

Solution 1: High-dimensional Gaussian Distributions

(a) In this case, we will use the linearity of the expectation.

$$\begin{aligned}\mathbb{E}[\|X\|_1] &= \mathbb{E}\left[\sum_{j=1}^p |x_j|\right] \\ &= \sum_{j=1}^p \underbrace{\mathbb{E}|x_j|}_{=\sqrt{\frac{2}{\pi}}} \\ &= \sqrt{\frac{2}{\pi}} \sum_{j=1}^p 1 \\ &= \sqrt{\frac{2}{\pi}} p\end{aligned}\tag{1}$$

(b) Considering that the coordinates X_1, \dots, X_p are independent and identically distributed, the variance of the sum equals the sum of the variance.

$$\begin{aligned}\text{Var}(\|X\|_1) &= \text{Var}\left(\sum_{j=1}^p |x_j|\right) \\ &= \sum_{j=1}^p \underbrace{\text{Var}(|x_j|)}_{=1-\frac{2}{\pi}} \\ &= \left(1 - \frac{2}{\pi}\right) \sum_{j=1}^p 1 \\ &= \left(1 - \frac{2}{\pi}\right) p\end{aligned}\tag{2}$$

(c) A random variable which is the subtraction of two normally distributed random variables is also normal, with the following parameters:

$$\begin{aligned}X - Y = Z &\sim \mathcal{N}(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2) \\ &\sim \mathcal{N}(0, 2)\end{aligned}\tag{3}$$

We also know that the variance of a random variable multiplied by a constant is equal to the variance of the random variable scaled by the square of the constant, consequently:

$$\begin{aligned}X - Y &\sim \mathcal{N}(0, 2) \\ \frac{X - Y}{\sqrt{2}} &\sim \mathcal{N}(0, 1)\end{aligned}\tag{4}$$

We will use the equations 3 and 4 to solve the exercise:

$$\begin{aligned}
\mathbb{E} [\|X - X'\|_1] &= \mathbb{E} \left[\sum_{j=1}^p |x_j - x'_j| \right] \\
&= \sum_{j=1}^p \mathbb{E} [|x_j - x'_j|] \\
&= \sum_{j=1}^p \mathbb{E} \left[\frac{\sqrt{2}}{\sqrt{2}} |x_j - x'_j| \right] \\
&= \sqrt{2} \sum_{j=1}^p \mathbb{E} \left[\underbrace{\left| \frac{x_j - x'_j}{\sqrt{2}} \right|}_{\sim \mathcal{N}(0,1)} \right] \\
&= \sqrt{2} \sqrt{\frac{2}{\pi}} \sum_{j=1}^p 1 \\
&= \frac{2p}{\sqrt{\pi}}
\end{aligned} \tag{5}$$

(d) Using equations 3 and 4 again, we get:

$$\begin{aligned}
\text{Var} (\|X - X'\|_1) &= \sum_{j=1}^p \text{Var} (|x_j - x'_j|) \\
&= \sum_{j=1}^p \left(\frac{\sqrt{2}}{\sqrt{2}} |x_j - x'_j| \right) \\
&= 2 \sum_{j=1}^p \text{Var} \left(\underbrace{\left| \frac{x_j - x'_j}{\sqrt{2}} \right|}_{\sim \mathcal{N}(0,1)} \right) \\
&= 2 \left(1 - \frac{\pi}{2} \right) \sum_{j=1}^p 1 \\
&= 2p \left(1 - \frac{\pi}{2} \right)
\end{aligned} \tag{6}$$

(e) Using the linearity of the expectation and the fact that \mathbf{x} is deterministic:

$$\begin{aligned}
\mathbb{E} [\langle X, \mathbf{x} \rangle] &= \mathbb{E} \left[\sum_{j=1}^p X_j x_j \right] \\
&= \sum_{j=1}^p \mathbb{E} [X_j x_j] \\
&= \sum_{j=1}^p x_j \mathbb{E} [X_j] \\
&= 0
\end{aligned} \tag{7}$$

We will again use the independency of the coordinates X_1, \dots, X_p and the fact that \mathbf{x} is deterministic:

$$\begin{aligned}
 \text{Var}(\langle X, \mathbf{x} \rangle) &= \text{Var} \left(\sum_{j=1}^p X_j x_j \right) \\
 &= \sum_{j=1}^p \text{Var}(X_j x_j) \\
 &= \sum_{j=1}^p x_j^2 \underbrace{\text{Var}(X_j)}_{=1} \\
 &= \sum_{j=1}^p x_j^2 \\
 &= \|\mathbf{x}\|_2^2
 \end{aligned} \tag{8}$$