Derivative Tests, Convexity & Inflection Points

Mathematics for ML

November 23, 2025

Outline

First-Derivative Test: Concept

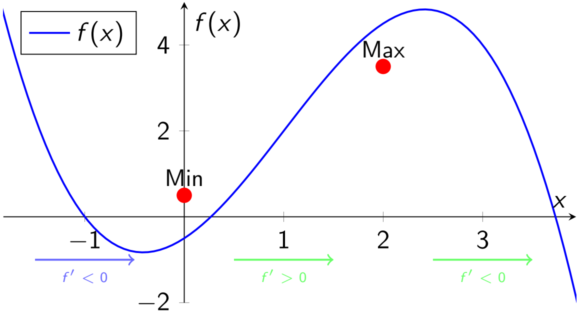
Purpose: Identify local extrema (maxima and minima) using the first derivative.

Critical Points: Points where $f'(x) = 0$ or $f'(x)$ does not exist.

First-Derivative Test:

- If $f'(x)$ changes from positive to negative at $x = c$, then f has a **local maximum** at c.
- If $f'(x)$ changes from negative to positive at $x = c$, then f has a **local minimum** at c.
- If $f'(x)$ does not change sign at $x = c$, then f has **no local extremum** at c.

First-Derivative Test: Visual Interpretation



Observation: Derivative changes sign at extrema.

Example 1: First-Derivative Test

Problem: Find and classify the critical points of $f(x) = x^3 - 3x^2 - 9x + 5$.

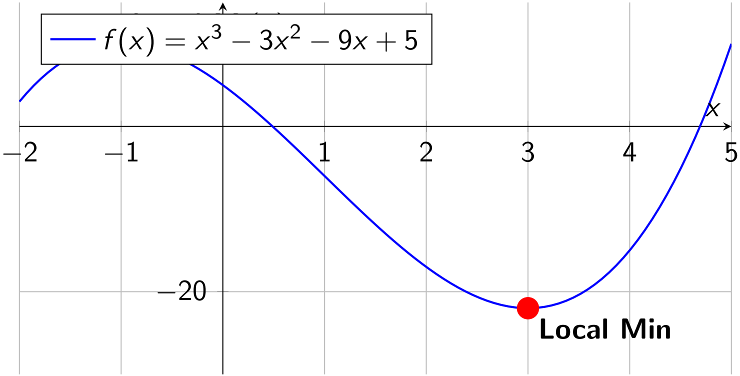
Solution:

1. Find $f'(x)$:

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$$

2. Critical points: $f'(x) = 0 \Rightarrow x = -1$ or $x = 3$
3. Test intervals:
    - $x < -1$: $f'(-2) = 15 > 0 \quad$ (increasing)
    - $-1 < x < 3$: $f'(0) = -9 < 0 \quad$ (decreasing)
    - $x > 3$: $f'(4) = 15 > 0 \quad$ (increasing)
4. **Conclusion:**
    - $x = -1$: Local maximum (sign changes $+$ to -)
    - $x = 3$: Local minimum (sign changes - to $+$)

Example 1: Visualization



Second-Derivative Test: Concept

Purpose: Classify critical points using the second derivative (faster than first-derivative test).

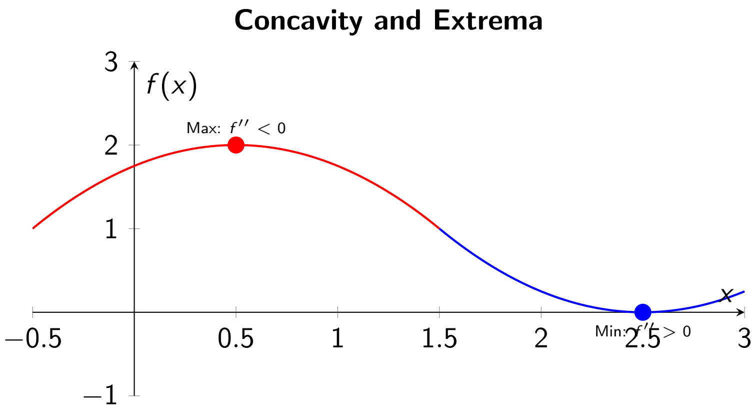
Second-Derivative Test:
Let c be a critical point where $f'(c) = 0$. Then:

- If $f''(c) > 0$, then f has a **local minimum** at c (concave up)
- If $f''(c) < 0$, then f has a **local maximum** at c (concave down)
- If $f''(c) = 0$, the test is **inconclusive** (use first-derivative test)

Intuition:

- $f''(x) > 0$: function is curving upward $\Rightarrow$ minimum
- $f''(x) < 0$: function is curving downward $\Rightarrow$ maximum

Second-Derivative Test: Visual



Example 2: Second-Derivative Test

Problem: Use the second-derivative test to classify critical points of $f(x) = x^4 - 4x^3 + 4x^2$.

