Introduction to Machine Learning and Data Science

Hayk Aprikyan, Hayk Tarkhanyan

What is Machine Learning?

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- It's about creating models that can generalize patterns from data.

Machine Learning as a School

Metaphor: A School for Machines

- Imagine building a school for machines.
- Our goal: Teach machines to learn and make decisions autonomously.
- Similar to how children come to school, machines learn from data.
- After training, machines leave with knowledge and competences.

Key Components of the "School"

- **Students (Machines):** Our machines are the students of the school.
- **Teachers (Data):** The data serves as the teachers, providing information for the machines to learn.
- **Curriculum (Algorithms):** Algorithms act as the curriculum, guiding the learning process.
- **Graduation (Model Deployment):** Once trained, machines graduate and are ready to apply their knowledge in real-world scenarios.

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- Handling large and diverse datasets.
- Making predictions and decisions in real-time.

Examples of Machine Learning Applications

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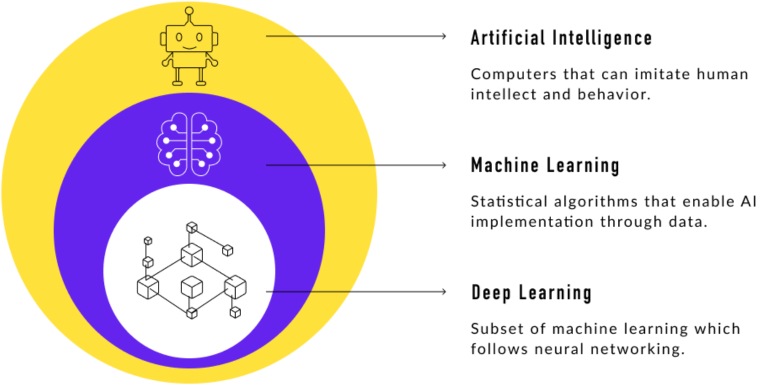
- Medical diagnosis and treatment planning
- & many more...

Difference between AI, ML, and DL



Vectors

Hayk Aprikyan, Hayk Tarkhanyan

- Communication:
    - For questions, memes, announcements, homeworks: Slack
    - For lecture slides and books: Google Drive, GitHub
- Main books (in English):
    - Poole, "Linear Algebra: A Modern Introduction"
    - Johnston, "Introduction to Linear and Matrix Algebra"
    - Stewart, "Calculus"
    - Blitzstein, Hwang, "Introduction to Probability"
    - Grimmett, Welsh, "Probability: An Introduction"
- Supplementary books (in Armenian):
    - Ohanyan, "Probability Theory", Lectures
    - Gevorgyan, Sahakyan, "Algebra and Elements of Mathematical Analysis 11, 12"
    - Musoyan, "Mathematical Analysis", parts 1-2
    - Movsisyan, "Higher Algebra and Number Theory"

Vectors

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20	1
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- Or by taking the two columns of the table: $\begin{bmatrix} 10 \\ 20 \\ 50 \\ 100 \\ 200 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Vectors

Definition

An ordered set of n real numbers is called a **vector** (or **column vector**) in $\mathbb{R}^n$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where $v_1, v_2, \ldots, v_n$ are the **components** of the vector.
A vector written horizontally is called a **row vector**:

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

We will denote $\mathbf{v} \in \mathbb{R}^n$ to indicate that $\mathbf{v}$ is a vector in $\mathbb{R}^n$.

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Vectors in $\mathbb{R}^1$

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We will denote $\mathbf{v} \in \mathbb{R}^n$ to indicate that $\mathbf{v}$ is a vector in $\mathbb{R}^n$.
Vectors in $\mathbb{R}^1$ are real numbers: $[v] \in \mathbb{R}$.

Examples of Vectors in $\mathbb{R}^n$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{(3-dimensional column vector)}$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(4-dimensional column vector)}$$

$$\mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \text{(2-dimensional column vector)}$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(Zero vector in 3-dimensional space)}$$

$$\mathbf{v}_5 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \quad \text{(3-dimensional row vector)}$$

Addition of vectors

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3×100 drams and 1×200 drams?

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We would have the following coins:

$$\mathbf{b} + \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 1+0 \\ 2+0 \\ 0+3 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

Addition of vectors

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To add two vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ in $\mathbb{R}^n$, add their corresponding components:

$$\mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

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Note that we can only add two vectors if they are of the same length!

Multiplication of vector by scalar

What if the money in our pockets doubled?

Multiplication of vector by scalar

What if the money in our pockets doubled? We would have:

$$2 \cdot \mathbf{b} = 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

from each coin.

Multiplication of vector by scalar

Definition

To multiply a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ by a scalar c in $\mathbb{R}^n$, multiply each component of the vector by the scalar:

$$c \cdot \mathbf{v} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_n \end{bmatrix}$$

Properties of Vectors

Associativity and Commutativity of Vector Addition

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the vector addition is commutative and associative:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

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Associativity and Commutativity of Scalar Multiplication

For any scalar c and vectors $\mathbf{v}$ and $\mathbf{u}$ in $\mathbb{R}^n$, scalar multiplication is associative and commutative:

$$c \cdot (\mathbf{v} + \mathbf{u}) = c \cdot \mathbf{v} + c \cdot \mathbf{u}$$

$$(c \cdot d) \cdot \mathbf{v} = c \cdot (d \cdot \mathbf{v})$$

Vector Subtraction

What if we buy something and spend 2 x 50 drams?

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Definition

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, the **negative** of $\mathbf{v}$, denoted as $-\mathbf{v}$, is obtained by negating each component:

$$-\mathbf{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$$

Vector Subtraction

The subtraction of vectors **u** and **v** in $\mathbb{R}^n$ is defined as the sum of **u** and the negative of **v**:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{bmatrix}$$

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Example

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-1 \\ -1-4 \\ 3-0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$$

Vector Transposition

Definition

For a column vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, the **transpose**, denoted as $\mathbf{v}^T$, is a row vector:

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

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For a row vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ in $\mathbb{R}^n$, the **transpose**, denoted as $\mathbf{u}^T$, is a column vector:

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Transpose Properties

- For any vector $\mathbf{v}$ in $\mathbb{R}^n$, $(\mathbf{v}^T)^T = \mathbf{v}$
- For any scalar c, $(c \cdot \mathbf{v})^T = c \cdot \mathbf{v}^T$

Dot Product of Vectors

In our example we had $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ coins of $\mathbf{a} = \begin{bmatrix} 10 \\ 20 \\ 50 \\ 100 \\ 200 \end{bmatrix}$ nominations (values) respectively.

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Definition

The **dot product** of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is:

$$\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

Dot Product of Vectors

Example

If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$, then:

$$\mathbf{u} \cdot \mathbf{v} = (2 \cdot 1) + (-1 \cdot 4) + (3 \cdot 0) = 2 - 4 + 0 = -2$$

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Going back to our example, we can calculate our money with the dot product of **a** and **b**:

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 20 \\ 50 \\ 100 \\ 200 \end{bmatrix} = 2 \cdot 10 + 1 \cdot 20 + 2 \cdot 50 + 0 \cdot 100 + 1 \cdot 200 = 340$$

Dot Product of Vectors

Remark 1

The dot product of two vectors is defined if and only if the vectors have the same number of components (i.e. are of the same length).

Remark 2

The dot product of two vectors is a *number* (scalar), not a vector.

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The dot product of two vectors is a *number* (scalar), not a vector.

This is why the dot product is often called **scalar product**.

Properties of Dot Product

Properties

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. The dot product has the following properties:

1. Commutativity:
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
2. Distributivity over Vector Addition:
$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$
3. Scalar Multiplication:
$$(c \cdot \mathbf{u}) \cdot \mathbf{v} = c \cdot (\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c \cdot \mathbf{v})$$
4. Non-negativity:
$$\mathbf{u} \cdot \mathbf{u} \geq 0 \text{ and } \mathbf{u} \cdot \mathbf{u} = 0 \text{ if and only if } \mathbf{u} = \mathbf{0}$$

Examples

Consider vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

Let's calculate $(5\mathbf{u} - \mathbf{v}) \cdot \mathbf{w}$:

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$$
\begin{aligned}
(5\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} &= \left(5 \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\
&= \left(\begin{bmatrix} 5 \\ -10 \\ 15 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ -14 \\ 16 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = 5 \cdot (-2) + (-14) \cdot 1 + 16 \cdot 2 = 8
\end{aligned}
$$

Geometric interpretation of vectors

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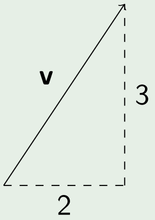
Geometric Interpretation

- In addition to their algebraic representation, vectors have a geometric interpretation.
- We can think of a vector $\mathbf{v}$ as a point in the 2d space,
- Or we can imagine it as an arrow in space, starting from the origin ($O(0,0)$) and pointing to the mentioned point.
- The components of $\mathbf{v}$ are the **coordinates** of the point in the plane.

Geometric interpretation of vectors

Example

- Consider the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
- This vector points to the point (2, 3) in the plane.



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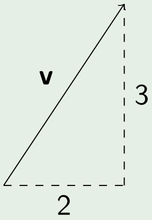
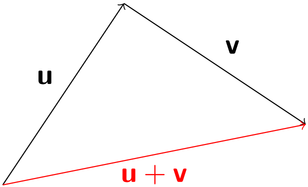


In general, the vector with coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$ is represented by the point with coordinates (x, y).
What do you think happens in the 3d space?

Addition of vectors

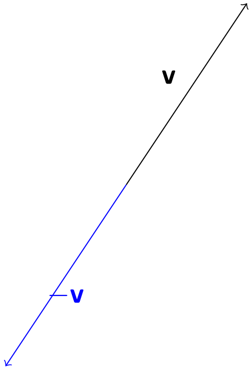
Let's interpret some of our vector operations geometrically.

- **Addition:** To add vectors $\mathbf{u}$ and $\mathbf{v}$, place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector pointing from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$.



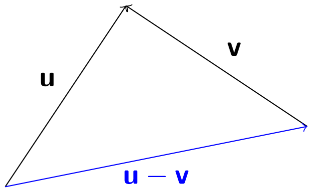
Negative of vectors

- **Negation:** The negative of a vector $\mathbf{v}$, denoted $-\mathbf{v}$, is a vector with the same magnitude but opposite direction.



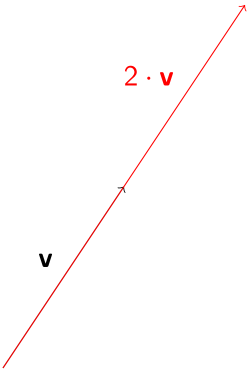
Subtraction of vectors

- **Subtraction:** To subtract $\mathbf{v}$ from $\mathbf{u}$, place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. The result $\mathbf{u} - \mathbf{v}$ is the vector pointing from the head of $\mathbf{v}$ to the head of $\mathbf{u}$.



Multiplication by scalar

- **Scalar Multiplication:** Scaling a vector **v** by a scalar c stretches or compresses the vector. The result $c \cdot \mathbf{v}$ has the same direction as **v** but a different magnitude.



Example

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Let $\mathbf{a} = [3, 2]$ and $\mathbf{b} = [2, 0]$. We want to find $3\mathbf{a} + \mathbf{b}$.

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Vector Operations

- $3\mathbf{a} + \mathbf{b} = 3 \cdot [3, 2] + [2, 0]$
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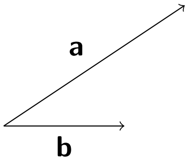
How can we interpret it geometrically?

Example



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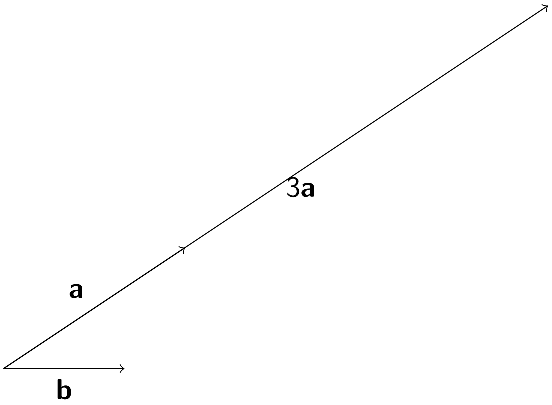

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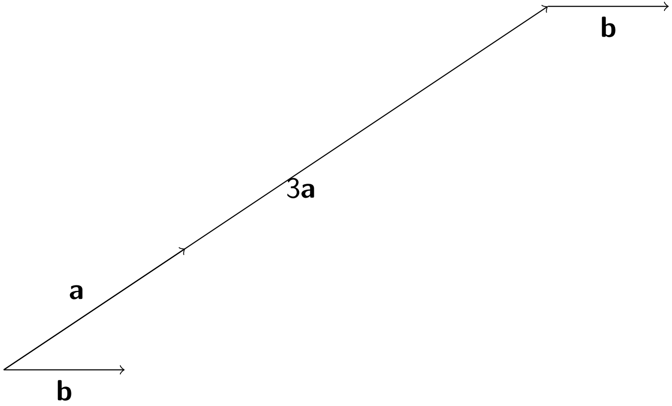

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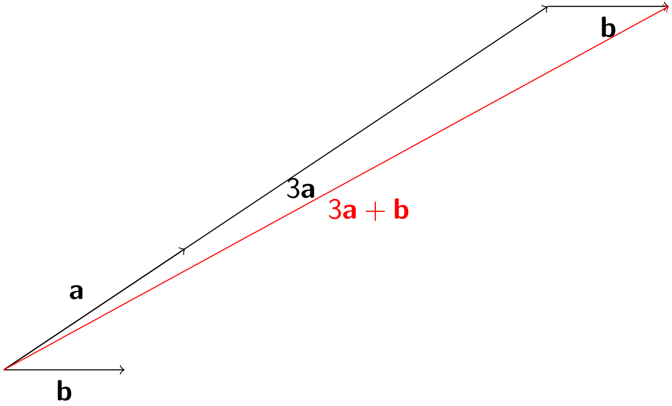
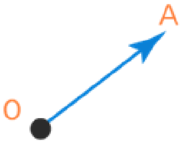


Geometry of Vectors, Matrices

Hayk Aprikyan, Hayk Tarkhanyan

Norm

What if we want to measure the length of some vector?



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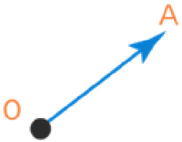

What we can say, is that

the length of the vector

$$=$$

the distance between O and A.

Norm

But how to measure distance?

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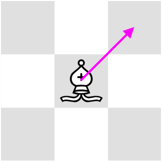
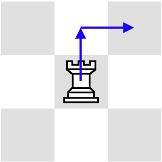


For a bishop, the distance to its upper-right neighbor is 1.



While for a rook, it is 2.

Norm

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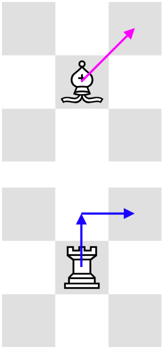

For a bishop, the distance to its upper-right neighbor is 1.

While for a rook, it is 2.

So there are different ways to measure distance and length.

Norm

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, its **Euclidean norm** or **L2 norm** is:

$$\|\mathbf{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

or, equivalently,

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

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Euclidean norm is the standard length we use in classic geometry.
Sometimes we omit the little "2" and just write $\|\mathbf{v}\|$ instead of $\|\mathbf{v}\|_2$.

Norm

For a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, its **Manhattan norm** or **L1 norm** is:

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Example

Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The Manhattan norm of $\mathbf{v}$ is:

$$\|\mathbf{v}\|_1 = |3| + |4| = 7$$

Norm

As we have seen, there are different types of norms (=many different ways to calculate the length of a vector), and one of them is chosen depending on the problem.

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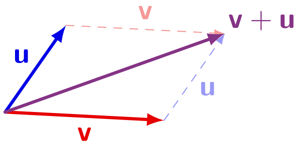
As we have seen, there are different types of norms (=many different ways to calculate the length of a vector), and one of them is chosen depending on the problem.

Notice, however, that independently of which one we take, all norms always satisfy the following three properties:

1. $\|\mathbf{v}\| \geq 0$, and equals 0 if only if $\mathbf{v} = \mathbf{0}$,
2. $\|c\mathbf{v}\| = |c| \cdot \|\mathbf{v}\|$,
3. $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$.



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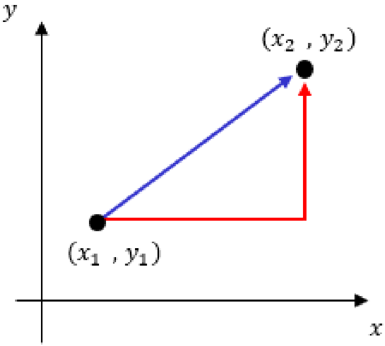

Manhattan Distance L^1

Euclidean Distance L^2

$$L^1 = |x_2 - x_1| + |y_2 - y_1|$$

$$L^2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Angle between vectors

Geometrically, a vector is an arrow in space, that is, it has both a length and direction.

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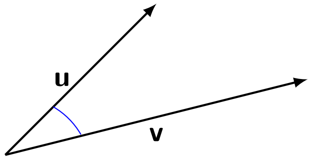

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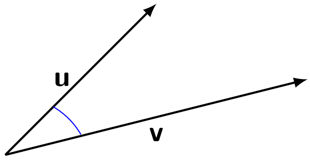

Remember the formula from high school geometry:
$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \alpha$, where α is the angle between **a** and **b**.

Angle between vectors

Definition

The angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is the angle $0 \leq \theta \leq \pi$ for which:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$$

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Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{(3 \cdot 7) + (4 \cdot 1)}{\sqrt{3^2 + 4^2} \cdot \sqrt{7^2 + 1^2}} = \frac{25}{\sqrt{25} \cdot \sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \quad \Rightarrow \quad \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4} = 45^\circ$$

Angle between vectors

Corollary 1

For any vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} \cdot \mathbf{u} = \|\mathbf{v}\| \cdot \|\mathbf{u}\| \cdot \cos\theta,$$

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Corollary 3

Any vector $\mathbf{v} \in \mathbb{R}^n$ forms an angle of 0° with itself and 180° with its negative.

Vector Space

Finally, we are left to notice two things. Take, for example,

- the set $D = \{0, 1, 2, \ldots, 9\}$ of digits, and
- the set $P = \{x \in \mathbb{R} : x > 0\}$ of positive numbers.

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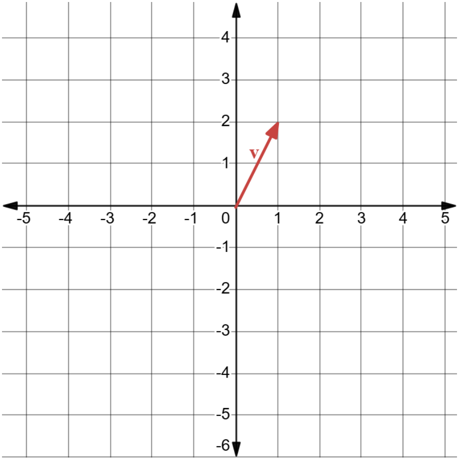
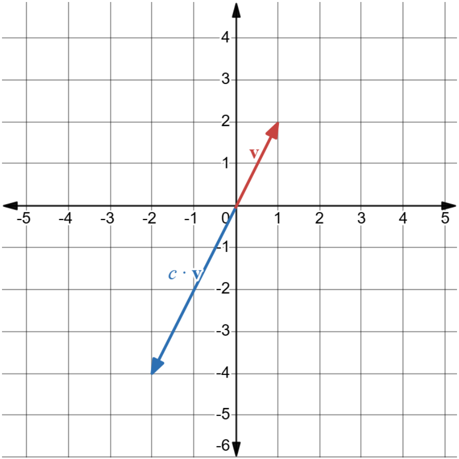
In this case we say that the set of vectors is **closed under addition and scalar multiplication**, while D or P are not (P is closed under addition only).

Furthermore, take the line $y = 2x$ and choose any vector on it:



After multiplying it with any number c, it will still stay on the line $y = 2x$:



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Similarly, if we add two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ which both lie on the line $y = 2x$, their sum would again be on the same line.

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In other words, the line $y = 2x$ is **closed under addition and scalar multiplication**, just like the whole set of vectors $\mathbb{R}^2$. This motivates us to give a special name to the good sets like the line $y = 2x$ and $\mathbb{R}^2$.

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We say that $\mathbb{R}^2$ is a **vector space**, and the set of vectors lying on the line $y = 2x$ are a **vector subspace** of $\mathbb{R}^2$.

Vector Space

Definition

A set V is called a **vector space** if

1. it is closed under addition and scalar multiplication,
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4. There exists a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
5. For every $\mathbf{v} \in V$, there exists a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
6. $(cd) \cdot \mathbf{v} = c \cdot (d \cdot \mathbf{v})$
7. $1 \cdot \mathbf{v} = \mathbf{v}$
8. $c \cdot (\mathbf{u} + \mathbf{v}) = c \cdot \mathbf{u} + c \cdot \mathbf{v}$
9. $(c + d) \cdot \mathbf{v} = c \cdot \mathbf{v} + d \cdot \mathbf{v}$

No need to memorize the properties–just the natural laws of addition and scalar multiplication.

