Math 1 – Tasks for the Tutorials and Finger Exercises

Inhaltsverzeichnis

1 — from October 13, 2025

In this week, no tutorial sessions are going to take place. Hence, there is no need (nor option) to submit problem sets for extra credit.

1.1 Finger exercises – starting next week

1.2 Tutorial – starting next week

2 — from October 20, 2025

2.1 Finger exercises – Summation and Product Notation

2.1.1 Summation notation I

Calculate the following sums:

a) $\displaystyle\sum_{j=2}^{5}(2\,j-3)^2$

b) $\displaystyle\sum_{i=0}^{2}\frac{i}{(i+1)(i+2)}$

c) $\displaystyle\sum_{k=-2}^{1}(k+3)^k$

d) $\displaystyle\sum_{k=0}^{2}(k+1)^{k-1}$

e) $\displaystyle\sum_{k=1}^{3}k^{k-2}$

f) $\displaystyle\sum_{k=1}^{3}(-1)^k\,k^{k/2}$

2.1.2 Summation notation II

Please rewrite the following sums using the summation notation:

a) $1+3+5+\cdots+99$

b) $1+\dfrac{x^3}{4}+\dfrac{x^6}{7}+\dfrac{x^9}{10}+\cdots+\dfrac{x^{27}}{28}$

c) $1+\dfrac{t}{3}+\dfrac{t^2}{5}+\dfrac{t^3}{7}+\cdots+\dfrac{t^8}{17}$

d) $\dfrac{a^3}{10\,b^2}+\dfrac{a^4}{13\,b^3}+\dfrac{a^5}{16\,b^4}+\cdots+\dfrac{a^9}{28\,b^8}$

2.1.3 Double sum

Please calculate the following double sums:

a) $\sum_{i=0}^{2}\sum_{j=0}^{2}(i+2\,j)^2$

b) $\sum_{i=0}^{2}\sum_{j=0}^{i}(i+2\,j)^2$

c) $\sum_{i=0}^{2}\sum_{j=i}^{2}(i+2\,j)^2$

d) $\sum_{i=0}^{2}\left(35-\sum_{j=0}^{2}(i+2\,j)^2\right)$

2.1.4 Product Notation

Please calculate the following products:

a) $\prod_{j=1}^{5} j$

b) $\prod_{j=0}^{5} j$

c) $\prod_{j=1}^{5}(x-3)$

d) $\prod_{j=0}^{5}(x-3)$

2.1.5 Summation and product notation

Please calculate the following expressions:

a) $\prod_{i=1}^{2}\sum_{j=1}^{2}(i+j)$

b) $\sum_{i=1}^{2}\prod_{j=1}^{2}(i+j)$

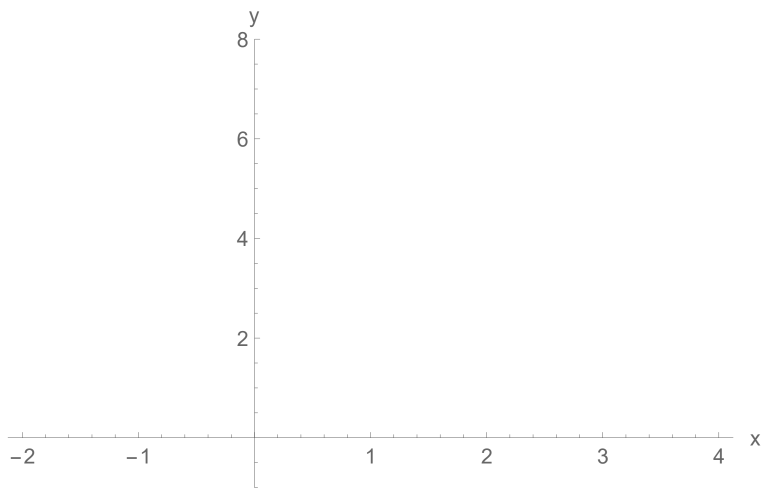
2.2 Tutorial – Graphs of Functions

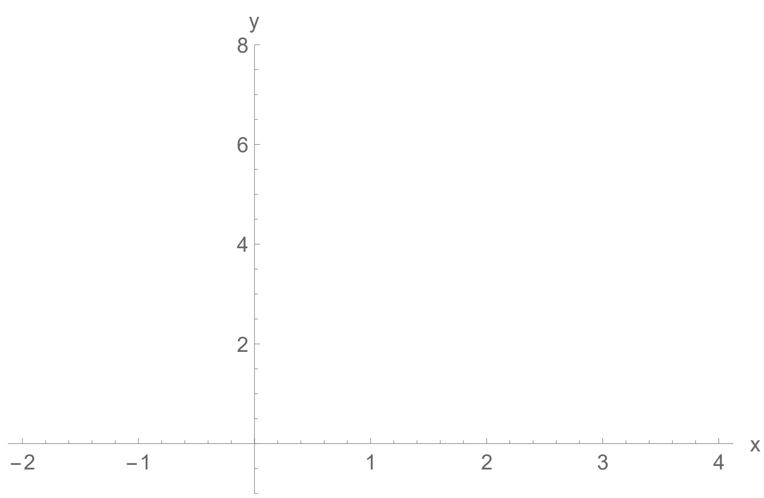
2.2.1 Composite functions

Let $f(x) = (x-1)^2$. Please give an expression for

$$\text{a) } f(x+1); \quad \text{b) } f(x)+1; \quad \text{c) } f(2\,x); \quad \text{d) } 2 \cdot f(x).$$

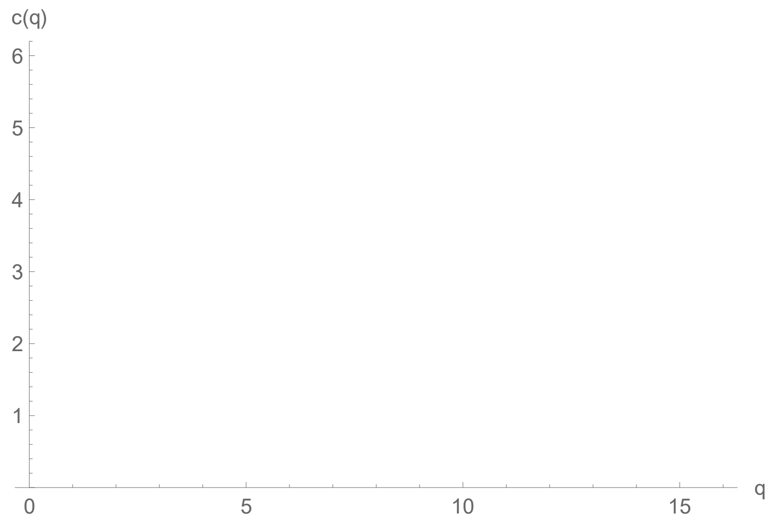
Please sketch the original function as well as the newly calculated functions. Can you name an economic interpretation for for each of the four newly calculated expressions?





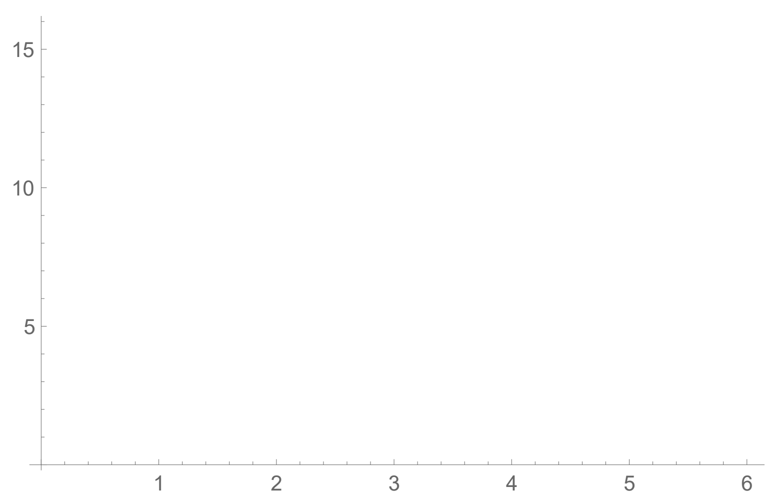
2.2.2 The inverse of a function

In order to produce a quantity q of some food, a firm faces the following cost function: $c(q) = 2 + \sqrt{q}$.
a) Sketch the function $c(q)$ for the case "many", where q is discrete and integral (e.g. bikes, shoes, ...), and for the case "much", where q is continuous (e.g. milk, flour, ...), respectively.



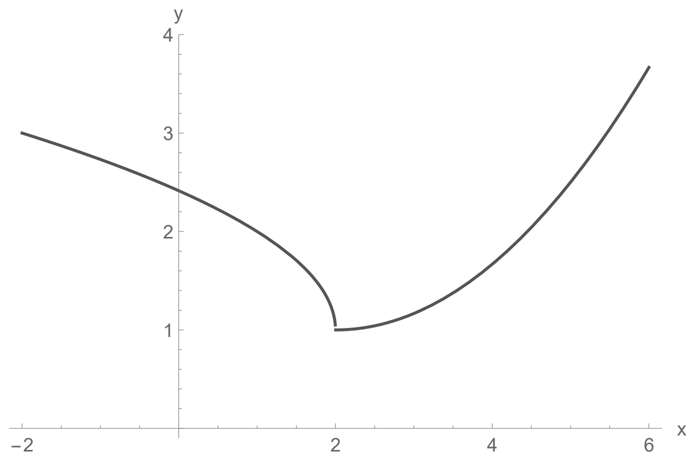
b) What is the inverse function of the two cases, respectively? Which (economic) relationship does an inverse describe?

c) Please sketch the inverse function for both cases.

