Solution 1: VC Dimension

(a) The two hypothesis spaces can be characterized as follows:

$$h_{a,b}(x) = \begin{cases} 1 & , x \in [a, b] \\ -1 & , x \notin [a, b] \end{cases} \qquad h_c(x) = \begin{cases} 1 & , x \in [c-1, c+1] \\ -1 & , x \notin [c-1, c+1] \end{cases} \tag{1}$$

- For any $c \in \mathbb{R}$, we can find a pair $\{a \in \mathbb{R}, b \in \mathbb{R} | a < b\}$ such that $h_{a,b}(x) = h_c(x)$

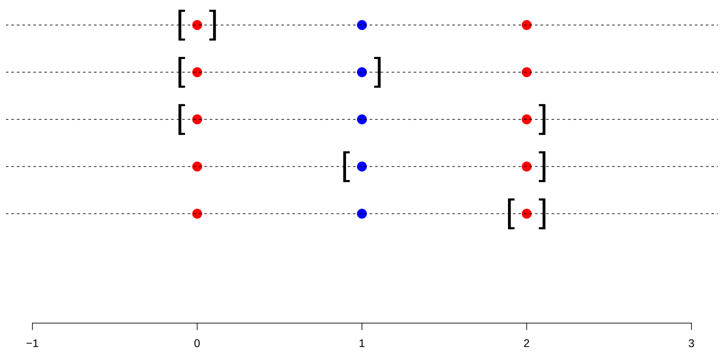
$$a = c - 1, \quad b = c + 1 \longrightarrow [a, b] = [c-1, c+1]$$

- However, for some combinations of $a \in \mathbb{R}$ and $b \in \mathbb{R}$, we can't find any $c \in \mathbb{R}$ where $h_{a,b}(x) = h_c(x)$ exists :

$$a = 0, \quad b = 3 \longrightarrow [a, b] \neq [c-1, c+1]$$

(b) The label space $\mathcal{Y} = \{-1, 1\}$ has two possible outcomes, and we have three points $\{x_1, x_2, x_3\}$. The number of assigments in this case is $2^3 = 8$.

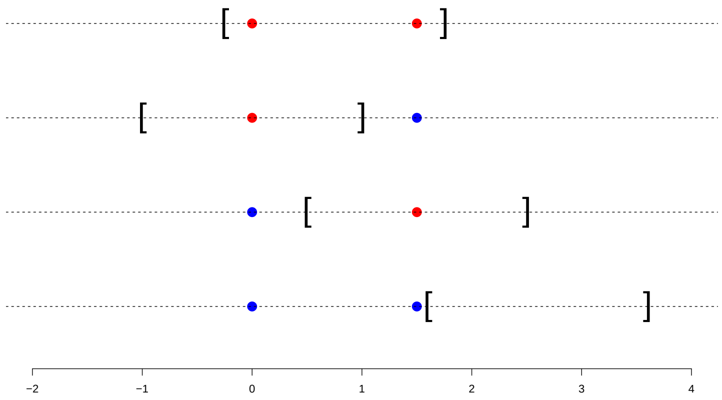
(c) We need to find an assignment $y = (y_1, y_2, y_3)^T$ such that no $h \in \mathcal{H} \rightarrow \hat{y} = (h_{ab}(x_1), h_{ab}(x_2), h_{ab}(x_3))^T$ can generate the assignment y. We can show that the assignment $y = (1, -1, 1)^T$ cannot be generated by any $h \in \mathcal{H}$.



(d) The latter tells us that the VC-dimension of the space $\mathcal{H}$, $VC(\mathcal{H}) < 3$. As we already proved that $\mathcal{H}' \subset \mathcal{H}$, we can deduce that $VC(\mathcal{H}') \leq VC(\mathcal{H}) < 3$.

(e) We need to find a pair of points $\{x_1, x_2\}$ such that any assignment $\{y_1, y_2\}$ can be shattered by $\mathcal{H}'$. The points $\{0, 1.5\}$ match these conditions:

$$\begin{cases} \{y_1, y_2\} = (1, 1)^T \rightarrow c = 0.75 \rightarrow h_c(0) = h_c(1.5) = 1 \\ \{y_1, y_2\} = (1, -1)^T \rightarrow c = 0 \rightarrow h_c(0) = 1, h_c(1.5) = -1 \\ \{y_1, y_2\} = (-1, 1)^T \rightarrow c = 1.5 \rightarrow h_c(0) = -1, h_c(1.5) = 1 \\ \{y_1, y_2\} = (-1, -1)^T \rightarrow c = 2.6 \rightarrow h_c(0) = h_c(1.5) = -1 \end{cases} \tag{2}$$



(f) From the previous question, we can deduce that $VC(\mathcal{H}') \geq 2$. As we already proved that $\mathcal{H}' \subset \mathcal{H}$, $VC(\mathcal{H}) \geq VC(\mathcal{H}') \geq 2$. Therefore, $VC(\mathcal{H}) = VC(\mathcal{H}') = 2$.