Solution:

1. Find $f'(x)$:

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$$

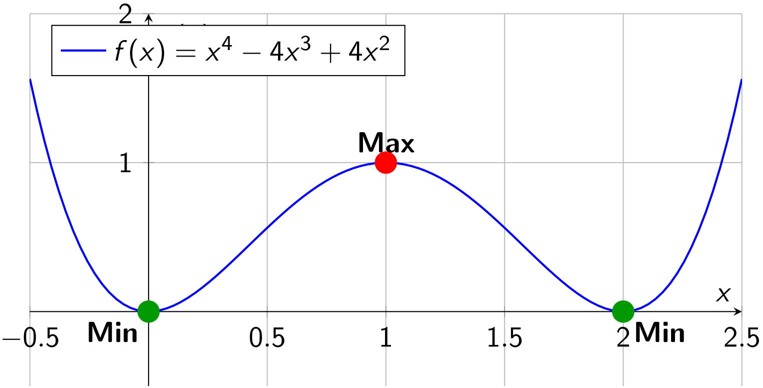
   Critical points: $x = 0, 1, 2$

2. Find $f''(x)$:

$$f''(x) = 12x^2 - 24x + 8$$

3. Evaluate f'' at critical points:
   - $f''(0) = 8 > 0 \quad \Rightarrow$ **Local minimum**
   - $f''(1) = 12 - 24 + 8 = -4 < 0 \quad \Rightarrow$ **Local maximum**
   - $f''(2) = 48 - 48 + 8 = 8 > 0 \quad \Rightarrow$ **Local minimum**

Example 2: Visualization



Convex and Concave Functions

Definitions:

- f is **concave up (convex)** on interval I if $f''(x) > 0$ for all $x \in I$
- f is **concave down (concave)** on interval I if $f''(x) < 0$ for all $x \in I$

Geometric Interpretation:

Convex (Concave Up):

- Curves upward
- Tangent lines below curve
- $f''(x) > 0$

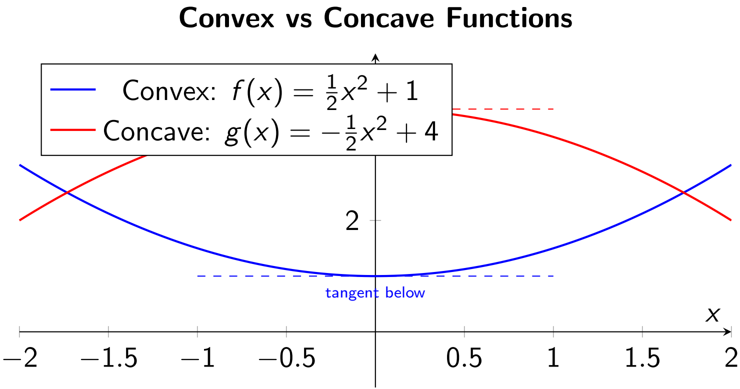
Concave (Concave Down):

- Curves downward
- Tangent lines above curve
- $f''(x) < 0$

Alternative Definition (Secant Line): f is convex if for all x_1, x_2 and $\lambda \in [0, 1]$:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

Convexity vs Concavity: Visual Comparison



Example 3: Determine Convexity

Problem: Determine where $f(x) = x^3 - 6x^2 + 9x + 1$ is convex and concave.