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Theorem

Assume V is a vector space, and U is a subset of V. Then U is a subspace of V if and only if

1. $\mathbf{x} + \mathbf{y} \in U$, for all $\mathbf{x}, \mathbf{y} \in U$,
2. $c\mathbf{x} \in U$, for all $\mathbf{x} \in U$ and $c \in \mathbb{R}$.

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2. $c\mathbf{x} \in U$, for all $\mathbf{x} \in U$ and $c \in \mathbb{R}$.

- So $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, ... are all vector spaces.
- The set of all vectors that lie on the same line (e.g. $y = kx$) form a subspace (on the condition that the line also contains the $\mathbf{0}$ vector).

Matrices

Definition

An $m \times n$ tuple A of elements a_{ij} ($i = 1, \ldots, m$ and $j = 1, \ldots, n$), is called a real-valued (m, n) **matrix**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R}.$$

The set of all real-valued (m, n) matrices is denoted by $\mathbb{R}^{m \times n}$.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3} \qquad B = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

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Note that the first number in (m, n) always shows rows, second: columns.

Matrix Addition

The vectors are practically 1-column matrices: $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. Similar to vectors, we define the following operations with the matrices:

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Definition

The sum of two matrices A and B, denoted as $A + B$, is obtained by adding corresponding elements. If A is of size $m \times n$ and B is of the same size, then $A + B$ is also of size $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

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Remark

Matrix addition is only defined for matrices of the same size.

Scalar Multiplication of a Matrix

Definition

The product of a scalar c and a matrix A, denoted as cA, is obtained by multiplying each element of the matrix by the scalar.

$$c \cdot A = c \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} & \dots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \dots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \dots & c \cdot a_{mn} \end{bmatrix}$$

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Scalar multiplication can be performed for any scalar c and any matrix A.

Negative of a Matrix

Definition

The negative of a matrix A, denoted as $-A$, is obtained by changing the sign of each element in the matrix.

$$-A = -\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} -a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & -a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{m1} & -a_{m2} & \dots & -a_{mn} \end{bmatrix}$$

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Remark

The negative of a matrix equals (-1) times the matrix.

Matrix Subtraction

Definition

The difference of two matrices A and B, denoted as $A - B$, is obtained by subtracting corresponding elements, or by adding A and $-B$. If A and B are both of size $m \times n$, then $A - B$ is also of size $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots & a_{2n} - b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \dots & a_{mn} - b_{mn} \end{bmatrix}$$

Matrix Subtraction

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Matrix subtraction is only defined for matrices of the same size.

Zero Matrix

Definition

The **zero matrix**, denoted as O or $O_{m \times n}$, is a matrix where all elements are zero.

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$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Remark

$A + O = O + A = A$ for any matrix A.

Transpose of a Matrix

Definition

The **transpose** of a matrix A, denoted as A^T, is obtained by swapping its rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

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$$A = \begin{bmatrix} 7 & 4 & 2 \\ 0 & 1 & -3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 2 & -3 \end{bmatrix}$$

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Remark

The transpose of an (m, n) matrix is an (n, m) matrix.

Matrices

Matrices can be added together and multiplied by numbers, and these operations share the same "good" properties (e.g. $A + B = B + A$) with vectors.

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In that sense, it is not difficult to prove that:

Theorem

For each $m, n \in \mathbb{N}$ the set of real-valued matrices $\mathbb{R}^{m \times n}$ forms a vector space.

Matrix-Vector Multiplication

Definition

Let A be an $m \times n$ matrix and $\mathbf{v}$ be a column vector of size $n \times 1$. The product $A\mathbf{v}$ is a column vector of size $m \times 1$ obtained by multiplying each row of A by the corresponding element of $\mathbf{v}$ and summing the results.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{bmatrix}$$

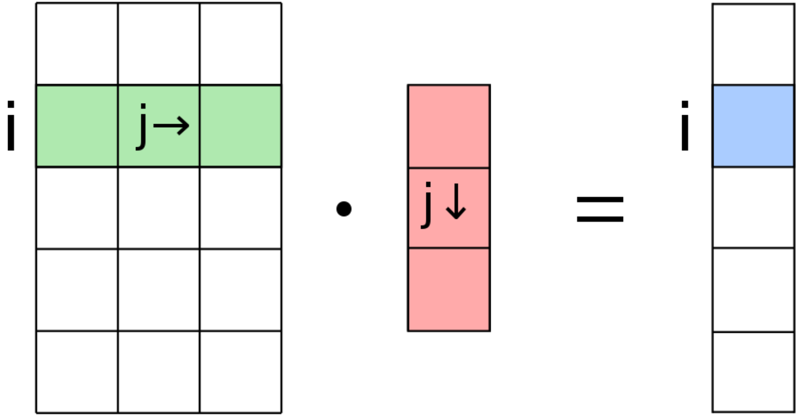
Matrix-Vector Multiplication

Or, in other words, if we denote the rows of A by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$, the product $A\mathbf{v}$ will be a column vector of size $m \times 1$ obtained by taking the dot product of each row of A with the vector $\mathbf{v}$:

$$A = \begin{bmatrix} \ldots & \mathbf{A}_1 & \ldots \\ \ldots & \mathbf{A}_2 & \ldots \\ & \vdots & \\ \ldots & \mathbf{A}_m & \ldots \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}v_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{v} \\ \mathbf{A}_2 \cdot \mathbf{v} \\ \vdots \\ \mathbf{A}_m \cdot \mathbf{v} \end{bmatrix}$$

Matrix-Vector Multiplication



Matrix-Vector Multiplication

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 3 \\ 4 \cdot 2 + 5 \cdot (-1) + 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}$$

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$$A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} (-2) \cdot 4 + 1 \cdot 2 \\ 0 \cdot 4 + 3 \cdot 2 \\ 1 \cdot 4 + (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix}$$

Matrix-Vector Multiplication

Matrix-vector multiplication shares properties with scalar multiplication and addition of vectors.

- **Distributive Property:**
  For a matrix A and vectors $\mathbf{v}$ and $\mathbf{w}$ of appropriate sizes:

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$$

- **Scalar Multiplication:**
  For a matrix A and a scalar c:

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Note that we can only multiply a matrix by a vector if the number of columns of the matrix equals the length of the vector.

Geometric Interpretation

Why do we define the matrix-vector multiplication this way? Turns out, it has a beautiful geometrical interpretation.

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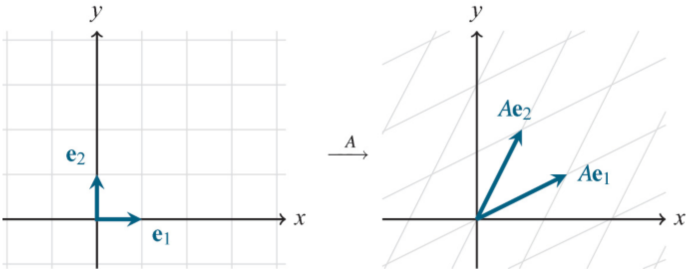
Think this way: when you multiply, say, a 2×2 matrix A by a vector $\mathbf{v} \in \mathbb{R}^2$, what you get is another vector $\mathbf{u} = A\mathbf{v} \in \mathbb{R}^2$. We call this $\mathbf{u}$ the **transformed version** of $\mathbf{v}$ (and we say that A is a linear transformation).



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As we will see later, the resulting "transformed version" $\mathbf{u}$ is just the same old $\mathbf{v}$ except it is **rotated** and **scaled** to become longer or shorter (and possibly, flipped).

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In this sense, all matrices are either just rotating vectors by some degree, or flipping them horizontally/vertically, or scale them, or do all three.

The key thing is: whatever a matrix "does" to one vector, it does the same to all other vectors too (when being multiplied with them).

Check different matrices yourself:
- `visualize-it.github.io/linear_transformations/simulation.html`
- `www.shad.io/MatVis`

We will learn more about this later–now back to matrices∼

Matrix Multiplication

Definition

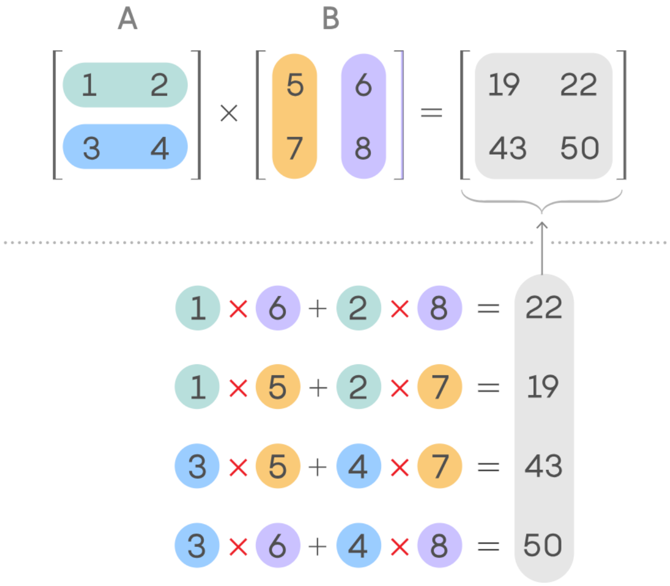
Let A be an $m \times n$ matrix, and let B be an $n \times k$ matrix. The product $C = AB$ is an $m \times k$ matrix, where each element c_{ij} is obtained by taking the dot product of the i-th row of A and the j-th column of B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix}$$

$$\text{where } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{p=1}^{n} a_{ip}b_{pj}$$

Matrix Multiplication



Matrix Multiplication

Matrix multiplication shares properties with scalar multiplication and addition of vectors, as well as matrix-vector multiplication.

- **Distributive Property:**
  For matrices A, B, and C of appropriate sizes:
  $$A(B + C) = AB + AC \quad \text{and} \quad (A + B)C = AC + BC$$
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Note that we can only multiply two matrices if the number of columns of the first matrix equals the number of rows of the second matrix: $(m \times n)$ with $(n \times k)$.

Matrix Multiplication

Example

Let

$$C = \begin{bmatrix} -1 & 0 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2} \qquad D = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & 7 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

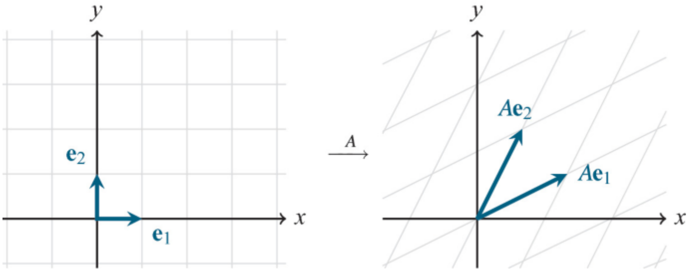
$$CD = \begin{bmatrix} -1\cdot 5 + 0\cdot 3 & -1\cdot(-2) + 0\cdot 0 & -1\cdot 1 + 0\cdot 7 \\ 2\cdot 5 + (-3)\cdot 3 & 2\cdot(-2) + (-3)\cdot 0 & 2\cdot 1 + (-3)\cdot 7 \\ 4\cdot 5 + 1\cdot 3 & 4\cdot(-2) + 1\cdot 0 & 4\cdot 1 + 1\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 & -1 \\ 1 & -4 & -19 \\ 23 & -8 & 11 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Inverse, Determinant

Hayk Aprikyan, Hayk Tarkhanyan

Recap:
When you multiply, say, a 2×2 matrix A by a vector $\mathbf{v} \in \mathbb{R}^2$, what you get is another vector $\mathbf{u} = A\mathbf{v} \in \mathbb{R}^2$. We call this $\mathbf{u}$ the **transformed version** of $\mathbf{v}$ (and we say that A is a linear transformation).



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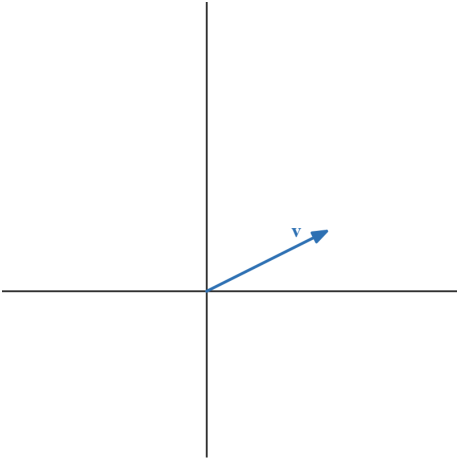
Now that we know what matrix $\times$ vector multiplication means, what about matrix $\times$ matrix multiplication? Why is it defined the way it is?

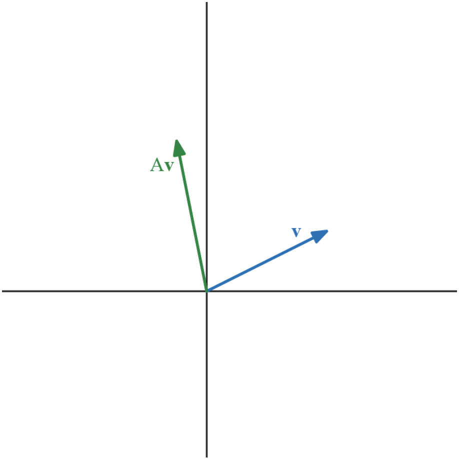
Geometric Interpretation

Suppose $\mathbf{v} \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 2}$:



Geometric Interpretation

If we apply A on $\mathbf{v}$, we get a transformed version of $\mathbf{v}$,



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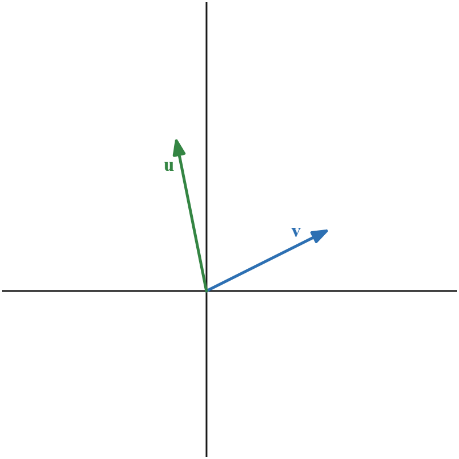
If we apply A on $\mathbf{v}$, we get a transformed version of $\mathbf{v}$, say $\mathbf{u}$:



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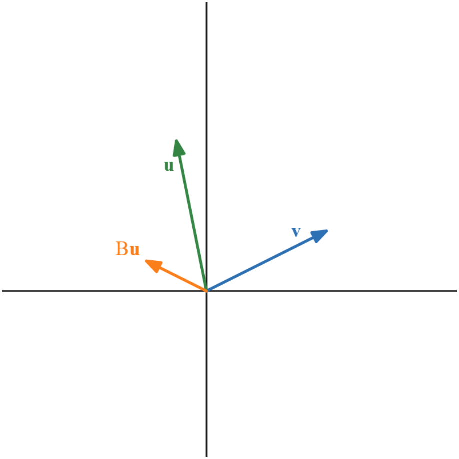
Now applying B on $\mathbf{u}$, we get a transformed version of $\mathbf{u}$, i.e. $B\mathbf{u}$



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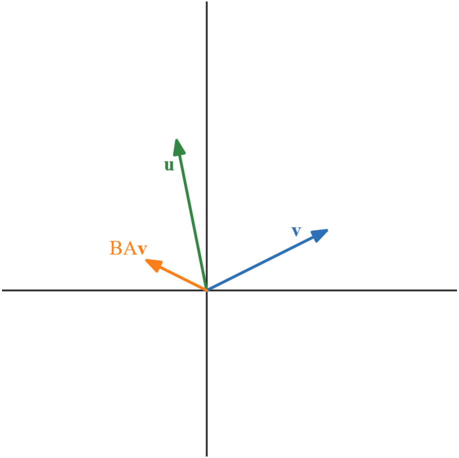
Now applying B on $\mathbf{u}$, we get a transformed version of $\mathbf{u}$, i.e. $B\mathbf{u} = BA\mathbf{v}$



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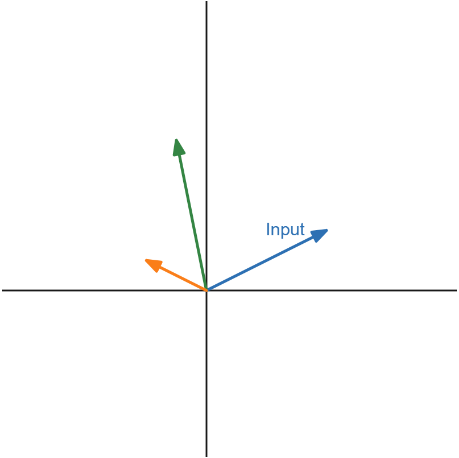
So what is the product BA? To get $(BA)(\mathbf{v})$, we do:



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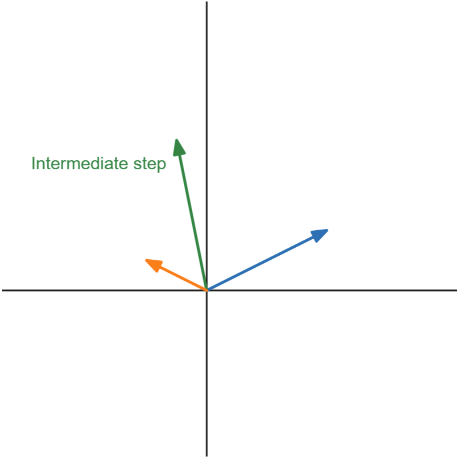

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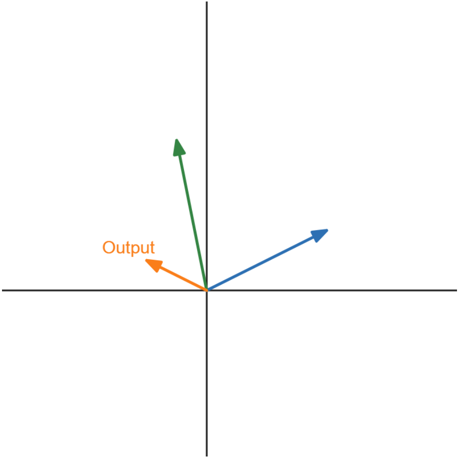

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Hence BA is the linear transformation that we get by **first applying** A, and **then applying** B!

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Suppose A is the matrix that rotates the vectors by 30°, B the one that rotates by 50°, and C by 260°.

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What would the product matrix BA be?

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Suppose A is the matrix that rotates the vectors by 30°, B the one that rotates by 50°, and C by 260°.
What would the product matrix BA be? What about CBA?

Which matrix leaves everything in its place (does not touch anything)?

Identity Matrix

Definition

A matrix is said to be **square** if it has the same number of rows and columns. In other words, an $n \times n$ matrix is a square matrix.

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Example

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 4 \\ 1 & 4 & 6 \end{bmatrix}$$

This matrix is both symmetric and (of course) square.

Identity Matrix

Definition

The **main diagonal** (or just the **diagonal**) of a matrix A are the terms a_{ii} for which the row and column indices are the same (a_{11}, a_{22}, $\ldots$), so from the upper left element to the lower right.

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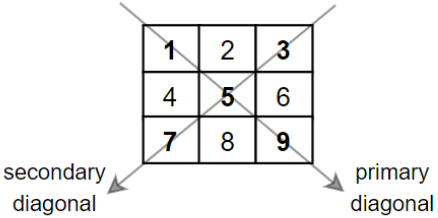

Identity Matrix

For example, here the main diagonal is marked with red:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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The secondary diagonal is marked with red:

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Identity Matrix

Definition

The **identity matrix** of $\mathbb{R}^{n \times n}$, denoted as I_n, is the square matrix with ones on the main diagonal and zeros elsewhere.

Applying the identity matrix on vectors does not change them.

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Therefore, we can say:

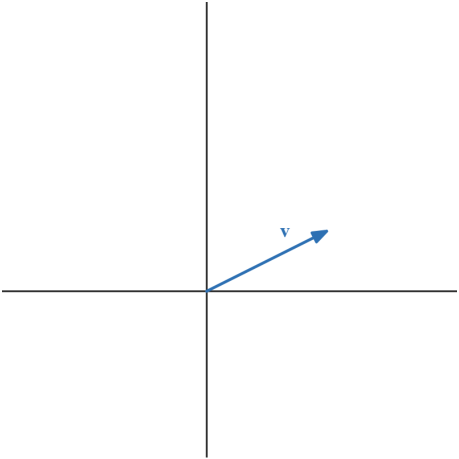
Property

For any matrix $A \in \mathbb{R}^{m\times n}$,

$$I_m A = A I_n = A$$

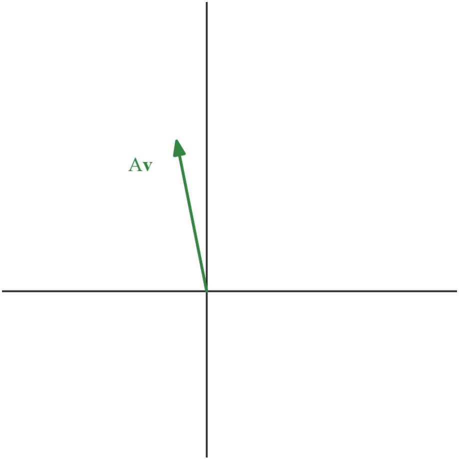
Inverse Matrix

Finally, what if we have a vector in $\mathbb{R}^n$,



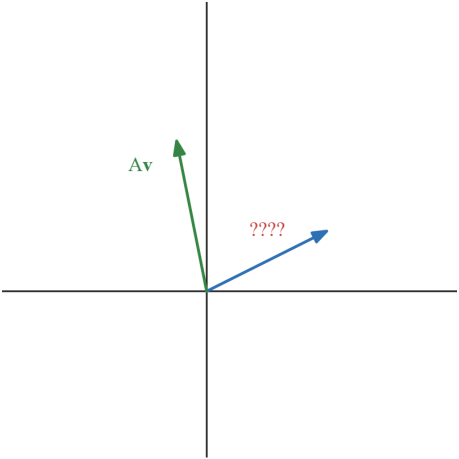
Inverse Matrix

Finally, what if we have a vector in $\mathbb{R}^n$, and we accidentally transform it?



