Exercise 1: High-dimensional Gaussian Distributions

Consider a random vector $X = (X_1, \ldots, X_p)^\top \sim \mathcal{N}(0, \boldsymbol{I})$, i.e., a multivariate normally distributed vector with mean vector zero and covariance matrix being the identity matrix of dimension $p \times p$. In this case, the coordinates $X_1, \ldots, X_p$ are i.i.d. each with distribution $\mathcal{N}(0, 1)$. Recall that the L_1-norm of a vector $\mathbf{x} = (x_1, \ldots, x_p)^\top \in \mathbb{R}^p$ is defined as

$$\|\mathbf{x}\|_1 = \sum\nolimits_{i=1}^p |x_i|.$$

(a) What is $\mathbb{E}[\|X\|_1]$? (Hint: $\mathbb{E}|X_i| = \sqrt{\frac{2}{\pi}}$.)

(b) What is $\mathsf{Var}[\|X\|_1]$? (Hint: $\mathsf{Var}|X_i| = 1 - \frac{2}{\pi}$.)

(c) Now let $X' = (X_1', \ldots, X_p')^\top \sim \mathcal{N}(0, \boldsymbol{I})$ be another multivariate normally distributed vector with mean vector zero and covariance matrix being the identity matrix of dimension $p \times p$. Further, assume that X and X' are independent. What is $\mathbb{E}(\|X - X'\|_1)$?

(d) What is $\mathsf{Var}(\|X - X'\|_1)$?

(e) Let $\mathbf{x} = (x_1, \ldots, x_p)^\top \in \mathbb{R}^p$ be some arbitrary deterministic vector. Compute $\mathbb{E}\langle X, \mathbf{x}\rangle$ and $\mathsf{Var}(\langle X, \mathbf{x}\rangle)$.