Solution 1: AdaBoost - Updates

(a) For the first case, only one sample is incorrect. The weighted in-sample misclassification rate can be calculated as:

$$\text{err}^{[m]} = \sum_{i=1}^{n} w^{[m](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[m]}(\mathbf{x}^{(i)})\}} = 0.01 \tag{1}$$

We can now calculate the weight of the base learner:

$$\hat{\beta}^{[m]} = \frac{1}{2} \log \left(\frac{1 - \text{err}^{[m]}}{\text{err}^{[m]}} \right) = 0.5 \cdot log \left(\frac{0.99}{0.01} \right) = log(\sqrt{99}) \tag{2}$$

The updated weights are then calculated as:

$$\begin{aligned} w^{[m+1](i)} &= w^{[m](i)} \cdot \exp \left(-\hat{\beta}^{[m]} \cdot \underbrace{y^{(i)} \cdot \hat{h}(\mathbf{x}^{(i)})}_{\{-1,1\}} \right) \\ &= \begin{cases} 0.01 \cdot \exp\left(-log(\sqrt{99}) \cdot 1 \cdot 1\right) & \text{if } i = 1, \ldots, 4, 6, \ldots, 10 \\ 0.01 \cdot \exp\left(-log(\sqrt{99}) \cdot 1 \cdot -1\right) & \text{if } i = 5 \\ 0.1 \cdot \exp\left(-log(\sqrt{99}) \cdot 1 \cdot 1\right) & \text{if } i = 11, \ldots, 19 \end{cases} \\ &= \begin{cases} 0.001 & \text{if } i = 1, \ldots, 4, 6, \ldots, 10 \\ 0.0995 & \text{if } i = 5 \\ 0.01 & \text{if } i = 11, \ldots, 19 \end{cases} \end{aligned} \tag{3}$$

(b) In the second case, five out of the 9 samples with weight 0.1 are misclassified. The weighted in-sample misclassification rate can be calculated as:

$$\text{err}^{[m]} = \sum_{i=1}^{n} w^{[m](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[m]}(\mathbf{x}^{(i)})\}} = 0.5 \tag{4}$$

We can now calculate the weight of the base learner:

$$\hat{\beta}^{[m]} = \frac{1}{2} \log \left(\frac{1 - \text{err}^{[m]}}{\text{err}^{[m]}} \right) = 0.5 \cdot log \left(\frac{0.5}{0.5} \right) = 0 \tag{5}$$

The updated weights are then calculated as:

$$\begin{aligned} w^{[m+1](i)} &= w^{[m](i)} \cdot \exp \left(-\hat{\beta}^{[m]} \cdot \underbrace{y^{(i)} \cdot \hat{h}(\mathbf{x}^{(i)})}_{\{-1,1\}} \right) \\ &= \begin{cases} 0.01 \cdot \exp\left(-0 \cdot 1 \cdot 1\right) & \text{if } i = 1, \ldots, 10 \\ 0.1 \cdot \exp\left(-0 \cdot 1 \cdot -1\right) & \text{if } i = 15, 16, 17, 18 \\ 0.1 \cdot \exp\left(-0 \cdot 1 \cdot 1\right) & \text{if } i = 11, \ldots, 14, 19 \end{cases} \\ &= \begin{cases} 0.01 & \text{if } i = 1, \ldots, 10 \\ 0.1 & \text{if } i = 15, 16, 17, 18 \\ 0.1 & \text{if } i = 11, \ldots, 14, 19 \end{cases} \end{aligned} \tag{6}$$

When the weighted in-sample misclassification rate is 0.5, the weight of the base learner is 0 and we don't do any updates to the weights.

