Exercise 1: Kernels

A **(Mercer) kernel** on a space $\mathcal{X}$ is a continuous function

$$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

of two arguments with the properties

- Symmetry: $k(\mathbf{x}, \tilde{\mathbf{x}}) = k(\tilde{\mathbf{x}}, \mathbf{x})$ for all $\mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{X}$.
- Positive definiteness: For each finite subset $\left\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\right\}$ the **kernel Gram matrix** $\boldsymbol{K} \in \mathbb{R}^{n \times n}$ with entries $K_{ij} = k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$ is positive semi-definite. More precisely, for any $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a}^T \boldsymbol{K} \mathbf{a} \geq 0$.

Which of the following functions are kernels? Justify your answer in each case.

(a) $k(\mathbf{x}, \tilde{\mathbf{x}}) = \mathbf{x}^\top \tilde{\mathbf{x}}$, where $\mathcal{X} \subset \mathbb{R}^p$.

(b) $k(x, \tilde{x}) = \cos(x + \tilde{x})$, where $\mathcal{X} \subset \mathbb{R}$.

(c) $k(x, \tilde{x}) = \max(x, \tilde{x})$, where $\mathcal{X} \subset \mathbb{R}_+$.

(d) $k(\mathbf{x}, \tilde{\mathbf{x}}) = \alpha k_1(\mathbf{x}, \tilde{\mathbf{x}}) + \beta k_2(\mathbf{x}, \tilde{\mathbf{x}})$, where k_1 and k_2 are kernels and $\alpha, \beta \geq 0$ as well as $\mathcal{X} \subset \mathbb{R}^p$.