Lecture 3: Properties of Estimators

Bias · Variance · MSE · Consistency · Sufficiency · Cramér–Rao

We use estimators every day. Are they any good?

We already use estimators (Lecture 1, plug-in principle):

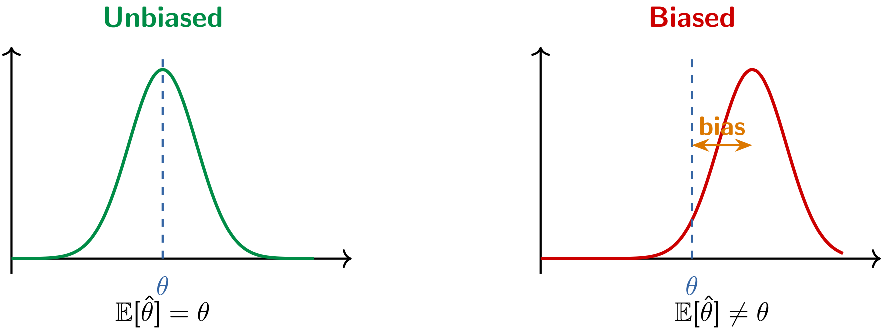
$\bar{X}$ for μ, S^2 for σ^2, $\hat{p} = \frac{\text{count}}{n}$ for p

But how do we **judge** an estimator?

Is it close to the truth? How much does it jump around? Can we do better?

Bias: Is the Estimator Centered on the Truth?

$$\text{Bias}(\hat{\theta}) \;=\; \mathbb{E}[\hat{\theta}] - \theta$$



If $\text{Bias}(\hat{\theta}) = 0$ for all θ, the estimator is **unbiased**.

Worked Example: Is $\bar{X}$ Unbiased for μ?

Let $X_1, \ldots, X_n$ be i.i.d. with $\mathbb{E}[X_i] = \mu$. Is $\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ unbiased?

Step 1: Compute $\mathbb{E}[\hat{\mu}]$:

$$\mathbb{E}[\bar{X}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}\cdot n\mu = \mu$$

Step 2: Check bias:

$$\text{Bias}(\bar{X}) = \mathbb{E}[\bar{X}] - \mu = \mu - \mu = 0 \quad \checkmark\ \text{Unbiased!}$$

Recipe for any estimator:
(1) Compute $\mathbb{E}[\hat{\theta}]$ $\rightarrow$ (2) Subtract the true θ $\rightarrow$ (3) If the result is 0, it's unbiased.

Worked Example: Why Dividing by n Is Biased

We want to estimate $\sigma^2 = \mathrm{Var}(X_i)$. Natural guess: $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$.

Trick: rewrite each $(X_i - \bar{X})$ by adding and subtracting the true mean μ:

$$X_i - \bar{X} = \underbrace{(X_i - \mu)}_{\text{deviation from truth}} - \underbrace{(\bar{X} - \mu)}_{\text{estimation error}}$$

Squaring and summing gives the **key identity**:

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n (X_i - \mu)^2 - n(\bar{X} - \mu)^2$$

Take expectations (using $\mathbb{E}[(X_i - \mu)^2] = \sigma^2$ and $\mathrm{Var}(\bar{X}) = \sigma^2/n$):

$$\mathbb{E}\left[\sum (X_i - \mu)^2\right] = n\sigma^2 \qquad (n \text{ terms, each } \sigma^2)$$

$$\mathbb{E}\left[n(\bar{X} - \mu)^2\right] = n \cdot \mathrm{Var}(\bar{X}) = n \cdot \frac{\sigma^2}{n} = \sigma^2 \qquad \text{(one ``lost'' degree of freedom)}$$

$$\Rightarrow \quad \mathbb{E}\left[\sum (X_i - \bar{X})^2\right] = n\sigma^2 - \sigma^2 = (n-1)\sigma^2$$

Bessel's Correction: The Fix

From the previous slide: $\mathbb{E}\left[\sum(X_i - \bar{X})^2\right] = (n-1)\sigma^2$, so:

$$\mathbb{E}[\hat{\sigma}_n^2] = \mathbb{E}\left[\frac{1}{n}\sum(X_i - \bar{X})^2\right] = \frac{(n-1)\sigma^2}{n} \neq \sigma^2 \quad \textbf{Biased!}$$

It **underestimates** by σ^2/n. Why? We used $\bar{X}$ instead of μ, "using up" one degree of freedom.

Bessel's correction: Divide by $n-1$ instead of n:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \mathbb{E}[S^2] = \sigma^2 \quad \checkmark \text{ Unbiased!}$$

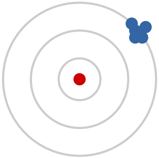
Intuition: We estimated μ from the same data, so the residuals $(X_i - \bar{X})$ are "too small" on average. Dividing by $n-1$ corrects for this.

