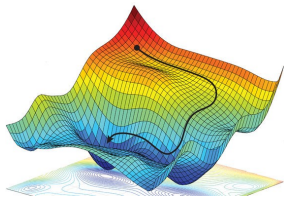


Introduction to Machine Learning

Advanced Risk Minimization

Risk Minimization Basics



Learning goals

- Risk minimization and ERM recap
- Bayes optimal model, Bayes risk
- Bayes regret, estimation and approximation error
- Optimal constant model
- Consistency

EMPIRICAL RISK MINIMIZATION

To learn a model, we usually do ERM:

$$\mathcal{R}_{\text{emp}}(f) = \sum_{i=1}^n L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right)$$

- observations $(\mathbf{x}^{(i)}, y^{(i)}) \in \mathcal{X} \times \mathcal{Y}$
- model $f_{\mathcal{H}} : \mathcal{X} \rightarrow \mathbb{R}^g$, from hypothesis space \mathcal{H} ; maps a feature vector to output score; often we omit \mathcal{H} in index
- loss $L : \mathcal{Y} \times \mathbb{R}^g \rightarrow \mathbb{R}$, measures error between label and prediction
- data generating process (DGP) \mathbb{P}_{xy} , we assume $(\mathbf{x}^{(i)}, y^{(i)}) \stackrel{\text{i.i.d.}}{\sim} \mathbb{P}_{xy}$

Minimizing theoretical risk, so expected loss over DGP, is major goal:

$$\mathcal{R}(f) := \mathbb{E}_{xy}[L(y, f(\mathbf{x}))] = \int L(y, f(\mathbf{x})) d\mathbb{P}_{xy}$$

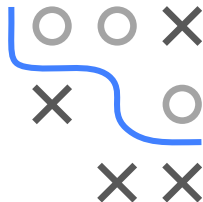


TWO SHORT EXAMPLES

Regression with linear model:

- Model: $f(\mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x} + \theta_0$
- Squared loss: $L(y, f(\mathbf{x})) = (y - f(\mathbf{x}))^2$
- Hypothesis space:

$$\mathcal{H}_{\text{lin}} = \left\{ \mathbf{x} \mapsto \boldsymbol{\theta}^\top \mathbf{x} + \theta_0 : \boldsymbol{\theta} \in \mathbb{R}^d, \theta_0 \in \mathbb{R} \right\}$$



Binary classification with shallow MLP:

- Model: $f(\mathbf{x}) = \pi(\mathbf{x}) = \sigma(\mathbf{w}_2^\top \text{ReLU}(\mathbf{W}_1 \mathbf{x} + \mathbf{b}_1) + b_2)$
- Bernoulli / Log / Cross-Entropy loss:
 $L(y, \pi(\mathbf{x})) = -(y \log(\pi(\mathbf{x})) + (1 - y) \log(1 - \pi(\mathbf{x})))$
- Hypothesis space:

$$\mathcal{H}_{\text{MLP}} = \left\{ \mathbf{x} \mapsto \sigma(\mathbf{w}_2^\top \text{ReLU}(\mathbf{W}_1 \mathbf{x} + \mathbf{b}_1) + b_2) : \mathbf{W}_1 \in \mathbb{R}^{h \times d}, \mathbf{b}_1 \in \mathbb{R}^h, \mathbf{w}_2 \in \mathbb{R}^h, b_2 \in \mathbb{R} \right\}$$

HYPOTHESIS SPACES AND PARAMETRIZATION

We often write $\mathcal{R}(f)$, but finding an optimal f is operationalized as finding optimal $\theta \in \Theta$ among a family of parametrized curves:

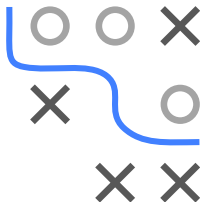
$$\mathcal{H} = \{f_{\theta} : f_{\theta} \text{ from functional family parametrized by } \theta\}$$



- Optimizing numeric vectors is more convenient than functions
- For some model classes, some parameters encode the same function (non-injective mapping, non-identifiability).
We don't care here, now.