(c) In the third case, all samples with the exception of one with a weight of 0.01 are misclassified. The weighted in-sample misclassification rate can be calculated as:

$$\text{err}^{[m]} = \sum_{i=1}^{n} w^{[m](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[m]}(\mathbf{x}^{(i)})\}} = 0.99 \tag{7}$$

We can now calculate the weight of the base learner:

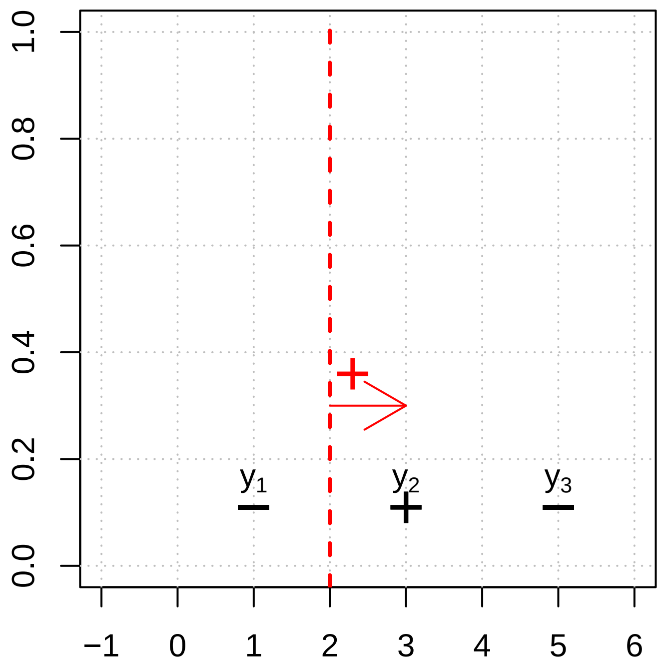
$$\hat{\beta}^{[m]} = \frac{1}{2} \log\left(\frac{1 - \text{err}^{[m]}}{\text{err}^{[m]}}\right) = 0.5 \cdot log\left(\frac{0.01}{0.99}\right) = log\left(\sqrt{\frac{1}{99}}\right) \tag{8}$$

The updated weights are then calculated as:

$$\begin{aligned} w^{[m+1](i)} &= w^{[m](i)} \cdot \exp\left(-\hat{\beta}^{[m]} \cdot \underbrace{y^{(i)} \cdot \hat{h}(\mathbf{x}^{(i)})}_{\{-1,1\}}\right) \\ &= \begin{cases} 0.01 \cdot \exp\left(-log\left(\sqrt{\frac{1}{99}}\right) \cdot 1 \cdot -1\right) & \text{if } i = 1, \dots, 9 \\ 0.01 \cdot \exp\left(-log\left(\sqrt{\frac{1}{99}}\right) \cdot 1 \cdot 1\right) & \text{if } i = 10 \\ 0.1 \cdot \exp\left(-log\left(\sqrt{\frac{1}{99}}\right) \cdot 1 \cdot -1\right) & \text{if } i = 11, \dots, 19 \end{cases} \\ &= \begin{cases} 0.001 & \text{if } i = 1, \dots, 9 \\ 0.0995 & \text{if } i = 10 \\ 0.01 & \text{if } i = 11, \dots, 19 \end{cases} \end{aligned} \tag{9}$$

Solution 2: AdaBoost - Decision Stump

(a) The initial weights all three points in the dataset is $\frac{1}{3}$. A decision boundary for the first decision stump could be the line $x = 2$.



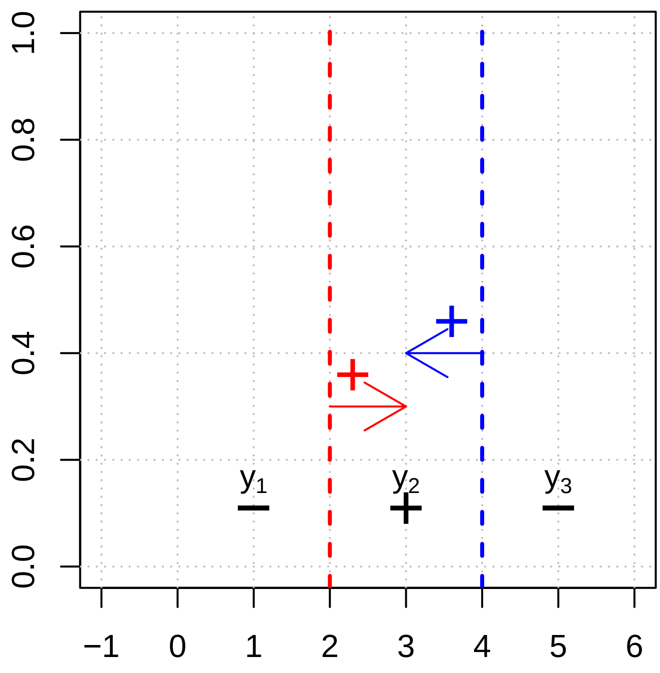
(b) The first stump makes two correct predictions and one incorrect prediction, we can then calculate the weighted in-sample misclassification rate, the weight for the stump and the new data points weights

