Derivatives: Essential Rules and Properties

Mathematics Lecture

Outline

Exponent Properties

For any real numbers $a, b > 0$ and real numbers m, n:

1. **Product:** $a^m \cdot a^n = a^{m+n}$
2. **Quotient:** $\dfrac{a^m}{a^n} = a^{m-n}$
3. **Power of power:** $(a^m)^n = a^{mn}$
4. **Product power:** $(ab)^n = a^n b^n$
5. **Quotient power:** $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
6. **Negative exponent:** $a^{-n} = \dfrac{1}{a^n}$
7. **Zero exponent:** $a^0 = 1$ (for $a \neq 0$)
8. **Fractional:** $a^{1/n} = \sqrt[n]{a}$

Exponent Examples

Example 1: Simplify $2^3 \cdot 2^5$

$$2^3 \cdot 2^5 = 2^{3+5} = 2^8 = 256$$

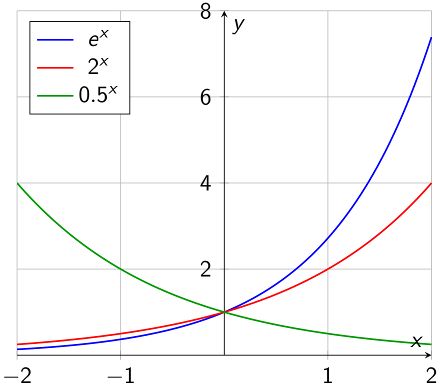
Example 2: Simplify $(x^2)^3$

$$(x^2)^3 = x^{2 \cdot 3} = x^6$$

Example 3: Simplify $8^{2/3}$

$$8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$$

Visualizing Exponential Functions



Notice: e^x and 2^x grow, while 0.5^x decays as x increases.

Logarithm Properties

The logarithm $\log_a x$ is the inverse of a^x. Natural log: $\ln x = \log_e x$

Key Identity: $a^{\log_a x} = x$ and $\log_a(a^x) = x$

For $a > 0$, $a \neq 1$, and $x, y > 0$:

1. **Product:** $\log_a(xy) = \log_a x + \log_a y$
2. **Quotient:** $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. **Power:** $\log_a(x^r) = r \log_a x$
4. **Change of base:** $\log_a x = \frac{\ln x}{\ln a}$
5. **Special values:** $\log_a 1 = 0$, $\log_a a = 1$

Logarithm Examples

Example 1: Simplify $\ln(e^{3x})$

$$\ln(e^{3x}) = 3x \ln e = 3x \cdot 1 = 3x$$

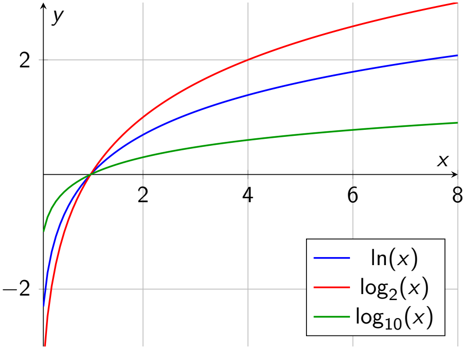
Example 2: Expand $\ln\left(\frac{x^2\sqrt{y}}{z^3}\right)$

$$\begin{aligned}\ln\left(\frac{x^2\sqrt{y}}{z^3}\right) &= \ln(x^2) + \ln(\sqrt{y}) - \ln(z^3) \\ &= 2\ln x + \frac{1}{2}\ln y - 3\ln z\end{aligned}$$

Example 3: Condense $3\ln x - 2\ln y$

$$3\ln x - 2\ln y = \ln(x^3) - \ln(y^2) = \ln\left(\frac{x^3}{y^2}\right)$$

Visualizing Logarithmic Functions



All logarithms pass through $(1, 0)$ and grow slowly for large x.

Sum Rule

Sum Rule

If $f(x)$ and $g(x)$ are differentiable:

$$(f+g)' = f' + g'$$

or equivalently:

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Example:

$$\frac{d}{dx}(x^3+5x^2-3x) = 3x^2+10x-3$$

Extends to multiple terms:

$$\frac{d}{dx}[f_1+f_2+\cdots+f_n] = f_1'+f_2'+\cdots+f_n'$$

Product Rule

Product Rule

If $f(x)$ and $g(x)$ are differentiable:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Mnemonic: "First times derivative of second, plus second times derivative of first"