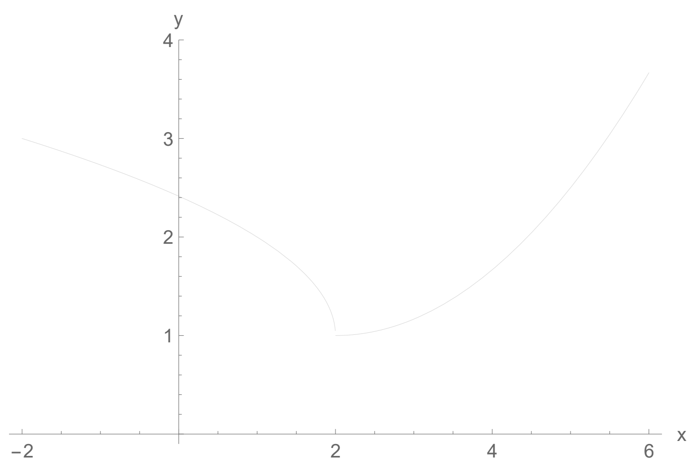


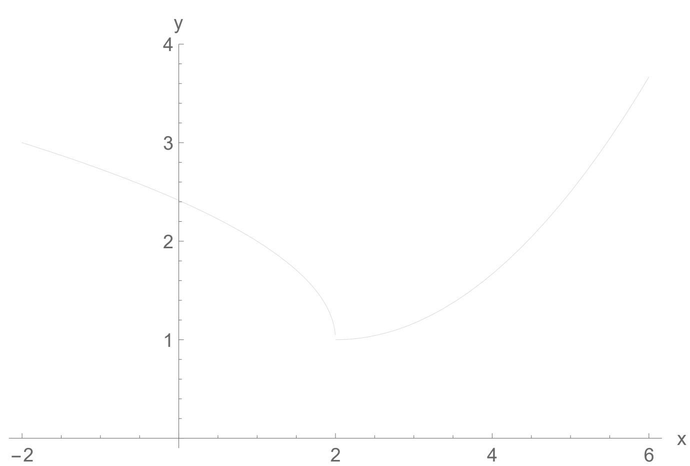
2.2.3 Shifting Graphs

Consider a given function $f(x)$ with the following graph:



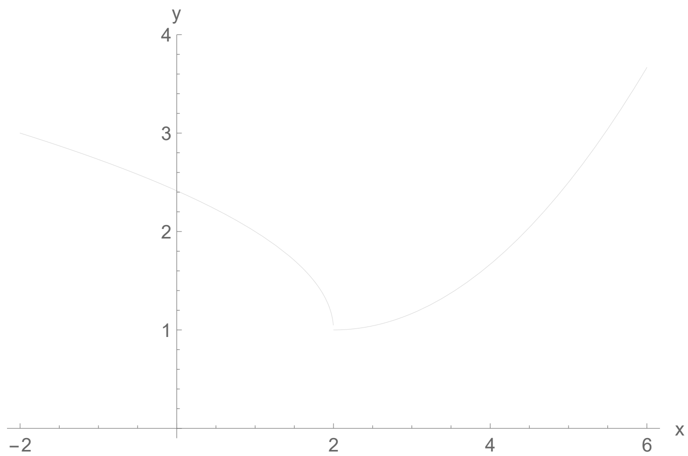
Sketch the following functions: a) $f(x+1)$; b) $f(x-2)$; c) $f(x)+1$; d) $f(x)-2$.

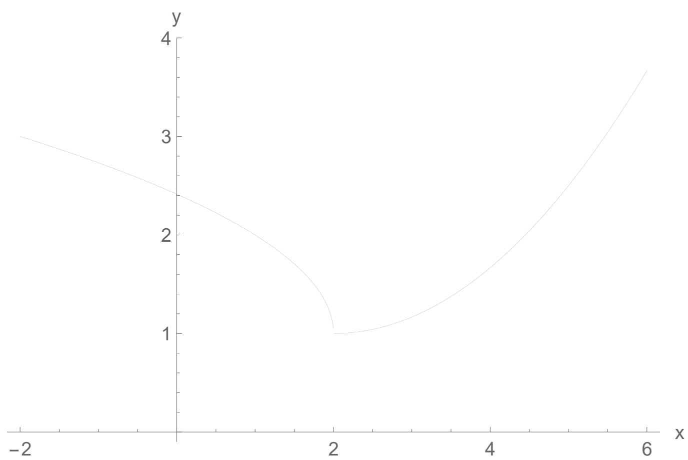


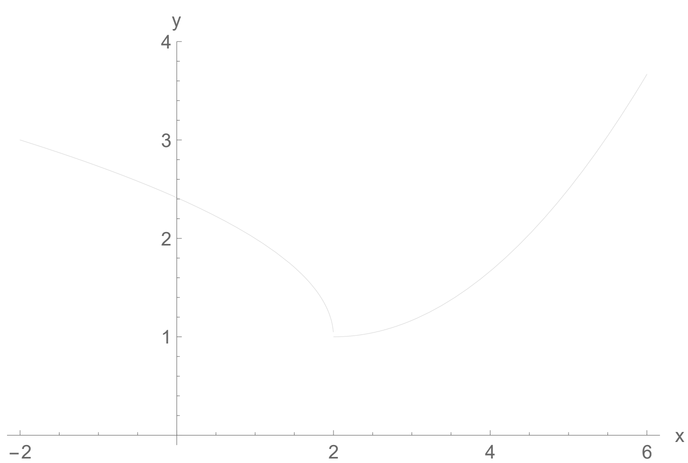


2.2.4 Stretching graphs

For the same initial function $f(x)$ as above, sketch a) $f(2\,x)$; b) $f(x/2)$; c) $2\,f(x)$; d) $f(x)/2$. In addition, sketch e) $f(0 \cdot x)$ and f) $0 \cdot f(x)$.





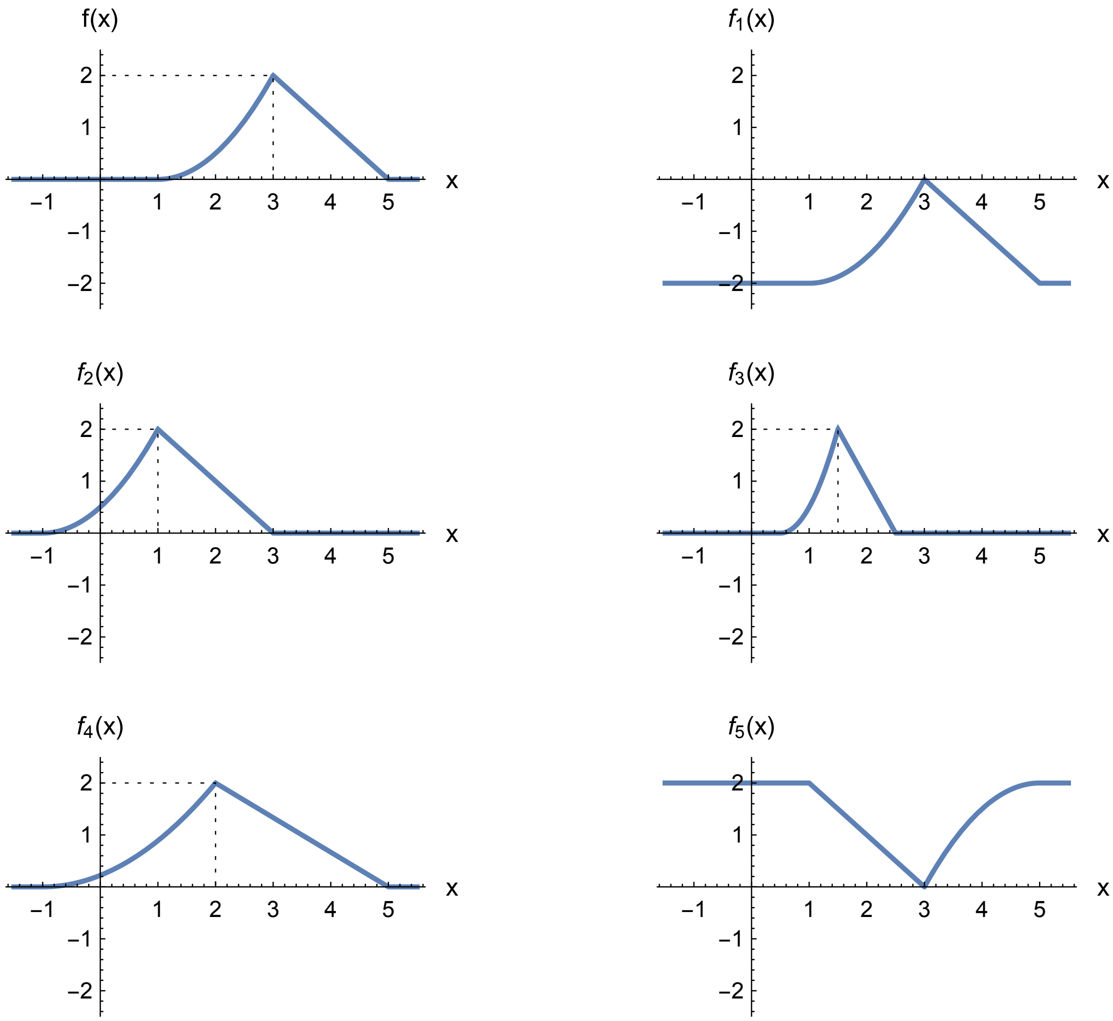


3 — from October 27, 2025

3.1 Finger exercises – Functions

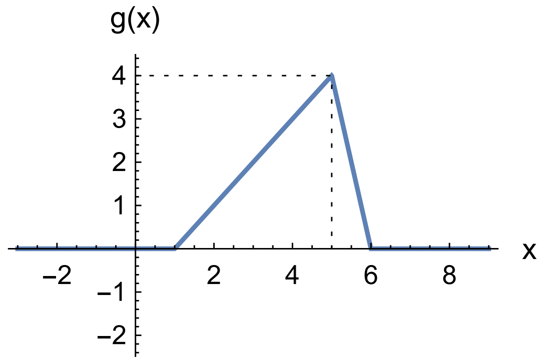
3.1.1 Recognizing shifted and stretched graphs

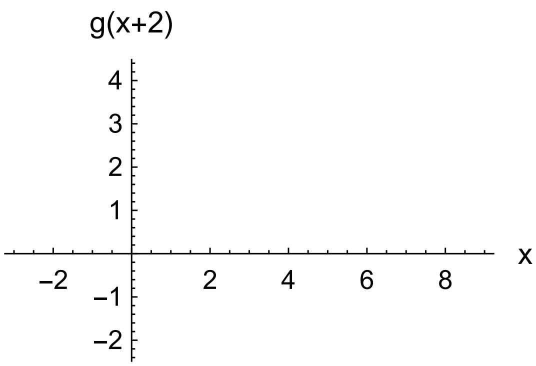
Consider a function $f(x)$. The functions $f_1(x)$ to $f_5(x)$ are shifted, stretched/compressed, and mirrored versions of $f(x)$. Explicitly state the respective relationship to $f(x)$, e.g. $f_1(x) = -f(x+2)+3$ or $f_2(x) = 3\,f(2\,x)$. Note: you may solve this problem using geometric arguments only, you do not have to calculate anything explicitly.

