

# Introduction to Machine Learning

## Differential Entropy



### Learning goals

- Know that the entropy expresses expected information for continuous RVs
- Know the basic properties of the differential entropy

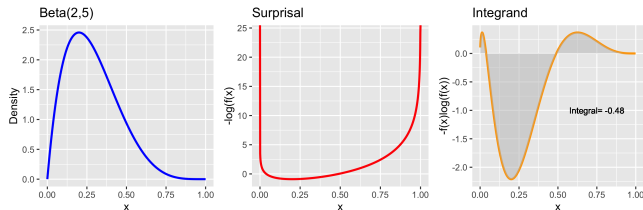


# DIFFERENTIAL ENTROPY

- For a continuous random variable  $X$  with density function  $f(x)$  and support  $\mathcal{X}$ , the analogue of entropy is **differential entropy**:

$$h(X) := h(f) := -\mathbb{E}[\log(f(x))] = - \int_{\mathcal{X}} f(x) \log(f(x)) dx$$

- The base of the log is again somewhat arbitrary, and we could either use 2 (and measure in bits) or e (to measure in nats).
- The integral above does not necessarily exist for all densities.
- Differential entropy lacks the non-negativeness of discrete entropy:  
 $h(X) < 0$  is possible as  $f(x) > 1$  is possible:



The diffent. is given  
by the integral:  
 $h(X) = -0.48$ .



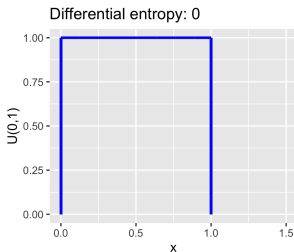
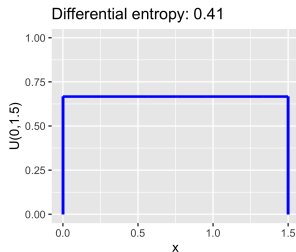
# DIFF. ENTROPY OF UNIFORM DISTRIBUTION

Let  $X$  be a uniform random variable on  $[0, a]$ .

$$\begin{aligned}h(X) &= - \int_0^a f(x) \log(f(x)) dx \\&= - \int_0^a \frac{1}{a} \log\left(\frac{1}{a}\right) dx = \log(a)\end{aligned}$$



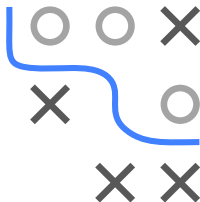
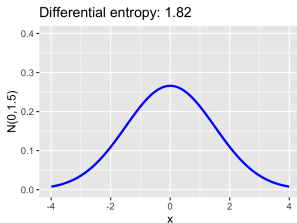
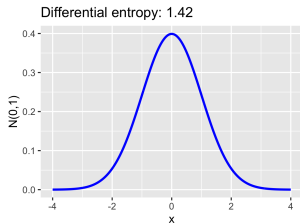
- For  $a < 1$ ,  $h(X) < 0$ .



## DIFF. ENTROPY OF GAUSSIAN

Let  $X \sim \mathcal{N}(\mu, \sigma^2)$  and let us measure in nats:

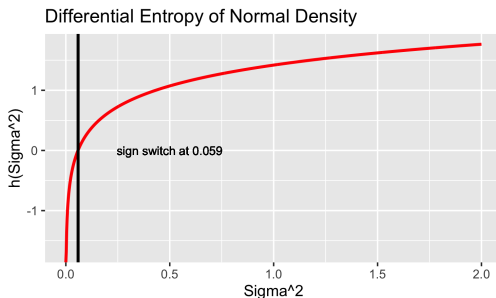
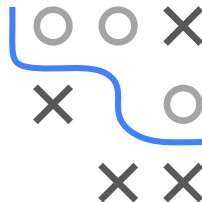
$$\begin{aligned} h(X) &= - \int_{\mathbb{R}} f(x) \log(f(x)) dx = - \int_{\mathbb{R}} f(x) \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx \\ &= - \int_{\mathbb{R}} f(x) \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right) dx + \int_{\mathbb{R}} f(x) \frac{(x-\mu)^2}{2\sigma^2} dx \\ &= - \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right) \underbrace{\int_{\mathbb{R}} f(x) dx}_{=1} + \frac{1}{2\sigma^2} \underbrace{\int_{\mathbb{R}} f(x) (x-\mu)^2 dx}_{=:\sigma^2} \\ &= \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2} = \log(\sigma\sqrt{2\pi e}) \end{aligned}$$



# DIFF. ENTROPY OF GAUSSIAN

$$h(X) = - \int_{\mathbb{R}} f(x) \log(f(x)) dx = \log(\sigma \sqrt{2\pi e})$$

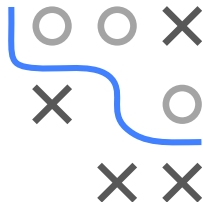
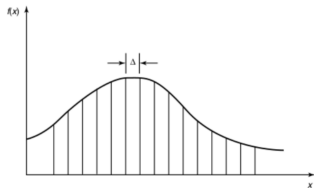
- $h(X)$  is not a function of  $\mu$  (see translation invariance later).
- As  $\sigma^2$  increases, the differential entropy also increases.
- For  $\sigma^2 < \frac{1}{2\pi e} \approx 0.059$ , it is negative.



# DIFF. ENTROPY VS. DISCRETE

It is not so simple as to characterize  $h(X)$  as a straightforward generalization of  $H(X)$  of a limiting process. Consider the quantized random variable  $X^\Delta$ , which is defined by

$$X^\Delta = x_i \quad \text{if} \quad i\Delta \leq X < (i+1)\Delta$$



If the density  $f(x)$  of the random variable  $X$  is Riemann-integrable, then

$$H(X^\Delta) + \log(\Delta) \rightarrow h(X) \text{ as } \Delta \rightarrow 0.$$

Thus, the entropy of an  $n$ -bit quantization of a continuous random variable  $X$  is approximately  $h(X) + n$ .

# JOINT DIFFERENTIAL ENTROPY

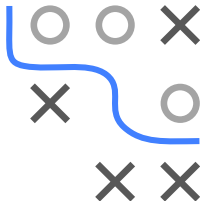
- For a continuous random vector  $X$  with density function  $f(x)$  and support  $\mathcal{X}$ , differential entropy is also defined as:

$$h(X) = h(X_1, \dots, X_n) = h(f) = - \int_{\mathcal{X}} f(x) \log(f(x)) dx$$

- Hence this also defines the joint differential entropy for a set of continuous RVs.

Entropy of a multivariate normal distribution: If  $X \sim N(\mu, \Sigma)$  is multivariate Gaussian, then

$$h(X) = \frac{1}{2} \log(2\pi e)^n |\Sigma| \quad (\text{nats})$$



# PROPERTIES OF DIFFERENTIAL ENTROPY

- 1  $h(f)$  can be negative.
- 2  $h(f)$  is additive for independent RVs.
- 3  $h(f)$  is maximized by the multivariate normal, if we restrict to all distributions with the same (co)variance, so
$$h(X) \leq \frac{1}{2} \log(2\pi e)^n |\Sigma|.$$
- 4  $h(f)$  is maximized by the continuous uniform distribution for a random variable with a fixed range.
- 5 Translation-invariant,  $h(X + a) = h(X)$ .
- 6  $h(aX) = h(X) + \log |a|$ .
- 7  $h(AX) = h(X) + \log |A|$  for random vectors and matrix  $A$ .



3) and 4) are slightly involved to prove, while the other properties are relatively straightforward to show