Bias: Summary

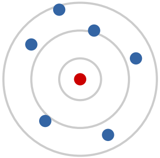
Estimator	**Bias**	**Unbiased?**
$\bar{X} = \frac{1}{n}\sum X_i$ for μ	0	**Yes**
$\hat{\sigma}_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$ for σ^2	$-\frac{\sigma^2}{n}$	**No**
$S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$ for σ^2	0	**Yes**
$\hat{p} = \frac{\sum X_i}{n}$ for p (Bernoulli)	0	**Yes**

Dividing by n instead of $n-1$ **underestimates** the true variance.
Bessel's correction ($n-1$) fixes this. Recall Lecture 2!

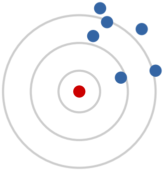
The Dartboard Analogy



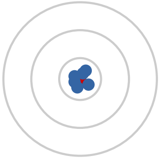
High bias, low var
Precise but inaccurate



Low bias, high var
Accurate but imprecise



High bias, high var
Worst of both worlds



Low bias, low var
The goal!

Bullseye = true θ. Blue dots = estimates from repeated samples.

Variance of an Estimator

The **variance** measures how much $\hat{\theta}$ wobbles across samples: $\mathrm{Var}(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2\right]$

Why is Var$(\bar{X}) = \sigma^2/n$ **and not** σ^2/n^2**?**

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \qquad \left(\frac{1}{n} \text{ comes out as } \frac{1}{n^2}\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) \qquad (\text{independent} \Rightarrow \text{variances } \textbf{add})$$

$$= \frac{1}{n^2} \cdot n\sigma^2 = \boxed{\frac{\sigma^2}{n}} \qquad (n \text{ terms cancel one } n)$$

$\mathrm{SE}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$ (**standard error** $= \sqrt{\mathrm{Var}}$)

Mean Squared Error: The Total Error

Bias tells us about the **aim**, variance about the **spread**. Can we combine them?

Mean Squared Error: $\text{MSE}(\hat{\theta}) \;=\; \mathbb{E}[(\hat{\theta} - \theta)^2]$

The average squared distance from the estimate to the truth.

The trick: add and subtract $\mathbb{E}[\hat{\theta}]$ to decompose the error:

$$\hat{\theta} - \theta = \underbrace{(\hat{\theta} - \mathbb{E}[\hat{\theta}])}_{\text{random fluctuation}} + \underbrace{(\mathbb{E}[\hat{\theta}] - \theta)}_{\text{bias (a constant!)}}$$

This splits the total error into two pieces: the **random part** (how much $\hat{\theta}$ moves around its own mean) and the **systematic part** (how far that mean is from the truth).

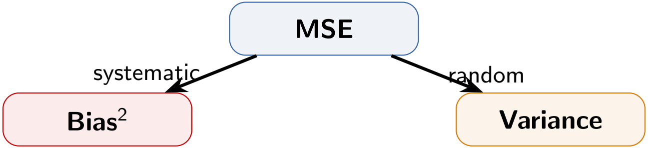
MSE $=$ Bias^2 $+$ Variance: The Proof

Now square $\hat{\theta} - \theta = (\hat{\theta} - \mathbb{E}[\hat{\theta}]) + (\mathbb{E}[\hat{\theta}] - \theta)$ and take expectations:

$$\mathbb{E}[(\hat{\theta} - \theta)^2] = \mathbb{E}\left[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2\right] + 2\,\underbrace{(\mathbb{E}[\hat{\theta}] - \theta)}_{\text{constant}} \cdot \underbrace{\mathbb{E}[\hat{\theta} - \mathbb{E}[\hat{\theta}]]}_{= 0 \text{ (always!)}} + (\mathbb{E}[\hat{\theta}] - \theta)^2$$

The cross term vanishes because $\hat{\theta} - \mathbb{E}[\hat{\theta}]$ has mean zero **by definition**.

$$\boxed{\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}(\hat{\theta}) + \mathrm{Bias}^2(\hat{\theta})}$$



Unbiased means MSE $=$ Var, but a biased estimator can still win if its variance is low enough.

When Biased Beats Unbiased

Example: Estimating σ^2 from $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.

Estimator	**Bias**	**Variance**	**MSE**
$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$	0	$\frac{2\sigma^4}{n-1}$	$\frac{2\sigma^4}{n-1}$
$\hat{\sigma}_n^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$	$-\frac{\sigma^2}{n}$	$\frac{2(n-1)\sigma^4}{n^2}$	$\frac{(2n-1)\sigma^4}{n^2}$

Compare: $\frac{2n-1}{n^2}$ vs $\frac{2}{n-1}$ $\Rightarrow$ $\hat{\sigma}_n^2$ has **lower MSE** for all $n \geq 2$!