Example 1: $\frac{d}{dx}(x^2 \sin x)$

$$\frac{d}{dx}(x^2 \sin x) = 2x \cdot \sin x + x^2 \cdot \cos x$$

Example 2: $\frac{d}{dx}[(3x+1)(x^2-2)]$

$$\begin{aligned}\frac{d}{dx}[(3x+1)(x^2-2)] &= 3(x^2-2) + (3x+1)(2x)\\ &= 3x^2 - 6 + 6x^2 + 2x = 9x^2 + 2x - 6\end{aligned}$$

Quotient Rule

If $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$:

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

Mnemonic: "Low dee-high minus high dee-low, over low-low"

Example: $\frac{d}{dx}\left(\frac{x^2}{x+1}\right)$

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2}{x+1}\right) &= \frac{2x(x+1) - x^2(1)}{(x+1)^2} \\ &= \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \end{aligned}$$

Chain Rule

Chain Rule

If $y = f(g(x))$, then:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Using Leibniz notation: if $y = f(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Intuition: Rate of change of y w.r.t. x = (rate of y w.r.t. u) $\times$ (rate of u w.r.t. x)

Chain Rule Examples

Example 1: $\frac{d}{dx}(3x+1)^5$
Let $u = 3x + 1$, then $y = u^5$

$$\frac{dy}{dx} = 5u^4 \cdot 3 = 15(3x+1)^4$$

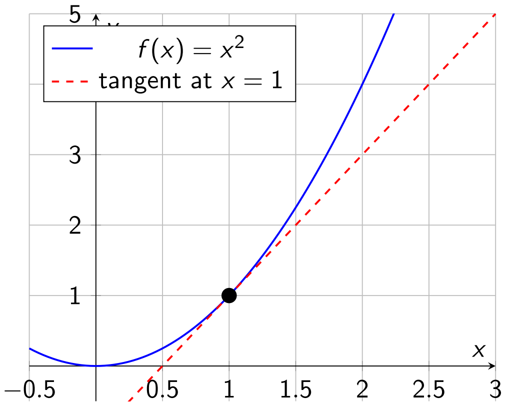
Example 2: $\frac{d}{dx}\sin(x^2)$

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

Example 3: $\frac{d}{dx}e^{x^3+2x}$

$$\frac{d}{dx}e^{x^3+2x} = e^{x^3+2x} \cdot (3x^2+2)$$

Geometric Interpretation of the Derivative



At $x = 1$: $f'(1) = 2$, so the tangent line has slope 2.

Combining Rules

Often we need multiple rules together!

Example: Find $\frac{d}{dx}\left[\frac{(x^2+1)^3}{x}\right]$
Using quotient rule + chain rule:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{(x^2+1)^3}{x}\right] &= \frac{x \cdot 3(x^2+1)^2 \cdot 2x - (x^2+1)^3 \cdot 1}{x^2} \\
&= \frac{6x^2(x^2+1)^2 - (x^2+1)^3}{x^2} \\
&= \frac{(x^2+1)^2[6x^2 - (x^2+1)]}{x^2} \\
&= \frac{(x^2+1)^2(5x^2-1)}{x^2}
\end{aligned}$$

Power Functions

$$\frac{d}{dx}(c) = 0 \quad \text{(constant)}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{(power rule)}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x$$

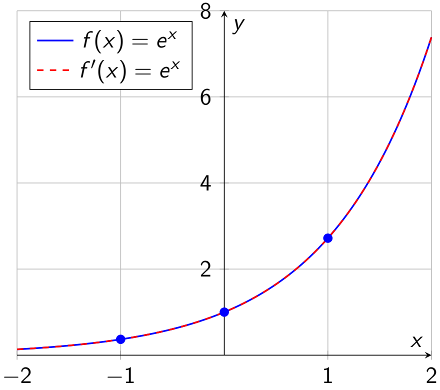
$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Note: The exponential function e^x is special—it's its own derivative!

Visualizing e^x and Its Derivative



The function and its derivative are identical! At any point, the slope equals the function value.