OPTIMAL LOSS VALUES – M-ESTIMATORS

- Assume some RV $z \sim Q, z \in \mathcal{Y}$ as target
- z not the same as y , as we want to fiddle with its distribution
- We now consider $\arg \min_c \mathbb{E}_{z \sim Q}[L(z, c)]$
What is the constant that approximates z with minimal loss?



3 cases for Q

- $Q = P_y$, distribution of labels y , marginal of \mathbb{P}_{xy}
optimal theoretical constant prediction
- $Q = P_n$, the empirical product distribution for data $y^{(1)}, \dots, y^{(n)}$
optimal empirical constant prediction
- $Q = P_{y|\mathbf{x}=\tilde{\mathbf{x}}}$, conditional label distribution at point $\mathbf{x} = \tilde{\mathbf{x}}$
Bayes optimal pointwise prediction / theoretical risk minimizer

OPTIMAL UNCONDITIONAL VALUES

- Associating such a

$$c = \arg \min_{c \in \mathbb{R}} \mathbb{E}_{z \sim Q}[L(z, c)]$$

with a distribution is called a “statistical functional”

- Such a loss-minimizing version, and especially its empirical version below, is called an **M-estimator**
- “M” can be read as “max-likelihood type”, or “minimizing”, I prefer the latter
- If we look at the empirical counterpart, with the empirical distribution, this is the so-called “plug-in” estimator

$$\arg \min_{c \in \mathbb{R}} \sum_{i=1}^n L(y^{(i)}, c)$$

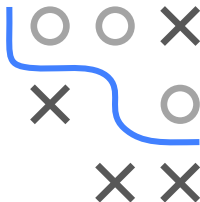
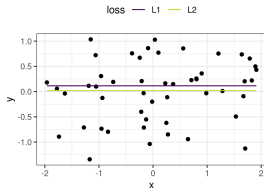


OPTIMAL CONSTANT MODEL

- Goal: loss optimal, constant baseline predictor
- “constant”: featureless ML model, always predicts same value
- “baseline”: more complex model has to be better
- Also useful as optimal intercept

$$f_c^* = \arg \min_{c \in \mathbb{R}} \mathbb{E}_{xy} [L(y, c)] = \arg \min_{c \in \mathbb{R}} \mathbb{E}_y [L(y, c)]$$

- Estimation via ERM: $\hat{f}_c = \arg \min_{c \in \mathbb{R}} \sum_{i=1}^n L(y^{(i)}, c)$



RISK MINIMIZER

- Assume, hypothesis space $\mathcal{H} = \mathcal{H}_{all}$ is unrestricted; contains any measurable $f : \mathcal{X} \rightarrow \mathbb{R}^g$
- We know \mathbb{P}_{xy}
- f with minimal risk across \mathcal{H}_{all} is called **risk minimizer, population minimizer** or **Bayes optimal model**



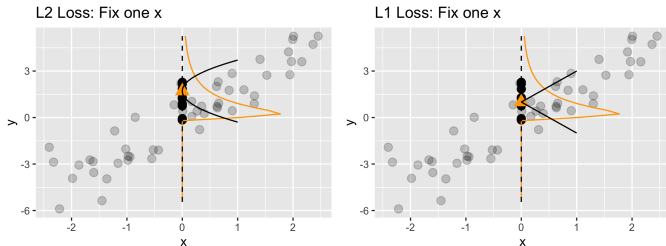
$$\begin{aligned} f_{\mathcal{H}_{all}}^* &= \arg \min_{f \in \mathcal{H}_{all}} \mathcal{R}(f) = \arg \min_{f \in \mathcal{H}_{all}} \mathbb{E}_{xy} [L(y, f(\mathbf{x}))] \\ &= \arg \min_{f \in \mathcal{H}_{all}} \int L(y, f(\mathbf{x})) d\mathbb{P}_{xy} \end{aligned}$$

- The resulting risk is called **Bayes risk**: $\mathcal{R}^* = \mathcal{R}(f_{\mathcal{H}_{all}}^*)$
- **Risk minimizer within** $\mathcal{H} \subset \mathcal{H}_{all}$ is $f_{\mathcal{H}}^* = \arg \min_{f \in \mathcal{H}} \mathcal{R}(f)$

