Intermediate Value Theorem, Sequences & L'Hôpital's Rule

Mathematics for ML

November 23, 2025

Outline

Intermediate Value Theorem: Statement

Theorem (Intermediate Value Theorem (IVT))

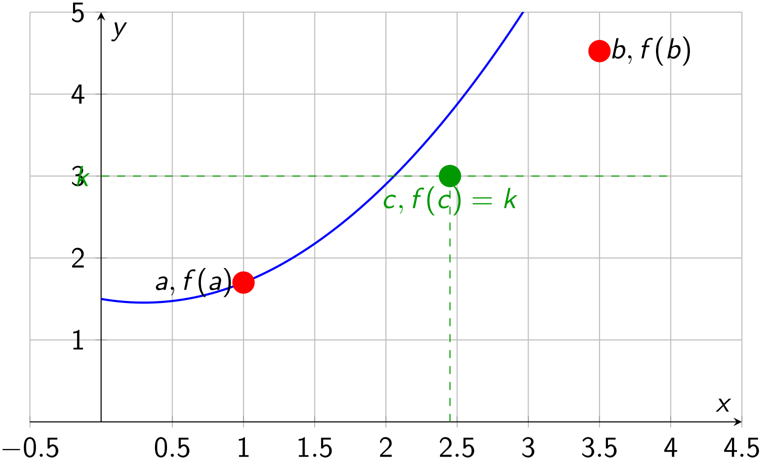
Let f be a **continuous** *function on the closed interval $[a, b]$.*
If k is any value between $f(a)$ and $f(b)$, then there exists at least one point $c \in (a, b)$ such that:

$$f(c) = k$$

Intuitive Meaning:

- A continuous function must pass through every value between $f(a)$ and $f(b)$
- You cannot "jump over" values
- Continuity is essential!

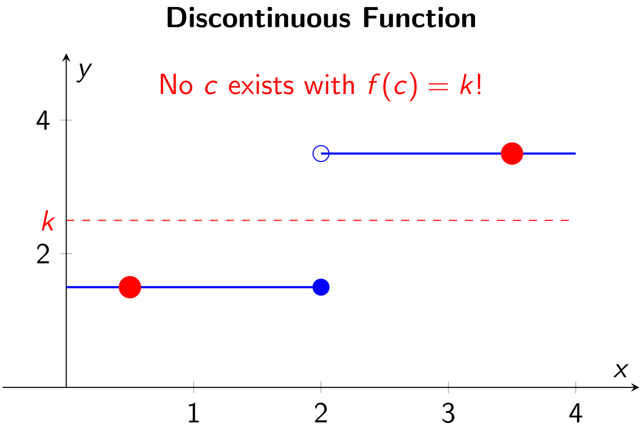
IVT: Visual Illustration



Since f is continuous and $f(a) < k < f(b)$, there exists c with $f(c) = k$.

Why Continuity Matters

Without continuity, IVT fails!



Proof of IVT (Sketch)

Proof Idea: Use the *bisection method* and completeness of $\mathbb{R}$.

Without loss of generality, assume $f(a) < k < f(b)$.

1. **Define sets:**
$$A = \{x \in [a, b] : f(x) \leq k\}$$
$$B = \{x \in [a, b] : f(x) \geq k\}$$
2. Note: $a \in A$ (since $f(a) < k$) and $b \in B$ (since $f(b) > k$)
3. A is non-empty and bounded above by b
4. By completeness of $\mathbb{R}$, $c = \sup A$ exists
5. By continuity: $f(c) = \lim_{x \to c} f(x)$
6. Since $c = \sup A$, we can find sequences $x_n \in A$ with $x_n \to c$
7. Thus $f(x_n) \leq k$ for all n, so $f(c) = \lim f(x_n) \leq k$

Proof of IVT (continued)

7. Similarly, for any $\epsilon > 0$, since c is least upper bound of A, there exists $y \in (c, c + \epsilon) \cap B$
8. For such y: $f(y) \geq k$
9. Taking $y \to c^+$, by continuity: $f(c) \geq k$
10. Combining: $f(c) \leq k$ and $f(c) \geq k$
11. Therefore: $f(c) = k$ $\square$

Key Ideas Used:

- Completeness of real numbers (supremum exists)
- Definition of continuity
- Properties of limits

Example 1: Root Finding

Problem: Prove that $f(x) = x^3 - x - 1$ has a root in $[1, 2]$.

