Exercise 1: Gaussian Processes - Prediction

Let $\mathcal{X} = \mathbb{R}$ and assume the following statistical model

$$y = f(x) + \epsilon, \qquad \epsilon \sim \mathcal{N}(0, \sigma^2),$$

where $f(x) \in \mathcal{GP}(0, k(x, x'))$. Suppose the covariance function of the GP is

$$k(x, x') = \mathbb{1}_{[|x-x'|<1]} \cdot (1 - |x - x'|)$$

and we have seen the training data:

As a test input we observe $x_* = 1.2$. Recall that the predictive distribution for $f(x_*)$ is

$$f(x_*) \mid \mathbf{X}, \boldsymbol{y}, x_* \sim \mathcal{N}(m_{\text{post}}, k_{\text{post}}).$$

with

$$\begin{array}{lll} m_{\mathrm{post}} & = & \boldsymbol{K}_{*}^{T} \left(\mathbf{K} + \sigma^{2} \cdot \boldsymbol{I} \right)^{-1} \boldsymbol{y} \\ k_{\mathrm{post}} & = & K_{**} - \boldsymbol{K}_{*}^{T} \left(\mathbf{K} + \sigma^{2} \cdot \boldsymbol{I} \right)^{-1} \boldsymbol{K}_{*}, \end{array}$$

Here,
$$\mathbf{K} = \left(k\left(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}\right)\right)_{i,j}, \ \mathbf{K}_* = \left(k\left(x_*, \mathbf{x}^{(1)}\right), ..., k\left(x_*, \mathbf{x}^{(n)}\right)\right)^{\top} \text{ and } K_{**} = k(x_*, x_*).$$

(a) Compute the predictive mean m_{post} .

- (b) Compute the predictive variance k_{post} .
- (c) Repeat the calculations from (a) and (b) by using as the test input $x_* = \mathbf{x}^{(i)}$ for each i = 1, 2, 3, 4, respectively.

(d) Based on your calculations so far, try to sketch the posterior Gaussian process.
(e) If the nugget σ^2 would be zero, how would the posterior Gaussian process (roughly) look like?
(f) Quiz time: Log in to Particify (https://partici.fi/63221686) and try to answer the questions for Week 13.