Mathematics 1 – Comprehensive Cheat Sheet

1. Summation and Product Notation

Definitions:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n \qquad \prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot \cdots \cdot a_n$$

Important Sums:

Name	Formula
Arithmetic	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Sum of squares	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
Geometric	$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ for $r \neq 1$
Constant	$\sum_{i=1}^{n} c = n \cdot c$

Properties: $\sum(a_i + b_i) = \sum a_i + \sum b_i$; $\sum c \cdot a_i = c \cdot \sum a_i$; $\prod(a_i \cdot b_i) = \prod a_i \cdot \prod b_i$

Template: *To convert to summation notation: (1) Identify the pattern, (2) Write general term a_i as function of index, (3) Determine start/end indices.*

2. Functions and Graph Transformations

Transform	Effect	Transform	Effect
$f(x) + c$	Shift UP by c	$c \cdot f(x)$	Vertical stretch by c
$f(x) - c$	Shift DOWN by c	$f(c \cdot x)$	Horizontal compress by c
$f(x + c)$	Shift LEFT by c	$-f(x)$	Reflect across x-axis
$f(x - c)$	Shift RIGHT by c	$f(-x)$	Reflect across y-axis

Key Concepts: Domain D_f = valid inputs; Image/Range W_f = actual outputs; Codomain = all possible outputs.

3. Exponential and Logarithmic Functions

3.1 Exponential Rules

$$e^{a+b} = e^a \cdot e^b \qquad e^{a-b} = \frac{e^a}{e^b} \qquad (e^a)^b = e^{ab} \qquad e^0 = 1 \qquad e^{\ln x} = x$$

3.2 Logarithm Rules

$$\ln(ab) = \ln a + \ln b \qquad \ln\left(\frac{a}{b}\right) = \ln a - \ln b \qquad \ln(a^n) = n \ln a$$

$$\ln 1 = 0 \qquad \ln e = 1 \qquad \ln(e^x) = x$$

Change of Base: $\log_a x = \frac{\ln x}{\ln a} = \frac{\log_b x}{\log_b a}$

Special: $\log_a a = 1$; $\log_a 1 = 0$; $\log_a a^n = n$; $a^{\log_a x} = x$

Solving: *For $e^{f(x)} = c$: take ln to get $f(x) = \ln c$. For $\ln(f(x)) = c$: exponentiate to get $f(x) = e^c$. Always check domain!*

4. Polynomial Long Division

Algorithm: (1) Arrange in descending powers, (2) Divide leading terms, (3) Multiply and subtract, (4) Repeat until deg(remainder) < deg(divisor).

Result: $P(x) = Q(x) \cdot D(x) + R(x)$

Applications: Finding roots of polynomials, simplifying rational functions, partial fraction decomposition.

5. Inverse Functions

Definition: f^{-1} is the inverse of f iff: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Existence: Function must be *bijective* (one-to-one AND onto). For continuous functions, *strictly monotonic* $\Rightarrow$ invertible.

Domain/Range swap: $\text{Dom}(f^{-1}) = \text{Range}(f)$; $\text{Range}(f^{-1}) = \text{Dom}(f)$.

Graphically: Graph of f^{-1} is reflection of f across line $y = x$.

Template: *(1) Write $y = f(x)$, (2) Swap x and y, (3) Solve for y, (4) Write $f^{-1}(x) = y$, (5) Determine domain.*

6. Derivatives – Basic Rules

Function	Derivative	Function	Derivative
c (constant)	0	$\sin x$	$\cos x$
x	1	$\cos x$	$-\sin x$
x^n	nx^{n-1}	$\ln x$	$1/x$
e^x	e^x	$\log_a x$	$\frac{1}{x \ln a}$
a^x	$a^x \ln a$	$\|x\|$	$\text{sgn}(x) = \frac{x}{\|x\|}$

6.1 Differentiation Rules

Rule	Formula
Sum/Difference	$(f \pm g)' = f' \pm g'$
Constant Multiple	$(cf)' = cf'$
Product Rule	$(fg)' = f'g + fg'$
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
Chain Rule	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Newton Quotient: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

7. Chain Rule and Composite Functions

Chain Rule: $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ or $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