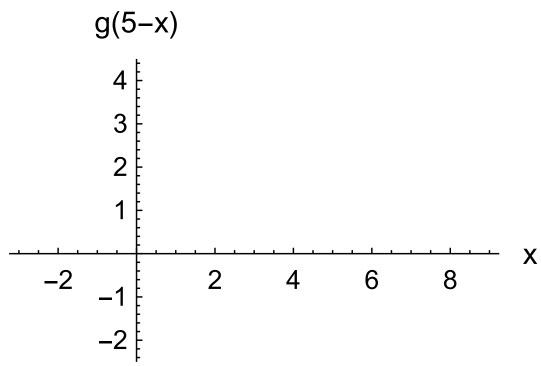


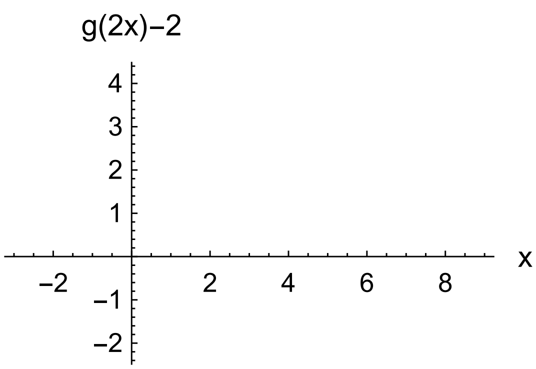
3.1.2 Sketching shifted and stretched functions

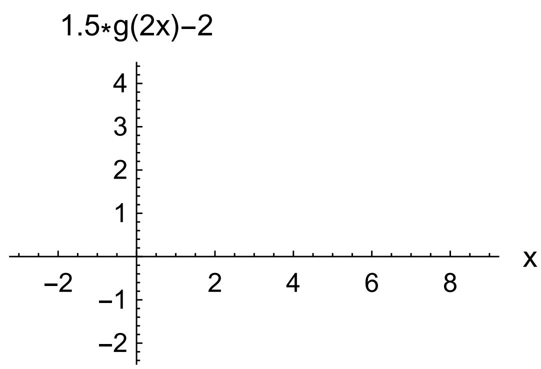
Let $g(x)$ be a function, whose graph is given below. Sketch the remaining indicated functions in the respective coordinate systems.

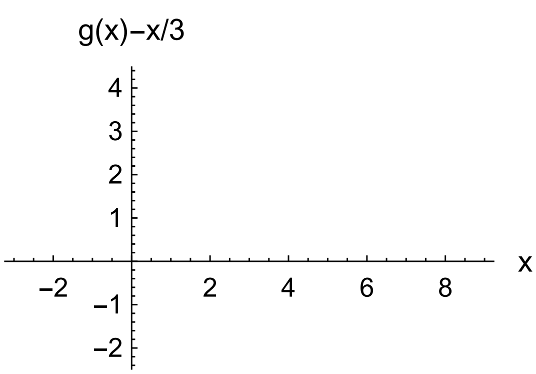


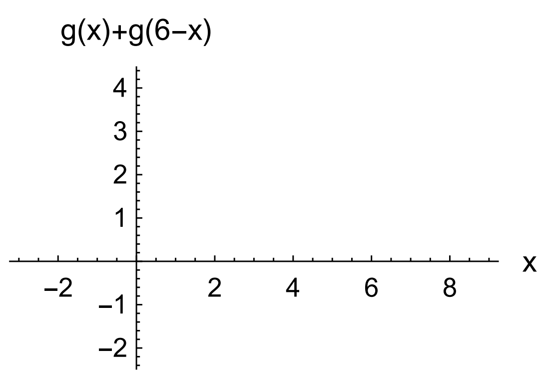












3.2 Tutorial

3.2.1 Functions

Functions can map from the real numbers into the real numbers, or from an interval into the real numbers, or from the natural numbers to the real numbers (in the latter case, the functions are then called "sequences"). These are the most common cases. We will also get to know functions with multiple input variables, or with multiple output variables. You will encounter both types quite often in economics. But the concept of a function is quite general. This shall be the focus of this task.

A Boolean variable can only take on two values, *true* oder *false*. Therefore, let $\mathbb{B} = \{\text{true}, \text{false}\}$.

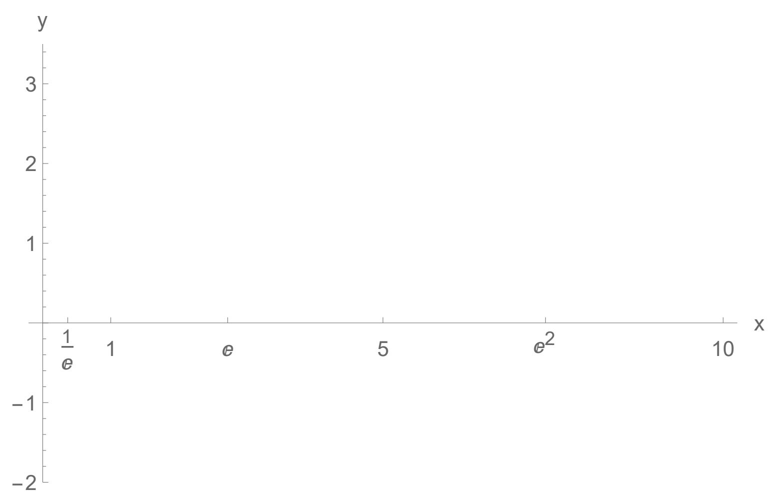
a) Are you able to think of a function $f : \mathbb{R} \to \mathbb{B}$? If so, write down / define such a function, possibly also with an economic/applied interpretation.

b) Are you able to think of a function $f : \mathbb{B} \to \mathbb{R}$? If so, write down / define such a function, possibly also with an economic/applied interpretation.

c) Are you able to think of a function $f : \mathbb{B} \to \mathbb{B}$? If so, write down / define such a function, possibly also with an economic/applied interpretation.

3.2.2 The exponential function and the logarithmic function

a) Sketch the graph of the natural logarithm $f(x) = \ln x$. Please also sketch the graphs of $f(x) + 1$ und $f(e\,x)$. Do you notice something? If so, are there other functions with a similar relationship?



b) Sketch the graph of the exponential function $f(x) = e^x$. Also sketch $f(x+1)$ and $e\,f(x)$. Notice anything? If yes, are there other functions with a similar relationship?

3.2.3 Monotonicity

a) Provide the definition of a (strictly) monotonically increasing and a (strictly) monotonically decreasing function. Note: such functions are called (strictly) monotone.

b) Give an example of a strictly monotonically increasing function. Argue mathematically that this function actually increases strictly monotonically.

c) Give an example of a non-monotonous function. Argue mathematically that this function is neither monotonically increasing nor monotonically decreasing (at least not on the entire support).

3.2.4 Polynomial long division

a) First of all, let's recap the algorithm of long division. Calculate the following expressions by hand, if necessary with a remainder (no decimal places):

i) 26 978 / 7

```
26978 : 7 =
```

ii) 2395 / 3 =

```
2395 : 3 =
```

b) Let us turn to polynomial long division now. Please calculate $(x^3 + 2\,x^2 - 5\,x - 6) : (x + 1)$.
Think about instances, in which polynomial long division might be useful.

