

### Solution 1: Hard Margin Classifier

(a) The dataset is linearly separable. A hard margin SVM is suitable as a solution.

- The safety margin  $\gamma$  should be as large as possible subject to the constraint equations:

$$y^{(i)} \left( \langle \boldsymbol{\theta}, \mathbf{x}^{(i)} \rangle + \theta_0 \right) \geq 1$$

We can derive graphically that the separating hyperplane lies between the points  $(4, 1)^T$  and  $(6, 1)^T$ . The maximum margin is achieved when the hyperplane lies exactly between these two points, with a value of  $\gamma = 1$ .

- The norm of  $\boldsymbol{\theta}$  can be calculated as :

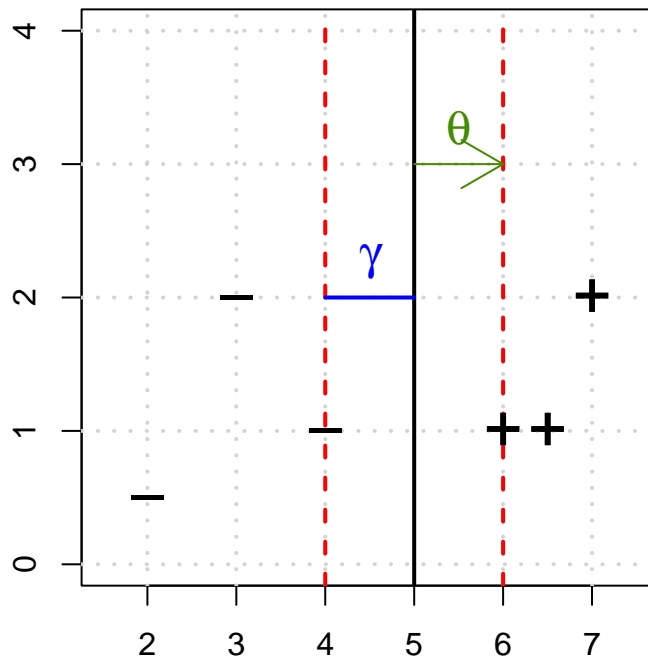
$$\|\boldsymbol{\theta}\| = \frac{1}{\gamma} = \frac{1}{1} = 1$$

- By inspection, the separating hyperplane equation is given by:

$$x_1 = 5$$

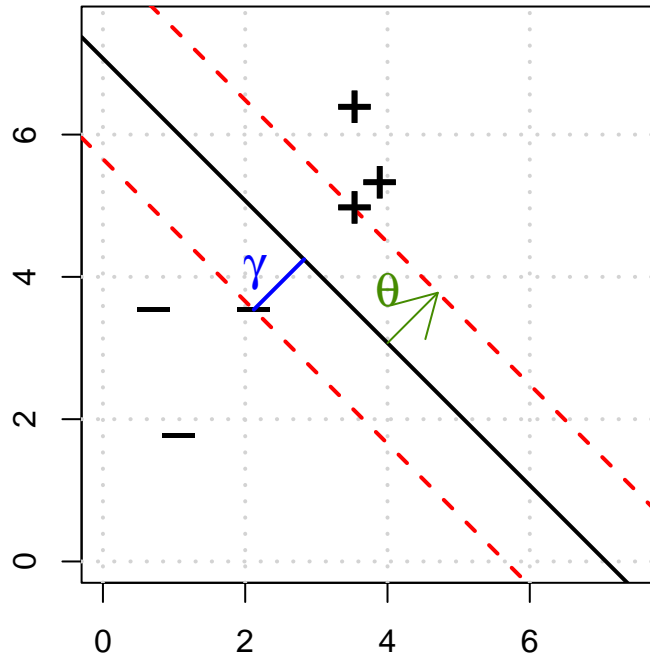
Using the formula  $\boldsymbol{\theta}^T \mathbf{x} + \theta_0 = 0$ , we obtain that  $\theta_1 = 1$ ,  $\theta_2 = 0$  and  $\theta_0 = -5$

- The support vectors are the ones that determine the margins, in this case:  $(4, 1)^T$  and  $(6, 1)^T$



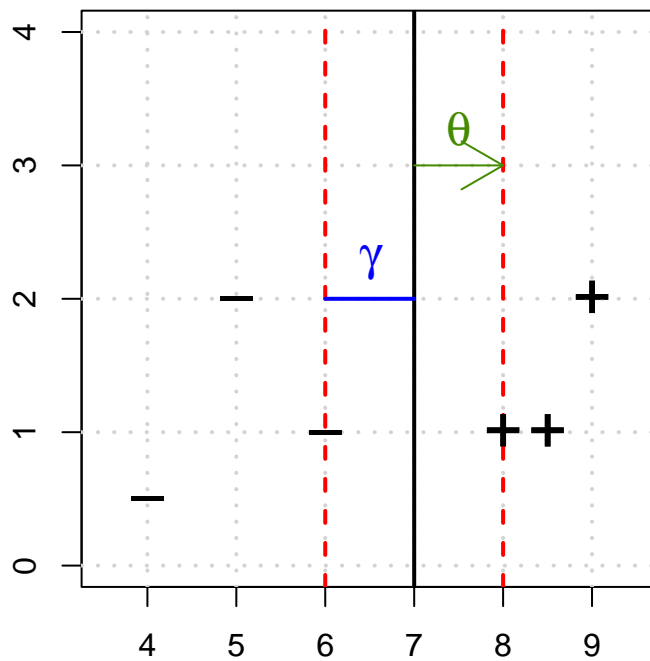
(b) Let's see what happens to the points calculated in (a) if we apply some changes to the problem.

