Foundations of Derivatives and Monotonicity

Lecture 1

Plan

1. **Function Limits** – Intuition and ε-δ definition
2. **Derivative Definition** – As a limit of difference quotient
3. **Monotonicity** – Using derivatives to determine increase/decrease
4. **Derivative Rules** – Sum, product, quotient rules
5. **Common Derivatives** – Essential derivative formulas

Learning Objectives

By the end of this lecture, you should be able to:

- Understand limits using the ε-δ definition
- Define and compute derivatives from first principles
- Identify whether a function is increasing or decreasing using its derivative
- Apply differentiation rules efficiently
- Recall and use common derivative formulas

Intuitive Idea of a Limit

Question: What value does $f(x)$ approach as x gets closer to a?

We write: $\lim_{x \to a} f(x) = L$

This means: as x approaches a, the function values $f(x)$ approach L.

Key insight: The limit describes behavior *near* a point, not necessarily *at* the point.

Example: Computing a Limit

Example:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

If we substitute $x = 2$ directly, we get $\frac{0}{0}$, which is indeterminate.

However, we can simplify:

$$\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2, \quad x \neq 2$$

Therefore,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The ε-δ Definition of Limit

Formal Definition: We say $\lim_{x \to a} f(x) = L$ if:

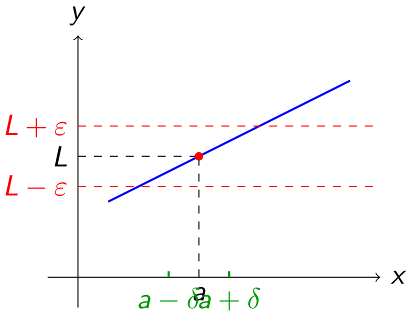
For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Interpretation:

- ε measures how close $f(x)$ is to L
- δ measures how close x is to a
- No matter how small ε is, we can find a δ that works

Visualizing ε-δ



When x is within δ of a, then $f(x)$ is within ε of L.

More Limit Examples

Example 1: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Factor: $\frac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$

Therefore: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$

Example 2: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (standard result)

These examples show that limits can be computed by algebraic simplification or known standard results.

Motivation: The Slope Problem

How do we find the slope of a curve at a single point?

For two points, the slope is:

$$\text{slope} = \frac{f(x+h) - f(x)}{h}$$

But what happens as the second point gets infinitely close to the first?

We take a **limit**!

Definition of the Derivative

The derivative of f at x is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Interpretation:

- Instantaneous rate of change of f at x
- Slope of the tangent line to the graph of f at point $(x, f(x))$

Example: Derivative of x^2 from First Principles

For $f(x) = x^2$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) = 2x \end{aligned}$$

Therefore: $(x^2)' = 2x$

Example: Derivative of e^x

For $f(x) = e^x$:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^x \cdot 1 = e^x
\end{aligned}
$$

Therefore: $(e^x)' = e^x$

(Using the standard result: $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$)

Monotonically Increasing and Decreasing Functions

A function f is said to be:

- **Monotonically increasing** on an interval if for all $x_1 < x_2$ in that interval, we have $f(x_1) < f(x_2)$
- **Monotonically decreasing** on an interval if for all $x_1 < x_2$ in that interval, we have $f(x_1) > f(x_2)$

Geometric interpretation:

- Increasing: graph rises as we move left to right
- Decreasing: graph falls as we move left to right

Connection to Derivatives

Key Theorem:

$$f'(x) > 0 \Rightarrow f \text{ is increasing}, \quad f'(x) < 0 \Rightarrow f \text{ is decreasing}$$

The derivative tells us whether the slope of the tangent line is positive (rising) or negative (falling).

Why? If $f'(x) > 0$, the function is going "uphill" at x.

Example: Monotonicity of $f(x) = x^2$

We computed $f'(x) = 2x$

Analysis:

- When $x < 0$: $f'(x) = 2x < 0$, so f is decreasing
- When $x > 0$: $f'(x) = 2x > 0$, so f is increasing
- At $x = 0$: $f'(x) = 0$ (critical point)

Conclusion: $f(x) = x^2$ decreases on $(-\infty, 0)$ and increases on $(0, \infty)$.