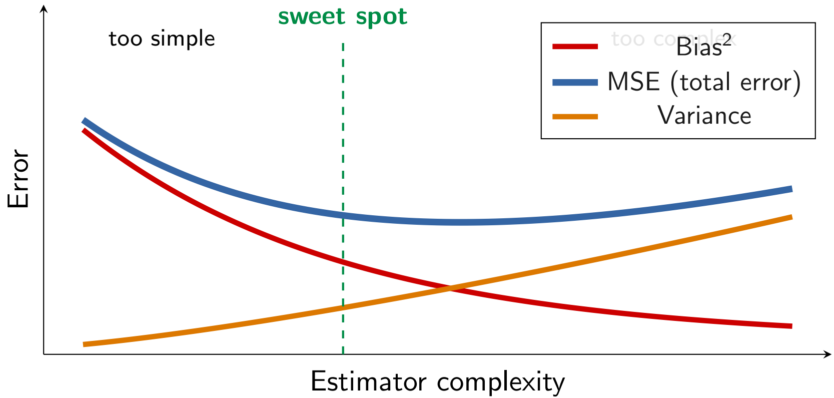
The biased estimator beats the unbiased S^2 because its variance reduction outweighs the small bias.

The Bias-Variance Tradeoff

You can't minimize bias and variance at the same time.

How do we find the **sweet spot**?

The Bias-Variance Tradeoff



Bias-Variance in Machine Learning

This tradeoff is **everywhere** in ML — it's the same principle in different disguises:

Setting	**Too simple (high bias)**	**Too complex (high var)**
Polynomial regression	Degree 1 (line)	Degree 20 (wiggly)
KNN	Large k (oversmoothed)	$k = 1$ (memorizes noise)
Decision tree	Shallow tree (underfits)	Deep tree (overfits)
Neural network	Too few neurons	Too many neurons
Regularization	Strong penalty (λ large)	No penalty ($\lambda = 0$)

Key insight: In all these cases, the total error (MSE, test loss) is minimized at an intermediate complexity. This is why we need **cross-validation**, **regularization**, and **held-out test sets** — to find the sweet spot empirically.

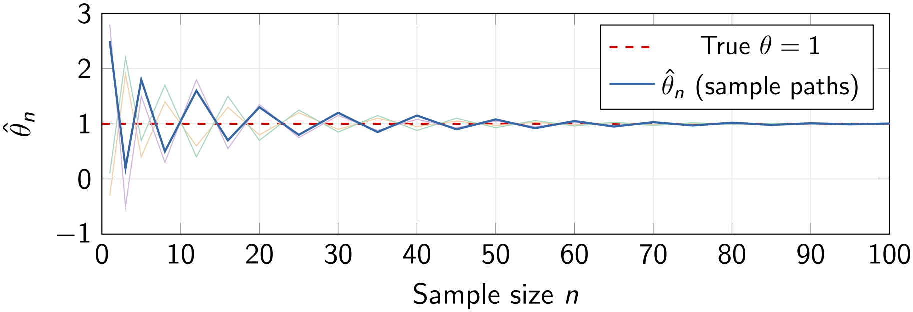
Consistency

Does our estimator converge to the truth
as we collect more and more data?

Consistency: Getting It Right Eventually

An estimator $\hat{\theta}_n$ is **consistent** if it converges to the truth as $n \to \infty$:

$$\hat{\theta}_n \xrightarrow{P} \theta \qquad \text{i.e.,} \quad \Pr\left(|\hat{\theta}_n - \theta| > \varepsilon\right) \to 0 \ \text{ for all } \varepsilon > 0$$



Consistent vs Inconsistent: A Contrast

Consistent: $\hat{\mu} = \bar{X}_n$

- $\mathbb{E}[\bar{X}_n] = \mu$ (unbiased)
- $\mathrm{Var}(\bar{X}_n) = \sigma^2/n \to 0$
- Uses **all** n observations
- More data $\Rightarrow$ more precise

Not consistent: $\tilde{\mu} = X_1$

- $\mathbb{E}[X_1] = \mu$ (also unbiased!)
- $\mathrm{Var}(X_1) = \sigma^2$ (constant!)
- Uses **only** the first observation
- Ignores all other data forever

Unbiased $\neq$ consistent. X_1 is unbiased but NOT consistent.
Consistent $\neq$ unbiased. $\hat{\sigma}_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$ is biased but IS consistent (because its bias $\to$ 0 and its variance $\to$ 0).

Sufficient Conditions for Consistency

Chebyshev's inequality gives us a concrete tool:

$$\Pr\left(|\hat{\theta}_n - \theta| \geq \varepsilon\right) \leq \frac{\mathbb{E}[(\hat{\theta}_n - \theta)^2]}{\varepsilon^2} = \frac{\mathrm{MSE}(\hat{\theta}_n)}{\varepsilon^2} = \frac{\mathrm{Bias}^2 + \mathrm{Var}}{\varepsilon^2}$$

$\mathrm{Bias}(\hat{\theta}_n) \to 0$ as $n \to \infty$

$\mathrm{Var}(\hat{\theta}_n) \to 0$ as $n \to \infty$

$\Downarrow$

$\mathrm{MSE} \to 0 \;\Rightarrow\; \Pr(|\hat{\theta}_n - \theta| \geq \varepsilon) \to 0 \;\Rightarrow$ **consistent!**

Example: $\bar{X}_n$ is consistent for μ: $\mathrm{Bias} = 0$, $\mathrm{Var} = \sigma^2/n \to 0$, so $\Pr(|\bar{X}_n - \mu| \geq \varepsilon) \leq \sigma^2/(n\varepsilon^2) \to 0$.