Inverse Matrix

How to get back to the original vector?



Inverse Matrix

In other words, in terms of what we learned about matrix multiplication,

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Question

Assume the matrix $A \in \mathbb{R}^{n \times n}$ does the following when applied on a vector:

1. scales the vector up 2 times in the horizontal direction,
2. then rotates it by 30° clockwise,
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Given $\mathbf{v} = A\mathbf{u}$, could we recover the original $\mathbf{u}$?

The answer is yes, i.e. the matrix A has an inverse. As we will see soon, only some square matrices actually have an inverse.

Trace

Definition

The **trace** of a square matrix A, denoted as $\text{tr}(A)$, is the sum of the elements on its main diagonal.

$$\text{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}$$

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Example

If

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & 4 \\ 7 & 2 & 6 \end{bmatrix}$$

then

$$\mathrm{tr}(A) = 2 + (-3) + 6 = 5$$

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Note that only square matrices have a trace.

Trace

Trace Properties

For any matrices A and B, and any scalar c, the trace of a matrix satisfies the following properties:

- $\mathrm{tr}(cA) = c \cdot \mathrm{tr}(A)$
- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

Determinant of a 2×2 Matrix

Determinant Formula

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is given by

$$\det(A) = ad - bc$$

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Example

For the matrix

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$$

the determinant is $\det(A) = (2)(4) - (5)(-3) = 8 + 15 = 23$.

Determinant of a 3×3 Matrix

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For a 3×3 matrix

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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$$\det(C) = aei + bfg + cdh - ceg - bdi - afh$$

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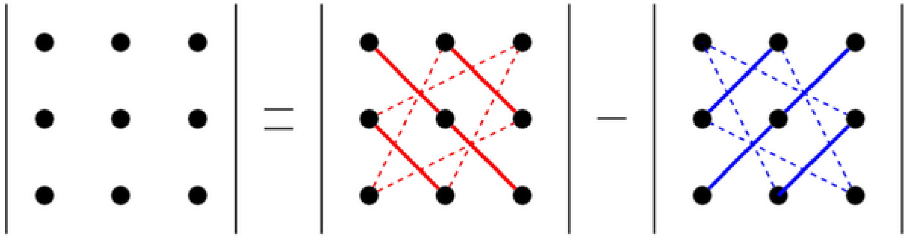
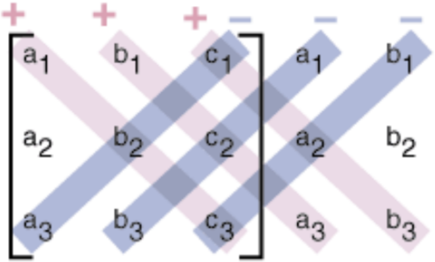
Forget that formula–remember the algorithm!

Determinant of a 3 × 3 Matrix



Determinant of a 3×3 Matrix

Alternatively,



$$\det A = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$

Determinant of a 3×3 Matrix

Example

For the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(C) = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 2 \cdot 4 \cdot 9 - 1 \cdot 6 \cdot 8 = 0$$

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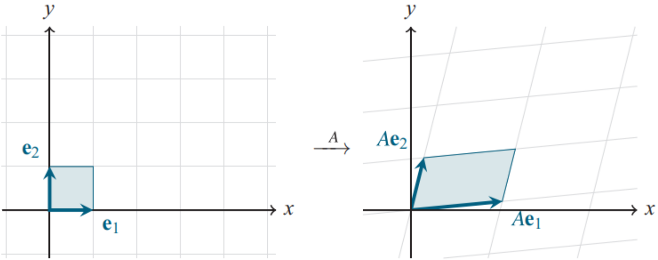
But what does the determinant show, and how do we need it?

Determinant

If we take, for example, the so-called "unit square" formed by the vectors $\mathbf{e}_1 = [1 \;\; 0]$ and $\mathbf{e}_2 = [0 \;\; 1]$, we can see that their transformed versions, $A\mathbf{e}_1$ and $A\mathbf{e}_2$, form a parallelogram:



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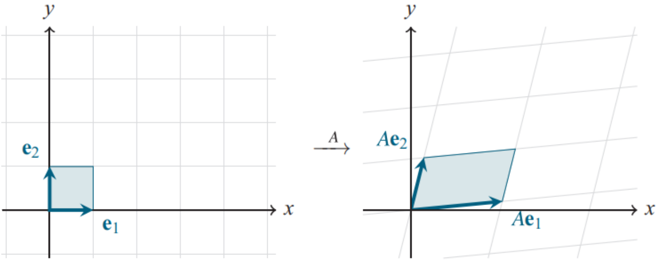
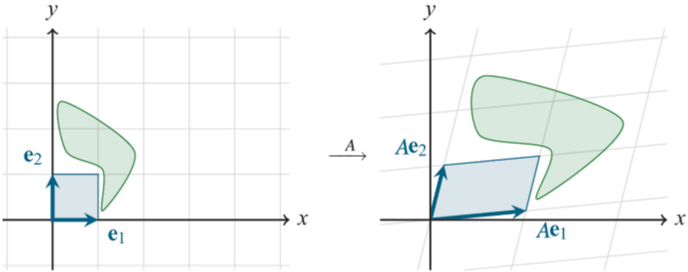


Then $\det(A)$ is the area of that parallelogram.

More generally, after we apply the transformation A (play that animation in your head), the area of *any shape* gets scaled by the factor of $\det(A)$:



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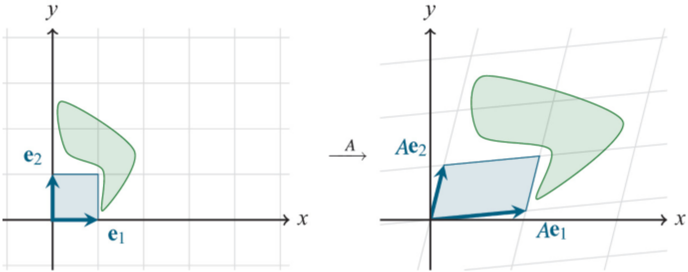

So the determinant shows how much the matrix scales up everything in average. Note that it is defined **only** for square matrices.

Determinant

Determinant Properties

Let $A, B \in \mathbb{R}^{n \times n}$ be square matrices of the same size, and let $c \in \mathbb{R}$ be any scalar. Then:

- $\det(cA) = c^n \cdot \det(A)$ *(where n is the size of the matrix)*
- $\det(AB) = \det(A) \cdot \det(B)$ *(multiplicativity)*
- $\det(I) = 1$
- If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(A^T) = \det(A)$ *(invariance under transpose)*
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It would be an exercise of huge importance to attempt proving these properties (except the last three) by playing the matrices in your head.

Determinant

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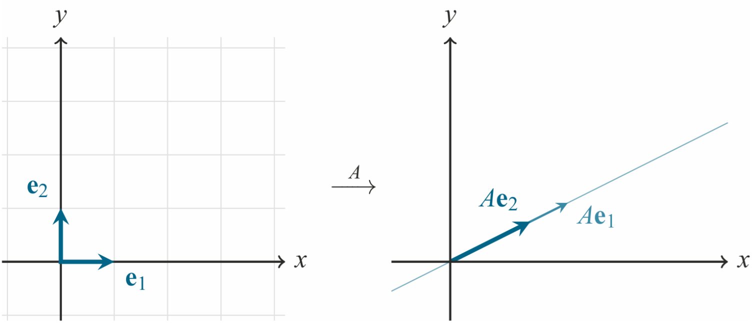

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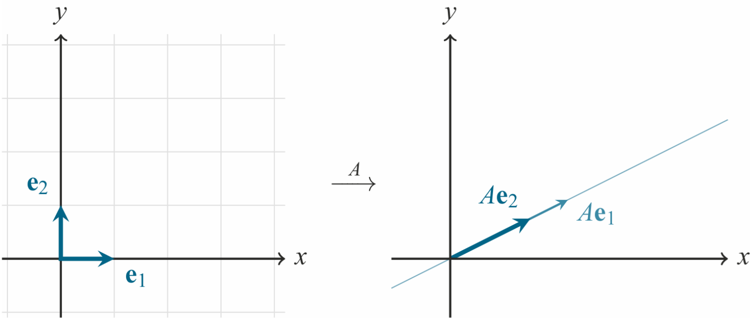

Theorem

A square matrix A has an inverse if and only if its determinant is not zero.

Inverse Matrix

Formula for 2x2

For a 2×2 invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse A^{-1} can be calculated using the formula:

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Example

Given $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ with $\det A = (2 \times 4) - (3 \times 1) = 5$, we can calculate the inverse as follows:

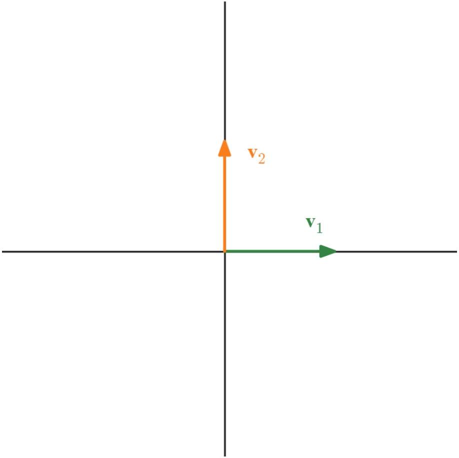
$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Basis, Eigenvalues and Eigenvectors

Hayk Aprikyan, Hayk Tarkhanyan

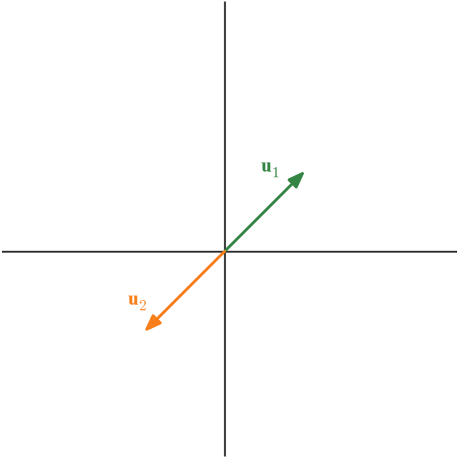
Motivation

When talking about vectors/matrices, why do we focus on these vectors?



Motivation

And not on these:



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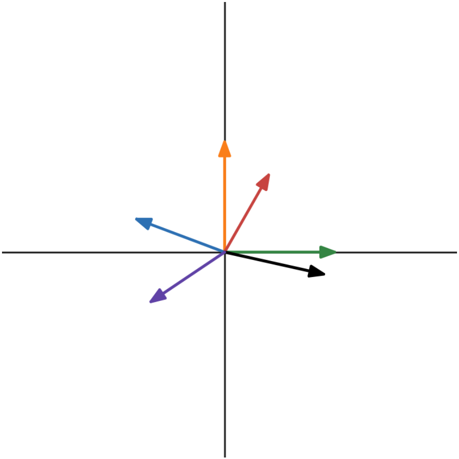

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We call expressions like these:

$$\text{something} \cdot \mathbf{v}_1 + \text{something} \cdot \mathbf{v}_2$$

the **linear combinations** of $\mathbf{v}_1$ and $\mathbf{v}_2$.

In our case, the vector $[4 \;\; 7]$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Span

More generally,

Definition

For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and for any scalars $c_1, c_2, \ldots, c_k$, the expression

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

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So in this sense, all vectors of $\mathbb{R}^2$ can be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$! In this case, we say that $\mathbb{R}^2$ is the **span** of $\mathbf{v}_1$ and $\mathbf{v}_2$:

Definition

The set of all possible linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called their **span**, i.e.

$$\text{span}\{\mathbf{v}_1,\mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}.$$

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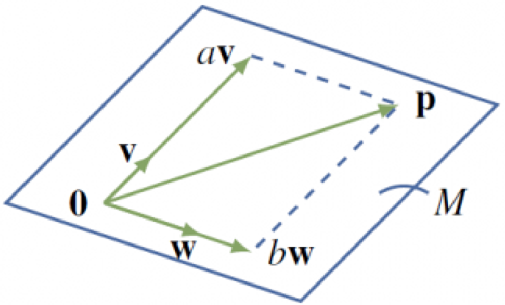
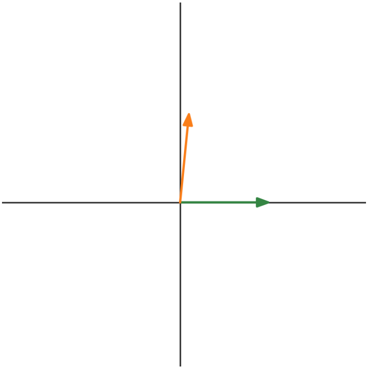


Span

Question

What is the span of vectors $[1 \quad 0]$ and $[0.1 \quad 1]$?



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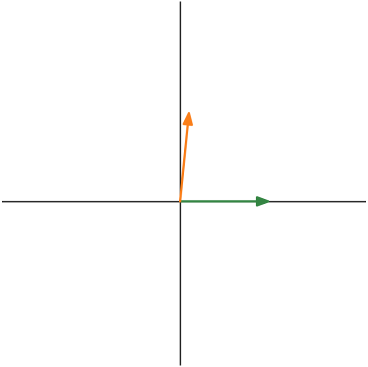
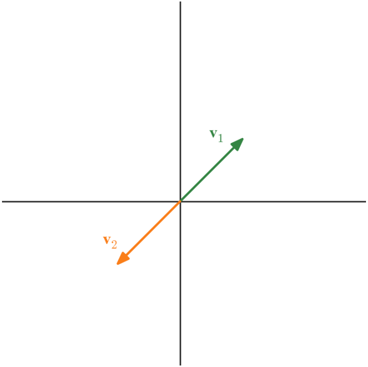


Again, it is the whole $\mathbb{R}^2$: We can express any vector using these two.

Span

Question

What is the span of vectors $[1 \quad 1]$ and $[-1 \quad -1]$?



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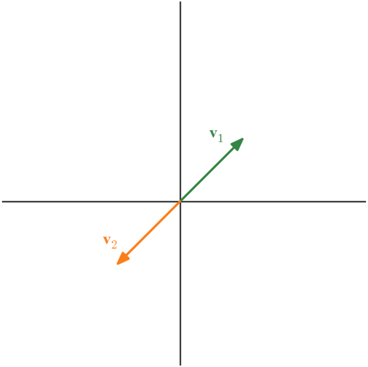
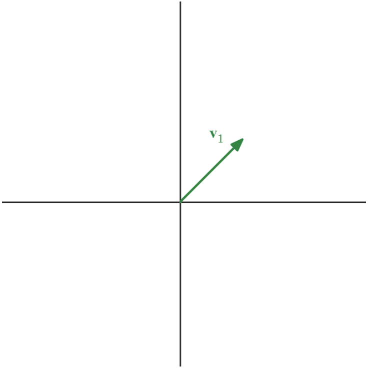


Since they both lie on the line $y = x$, their span is the line $y = x$ itself.

Span

Question

What about the span of the single vector $[1 \quad 1]$?



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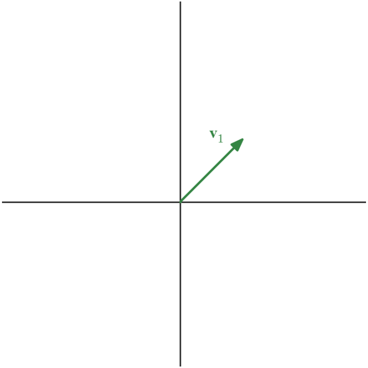

Again, the span of $[1 \quad 1]$ is the line $y = x$.

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But what is the reason that

- in one case (e.g. $\mathbf{v}_1 = [1 \ \ 0]$ and $\mathbf{v}_2 = [0 \ \ 1]$) the span is the whole $\mathbb{R}^2$,
- but in another case (e.g. $\mathbf{u}_1 = [1 \ \ 1]$ and $\mathbf{u}_2 = [-1 \ \ -1]$) the span is only a line?

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Because in the second case, one of the vectors can be expressed by another!

Linear Independence

Indeed, you can express $\mathbf{u}_2$ with $\mathbf{u}_1$:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \cdot \mathbf{u}_1,$$

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In this case, we say that

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Definition

The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are called **linearly independent** if none of them can be written as a linear combination of the others.

Indeed, you can express $\mathbf{u}_2$ with $\mathbf{u}_1$:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1) \cdot \mathbf{u}_1,$$

but you **cannot express** $\mathbf{v}_2$ with $\mathbf{v}_1$ (or vice versa).

In this case, we say that

- the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are **linearly dependent,**
- while the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent**.

More generally,

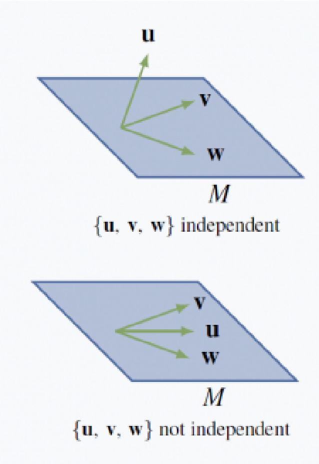
Definition

The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are called **linearly independent** if none of them can be written as a linear combination of the others.

And we say that they are linearly dependent if one of them, say $\mathbf{v}_n$, can be written as

$$\mathbf{v}_n = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

Linear Independence



Check these animations:

- www.desmos.com/calculator/9rnbn0ycdd
- www.desmos.com/calculator/aje8cboe0j

Linear Independence

Geometrically,

- Two vectors are linearly dependent if they lie on the same line,
- Three vectors are linearly dependent if they lie on the same plane,

and so on.

Linear Independence

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and so on.

There is also another characterization of linear independence (try to prove it by yourself):

Theorem

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if and only if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

is *only* true if $c_1 = c_2 = \ldots = c_n = 0$.

(i.e. if you plug in any numbers other than 0, the sum will not be $\mathbf{0}$).

Basis

So now we can say that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are linearly independent and their span is $\mathbb{R}^2$.

We call the pairs of vectors like $\mathbf{v}_1$ and $\mathbf{v}_2$ the **basis** of the space $\mathbb{R}^2$.

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called a **basis** of the vector space V if:

1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent,
2. V is equal to the span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

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2. V is equal to the span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

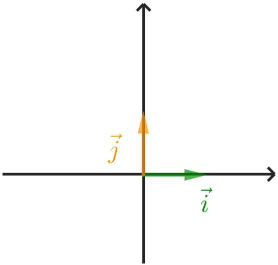
In other words, there are no "irrelevant", "redundant" vectors, and any vector of V can be expressed with $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
(In fact, such representation is always unique, i.e. there is only one way to express $[3 \;\; 4]$ with $\mathbf{v}_1 = [1 \;\; 0]$ and $\mathbf{v}_2 = [0 \;\; 1]$: $\quad 3 \cdot \mathbf{v}_1 + 4 \cdot \mathbf{v}_2$)

Example

The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^2$ and are called the **standard basis**. They are often denoted $\hat{i}, \hat{j}$.



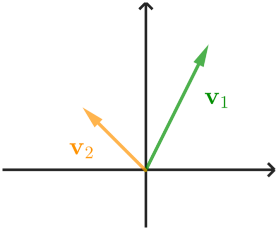
Basis

Example

The linearly independent vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^2$ as these vectors are linearly independent and their span is $\mathbb{R}^2$.

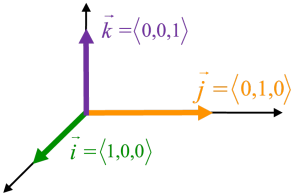


Example

The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$ and are called the **standard basis**. They are often denoted $\hat{i}, \hat{j}, \hat{k}$.



Basis

Example (too many vectors)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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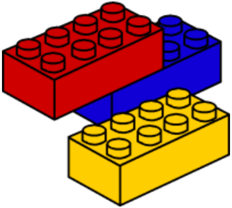
The same is true for any vector space.

The basis is our Lego set of "building blocks" out of which we build our castle (i.e. the vector space). Since they all have the same number of vectors, we call that number the **dimension** of the space.



