Email training, N1

Problem 5.1. Show that for positive reals a, b, c we have $abc = 1$ if and only if

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} = 1.$$

Solution 5.1. First, assume that $abc = 1$. Let x, y, z be positive real numbers such that $a = y/x$, $b = z/y$ and $c = x/z$. Then we have

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} =$$

$$\frac{1}{1+\frac{y}{x}+\frac{z}{x}} + \frac{1}{1+\frac{z}{y}+\frac{x}{y}} + \frac{1}{1+\frac{x}{z}+\frac{y}{z}} =$$

$$\frac{x}{x+y+z} + \frac{y}{x+y+z} + \frac{z}{x+y+z} = 1.$$

Now assume we have

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} = 1.$$

This is equivalent to

$$\frac{1}{1+a+ab} + \frac{a}{a+ab+abc} + \frac{ab}{ab+abc+a^2bc} = 1.$$

Subtracting this with the following identity

$$\frac{1}{1+a+ab} + \frac{a}{1+a+ab} + \frac{ab}{1+a+ab} = 1$$

we get

$$\frac{a(abc-1)}{(1+a+ab)(a+ab+abc)} + \frac{ab(a^2bc+abc-a-1)}{(1+a+ab)(ab+abc+a^2bc)} = 0.$$

This means

$$(abc-1)\Big(\frac{a}{(1+a+ab)(a+ab+abc)} + \frac{ab(a+1)}{(1+a+ab)(ab+abc+a^2bc)}\Big) = 0.$$

Since the second multiplier is positive, then $abc - 1 = 0$.

Problem 5.2. Let $p > 3$ be a prime such that $p \equiv 3[4]$. Given a positive integer a_0 define the sequence $a_0, a_1, \dots$ of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \dots$. Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \dots$ is not constant modulo p for any positive integer N.

Solution 5.2. Let p be a prime with residue 3 modulo 4 and $p > 3$. Then $p-1 = 2u$, where $u > 1$ is odd. Choose $a_0 = 2$. The order of 2 modulo p is a divisor of $\phi(p) = p - 1 = 2u$, but not a divisor of 2 since $1 < 2^2 < p$. Hence the order of 2 modulo p is not a power of 2.

By definition we see that $a_n = a^{2^{1+2+\Delta\Delta\Delta+n}}$ and since the order of $a_0 = 2$ modulo p is not a power of 2, we know that

$$a_n \neq 1[p]$$

for all $n = 1, 2, 3, ...$ We proof the statement by contradiction. Assume there exists a positive integer N such that $a_n \equiv a_N[p]$ for all $n \geq N$. Let $d > 1$ be the order of a_N modulo p. Then

$$a + N \equiv a_n \equiv a_{n+1} = a_n^{2^{n+1}} \equiv a_N^{2^{n+1}}[p],$$

and hence

$$a_N^{2^{n+1}-1} \equiv 1[p]$$

for all $n \geq N$. Now d divides $2^{n+1} - 1$ for all $n \geq N$, but this is a contradiction since $gcd(2^{n+1} - 1, 2^{n+2} - 1) = 1$. Hence there does not exist such an N.

Problem 5.3. Three pairwise distinct positive integers a, b, c with $gcd(a; b; c) = 1$, satisfy

$$a \mid (b - c)^2; \quad b \mid (c - a)^2 \quad \text{and} \quad c \mid (a - b)^2.$$

Prove that there does not exist a non-degenerate triangle with side lengths a, b, c.

Solution 5.3. First observe that these numbers are pairwise coprime. Indeed, if, say, a and b are divisible by a prime p, then p divides b, which divides $(a - c)^2$, hence p divides $a - c$, and therefore p divides c. Thus, p is a common divisor of these three numbers, a contradiction. Now consider the number

$$M = 2ab + 2bc + 2ac - a^2 - b^2 - c^2.$$

It is clear from the problem condition that M is divisible by a, b, c, and therefore M is divisible by abc. Assume that a triangle with sides a, b, c exists. Then $a < b + c$, and so $a^2 < ab + ac$. Analogously, we have $b^2 < bc + ba$ and $c^2 < ca + cb$. Summing these three inequalities leads to $M > 0$, and hence $M \geq abc$.