$$
\begin{aligned}
\mathrm{err}^{[0]} &= \sum_{i=1}^{n} w^{[0](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[0]}(\mathbf{x}^{(i)})\}} = 0.33 \\
\hat{\beta}^{[0]} &= \frac{1}{2} \log \left(\frac{1 - \mathrm{err}^{[0]}}{\mathrm{err}^{[0]}} \right) = 0.5 \cdot log \left(\frac{0.67}{0.33} \right) \approx 0.35 \\
w^{[1](1)} &= w^{[0](1)} \cdot \exp \left(-\hat{\beta}^{[0]} \cdot y^{(1)} \cdot \hat{b}^{[0]}(\mathbf{x}^{(1)}) \right) = 0.33 \cdot \exp\left(-0.35 \cdot -1 \cdot -1\right) \approx 0.23 \\
w^{[1](2)} &= w^{[0](2)} \cdot \exp \left(-\hat{\beta}^{[0]} \cdot y^{(2)} \cdot \hat{b}^{[0]}(\mathbf{x}^{(2)}) \right) = 0.33 \cdot \exp\left(-0.35 \cdot 1 \cdot 1\right) \approx 0.23 \\
w^{[1](3)} &= w^{[0](3)} \cdot \exp \left(-\hat{\beta}^{[0]} \cdot y^{(3)} \cdot \hat{b}^{[0]}(\mathbf{x}^{(3)}) \right) = 0.33 \cdot \exp\left(-0.35 \cdot 1 \cdot -1\right) \approx 0.47
\end{aligned}
\tag{10}
$$

We need to normalize the weights so that they sum up to one:

$$
\begin{aligned}
w^{[1](1)} &= \frac{w^{[1](1)}}{\sum\limits_{i=1}^{n} w^{[1](i)}} = \frac{0.23}{0.23 + 0.23 + 0.47} \approx 0.25 \\
w^{[1](2)} &= \frac{w^{[1](2)}}{\sum\limits_{i=1}^{n} w^{[1](i)}} = \frac{0.23}{0.23 + 0.23 + 0.47} \approx 0.25 \\
w^{[1](3)} &= \frac{w^{[1](3)}}{\sum\limits_{i=1}^{n} w^{[1](i)}} = \frac{0.47}{0.23 + 0.23 + 0.47} \approx 0.50
\end{aligned}
\tag{11}
$$

(c) As the training error is not yet 0, we do a second stump using the new weights, The decision boundary is x=4. The stump makes two correct predictions and one incorrect prediction:



We calculate the weighted in-sample misclassification rate and the weight for the stump.

$$
\begin{aligned}
\mathrm{err}^{[1]} &= \sum_{i=1}^{n} w^{[1](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[1]}(\mathbf{x}^{(i)})\}} = 0.25 \\
\hat{\beta}^{[1]} &= \frac{1}{2} \log \left(\frac{1 - \mathrm{err}^{[1]}}{\mathrm{err}^{[1]}} \right) = 0.5 \cdot log \left(\frac{0.75}{0.25} \right) \approx 0.54
\end{aligned} \tag{12}
$$