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The dimension describes how **big** our vector space is. It shows how many linearly independent vectors are there in that vector space at most.

Geometric Interpretation of Matrices (last time)

Getting back to our matrices, our main question remains:

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How can we tell which transformation a matrix represents by just looking at it?

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Turns out, the answer is hidden in the concept of basis.

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$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

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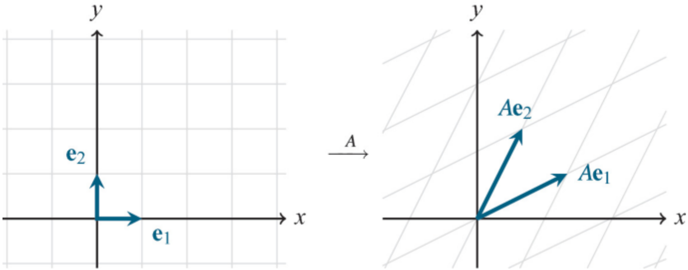

As we see, applying a matrix **transforms the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$** into the **columns of the matrix**:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \text{1st column of } A$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \text{2nd column of } A$$

Geometric Interpretation of Matrices (last time)

Therefore any vector with coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$ is transformed into

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

i.e. the linear transformation sends our basis vectors to its columns, which become a new basis for our space.

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More precisely, if the columns of A are linearly independent, then they form a basis.

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The number of linearly independent columns of the matrix A is called the **rank** of A.

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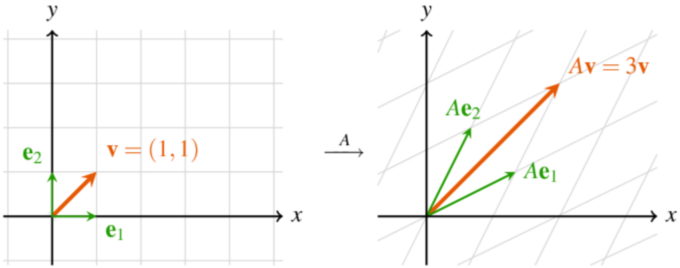
Finally, peace.

Geometric Interpretation of Matrices (last time)

Sometimes the matrix also has some vectors which **do not change their direction** when being multiplied (transformed) by that matrix, rather they get scaled by some number:



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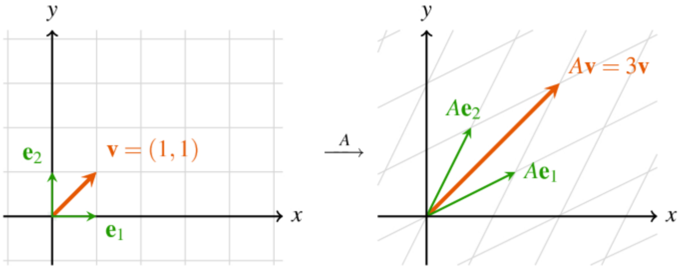

Vectors like these are of special interest to us, and we call them *eigenvectors*.

Definition

If for some number λ and some non-zero vector $\mathbf{v}$

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say

- λ is an **eigenvalue** of A,
- $\mathbf{v}$ is an **eigenvector** of A corresponding to the eigenvalue λ.

Eigenvalues and Eigenvectors

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Example

For the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$,

$$A\mathbf{v} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4\mathbf{v}$$

so $\lambda = 4$ is an eigenvalue of A with eigenvector $\mathbf{v}$.

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so $\lambda = 4$ is an eigenvalue of A with eigenvector $\mathbf{v}$. What about $-3\mathbf{v}$?

Eigenvalues and Eigenvectors

Remark

If $\mathbf{v}$ is an eigenvector of A, then for any scalar $c \neq 0$, $c\mathbf{v}$ is also an eigenvector for A.

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Definition

The set of all eigenvalues of A is called the **spectrum** of A.

How can we find the eigenvalues and eigenvectors of a given matrix?

Eigenvalues and Eigenvectors

Suppose $A \in \mathbb{R}^{n \times n}$ is a matrix, and we want to find $x \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ such that:

$$
\begin{aligned}
A\mathbf{v} = x\mathbf{v} = I(x\mathbf{v}) &= xI\mathbf{v} \\
A\mathbf{v} - xI\mathbf{v} &= \mathbf{0} \\
(A - xI)\mathbf{v} &= \mathbf{0}
\end{aligned}
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The polynomial above is called the **characteristic polynomial** of A. Its roots are the eigenvalues of A.

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$$p_A(x) = \det(A - xI) = (3-x)(-1-x) - 5 = x^2 - 2x - 8$$

Eigenvalues and Eigenvectors

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Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$.

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Example

Hence, the roots of $p_A(x)$ are $\lambda = 4$ and $\lambda = -2$, so these are the eigenvalues of A. The spectrum of A is $\{4, -2\}$. For $\lambda = 4$,

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Setting $v_2 = a$ and $v_1 = 5a$ for any scalar $a \in \mathbb{R}$, we will get the solution. There are infinite solutions which are all multiplies of each other:

$$\mathbf{v} = a \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors

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Similarly we get $E_{-2} = \{a \cdot [-1 \quad 1]^T \mid \text{for any } a \in \mathbb{R}\}$.

Eigenvalues and Eigenvectors

Again, what we are concerned with, is the *concept* and not the computation.

Eigenvalues and Eigenvectors

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As a bonus, we have a surprising theorem:

Theorem

The determinant of a matrix is equal to the product of its eigenvalues:

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

and the trace of a matrix is equal to the sum of its eigenvalues:

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

Systems of Linear Equations

Let us consider one last application of the matrices.

Question

Imagine scrolling Facebook, when you suddenly see the following problem: You have 2 types of fruits, apples and oranges. You buy 2 apples and 3 oranges for a total cost of 11 dollars. Additionally, you buy 1 apple and 4 oranges for a total cost of 7 dollars.
Only people with 140 IQ can find the prices of apples and oranges.

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Only people with 140 IQ can find the prices of apples and oranges.

Let x be the cost of one apple and y be the cost of one orange. The problem can be represented as a 2×2 system of linear equations:

$$\begin{cases} 2x + 3y & = 11 \\ x + 4y & = 7 \end{cases}$$

Solving this system will give us the prices of apples (x) and oranges (y).

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A **system of linear equations** is a collection of two or more linear equations (all with the same variables).

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A system of m linear equations with n variables can be written as:

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\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \dots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m
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Definition

A **particular solution** to the system is a set of values for the variables $(x_1, x_2, \dots, x_n)$ that satisfies all equations simultaneously. The collection of all particular solutions is called the **general solution**.

Systems of Linear Equations

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$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

So Facebook is just asking: On which vector should you apply this matrix to get $[11 \quad 7]$?

Systems of Linear Equations

Let's consider three systems of linear equations:

a)

$$\begin{cases} 2x + 3y & = 7 \\ 4x - y & = 5 \end{cases}$$

b)

$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 14 \end{cases}$$

c)

$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 15 \end{cases}$$

a)

$$\begin{cases} 2x + 3y & = 7 \\ 4x - y & = 5 \end{cases}$$

$$2x + 3y = 7 \Rightarrow 2x = 7 - 3y \Rightarrow x = \frac{7 - 3y}{2}$$

Systems of Linear Equations

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$$\begin{cases} 2x + 3y &= 7 \\ 4x - y &= 5 \end{cases}$$

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$$4\left(\frac{7 - 3y}{2}\right) - y = 5 \Rightarrow 7 - 3y - y = 5 \Rightarrow -4y = -2 \Rightarrow y = \frac{1}{2}$$

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$$x = \frac{7 - 3(\frac{1}{2})}{2} = 2$$

Solution: $x = 2, \quad y = \frac{1}{2}$

Systems of Linear Equations

b)

$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 14 \end{cases}$$

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$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 14 \end{cases}$$

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Infinite solutions: $x = \frac{7-3y}{2}, \quad \text{for any } y \in \mathbb{R}$

Systems of Linear Equations

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$$\begin{cases} 2x + 3y & = 7 \\ 4x + 6y & = 15 \end{cases}$$

Multiplying the first equation by 2 gives:

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which contradicts the second equation.

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$$\begin{cases} 2x + 3y &= 7 \\ 4x + 6y &= 15 \end{cases}$$

Multiplying the first equation by 2 gives:

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which contradicts the second equation.
No solution.

Systems of Linear Equations

So as we saw, in general a system of linear equations can have a *unique solution*, *no solution*, or *infinitely many solutions*.

Definition

A system of linear equations is **consistent** if it has at least one solution. A system is *inconsistent* if it has no solutions.

Systems of Linear Equations

Consider the system of three linear equations:

$$
\begin{cases}
2x + y - z &= 5 \\
-3x - 2y + 2z &= -8 \\
x + 4y - 3z &= 1
\end{cases}
$$

Systems of Linear Equations

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$$\begin{cases} 2x + y - z & = 5 \\ -3x - 2y + 2z & = -8 \\ x + 4y - 3z & = 1 \end{cases}$$

We can write it in the form:

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -2 & 2 \\ 1 & 4 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix}$$

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Theorem (very fundamental)

The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b} \in \mathbb{R}^n$, if and only if $\det A \neq 0$ (i.e. A is invertible).

Limit, Derivative, Extrema of a Function

Hayk Aprikyan, Hayk Tarkhanyan

Motivation

Suppose you run a supermarket and your profits from the sales of apples vary like this:

$$f(x) = 20\sqrt{x} - 3x^3$$

where x is the price of apples in dollars.

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$$f(x,y,z,t) = 3xy^2 - y\log t - (1-y)\log(1-t) + \frac{z^3}{t}$$

with real-time values $x = 4$, $y = 0.4$, $z = 0.8$, $t = 55$, and you should decide whether to increase or decrease each of x, y, z, t (and how much).

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with real-time values $x = 4$, $y = 0.4$, $z = 0.8$, $t = 55$, and you should decide whether to increase or decrease each of x, y, z, t (and how much).
In machine learning you often have $1.000.000+$ such parameters.

Limit of a Sequence

To begin our journey, let's start with the definition of a sequence of numbers.

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Example

- $1, 2, 3, 4, 5, \ldots$
- $1, -1, 1, -1, 1, \ldots$
- $0, 0.2, 0.4, 0.6, 0.8, \ldots$
- $6, 6, 6, 6, 6, \ldots$

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We usually fix a letter, say a, and denote the first term by a_1, the second term by a_2, and so on. In general, for the n^{th} term we write a_n, and to denote the whole sequence we use $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

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Sometimes it also comes in handy to give the formula of the general n^{th} term, e.g. $a_n = n^2$ or $\{a_n\} = \{n^2\}$, which means:

$$a_1 = 1, \qquad a_2 = 4, \qquad a_3 = 9, \qquad \ldots$$

Limit of a Sequence

There are many interesting examples of sequences. Take, for example, the sequence $a_n = \frac{1}{n}$:

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Question

Does the sequence become equal to 0 at some point?

Interestingly, it does not: The numbers come arbitrarily close to 0 but they never actually become 0. This shows that the sequence may or may not eventually equal to its limit.

Limit of a Sequence

Definition

We say that $\{a_n\}$ **converges** to the number L (or that the number L is its **limit**), denoted as

$$\lim_{n\to\infty} a_n = L \qquad (\text{or } a_n \to L)$$

if as n becomes large enough, all terms a_n become arbitrarily close to L.

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and then whatever number you say (e.g. "not further than 0.002"), we can point out some number N (say, $N = 1000$) such that after the N^{th} term, all others are close to L by 0.002, i.e.

$$|a_N - L| < 0.002, \quad |a_{N+1} - L| < 0.002, \quad |a_{N+2} - L| < 0.002, \quad \ldots$$

Limit of a Sequence

So more technically, $\lim_{n \to \infty} a_n = L$ means that

- for any positive number $\varepsilon > 0$
- there exists N large enough
- such that $|a_n - L| < \varepsilon$ for all $n = N, \; N+1, \; N+2, \; \ldots$

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Definition

If $\{a_n\}$ has a *finite* limit, we say that it is **convergent**, otherwise it is **divergent**.

Limit of a Sequence

Example

In the previous examples,

- $1, \ 2, \ 3, \ 4, \ 5, \ \ldots$

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- $c^n \to +\infty$ if $c > 1$, but $c^n \to 0$ if $|c| < 1$
- If a sequence consists of the same number (or if it becomes constant starting from some point), the limit is that number.

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More examples (we will not go further into details):

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Consider the sequence $\{\frac{1}{n}\}$. We claim that $\lim\limits_{n \to \infty} \frac{1}{n} = 0$.

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Example

Consider the sequence $\{0.3n\}$. $\lim\limits_{n\to\infty} 0.3n = \infty$ (it is divergent).

Limit of a Sequence

Properties

1. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

$$\lim_{n\to\infty} (a_n + b_n) = L + M$$

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Limit of a Function

Now that we have the notion of

$$\lim_{n\to\infty}\left(2+\frac{1}{n}\right)^3$$

what do you think the expression

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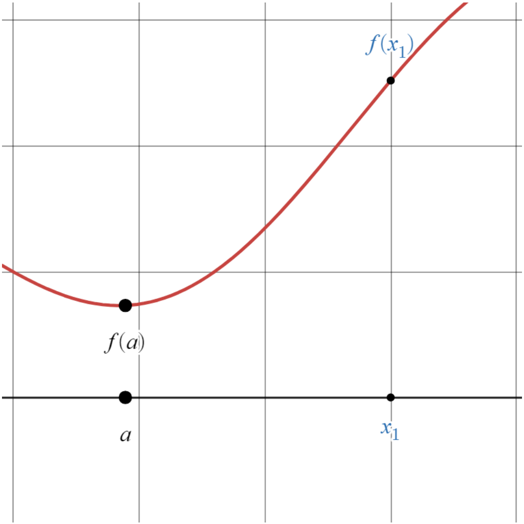
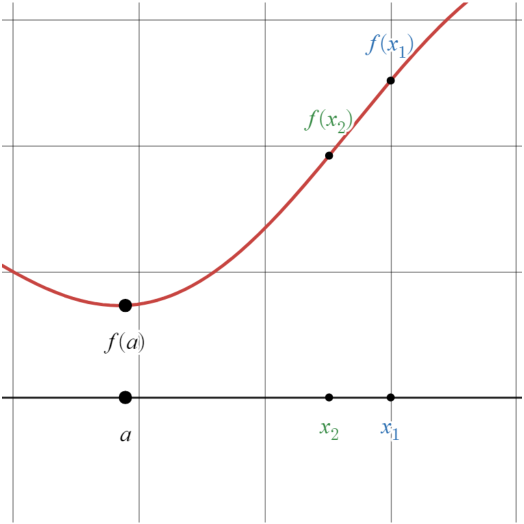
How do we do that? We can say: take any sequence x_n that converges to a, calculate the values of $f(x)$ at $x_1, x_2, \ldots,$ and see what happens.

Limit of a Function



Limit of a Function



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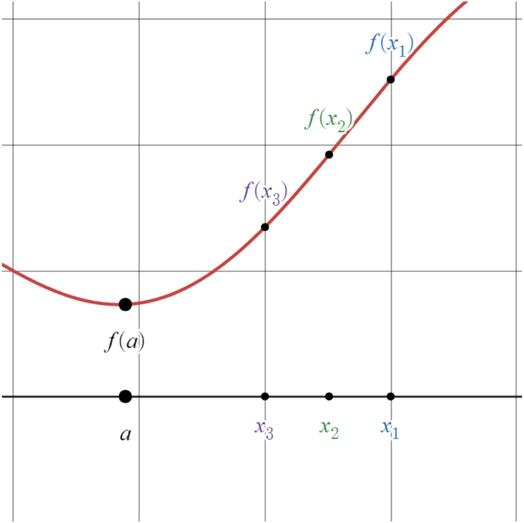

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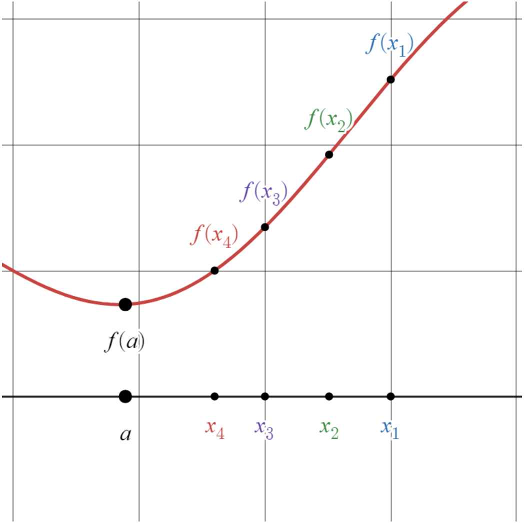

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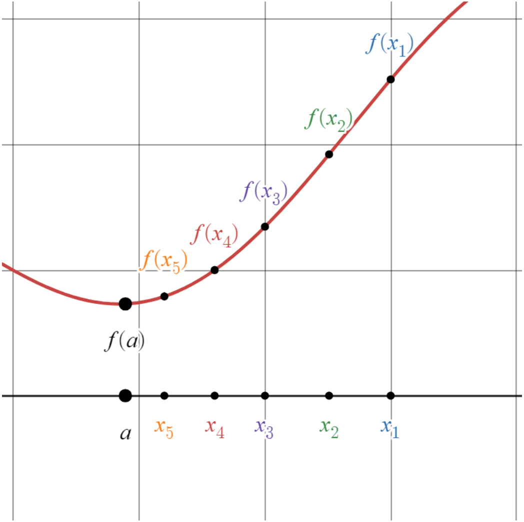
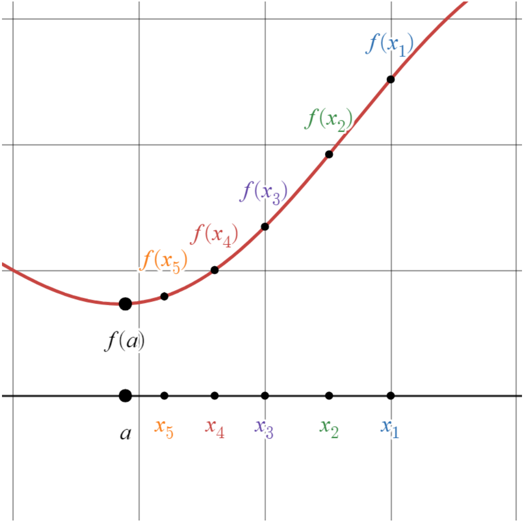
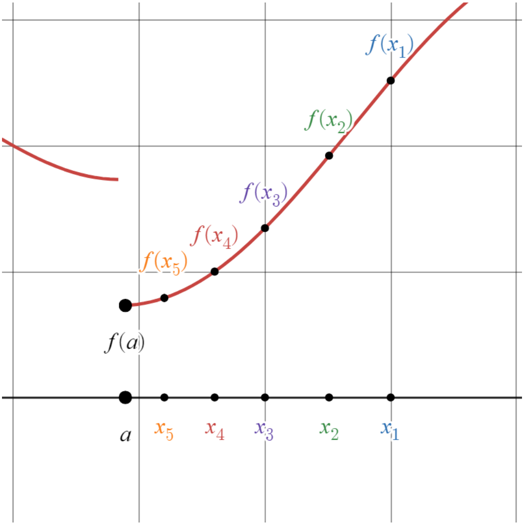




If numbers $f(x_1)$, $f(x_2)$, $\ldots$ approach some limit L, then $\lim_{x \to a} f(x) = L$

Limit of a Function



It may happen that if $x_n \to a$ from the other side, we get another "limit".

Limit of a Function

In the second case, we say that the limit does not exist and the function is **discontinuous** at that point.

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If for **all** sequences $x_n \to a$ (no matter from left or right), the sequence

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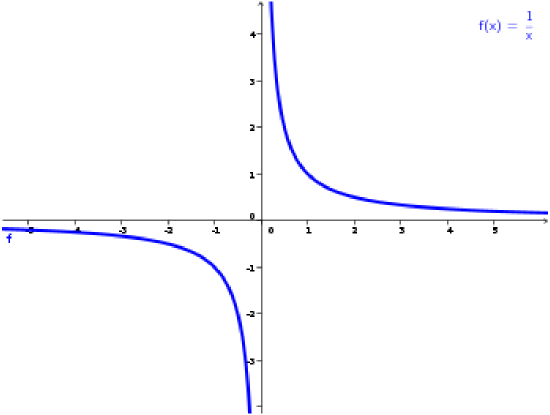
In other words, if the value of $f(x)$ always approaches L, as its input approaches a.

