

### Solution 1: Kernelized Multiclass SVM

(a) We consider the following constrained optimization problem:

$$\begin{aligned} \min_{\boldsymbol{\theta}, \theta_0, \zeta^{(i)}} \quad & \frac{1}{2} \boldsymbol{\theta}^\top \boldsymbol{\theta} + C \sum_{i=1}^n \zeta^{(i)} \\ \text{s.t.} \quad & y^{(i)} \left( \langle \boldsymbol{\theta}, \phi(\mathbf{x}^{(i)}) \rangle + \theta_0 \right) \geq 1 - \zeta^{(i)} \quad \forall i \in \{1, \dots, n\}, \\ \text{and} \quad & \zeta^{(i)} \geq 0 \quad \forall i \in \{1, \dots, n\}. \end{aligned}$$

In the optimum, the inequalities will hold with equality (as we minimize the slacks), so  $\zeta^{(i)} = 1 - y^{(i)} (\langle \boldsymbol{\theta}, \phi(\mathbf{x}^{(i)}) \rangle + \theta_0)$ , but the lowest value  $\zeta^{(i)}$  can take is 0 (we do not get a bonus for points beyond the margin on the correct side). So we can rewrite the above:

$$\frac{1}{2} \|\boldsymbol{\theta}\|^2 + C \sum_{i=1}^n \max(1 - y^{(i)} (\boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}) + \theta_0), 0).$$

Note that this is essentially the same argument we used in the linear SVM case to write it as the regularized ERM problem with the hinge loss without using a feature map.

(b) Let  $\psi(\mathbf{x}, y) = \frac{1}{2} y \phi(x)$ , where  $\phi$  is the feature map of the regularized binary ERM problem in (a). Now, if  $y \neq y^{(i)}$  it holds that  $y = -y^{(i)}$ , so that

$$\begin{aligned} 1 + \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y) - \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y^{(i)}) &= 1 + \frac{1}{2} y \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}) - \frac{1}{2} y^{(i)} \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}) && \text{(Definition of } \psi) \\ &= 1 + \frac{1}{2} (y - y^{(i)}) \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}) && \text{(Distributivity)} \\ &= \begin{cases} 1 + \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}), & \text{if } y^{(i)} = -1 \\ 1 - \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}), & \text{if } y^{(i)} = +1 \end{cases} && \text{(Since } y = -y^{(i)}) \\ &= 1 - y^{(i)} \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) &= \frac{1}{2} \|\boldsymbol{\theta}\|^2 + C \sum_{i=1}^n \sum_{y \neq y^{(i)}} \max(1 + \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y) - \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y^{(i)}), 0) \\ &= \frac{1}{2} \|\boldsymbol{\theta}\|^2 + C \sum_{i=1}^n \max(1 + \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, -y^{(i)}) - \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y^{(i)}), 0) \quad (y \neq y^{(i)} \text{ implies } y = -y^{(i)}) \\ &= \frac{1}{2} \|\boldsymbol{\theta}\|^2 + C \sum_{i=1}^n \max(1 - y^{(i)} \boldsymbol{\theta}^\top \phi(\mathbf{x}^{(i)}), 0). \end{aligned}$$

(c) The representer theorem tells us that for the solution  $\boldsymbol{\theta}^*$  (if it exists) of  $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$  it holds that  $\boldsymbol{\theta}^* \in \text{span}\{(\psi(\mathbf{x}^{(i)}, y))_{i=1, \dots, n, y=1, \dots, g}\}$ . This means that  $\boldsymbol{\theta}$  has to be a linear combination of

$(\psi(\mathbf{x}^{(i)}, y))_{i=1, \dots, n, y=1, \dots, g}$ , so that we can write  $\boldsymbol{\theta} = \mathbf{X}^\top \boldsymbol{\beta}$  for  $\boldsymbol{\beta} \in \mathbb{R}^{ng}$  and

$$\mathbf{X} = \begin{pmatrix} \psi(\mathbf{x}^{(1)}, 1)^\top \\ \psi(\mathbf{x}^{(1)}, 2)^\top \\ \vdots \\ \psi(\mathbf{x}^{(1)}, g)^\top \\ \psi(\mathbf{x}^{(2)}, 1)^\top \\ \vdots \\ \psi(\mathbf{x}^{(n)}, g)^\top \end{pmatrix}.$$

For  $\mathbf{K} = \mathbf{X}\mathbf{X}^\top$  we obtain that

$$\|\boldsymbol{\theta}\|^2 = \boldsymbol{\theta}^\top \boldsymbol{\theta} = (\mathbf{X}^\top \boldsymbol{\beta})^\top \mathbf{X}^\top \boldsymbol{\beta} = \boldsymbol{\beta}^\top \mathbf{X}\mathbf{X}^\top \boldsymbol{\beta} = \boldsymbol{\beta}^\top \mathbf{K} \boldsymbol{\beta}.$$

