Convergence, LLN and CLT

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Convergence of Functions

In calculus, we used to work with *sequences of numbers:*

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots$$

and in case the sequence got closer and closer to some number x as n gets larger and larger, we said that the sequence *converges* to x:

$$\lim_{n\to\infty} x_n = x$$

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If instead, we had a *sequence of functions:*

$$f_1(a),\ f_2(a),\ f_3(a),\ \ldots,\ f_n(a),\ \ldots$$

and at some point a, their values got closer to some number b as $n \to \infty$, we say that the sequence $\{f_n\}$ *converges* to b **at the point** a:

$$\lim_{n\to\infty} f_n(a) = b$$

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As we see, this sequence of functions converges to 0 at **every point** $a \in \mathbb{R}$:

$$\lim_{n \to \infty} f_n(a) = 0 \qquad \text{for all } a \in \mathbb{R}$$

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$$f_n(a) \to g(a) \qquad \text{for all } a \in \mathbb{R}$$

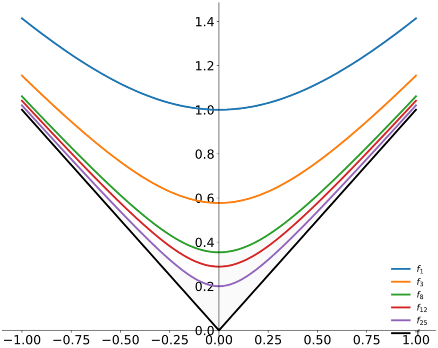
or simply that the sequence of functions $\{f_n\}$ *converges* to the function g.

Convergence of Functions

Definition

A sequence of functions $\{f_n\}$ *converges* to a function g, if for every point a and for every $\varepsilon > 0$, there exists a natural number N such that for all $n \geq N$,

$$|f_n(a) - g(a)| < \varepsilon$$



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i.e.

```
X = lambda ω:  the height of ω
X(Arthur Abraham)
>> 1.75
```

Random Variable as a Function

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If X denotes the outcome of tossing a coin, we can say:

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So can we talk about the convergence of random variables?

As it turns out, the answer is yes.

Convergence of Random Variables

Everyday Anahit randomly chooses a number from the interval $[0, 1]$ and multiplies the number by itself n times, where n is the number of days since she started doing this.

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- If on the first day, she chooses 0.5, then $X_1 = 0.5^1 = 0.5$

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Question

What happens to X_n as n gets larger and larger? Does it converge to some number?

Definition

If something happens with probability 1 (i.e. it can also not happen, but this event has probability zero) we say that it happens *almost surely (a.s.)*.

We distinguish between three possible ways how a sequence of random variables $\{X_n\}$ can converge to a random variable X:

- The strongest case: $X_n \to X$ *almost surely*

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- The weakest case: $X_n \xrightarrow{d} X$ *in distribution*
  when just the PMF/PDFs of X_n converge to the PMF/PDF of X – but the values of X_n do not necessarily get closer to the value of X.

Convergence of Random Variables

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Almost sure $\Rightarrow$ in probability $\Rightarrow$ in distribution

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Let X_n be the random variable defined as follows:

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What can we say about the convergence of X_n?

Law of Large Numbers

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Law of Large Numbers (LLN)

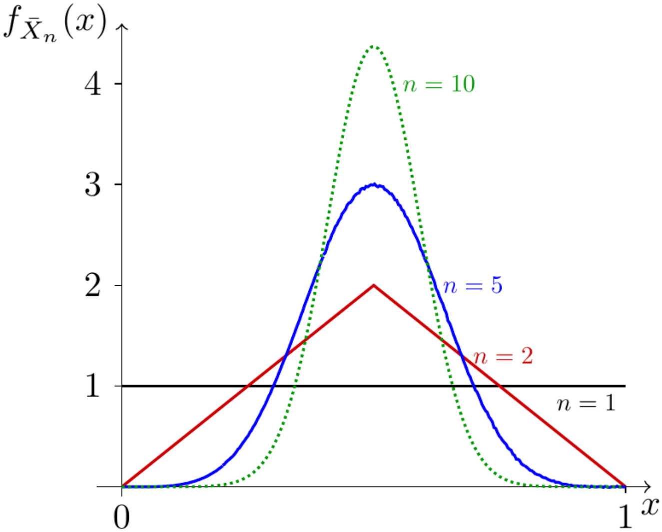
Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with the same expected value:

$$\mathbb{E}[X_k] = m$$

and with finite variance (again, same). Then their average

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{p} m \qquad \text{as } n \to \infty$$

Law of Large Numbers



Central Limit Theorem

Ok, so eventually, the average of i.i.d. random variables converges to their expected value. But what about the distribution of the average?

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Central Limit Theorem (CLT)

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables with the same expected value:

$$\mathbb{E}[X_k] = m$$

and with finite variance

$$\mathrm{Var}[X_k] = \sigma^2$$

Then their normalized average

$$\sum_{k=1}^{n} \frac{X_k - m}{\sigma\sqrt{n}} \xrightarrow{d} \mathrm{N}(0,1) \qquad \text{as } n \to \infty$$