Limit of a Function



Example

$$\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$$

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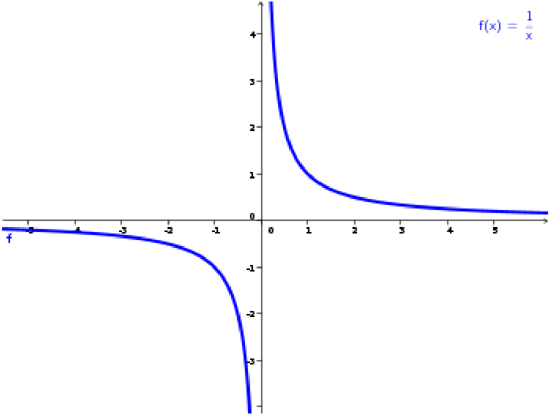

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$$\lim_{x\to 3}\frac{1}{x}=\frac{1}{3} \qquad \text{but} \qquad \lim_{x\to 0}\frac{1}{x}$$

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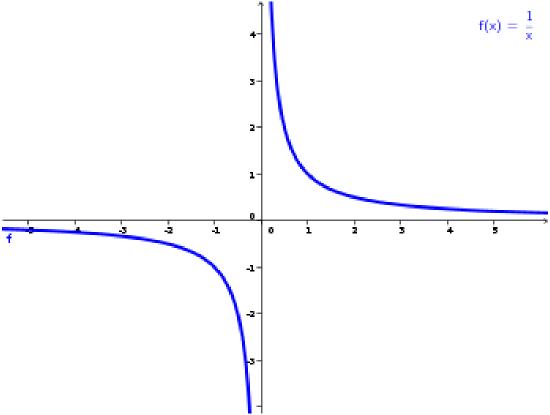

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Notice how the graph of $f(x) = \dfrac{1}{x}$ looks like it consists of two separate graphs put together at $x = 0$.

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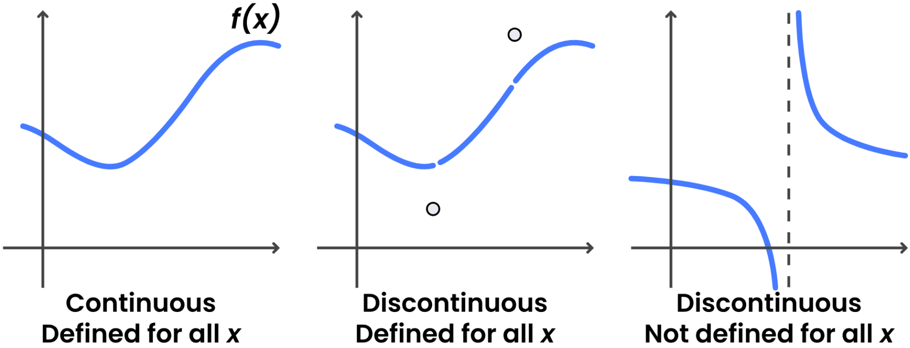
If a function is continuous at all points, it is called a **continuous function**.

Continuity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In fact, most "good" functions are continuous (in their domains!):

- Polynomials (e.g. x^2+7x-1, $xy-y^4+z$)
- Root functions (e.g. $\sqrt{x}$, $\sqrt[5]{x}$)
- Exponential and logarithmic functions (e.g. 2^x, e^{3x}, $\ln x$)
- Trigonometric functions and their inverses (e.g. $\cos(3x)$, $\arcsin x$)

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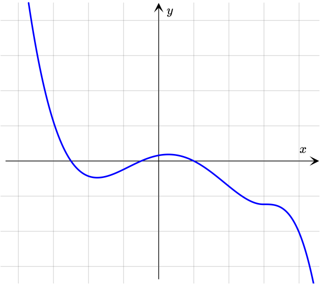

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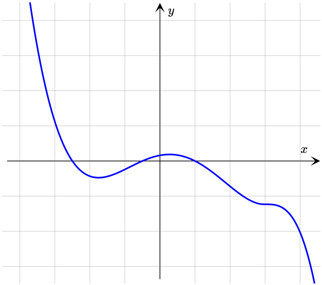

Now assume we want to maximize or minimize this function.
Notice how at some points it changes "faster" than at the others.
How can we measure that?

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Definition

We say $f(x)$ is **differentiable at a point** a, if the following limit

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Note that $f'(x)$ is a **function** itself and not a fixed number!

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Consider the function $f(x) = x^2$. The derivative of $f(x)$ is:

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If a function is differentiable at some point, then it is also continuous, but the reverse is not always true.

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The function $f(x) = |x|$ is continuous but it is not differentiable at point $x = 0$.

Derivative

Similarly, we can compute the derivative of $f'(x)$ itself (it will show the speed of the speed of $f(x)$, i.e. its *acceleration*).

We denote the derivative of $f'(x)$ by $f''(x)$, that of $f''(x)$ by $f'''(x)$, and so on. The derivative taken of $f(x)$ n times is also denoted by $f^{(n)}(x)$.

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Derivatives tell about amazingly many interesting properties of the function.

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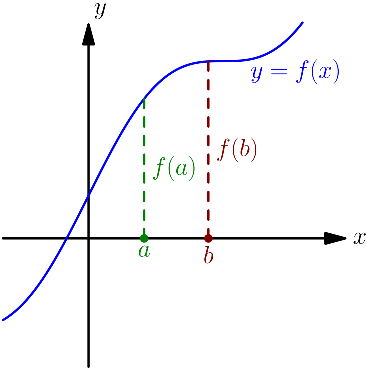

We say that a function $f(x)$ is **increasing** at the point a if

$$f(c) < f(a) < f(b) \qquad \text{when} \qquad c < a < b$$

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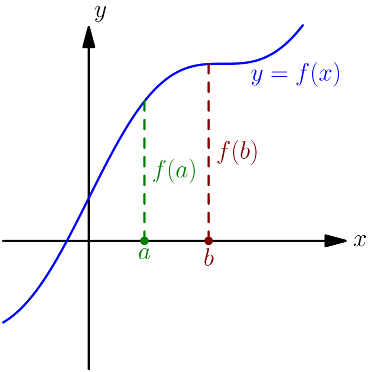

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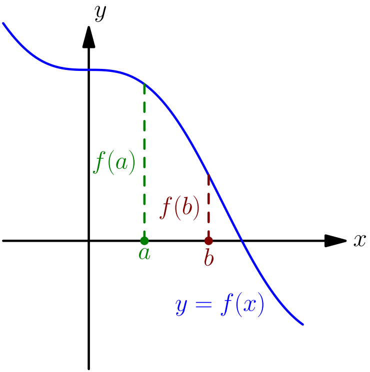

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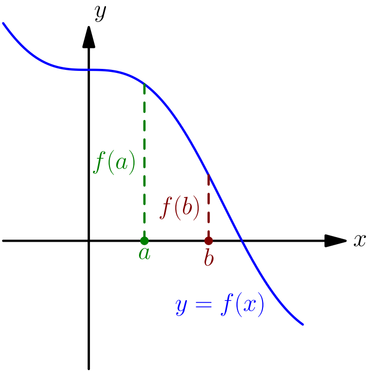

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Question

What if $f'(a) = 0$?

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Finally, what about the largest/smallest values of the function?

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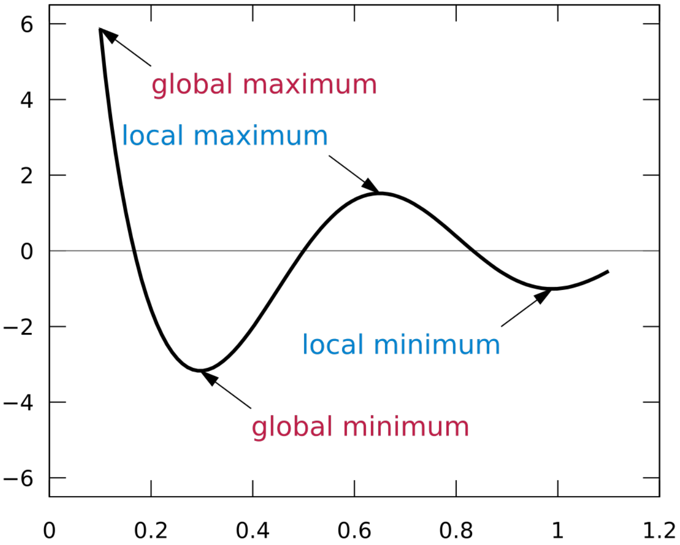
Theorem

Every continuous function f has both a global maximum and a global minimum on any **closed** interval $[a, b]$.

Extrema of a Function



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Hence, the condition $f'(x) = 0$ is necessary but *not sufficient*.

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How can we tell if a critical point is a local minimum/maximum point?

Extrema of a Function

Theorem 1 (f'' at one point)

If $f'(x_0) = 0$ and there exists finite $f''(x_0)$, then

1. If $f''(x_0) > 0$, then x_0 is a local minimum point,
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Theorem 2 (f' at multiple points)

If for some $\delta > 0$, f is differentiable in the intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$ and continuous at x_0, then

1. If $f'(x) > 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) < 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a local maximum point.
2. If $f'(x) < 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) > 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a local minimum point.
3. If $f'(x)$ doesn't change its sign, then x_0 is not an extremum point.

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Step 3: a) If there exists finite $f''(x_0) \neq 0$, use Theorem 1.
b) If you find the sign of $f'(x)$ on left and right "sides" of x_0, use Theorem 2.

Taylor Series, Integral

Hayk Aprikyan, Hayk Tarkhanyan

Convex and Concave Functions

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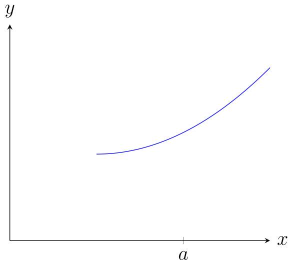

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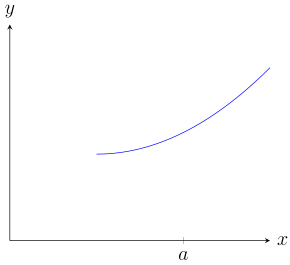

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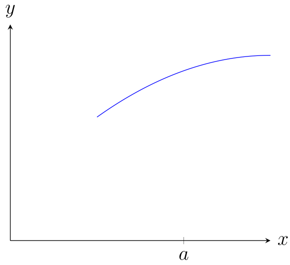

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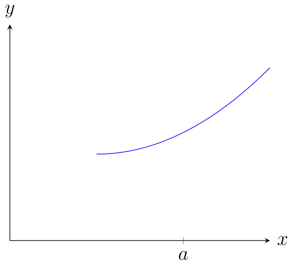

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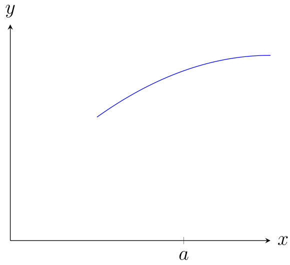
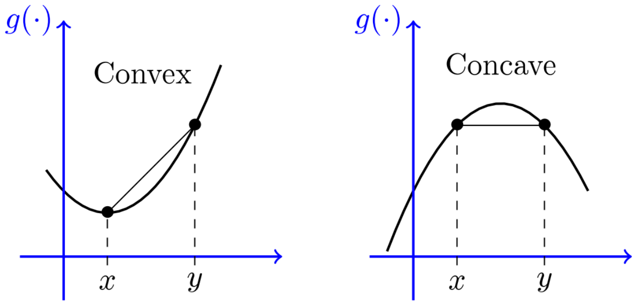


How can you determine which way it is?

Convex and Concave Functions



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We say that $f(x)$ is **convex** on some interval if for any two points on its graph, the line connecting them always lies **above** the graph of $f(x)$.

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More technically, a function is

- convex if

$$f(\alpha p + (1 - \alpha)q) \leq \alpha f(p) + (1 - \alpha)f(q)$$

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We will not prove this

Convex and Concave Functions

More technically, a function is

- convex if
$$f(\alpha p + (1-\alpha)q) \leq \alpha f(p) + (1-\alpha) f(q)$$
- concave if
$$f(\alpha p + (1-\alpha)q) \geq \alpha f(p) + (1-\alpha) f(q)$$

for any two points p and q, and for any number $0 \leq \alpha \leq 1$.

We will not prove this but open the link and play with α to get a feeling of what this means:

- www.desmos.com/calculator/ujoh0mh59d

Convex and Concave Functions

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Convex and Concave Functions

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Again, derivative is all you need!

Theorem

If $f(x)$ is twice-differentiable (i.e. there exists $f''(x)$), then:

- f is convex if and only if $f''(x) \geq 0$
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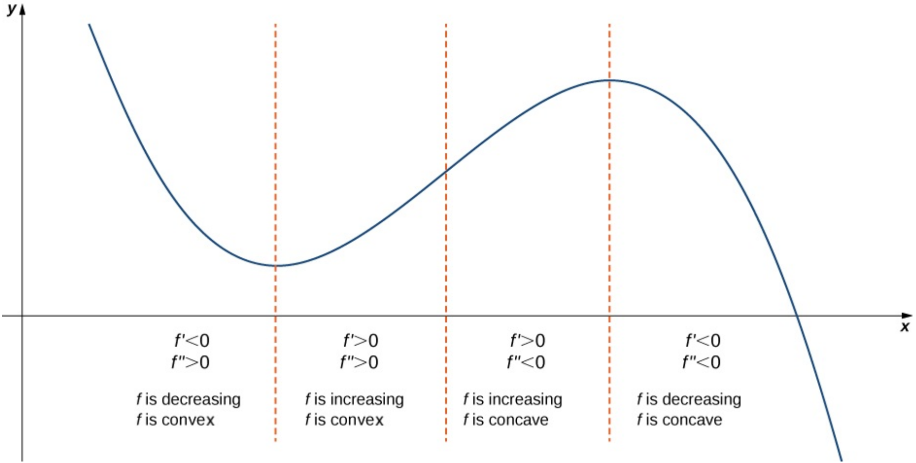
Examples

1. $f(x) = x^2$ is convex on $(-\infty, \infty)$.
   $f''(x) = 2 > 0$
2. $f(x) = -x^2$ is concave on $(-\infty, \infty)$.
   $f''(x) = -2 < 0$
3. $f(x) = x$ is both convex and concave on $(-\infty, \infty)$.
   $f''(x) = 0$

Convex and Concave Functions



Taylor Series

As we saw, the derivatives of a function tell so much about them. Can they be used to approximate the function?

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- ...

Taylor Series

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Suppose we have a function $f(x)$. Let us construct another function $g(x)$ which, at some point a has (or, equivalently):

- $g(a) = f(a)$
- $g'(a) = f'(a)$
- $g''(a) = f''(a)$
- $\ldots$

Taylor Series

For example, suppose $f(x) = e^x$. Then $f(0) = 1$ and $f'(0) = 1$.

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If there exist derivatives f', f'', $\dots$, $f^{(k)}$, then the polynomial

$$P_k(x) = f(a)$$

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is called the **Taylor polynomial** of order k of the function $f(x)$ about the point a.

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If we take $k = \infty$, we will get an infinite sum which is called the **Taylor series** of $f(x)$ about a.

Taylor Series

Taylor's Theorem

If a function $f(x)$ is k times differentiable at the point a, then

$$P_k(x) \to f(x), \qquad k \to \infty$$

around some (maybe small) interval around a.

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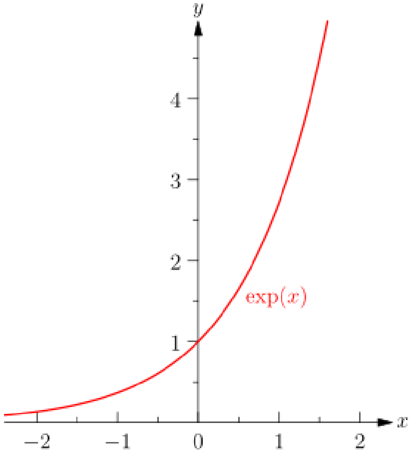
Example

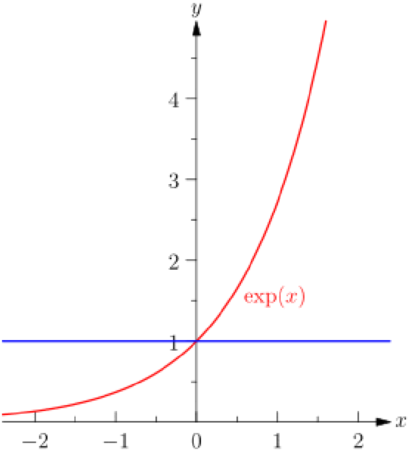
- The Taylor series for any polynomial is the polynomial itself.
- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$
- $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$
- $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

Taylor Series



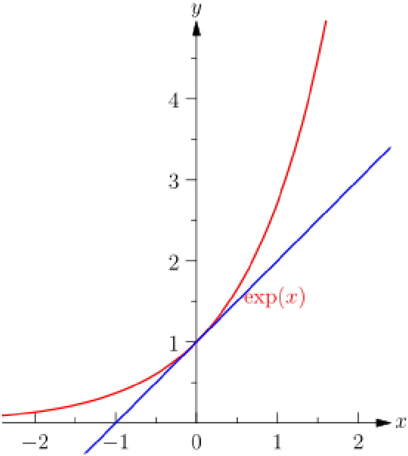
Taylor Series

1



Taylor Series

$1 + x$



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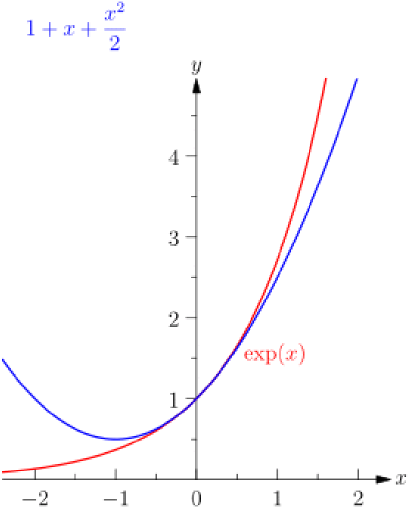

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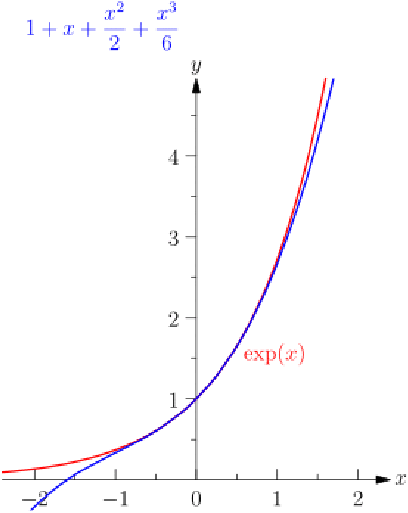

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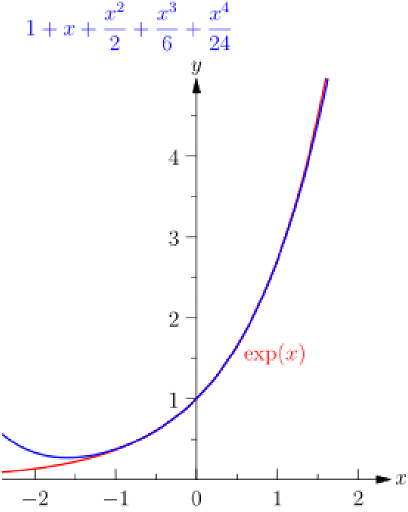
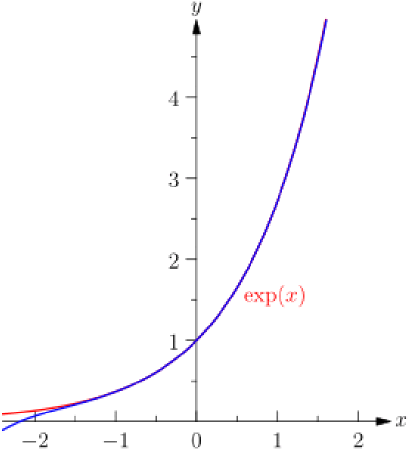


Taylor Series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{5!}$$



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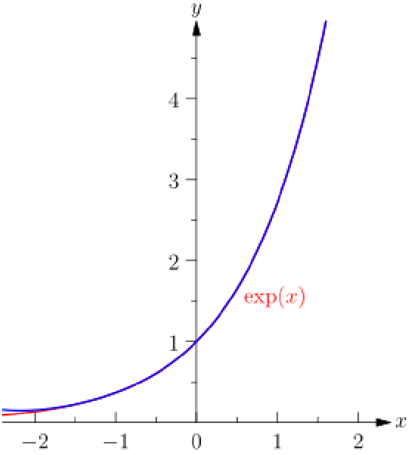

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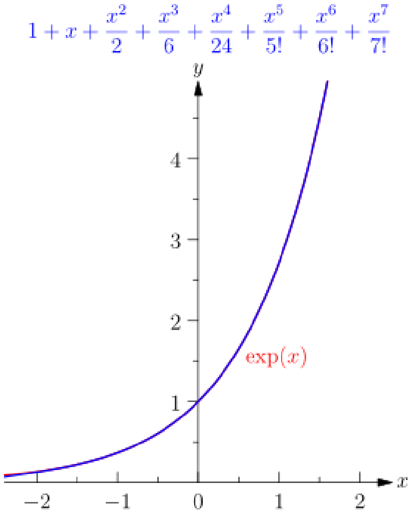

- Try another function yourself!

Indefinite Integral

$2x$ is the derivative of x^2

Indefinite Integral

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So what is x^2?