This is precisely the **(Weak) Law of Large Numbers**: $\bar{X}_n \xrightarrow{P} \mu$.

Sufficiency

We have n data points. Do we really need **all** of them?

Can we **compress** without losing information?

Sufficiency: Can We Compress the Data?

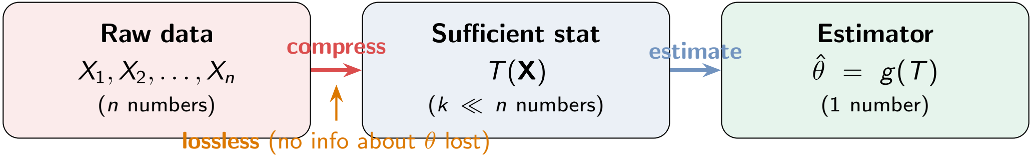
Example: $X_1, \ldots, X_n \sim \text{Bern}(p)$. To estimate p:

- We only need $T = \sum X_i$ (total number of successes)
- The specific order (HHTHT vs THHTH) tells us nothing more about p

Definition: A statistic $T(\mathbf{X})$ is **sufficient** for θ if
the conditional distribution of $\mathbf{X} \mid T(\mathbf{X})$ does not depend on θ.

Intuition: Once you know T, the remaining randomness in the data is just noise — it carries **no information** about θ. T is a "lossless summary."

Sufficiency as Data Compression



Example **Bernoulli**

$$0, 1, 1, 0, 1, 1, 1, 0, 1, 0 \longrightarrow T = \sum X_i = 6 \longrightarrow \hat{p} = 6/10 = 0.6$$

The order $(0, 1, 1, 0, 1, \ldots)$ doesn't matter for estimating p — only the **total count** matters.

How to Check: Fisher–Neyman Factorization

Theorem: $T(\mathbf{X})$ is sufficient for θ if and only if:

$$f(\mathbf{x} \mid \theta) = g(T(\mathbf{x}),\ \theta) \cdot h(\mathbf{x})$$

where g depends on the data **only through** T, and h does not depend on θ.

Bernoulli worked example: $X_1, \ldots, X_n \sim \text{Bern}(p)$, let $T = \sum X_i$.

$$f(\mathbf{x} \mid p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = \underbrace{p^{\sum x_i}(1-p)^{n-\sum x_i}}_{g(T,\ p)} \cdot \underbrace{1}_{h(\mathbf{x})}$$

Model	Sufficient statistic	Intuition
$\text{Bern}(p)$	$T = \sum X_i$	1 number for 1 parameter
$N(\mu, \sigma_0^2)$ (σ_0^2 known)	$T = \bar{X}$	1 number for 1 parameter
$N(\mu, \sigma^2)$ (both unknown)	$T = (\bar{X},\ S^2)$	2 numbers for 2 parameters

Minimal Sufficiency

The full data $\mathbf{X}$ is always trivially sufficient. But can we compress **further**?

> A sufficient statistic is **minimal** if it is a function of every other sufficient statistic.
>
> It achieves the **maximum compression** without losing information about θ.

Example: For $X_1, \dots, X_n \sim N(\mu, \sigma_0^2)$ with σ_0^2 known:

$\mathbf{X} = (X_1, \dots, X_n)$	$\xrightarrow{\text{compress}}$	$(\bar{X}, S^2)$	$\xrightarrow{\text{compress more}}$	$\bar{X}$ alone
sufficient, no compression		sufficient, partial		sufficient **and minimal**

Since only μ is unknown, S^2 carries no extra information — $\bar{X}$ alone is enough.

The Rao–Blackwell Theorem

Why does sufficiency matter for estimation? Because it lets us **improve** any estimator:

Rao–Blackwell Theorem: Given *any* unbiased estimator $\tilde{\theta}$ and a sufficient statistic T, define $\hat{\theta} = \mathbb{E}[\tilde{\theta} \mid T]$. Then:

(1) $\hat{\theta}$ is still **unbiased**: $\mathbb{E}[\hat{\theta}] = \mathbb{E}[\tilde{\theta}] = \theta$

(2) $\hat{\theta}$ has **lower or equal variance**: $\mathrm{Var}(\hat{\theta}) \leq \mathrm{Var}(\tilde{\theta})$

Conditioning on a sufficient statistic **never hurts, often helps**.

