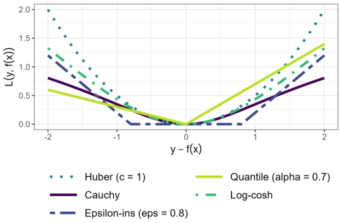
Introduction to Machine Learning

Advanced Risk Minimization
Advanced Regression Losses

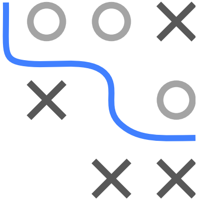


Learning goals

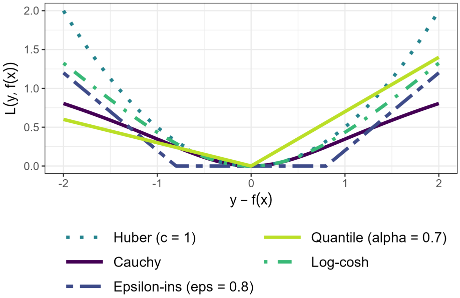
- Huber loss
- Log-Cosh loss
- Cauchy loss
- ϵ-Insensitive loss
- Quantile loss

ADVANCED LOSS FUNCTIONS

▸ Wang et al. 2020

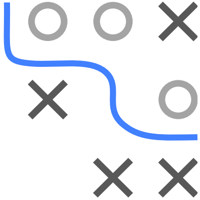


- Handle errors in custom fashion
- Model other error distributions (see section on max. likelihood)
- Induce properties like robustness
- Handle other predictive tasks

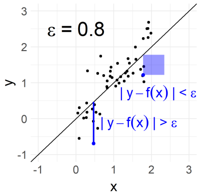


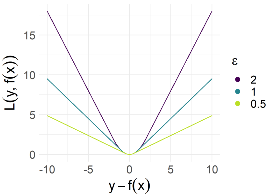
HUBER LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} \frac{1}{2}(y - f(\mathbf{x}))^2 & \text{if } |y - f(\mathbf{x})| \le \epsilon \\ \epsilon |y - f(\mathbf{x})| - \frac{1}{2}\epsilon^2 & \text{otherwise} \end{cases} \qquad \epsilon > 0$$



- Piece-wise combination of $L1/L2$ to have robustness/smoothness
- Analytic properties: convex, differentiable (once)





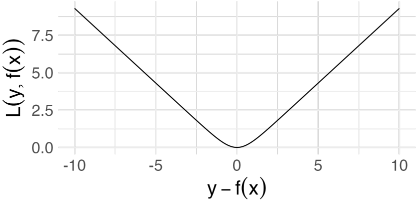
- No closed-form solution even for constant or linear model
- Solution behaves like **trimmed mean**:
  a (conditional) mean of two (conditional) quantiles

LOG-COSH LOSS

R. A. Saleh and A. Saleh 2022

$$L\left(y, f(\mathbf{x})\right) = \log\left(\cosh(|y - f(\mathbf{x})|)\right) \qquad \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

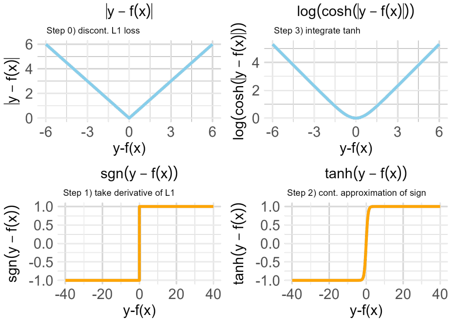
- Approx. $0.5(|y - f(\mathbf{x})|)^2$ for small residuals; $|y - f(\mathbf{x})| - \log 2$ for large residuals
- Smoothed combo of $L1$ / $L2$ loss
- Similar to Huber, but twice differentiable



LOG-COSH LOSS ▸ R. A. Saleh and A. Saleh 2022

Essential idea:

1. Derivative of $L1$ w.r.t. residual
2. Approx. sign with tanh
3. Integrate "up again"



Same trick can be used to get differentiable pinball losses

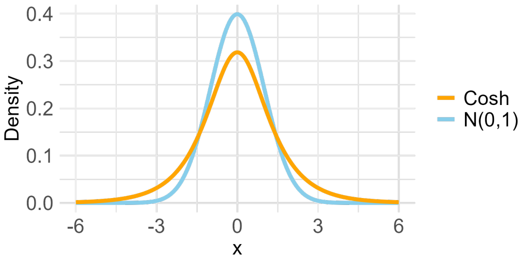
LOG-COSH LOSS ▸ R. A. Saleh and A. Saleh 2022

cosh(θ, σ) distribution:

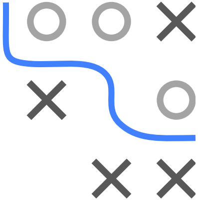
- Normalized reciprocal $\cosh(x)$ is pdf: positive and $\int_{-\infty}^{\infty} \frac{1}{\pi \cosh(x)} dx = 1$
- Location-scale type (θ, σ) resembling Gaussian with heavy tails
- ERM using log-cosh is equivalent to MLE of $\cosh(\theta, 1)$ distribution