Indefinite Integral

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F is called an **antiderivative** or **indefinite integral** of f if $F'(x) = f(x)$.

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The indefinite integral is denoted like this:

$$\int f(x)\,dx$$

Indefinite Integral

So in this notation,

$$\int 2x\,dx = x^2$$

Can you name another function that also has derivative $2x$?

Indefinite Integral

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If $F'(x) = f(x)$, then $(F(x) + c)' = f(x)$ for any number c, so actually:

$$\int 2x\,dx = x^2 + C$$

for any number C.

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$\int f(x)\,dx$ is not one function, but a *set* of functions.

Indefinite Integral

Properties

1. $$\int af(x)\,dx = a\int f(x)\,dx$$

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   This can also be written as:

   $$\int f\,dg = fg - \int g\,df,$$

   where $\displaystyle\int f\,dg = \int fg'\,dx$.

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- The antiderivative of $f(x) = \frac{1}{x}$ is:

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

Indefinite Integral

- The antiderivative of $f(x) = e^x$ is:

$$\int e^x \, dx = e^x + C$$

Indefinite Integral

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- The antiderivative of $f(x) = \frac{1}{1+x^2}$ is:

$$\int \frac{1}{1+x^2} \, dx = \operatorname{arctg} x + C$$

Example

Which function should you differentiate to get $x \cos x$?

Indefinite Integral

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Notice that $\cos x = (\sin x)'$, so by the Property 3,

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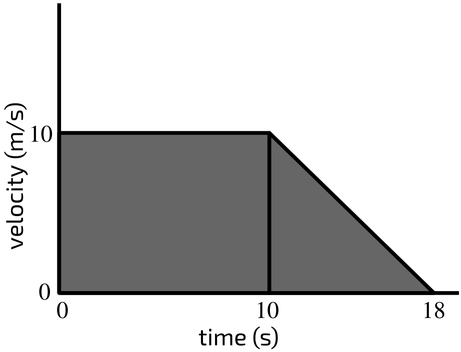
All of this is, of course, computation and tricks (no need to memorize). How about we make an *actual* use of this?

Suppose we are given the velocity of a car at each timepoint. How can we calculate the distance travelled by the car?



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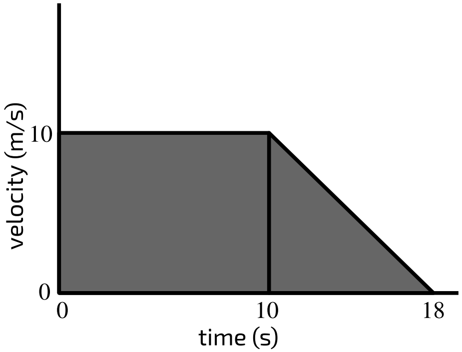

According to physics, distance = area under the velocity curve.

Definite Integral

Question

Suppose you have a continuous function f. How can we calculate the area under its graph?

Definite Integral

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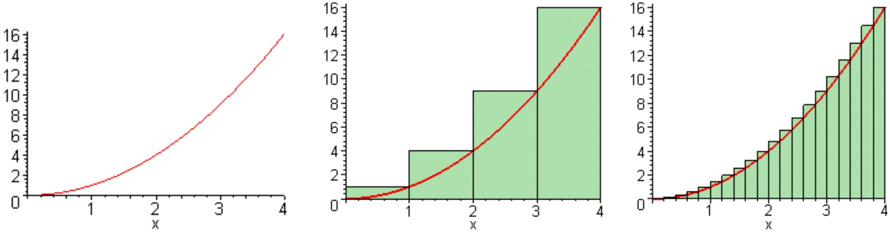
Suppose you have a continuous function f. How can we calculate the area under its graph?

By dividing it into tiny rectangles and adding up their areas.



Definite Integral

Take the interval $[a, b]$ and divide (partition) it into n small parts with points $\{x_0, x_1, \ldots, x_n\}$. Let $\Delta x_i = x_i - x_{i-1}$ denote the length of $[x_{i-1}, x_i]$.

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Definition

The **Riemann sum** of a function $f(x)$ is given by:

$$R_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is any point from $[x_{i-1}, x_i]$.

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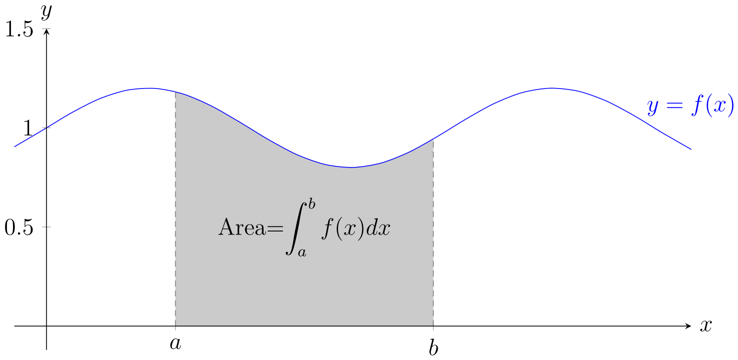
Definition

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

is called the **definite integral** of the function $f(x)$ on $[a, b]$.

Definite Integral

The definite integral $\int_a^b f(x)\,dx$ represents the **signed area** between the graph of $f(x)$ and the x-axis over the interval $[a, b]$.



- Play with Riemann sums!

Definite Integral

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Definite Integral

How can we calculate the definite integral without limits?

Theorem (very important)

Suppose that $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is an antiderivative of $f(x)$, then

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Example

- $$\int_0^2 x^2\,dx = \frac{1}{3}\cdot x^3\Big|_0^2 = \frac{1}{3}(2^3 - 0^3) = \frac{8}{3}$$
- $$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = 2$$

Definite Integral

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \qquad \int_a^a f(x)\,dx = 0$$

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3. $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Definite Integral

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i.e. the name of the variable does not matter.

Functions of Several Variables

Hayk Aprikyan, Hayk Tarkhanyan

Functions of Several Variables

At the moment, we know how to be an effective apple salesperson. What about selling more than one item:

$$f(x, y) = x^2 + y^2 + e^{y-x}$$

i.e. how to deal with functions with more than one variable?

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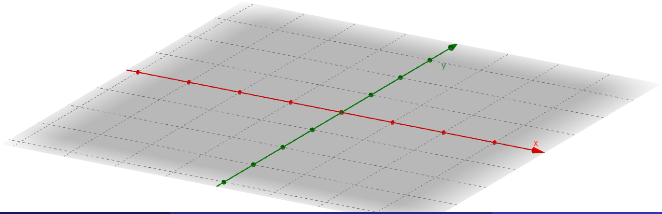

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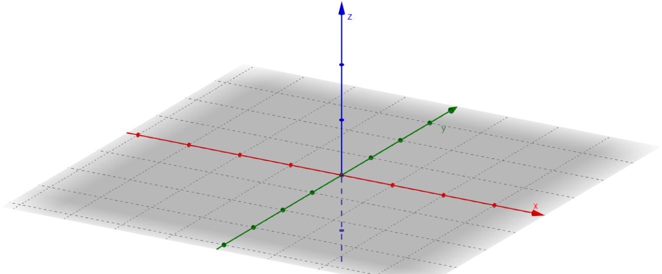

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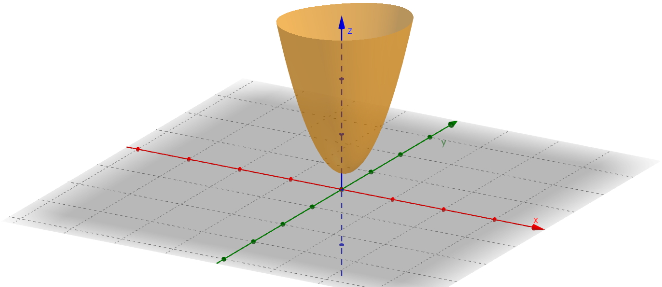

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Partial Derivative

Again, suppose x and y are the costs of apples and peaches, and your profit is given by:

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How can you measure the effect of increasing the cost of apple by a little (i.e. how quickly will f change if x changes)?

By fixing y and then doing the usual derivative stuff with x!

Partial Derivative

Definition

If there exists a finite limit

$$f_x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

then it is called the **partial derivative** of $f(x,y)$ with respect to x, and denoted by f_x or $\frac{\partial f}{\partial x}$.

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Example

If $f(x, y) = x^2 + y^2$, then:

$$f_x = 2x \qquad \text{and} \qquad f_y = 2y$$

Partial Derivative

The partial derivative is the rate of the function change with respect to only one of the variables, while the others are being kept unchanged (constant).

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So naturally, instead of 1 object to describe the speed, we have 2 objects. We can combine them together in one vector:

Definition

The vector consisting of the partial derivatives of $f(x, y)$:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

is called the **gradient** of $f(x, y)$.

In the previous example, $\nabla f = [2x \quad 2y]$.

Partial Derivative

Similarly, for a function of n variables, $f(x_1, \dots, x_n) = f(\mathbf{x})$ we define partial derivatives as:

$$f_{x_1}(\mathbf{x}) = \frac{\partial f}{\partial x_1}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h},$$

$$f_{x_2}(\mathbf{x}) = \frac{\partial f}{\partial x_2}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, x_2 + h, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h},$$

$$\vdots$$

$$f_{x_n}(\mathbf{x}) = \frac{\partial f}{\partial x_n}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_n + h) - f(x_1, x_2, \dots, x_n)}{h}.$$

And the gradient of $f(\mathbf{x})$ as:

$$\nabla f = \frac{df}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \frac{\partial f(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{1 \times n}$$

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Partial Derivative

Example

Let $f(x,y) = 2x^2$ and $g(x,y) = 4x + 6y$.

$$(f \cdot g)_x = 2x^2(4) + (4x + 6y)(4x) = 24x^2 + 24xy$$
$$(f \cdot g)_y = 2x^2(6) + (4x + 6y)(0) = 12x^2$$

Chain Rule

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Assume you're running a supermarket with the profit function

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How does a change of temperature affect your profit?

Chain Rule

In other words,

- if f depends on x and y
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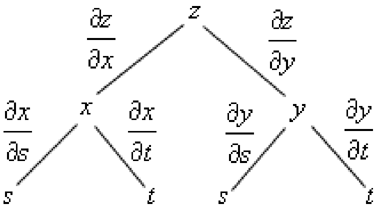

Chain Rule

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Let $z = \sin(x^2 + y^2),\ x = t^2 + 3,\ y = t^3.$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x\cos(x^2+y^2))\cdot(2t) + (2y\cos(x^2+y^2))\cdot(3t^2)$$
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Let $z(x) = x^2 + 4x$, $x(t) = 5t^3 + 2t$. We can again use the chain rule:

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} = (2x + 4) \cdot (15t^2 + 2) = (2 \cdot (5t^3 + 2t) + 4) \cdot (15t^2 + 2)$$

$$= 150t^5 + 80t^3 + 60t^2 + 8t + 8$$

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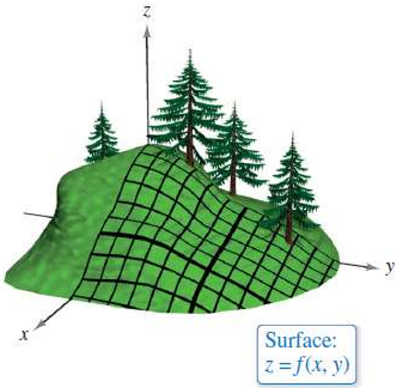
Note that the directional derivative is a *number* (like partial derivatives), not a vector.

Directional Derivative

The directional derivative shows how much our function changes if we "walk" not only along the x or y-axis, but by an arbitrary direction of our choice.



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- Play with directional derivative

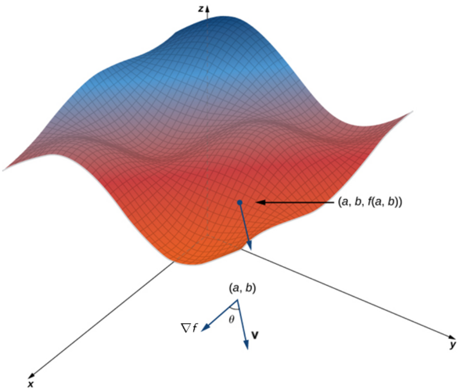
A particularly important question you might ask is:

Question

By which direction should I move, so the function increases the most?

In other words, along which direction does $\nabla_{\mathbf{v}} f$ take its highest value?

Directional Derivative



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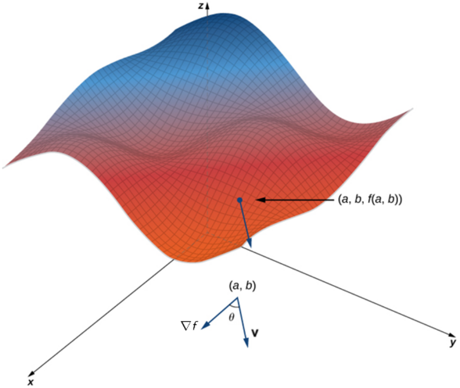

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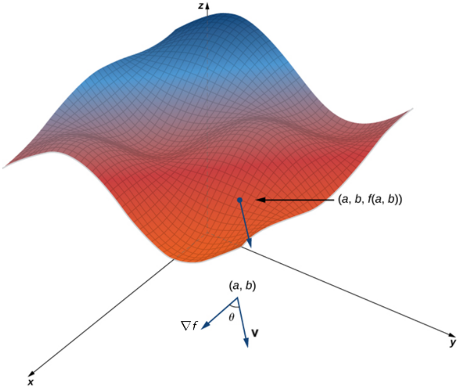

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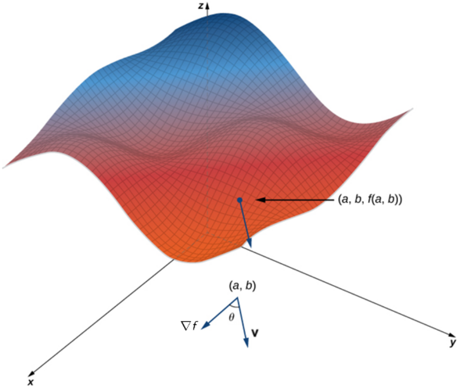

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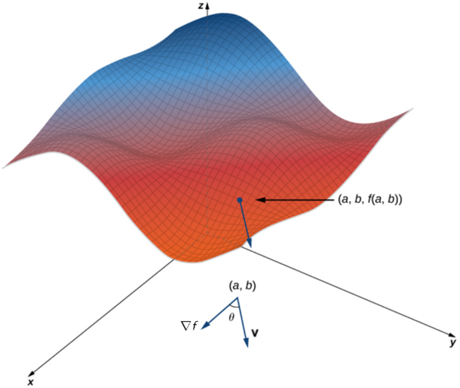

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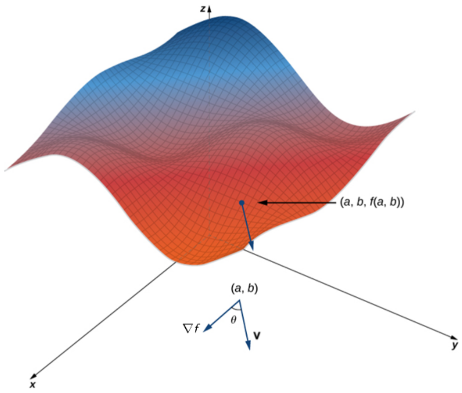

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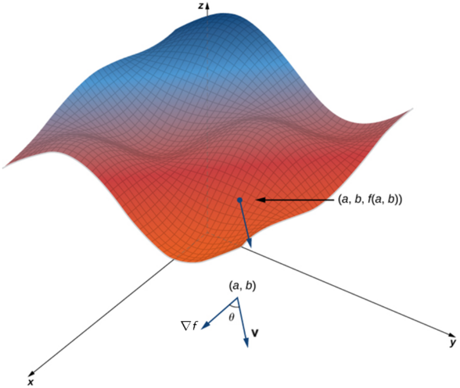

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- When does that happen?

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$$\nabla_{\mathbf{v}} f = \|\nabla f\| \cos\theta$$

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The gradient is the **fastest increasing direction** of the function.

Similarly, $-\nabla f$ is the fastest decreasing direction of the function.

Extrema of a Function

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$\mathbf{x}_0$ is called a **local maximum (minimum)** point of f if there exists a positive number $\delta > 0$ such that for all $\mathbf{x}$, if $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then $f(\mathbf{x}) \leq f(\mathbf{x}_0) \quad (f(\mathbf{x}) \geq f(\mathbf{x}_0))$.

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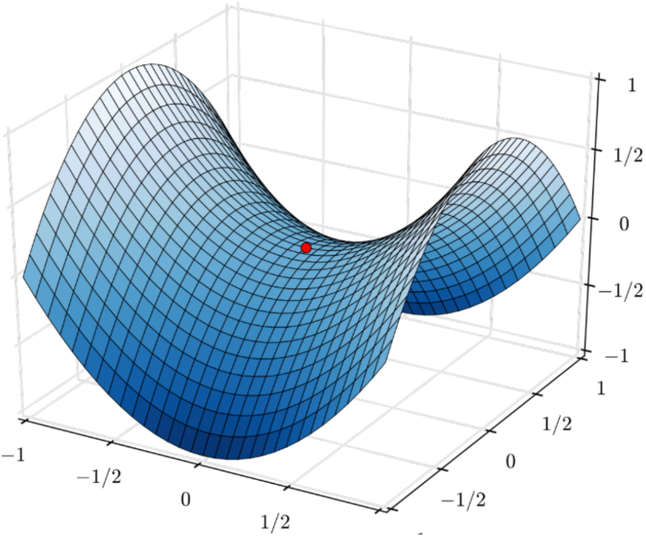
Extrema of a Function



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For the functions of one variable we looked at the sign of f''.
In case of two variables, we look at:

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If $\nabla f(a, b) = \mathbf{0}$ at some point (a, b), and

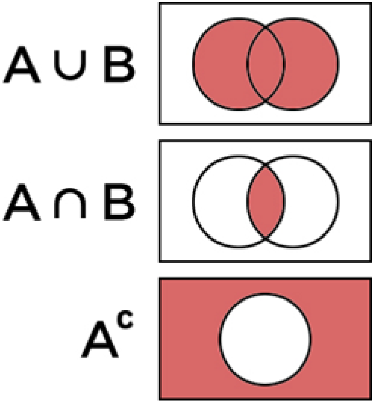
- $D > 0$ and $f_{xx} > 0$ $\Rightarrow$ local minimum
- $D > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum
- $D < 0$ $\Rightarrow$ saddle point

Probability, Independence, Bayes Rule

Hayk Aprikyan, Hayk Tarkhanyan

Preliminaries

Recall the set operations:



Union $A \cup B$:
All elements that belong to A or B or both

Intersection $A \cap B$:
All elements that belong to *both* A and B

Complement A^c:
All elements that do not belong to A

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- Oftentimes, we encounter a situation where we do not know the precise value or outcome of a particular event or circumstance.
- E.g. predicting the score of a football match, or the outcome of a tossed coin.
- While we cannot tell the exact outcome of such an event, we can still speculate about the likely outcomes (e.g. which outcome is more likely than the others).
- The mathematical notion associated with the likeliness of a particular output to happen is called **probability**.

Outcomes

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We denote the sample space with the letter Ω, so

$$\Omega = \{\text{all possible outcomes}\}$$

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Example

A football match is a random experiment, where:

- Outcome is the score of the game.
- For example, one possible outcomes is "Pyunik 2 - 1 Alashkert". One way to denote it is $(2, 1)$.
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Informally, we can say that both outcomes are equally likely: $50/50$.

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Whenever Ω is an interval (e.g. $(0, 1), [1, 10], [0, +\infty)$), the probability of each outcome is 0.
(of course, this does not mean that they are impossible – watch this!)

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- The set of all events is called the **event space** and denoted by $\mathcal{F}$.

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When rolling a die, the events $A = \{1, 4\}$ and $B = \{2, 5\}$ are disjoint, while A and $C = \{3, 4, 5\}$ are not.

Example

When waiting for a bus, the events $A = [0, 20]$ and $B = [30, 40]$ are disjoint, but none of them is disjoint with $C = [10, 40]$.

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Experiments like this (where all outcomes have the same probability of occuring) are called **equiprobable**.

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As it shows, probability can also be not equiprobable.

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So in general, in different problems the *probability measure* $\mathbb{P}$ can be different, and it depends on the specifics of the problem how it is computed.