Worked example: $X_1, \ldots, X_n \sim \mathrm{Bern}(p)$, sufficient stat $T = \sum X_i$.

$$\underbrace{\tilde{p} = X_1}_{\text{naive: unbiased, Var}=p(1-p)} \xrightarrow{\mathbb{E}[\,\cdot\,|\,T\,]} \underbrace{\hat{p} = \mathbb{E}[X_1 \mid T] = T/n = \bar{X}}_{\text{improved: unbiased, Var}=p(1-p)/n}$$

×n better!

Finding Minimal Sufficient Statistics

Theorem (Likelihood Ratio Criterion): $T(\mathbf{X})$ is minimal sufficient iff for all $\mathbf{x}, \mathbf{y}$:

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \Longleftrightarrow \quad \frac{f(\mathbf{x} \mid \theta)}{f(\mathbf{y} \mid \theta)} \text{ does not depend on } \theta$$

Bernoulli example: $X_1, \ldots, X_n \sim \mathrm{Bern}(p)$.

$$\frac{f(\mathbf{x} \mid p)}{f(\mathbf{y} \mid p)} = \frac{p^{\sum x_i}(1-p)^{n-\sum x_i}}{p^{\sum y_i}(1-p)^{n-\sum y_i}} = \left(\frac{p}{1-p}\right)^{\sum x_i - \sum y_i}$$

Free of $p \iff \sum x_i = \sum y_i$. So $T = \sum X_i$ is **minimal sufficient** for p. ✓

Recipe: Write the likelihood ratio $f(\mathbf{x} \mid \theta)/f(\mathbf{y} \mid \theta)$.
Find which function of the data must match for the ratio to lose its θ-dependence.
That function is the minimal sufficient statistic.

The Exponential Family: A Unifying Framework

All our examples — Bernoulli, Normal, Poisson, Exponential — share one structure:

$$\boxed{f(x \mid \theta) = h(x)\ \exp\Big(\eta(\theta)\, T(x) - A(\theta)\Big)}$$

Distribution	**Natural param** $\eta(\theta)$	$T(x)$	**Suff. stat** (n obs)
Bern(p)	$\log \frac{p}{1-p}$	x	$\sum X_i$
$N(\mu, \sigma_0^2)$ (σ_0^2 known)	μ/σ_0^2	x	$\sum X_i$
Pois(λ)	$\log \lambda$	x	$\sum X_i$
Exp(λ)	$-\lambda$	x	$\sum X_i$

Pattern: For single-parameter families, $T(x) = x$. The sufficient statistic for n observations is always $\sum T(X_i)$ — straight from the factorization theorem!

Why Exponential Families Are Special

Nearly every nice property we've discussed is **automatic** in exponential families:

Sufficiency: $T(\mathbf{X}) = \sum T(X_i)$ is sufficient **and minimal**

Completeness: the natural sufficient statistic is **complete** (see below)

Regularity: all conditions for the Cramér–Rao bound (coming soon) are satisfied

Optimal estimators exist: we'll see this when we reach the CR bound

Completeness means: if $\mathbb{E}_\theta[g(T)] = 0$ for all θ, then $g(T) = 0$ a.s. $\rightarrow$ **no non-trivial unbiased estimator of zero** based on T.

Lehmann–Scheffé: An unbiased estimator based on a **complete** sufficient statistic is the **unique best** unbiased estimator (UMVUE). For exp. families, $\sum T(X_i)$ is always complete $\Rightarrow$ UMVUE exists!

Can We Do Better? The Fundamental Question

We know $\mathrm{Var}(\bar{X}) \;=\; \sigma^2/n$ for estimating the mean.

Can **any** unbiased estimator have **lower** variance?

Or is $\bar{X}$ already the best we can do?

To answer this, we need to measure **how much information** one observation carries about θ.

Roadmap:

Why log? $\rightarrow$ **Score function** (sensitivity of the model to θ) $\rightarrow$ **Fisher information** $\rightarrow$ **Cramér–Rao bound** (the variance floor)

Why the Logarithm? From Products to Sums

The likelihood for i.i.d. data is a **product**:

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

Taking the log turns this into a **sum**:

$$\ell(\theta) = \log L(\theta) = \sum_{i=1}^{n} \log f(X_i \mid \theta)$$

Products are painful:

- Multiplying tiny numbers $\rightarrow$ underflow
- Product rule for derivatives is messy
- Hard to work with analytically

Sums are friendly:

- Numerically stable
- Derivative of a sum $=$ sum of derivatives
- LLN, CLT apply directly

Key fact: log is monotonically increasing, so
$\arg\max_\theta L(\theta) = \arg\max_\theta \ell(\theta)$. Same maximizer!

The Score Function: How Sensitive Is the Model?

