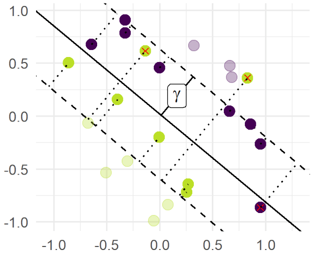
Introduction to Machine Learning

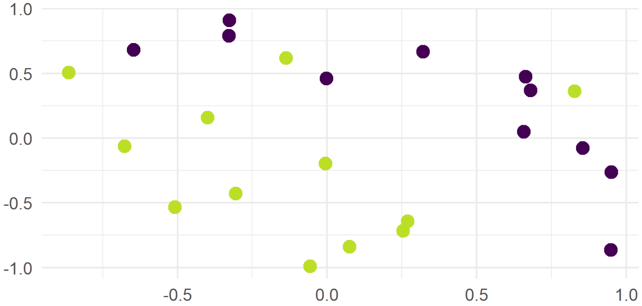
Soft-Margin SVM



Learning goals

- Understand that the hard-margin SVM problem is only solvable for linearly separable data
- Know that the soft-margin SVM problem therefore allows margin violations
- The degree to which margin violations are tolerated is controlled by a hyperparameter

NON-SEPARABLE DATA



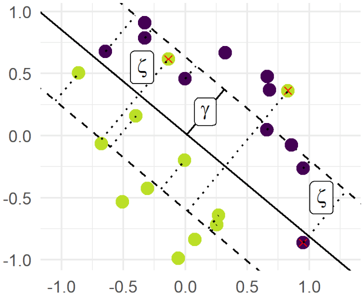
- Assume that dataset $\mathcal{D}$ is not linearly separable.
- Margin maximization becomes meaningless because the hard-margin SVM optimization problem has contradictory constraints and thus an empty **feasible region**.

MARGIN VIOLATIONS

- We still want a large margin for most of the examples.
- We allow violations of the margin constraints via slack vars $\zeta^{(i)} \geq 0$

$$y^{(i)} \left(\left\langle \boldsymbol{\theta}, \mathbf{x}^{(i)} \right\rangle + \theta_0 \right) \geq 1 - \zeta^{(i)}$$

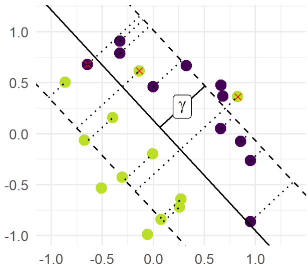
- Even for separable data, a decision boundary with a few violations and a large average margin may be preferable to one without any violations and a small average margin.



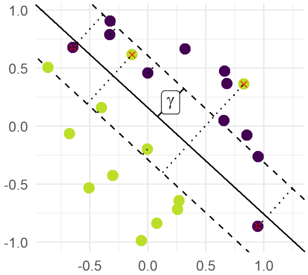
We assume $\gamma = 1$ to not further complicate presentation.

MARGIN VIOLATIONS

- Now we have two distinct and contradictory goals:
  1. Maximize the margin.
  2. Minimize margin violations.
- Let's minimize a weighted sum of them: $\frac{1}{2}\|\boldsymbol{\theta}\|^2 + C\sum_{i=1}^{n}\zeta^{(i)}$
- Constant $C > 0$ controls the relative importance of the two parts.



C = 0.5



C = 100

SOFT-MARGIN SVM

The linear **soft-margin** SVM is the convex quadratic program:

$$
\begin{aligned}
\min_{\boldsymbol{\theta},\boldsymbol{\theta}_0,\zeta^{(i)}} \quad & \frac{1}{2}\|\boldsymbol{\theta}\|^2 + C\sum_{i=1}^{n}\zeta^{(i)} \\
\text{s.t.} \quad & y^{(i)}\left(\left\langle \boldsymbol{\theta}, \mathbf{x}^{(i)} \right\rangle + \boldsymbol{\theta}_0\right) \geq 1 - \zeta^{(i)} \quad \forall\, i \in \{1, \dots, n\}, \\
\text{and} \quad & \zeta^{(i)} \geq 0 \quad \forall\, i \in \{1, \dots, n\}.
\end{aligned}
$$

This is called "soft-margin" SVM because the "hard" margin constraint is replaced with a "softened" constraint that can be violated by an amount $\zeta^{(i)}$.

SOFT-MARGIN SVM DUAL FORM

Can be derived exactly as for the hard margin case.

$$
\begin{aligned}
\max_{\boldsymbol{\alpha}\in\mathbb{R}^n} \quad & \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j y^{(i)} y^{(j)} \left\langle \mathbf{x}^{(i)}, \mathbf{x}^{(j)} \right\rangle \\
\text{s.t.} \quad & 0 \leq \alpha_i \leq C, \\
& \sum_{i=1}^{n} \alpha_i y^{(i)} = 0,
\end{aligned}
$$

or, in matrix notation:

$$
\begin{aligned}
\max_{\boldsymbol{\alpha}\in\mathbb{R}^n} \quad & \mathbf{1}^T\boldsymbol{\alpha} - \frac{1}{2}\boldsymbol{\alpha}^T \operatorname{diag}(\mathbf{y})\boldsymbol{K}\operatorname{diag}(\mathbf{y})\boldsymbol{\alpha} \\
\text{s.t.} \quad & \alpha^T\mathbf{y} = 0, \\
& 0 \leq \boldsymbol{\alpha} \leq C,
\end{aligned}
$$

with $\boldsymbol{K} := \mathbf{X}\mathbf{X}^T$.

COST PARAMETER C

- The parameter C controls the trade-off between the two conflicting objectives of maximizing the size of the margin and minimizing the frequency and size of margin violations.
- It is known under different names, such as “trade-off parameter”, “regularization parameter”, and “complexity control parameter”.
- For sufficiently large C margin violations become extremely costly, and the optimal solution does not violate any margins if the data is separable. The hard-margin SVM is obtained as a special case.

SUPPORT VECTORS

There are three types of training examples:

- Non-SVs have a margin $>$ 1 and can be removed from the problem without changing the solution.
- Some SVs are located exactly on the margin and have $yf(\mathbf{x}) = 1$.
- Other SVs are margin violators, with $yf(\mathbf{x}) < 1$, and have an associated positive slack $\zeta^{(i)} > 0$. They are misclassified if $\zeta^{(i)} \geq 1$.