Solution:

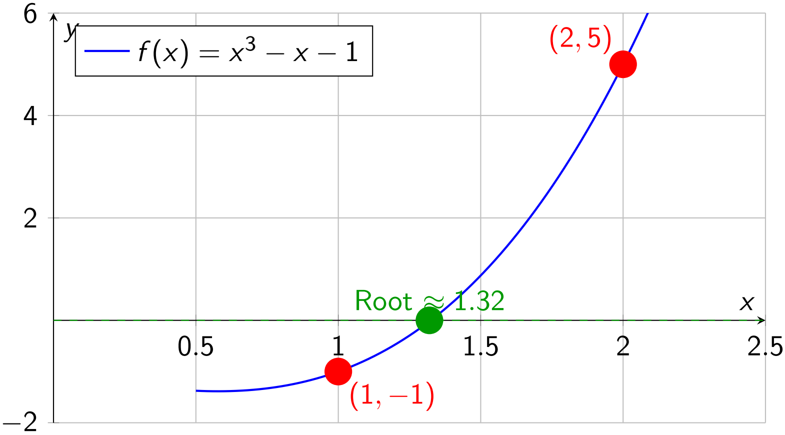
1. Check that f is continuous (polynomial $\Rightarrow$ continuous)
2. Evaluate at endpoints:

$$f(1) = 1^3 - 1 - 1 = -1 < 0$$
$$f(2) = 2^3 - 2 - 1 = 5 > 0$$

3. Since $f(1) < 0 < f(2)$ and f is continuous on $[1, 2]$
4. By IVT: $\exists c \in (1, 2)$ such that $f(c) = 0$

Conclusion: The equation $x^3 - x - 1 = 0$ has at least one solution in $(1, 2)$.

Example 1: Visualization



Example 2: Fixed Point Theorem

Problem: Show that $g(x) = \cos(x)$ has a fixed point in $[0, \pi/2]$.
(A **fixed point** means $g(c) = c$ for some c)

Solution:

1. Define $f(x) = g(x) - x = \cos(x) - x$
2. f is continuous on $[0, \pi/2]$ (difference of continuous functions)
3. Evaluate at endpoints:

$$f(0) = \cos(0) - 0 = 1 - 0 = 1 > 0$$
$$f(\pi/2) = \cos(\pi/2) - \pi/2 = 0 - \pi/2 < 0$$

4. Since $f(0) > 0 > f(\pi/2)$, by IVT: $\exists c \in (0, \pi/2)$ with $f(c) = 0$
5. Thus $\cos(c) - c = 0$, i.e., $\cos(c) = c$

Sequences: Definition

Definition

An **infinite sequence** is a function from $\mathbb{N}$ to $\mathbb{R}$:

$$a : \mathbb{N} \to \mathbb{R}, \quad n \mapsto a_n$$

We write: $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$

Examples:

1. $a_n = \frac{1}{n}$: $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
2. $a_n = (-1)^n$: $\quad -1, 1, -1, 1, -1, \ldots$
3. $a_n = \frac{n}{n+1}$: $\quad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
4. $a_n = n^2$: $\quad 1, 4, 9, 16, 25, \ldots$

Convergence of Sequences

Definition

A sequence $\{a_n\}$ **converges** to L if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N : |a_n - L| < \epsilon$$

We write: $\lim_{n \to \infty} a_n = L$ or $a_n \to L$

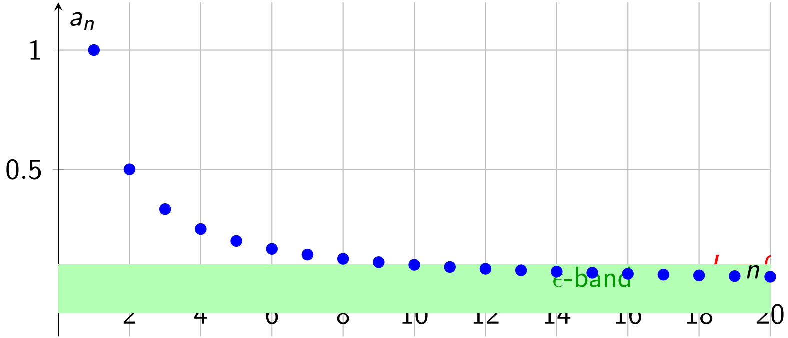
Intuitive Meaning:

- Terms get arbitrarily close to L as n increases
- For any tolerance ϵ, eventually all terms are within ϵ of L
- The sequence "settles down" to L

Divergence: If no such L exists, the sequence **diverges**.