7.1 Common Patterns

Function	Derivative
$e^{g(x)}$	$e^{g(x)} \cdot g'(x)$
$\ln(g(x))$	$\frac{g'(x)}{g(x)}$
$[g(x)]^n$	$n[g(x)]^{n-1} \cdot g'(x)$
$f(ax+b)$	$a \cdot f'(ax+b)$
$[g(x)]^{h(x)}$	$[g(x)]^{h(x)} \left[h'(x) \ln g(x) + h(x) \frac{g'(x)}{g(x)}\right]$

7.2 Maximum Location Under Transformations

If $f(x)$ has maximum at x^*:

New Function	Max Location
$f(x) + c$	x^* (unchanged)
$f(x+c)$	$x^* - c$
$c \cdot f(x)$, $c > 0$	x^* (unchanged)
$f(cx)$, $c > 0$	x^*/c
$h(f(x))$, h increasing	x^* (unchanged)
$f(h(x))$, h increasing	$h^{-1}(x^*)$

8. Convexity and Concavity

Convex (concave up): $f''(x) > 0$ – "holds water", "smiling"

Concave (concave down): $f''(x) < 0$ – "sheds water", "frowning"

Inflection point: Where $f''(x) = 0$ AND concavity changes.

Formal Definition: f is convex on I if for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Key: *Convex functions: local min = global min. Concave functions: local max = global max.*

9. Sequences and Series

Type	Closed Form	Recursive
Arithmetic	$s_n = a + (n-1)d$	$s_{n+1} = s_n + d$
Geometric	$s_n = ar^{n-1}$	$s_{n+1} = r \cdot s_n$

Series Formulas:

- Arithmetic (finite): $\displaystyle\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2} = \frac{n}{2}(2a + (n-1)d)$
- Geometric (finite): $\displaystyle\sum_{i=0}^{n} ar^i = a \cdot \frac{1 - r^{n+1}}{1 - r}$
- Geometric (infinite, $|r| < 1$): $\displaystyle\sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r}$

10. Mathematical Induction

Step 1 (Base Case): Prove $P(n_0)$ is true.
Step 2 (Inductive Step): Assume $P(k)$ is true (inductive hypothesis). Then prove $P(k+1)$ is true.
Conclusion: By induction, $P(n)$ holds for all $n \geq n_0$.

11. L'Hôpital's Rule and Limits

L'Hôpital's Rule: If $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(provided the right side exists)

11.1 Indeterminate Forms

Form	Strategy
$\frac{0}{0}$ or $\frac{\infty}{\infty}$	L'Hôpital directly
$0 \cdot \infty$	Rewrite as $\frac{0}{1/\infty}$ or $\frac{\infty}{1/0}$
$\infty - \infty$	Combine fractions or factor
1^∞, 0^0, ∞^0	Take logarithm first

Important Limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty \qquad \lim_{x \to \infty} \frac{x^n}{e^x} = 0 \qquad \lim_{x \to \infty} x^n e^{-x} = 0$$

12. Curve Sketching – Systematic Approach

Step	Find	Method
1. Domain	Where $f(x)$ defined	Check denominators, roots, logs
2. Intercepts	y: $f(0)$; x: $f(x) = 0$	Direct evaluation
3. Symmetry	Even/Odd	$f(-x) =?$
4. Asymptotes	Vertical, Horizontal, Oblique	Limits at boundaries
5. $f'(x)$	Critical pts, inc/dec	$f' = 0$, sign analysis
6. $f''(x)$	Inflection pts, convexity	$f'' = 0$, sign analysis
7. Extrema	Max/min classification	$f' = 0$ with f'' test

Example – Logistic Function: $S(t) = \frac{1}{1+e^{-\mu t}}$ with $\mu > 0$
Domain: all $\mathbb{R}$; Limits: $\lim_{t \to -\infty} S(t) = 0$, $\lim_{t \to \infty} S(t) = 1$; Always increasing (S-shaped); Inflection at $t = 0$.

13. Taylor Expansion

Taylor's Formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

Remainder (Lagrange): $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ for some ξ between a and x.