Given a model $f(x \mid \theta)$, the **score** measures how the log-probability changes with θ:

$$s(\theta) = \frac{\partial}{\partial \theta} \log f(X \mid \theta)$$

Concrete example: $X \sim \text{Bernoulli}(p)$.
$\log f(x \mid p) = x \log p + (1-x) \log(1-p)$

$$s(p) = \frac{\partial}{\partial p} [x \log p + (1-x) \log(1-p)] = \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)}$$

- If we observe $x = 1$ and p is small, the score is **large positive** $\rightarrow$ "p should be higher"
- If we observe $x = 0$ and p is large, the score is **large negative** $\rightarrow$ "p should be lower"
- On average: $\mathbb{E}[s(p)] = 0$ — the score points in the right direction but **averages out**

Fisher Information: How Informative Is One Observation?

The score averages to zero, but it **varies**. More variation = more information:

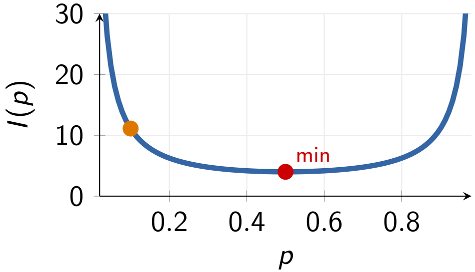
$$\boxed{I(\theta) = \mathrm{Var}[s(\theta)] = \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X \mid \theta)\right)^2\right]}$$

Bernoulli derivation: We found $s(p) = \frac{X-p}{p(1-p)}$.

Since $\mathbb{E}[s] = 0$:

$$I(p) = \mathbb{E}[s^2] = \mathbb{E}\left[\frac{(X-p)^2}{p^2(1-p)^2}\right]$$

$$= \frac{\mathrm{Var}(X)}{p^2(1-p)^2} = \frac{p(1-p)}{p^2(1-p)^2} = \boxed{\frac{1}{p(1-p)}}$$



p near 0 or 1: very informative. $p = 0.5$: max noise, min info.

Fisher Information: Two Equivalent Forms

Under regularity conditions, there is an equivalent formula that's often easier to compute:

$$\boxed{I(\theta) = \mathbb{E}\left[s(\theta)^2\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X \mid \theta)\right]}$$

Why are these the same? Start from $\mathbb{E}[s(\theta)] = 0$ and differentiate both sides w.r.t. θ:

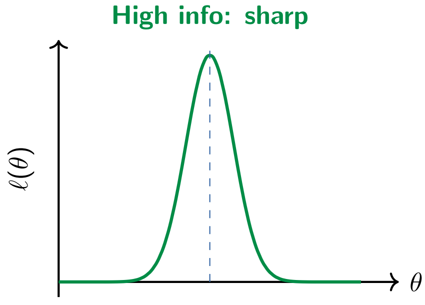
$$0 = \frac{\partial}{\partial\theta}\mathbb{E}[s] = \mathbb{E}\left[\frac{\partial s}{\partial\theta}\right] + \mathbb{E}[s \cdot s] = \mathbb{E}[\ell''] + \mathbb{E}[s^2]$$

So: $\mathbb{E}[s^2] = -\mathbb{E}[\ell'']$. ✓

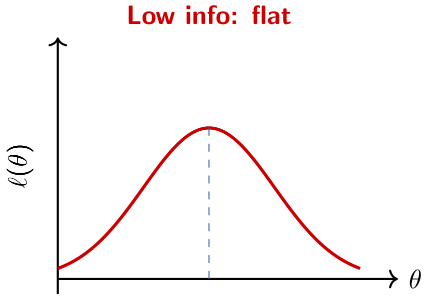
Verify for Bernoulli: $\ell(p) = x\log p + (1-x)\log(1-p)$

$$\ell''(p) = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2} \quad \Rightarrow \quad -\mathbb{E}[\ell''] = \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)} \quad ✓$$

Intuition: Sharp vs Flat Log-Likelihood



Precise estimate



Uncertain estimate

$I(\theta)$ measures the **curvature** of the log-likelihood at the true θ.

Sharp curve $\Rightarrow$ high $I(\theta)$ $\Rightarrow$ data is very informative $\Rightarrow$ estimator is precise.

This connects the two forms: $I(\theta) = -\mathbb{E}[\ell'']$ is literally the expected curvature.

Cramér–Rao Lower Bound

Now we can answer the fundamental question. For any **unbiased** estimator $\hat{\theta}$ based on n i.i.d. observations:

$$\boxed{\mathrm{Var}(\hat{\theta}) \;\geq\; \frac{1}{n \cdot I(\theta)}}$$

Intuition: Why $\frac{1}{n \cdot I(\theta)}$?

- **More observations (n large)** $\Rightarrow$ bound gets smaller $\Rightarrow$ can estimate more precisely
- **More informative data ($I(\theta)$ large)** $\Rightarrow$ bound gets smaller $\Rightarrow$ each observation tells us more
- The bound is **tight** for many models — it's the actual achievable precision

Verify for Bernoulli:

$$I(p) = \frac{1}{p(1-p)} \quad \Rightarrow \quad \text{CR bound: } \mathrm{Var}(\hat{p}) \geq \frac{1}{n \cdot \frac{1}{p(1-p)}} = \frac{p(1-p)}{n}$$

Actual variance of $\hat{p} = \bar{X}$: $\mathrm{Var}(\hat{p}) = \frac{p(1-p)}{n}$ ✓ Hits the bound exactly!