Note: As you know, you get the bonus points for trying to come up with a solution, not for necessarily providing the correct solution. The topic of polynomial division is taught in most high schools in Germany, but not in all. Throughout this course, it therefore makes no sense at all if you copy a solution from one of your peers. If you lack the background knowledge for tackling a certain topic, I urge you to rather state that in your submission. It is only with this information that the tutor can then go into more detail about specific issues. Alternatively, look at the topic together with your learning group. But: please don't copy from eachother, that doesn't do anyone any good (and the tutor will think, "All right, everyone was able to do that, there is no need to discuss this any further.")

```
(x^3 + 2 x^2 - 5 x - 6)  /  (x + 1) =
```

c) Finally, polynomial long division with a remainder term: please calculate $(x^3 + 2\,x^2 - 5\,x - 5) : (x + 2)$

```
(x^3 + 2 x^2 - 5 x - 5) : (x + 2) =
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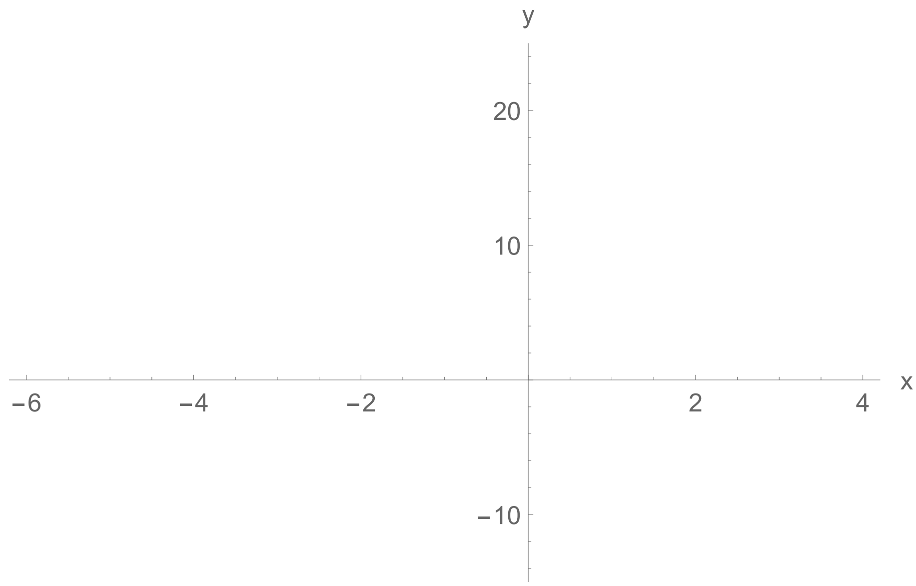
Sketch the function $f(x) = \frac{x^3+2x^2-5x-5}{x+2}$, sketch the resulting polynomial $g(x)$ after performing polynomial long division (disregarding the remainder term), and sketch their difference $f(x) - g(x)$.



4 — from November 3, 2025

4.1 Finger exercises – Polynomial Long Division and Logarithms

4.1.1 Polynomial long division I

Calculate the following division, possibly with a resulting remainder term:

a) $2\,x^3 + 3\,x^2 - 8\,x + 3 \quad \div \quad (x+3)$
b) $3\,x^3 + 5\,x^2 - 11\,x + 3 \quad \div \quad (3\,x - 3)$
c) $x^3 - a\,x^2 - x + a \quad \div \quad (x+1)$
d) $x^3 + a^2\,x^2 - 2\,x - 2\,a^2 \quad \div \quad (x + a^2)$

Bonus question: What are the roots of $f(x) = x^3 + a^2\,x^2 - 2\,x - 2\,a^2$?

4.1.2 Polynomial long division II

For which $k \in \mathbb{Z}$ is the polynomial $x^3 - 7x^2 + 2x + k$ divisible without remainder term by $x + 1$?

4.1.3 Logarithms I

Calculate the following logarithms:

a) $\ln e^2$
b) $\ln 1 + \log_{10} 1$
c) $\log_9 27$
d) $\log_2 32 + \log_3 3 + \log_4 1$
e) $5^{3\log_5(2)}$
f) $\log_3(81)$
g) $\log_2(2^3 \cdot 2^4)$

4.1.4 Logarithms II

Determine the value of t for which the following equation holds. Do not explicitly calculate any values of the ln or e function.

a) $e^{-2t} = \frac{1}{2}$
b) $\ln(4\,t) = 3$
c) $\ln(4\,t - 13) = 1$
d) $2e^t - e^{-2\,t} = 0$

4.2 Tutorial

4.2.1 Inverse functions I

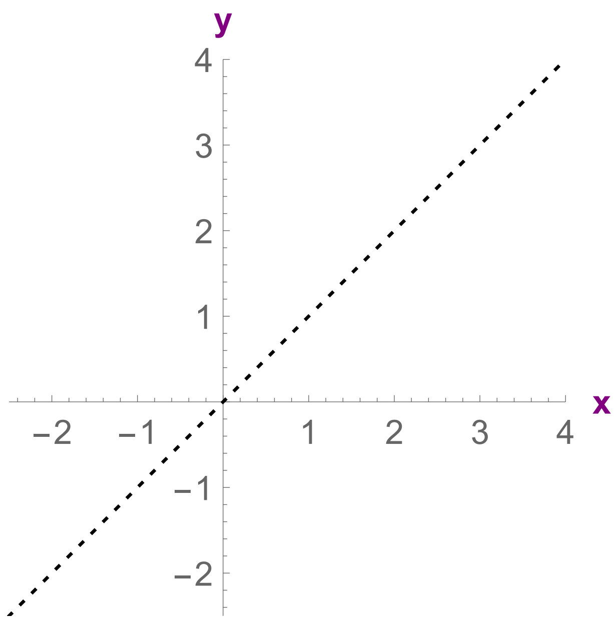
Let

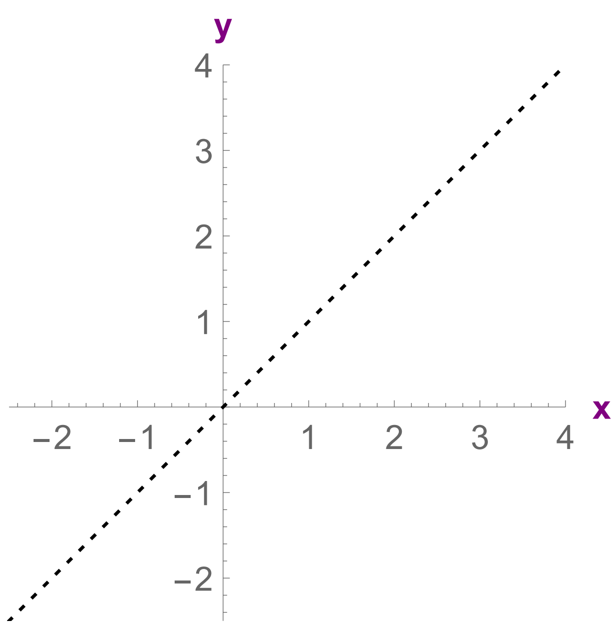
$$f(x) = \begin{cases} x-2, & x \geq 2 \\ \sqrt{x+2}-2, & -2 \leq x < 2 \end{cases}$$

a) What are the domain D_f and the image W_f of the function $f(x)$?

Definition: W_f = image of f = set of points, which are mapped into by the function. This is in contrast with the target (space) = codomain = set of points, which are an admissible target of a function's mapping. Example: the target of $f(x) = x^2$ is the set of real numers $\mathbb{R}$, whereas the image is merely a subset of the target space containing non-negative real numbers, i.e. $\mathbb{R}_{\geq 0} = \mathbb{R}^+ \subset \mathbb{R}$.

b) Sketch the function $f(x)$ from above. Does an inverse function exist? If so, provide a sketch of the inverse function as well. What are domain and image of the inverse function?





c) What is the formal definition of the inverse function (provide a formula)?

4.2.2 Inverse functions II

a) A family eats more apples the lower the price. Specifically, consumption per month is given by

$$C(p) = \frac{20}{p+4},$$

where p denotes the price of a kilogram (kg) of apples in Euro. Hence, a little more precise version of the consumption function would be

$$C(p) = \frac{20 \text{ Euro}}{p + 4 \text{ Euro/kg}}.$$

What is a sensible set for the domain of the function? What about the image? What does the inverse function look like defined over appropriate sets? The answer to which economically relevant question does the inverse function provide?

b) Depending on the price P of some good, a company produces the quantity $Q(P) = 100\, e^{-P/100}$. Below a critical price $P_0 = 5$, the company stops production entirely. What is a formal representation of the production function? What are it's domain and image, respectively? What economic question does the inverse function answer? Formally, what does the inverse function look like?

4.2.3 Derivatives I

a) The simplest function in the world is probably $f(x) = x$, right? Okay, admittedly, $g(x) = 0$ is perhaps even simpler... What is the derivative of $g(x) = 0$? “Prove” your “conjecture”. What is the derivative of $f(x) = x$? “Prove” your “conjecture”.

b) Someone claims “If you shift a function upward by a constant, then the derivative remains the same.” What is the statement formally (i.e. as a formula)? “Prove” this “formula”.

4.2.4 Derivatives II

Using the Newton quotient, calculate the derivative of $f(x) = \sqrt{x}$.

Hint: As usual, there are alternating ways to approach this problem. One of them consists of expanding the fraction appropriately.

5 — from November 10, 2025

5.1 Finger exercises – More on Logarithms

5.1.1 Terms incl. logarithms

Simplify the following terms as much as possible. *Note: The tasks are intentionally not sorted by difficulty.*

a) $\log_{10} 2 + \log_{10} 5$
b) $\log_{a\,b} a + \log_{a\,b} b$
c) $\log_b a + \log_a b$
d) $\log_2 4 + \log_4 2$
e) $\log_a 1$
f) $\log_1 a$
g) $\log_{a^2} a$
h) $\log_{\sqrt{a}} a$
i) $\ln(e \ln(e))$
j) $\ln(\ln(e)/e)$
k) $\ln(\ln(e^a)/e^b)$
l) $\log_{1/z} z^2$
m) $\log_3 5 + \log_3 15 - \log_3 25$
n) $\sqrt{a}^{\log_a 5}$
o) $3 \log_a b + 2 \log_a c - 4 \log_a d$
p) $\log_a a$
q) $\log_a a^n$
r) $\log_a \sqrt{a^3}$
s) $\log_a \sqrt[5]{a}$
t) $\log_a 1/\sqrt[3]{a^2}$

5.1.2 Equations with logarithms

Determine the value of x (possibly as a function of other unknown parameters).

a) $\log_2 x = 8$
b) $\log_{0.2} 5 = x$
c) $\log_x \sqrt{3} = 0.25$
d) $\log_2 7x = 2/3$
e) $\log_{10} 10^8 = x$
f) $\ln(\ln x) = 1$
g) $\log_4 2\,x = 4$
h) $\log_2(\log_2 x) = 3$
i) $\log_x 2 = 0.5$
j) $\log_x 4 = -0.5$
k) $\log_x a^n = 2\,n$

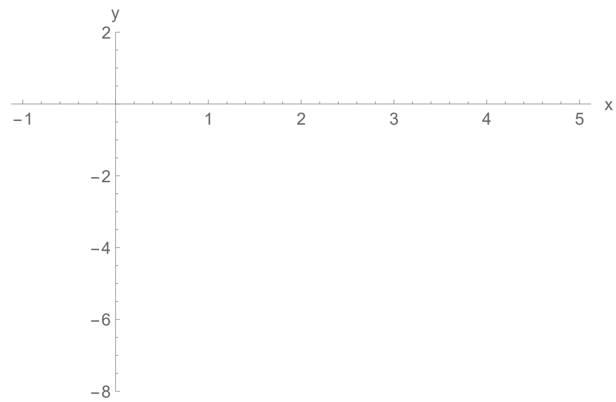
5.2 Tutorial

5.2.1 Compositions of Functions I (Verkettete Funktionen)

The function $f(x)$ reaches its maximum at $x^* = 3$.

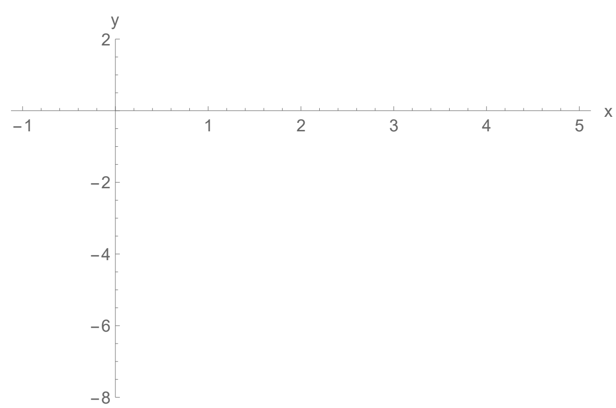
a) What is the maximum point of $g(x) = f(x) + 2$? First, try to get some intuition sketching the two functions. Argue, why you obtain the relationship made out in the sketch. Does this result depend on whether the function is differentiable (i.e. smooth)?

Provide an economic interpretation for the resulting relationship.



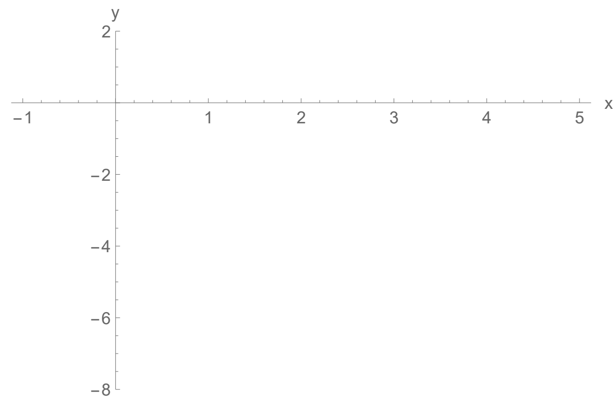
b) Where does the function $g(x) = f(x + 2)$ have a maximum? First, try to get some intuition sketching the two functions. Argue, why you obtain the relationship made out in the sketch. Does this result depend on whether the function is differentiable (i.e. smooth)?

Provide an economic interpretation for the resulting relationship.