On the other hand, $a^2 + b^2 + c^2 > ab + bc + ac$, and therefore $M < ab + bc + ac$. Supposing, with no loss of generality, $a > b > c$, we must have $M < 3ab$. Taking into account the inequality $M \geq abc$, we conclude that $c = 1$ or $c = 2$ are the only possibilities.

For $c = 1$ we have $b < a < b + 1$ (the first inequality is our assumption, the second is the triangle inequality), a contradiction. For $c = 2$ we have $b < a < b + 2$, i.e. $a = b + 1$. But then $1 = (a - b)^2$ is not divisible by $c = 2$.

Problem 5.4. Prove that any sequence of $n^2 + 1$ real numbers contains a subsequence of length $n + 1$ which is either increasing or decreasing.

Solution 5.4. Let the sequence $a_1, a_2, \ldots, a_{n^2+1}$ is given. For any integer $1 \leq i \leq n^2 + 1$ denote by x_i the length of the longest increasing subsequence that the last term is a_i. For example $x_1 = 1$, also $x_2 = 2$ if $a_2 \geq a_1$ and $x_2 = 1$ if $a_2 < a_1$. Analogously denote by y_i the length of the longest decreasing subsequence that the last term is a_i. Note, that if $a_i \geq a_j$ with $i > j$ then then $x_i > x_j$, otherwise $y_i > y_j$. This means $(x_i, y_i) \neq (y_i, y_j)$ whenever $i \neq j$. Now consider the sequence of pairs

$$(x_1, y_1), \ (x_2, y_2), \ (x_3, y_3), \ \ldots, (x_{n^2+1}, y_{n^2+1}).$$

Since all of them are different and x_i, y_i are positive integers, then by Pigeonhole principle some pair contains a number which is bigger than n. So there exists a monotone subsequence having more than n terms.

Problem 5.5. There are n integers, each of them equal to 1 written on a blackboard. At each step, you erase any two numbers a and b and replace them with $\frac{a+b}{4}$. After $n-1$ steps, there is only one number left on the blackboard. Prove that this number is at least $\frac{1}{n}$.

Solution 5.5. Let S be the sum of the reciprocals of the numbers. We claim that S is non-increasing. When we replace a and b with $\frac{a+b}{4}$, the value of S decreases by

$$\frac{1}{a} + \frac{1}{b} - \frac{4}{a+b} = \frac{(a-b)^2}{ab(a+b)},$$

which is nonnegative. Initially S is equal to n. Therefore, if the last number is x, then x must satisfy $\frac{1}{x} \leq n$, so $x \geq \frac{1}{n}$.

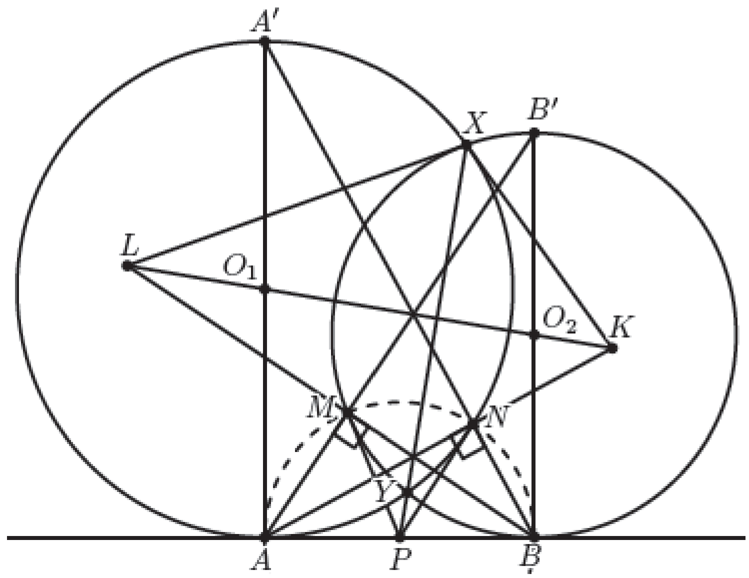
Problem 5.6. Is it true that in any convex n-gon with $n > 3$, there exists a vertex and a diagonal passing through this vertex such that the angles of this diagonal with both sides adjacent to this vertex are acute?

