Trigonometric Functions & Domain/Range

Mathematics for ML

November 23, 2025

Outline

Domain and Range: Definitions

Definition (Domain)

The **domain** of a function $f : A \to B$ is the set of all possible input values x for which $f(x)$ is defined.

$$\text{Domain}(f) = \{x \in A : f(x) \text{ exists}\}$$

Definition (Range (Image))

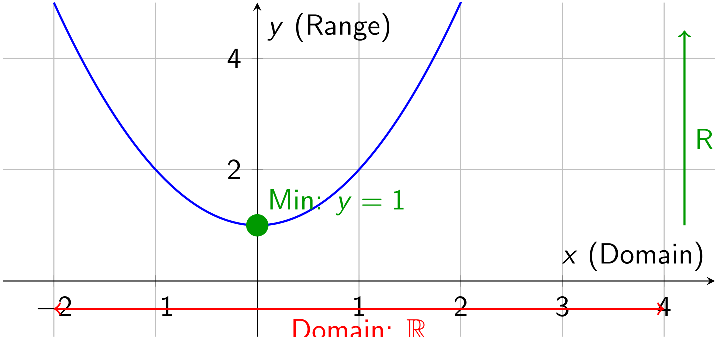
The **range** (or **image**) of a function f is the set of all possible output values.

$$\text{Range}(f) = \{y \in B : \exists x \in \text{Domain}(f) \text{ such that } f(x) = y\}$$

Key Point: Range $\subseteq$ Codomain, but they may not be equal!

Visual Understanding

Example: $f(x) = x^2 + 1$



Finding Domain: Common Restrictions

What restricts the domain?

1. **Division by zero:** $f(x) = \frac{1}{x-2}$ Domain: $x \neq 2$
2. **Square roots (even roots):** $f(x) = \sqrt{x-3}$ Domain: $x \geq 3$
3. **Logarithms:** $f(x) = \ln(x)$ Domain: $x > 0$
4. **Combinations:** $f(x) = \frac{\sqrt{4-x^2}}{x-1}$ Domain: $[-2, 1) \cup (1, 2]$
   - Need $4 - x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$
   - Need $x \neq 1$

Rule: Domain is all real numbers *except* where function is undefined.

Finding Range: Strategies

Methods to find range:

1. **Graphical:** Sketch the function and observe y-values
2. **Algebraic:** Solve $y = f(x)$ for x in terms of y
   - If x exists for all y, then y is in range
   - Look for restrictions on y
3. **Calculus:** Find critical points, limits at boundaries
   - Local extrema give range boundaries
   - Check behavior as $x \to \pm\infty$
4. **Known functions:** Use standard ranges
   - x^2: Range $[0, \infty)$
   - e^x: Range $(0, \infty)$
   - $\ln(x)$: Range $\mathbb{R}$

Example 1: Finding Domain and Range

Problem: Find domain and range of $f(x) = \frac{x+1}{x-2}$

Solution:

Domain:

- Denominator cannot be zero: $x - 2 \neq 0$
- Domain: $\mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2, \infty)$

Range: Solve for x in terms of y:

$$\begin{aligned} y &= \frac{x+1}{x-2} \\ y(x-2) &= x+1 \\ yx - 2y &= x+1 \\ yx - x &= 1+2y \\ x(y-1) &= 1+2y \\ x &= \frac{1+2y}{} \end{aligned}$$

Example 2: With Square Root

Problem: Find domain and range of $f(x) = \sqrt{9 - x^2}$

Solution:
Domain: Need $9 - x^2 \geq 0$

$$\begin{aligned} 9 - x^2 &\geq 0 \\ 9 &\geq x^2 \\ -3 \leq x &\leq 3 \end{aligned}$$

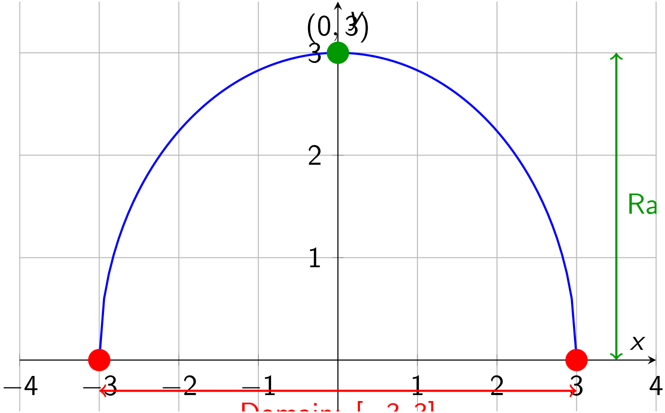
Domain: $[-3, 3]$

Range:

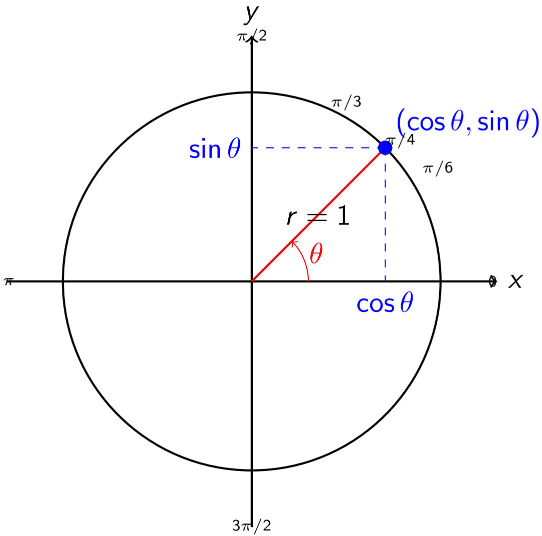
- This is the upper half of a circle with radius 3
- Minimum: $f(\pm 3) = 0$
- Maximum: $f(0) = \sqrt{9} = 3$
- Range: $[0, 3]$

Example 2: Visualization

$$f(x) = \sqrt{9 - x^2}$$



The Unit Circle



Basic Trigonometric Functions

Primary Functions:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}}$$

Reciprocal Functions:

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Fundamental Identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Special Angle Values

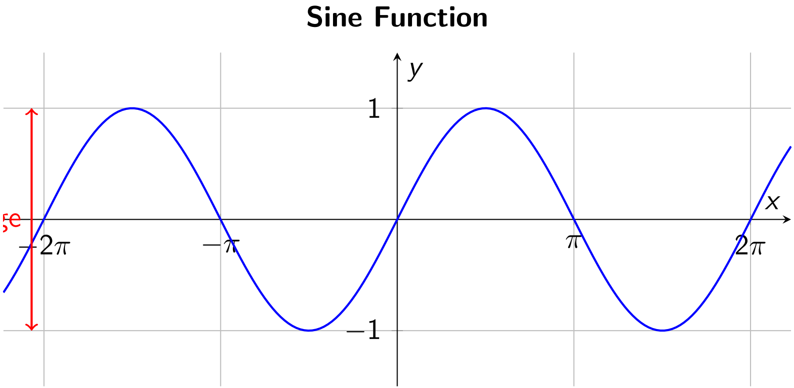
Angle	0	$\pi/6$ (30°)	$\pi/4$ (45°)	$\pi/3$ (60°)	$\pi/2$ (90°)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Memory Aid:

$$\sin(0, 30, 45, 60, 90) = \frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$$

$$\cos(0, 30, 45, 60, 90) = \frac{\sqrt{4}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{0}}{2}$$

Sine Function: $y = \sin(x)$

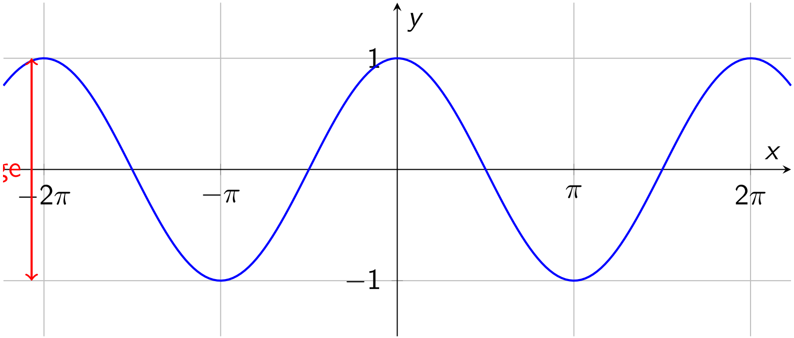


Properties:

- Domain: $\mathbb{R}$
- Range: $[-1, 1]$
- Period: 2π

Cosine Function: $y = \cos(x)$

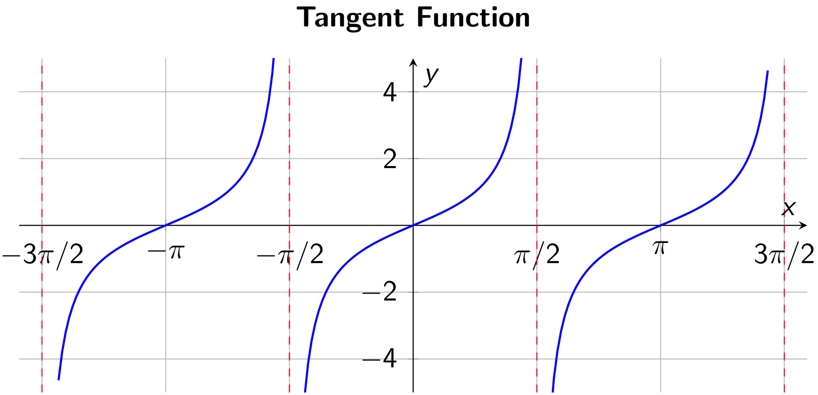
Cosine Function



Properties:

- Domain: $\mathbb{R}$
- Range: $[-1, 1]$
- Period: 2π

Tangent Function: $y = \tan(x)$



Properties:

- Domain: $\mathbb{R} \setminus \{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\}$
- Range: $\mathbb{R}$
- Period: π

Domain and Range of Trig Functions

Function	**Domain**	**Range**	**Period**
$\sin(x)$	$\mathbb{R}$	$[-1, 1]$	2π
$\cos(x)$	$\mathbb{R}$	$[-1, 1]$	2π
$\tan(x)$	$\mathbb{R} \setminus \{\frac{\pi}{2} + n\pi\}$	$\mathbb{R}$	π
$\csc(x)$	$\mathbb{R} \setminus \{n\pi\}$	$(-\infty, -1] \cup [1, \infty)$	2π
$\sec(x)$	$\mathbb{R} \setminus \{\frac{\pi}{2} + n\pi\}$	$(-\infty, -1] \cup [1, \infty)$	2π
$\cot(x)$	$\mathbb{R} \setminus \{n\pi\}$	$\mathbb{R}$	π

Key Observations:

- sin and cos are bounded: $|\sin x| \leq 1$, $|\cos x| \leq 1$
- tan and cot are unbounded
- csc and sec cannot have values in $(-1, 1)$

Inverse Trigonometric Functions

To define inverses, restrict domain:

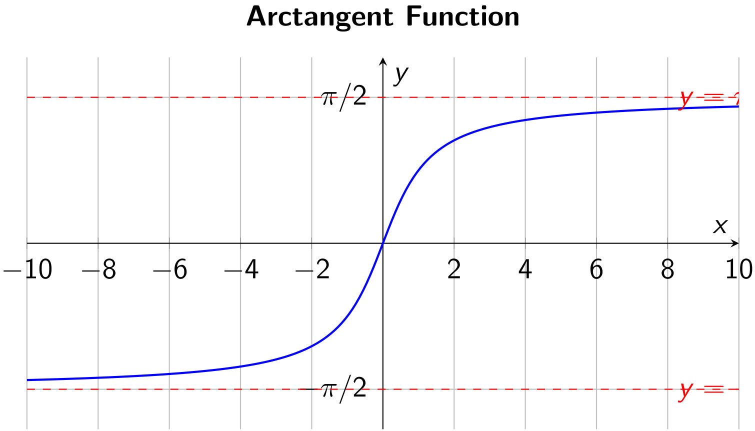
Function	Domain	Range
$\arcsin(x)$ or $\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\arccos(x)$ or $\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$ or $\tan^{-1}(x)$	$\mathbb{R}$	$(-\pi/2, \pi/2)$

Properties:

- $\sin(\arcsin(x)) = x$ for $x \in [-1, 1]$
- $\arcsin(\sin(x)) = x$ for $x \in [-\pi/2, \pi/2]$
- Similar identities hold for arccos and arctan

Note: Domain of inverse = Range of original (restricted)

Arctangent Function: $y = \arctan(x)$



Domain: $\mathbb{R}$, Range: $(-\pi/2, \pi/2)$
Horizontal asymptotes at $y = \pm\pi/2$

Trigonometric Identities

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Angle Addition:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double Angle:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

Example 3: Domain of Composite Function

Problem: Find the domain of $f(x) = \arcsin\left(\frac{x-1}{2}\right)$

Solution:
For $\arcsin(u)$ to be defined, we need $u \in [-1, 1]$.
Here $u = \frac{x-1}{2}$, so:

$$\begin{aligned} -1 &\leq \frac{x-1}{2} \leq 1 \\ -2 &\leq x - 1 \leq 2 \\ -1 &\leq x \leq 3 \end{aligned}$$

Domain: $[-1, 3]$

Range: Since $\frac{x-1}{2}$ covers all values in $[-1, 1]$ as x varies over $[-1, 3]$, and arcsin has range $[-\pi/2, \pi/2]$:
Range: $[-\pi/2, \pi/2]$

Example 4: Solving Trig Equations

Problem: Solve $2\sin^2(x) - \sin(x) - 1 = 0$ for $x \in [0, 2\pi]$

Solution:
This is a quadratic in $\sin(x)$. Let $u = \sin(x)$:

$$2u^2 - u - 1 = 0$$
$$(2u + 1)(u - 1) = 0$$

So $u = -\frac{1}{2}$ or $u = 1$.

