

### Exercise 1: VC Dimension

Consider a binary classification learning problem with feature space  $\mathcal{X} = \mathbb{R}$  and label space  $\mathcal{Y} = \{-1, 1\}$ . Moreover, let

$$\mathcal{H} = \{h_{a,b} : \mathcal{X} \rightarrow \mathcal{Y} \mid a, b \in \mathbb{R}, a \leq b\}$$

be the hypothesis space of interval classifiers on the reals, where  $h_{a,b}(x) = 1$  for  $x \in [a, b]$  and  $= -1$  otherwise, and

$$\mathcal{H}' = \{h_c : \mathcal{X} \rightarrow \mathcal{Y} \mid c \in \mathbb{R}\}$$

be the hypothesis space of neighborhood classifiers, where  $h_c(x) = 1$  for  $x \in [c - 1, c + 1]$  and  $= -1$  otherwise.

An assignment for a set of points  $x_1, \dots, x_N \in \mathcal{X}$  by means of  $h \in \mathcal{H}$  (or  $h \in \mathcal{H}'$ ) is the vector  $(h(x_1), \dots, h(x_N))^T \in \mathcal{Y}^N$ . A set of points is shattered by  $\mathcal{H}$  (or  $\mathcal{H}'$ ) if we can find for any  $(y_1, \dots, y_N)^T \in \mathcal{Y}^N$  a hypothesis  $h \in \mathcal{H}$  (or  $h \in \mathcal{H}'$ ) such that

$$(y_1, \dots, y_N)^T = (h(x_1), \dots, h(x_N))^T.$$

The maximal number of points  $N$  which can be shattered by  $\mathcal{H}$  (or  $\mathcal{H}'$ ) is the VC-dimension of  $\mathcal{H}$  (or  $\mathcal{H}'$ ) and denoted by  $VC_p(\mathcal{H})$  (or  $VC_p(\mathcal{H}')$ ), where  $p$  is the dimension of  $\mathcal{X}$ .

- (a) Show that  $\mathcal{H}$  is “richer” than  $\mathcal{H}'$  in the sense that  $\mathcal{H}' \subseteq \mathcal{H}$  but  $\mathcal{H} \not\subseteq \mathcal{H}'$ .
- (b) Consider three (arbitrary) points  $x_1, x_2, x_3 \in \mathcal{X}$ . How many assignments are in general possible for these three points?
- (c) Now assume that the points are such that  $x_1 < x_2 < x_3$ . Is there an assignment which cannot be generated by some  $h \in \mathcal{H}$ ?

- (d) What does the latter mean for the VC-dimension of  $\mathcal{H}$  and  $\mathcal{H}'$ ?
- (e) Specify two points  $x'_1, x'_2 \in \mathcal{X}$  such that they can be shattered by  $\mathcal{H}'$ .
- (f) What does the latter mean for the VC-dimension of  $\mathcal{H}'$  and  $\mathcal{H}$ ?
- (g) Quiz time: Log in to Particify (<https://partici.fi/63221686>) and try to answer the questions for **Week 8**.