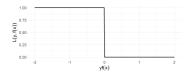
Introduction to Machine Learning

0-1-Loss



Learning goals

- Derive the risk minimizer of the 0-1-loss
- Derive the optimal constant model for the 0-1-loss



0-1-LOSS

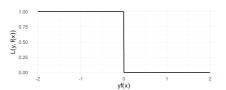
- Let us first consider a discrete classifier $h(\mathbf{x}): \mathcal{X} \to \mathcal{Y}$.
- The most natural choice for L(y, h(x)) is the 0-1-loss

$$L(y, h(\mathbf{x})) = \mathbb{1}_{\{y \neq h(\mathbf{x})\}} = \begin{cases} 1 & \text{if } y \neq h(\mathbf{x}) \\ 0 & \text{if } y = h(\mathbf{x}) \end{cases}$$

• For the binary case (g = 2) we can express the 0-1-loss for a scoring classifier $f(\mathbf{x})$ based on the margin $\nu := yf(\mathbf{x})$

$$L(y, f(\mathbf{x})) = \mathbb{1}_{\{\nu < 0\}} = \mathbb{1}_{\{yf(\mathbf{x}) < 0\}}.$$

 Analytic properties: Not continuous, even for linear f the optimization problem is NP-hard and close to intractable.





0-1-LOSS: RISK MINIMIZER

By the law of total expection we can in general rewrite the risk as (this all works for the multiclass case with 0-1)

$$\mathcal{R}(f) = \mathbb{E}_{xy} [L(y, f(\mathbf{x}))] = \mathbb{E}_{x} [\mathbb{E}_{y|x} [L(y, f(\mathbf{x}))]]$$
$$= \mathbb{E}_{x} \left[\sum_{k \in \mathcal{Y}} L(k, f(\mathbf{x})) \mathbb{P}(y = k \mid \mathbf{x} = \mathbf{x}) \right],$$

with $\mathbb{P}(y = k | \mathbf{x} = \mathbf{x})$ the posterior probability for class k. For the binary case we denote $\eta(\mathbf{x}) := \mathbb{P}(y = 1 \mid \mathbf{x} = \mathbf{x})$ and the expression becomes

$$\mathcal{R}(f) = \mathbb{E}_{\mathbf{x}} \left[L(1, \pi(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(0, \pi(\mathbf{x})) \cdot (1 - \eta(\mathbf{x})) \right].$$



0-1-LOSS: RISK MINIMIZER / 2

We compute the point-wise optimizer of the above term for the 0-1-loss (defined on a discrete classifier $h(\mathbf{x})$):

$$h^*(\mathbf{x}) = \arg\min_{l \in \mathcal{Y}} \sum_{k \in \mathcal{Y}} L(k, l) \cdot \mathbb{P}(y = k \mid \mathbf{x} = \mathbf{x})$$

$$= \arg\min_{l \in \mathcal{Y}} \sum_{k \neq l} \mathbb{P}(y = k \mid \mathbf{x} = \mathbf{x})$$

$$= \arg\min_{l \in \mathcal{Y}} 1 - \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x})$$

$$= \arg\max_{l \in \mathcal{Y}} \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x}),$$

which corresponds to predicting the most probable class.

Note that sometimes $h^*(\mathbf{x}) = \arg\max_{l \in \mathcal{Y}} \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x})$ is referred to as the **Bayes optimal classifier** (without closer specification of the the loss function used).



0-1-LOSS: RISK MINIMIZER / 3

The Bayes risk for the 0-1-loss (also: Bayes error rate) is

$$\mathcal{R}^* = 1 - \mathbb{E}_x \left[\max_{l \in \mathcal{Y}} \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x}) \right].$$



In the binary case (g = 2) we can write risk minimizer and Bayes risk as follows:

$$h^*(\mathbf{x}) = \begin{cases} 1 & \eta(\mathbf{x}) \geq \frac{1}{2} \\ 0 & \eta(\mathbf{x}) < \frac{1}{2} \end{cases}$$

$$\mathcal{R}^* = \mathbb{E}_x \left[\min(\eta(\mathbf{x}), 1 - \eta(\mathbf{x})) \right] = 1 - \mathbb{E}_x \left[\max(\eta(\mathbf{x}), 1 - \eta(\mathbf{x})) \right].$$

0-1-LOSS: RISK MINIMIZER / 4

Example: Assume that $\mathbb{P}(y=1)=\frac{1}{2}$ and

$$\mathbb{P}(x \mid y) = \begin{cases} \phi_{\mu_1, \sigma^2}(x) & \text{for } y = 0\\ \phi_{\mu_2, \sigma^2}(x) & \text{for } y = 1 \end{cases}$$

The decision boundary of the Bayes optimal classifier is shown in orange and the Bayes error rate is highlighted as red area.

