Solution 1: Hard Margin Classifier

- (a) The dataset is linearly separable. A hard margin SVM is suitable as a solution.
 - The safety margin γ should be as large as possible subject to the constraint equations:

$$y^{(i)}\left(\left\langle \boldsymbol{\theta}, \mathbf{x}^{(i)} \right\rangle + \theta_0\right) \ge 1$$

We can derive graphically that the separating hyperplane lies between the points $(4,1)^T$ and $(6,1)^T$. The maximum margin is achieved when the hyperplane lies exactly between these two points, with a value of $\gamma = 1$.

 \bullet The norm of $\pmb{\theta}$ can be calculated as :

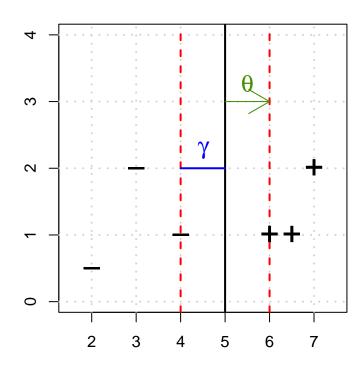
$$\|\boldsymbol{\theta}\| = \frac{1}{\gamma} = \frac{1}{1} = 1$$

• By inspection, the separating hyperplane equation is given by:

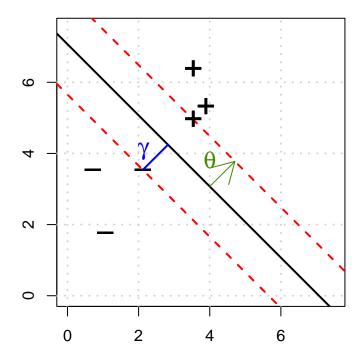
$$x_1 = 5$$

Using the formula $\theta^T \mathbf{x} + \theta_0 = 0$, we obtain that $\theta_1 = 1$, $\theta_2 = 0$ and $\theta_0 = -5$

• The support vectors are the ones that determine the margins, in this case: $(4,1)^T$ and $(6,1)^T$

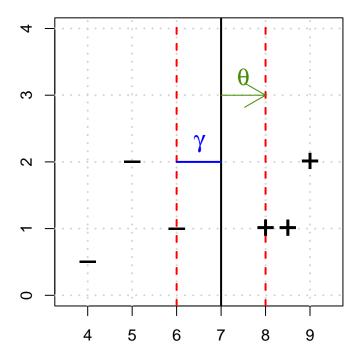


- (b) Let's see what happens to the points calculated in (a) if we apply some changes to the problem.
 - If the points are rotated 45 degrees counterclockwise, the problem now looks like this:



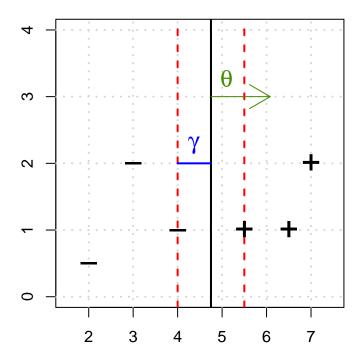
The safety margin γ and thus the norm of θ remain the same. However, the direction of θ changed because of the rotation. The support vectors are still the same as before, but rotated.

• We are now gonna shift all points by 2 to the right:



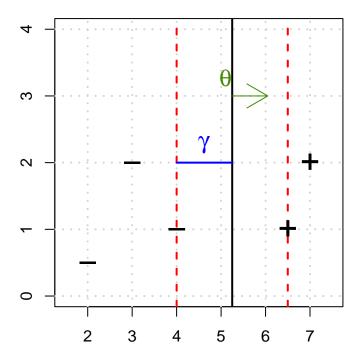
All magnitudes remain the same with the exception of θ_0 , which is now -7. The support vectors are now shifted by 2 to the right.

 \bullet If one SV moves closer to the boundary $(6,1) \to (5.5,1)$:



The margin γ is now smaller, and thus $\|\boldsymbol{\theta}\|$ is bigger. The direction of $\boldsymbol{\theta}$ and θ_0 remain the same. The support vectors are now $(4,1)^T$ and $(5.5,1)^T$.

• If we remove a support vector such as $(6,1)^T$:



The margin γ increases, and thus $\|\boldsymbol{\theta}\|$ will decrease. The direction of $\boldsymbol{\theta}$ remains the same but θ_0 changes. The new Support vectors are $(4,1)^T$ and $(6.5,1)^T$.

Summarizing the four cases in one table:

| γ | $\ oldsymbol{	heta}\ $ | θ | $	heta_0$ | SV |
|----------|------------------------|-----------------------------|-----------|-------------------------------|
| 1 | 1 | $\frac{1}{\sqrt{2}}(1,1)^T$ | -5 | Rotation $((4,1)^T, (6,1)^T)$ |
| 1 | 1 | $(1,0)^T$ | -7 | $(6,1)^T, (8,1)^T$ |
| 0.75 | 4/3 | $(4/3,0)^T$ | -6.33 | $(4,1)^T, (5.5,1)^T$ |
| 1.25 | 4/5 | $(4/5,0)^T$ | -4.2 | $(4,1)^T, (6.5,1)^T$ |