- $p(x|\theta, \sigma) = \frac{1}{\pi\sigma \cosh\left(\frac{x-\theta}{\sigma}\right)}$
- $\mathbb{E}_{x \sim p}[x] = \theta$
- $\text{Var}_{x \sim p}[x] = \frac{1}{4}\pi^2\sigma^2$
- $\hat{\theta}^{MLE} = \arg\max_\theta \prod_{i=1}^{n} \frac{1}{\pi \cosh\left(y^{(i)} - \theta\right)} =$

  $\hat{\theta} = \arg\min_\theta \sum_{i=1}^{n} \log(\cosh(y^{(i)} - \theta))$

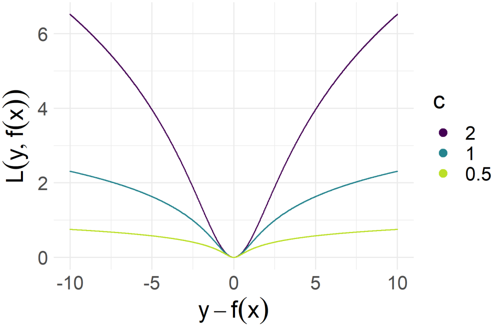


CAUCHY LOSS



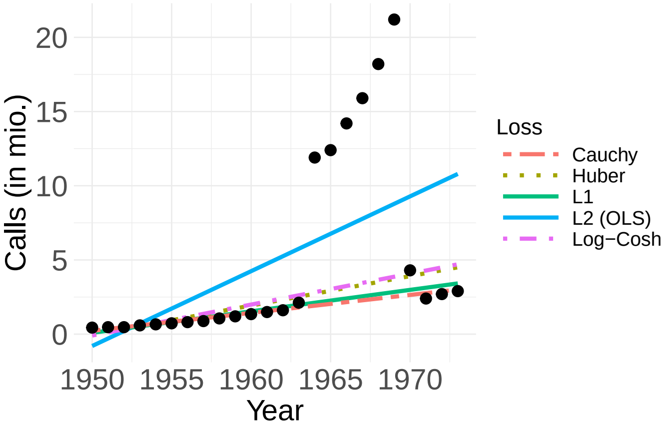
$$L(y, f(\mathbf{x})) = \frac{c^2}{2} \log\left(1 + \left(\frac{|y - f(\mathbf{x})|}{c}\right)^2\right), \quad c \in \mathbb{R}$$

- Particularly robust toward outliers (controllable via c)
- Analytic properties: differentiable, but not convex

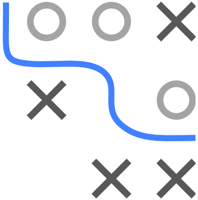


TELEPHONE DATA

- Illustrate the effect of robust losses on telephone data set
- Nr. of calls (in 10mio units) in Belgium 1950-1973
- Outliers due to a change in measurement without re-calibration

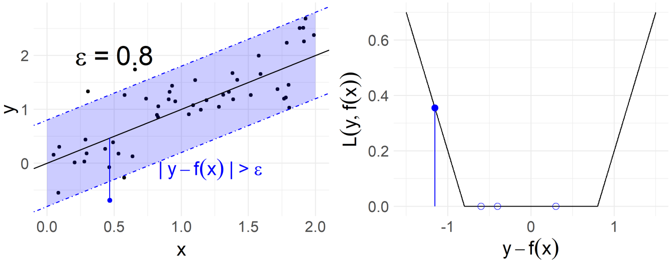


ϵ-INSENSITIVE LOSS



$$L(y, f(\mathbf{x})) = \begin{cases} 0 & \text{if } |y - f(\mathbf{x})| \leq \epsilon \\ |y - f(\mathbf{x})| - \epsilon & \text{otherwise} \end{cases}, \quad \epsilon \in \mathbb{R}_+$$

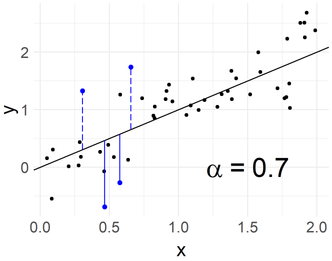
- Modification of $L1$, errors below ϵ get no penalty
- Used in SVM regression
- Properties: convex, not differentiable for $y - f(\mathbf{x}) \in \{-\epsilon, \epsilon\}$

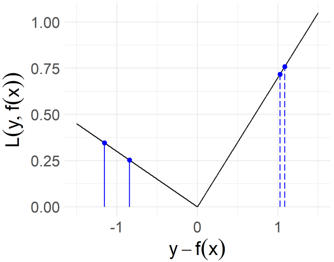


QUANTILE LOSS / PINBALL LOSS

$$L\left(y, f(\mathbf{x})\right) = \begin{cases} (1-\alpha)(f(\mathbf{x}) - y) & \text{if } y < f(\mathbf{x}) \\ \alpha(y - f(\mathbf{x})) & \text{if } y \geq f(\mathbf{x}) \end{cases}, \quad \alpha \in (0, 1)$$

- Extension of $L1$ loss (equal to $L1$ for $\alpha = 0.5$).
- Penalizes either over- or under-estimation more
- Risk minimizer is (conditional) α-quantile (median for $\alpha = 0.5$)





QUANTILE LOSS / PINBALL LOSS

- Simulate $n = 200$ samples from heteroskedastic LM
- $y = 1 + 0.2x + \varepsilon; \quad \varepsilon \sim \mathcal{N}(0, 0.5 + 0.5x); \quad x \sim \mathcal{U}[0, 10]$
- Fit LM with pinball losses to estimate α-quantiles