Solution:

1. Find $f''(x)$:
$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12 = 6(x - 2)$$
2. Find where $f''(x) = 0$:
$$6(x - 2) = 0 \Rightarrow x = 2$$
3. Test intervals:
   - For $x < 2$: $f''(1) = -6 < 0 \quad \Rightarrow$ **Concave down**
   - For $x > 2$: $f''(3) = 6 > 0 \quad \Rightarrow$ **Concave up (Convex)**

Conclusion:

- Concave on $(-\infty, 2)$
- Convex on $(2, \infty)$
- Point $x = 2$ is an **inflection point**

Inflection Points: Definition

Definition: A point $x = c$ is an **inflection point** if:

1. The concavity of f changes at $x = c$
2. Equivalently: $f''(x)$ changes sign at $x = c$

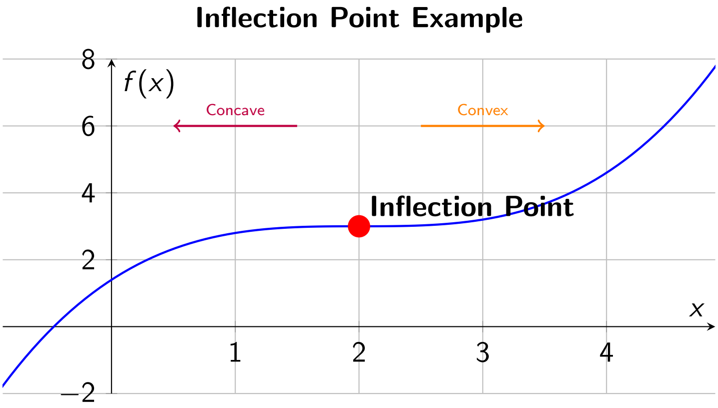
Finding Inflection Points:

1. Find $f''(x)$
2. Solve $f''(x) = 0$ (or find where f'' doesn't exist)
3. **Verify** that f'' changes sign at these points

Warning: $f''(c) = 0$ does NOT guarantee an inflection point!

- Example: $f(x) = x^4$ has $f''(0) = 0$, but $x = 0$ is **not** an inflection point
- $f''(x) = 12x^2 \geq 0$ for all x (no sign change)

Inflection Points: Visual



At the inflection point, the curve changes from concave down to concave up (or vice versa).

Example 4: Finding Inflection Points

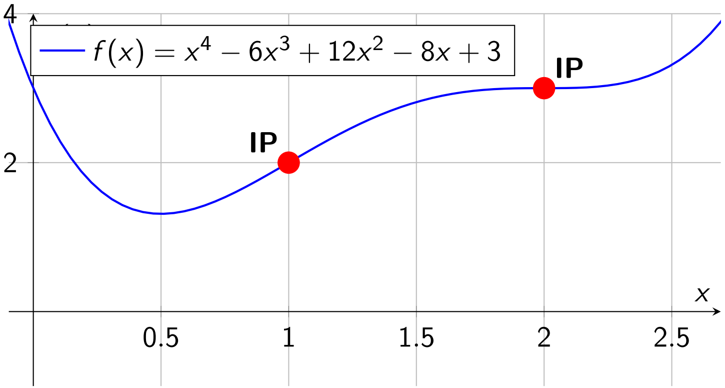
Problem: Find all inflection points of $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$.

Solution:

1. Find $f''(x)$:
$$f'(x) = 4x^3 - 18x^2 + 24x - 8$$
$$f''(x) = 12x^2 - 36x + 24 = 12(x^2 - 3x + 2) = 12(x-1)(x-2)$$
2. Solve $f''(x) = 0$:
$$12(x-1)(x-2) = 0 \Rightarrow x = 1 \text{ or } x = 2$$
3. Check sign changes:
   - $x < 1$: $f''(0) = 24 > 0 \quad$ (convex)
   - $1 < x < 2$: $f''(1.5) = -3 < 0 \quad$ (concave)
   - $x > 2$: $f''(3) = 24 > 0 \quad$ (convex)

Conclusion: Inflection points at $x = 1$ and $x = 2$ (both have sign changes).