13.1 Maclaurin Series ($a = 0$)

Function	Expansion
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
$(1+x)^n$	$1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

Linear: $f(x) \approx f(a) + f'(a)(x-a)$ **Quadratic:** $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

14. Elasticities

Definition:

$$\varepsilon_f = \frac{x}{f(x)} \cdot f'(x) = \frac{d(\ln f)}{d(\ln x)}$$

Interpretation: Percent change in f per percent change in x.
Rules:

Operation	Elasticity Rule
Product: $f \cdot g$	$\varepsilon_{f\cdot g} = \varepsilon_f + \varepsilon_g$
Quotient: f/g	$\varepsilon_{f/g} = \varepsilon_f - \varepsilon_g$
Power: f^n	$\varepsilon_{f^n} = n \cdot \varepsilon_f$

Economics: $|\varepsilon| > 1$: elastic; $|\varepsilon| < 1$: inelastic; $|\varepsilon| = 1$: unit elastic.

15. Implicit Differentiation

Method: Given $F(x, y) = 0$ where $y = y(x)$:

1. Differentiate both sides with respect to x
2. Apply chain rule: $\frac{d}{dx}[g(y)] = g'(y) \cdot \frac{dy}{dx}$
3. Solve for $\frac{dy}{dx}$

Formula: If $F(x, y) = 0$, then:

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

Second Derivative: Differentiate $\frac{dy}{dx}$ implicitly again, substituting the first derivative.

16. Optimization (Single Variable)

Necessary condition: $f'(x^*) = 0$ (critical point)
Second Derivative Test:

Condition	Conclusion
$f''(x^*) > 0$	Local minimum
$f''(x^*) < 0$	Local maximum
$f''(x^*) = 0$	Inconclusive

Global Extrema on $[a, b]$: (1) Find critical points in (a, b), (2) Evaluate f at critical points AND endpoints, (3) Compare values.
Extreme Value Theorem: If f is continuous on $[a, b]$, then f attains both a max and min on $[a, b]$.

17. Multivariate Functions

Partial Derivatives:

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{(treat } y \text{ as constant)}$$

Second Partials: f_{xx}, f_{yy}, f_{xy}, f_{yx}
Schwarz's Theorem: If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.
Gradient: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ – perpendicular to level curves.
Level Curves: Set of points where $f(x,y) = c$ (constant).

18. Multivariate Optimization

First-Order Conditions: $f_x(x^*,y^*) = 0$ AND $f_y(x^*,y^*) = 0$
Hessian Matrix:

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Determinant: $D = \det(H) = f_{xx} \cdot f_{yy} - (f_{xy})^2$
Second-Order Conditions:

D	f_{xx}	**Conclusion**
$D > 0$	$f_{xx} > 0$	Local minimum
$D > 0$	$f_{xx} < 0$	Local maximum
$D < 0$	any	Saddle point
$D = 0$	any	Inconclusive

19. Lagrange Multipliers

Problem: Optimize $f(x,y)$ subject to $g(x,y) = c$
Lagrangian:

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$$

First-Order Conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = f_x - \lambda g_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = f_y - \lambda g_y = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(g(x,y) - c) = 0$$

Interpretation of λ: $\lambda = \frac{df^*}{dc}$ = marginal value of relaxing constraint (shadow price).

Template: *(1) Set up Lagrangian, (2) Take partial derivatives and set = 0, (3) Solve system, (4) Check boundaries if domain bounded, (5) Verify max/min.*

20. Integration

20.1 Basic Integrals

Function	**Integral**	**Function**	**Integral**
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$	e^x	$e^x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	a^x	$\frac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$	$\cos x$	$\sin x + C$

Fundamental Theorem: $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$

20.2 Integration Techniques

By Parts: $\int u\,dv = uv - \int v\,du$
LIATE rule for choosing u: Logs, Inverse trig, Algebraic, Trig, Exponential
Substitution: $\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$
Rational Functions: (1) Long division if $\deg(P) \geq \deg(Q)$, (2) Partial fractions, (3) Integrate terms.

Quick Reference: Economic Applications

Concept	**Formula/Description**
Cost function $C(q)$	Typically increasing, often convex
Revenue $R(q)$	$R = p \cdot q$
Profit $\Pi(q)$	$\Pi = R(q) - C(q)$
Profit max condition	$MR = MC$ (marginal revenue = marginal cost)
Utility max condition	$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$ (MRS = price ratio)
Cobb-Douglas $Q = AL^\alpha K^\beta$	CRS: $\alpha + \beta = 1$; IRS: $\alpha + \beta > 1$; DRS: $\alpha + \beta < 1$