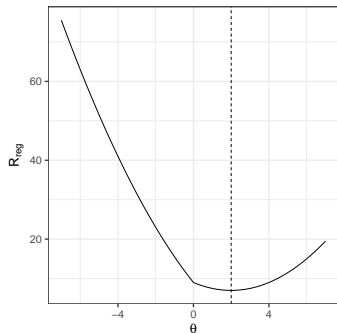


# Introduction to Machine Learning

## Regularization

### Soft-thresholding and lasso (Deep-Dive)



#### Learning goals

- Understand the relationship between soft-thresholding and L1 regularization

# SOFT-THRESHOLDING AND L1 REGULARIZATION

In the lecture, we wanted to solve

$$\min_{\theta} \tilde{\mathcal{R}}_{\text{reg}}(\theta) = \min_{\theta} \mathcal{R}_{\text{emp}}(\hat{\theta}) + \sum_j \left[ \frac{1}{2} H_{j,j} (\theta_j - \hat{\theta}_j)^2 \right] + \sum_j \lambda |\theta_j|$$

with  $H_{j,j} \geq 0, \lambda > 0$ . Note that we can separate the dimensions, i.e.,

$$\tilde{\mathcal{R}}_{\text{reg}}(\theta) = \sum_j z_j(\theta_j) \text{ with } z_j(\theta_j) = \frac{1}{2} H_{j,j} (\theta_j - \hat{\theta}_j)^2 + \lambda |\theta_j|.$$

Hence, we can minimize each  $z_j$  separately to find the global minimum.

If  $H_{j,j} = 0$ , then  $z_j$  is clearly minimized by  $\hat{\theta}_{\text{lasso},j} = 0$ . Otherwise,  $z_j$  is strictly convex since  $\frac{1}{2} H_{j,j} (\theta_j - \hat{\theta}_j)^2$  is strictly convex and the sum of a strictly convex function and a convex function is strictly convex.



# SOFT-THRESHOLDING AND L1 REGULARIZATION

For strictly convex functions, there exists only one unique minimum and for convex function a stationary point (if it exists) is a minimum.

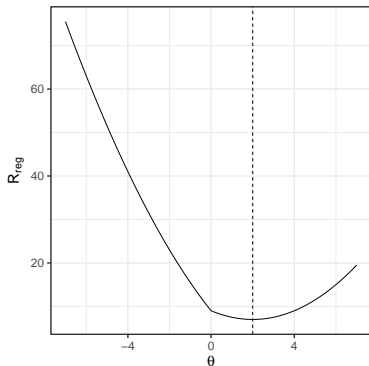
To find this minimum, we will first investigate  $z_j$  when  $\theta_j > 0, \theta_j < 0$ .

Note that in these cases  $z_j$  is differentiable (with possible stationary point) since

- for  $\theta_j > 0 : \frac{d}{d\theta_j} |\theta_j| = \frac{d}{d\theta_j} \theta_j = 1$ ,
- for  $\theta_j < 0 : \frac{d}{d\theta_j} |\theta_j| = \frac{d}{d\theta_j} (-\theta_j) = -1$ .



## SOFT-THRESHOLDING AND L1 REGULARIZATION

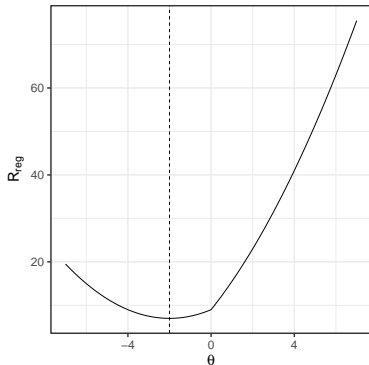


This minimum is only valid if  $\hat{\theta}_{\text{lasso},j} > 0$  and so it must hold that



# SOFT-THRESHOLDING AND L1 REGULARIZATION

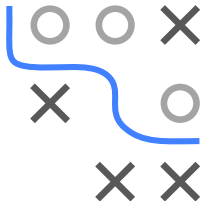
2)  $\hat{\theta}_{\text{lasso},j} < 0$  :



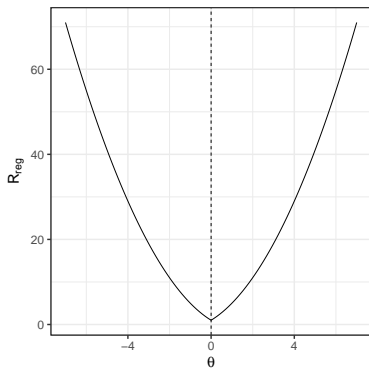
$$\frac{d}{d\theta_j} z_j(\theta_j) = H_{j,j}\theta_j - H_{j,j}\hat{\theta}_j - \lambda \stackrel{!}{=} 0$$

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \hat{\theta}_j + \frac{\lambda}{H_{j,j}} < 0$$

$$\Leftrightarrow \hat{\theta}_j < -\frac{\lambda}{H_{j,j}}$$



# SOFT-THRESHOLDING AND L1 REGULARIZATION



$\Rightarrow$  If  $\hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}]$  then  $z_j$  has no stationary point with

$$\hat{\theta}_{\text{lasso},j} < 0 \text{ or } \hat{\theta}_{\text{lasso},j} > 0.$$

However, a unique stationary point must exist since  $z_j$  is strictly convex for  $H_{j,j} > 0$ . This means the only possible minimizer of  $z_j$  is  $\hat{\theta}_{\text{lasso},j} = 0$ .

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \begin{cases} \hat{\theta}_j + \frac{\lambda}{H_{j,j}} & , \text{ if } \hat{\theta}_j < -\frac{\lambda}{H_{j,j}} \text{ and } H_{j,j} > 0 \\ 0 & , \text{ if } \hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}] \text{ or } H_{j,j} = 0 \\ \hat{\theta}_j - \frac{\lambda}{H_{j,j}} & , \text{ if } \hat{\theta}_j > \frac{\lambda}{H_{j,j}} \text{ and } H_{j,j} > 0 \end{cases}$$