c) Where does the function $g(x) = 2\,f(x)$ have a maximum? First, try to get some intuition sketching the two functions. Argue, why you obtain the relationship made out in the sketch. Does this result depend on whether the function is differentiable (i.e. smooth)?

Provide an economic interpretation for the resulting relationship.

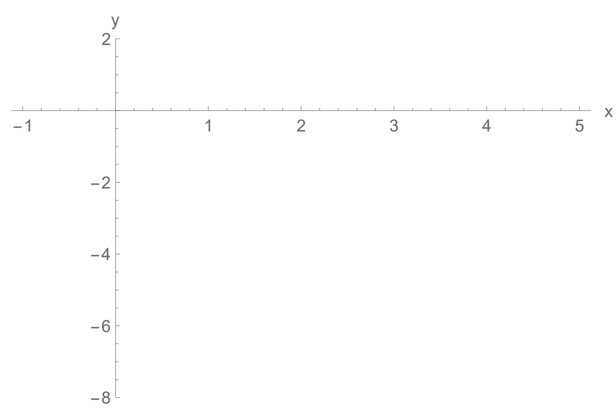


5.2.2 Compositions of Functions II (Verkettete Funktionen)

This section can be seen as a second part to the section above and will, hence, provide an additional bonus point if attempted and submitted.

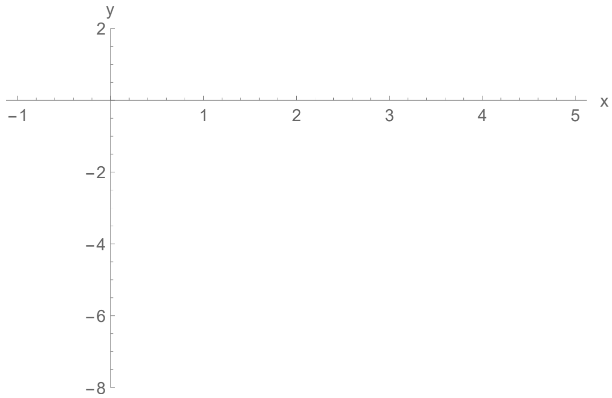
d) Where does the function $g(x) = f(2\,x)$ have a maximum? First, try to get some intuition sketching the two functions. Argue why you obtain the relationship made out in the sketch. Does this result depend on whether the function is differentiable (i.e. smooth)?

Provide an economic interpretation for the resulting relationship.



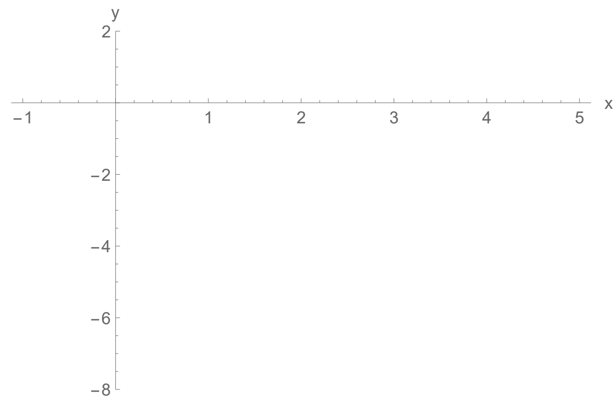
e) Let $h(x)$ be an arbitrary monotonically increasing function. Where does the function $g(x) = h(f(x))$ achieve its maximum? First, provide a sketch. Argue why you obtain the relationship made out in the sketch. Does this result depend on whether the function is differentiable (i.e. smooth)?

Provide an economic interpretation for the resulting relationship.


y
2
−2
−4
−6
−8
x
−1
1
2
3
4
5

f) Where does the function $g(x) = f(h(x))$ achieve its maximum? First, provide a sketch. Argue why you obtain the relationship made out in the sketch. Does this result depend on whether the function is differentiable (i.e. smooth)?

Provide an economic interpretation for the resulting relationship.



5.2.3 The chain rule

Calculate the (first) derivatives of the following functions.

a) $f(x\, g(x))$

Think of a specific example and check whether the formula is correct.

b) $x\, f(g(x))$

c) $f(f(x))$

Think of a specific example and check whether the formula is correct.

d) $f(g(x) + h(x))$

e) $f(g(x)\, h(x))$

5.2.4 Convexity

a) What is the definition of a convex function? What about a concave function?

b) Suppose a function is twice differentiable and that you know the second derivative. What can you say about convexity/concavity of that function?

c) Provide an example of a function $f : \mathbb{R} \to \mathbb{R}$, which is concave on the interval $(-\infty, 1) \subset D_f$ and convex on $(1, \infty)$.

d) Consider a function $f(x)$ with $f'(x) > 0$ and $f''(x) \leq 0$ for all x. This implies that the function is concave. But now consider the function $f(\sqrt{x})$ instead. Is this function also (necessarily) concave? If not, provide a counterexample. If yes, try to prove this. *As always: discuss in your study group. If you get stuck, calculate examples and try to get a feeling for the situation at hand.*

Now consider the opposite scenario: suppose you know that $f(\sqrt{x})$ is concave and $f'(x) > 0$. Is it true, that f is necessarily concave as well? If not, provide a counterexample. If yes, try to prove this.

6 — from November 17, 2025

6.1 Finger exercises

6.1.1 ...logarithmic functions over and over again

Determine x.

a) $\log_4(3\,x+1)=3$

b) $\log_{10}(12\,x-8)=2$

c) $\log_2 \frac{2\,x-6}{x-3}=4$

d) $\log_2(x^2-1)+5=8$

e) $\log_3(2\,x-5)-\log_3(x-1)=3$

f) $\log_3(x+3)+\log_3 6=\log_3(x-4)$

g) $(\ln x)^2+\ln x=6$

6.1.2 Equations with logarithms

Approximately calculate the following expressions without using a calculator:

a) $\log_7 50$

b) $\log_3 80$

6.1.3 Logarithmen umformen

Suppose we know, that $\ln(a+b)=2$. Simplify:

$$\ln(a-b)+\ln\sqrt{a+b}-\ln\frac{a^2-b^2}{a^2+2\,a\,b+b^2}$$

6.2 Tutorial – Sequences and Series

Compared to the other tutorials, this tutorial stands out a bit: it only covers a relatively short chapter of the Sydsæter (currently 7.11). However, these topics (sequences, series and their convergence) are treated much more extensively in many other textbooks. We have therefore decided to give the topics a little more weight, at least in this tutorial.

6.2.1 Sequences and series I

a) Write down the first five elements (terms) of the following sequences:

i) $s_n = (-1)^n$ for all natural numbers, i.e. $n \in \mathbb{N}$

ii) $s_n = n\,(n+1)/2$ for $n \in \mathbb{N}$

iii) $s_n = 2\,n - 1$ for $n \in \mathbb{N}$

iv) $s_n = \alpha\, n$, where $\alpha > 0$ positive and for $n \in \mathbb{N}$

v) $s_n = \alpha\, n$, where $\alpha < 0$ negative and for $n \in \mathbb{N}$

b) Write down the first five elements (terms) of the following (recursively defined) sequences:

i) $s_1 = 1$ and $s_{n+1} = s_n + (n+1)$ for all natural numbers, i.e. $n \in \mathbb{N}$

ii) $s_1 = 1$ and $s_{n+1} = s_n + 2$ for all natural numbers, i.e. $n \in \mathbb{N}$

iii) $s_1 = 1$ and $s_{n+1} = 2\,s_n$ for all natural numbers, i.e. $n \in \mathbb{N}$

iv) $s_1 = 1$ and $s_{n+1} = \alpha\, s_n$ for all natural numbers, i.e. $n \in \mathbb{N}$

6.2.2 Sequences and series II

a) Which of the following (recursively defined) sequences can be expressed in closed form?

i) $s_1 = 1$ and $s_{n+1} = s_n + (n+1)$ for all natural numbers, i.e. $n \in \mathbb{N}$

ii) $s_1 = 1$ and $s_{n+1} = 2 - s_n/2$ for all natural numbers, i.e. $n \in \mathbb{N}$

b) For which of the following sequences can you provide a recursively defined formula?

i) $s_n = (-1)^n$ for all natural numbers, i.e. $n \in \mathbb{N}$

ii) $s_n = 1/n$ for all natural numbers, i.e. $n \in \mathbb{N}$

6.2.3 Mathematical induction I

a) Show: if $s_1 = 1$ and $s_{n+1} = s_n + (n+1)$ for all natural numbers, i.e. $n \in \mathbb{N}$, then it holds that $s_n = n\,(n+1)/2$.

Hint: at this point, some knowledge of mathematical induction would be useful (Sydsæter chapter 1.4).

b) Also show the reverse: if $s_n = n\,(n+1)/2$ for all natural numbers, i.e. $n \in \mathbb{N}$, it then holds that $s_1 = 1$ and $s_{n+1} = s_n + (n+1)$.

6.2.4 Mathematical induction II

a) Show: if $s_0 = 0$ and

$$s_{t+1} = s_t + \frac{1}{(1+r)^{t+1}}$$

for all natural numbers, i.e. $t \in \mathbb{N}$, it then holds that $s_t = \big(1 - (1+r)^{-t}\big)/r$, with $r \in \mathbb{R}_{>0}$.

b) Also show the reverse: if $s_t = \big(1 - (1+r)^{-t}\big)/r$ for all natural numbers, i.e. $n \in \mathbb{N}$, it then holds that $s_0 = 0$ and $s_{t+1} = s_t + \frac{1}{(1+r)^{t+1}}$.

7 — from November 24, 2025

7.1 Finger exercises – Fractions and Exponentiation

7.1.1 Sequences and series

i) Write in closed form:

$1, -2, 3, -4, 5, -6, \ldots$
$1, 2+\alpha, 3+2\,\alpha, 4+3\,\alpha, \ldots$
$(1+r), (1+r)^2\,\rho, (1+r)^3\,\rho^2, (1+r)^4\,\rho^3, \ldots$
$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$

ii) Write down the first four terms of the following sequences and simplify:

$s_n = \frac{(1+\alpha)^{n+1}}{(1+\alpha)^n}$
$s_n = \frac{\alpha^n - 1}{\alpha - 1}$
$s_n = \frac{n^\alpha - 1}{n - 1}$

iii) Write down the first four terms of the following sequences and simplify:

$s_n = \sum_{i=1}^{n} \text{i}$
$s_n = \sum_{i=1}^{n} \alpha$
$s_n = \sum_{i=1}^{n} \text{n}$

7.1.2 Exponentiation