Case 1: $\sin(x) = 1 \quad \Rightarrow \quad x = \frac{\pi}{2}$
Case 2: $\sin(x) = -\frac{1}{2} \quad \Rightarrow \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}$

Solutions: $x \in \left\{ \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$

Transformations of Trig Functions

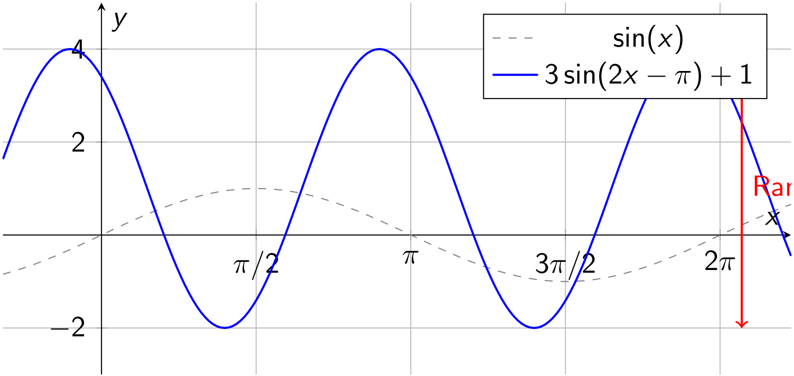
General Form: $y = A\sin(B(x - C)) + D$ or $y = A\cos(B(x - C)) + D$

Parameters:

- A: **Amplitude** (vertical stretch/compression)
  - Range becomes $[D - |A|, D + |A|]$
- B: Affects **period**
  - New period: $\frac{2\pi}{|B|}$
- C: **Phase shift** (horizontal shift)
  - Shifts graph C units to the right
- D: **Vertical shift**
  - Shifts graph D units up

Example: $y = 3\sin(2x - \pi) + 1$ has amplitude 3, period π, phase shift $\pi/2$, vertical shift 1.

Example 5: Transformed Function



$y = 3\sin(2x - \pi) + 1$: Amplitude 3, Period π, Shift right $\pi/2$, Up 1

Applications in Calculus

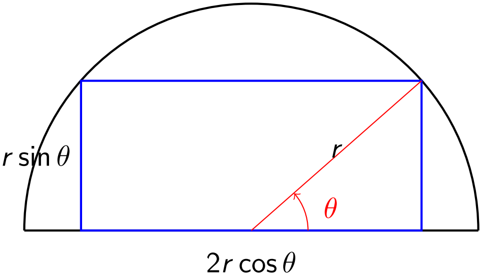
Derivatives:

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Note: Domains of derivatives match domains of original functions!

Applications in Optimization

Example: Maximize area of rectangle inscribed in semicircle of radius r.



Solution: $A(\theta) = 2r\cos\theta \cdot r\sin\theta = r^2\sin(2\theta)$
Domain: $\theta \in [0, \pi/2]$, Maximum at $\theta = \pi/4$, giving $A_{\max} = r^2$

Summary: Domain and Range

Key Takeaways:

- **Domain:** All possible inputs where function is defined
  - Check: division by zero, square roots, logarithms
- **Range:** All possible outputs
  - Methods: graphing, algebra, calculus
- **Inverse functions:** Domain and range swap
- **Compositions:** Inner function's range must fit outer function's domain

Remember: Always verify both domain AND range when working with functions!

Summary: Trigonometric Functions

Essential Knowledge:

1. **Unit Circle:** $(\cos\theta, \sin\theta)$ for angle θ
2. **Six Functions:** $\sin, \cos, \tan, \csc, \sec, \cot$
3. **Domains & Ranges:**
   - $\sin, \cos$: Domain $\mathbb{R}$, Range $[-1, 1]$
   - $\tan, \cot$: Domain has gaps, Range $\mathbb{R}$
4. **Identities:** $\sin^2 + \cos^2 = 1$, angle addition, double angle
5. **Inverse Functions:** Restricted domains to be one-to-one
6. **Transformations:** Amplitude, period, phase shift, vertical shift

Practice Problems

Domain and Range:

1. Find domain and range of $f(x) = \sqrt{x^2 - 4}$
2. Find domain of $g(x) = \ln(3 - x) + \frac{1}{\sqrt{x-1}}$
3. Find range of $h(x) = \frac{2x+1}{x+3}$

Trigonometry:

1. Solve $\cos(2x) = \frac{1}{2}$ for $x \in [0, 2\pi]$
2. Find amplitude, period, and phase shift of $y = -2\cos(3x + \pi) - 1$
3. Simplify $\sin(\arccos(x))$ for $x \in [-1, 1]$
4. Find domain and range of $f(x) = 2\arctan(x - 1) + \pi$

Thank You!

Questions?

Master these fundamentals — they appear everywhere in calculus, optimization, and machine learning!