Solution 5.6. Suppose the answer is no. Given a convex n-gon ($n > 3$), consider its longest diagonal AD (if the longest diagonal is not unique, choose an arbitrary on among them). Let B and C be the vertices neighboring to A. Without loss of generality assume that $\angle BAD \geq 90^o$. This means $BD > AD$, so BD is not a diagonal and hence is a side of the n-gon. Furthermore, $\angle ADB < 90^o$. Let C' be the vertex neighboring to D and distinct from B. Then $\angle ADC' \geq 90^o$. Similarly, $AC' > AD$, so AC' is a side, $C' = C$ and $n = 4$. Angles BAC and BDC are obtuse, so BC is longer than AC and BD, hence $BC > AD$ and AD is not the longest diagonal, a contradiction. Hence the statement is true.

Problem 5.7. Circles ω_1 and ω_2 have centres O_1 and O_2, respectively. These two circles intersect at points X and Y. AB is common tangent line of these two circles such that A lies on ω_1 and B lies on ω_2. Let tangents to ω_1 and ω_2 at X intersect O_1O_2 at points K and L, respectively. Suppose that line BL intersects ω_2 for the second time at M and line AK intersects ω_1 for the second time at N. Prove that lines AM, BN and O_1O_2 concur.

Solution 5.7. -

Let P be the midpoint of AB; Since P has the same power with respect to both circles, it lies on the radical axis of them, which is line XY.

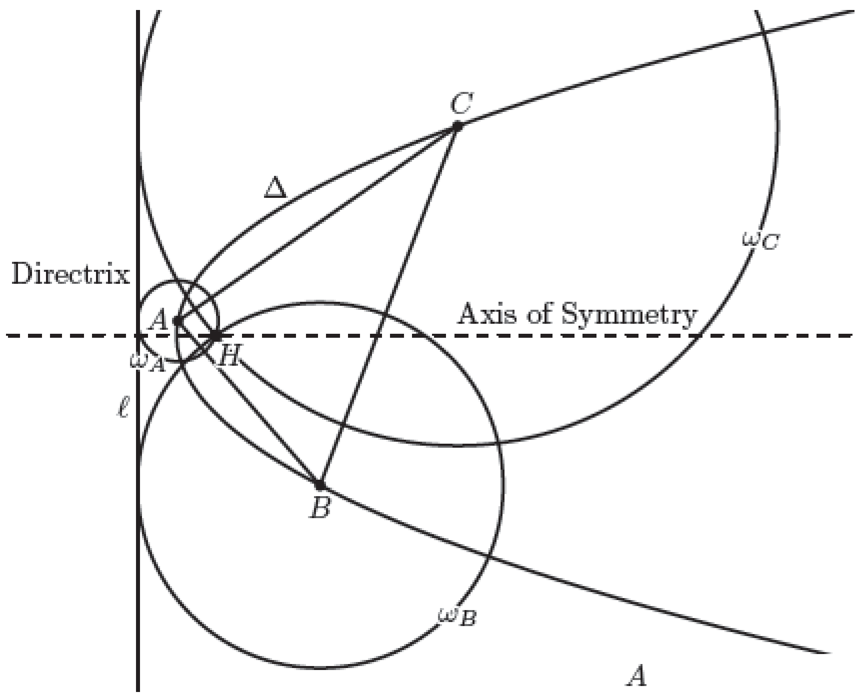


According to the symmetry, KY is tangent to ω_1, therefore XY is the polar of K with respect to ω_1. Since P lies on XY, the polar of P passes through K, and similarly, it also passes through A; Meaning AK is the polar of P with respect to ω_1 and PN is tangent to ω_1. Similarly, PM is tangent to ω_2; Thus points A, B, M and N lie on a circle with center P and $\angle AMB = \angle ANP = 90°$. Let A' be the antipode of A in circle ω_1, and let B' be the antipode of B. Line BN passes through A' and line AM passes through B'. Note that $AA'B'B$ is a trapezoid and O_1 and O_2 are the midpoints of its bases; Hence $A'B$, $B'A$ and O_1O_2 are concurrent, resulting in the claim of the problem.