Calculate/simplify without using a calculator.*Please remember: first and foremost, you should complete this task if you feel unsure about the area of "taking roots and exponentiating". Otherwise, please concentrate on the tasks covered the tutorial.*

a) 0.1^{-3}
b) $(-2/3)^{-3}$
c) $0.5^{-3/2}$
d) $\sqrt[10]{1024}$
e) $\sqrt[5]{1024}$
f) $(27/125)^{2/3}$
g) $(25/49)^{-3/2}$

7.1.3 Simplifying terms I

Simplify as much as possible.

a) $\frac{4}{x^2 - y^2} \cdot (\frac{2}{x+y})^{-2}$
b) $(2\,a - 5\,b)^{-2} \cdot (8\,a^2 - 50\,b^2)$

7.1.4 Fractions

Eliminate the fractions by using negative exponents.

a) $\dfrac{2}{z^{-2}} - \dfrac{1}{z}$

b) $\dfrac{5\,c}{(a+b)^2}$

7.1.5 Simplifying terms II

Simplify as much as possible.

a) $(2\,a^5 + 3\,b^3) \cdot (2\,a^3 - 2\,b^4)$

b) $\dfrac{k^{2\,m}}{k^3}$

c) $\dfrac{15\,x^{-5}\,b^8}{35\,a^7\,b^{-2}} \cdot \dfrac{21\,x^3\,y^4}{9\,x^{-2}\,a^{-3}\,b^{10}}$

d) $(4\,a^2\,b^3 - b^5) \cdot (2\,a\,b + b^2)^{-1}$

e) $\dfrac{r^3\,s^2 + 2\,r^4\,s^4 + r^5\,s^6}{r^3\,s^3 + r^4\,s^5} : \dfrac{r^2\,s - r^3\,s^3}{r^2\,s^2 - 2\,r^3\,s^4 + r^4\,s^6}$

f) $\dfrac{(16\,x^4 - 25\,y^{-2})^n}{(4\,x^2 - 5/y)^n}$

g) $\dfrac{(2\,a + 3\,b)^{-5}}{(4\,a^2 - 9\,b^2)^{-5}}$

h) $\dfrac{(16\,r^4 - 24\,r^2\,s^3 + 9\,s^6)^4}{(16\,r^4 - 9\,s^6)^4}$

i) $\sqrt[7]{5^5} : \sqrt[3]{5^2}$

j) $\sqrt{\sqrt[4]{x}}$

k) $\sqrt{\sqrt[3]{16} \cdot \sqrt[9]{64}}$

l) $\sqrt{a + \sqrt{4\,a\,x} + x}$

m) $\sqrt[3\,n]{(a - 4\,b)^{-3} \cdot (a + 4\,\sqrt{a\,b} + 4\,b)^3}$

n) $\sqrt[3]{a^2\,b} \cdot \sqrt[3]{b^2\,a}$

o) $\dfrac{x^3}{\sqrt[3]{x}} \cdot \dfrac{3\,x^{5/3}}{x\,\sqrt[3]{x}}$

p) $\sqrt{\sqrt{x} - \sqrt{y}} \cdot \sqrt{\sqrt{x} + \sqrt{y}}$

q) $\sqrt{2\,v^2 - v\,\sqrt{6\,v^2 - (\sqrt{2}\,v)^2}}$

r) $\dfrac{1/x - 1/y}{1/\sqrt{x} - 1/\sqrt{y}}$

s) $(u - v)\,\sqrt{1 + \dfrac{4\,u\,v}{(u - v)^2}}$

7.2 Tutorial

7.2.1 L'Hôpital's rule

Let $f(x) = \dfrac{e^x - a}{x + b}$. Calculate the limit of $f(x)$ as x tends to 0, i.e. $\lim_{x\to 0} f(x)$. $a, b \in \mathbb{R}$ are arbitrary parameters.

If you cannot solve the problem for arbitrary parameters a and b, or if you want to acclimate yourself to the problem first, set $a = 0$ or $a = 1$, and also $b = 0$ or $b = 1$. You then end up with four cases to consider.

7.2.2 Sketching a Curve

a) Let $f : \mathbb{R}_+ \to \mathbb{R}$, specifically

$$f(x) = x^n / e^x = x^n \, e^{-x},$$

where $n \in \mathbb{N}$. Determine the image of f as well as all local and global extreme points.

Here, $\mathbb{R}_+$ is the set of all real numbers ≥ 0, and $\mathbb{N}$ denotes the set of the natural numbers without 0. The image is the set of all numbers y, which are mapped into at some point as a function value, i.e. image$(f) = \{y \in \mathbb{R} : \exists x \in \mathbb{R}_+$ with $f(x) = y\}$.

7.2.3 The logistic function

The logistic function has many applications, for example in the context of neural networks (as a special case of a sigmoid function, which is just another word for S-shaped function), or in logit regression (which involves estimating probabilities). Here is an example from business administration/economics.

Suppose a market has a maximum size of M. At time t, the saturation of the market (in %) can be described by

$$S(t) = \frac{1}{1 + e^{-\mu t}} \qquad \text{with } \mu > 0.$$

a) Use the technique of curve sketching to answer the following questions about $S(x)$: Over which interval is the function defined? How does the function behave at the boundaries (limits)? On which part of the domain is the function monotonically falling or monotonically increasing, respectively? On which part is the function concave or convex, respectively? Once you have answered these questions, provide a sketch of the function.

For example, you could imagine that a company enters a new, uncontested market, which implies that the market saturation is negligible in the beginning. With increasing competition however, it reaches 100% at some point. The growth of the market is the described by the logistic function. Growth rates are low at the beginning, high in the "medium term", and low again when the market is close to fully saturated.

b) Let $Z(t) = \dot{S}(t) = S'(t)$. Can you provide an economic interpretation of $Z(t)$? How would you interpret $\dot{Z}(t)$ and $\ddot{Z}(t)$?

7.2.4 Derivatives (lots of!) ✓

Differentiate the following functions with respect to x. Don't forget (and this is something you should generally keep in mind) to first check if you can simplify some of the expressions.

a) $f(x) = \ln(\alpha\sqrt{x})$

b) $f(x) = e^{\sqrt{x}}$

c) $f(x) = \ln\left(\alpha\, e^{\sqrt{-x}}\right)$

d) $f(x) = \log_a\left(x^a\right)$

e) $f(x) = \log_{e^x}(b)$

f) $f(x) = \ln(b + e^{x-b})$

g) $f(x) = \ln\left(\frac{\sqrt[3]{x+1}}{\sqrt[5]{x-1}}\right)$

h) $f(x) = \ln(x^x)$

i) $f(x) = \ln|g(x)|$

j) $f(x) = e^{g(x)\,h(x)}$

k) $f(x) = g(x)^{\ln(\alpha+\beta)}$

l) $f(x) = \sqrt[g(x)]{\alpha}$

8 — from December 1, 2025

8.1 Finger exercises – Derivatives

Differentiate the following functions with respect to x. Greek letters (α, β) denote constants, but y and z are functions of x, i.e. $y(x)$ and $z(x)$.

a) $f(x) = \ln(\sqrt[3]{x^2+1})$

b) $f(x) = \ln\left(\frac{1}{1/3\sqrt[3]{\alpha\, x^2+1}}\right)$

c) $f(x) = \ln(\sqrt{e^{\alpha\, x}})$

d) $f(x) = \ln(\ln(y))$

e) $f(x) = e^{\ln(x)\, e^x}$

f) $f(x) = x^{-x}$

g) $f(x) = \log_\alpha(\log_\beta(x))$

h) $f(x) = \log_x(\beta)$

i) $f(x) = \log_y(y^2)$

j) $f(x) = e^x\, e^\alpha$

k) $f(x) = e^{x+\beta} \ln(x-\beta)$

l) $f(x) = e^{\beta+\ln(x-\beta)}$

m) $f(x) = (e^x)^{1/5}$

n) $f(x) = (\ln(x))^{-1/5}$

o) $f(x) = e^{\ln(e^y)}$

p) $f(x) = \ln\left(\frac{x+1}{x-1}\right)$

q) $f(x) = \ln\left(\frac{\sqrt[3]{x}+1}{\sqrt[5]{x}-1}\right)$

r) $f(x) = \ln(\alpha^3\, \beta^2\, \sqrt[3]{c}\sqrt[5]{x})$

s) $f(x) = x\, \ln y$

8.2 Tutorial

8.2.1 The standard normal distribution

a) Consider the function

$$f(x) = e^{-x^2/2} = \exp(-x^2/2).$$

Use the technique of curve sketching to answer the following questions about f: On which part of the domain is f monotonically increasing, where decreasing? Where is it convex or concave? Where are minimum points, maximum points, saddle points, turning points located? What is image of f?

b) Now consider

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2).$$

Answer the same questions as in a).

8.2.2 The normal distribution

a) Consider the function

$$h(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2\,\sigma^2}\Big).$$

As in the previous exercise, discuss the features of this function applying the technique of curve sketching.

b) Consider the following function:

$$k(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2}\big(\frac{x-\mu}{\sigma}\big)^2\Big).$$

This is again an importan function on relation to the normal distribution. To be precise, k is the probability density function of the normal distribution with describing parameters μ and σ (π is a fixed irrational number and therefore not a parameter). As in the previous exercise, discuss the features of this function applying the technique of curve sketching.

8.2.3 Taylor expansion

a) Provide Taylor's formula of degree n, including the remainder term.