There are, however, three properties which the probability measure $\mathbb{P}$ always satisfies:

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$$\mathbb{P}\left(A_1 \cup A_2 \cup \ldots\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots$$

i.e.

$$\mathbb{P}\left(\bigcup_n A_n\right) = \sum_n \mathbb{P}(A_n)$$

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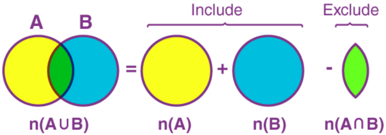
5. **(Inclusion-Exclusion Rule)** For any events A and B:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$



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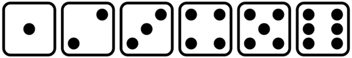

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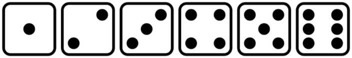

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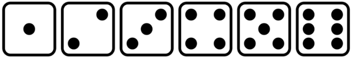

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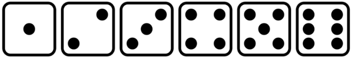

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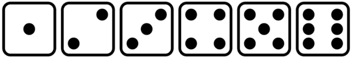

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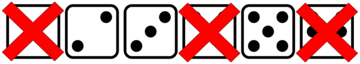

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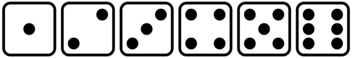

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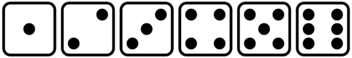

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Definition

For any events A and B (if $\mathbb{P}(B) \neq 0$), the following number:

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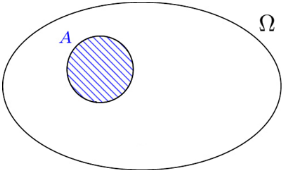

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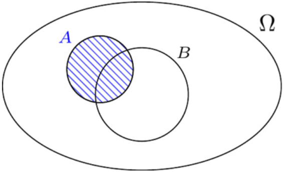

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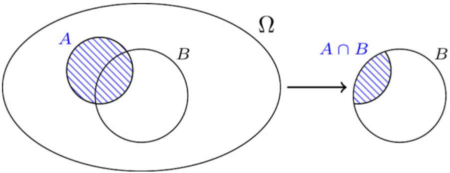

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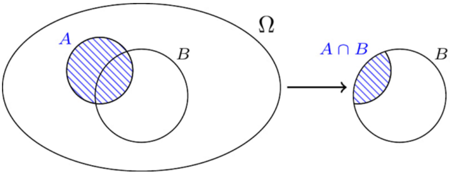

In our problem, if A means being even and B means being prime, we had:

$$\mathbb{P}(\text{even} \mid \text{given that prime}) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

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Example

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2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Usually, there would be 36 total outcomes, but since we *know* the sum is 5, they are only 10 possible outcomes left. Out of them only three outcomes ("2-1", "2-2" and "2-3") are desired. So the probability is $\frac{3}{10}$.

Law of Total Probability

Question

In some university, 3/5 of all students are women and the rest are men. It is known that 15% of men are over left-handed, while only 10% of women are. If you choose a random student, what is the probability of the student being left-handed?

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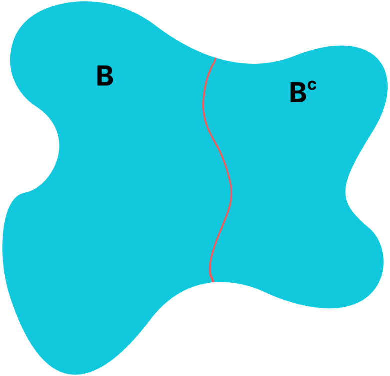
Let B denote the event that the randomly selected student is a woman, and B^c that he is a man.

With another letter, say A, let us denote the event of being left-handed.

Law of Total Probability



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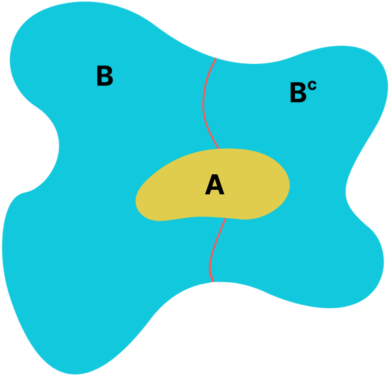

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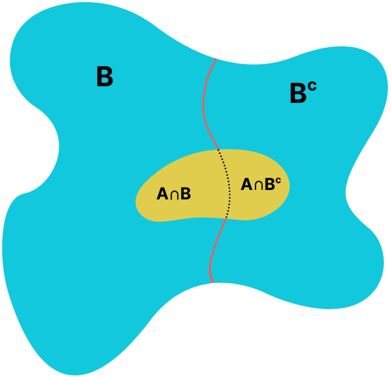

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) =$$

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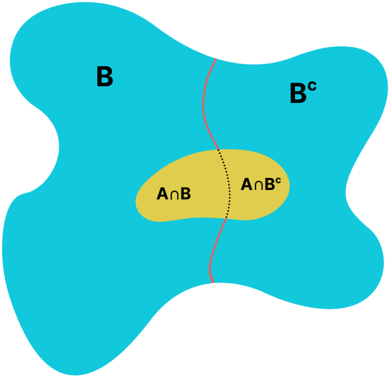

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c)$$

Law of Total Probability

What we get is known as the Law of Total Probability:

Theorem

If A and B are some events such that $\mathbb{P}(B) \neq 0$, then

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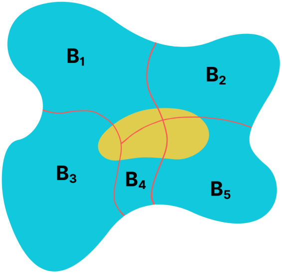

We can also generalize it to the case of three or more subgroups:

Theorem

If $B_1, B_2, \ldots, B_n$ are some disjoint events such that $A \subset \bigcup_{k=1}^{n} B_k$, then

$$\mathbb{P}(A) = \mathbb{P}(B_1) \cdot \mathbb{P}(A|B_1) + \cdots + \mathbb{P}(B_n) \cdot \mathbb{P}(A|B_n)$$

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Example

There are 52 cards in a deck. One of the cards is randomly removed from the deck, 51 are left. What is the probability that if we take a card, it will be a diamond?

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If we denote by A the probability of taking a diamond, and by B the probability that the removed card was a diamond, then either:

- the taken card was a diamond, and there are 12 left,
- the taken card wasn't a diamond, and there are 13 left.

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The total probability will be:

$$\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c)$$

$$\mathbb{P}(A) = \frac{13}{52} \cdot \frac{12}{51} + \frac{39}{52} \cdot \frac{13}{51} = \frac{1}{4}$$

Bayes Rule

One more simple yet powerful tool is the so called Bayes Rule:

Theorem

If A and B are some events (with non-zero probabilities), then

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Let F denote dangerous fire, S smoke. Then:

$$\mathbb{P}(F|S) = \frac{\mathbb{P}(F) \cdot \mathbb{P}(S|F)}{\mathbb{P}(S)} = \frac{1}{100} \cdot \frac{90}{100} : \frac{10}{100} = 0.09$$

Independence

Now let's consider another simple experiment: flipping a fair coin and rolling a six-sided die. The probability of getting **Heads** on the coin flip is independent of the probability of rolling a specific number on the die, as these two events *do not affect* each other.

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This definition makes sense because it means that the probability of both A and B occurring together is simply the product of their individual probabilities: the outcome of one event has no effect on the other.

Independence

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
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$$\mathbb{P}(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

Independence

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
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Suppose we are rolling two fair dice (for the last time).
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When solving a problem, principles like total probability, Bayes rule, and independence help us a lot. Let's explore one more useful problem solving technique.

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Your bus is coming at a random time between 12 pm and 1 pm. If you show up at 12:30 pm, how likely are you to catch the bus?

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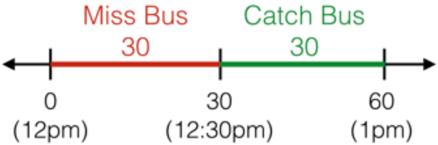


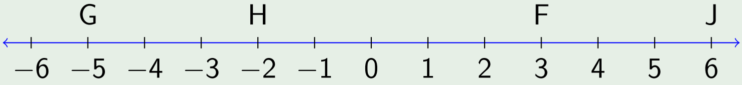
If we draw the hours on a number line and mark the time points for which we will miss or catch the bus, we see that the *length* of the segment for catching the bus is half the total length:

$$\mathbb{P}(\text{catching the bus}) = \frac{30}{30 + 30} = \frac{1}{2}$$

Geometric Probability

Example

A selection is to be made between points G and J as seen below



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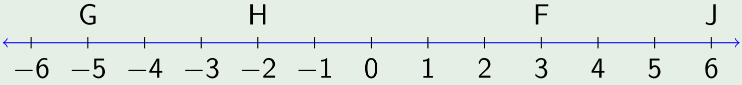

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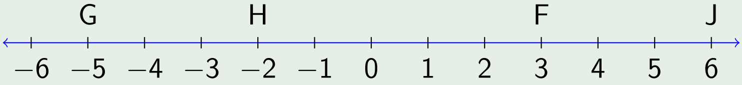

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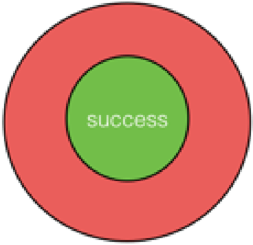
So in general, we draw the sample space and the set of desired outcomes as lines or line segments, and divide the length of the "desired outcomes" by the length of the "sample space" to get the probability.

Geometric Probability

Imagine now that we are playing darts with two circles, and the smaller circle has two times smaller radius than the bigger one.



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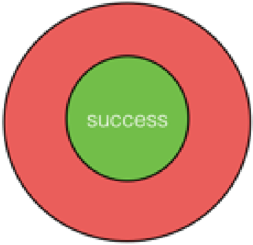

A dart is thrown at a random. What is the probability that it lands in the smaller circle?

In this case, since the sample space and the set of desired outcomes is 2-dimensional, we should divide the *area* of the smaller circle by the *area* of the bigger circle.

The probability that the dart falls in the smaller circle, is $\frac{\pi r^2}{\pi(2r)^2} = \frac{1}{4}$.

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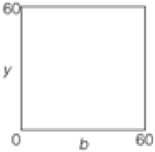

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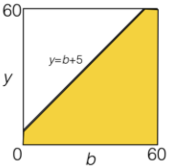
Then, we need to determine the region of "success"; that is, the points where we catch the bus. Since the bus will wait for 5 minutes, you need to arrive within 5 minutes of the bus' arrival, so $y \leq b + 5$.



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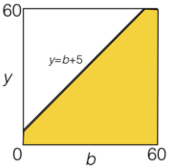
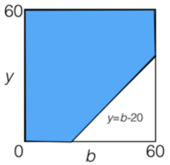


However, you only wait for 20 minutes, so you can't arrive more than 20 minutes before the bus, so $y \geq b - 20$.



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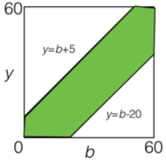
Combining our two conditions, we can draw the region of success:



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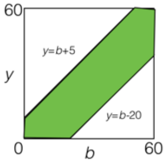

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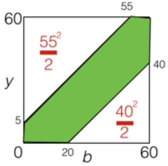

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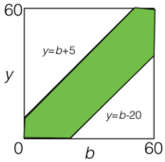

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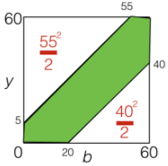

$$\mathbb{P}(\text{catching the bus}) = \frac{60^2 - \frac{55^2}{2} - \frac{40^2}{2}}{60^2} = \frac{103}{288}$$

Random Variables

Hayk Aprikyan, Hayk Tarkhanyan

April 15, 2025

Random Variables

Recap:

During the previous lecture we had a problem like this:

Example

Both the bus and you get to the bus stop at random times between 12 pm and 1 pm. When the bus arrives, it waits for 5 minutes before leaving. When you arrive, you wait for 20 minutes before leaving if the bus doesn't come. What is the probability that you catch the bus?

Here we did not know the **exact values** of the arrival times, so in order to compare them, we **denoted them** by y and b – and calculated their probabilities.

In this case, we say that y and b are **random variables**.

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You throw a fair dice and win \$1 if it is prime, lose \$1 if it is composite (4 or 6), and stay even otherwise. In other words, if X is your profit, then

$$X = \begin{cases} 1, & \text{if } \omega \in \{2,3,5\} \\ -1, & \text{if } \omega \in \{4,6\} \\ 0, & \text{if } \omega = 1 \end{cases}$$

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What is the probability that $X = 1$? (We'll get to this later.)

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What values can each of the random variables above take?

Random Variables (optional)

Not a random variable

Two fair dice are thrown,

- who threw the dice,
- the color of the sky,
- what will show up if you had thrown a third die

are not random variables, as their values *do not depend* on the outcome of the dice (even if you knew the outcome, you could not answer these).

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In practice, every unknown thing in the given problem is a random variable.

Technically speaking, although, the value of the random variable should somehow depend on the outcome ω of the experiment. In other words, if you ask "When is $X = 4$?", there should be a way to measure that probability, i.e. $\{X = 4\}$ should be an event.

In your textbook you will find it as $\{\omega \text{ for which } X = 4\} \in \mathcal{F}$, but we skip the technicalities.

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Notice that in some cases the possible values look like $\{2, 3, 4, \ldots, 12\}$ or $\{0, 1, 2, \ldots\}$, while in others they are intervals like $(0, +\infty)$.

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Definition

If the values of X cannot be represented as a list (e.g. they are an interval), it is called a **continuous random variable.**

PMF

Of course, what we are interested in the most, is the probability that X takes a certain value.

For example:

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The thing above, $\mathbb{P}(X = \text{something})$, is called the **probability mass function** or just the **PMF** of X.

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A pitiful student knows 4 out of 15 topics, and there will be 2 questions on the exam.

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A pitiful student knows 4 out of 15 topics, and there will be 2 questions on the exam. Let X = number of questions the student gets correct.

$$\mathbb{P}(X=0) = \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{21} \qquad \text{(rip student)}$$

$$\mathbb{P}(X=1) = \frac{4}{15} \cdot \frac{11}{14} + \frac{11}{15} \cdot \frac{4}{14} = \frac{44}{105}$$

$$\mathbb{P}(X=2) = \frac{4}{15} \cdot \frac{3}{14} = \frac{6}{105} \qquad \left(\text{or just } 1 - \frac{11}{21} - \frac{44}{105}\right)$$

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k (right answers)	$\mathbb{P}(X = k)$
0	$11/21$
1	$44/105$
2	$6/105$

CDF

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- Not only the probabilities that $X = 0$ or $X = 1$ or $X = 2$,

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If there were, say, 5 questions and more than 3 points needed to pass, then we would be interested in

$$\mathbb{P}(X \leq 3) \qquad \text{(probability of failing)}$$

or

$$\mathbb{P}(X > 3) \qquad \text{(probability of passing)}$$

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The quantity

$$\mathbb{P}(X \leq k) \qquad \text{(where } k \text{ is any number)}$$

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- $0 \le F_X(x) \le 1, \;$ for any $x \in \mathbb{R}$
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Example

Let X indicate the number on a fair die. Then

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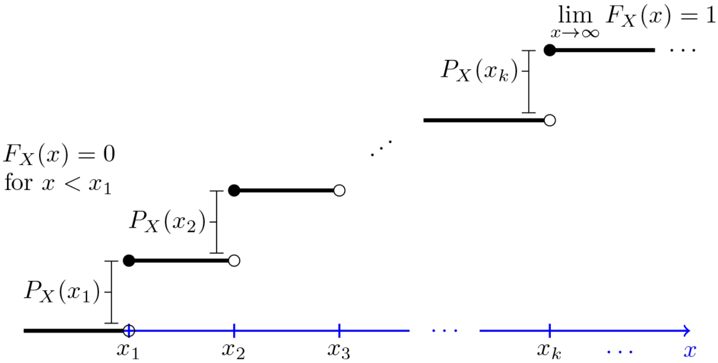
$$F_X(x) = \begin{cases} 0, & \text{if } x < 1, \\ \frac{1}{6}, & \text{if } 1 \le x < 2, \\ \frac{2}{6}, & \text{if } 2 \le x < 3, \\ \frac{3}{6}, & \text{if } 3 \le x < 4, \\ \frac{4}{6}, & \text{if } 4 \le x < 5, \\ \frac{5}{6}, & \text{if } 5 \le x < 6, \\ 1, & \text{if } x \ge 6 \end{cases}$$

CDF

Graphs of CDFs usually usually look like this:



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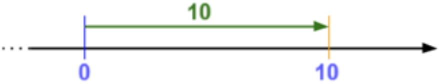
PDF

Finally, what if X is a continuous random variable?



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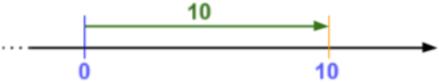

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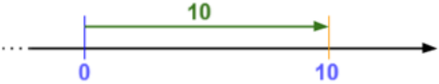

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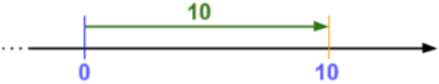

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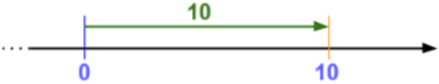

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So another question we might ask is:

Question

What do you think $\mathbb{P}(X \leq 5)$ is?

PDF

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Definition

If X is a continuous random variable, then there exists a nonnegative function $f(x)$ such that for any $c \in \mathbb{R}$,

$$F_X(c) = \mathbb{P}(X \leq c) = \int_{-\infty}^{c} f(t)\,dt$$

The function f is called the **probability density function** or **PDF** of X.

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The PDF of a continuous random variable plays essentially the same role as the PMF of a discrete random variable.

PDF

Just like the density of an object measures the concentration of mass (per unit volume), the probability density function captures the density of *probability* at point x:

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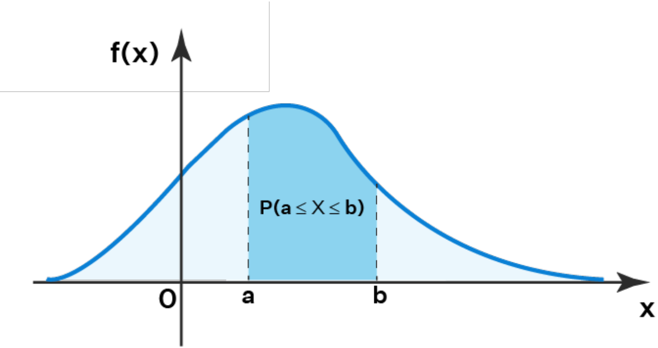

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5. $\mathbb{P}(X = a) = \mathbb{P}(X \leq a) - \mathbb{P}(X < a)$

PDF

Example

Ani chooses a random real number X uniformly from the interval $[a, b]$.

By "uniformly" we mean that for any two intervals of the same length (e.g. $(1.3, 1.5)$ and $(4.7, 4.9)$) X can belong to them with the same probability.

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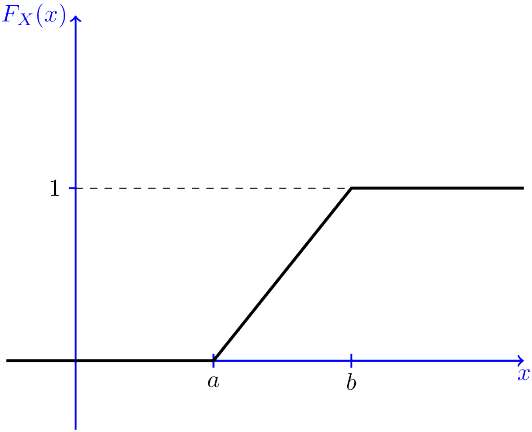
For $a \leq x \leq b$, we have:

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \in [a, x]) = \frac{x - a}{b - a}$$

PDF



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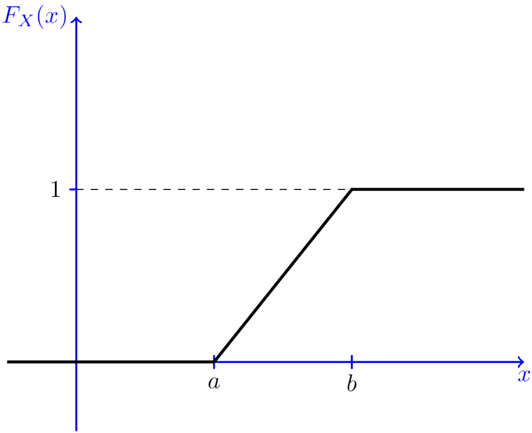

Thus,

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

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To find $f_X(x)$, we take the derivative of $F_X(x)$:

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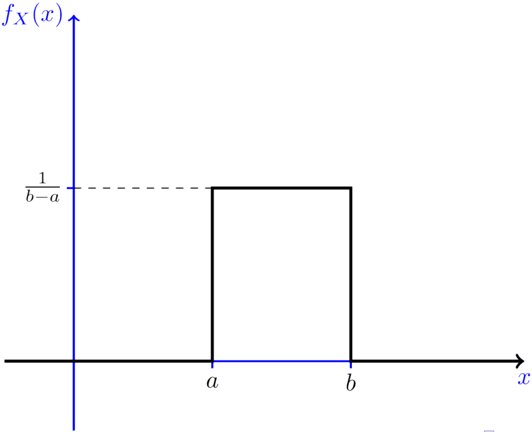

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Definition

X and Y are called **independent** if

$$\mathbb{P}(X \leq a \text{ and } Y \leq b) = \mathbb{P}(X \leq a) \cdot \mathbb{P}(Y \leq b)$$

for any $a, b \in \mathbb{R}$.

So the probability of both X and Y simultaneously being less than some numbers is just their *separate* probabilities multiplied together.