Problem 5.8. Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remains at H, inradius of triangle ABC remains unchanged.

Solution 5.8. -

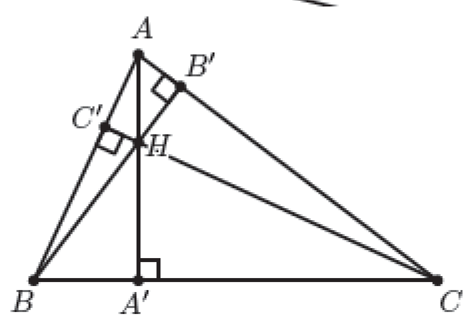
Since H coincides with the focus of parabola Δ, the circles $w_A = (A, AH)$, $w_B = (B, BH)$ and $w_C = (C, CH)$ are tangent to line ℓ, the directrix of Δ.



Now consider triangle ABC.

It is well-known that

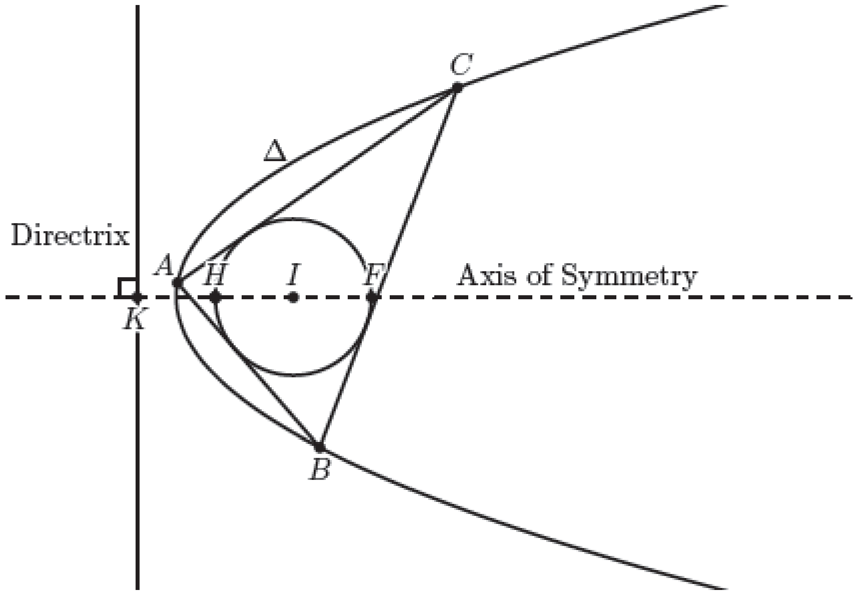
$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC' = t.$$



Also

$$\left.\begin{array}{l} HA = 2R\cos A \\ HA' = 2R\cos B\cos C \end{array}\right\} \Longrightarrow t = 4R^2 \cos A \cos B \cos C. \qquad (1)$$

Inversion with center H and inversion radius $-2t$, inverts the three circle w_A, w_B and w_C to lines BC, AC and AB respectively. In this inversior line ℓ inverts to incircle of triangle ABC. Therefore $IH \perp \ell$, thus point lies on axis of symmetry of Δ. Also point H lies on the incircle of triangl ABC. Hence $HI = r$.



As a result, if orthocenter of ABC lies on its incircle; Also

$$\begin{aligned} HI^2 &= 2r^2 - 4R^2 \cos A \cos B \cos C \\ \implies r^2 &= 2r^2 - 4R^2 \cos A \cos B \cos C \\ \implies r^2 &= 4R^2 \cos A \cos B \cos C \end{aligned}$$

According to (1), it is implied that $r^2 = t = HA \cdot HA'$. In inversion, points K and F are invert points, thus

$$\overrightarrow{HK} \cdot \overrightarrow{HF} = -2t = -2r^2 \implies HK = r.$$

Which gives the result that inradius of triangle ABC is constant.