Exercise: Monotonicity of $f(x) = x^3 - 3x$

Determine where the function $f(x) = x^3 - 3x$ is increasing and where it is decreasing.

Solution:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$$

Critical points: $x = -1$ and $x = 1$

Sign analysis:

- $(-\infty, -1)$: $f'(x) > 0$ (increasing)
- $(-1, 1)$: $f'(x) < 0$ (decreasing)
- $(1, \infty)$: $f'(x) > 0$ (increasing)

Proof Outline: Sum Rule

Using the definition of the derivative and linearity of limits:

$$(f+g)'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x) + g'(x)$$

Proof Outline: Product Rule

Start from the difference quotient and add–subtract $f(x)g(x+h)$:

$$
\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right] \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

Proof Outline: Quotient Rule

Write $\dfrac{f}{g} = f \cdot g^{-1}$ and use the product rule with $(g^{-1})' = -g'/g^2$:

$$\left(\frac{f}{g}\right)' = f'g^{-1} + f \cdot (g^{-1})' = \frac{f'}{g} - f \cdot \frac{g'}{g^2} = \frac{f'g - fg'}{g^2}$$

Sum Rule

$$(f+g)' = f' + g'$$

Example: If $f(x) = x^2$ and $g(x) = e^x$, then

$$(f+g)' = 2x + e^x$$

Product Rule

$$(fg)' = f'g + fg'$$

Example: $f(x) = x^2 e^x$

$$f'(x) = 2xe^x + x^2 e^x = e^x(2x + x^2)$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Example: $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

Power Functions

Power Rule:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Examples:

$$\frac{d}{dx}x^3 = 3x^2$$

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

Exponential and Logarithmic Functions

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a \quad (a > 0)$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}\cot x = -\csc^2 x = -\frac{1}{\sin^2 x}$$

More Trigonometric Functions

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Constants and Linear Functions

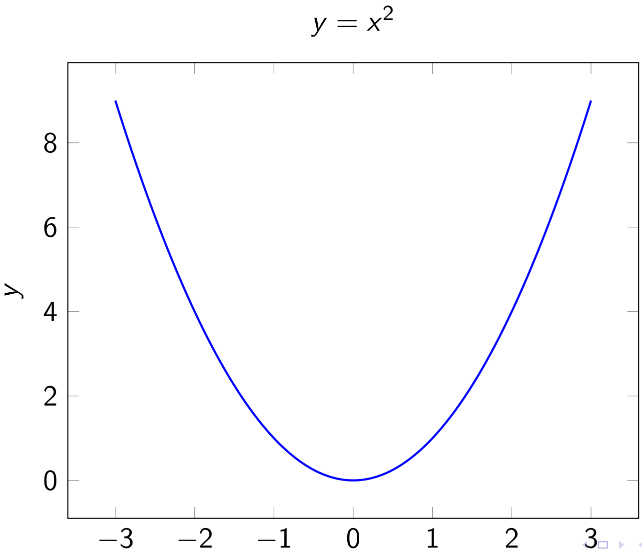
$$\frac{d}{dx} c = 0 \quad \text{(constant)}$$

$$\frac{d}{dx} cx = c$$

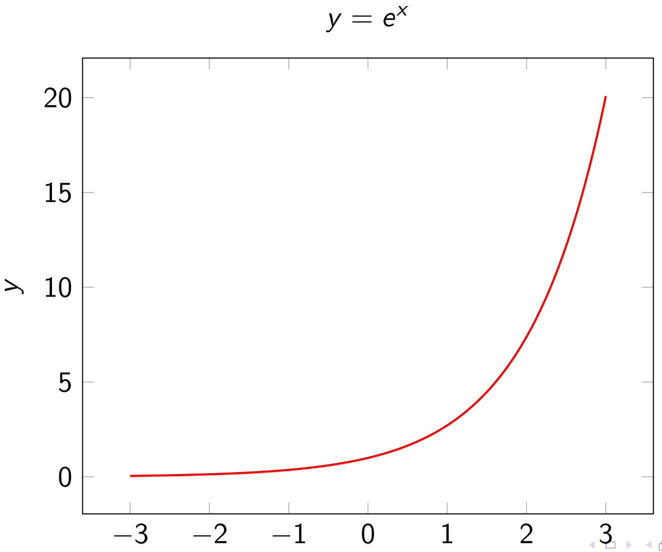
$$\frac{d}{dx} (mx + b) = m$$

Remember: The derivative of a constant is always zero!