Further, it holds that

$$\begin{aligned} \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y) - \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y^{(i)}) &= \boldsymbol{\beta}^\top \mathbf{X} \psi(\mathbf{x}^{(i)}, y) - \boldsymbol{\beta}^\top \mathbf{X} \psi(\mathbf{x}^{(i)}, y^{(i)}) \\ &\stackrel{(*)}{=} (\mathbf{K} \boldsymbol{\beta})_{(i-1)g+y} - (\mathbf{K} \boldsymbol{\beta})_{(i-1)g+y^{(i)}}. \end{aligned}$$

In order to see  $(*)$  note that  $\psi(\mathbf{x}^{(i)}, y)$  corresponds to the  $((i-1)g+y)$ -th row of  $\mathbf{X}$  and  $\psi(\mathbf{x}^{(i)}, y^{(i)})$  corresponds to the  $((i-1)g+y^{(i)})$ -th row of  $\mathbf{X}$ . Thus, the matrix-vector product  $\mathbf{X} \psi(\mathbf{x}^{(i)}, y)$  corresponds to the  $((i-1)g+y)$ -th column/row of  $\mathbf{K} = \mathbf{X}\mathbf{X}^\top$  and the matrix-vector product  $\mathbf{X} \psi(\mathbf{x}^{(i)}, y^{(i)})$  corresponds to the  $((i-1)g+y^{(i)})$ -th column/row of  $\mathbf{K} = \mathbf{X}\mathbf{X}^\top$  (keep in mind that  $\mathbf{K}$  is symmetric). Finally, computing the inner product of  $\boldsymbol{\beta}$  with  $\mathbf{X} \psi(\mathbf{x}^{(i)}, y)$  (or  $\mathbf{X} \psi(\mathbf{x}^{(i)}, y^{(i)})$ ) is the same as computing first the matrix-vector product  $\mathbf{K} \boldsymbol{\beta}$  and then projecting onto the  $((i-1)g+y)$ -th entry (or the  $((i-1)g+y^{(i)})$ -th entry).

With this,

$$\begin{aligned} \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) &= \frac{1}{2} \|\boldsymbol{\theta}\|^2 + C \sum_{i=1}^n \sum_{y \neq y^{(i)}} \max(1 + \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y) - \boldsymbol{\theta}^\top \psi(\mathbf{x}^{(i)}, y^{(i)}), 0) \\ &= \frac{1}{2} \boldsymbol{\beta}^\top \mathbf{K} \boldsymbol{\beta} + \sum_{i=1}^n \sum_{y \neq y^{(i)}} \max(1 + (\mathbf{K} \boldsymbol{\beta})_{(i-1)g+y} - (\mathbf{K} \boldsymbol{\beta})_{(i-1)g+y^{(i)}} , 0). \end{aligned}$$

## Solution 2: Kernel Trick

The polynomial kernel is defined as

$$k(x, \tilde{x}) = (x^\top \tilde{x} + b)^d.$$

Furthermore, assume  $x \in \mathbb{R}^2$  and  $d = 2$ .

- (a) Derive the explicit feature map  $\phi$  taking into account that the following equation holds:

$$k(x, \tilde{x}) = \langle \phi(x), \phi(\tilde{x}) \rangle$$

**Solution:**

$$\begin{aligned}
k(x, \tilde{x}) &= \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + b \right)^2 \\
&= (x_1 \tilde{x}_1 + x_2 \tilde{x}_2 + b)^2 \\
&= (x_1 \tilde{x}_1 + x_2 \tilde{x}_2)^2 + 2(x_1 \tilde{x}_1 + x_2 \tilde{x}_2)b + b^2 \\
&= x_1^2 \tilde{x}_1^2 + 2x_1 \tilde{x}_1 x_2 \tilde{x}_2 + x_2^2 \tilde{x}_2^2 + 2bx_1 \tilde{x}_1 + 2bx_2 \tilde{x}_2 + b^2 \\
&= \left\langle \begin{pmatrix} x_1^2 \\ \sqrt{2}x_1 x_2 \\ x_2^2 \\ \sqrt{2}bx_1 \\ \sqrt{2}bx_2 \\ b \end{pmatrix}, \begin{pmatrix} \tilde{x}_1^2 \\ \sqrt{2}\tilde{x}_1 \tilde{x}_2 \\ \tilde{x}_2^2 \\ \sqrt{2}b\tilde{x}_1 \\ \sqrt{2}b\tilde{x}_2 \\ b \end{pmatrix} \right\rangle \\
&= \langle \phi(x), \phi(\tilde{x}) \rangle
\end{aligned}$$

(b) Describe the main differences between the kernel method and the explicit feature map.

**Solution:**

Using the kernel method reduces the computational costs of computing the scalar product in the higher-dimensional features space after calculating the feature map.