Solution 1: Filter problems

Let $f(x_1, x_2|\boldsymbol{\mu})$ be the density function of the bivariate Normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{I}_2$. You are given the following data generating process (DGP):

- the target $Y \sim \texttt{Bernoulli}(0.5)$,
- the conditional density $p(x_1, x_2|Y = 1) = 0.5\left(f(x_1, x_2|(1, -1)^\top) + f(x_1, x_2|(-1, 1)^\top)\right)$,
- the conditional density $p(x_1, x_2|Y = 0) = 0.5\left(f(x_1, x_2|(1, 1)^\top) + f(x_1, x_2|(-1, -1)^\top)\right)$.

(a) Sketch the DGP

(b) Compute $\mathbb{P}(Y = 1|x_1 = \widetilde{x}_1), \mathbb{P}(Y = 1|x_2 = \widetilde{x}_2)$

(c) Compute $\mathbb{P}(Y = 1|x_1 = 1, x_2 = 1)$

(d) Explain what happens if we apply mutual information as filter in this scenario