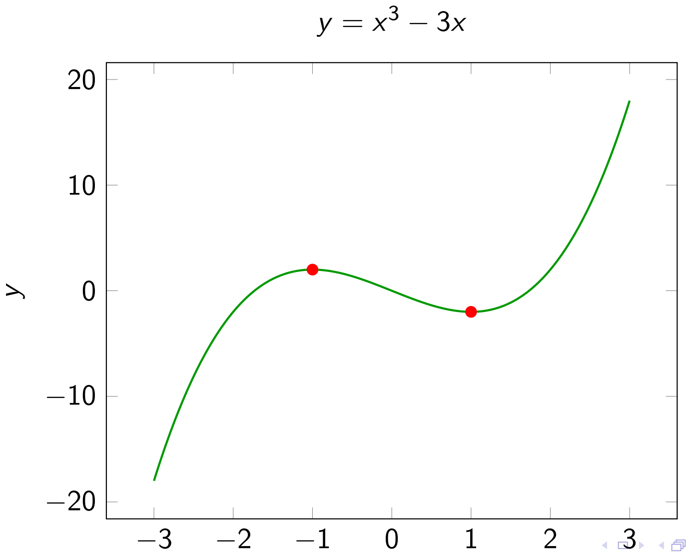
Graph of $y = x^2$



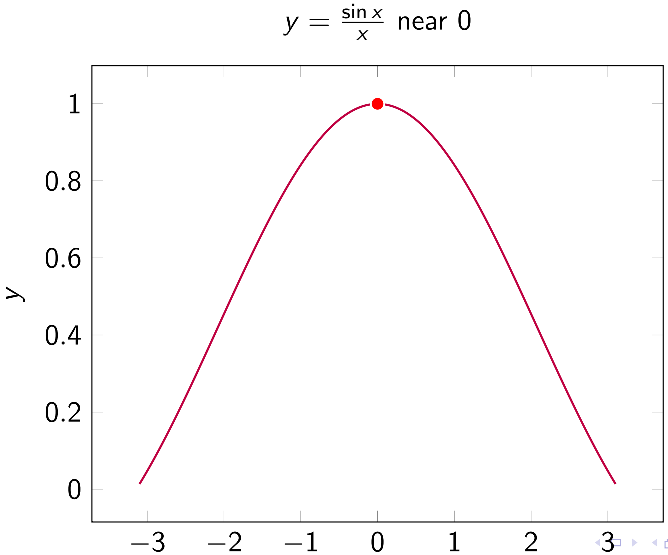
Graph of $y = e^x$



Graph of $y = x^3 - 3x$



Graph of $y = \frac{\sin x}{x}$ near 0



Summary

1. **Limits:** Defined using ε-δ; describes function behavior near a point
2. **Derivatives:** Defined as $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$; measures instantaneous rate of change
3. **Monotonicity:** $f'(x) > 0$ means increasing; $f'(x) < 0$ means decreasing
4. **Rules:** Sum, product, and quotient rules for efficient differentiation
5. **Common Derivatives:** Power rule, exponential, logarithmic, trigonometric functions

Homework Problem 1

Find intervals of increase/decrease for $f(x) = x^3 - 6x^2 + 9x$.

Homework Problem 2

Compute the following limits:

1. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$
2. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

Homework Problem 3

Differentiate:

1. $f(x) = e^x + x^3$
2. $f(x) = x^2 e^x$
3. $f(x) = \frac{1}{x^2+1}$

Thank You!

Questions?