OPTIMAL POINT-WISE PREDICTIONS

- To derive the RM, by law of total expectation

$$\mathcal{R}(f) = \mathbb{E}_{xy} [L(y, f(\mathbf{x}))] = \mathbb{E}_x [\mathbb{E}_{y|x} [L(y, f(\mathbf{x})) \mid \mathbf{x}]]$$



THEORETICAL AND EMPIRICAL RISK

- Bayes risk minimizer is mainly a theoretical tool
- In practice, need to restrict \mathcal{H} for efficient search
- We don't normally know \mathbb{P}_{xy} . Instead, use ERM.

$$\hat{f}_{\mathcal{H}} = \arg \min_{f \in \mathcal{H}} \mathcal{R}_{\text{emp}}(f) = \arg \min_{f \in \mathcal{H}} \sum_{i=1}^n L(y^{(i)}, f(\mathbf{x}^{(i)}))$$

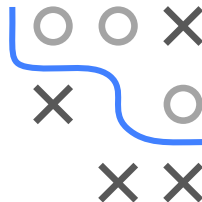
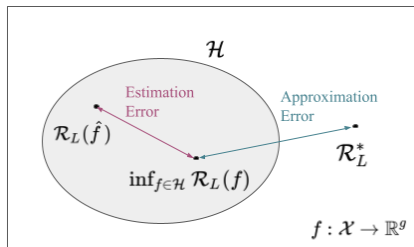
- Due to **law of large numbers**, empirical risk for fixed model converges to true risk, so consistent estimator

$$\bar{\mathcal{R}}_{\text{emp}}(f) = \frac{1}{n} \sum_{i=1}^n L(y^{(i)}, f(\mathbf{x}^{(i)})) \xrightarrow{n \rightarrow \infty} \mathcal{R}(f)$$

- Still, that does not imply that the selected ERM minimizer converges to f^* , due to overfitting or lack of uniform convergence
- Would need more assumptions / math. machinery for this, will not pursue this here



ESTIMATION AND APPROXIMATION ERROR



$$\mathcal{R}(\hat{f}_{\mathcal{H}}) - \mathcal{R}^* = \underbrace{\left[\mathcal{R}(\hat{f}_{\mathcal{H}}) - \inf_{f \in \mathcal{H}} \mathcal{R}(f) \right]}_{\text{estimation error}} + \underbrace{\left[\inf_{f \in \mathcal{H}} \mathcal{R}(f) - \mathcal{R}^* \right]}_{\text{approximation error}}$$

- **Estimation error:** We fit $\hat{f}_{\mathcal{H}}$ via ERM on finite data, so we don't find best $f \in \mathcal{H}$
- **Approximation error:** \mathcal{H} will often not contain Bayes optimal f^*

(UNIVERSALLY) CONSISTENT LEARNERS

► Stone 1977

Consistency is an asymptotic property of a learning algorithm, which ensures the algorithm returns **the correct model** when given **unlimited data**.

Let $\mathcal{I} : \mathbb{D} \rightarrow \mathcal{H}$ be a learning algorithm that takes a training set $\mathcal{D}_{\text{train}} \sim \mathbb{P}_{xy}$ of size n_{train} and estimates a model $\hat{f} : \mathcal{X} \rightarrow \mathbb{R}^g$.

The learning method \mathcal{I} is said to be **consistent** w.r.t. a certain distribution \mathbb{P}_{xy} if the risk of the estimated model \hat{f} converges in probability (“ \xrightarrow{p} ”) to the Bayes risk \mathcal{R}^* when n_{train} goes to ∞ :

$$\mathcal{R}(\mathcal{I}(\mathcal{D}_{\text{train}})) \xrightarrow{p} \mathcal{R}^* \quad \text{for } n_{\text{train}} \rightarrow \infty$$



(UNIVERSALLY) CONSISTENT LEARNERS

► Stone 1977

Consistency is defined w.r.t. a particular distribution \mathbb{P}_{xy} . But since we usually don't know \mathbb{P}_{xy} , consistency does not offer much help to choose an algorithm for a specific task.

More interesting is the stronger concept of **universal consistency**: An algorithm is universally consistent if it is consistent for **any** distribution.

In Stone's famous consistency theorem (1977), the universal consistency of a weighted average estimator such as KNN was proven. Many other ML models have since then been proven to be universally consistent (SVMs, ANNs, etc.).