b) Calculate $\ln\sqrt{3/2}$ using a Taylor's formula of degree 2. Can you tell what the margin of error is here? I.e. in which interval does the true function value lie? Check your solution using a calculator.

8.2.4 Linear approximations

You are interested in the relationship of the two economic variables p (price) and q (quantity). In particular, you know that the price p is typically relatively small. Unfortunately, you can only determine an implicit relation between p and q, which is given by

$$g(p\,q^2) - q\,e^{\alpha\,p} = 1.$$

The only things that you know to hold with respect to g are that $g(0) = 2$ and $g'(0) = g''(0) = 0$.

a) Provide a linear approximation of the function $q(p)$ about $p_0 = 0$.

b) Can you also provide a quadratic approximation of the function $q(p)$ about $p_0 = 0$?

9 — from December 8, 2025

9.1 Finger exercises – More derivatives

Differentiate the following functions with respect to x. Greek letters (α, β) are constants, but y and z are functions of x, i.e. $y(x)$ and $z(x)$.

a) $f(x) = \dfrac{\ln(x)}{e^x}$

b) $f(x) = x\,e^x$

c) $f(x) = g(x^x)$

d) $f(x) = x^{g(x)}$

e) $f(x) = g(e^x)$

f) $f(x) = \ln(g(x)\,h(x))$

g) $f(x) = \ln\left(\dfrac{g(x)}{h(x)}\right)$

h) $f(x) = e^{g(x)+h(x)}$

i) $f(x) = \beta^{g(x)}$

j) $f(x) = g(x)^\alpha$

k) $f(x) = g(x)^{h(x)}$

l) $f(x) = \log_{g(x)}(h(x))$

m) $f(x) = \sqrt[g(x)]{-x}$

n) $f(x) = \sqrt[g(x)]{h(x)}$

o) $f(x) = \dfrac{\ln(x)}{\sqrt{g(x)}}$

p) $f(x) = \dfrac{g(x)}{\sqrt{h(\alpha x)}}$

9.2 Tutorial

9.2.1 Derivatives

Let $x \in \mathbb{R}_{\neq 0}$. Differentiate the following functions with respect to x. If you want, you may use the sign function sgn (with $\text{sgn}(x) = 1$ for positive x, and -1 for negative x, and undefined for $x = 0$).

a) $f(x) = |x|$

b) $f(x) = e^{|x|}$

c) $f(x) = g(|x| - 1)$

9.2.2 Elasticities

a) Please provide the definition of an elasticity and use an example to describe what statement the elasticity gives.

b) Suppose you have a function, which is the product of two functions, i.e. $f(x) = g(x) \cdot h(x)$. What can you say about the elasticity of f?

c) Can you think of a good example from economics for the formula derived in part **b)**?

d) Suppose you have a function, which can be represented by the fraction of two functions? What can you say about the elasticity of such a function?

e) Can you think of a good example from economics for the formula derived in part **d)**?

9.2.3 Implicit differentiation

Suppose there is an implicit relationship between x and y given by

$$(x-1)^2 + y^2 = 2.$$

a) For $x = 0$, what is the value of y?

b) For $x = 0$, what is the value of y'?

c) What is the equation of the linear function tangent to $y(x)$ at $x = 0$?

d) Determine the second derivative y'' at $x = 0$.

9.2.4 Limits

Calculate the following limits.

a) $\lim\limits_{x \to 0} \dfrac{2^x - 1}{x}$

b) $\lim\limits_{x \to 0} \dfrac{2^x - 2}{x}$

c) $\lim\limits_{x \to \infty} x\, e^{-x}$

d) $\lim\limits_{x \to 0} x\, e^{-x}$

e) $\lim\limits_{x \to \infty} \dfrac{a^x}{b^x}$ mit $a, b > 0$.

f) $\lim\limits_{x \to 1} \dfrac{x^2 - 4\,x + 3 + 2\,\ln x}{2\,(x-1)^3}$

10 — from December 22, 2025

10.1 Finger exercises – Derivatives, chained functions, and inequalities

10.1.1 Ableitungen

Differentiate the following functions with respect to x.

a) $f(x) = \dfrac{e^{\alpha x}}{\ln(x)}$

b) $f(x) = e^{\frac{-x}{\ln(x-\mu)}}$

c) $f(x) = |x^2 - 1|$

d) $f(x) = \log_{10}|x|$

e) $f(x) = |g(x)| \cdot h(x)$

f) $f(x) = g(|x|)\, h(x)$

g) $f(x) = x^2 - \alpha\, x - 1$ für $x \geq 5$, sonst $f(x) = 0$

10.1.2 Chained functions

a) $f(2)$

b) $g(\alpha)$

c) $f(g(-1))$

d) $h(\kappa) + g(2)$

e) $g'(h(8))$

f) $2^{g(1)}$

g) $g'(h'(\lambda))$

10.1.3 Inequalities

For which x do the following inequalities hold?

a) $x^2 - 2\,x - 8 \geq 0$?

b) $x^2 - 2\,|x| - 8 \geq 0$?

c) $x - 2\,|x| - 3 \geq 0$?

d) $x - 2\,|x| + 3 \geq 0$?

10.2 Tutorial

10.2.1 Minimum points and maximum points

Consider the following function

$$f(x) = e^{\alpha x} + x + \alpha$$

with $\alpha > 0$.

a) Show that f has exactly one (unique) root. Please state, which tools you require in order to make a case for this argument (e.g. continuity, the intermediate value theorem, the mean value theorem, etc.). *As usual: if you cannot come up with a solution yourself, please discuss in your study groups, consider other examples first, etc.*

b) Argue that f has a slope of $f'(x) = 2$ at exactly one (unique) point. As above, be sure to state the theorems that apply.

10.2.2 Optimization I

If possible, here and in the next exercise, try to avoid differentiating the function when making an argument. *In real economic examples, calculating higher-order deriavtives of functions often leads to very complex, possibly even uninterpretable, terms. If you can you should avoid them, i.e. if an alternative line of argument is available.*

a) At which values is the function $f(x) = \sqrt{\mu - x}$, defined over the interval $D = [0; \mu]$, maximized and at which values minimized? Sketch the function. You may assume that $\mu > 0$.

b) Show that in case of the function $f(x) = \lambda/(\gamma - x)$, defined on $D = (0; \gamma)$, there does not exist a global minimum point nor global maximum point. Again, sketch the function. Assume that $\lambda > 0$ and $\gamma > 0$.

10.2.3 Optimization II

Where does the function $f(x) = e^{-(x-1)(x+2)} + \beta$ take on (local and global) maxima and minima? Let $\beta > 0$. Sketch the function.

10.2.4 Optimization III

Can you find all (local and global) minimum and maximum points of the function $f(x) = \vartheta\, x + |x|$ defined on the interval $D = [-1, 1]$ with $\vartheta > 0$? Sketch the function twice: once for $\vartheta < 1$ and once for $\vartheta > 1$.

11 — from January 12, 2026

11.1 Finger exercises – Fractions, Function Value

11.1.1 Fractions

Rearrange:

a) $\dfrac{1}{a^2} + \dfrac{a}{1-a}$

b) $\dfrac{1}{f(x)} + \dfrac{\gamma}{g(x)}$

c) $\dfrac{\mu}{\lambda} - \dfrac{\lambda}{\mu}$

11.1.2 Function values of multivariate functions

Let$f(x,y) = 2\,x + 3\,y$ and $g(x,y) = x^2 - 2\,y^2 - 1$. Further, let $k(x) = \ln(x+1)$ and $m(x) = \sqrt{x-\phi}$. Calculate:

a) $f(2,1)$

b) $g(-2,-1)$

c) $f(\alpha,\alpha)$

d) $g(\alpha,x)$

e) $f(1/x,1/y)$

f) $f(-x,2\,y)$

g) $f(k(x),y)$

g) $f(g(1,2))$

i) $f(f(a,b),c)$

j) $f(a,f(b,c))$

k) $g(m(x),m(y-\phi))$

l) $f(k'(x),c)$

m) $f(x,m^{-1}(y))$

n) $g(k^{-1}(x),1)$

11.2 Tutorial

11.2.1 Bivariate functions

Determine the partial derivatives for the following functions. For which points (x, y) we have both $f'_x(x, y) = 0$ and $f'_y(x, y) = 0$?

a) $f(x, y) = x^2 + x\,y + y^2/2 + 2\,x$

b) $f(x, y) = e^{2\,x+y} - x^2 - 2\,y$

c) $f(x, y) = \ln(y + \alpha\,x) - y$ with $\alpha > 0$

11.2.2 Level curves

Determine the level curves of the function $f(x, y) = (x-1)^2 + y^2$ at the levels $h = 1, 2, 3$ and sketch them.

11.2.3 Level curves II

a) Suppose you are given a general function $f(x, y)$ and a strictly monotonically increasing function $k(x)$. Show that a level curve of $f(x, y)$ is also automatically a level curve of $g(x, y) := k(f(x, y))$.