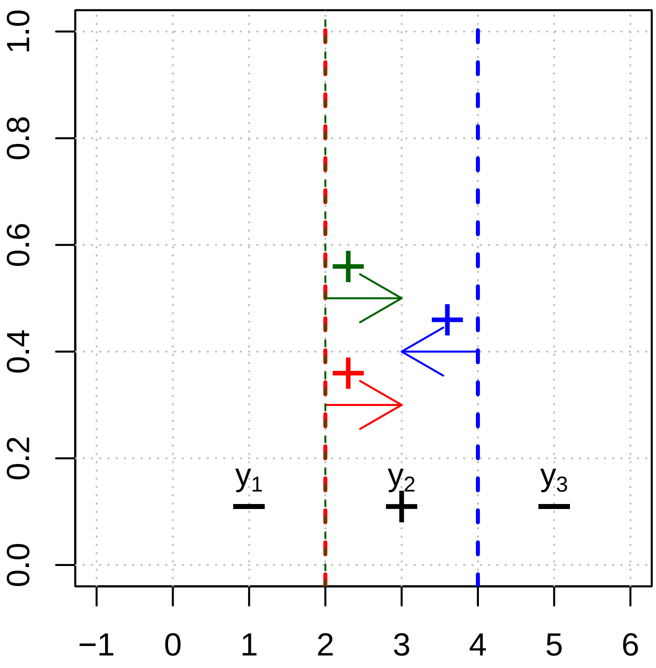
We can see that for the left-most point and the right-most point, there is a disagreement between our two stumps. In the case of the right-most point, the second stump has a bigger weight than the first one, so we will classify it correctly as $y = -1$. Unfortunately, in the case of the left-most point, we will classify it incorrectly as $y = 1$. Let's calculate the new weights:

$$
\begin{aligned}
w^{[2](1)} &= w^{[1](1)} \cdot \exp\left(-\hat{\beta}^{[0]} \cdot y^{(1)} \cdot \hat{b}^{[1]}(\mathbf{x}^{(1)})\right) = 0.25 \cdot \exp\left(-0.54 \cdot 1 \cdot -1\right) \approx 0.42 \\
w^{[2](2)} &= w^{[1](2)} \cdot \exp\left(-\hat{\beta}^{[0]} \cdot y^{(2)} \cdot \hat{b}^{[1]}(\mathbf{x}^{(2)})\right) = 0.25 \cdot \exp\left(-0.54 \cdot 1 \cdot 1\right) \approx 0.15 \\
w^{[2](3)} &= w^{[1](3)} \cdot \exp\left(-\hat{\beta}^{[0]} \cdot y^{(3)} \cdot \hat{b}^{[1]}(\mathbf{x}^{(3)})\right) = 0.5 \cdot \exp\left(-0.54 \cdot -1 \cdot -1\right) \approx 0.29
\end{aligned} \tag{13}
$$

We need to normalize the weights so that they sum up to one:

$$
\begin{aligned}
w^{[2](1)} &= \frac{w^{[2](1)}}{\sum\limits_{i=1}^{n} w^{[2](i)}} = \frac{0.42}{0.42 + 0.15 + 0.29} \approx 0.49 \\
w^{[2](2)} &= \frac{w^{[2](2)}}{\sum\limits_{i=1}^{n} w^{[2](i)}} = \frac{0.23}{0.42 + 0.15 + 0.29} \approx 0.17 \\
w^{[2](3)} &= \frac{w^{[2](3)}}{\sum\limits_{i=1}^{n} w^{[2](i)}} = \frac{0.47}{0.42 + 0.15 + 0.29} \approx 0.34
\end{aligned} \tag{14}
$$

We will start now with the third iteration. Unfortunately, we can't have open end nodes in a stump because doing such a split does not improve the splitting criterion. The best we can do is to add a new stump in the same place as the first one:



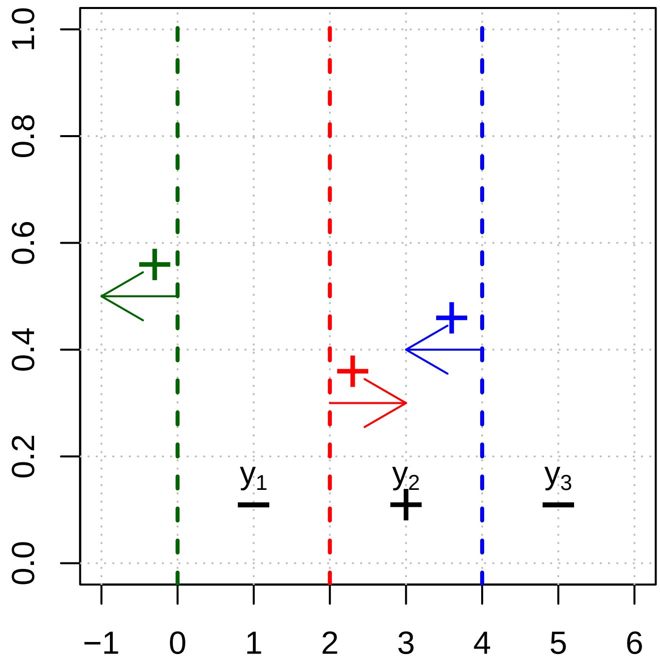
We calculate the weighted in-sample misclassification error for this stump:

$$
\begin{aligned}
\text{err}^{[2]} &= \sum_{i=1}^{n} w^{[2](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[2]}(\mathbf{x}^{(i)})\}} = 0.34 \\
\hat{\beta}^{[2]} &= \frac{1}{2} \log\left(\frac{1-\text{err}^{[2]}}{\text{err}^{[2]}}\right) = 0.5 \cdot log\left(\frac{0.66}{0.34}\right) \approx 0.35
\end{aligned}
\tag{15}
$$

We can see that we are going back to a similar case as the first iteration and now the rounding errors could play a role in our calculations. Let's build a model to see what happens:

```
df = data.frame(x1,x2,y)
for (i in 1:6){
  # Train the model with the amount of iterations
  model= sboost::sboost(df[1:2],df$y,iterations = i)
  # Assess the performance
  model_assessment=sboost::assess(model,features = df[1:2],outcomes = df$y)
  cat("Iteration number :" , i,
      ". model accuracy: ", model_assessment$performance[[6]],"\n"  )
  }

## Iteration number : 1 . model accuracy:  0.6666667
## Iteration number : 2 . model accuracy:  0.6666667
## Iteration number : 3 . model accuracy:  0.6666667
## Iteration number : 4 . model accuracy:  0.6666667
## Iteration number : 5 . model accuracy:  0.6666667
## Iteration number : 6 . model accuracy:  0.6666667

cat("Voting power of each stump: \n", model$classifier$vote)

## Voting power of each stump:
##  0.3465736 0.5493061 0.3465736 0.2554128 0.2027326 0.1682361

cat("Where each stump is located: ", model$classifier$split )

## Where each stump is located:  2 4 2 4 2 4
```

Considering the restrictions in the stump training, this model will never reach a training error of zero. But what would happen if we allow empty end nodes? We could add a stump at $x_1 = 0$ in the third iteration:



We calculate the weighted in-sample misclassification rate and the weight for the stump.

$$
\begin{aligned}
\mathrm{err}^{[2]} &= \sum_{i=1}^{n} w^{[2](i)} \cdot \mathbb{1}_{\{y^{(i)} \neq \hat{b}^{[2]}(\mathbf{x}^{(i)})\}} = 0.17 \\
\hat{\beta}^{[2]} &= \frac{1}{2} \log \left(\frac{1-\mathrm{err}^{[2]}}{\mathrm{err}^{[2]}} \right) = 0.5 \cdot log \left(\frac{0.83}{0.17} \right) \approx 0.79
\end{aligned} \tag{16}
$$

Let's check how the points are classified

$$
\begin{aligned}
\hat{y_1} &= \mathrm{sign}\left(\sum_{i=1}^{m} \hat{\beta}^{[i]} \hat{b}^{[i]}(\mathbf{x}^{(1)}) \right) = \mathrm{sign}(-0.35 + 0.54 - 0.79) = -1 \\
\hat{y_2} &= \mathrm{sign}\left(\sum_{i=1}^{m} \hat{\beta}^{[i]} \hat{b}^{[i]}(\mathbf{x}^{(2)}) \right) = \mathrm{sign}(+0.35 + 0.54 - 0.79) = 1 \\
\hat{y_3} &= \mathrm{sign}\left(\sum_{i=1}^{m} \hat{\beta}^{[i]} \hat{b}^{[i]}(\mathbf{x}^{(3)}) \right) = \mathrm{sign}(+0.35 - 0.54 - 0.79) = -1
\end{aligned} \tag{17}
$$

In this case, we reached zero error in training in the third iteration.