Exponential Examples

Example 1: $\frac{d}{dx}(e^{3x})$

$$\frac{d}{dx}(e^{3x}) = e^{3x} \cdot 3 = 3e^{3x}$$

Example 2: $\frac{d}{dx}(xe^{x})$ (product rule)

$$\frac{d}{dx}(xe^{x}) = 1 \cdot e^{x} + x \cdot e^{x} = e^{x}(1 + x)$$

Example 3: $\frac{d}{dx}(2^{x})$

$$\frac{d}{dx}(2^{x}) = 2^{x} \ln 2$$

Logarithm Examples

Example 1: $\frac{d}{dx}[\ln(x^2+1)]$

$$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

Example 2: $\frac{d}{dx}[x \ln x]$ (product rule)

$$\frac{d}{dx}[x \ln x] = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Example 3: $\frac{d}{dx}(\log_{10} x)$

$$\frac{d}{dx}(\log_{10} x) = \frac{1}{x \ln 10}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

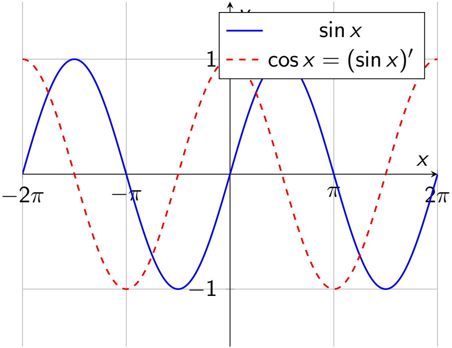
$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Visualizing $\sin x$ and Its Derivative



The derivative $\cos x$ represents the slope of $\sin x$ at each point.

Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

What are Higher-Order Derivatives?

The derivative of a derivative!

Notation:

- **First derivative:** $f'(x)$ or $\frac{df}{dx}$
- **Second derivative:** $f''(x)$ or $\frac{d^2 f}{dx^2}$
- **Third derivative:** $f'''(x)$ or $\frac{d^3 f}{dx^3}$
- ***n*-th derivative:** $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$

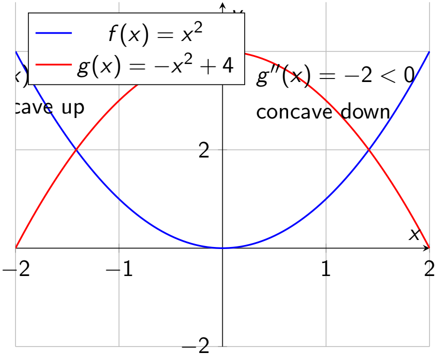
Physical Interpretation

If $f(t)$ represents position at time t:

- $f'(t) =$ **velocity** (rate of change of position)
- $f''(t) =$ **acceleration** (rate of change of velocity)
- $f'''(t) =$ **jerk** (rate of change of acceleration)

Higher derivatives tell us about the *curvature* and behavior of functions!

Visualizing Concavity with Second Derivative



$f'' > 0$: concave up (U-shaped). $f'' < 0$: concave down ($\cap$-shaped).

Example 1: Polynomial

Find all derivatives of $f(x) = x^4 - 3x^3 + 2x - 5$

$$\begin{aligned}
f(x) &= x^4 - 3x^3 + 2x - 5 \\
f'(x) &= 4x^3 - 9x^2 + 2 \\
f''(x) &= 12x^2 - 18x \\
f'''(x) &= 24x - 18 \\
f^{(4)}(x) &= 24 \\
f^{(5)}(x) &= 0
\end{aligned}$$

All subsequent derivatives are zero!

Example 2: Trigonometric

Find the first few derivatives of $f(x) = \sin x$

$$\begin{aligned}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f'''(x) &= -\cos x \\
f^{(4)}(x) &= \sin x
\end{aligned}$$

Notice the pattern repeats every 4 derivatives!

Example 3: Exponential

Find the first few derivatives of $f(x) = e^x$

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ f'''(x) &= e^x \\ f^{(n)}(x) &= e^x \quad \text{for all } n \end{aligned}$$

The exponential function is its own derivative at *every order*!

Practice Problem

Find $f''(x)$ for $f(x) = x^2 e^x$

Solution:
First derivative (product rule):

$$f'(x) = 2x \cdot e^x + x^2 \cdot e^x = e^x(2x + x^2)$$

Second derivative (product rule again):

$$\begin{aligned} f''(x) &= e^x(2x + x^2) + e^x(2 + 2x) \\ &= e^x(2x + x^2 + 2 + 2x) \\ &= e^x(x^2 + 4x + 2) \end{aligned}$$

Summary

- **Exponent/Log Properties:** Essential for simplification
- **Sum Rule:** $(f+g)' = f' + g'$
- **Product Rule:** $(fg)' = f'g + fg'$
- **Quotient Rule:** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- **Chain Rule:** $(f \circ g)' = f'(g(x)) \cdot g'(x)$
- **Key Derivatives:** e^x, $\ln x$, $\sin x$, $\cos x$, x^n
- **Higher-Order:** Derivatives of derivatives reveal function behavior

These tools are fundamental for calculus, optimization, and machine learning!