- If the points are rotated 45 degrees counterclockwise, the problem now looks like this:



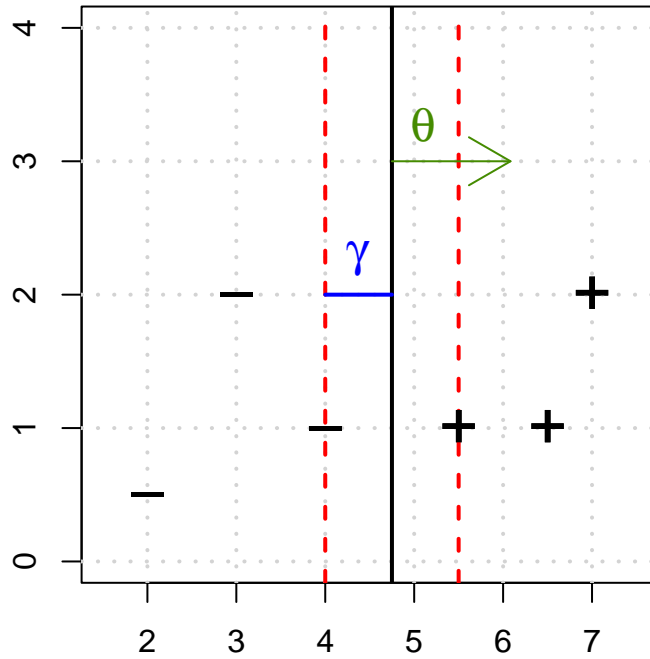
The safety margin  $\gamma$  and thus the norm of  $\theta$  remain the same. However, the direction of  $\theta$  changed because of the rotation. The support vectors are still the same as before, but rotated.

- We are now gonna shift all points by 2 to the right:



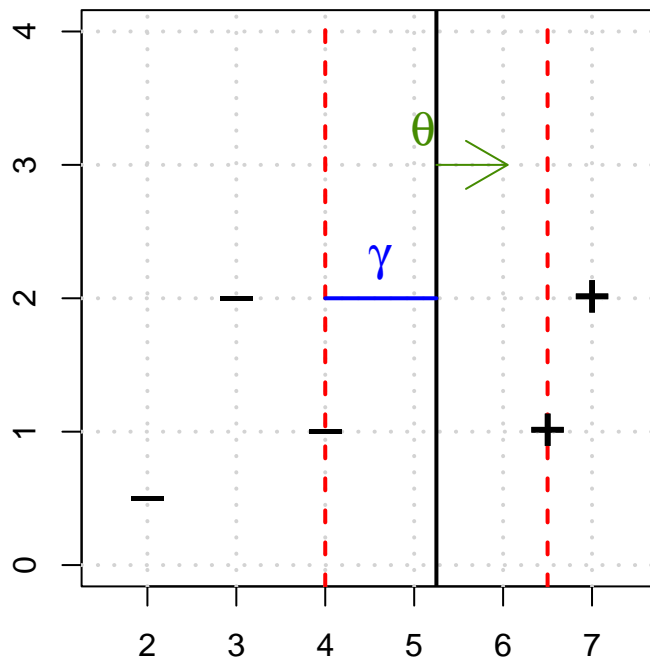
All magnitudes remain the same with the exception of  $\theta_0$ , which is now  $-7$ . The support vectors are now shifted by 2 to the right.

- If one SV moves closer to the boundary  $(6, 1) \rightarrow (5.5, 1)$  :



The margin  $\gamma$  is now smaller, and thus  $\|\theta\|$  is bigger. The direction of  $\theta$  and  $\theta_0$  remain the same. The support vectors are now  $(4, 1)^T$  and  $(5.5, 1)^T$ .

- If we remove a support vector such as  $(6, 1)^T$ :



The margin  $\gamma$  increases, and thus  $\|\theta\|$  will decrease. The direction of  $\theta$  remains the same but  $\theta_0$  changes. The new Support vectors are  $(4, 1)^T$  and  $(6.5, 1)^T$ .

Summarizing the four cases in one table:

$\gamma$	$\ \boldsymbol{\theta}\ $	$\boldsymbol{\theta}$	$\theta_0$	SV
1	1	$\frac{1}{\sqrt{2}}(1, 1)^T$	-5	Rotation $((4, 1)^T, (6, 1)^T)$
1	1	$(1, 0)^T$	-7	$(6, 1)^T, (8, 1)^T$
0.75	4/3	$(4/3, 0)^T$	-6.33	$(4, 1)^T, (5.5, 1)^T$
1.25	4/5	$(4/5, 0)^T$	-4.2	$(4, 1)^T, (6.5, 1)^T$