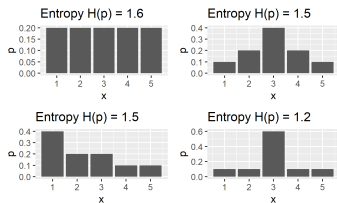
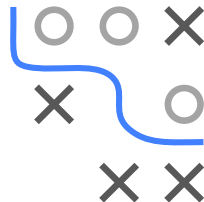


Introduction to Machine Learning

Joint Entropy and Mutual Information I



Learning goals

- Know the joint entropy
- Know conditional entropy as remaining uncertainty
- Know mutual information as the amount of information of an RV obtained by another

JOINT ENTROPY

- Recap: The **joint entropy** of two discrete RVs X and Y with joint pmf $p(x, y)$ is:

$$H(X, Y) = - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log(p(x, y)),$$

which can also be expressed as

$$H(X, Y) = -\mathbb{E} [\log(p(X, Y))].$$

- For continuous RVs X and Y with joint density $p(x, y)$, the differential joint entropy is:

$$h(X, Y) = - \int_{\mathcal{X} \times \mathcal{Y}} p(x, y) \log p(x, y) dx dy$$

For the rest of the section we will stick to the discrete case. Pretty much everything we show and discuss works in a completely analogous manner for the continuous case - if you change sums to integrals.



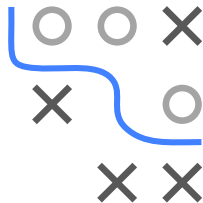
CONDITIONAL ENTROPY

- The **conditional entropy** $H(Y|X)$ quantifies the uncertainty of Y that remains if the outcome of X is given.
- $H(Y|X)$ is defined as the expected value of the entropies of the conditional distributions, averaged over the conditioning RV.
- If $(X, Y) \sim p(x, y)$, the conditional entropy $H(Y|X)$ is defined as

$$\begin{aligned} H(Y|X) &= \mathbb{E}_x[H(Y|X=x)] = \sum_{x \in \mathcal{X}} p(x) H(Y|X=x) \\ &= - \sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p(y|x) \log p(y|x) \\ &= - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x) \\ &= -\mathbb{E}[\log p(Y|X)]. \end{aligned}$$

- For the continuous case with density f we have

$$h(Y|X) = - \int f(x, y) \log f(x|y) dx dy.$$



CHAIN RULE FOR ENTROPY

The **chain rule for entropy** is analogous to the chain rule for probability and derives directly from it.

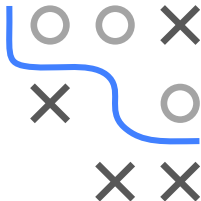
$$H(X, Y) = H(X) + H(Y|X)$$

Proof:

$$\begin{aligned} H(X, Y) &= - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y) \\ &= - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x) p(y|x) \\ &= - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x) - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x) \\ &= - \sum_{x \in \mathcal{X}} p(x) \log p(x) - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x) \\ &= H(X) + H(Y|X) \end{aligned}$$

n-variable version:

$$H(X_1, X_2, \dots, X_n) = \sum_{i=1}^n H(X_i | X_{i-1}, \dots, X_1).$$



JOINT AND CONDITIONAL ENTROPY

The following relations hold:

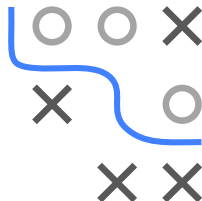
$$H(X, X) = H(X)$$

$$H(X|X) = 0$$

$$H((X, Y)|Z) = H(X|Z) + H(Y|(X, Z))$$

Which can all be trivially derived from the previous considerations.

Furthermore, if $H(X|Y) = 0$, then X is a function of Y , so for all y with $p(y) > 0$, there is only one x with $p(x, y) > 0$. Proof is not hard, but also not completely trivial.



MUTUAL INFORMATION

- The MI describes the amount of info about one RV obtained through another RV or how different their joint distribution is from pure independence.
- Consider two RVs X and Y with a joint pmf $p(x, y)$ and marginal pmfs $p(x)$ and $p(y)$. The MI $I(X; Y)$ is the Kullback-Leibler Divergence between the joint distribution and the product distribution $p(x)p(y)$:

$$\begin{aligned} I(X; Y) &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} \\ &= D_{KL}(p(x, y) \| p(x)p(y)) \\ &= \mathbb{E}_{p(x, y)} \left[\log \frac{p(X, Y)}{p(X)p(Y)} \right]. \end{aligned}$$

- For two continuous random variables with joint density $f(x, y)$:

$$I(X; Y) = \int f(x, y) \log \frac{f(x, y)}{f(x)f(y)} dx dy.$$

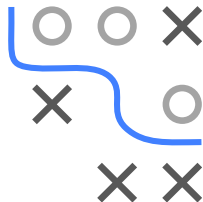


MUTUAL INFORMATION

We can rewrite the definition of mutual information $I(X; Y)$ as

$$\begin{aligned} I(X; Y) &= \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} \\ &= \sum_{x,y} p(x, y) \log \frac{p(x|y)}{p(x)} \\ &= - \sum_{x,y} p(x, y) \log p(x) + \sum_{x,y} p(x, y) \log p(x|y) \\ &= - \sum_x p(x) \log p(x) - \left(- \sum_{x,y} p(x, y) \log p(x|y) \right) \\ &= H(X) - H(X|Y). \end{aligned}$$

So, $I(X; Y)$ is reduction in uncertainty of X due to knowledge of Y .



MUTUAL INFORMATION

The following relations hold:

$$I(X; Y) = H(X) - H(X|Y)$$

$$I(X; Y) = H(Y) - H(Y|X)$$

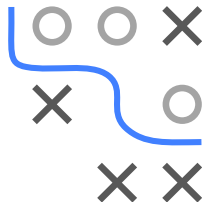
$$I(X; Y) \leq \min\{H(X), H(Y)\} \text{ if } X, Y \text{ are discrete RVs}$$

$$I(X; Y) = H(X) + H(Y) - H(X, Y)$$

$$I(X; Y) = I(Y; X)$$

$$I(X; X) = H(X)$$

All of the above are trivial to prove.



MUTUAL INFORMATION - EXAMPLE

Let X, Y have the following joint distribution:

	X_1	X_2	X_3	X_4
Y_1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$
Y_2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$
Y_3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
Y_4	$\frac{1}{4}$	0	0	0



Marginal distribution of X is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ and marginal distribution of Y is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and hence $H(X) = \frac{7}{4}$ bits and $H(Y) = 2$ bits.

MUTUAL INFORMATION - EXAMPLE

The conditional entropy $H(X|Y)$ is given by:

$$\begin{aligned} H(X|Y) &= \sum_{i=1}^4 p(Y=i) H(X|Y=i) \\ &= \frac{1}{4} H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right) \\ &\quad + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4} H(1, 0, 0, 0) \\ &= \frac{1}{4} \cdot \frac{7}{4} + \frac{1}{4} \cdot \frac{7}{4} + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 0 \\ &= \frac{11}{8} \text{ bits.} \end{aligned}$$

Similarly, $H(Y|X) = \frac{13}{8}$ bits and $H(X, Y) = \frac{27}{8}$ bits.