Example 4: Visualization



Comprehensive Example

Problem: Analyze $f(x) = x^3 - 3x + 1$ completely:

- Find critical points and classify them
- Determine concavity
- Find inflection points

Solution:

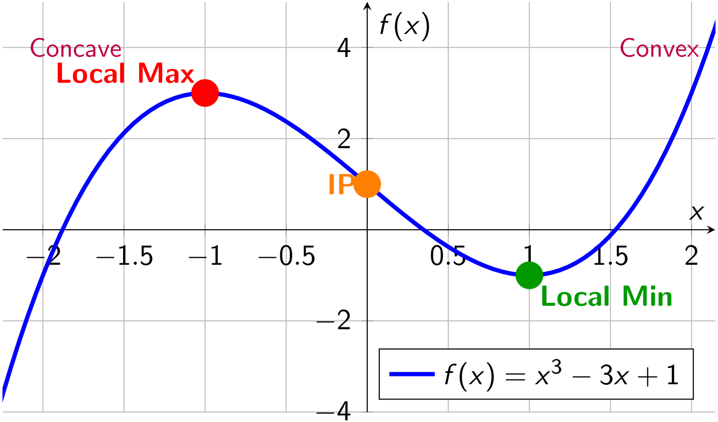
1. **First derivative:** $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$
   - Critical points: $x = -1, 1$
2. **Second derivative:** $f''(x) = 6x$
3. **Classify critical points:**
   - $f''(-1) = -6 < 0 \Rightarrow$ local maximum at $x = -1$, $f(-1) = 3$
   - $f''(1) = 6 > 0 \Rightarrow$ local minimum at $x = 1$, $f(1) = -1$

4. **Concavity:**
    - $f''(x) = 6x$
    - $x < 0$: $f''(x) < 0$ (concave down)
    - $x > 0$: $f''(x) > 0$ (concave up / convex)
5. **Inflection point:**
    - $f''(x) = 0 \Rightarrow x = 0$
    - Sign changes from negative to positive
    - Inflection point at $(0, f(0)) = (0, 1)$

Summary:

- Local max at $(-1, 3)$
- Local min at $(1, -1)$
- Inflection point at $(0, 1)$
- Concave on $(-\infty, 0)$, convex on $(0, \infty)$

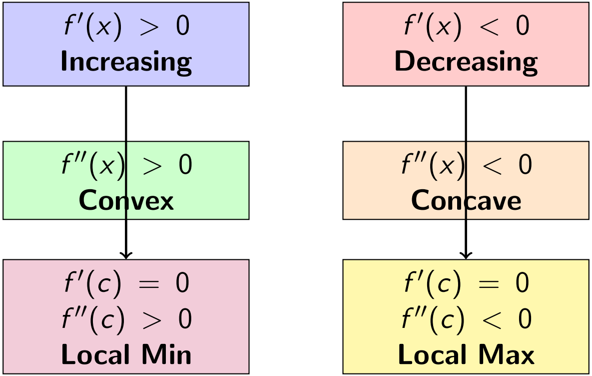
Comprehensive Example: Complete Graph



Summary Table

Concept	Condition	Interpretation
First-Derivative Test		
Critical point	$f'(c) = 0$	Potential extremum
Local maximum	f' changes + to -	Top of hill
Local minimum	f' changes - to +	Bottom of valley
Second-Derivative Test		
Local maximum	$f'(c) = 0, f''(c) < 0$	Concave down at c
Local minimum	$f'(c) = 0, f''(c) > 0$	Concave up at c
Inconclusive	$f'(c) = 0, f''(c) = 0$	Use 1st derivative test
Concavity		
Convex (Concave up)	$f''(x) > 0$	Curves upward
Concave (Concave down)	$f''(x) < 0$	Curves downward
Inflection Points		
Inflection point	f'' changes sign	Concavity changes

Key Relationships



Practice Problems

Try these on your own:

1. Find and classify all critical points of $f(x) = x^4 - 4x^3 + 10$
2. Determine the intervals where $g(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$ is convex and concave
3. Find all inflection points of $h(x) = x^5 - 5x^4 + 5x^3$
4. For $f(x) = xe^{-x}$:
   - Find critical points
   - Determine concavity
   - Sketch the graph
5. Prove that $f(x) = \ln(x)$ is concave on $(0, \infty)$

Applications

Real-World Applications:

- **Economics:**
  - Profit maximization (find where $\text{Profit}'(x) = 0$)
  - Diminishing marginal returns (concavity of production functions)
  - Cost minimization
- **Machine Learning:**
  - Loss function optimization
  - Convex optimization ensures global minimum
  - Second-order methods (Newton's method)
- **Physics:**
  - Maximum height of projectile
  - Minimum energy states
- **Data Analysis:**
  - Finding peaks and valleys in time series
  - Change point detection (inflection points)

Thank You!

Questions?

Remember: Derivatives tell us about rate of change,
second derivatives tell us about the rate of change of the rate of change!