Solution 1: High-dimensional Gaussian Distributions

(a) In this case, we will use the linearity of the expectation.

$$\begin{aligned}\mathbb{E}\left[\|X\|_1\right] &= \mathbb{E}\left[\sum_{j=1}^{p}\|x_j\|\right] \\ &= \sum_{j=1}^{p}\underbrace{\mathbb{E}\|x_j\|}_{=\sqrt{\frac{2}{\pi}}} \\ &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{p}1 \\ &= \sqrt{\frac{2}{\pi}}\,p\end{aligned} \tag{1}$$

(b) Considering that the coordinates $X_1, \ldots, X_p$ are independent and identically distributed, the variance of the sum equals the sum of the variance.

$$\begin{aligned}\mathsf{Var}\left(\|X\|_1\right) &= \mathsf{Var}\left(\sum_{j=1}^{p}\|x_j\|\right) \\ &= \sum_{j=1}^{p}\underbrace{\left(\mathsf{Var}\left(\|x_j\|\right)\right)}_{=1-\frac{2}{\pi}} \\ &= \left(1-\frac{2}{\pi}\right)\sum_{j=1}^{p}1 \\ &= \left(1-\frac{2}{\pi}\right)p\end{aligned} \tag{2}$$

(c) A random variable which is the substraction of two normally distributed random variable is also normal, with the following parameters:

$$\begin{aligned}X - Y = Z &\sim \mathcal{N}(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2) \\ &\sim \mathcal{N}(0, 2)\end{aligned} \tag{3}$$

We also know that the variance of a random variable multiplied by a constant is equal to the variance of the random variable scaled by the square of the constant, consequently:

$$\begin{aligned}X - Y &\sim \mathcal{N}(0, 2) \\ \frac{X-Y}{\sqrt{2}} &\sim \mathcal{N}(0, 1)\end{aligned} \tag{4}$$

We will use the equations 3 and 4 to solve the exercise:

$$
\begin{aligned}
\mathbb{E}\left[\|X - X'\|_1\right] &= \mathbb{E}\left[\sum_{j=1}^{p} \|x_j - x'_j\|\right] \\
&= \sum \mathbb{E}\left[\|x_j - x'_j\|\right] \\
&= \sum \mathbb{E}\left[\frac{\sqrt{2}}{\sqrt{2}}\|x_j - x'_j\|\right] \\
&= \sqrt{2}\sum_{j=1}^{p} \mathbb{E}\left[\|\underbrace{\frac{x_j - x'_j}{\sqrt{2}}}_{\sim\mathcal{N}(0,1)}\|\right] \\
&= \sqrt{2}\sqrt{\frac{2}{\pi}}\sum_{j=1}^{p} 1 \\
&= \frac{2p}{\sqrt{\pi}}
\end{aligned}
\tag{5}
$$

(d) Using equations 3 and 4 again, we get:

$$
\begin{aligned}
\mathsf{Var}\left(\|X - X'\|_1\right) &= \sum_{j=1}^{p} \mathsf{Var}\left(\|x_j - x'_j\|\right) \\
&= \sum_{j=1}^{p} \left(\frac{\sqrt{2}}{\sqrt{2}}\|x_j - x'_j\|\right) \\
&= 2\sum_{j=1}^{p} \mathsf{Var}\left(\|\underbrace{\frac{x_j - x'_j}{\sqrt{2}}}_{\sim\mathcal{N}(0,1)}\|\right) \\
&= 2\left(1 - \frac{\pi}{2}\right)\sum_{j=1}^{p} 1 \\
&= 2p\left(1 - \frac{\pi}{2}\right)
\end{aligned}
\tag{6}
$$

(e) Using the linearity of the expectation and the fact that $\mathbf{x}$ is deterministic:

$$
\begin{aligned}
\mathbb{E}\left[\langle X, \mathbf{x}\rangle\right] &= \mathbb{E}\left[\sum_{j=1}^{p} X_j x_j\right] \\
&= \sum_{j=1}^{p} \mathbb{E}\left[X_j x_j\right] \\
&= \sum_{j=1}^{p} x_i \mathbb{E}\left[X_i\right] \\
&= 0
\end{aligned}
\tag{7}
$$

We will again use the independency of the coordinates $X_1, \dots, X_p$ and the fact that $\mathbf{x}$ is deterministic:

$$\begin{aligned} \mathsf{Var}\left(\langle X, \mathbf{x} \rangle\right) &= \mathsf{Var}\left(\sum_{j=1}^{p} X_j x_j\right) \\ &= \sum_{j=1}^{p} \mathsf{Var}\left(X_j x_j\right) \\ &= \sum_{j=1}^{p} x_j^2 \underbrace{\mathsf{Var}\left(X_j\right)}_{=1} \\ &= \sum_{j=1}^{p} x_j^2 \\ &= \|x\|_2{}^2 \end{aligned} \tag{8}$$