Expected Value, Variance, Distributions

Hayk Aprikyan, Hayk Tarkhanyan

April 18, 2025

Expected Value

Suppose you are playing roulette, with numbers 1 to 38 on it. You bet a number and spin it.

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Would you play this game? What if instead of $36, you won $150 if it fell on 8? How about $37.01?

Expected Value

Since the chance of winning is only $\frac{1}{38}$, if you play it a couple of thousands times (say 38000), then you can expect to win about ~ 1000 times and lose ~ 37000 times. Your net revenue would then be:

$$1000 \cdot (+37.01) + 37000 \cdot (-1) > 0 \qquad \Rightarrow \qquad \$\$$$

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These examples motivate the notion of the **mean** or **expected value** of a random variable.

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In words, the expected value is the weighted average of all its possible values, each of the values being weighted by its probability.

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Definition

If X is a random variable and $g(x)$ is a continuous function, then $g(X)$ is a random variable too.

Theorem

If X is a continuous random variable, then for any continuous function g,

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

If X is a discrete random variable, then for any continuous function g,

$$\mathbb{E}(g(X)) = \sum_{x_i} g(x_i) \cdot \mathbb{P}(X = x_i)$$

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Theorem

If X and Y are independent random variables, then

$$\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

The converse is not necessarily true.

Variance

Now assume you are offered to play one of these two games:

- You toss a coin and win \$1 if it is Heads, otherwise you lose \$1,
- You toss a coin and win \$10000 if it is Heads, otherwise you lose \$10000.

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In both games, your overall winnings is on average \$0. However in the first game the amounts differ from \$0 by a small amount, while in the second game, they differ by such a large amount you might not take a risk of playing at all.

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In this case, we say that the winnings of the second game have a **higher variance** than those of the first one.

Variance

Definition

The **variance** of a random variable X, denoted by $\mathrm{Var}(X)$, is defined by:

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$\sqrt{\mathrm{Var}(X)}$ is called the **standard deviation** of X.

The standard deviation shows how much, in average, do the values of the random variable deviate from their average ($\mathbb{E}(X)$).

Variance

Example

Let X denote the number on a fair die. Then,

$$\mathbb{E}(X) =$$

Variance

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Let X denote the number on a fair die. Then,

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$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \big(\mathbb{E}(X)\big)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} \approx 2.92$$

Variance

Properties

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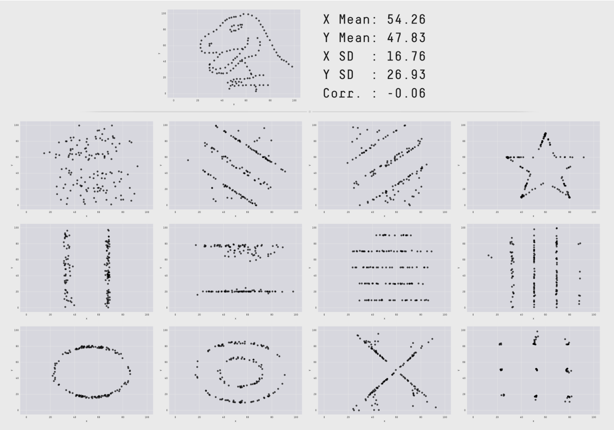
Why do you think the 4th point makes sense?

Variance

Warning

Expected value and variance are very useful to describe random variables, **but they are not everything!** They do not replace CDF/PDF/PMF!



[source]

Covariance

Suppose X is the stock price of Tesla, Y is the stock price of Yeraz, and you have some Yeraz stocks.

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Due to some reasons, the stock of Tesla starts to decrease. Naturally, you are interested in how could that affect your Yeraz stocks, i.e.

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How would a change of X affect Y?

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How would a change of X affect Y?

More specifically, if X goes up by 1 unit, how much would Y change?

Covariance

Definition

The **covariance** between random variables X and Y is defined by

$$\mathrm{Cov}(X, Y) = \mathbb{E}\big((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big)$$

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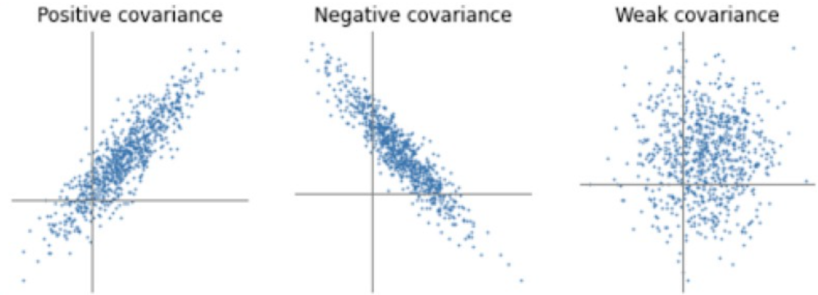
Covariance shows how much the linear growth of one RV is related to the linear growth of the other RV. It is very similar to the concept of dot product of the two vectors.

Covariance


Positive covariance
Negative covariance
Weak covariance

Covariance

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$\mathrm{Cov}(X, Y)$ can be *any* number (positive/negative, large/small, zero, etc). What if we want a normalized, universal method to measure the relatedness level of two random variables?

Correlation

Definition

For two non-constant ($\mathrm{Var}(X), \mathrm{Var}(Y) \neq 0$) random variables X and Y, the **correlation** (or **Pearson correlation coefficient**) between them is defined as:

$$\rho(X, Y) = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}$$

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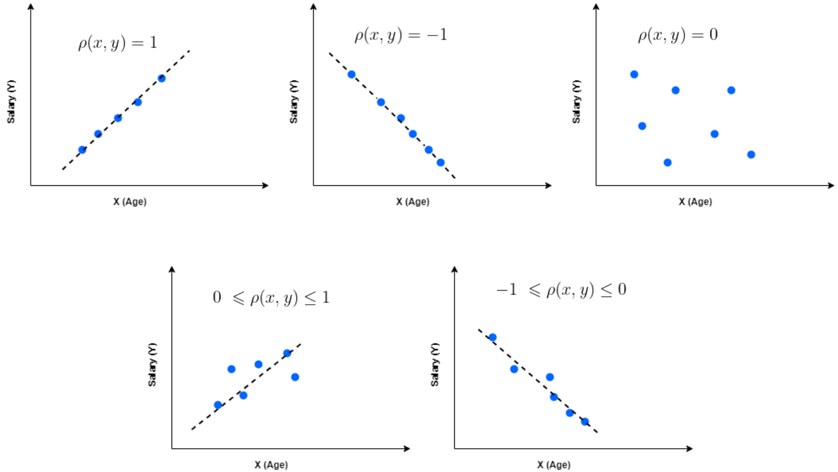
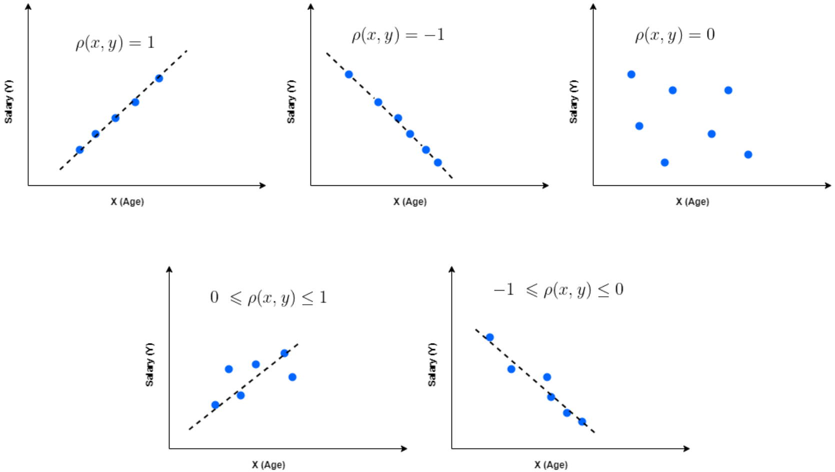
Properties

1. $\rho(X, Y) = \rho(Y, X)$,
2. $-1 \leq \rho(X, Y) \leq 1$,
3. $Y = aX + b$ for some constants a, b if and only if $\rho(X, Y) = \pm 1$.

Correlation

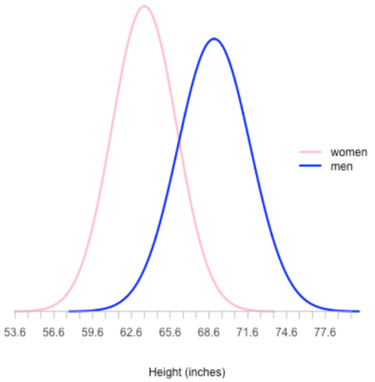


Correlation



- Play with this correlation visualization!

Very often in practice, many random variables share similar properties. In particular, the probabilities of their values seem to follow a common pattern, i.e. their CDFs (or PMFs/PDFs) are similar to each other:



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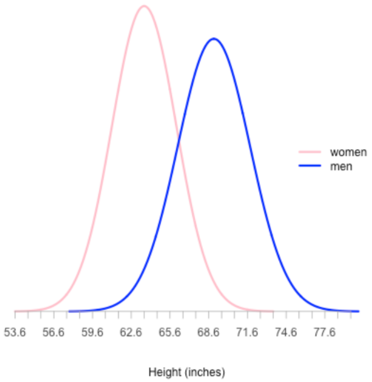

The way the values of an RV are distributed is called a *distribution*.

Bernoulli Distribution

Consider these situations:

Bernoulli Distribution

Consider these situations:

- You take a pass-fail exam. You either pass or fail.

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Definition

A random variable X is said to be a **Bernoulli** random variable with parameter p, denoted by $X \sim Bernoulli(p)$, if it only takes two values and its PMF is given by:

$$\mathbb{P}(X = x) = \begin{cases} p & \text{for } x = 1 \\ 1 - p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$.

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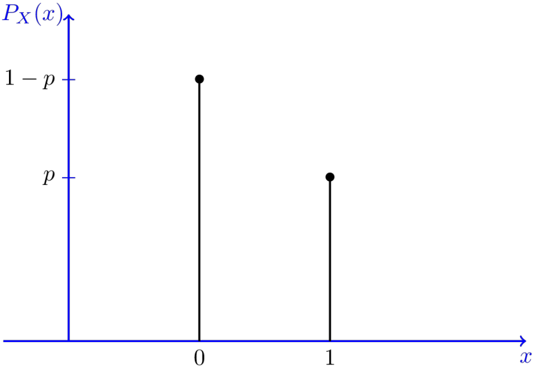
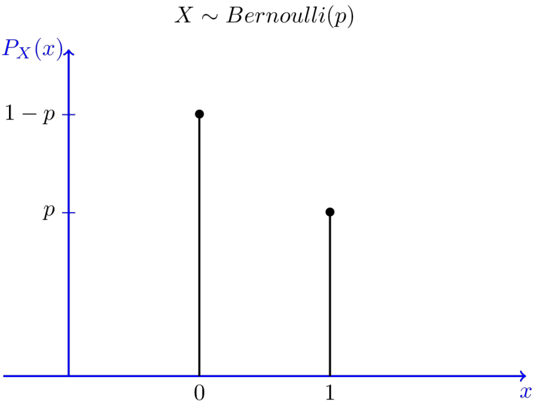
A series of n independent experiments all following Bernoulli distribution $Bernoulli(p)$, is called **Bernoulli trials**.

Bernoulli Distribution

$$X \sim Bernoulli(p)$$



Bernoulli Distribution



If $X \sim Bernoulli(p)$,

$$\mathbb{E}(X) = p, \qquad \mathrm{Var}(X) = p(1-p)$$

Geometric Distribution

Suppose we have a coin with $\mathbb{P}(\text{Heads}) = p$ and we toss it until we observe the first Heads, after which we stop. We define X as the total number of coin tosses in this experiment. Then X is said to have *geometric distribution* with parameter p.

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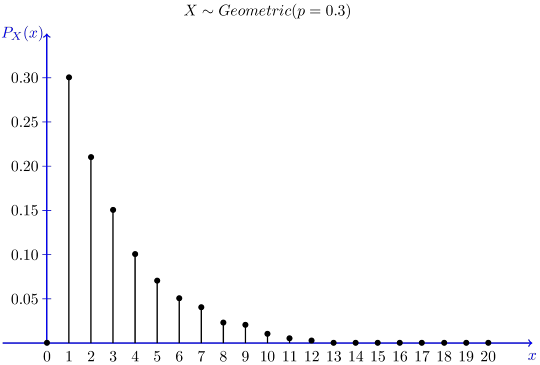
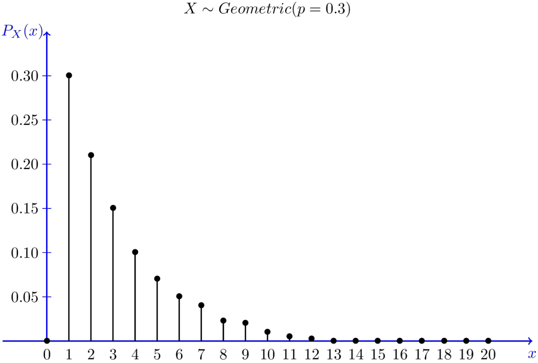
Geometric Distribution



Geometric Distribution



If $X \sim Geo(p)$,

$$\mathbb{E}(X) = \frac{1}{p}, \qquad \mathrm{Var}(X) = \frac{1-p}{p^2}$$

Binomial Distribution

Suppose we have a coin with $\mathbb{P}(\text{Heads}) = p$ and we toss it n times. We define X to be the total number of Heads observed. Then X is said to have *binomial distribution* with parameter n and p.

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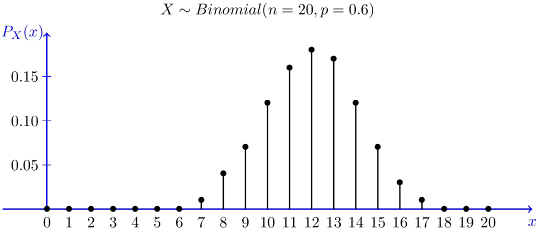
Definition

A random variable X is said to be a **binomial** random variable with parameters n and p, denoted by $X \sim B(n, p)$, if its PMF is given by:

$$\mathbb{P}(X = k) = \begin{cases} C_n^k p^k (1-p)^{n-k} & \text{for } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$.

Binomial Distribution



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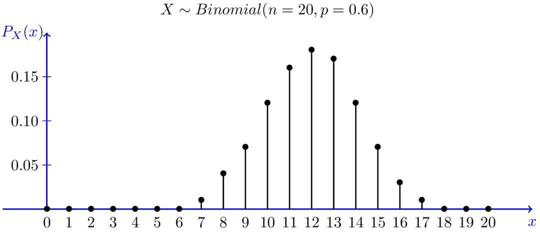

If $X \sim B(n, p)$,

$$\mathbb{E}(X) = np, \qquad \mathrm{Var}(X) = np(1-p)$$

Poisson Distribution

The Poisson distribution is one of the most widely used probability distributions. It is usually used in scenarios where we are counting the occurrences of certain events in an interval of time or space. In practice, it is often an approximation of a real-life random variable.

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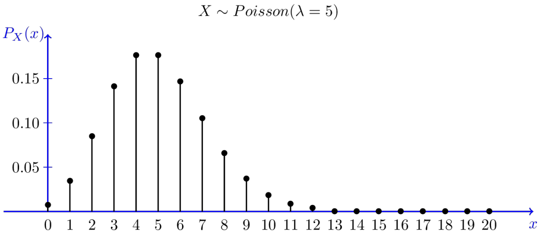
Definition

A random variable X is said to be a **Poisson** random variable with parameter λ, denoted by $X \sim Poisson(\lambda)$, if its PMF is given by:

$$\mathbb{P}(X = k) = \begin{cases} \dfrac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Poisson Distribution



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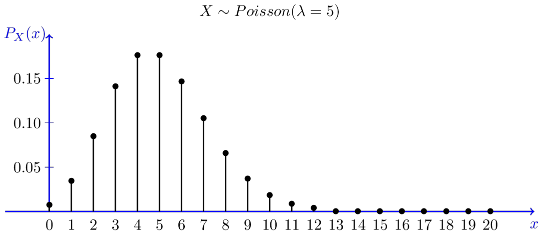

If $X \sim Poisson(\lambda)$,

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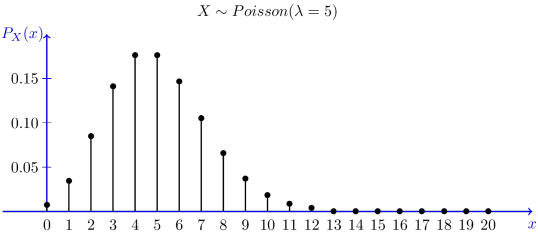

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Remark

If $X \sim B(n, p)$, then

$$\mathbb{P}(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda}, \qquad \text{where } \lambda = np$$

Uniform Distribution

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A random variable X is said to be a **uniform** random variable over the interval $[a, b]$, denoted by $X \sim U(a, b)$, if its PDF is given by:

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b) \\ 0 & x \notin (a, b) \end{cases}$$

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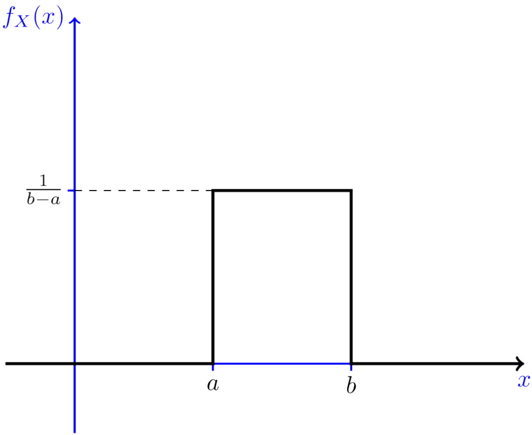
$$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$

Uniform Distribution



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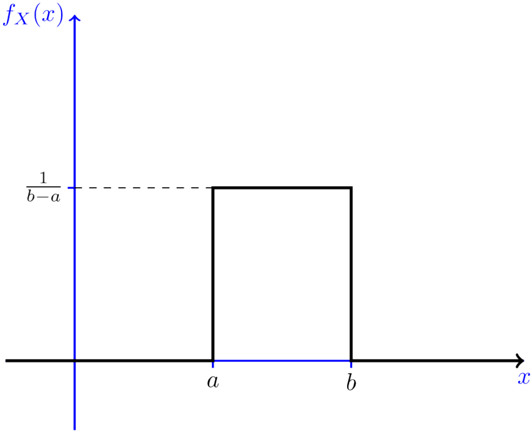

If $X \sim U(a, b)$,

$$\mathbb{E}(X) = \frac{a+b}{2}, \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential Distribution

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Definition

A random variable X is said to be a **exponential** random variable with parameter $\lambda > 0$, denoted by $X \sim Exp(\lambda)$, if its PDF is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

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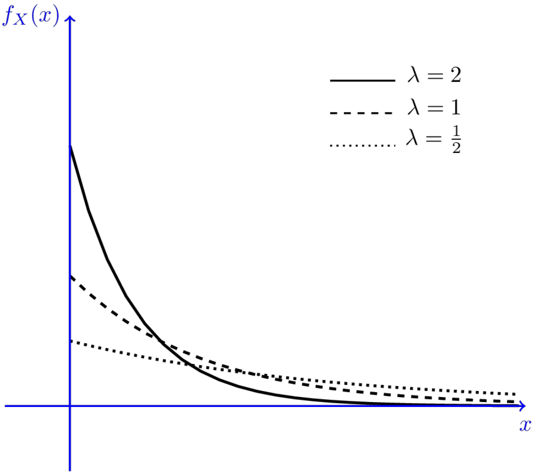
Remark

If $X \sim Exp(\lambda)$, then X is a **memoryless** random variable, that is

$$\mathbb{P}(X > x + a \mid X > a) = \mathbb{P}(X > x), \quad \text{for } a, x \geq 0.$$

Exponential Distribution



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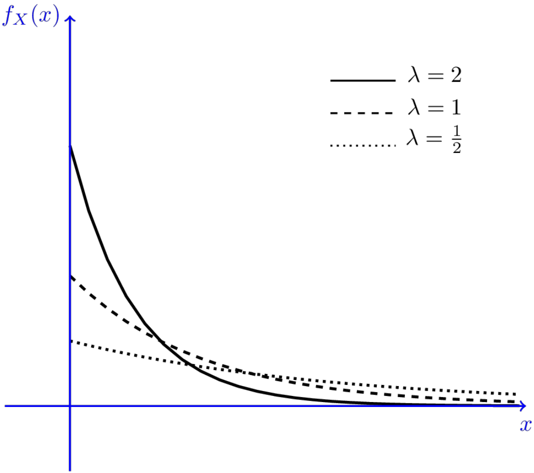

If $X \sim Exp(\lambda)$,

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Normal Distribution

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Definition

A random variable X is said to be a **normal** random variable with mean μ and variance σ^2, denoted by $X \sim N(\mu, \sigma^2)$, if its PDF is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}$$

Normal Distribution