrow$ LM is special case: Gaussian distribution for $y \mid \mathbf{x}$ with $g$ as identity function

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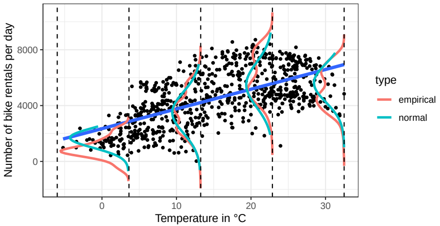


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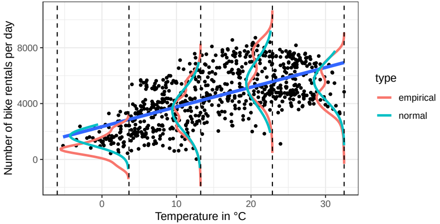


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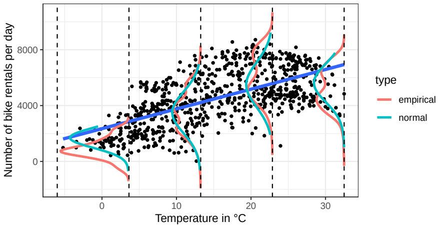


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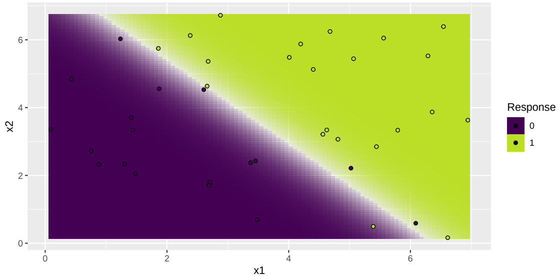
- Link function $g$ and distribution need to be specified
- High-order and interaction effects can be manually added as in LMs
- Note: Interpretation of weights depend on link function and distribution

# GLM - LOGISTIC REGRESSION

- Logistic regression $\hat{=}$ GLM with Bernoulli distribution and logit link function:

$$g(x) = \log\left(\frac{x}{1-x}\right)$$

$$\Rightarrow \quad g^{-1}(x) = \frac{1}{1+\exp(-x)}$$

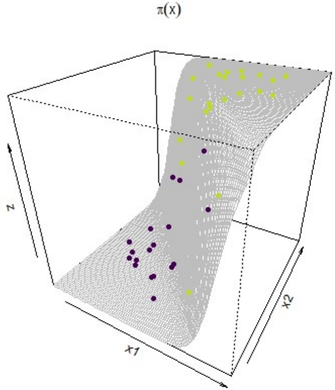


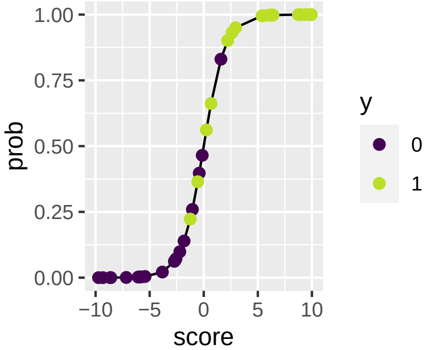

- Models probabilities for binary classification by

$$\mathbb{E}(y \mid \mathbf{x}) = P(y=1) = g^{-1}(\mathbf{x}^\top\theta) = \frac{1}{1+\exp(-\mathbf{x}^\top\theta)}$$

# GLM - LOGISTIC REGRESSION

- Typically, we set the threshold to 0.5 to predict classes, e.g.,
  - Class 1 if $P(y = 1) > 0.5$
  - Class 0 if $P(y = 1) \leq 0.5$



# GLM - LOGISTIC REGRESSION - INTERPRETATION

- **Recall:** Odds is the quotient of two probabilities, odds ratio compares the ratio of two odds
- Weights $\theta_j$ are interpreted linear as in LM (but w.r.t. log-odds) $\rightsquigarrow$ difficult to comprehend

$$\textit{log-odds} = \log\left(\frac{P(y=1)}{1-P(y=1)}\right) = \log\left(\frac{P(y=1)}{P(y=0)}\right) = \theta_0 + \theta_1 x_1 + \ldots + \theta_p x_p$$

**Interpretation:** Changing $x_j$ by one unit, changes log-odds of class 1 compared to class 0 by $\theta_j$

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- Odds for class 1 vs. class 0: $\textit{odds} = \dfrac{P(y=1)}{P(y=0)} = \exp(\theta_0 + \theta_1 x_1 + \ldots + \theta_p x_p)$
- Instead of interpreting changes w.r.t. log-odds, it is more common to use odds ratios:

$$\textit{odds ratio} = \frac{\textit{odds}_{x_j+1}}{\textit{odds}} = \frac{\exp(\theta_0 + \theta_1 x_1 + \ldots + \theta_j(x_j+1) + \ldots + \theta_p x_p)}{\exp(\theta_0 + \theta_1 x_1 + \ldots + \theta_j x_j + \ldots + \theta_p x_p)} = \exp\left(\theta_j\right)$$

**Interpretation**: Changing $x_j$ by one unit, changes the **odds ratio** for class 1 (compared to class 0) by the **factor** $\exp(\theta_j)$

# GLM - LOGISTIC REGRESSION - EXAMPLE

- Create a binary target variable for bike rental data:
  - Class 1: “high number of bike rentals” - more than the 70% quantile (i.e., `cnt` > 5531)
  - Class 0: “low to medium number of bike rentals” (i.e., `cnt` $\leq$ 5531)
- Fit a logistic regression model (GLM with Bernoulli distribution and logit link)

| | Weights | SE | p-value |
|---:|---:|---:|---:|
| (Intercept) | -8.52 | 1.21 | 0.00 |
| seasonSPRING | 1.74 | 0.60 | 0.00 |
| seasonSUMMER | -0.86 | 0.77 | 0.26 |
| seasonFALL | -0.64 | 0.55 | 0.25 |
| temp | 0.29 | 0.04 | 0.00 |
| hum | -0.06 | 0.01 | 0.00 |
| windspeed | -0.09 | 0.03 | 0.00 |
| days_since_2011 | 0.02 | 0.00 | 0.00 |

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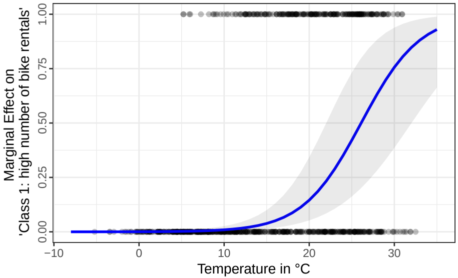


## Interpretation

- If `temp` increases by $1^\circ C$, odds ratio for class 1 increases by factor $\exp(0.29) = 1.34$ compared to class 0, c.p. ($\hat{=}$ “high number of bike rentals” become 1.34 times more likely)