b) Suppose you are given a general function $f(x, y)$ and a strictly monotonically increasing function $k(x)$. Find a counterexample to the following statement: A level curve of $f(x, y)$ is also automatically a level curve of $g(x, y) := f(k(x), k(y))$.

11.2.4 Multivariate optimization I

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Provide the necessary and sufficient conditions for global extremum points.

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Provide the necessary and sufficient conditions for local extremum points.

c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Characterize a saddle point.

d) Let $f : S \subset \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable with S closed and bounded. Describe a procedure of how to find minimum and maximum points of f.

12 — from January 19, 2026

12.1 Finger exercises – Partial Derivatives

Calculate the partial derivatives of $f(x,y)$ with respect to x and y. Let $g(x)$ and $h(x)$ be general differentiable functions.

a) $f(x,y) = x/y$

b) $f(x,y) = y/x$

c) $f(x,y) = x^2 + 2\ln(2\,x + y)$

d) $f(x,y) = e^{y+2\,\ln y}$

e) $f(x,y) = g(x+y) + g(x-y)$

f) $f(x,y) = \sqrt[y]{x}$

g) $f(x,y) = \dfrac{g(x)}{h(y)}$

h) $f(x,y) = \alpha\,\ln x + \beta\,\ln y$

i) $f(x,y) = \alpha^x \cdot \beta^y$

12.2 Tutorial

12.2.1 Multivariate optimization II

a) Determine the global extremum points of $f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = 3xy - x^2 - y^2$

b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = 5 - x^2 + 6x - 2y^2 + 8y$ and classify all stationary points (lokal minimum points, local maximum points, saddle points).

c) Provide an example of a function $f(x, y)$ that has a saddle point.

d) Let $f : \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} \to \mathbb{R},\ f(x, y) = x^2 + y^2 + y - 1$. Determine all extremum points and corresponding function values.

12.2.2 Continuity and differentiability

Let

$$f(x) = \begin{cases} x + \alpha & : \ x \leq 0 \\ \ln(x + \beta) & : \ x > 0 \end{cases}$$

For which combinations of α and β is the function continuous? For which combinations of α and β is the function differentiable?

12.2.3 Stationary points and concavity I

For the function $f(x, y) = e^x + e^{-x} + e^y + e^{-y} - 2\,x + 2\,y$, find all stationary points. Are these (local/global) maxima or minima?

12.2.4 Stationary points and concavity II

Calculate all stationary points for the function $\Pi(L, K) = L^\alpha\, K^\beta - L - K$. Let $0 < \alpha$, $0 < \beta$ and $\alpha + \beta < 1$. Are these (local/global) maxima or minima? Calculate the stationary point and the function value specifically for $\alpha = \beta = 1/3$.

$\Pi(L, K)$ corresponds to the profit function of an entrepreneur with production function $L^\alpha\, K^\beta$, labor costs of 1 and capital costs of 1.

13 — from January 26, 2026

13.1 Finger exercises – Partial derivatives

Calculate the partial derivatives of $f(x, y)$ with respect to x and y. Let $g(x, y)$ and $h(x)$ be general differentiable functions.

a) $f(x, y) = x + h(y)$

b) $f(x, y) = x\, h(y)$

c) $f(x, y) = g(x, 2\, y)$

d) $f(x, y) = h(x + 2\, y)$

e) $f(x, y) = g(1/x, \alpha/y)$

f) $f(x, y) = g(x + y, x - y)$

g) $f(x, y) = g(x, h(y))$

h) $f(x, y) = g(x^2 + y, h(y^2 + x))$

i) $f(x, y) = g(e^x, \ln y)$

j) $f(x, y) = h(g(x, y))$

13.2 Tutorial

13.2.1 Lagrange I

a) Solve the following problem using the Lagrange multiplier method:

max $f(x,y) = 3x + 4y$ unter der NB $g(x,y) = x^2 + y^2 = 225$

b) Assume that the right side of the constraint changes from 225 to 224. Calculate the *approximate* change of the optimal value of f.

13.2.2 Lagrange II

An individual has a utility function $u(x,y) = y\, e^x$. Suppose that she possesses four units of money, implying a budget constraint $2\,x + y = 4$ and non-negativity constraints $x, y \geq 0$.

a) Write down the Langrangian function and determine all stationary points.

b) Find the global maximum and minimum of $u(x,y)$ under the budget constraint above (in addition to $x, y \geq 0$). Argue in detail (do not just calculate them). Are there other local extremum points?

c) What is the consumer's utility at the optimum?

d) *Approximately*, what would be the optimal utility if the consumer had 4.20 (instead of 4.00) at his disposal?

13.2.3 Langrange III – Part 1

A consumer has the utility function $u(x_1, x_2) = \ln x_1 + 2 \ln x_2$. She has a budget of b, her budget constraint is $p_1\, x_1 + p_2\, x_2 \leq b$, and $x_1, x_2 \geq 0$. Let $p_1, p_2 > 0$ be the prices of the two goods, and let x_1, x_2 be the quantities consumed.

a) Write down the Langrangian function and determine all stationary points.

b) Find the global maximum and minimum of $u(x_1, x_2)$ subject to the constraints. Also determine the corresponding λ. Argue in detail (do not just calculate blindly). Are there other local extrema?

13.2.4 Langrange III – Part 2

c) Determine the optimal value function, i.e., the function value at the optimal point. If you calculated correctly, this should be a function depending on three parameters (p_1, p_2, b).

d) Interpret the derivative of the optimal value function with respect to b. Does the value match λ? If so, why? If not, why not?

e) Interpret the derivative of the optimal value function with respect to p_1 and p_2. Does the envelope theorem apply here? If so, why? If not, why not?

14 — from February 2, 2026

14.1 Finger exercises – Stationary Points and Concavity

Consider the respective function below in each subtask and complete the following tasks. 1) Find all stationary points. 2) Check whether the function is concave or convex in the x direction or in the y direction, respectively, and whether it is concave or convex overall (that is, whether the cross derivative dominates or not). 3) Is it the stationary point and a global or local minimum or maximum? 4) If possible, consider whether there are other, simpler arguments in addition to the second-order condition.

a) $f(x, y) = 3 - x^2 - y^2 - x\,y + 3\,x$.

b) $f(x, y) = e^{x\,y} - x$.

c) $f(x, y) = \ln(x^y) - \ln(\alpha\,x) - \ln(\beta\,y)$ with $\alpha, \beta > 0$.

14.2 Tutorial

14.2.1 Integrals

Please solve the following integrals.

a) $\int_1^2 x^2 \, dx$

b) $\int_1^2 2^x \, dx$

c) $\int_1^2 \frac{1}{x^2} \, dx$

d) $\int_1^2 \frac{1}{x} \, dx$

14.2.2 Integration by parts

Please solve the following integrals. The questions are partly the same as in the lecture, because we want to practice with simple functions.

a) $\int_1^2 x \, e^x \, dx$

b) $\int_1^2 e^x \, x^2 \, dx$

c) $\int x^2 \, \ln x \, dx$

14.2.3 Integration by substitution

Please solve the following integrals.

a) $\int_1^2 2 \, x \, e^{x^2} \, dx$.

b) $\int 2 \, x \, f'(x^2)$.

c) $\int 2 \, g'(x) \, g(x) \, dx$.

d) $\int \frac{x}{\sqrt{x^2+2}} dx$.

14.2.4 Rational Functions

Please solve the following integrals.

a) $\int \frac{2}{x+3} \, dx$.

b) $\int \frac{x+2}{x+3} \, dx$.

c) $\int \frac{x^2+2}{x+3} \, dx$.

d) $\int \frac{x^3+2}{x+3} \, dx$.