Cramér–Rao: Efficiency and Practical Use

What it says:
There is a **floor** on
how precise any unbi-
ased estimator can be

Efficient estimator:
Achieves the bound —
the **best possible**

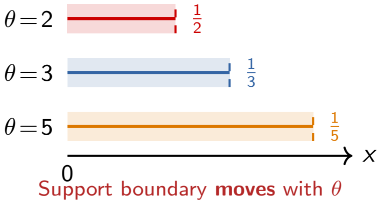
Practical use:
Tells you whether to keep
searching for a
better estimator

Model	**Estimator**	$\mathrm{Var}(\hat{\theta})$	**CR bound**	**Efficient?**
$\mathrm{Bern}(p)$	$\hat{p} = \bar{X}$	$\frac{p(1-p)}{n}$	$\frac{p(1-p)}{n}$	**Yes**
$N(\mu, \sigma_0^2)$	$\hat{\mu} = \bar{X}$	$\frac{\sigma_0^2}{n}$	$\frac{\sigma_0^2}{n}$	**Yes**
$\mathrm{Exp}(\lambda)$	$\hat{\lambda} = 1/\bar{X}$	$\frac{\lambda^2}{n}$	$\frac{\lambda^2}{n}$	**Yes**

Regularity Conditions: When Does CR Apply?

The Cramér–Rao bound requires **regularity conditions**:

1. **Support** of $f(x \mid \theta)$ doesn't depend on θ
2. θ in the **interior** of the parameter space
3. Can **differentiate under the integral** sign (swap $\frac{\partial}{\partial\theta}$ and $\int$)
4. $0 < I(\theta) < \infty$ (finite, positive info)



Counterexample: Uniform$(0, \theta)$

- Support $[0, \theta]$ depends on θ! (violates #1)
- Suff. stat: $X_{(n)} = \max_i X_i$
- $\mathrm{Var}(X_{(n)}) \sim 1/n^2$ — **faster** than CR!
  (CR would give $1/n$, but $1/n^2$ is possible here)

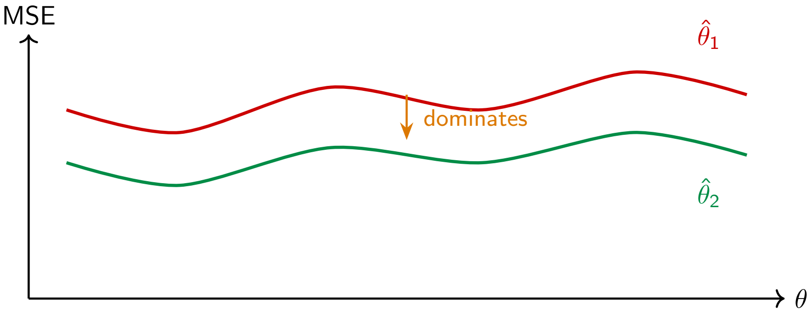
Good news: All exponential family distributions automatically satisfy
the regularity conditions. The CR bound always applies to them.

Admissibility

Definition: $\hat{\theta}_1$ is **inadmissible** if $\exists\, \hat{\theta}_2$ that **dominates** it:

$$\text{MSE}(\hat{\theta}_2, \theta) \leq \text{MSE}(\hat{\theta}_1, \theta) \;\; \forall\, \theta, \quad \text{with strict inequality for some } \theta$$

An estimator is **admissible** if no other estimator dominates it.



$\hat{\theta}_1$ is **inadmissible** — $\hat{\theta}_2$ is at least as good everywhere, and strictly better somewhere.

Stein's Paradox (1956)

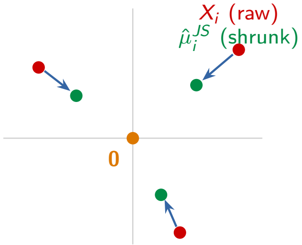
Surprising fact:

When estimating $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ from $X_i \sim N(\mu_i, 1)$,
the sample mean $\hat{\mu}_i = X_i$ is **inadmissible** when $d \geq 3$!

The **James–Stein estimator** dominates it:

$$\hat{\mu}_i^{JS} = \left(1 - \frac{d-2}{\|\mathbf{X}\|^2}\right) X_i$$

- **Shrinks** each X_i toward 0
- Works even if μ_i's are unrelated!
- A little bias buys a lot of variance reduction



Paradox: estimating the average temperature in Yerevan *improves* if you jointly estimate it with the price of tea in China and the height of the Eiffel Tower.

Why Does Stein's Paradox Work?

The MSE comparison tells the whole story:

$\mathrm{MSE}(\mathbf{X}) = d$
(1 per coordinate)

$\xrightarrow{\text{shrinkage helps}}$

$\mathrm{MSE}(\hat{\boldsymbol{\mu}}^{JS}) < d$
(always, when $d \geq 3$)

Why $d \geq 3$? The shrinkage factor $\frac{d-2}{\|\mathbf{X}\|^2}$ needs to be estimated from data.

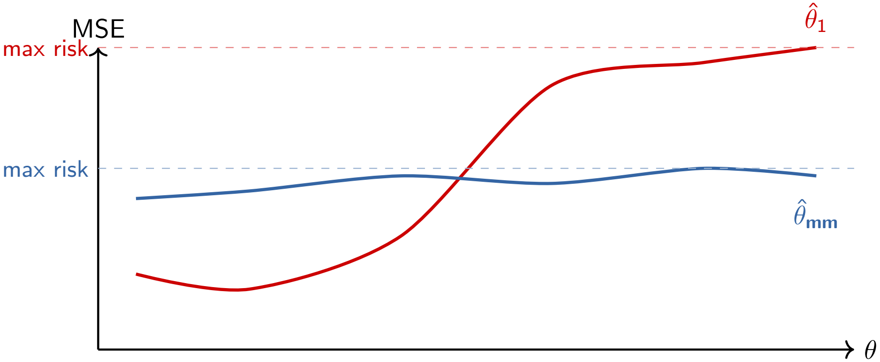
- In $d = 1$ or 2: not enough "room" — estimation error of the shrinkage factor wipes out the gain
- In $d \geq 3$: $\|\mathbf{X}\|^2$ concentrates well enough $\rightarrow$ shrinkage factor is accurate $\rightarrow$ net win

Connection to ML: James–Stein shrinkage is an early form of **regularization**.
Ridge regression (L^2 penalty) does the same thing: shrink coefficients toward zero. The bias-variance tradeoff in action: a little bias buys a lot of variance reduction.

Minimax Estimators

A **minimax** estimator minimizes the **worst-case** risk:

$$\hat{\theta}_{\text{minimax}} = \arg\min_{\hat{\theta}} \max_{\theta} \text{MSE}(\hat{\theta}, \theta)$$



Minimax = **conservative**: protects against the worst θ. Minimax hedges.

Three Philosophies of Estimation

Plug-in (unbiased)	Shrinkage	Minimax
Use sample statistic directly ($\bar{X}$, S^2, $\hat{p}$) Admissible in $d = 1$ Inadmissible in $d \geq 3$	Pull estimates toward a central value (e.g. 0) Biased but lower MSE (James–Stein)	Minimize worst-case risk Conservative guarantee No single θ can hurt you badly

Takeaway: In high dimensions ($d \geq 3$), shrinkage estimators are provably better
than using each sample statistic on its own. We'll see more of this in later lectures.

What We Haven't Covered (Yet)

This lecture focused on **point estimation** — producing a single "best guess" for θ. But there's much more to statistical inference:

Confidence intervals: How uncertain is our estimate? (Lectures 5–6)

Hypothesis testing: Is the effect real or just noise? (Lectures 7–8)

Bayesian estimation: Incorporating prior beliefs (Lecture 5)

Bootstrap: Resampling to estimate uncertainty without formulas (Lecture 6)

Asymptotic theory: What happens as $n \to \infty$ in general? (Lecture 5)

Nonparametric estimation: What if we don't assume a distribution at all?

Today's tools (bias, MSE, CR bound, sufficiency) will be the **foundation** for all of these.

Summary: How to Judge an Estimator

Bias: $\mathbb{E}[\hat{\theta}] - \theta$. Does it aim at the right place?

Variance: $\mathrm{Var}(\hat{\theta})$. How much does it jump around?

MSE $= \mathrm{Bias}^2 + \mathrm{Var}$. Total error. Biased can beat unbiased!

Consistency: $\hat{\theta}_n \xrightarrow{P} \theta$. Converges to truth with enough data.

Sufficiency: $T(\mathbf{X})$ captures everything about θ. Compress without loss.

Cramér–Rao: $\mathrm{Var} \geq 1/(n \cdot I(\theta))$. The efficiency floor.

Admissibility: No other estimator dominates it everywhere.

Minimax: Best worst-case guarantee. Shrinkage often wins.

Homework

1. Show that $\bar{X}$ is unbiased for μ and compute its MSE.
2. Show that $\hat{\sigma}_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$ is biased for σ^2. Find the bias.
3. Compute the Fisher information $I(\theta)$ for Poisson(λ).
   Use it to find the Cramér–Rao lower bound for estimating λ.
   Is $\hat{\lambda} = \bar{X}$ efficient?
4. Suppose you shrink $\bar{X}$ toward 0: $\hat{\mu}_c = c\bar{X}$ for $0 < c < 1$.
   Find the bias, variance, and MSE as functions of c.
   For what value of c is MSE minimized? Is the optimal estimator biased?
5. Use the factorization theorem to show that $T = \sum X_i$ is a sufficient statistic for λ when $X_1, \ldots, X_n \sim$ Poisson(λ).

Questions?