Convergence: Visual

Convergent Sequence: $a_n = \frac{1}{n}$



Important Sequence Limits

Common Limits:

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad$ for any $p > 0$
2. $\lim_{n\to\infty} r^n = 0 \quad$ if $|r| < 1$
3. $\lim_{n\to\infty} r^n = \infty \quad$ if $r > 1$
4. $\lim_{n\to\infty} r^n$ does not exist if $r \leq -1$
5. $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$
6. $\lim_{n\to\infty} \frac{n}{n+1} = 1$
7. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} \quad$ (if limits exist and $\lim b_n \neq 0$)

Properties of Limits

Theorem

If $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$, *then:*

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
2. $\lim_{n\to\infty}(c \cdot a_n) = c \cdot L \quad$ *for any constant* c
3. $\lim_{n\to\infty}(a_n \cdot b_n) = L \cdot M$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \quad$ *if* $M \neq 0$
5. *If* $a_n \leq b_n$ *for all* n, *then* $L \leq M$

Squeeze Theorem: If $a_n \leq c_n \leq b_n$ and $\lim a_n = \lim b_n = L$, then $\lim c_n = L$.

Example 3: Prove Convergence

Problem: Prove that $\lim_{n\to\infty} \frac{2n+3}{n+1} = 2$.

Proof:

Given $\epsilon > 0$, we need to find N such that for all $n > N$:

$$\left|\frac{2n+3}{n+1} - 2\right| < \epsilon$$

$$\begin{aligned}
\left|\frac{2n+3}{n+1} - 2\right| &= \left|\frac{2n+3-2(n+1)}{n+1}\right| \\
&= \left|\frac{2n+3-2n-2}{n+1}\right| \\
&= \left|\frac{1}{n+1}\right| \\
&= \frac{1}{n+1} < \epsilon
\end{aligned}$$

Monotone Convergence Theorem

Theorem (Monotone Convergence Theorem)

1. *If* $\{a_n\}$ *is* **increasing** *and* **bounded above**, *then* $\{a_n\}$ *converges to* $\sup\{a_n\}$
2. *If* $\{a_n\}$ *is* **decreasing** *and* **bounded below**, *then* $\{a_n\}$ *converges to* $\inf\{a_n\}$

Proof Sketch (for increasing case):

1. Let $L = \sup\{a_n\}$ (exists by completeness)
2. For any $\epsilon > 0$, $L - \epsilon$ is not an upper bound
3. So $\exists N$ with $a_N > L - \epsilon$
4. Since $\{a_n\}$ is increasing: $a_n \geq a_N > L - \epsilon$ for all $n \geq N$
5. Also $a_n \leq L < L + \epsilon$ (since L is supremum)
6. Thus $|a_n - L| < \epsilon$ for all $n \geq N$ □

L'Hôpital's Rule: Statement

Theorem (L'Hôpital's Rule)

Suppose f and g are differentiable near a (except possibly at a), and $g'(x) \neq 0$ near a.
If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or* $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$
and *if* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists (or is $\pm\infty$), then:*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note: This also works for:

- One-sided limits ($x \to a^+$ or $x \to a^-$)
- Limits at infinity ($x \to \infty$ or $x \to -\infty$)

Indeterminate Forms

L'Hôpital's Rule applies to **indeterminate forms**:

Direct Application:

- $\frac{0}{0}$ form: Both numerator and denominator approach 0
- $\frac{\infty}{\infty}$ form: Both approach infinity

Can be converted:

- $0 \cdot \infty$ form: Rewrite as $\frac{0}{1/\infty}$ or $\frac{\infty}{1/0}$
- $\infty - \infty$ form: Combine into single fraction
- 0^0, 1^∞, ∞^0 forms: Use logarithms

Warning: L'Hôpital's Rule does NOT apply if the limit is not indeterminate!

Proof of L'Hôpital's Rule (0/0 case)

Proof for $\frac{0}{0}$ form at $x = a$:

1. Assume $f(a) = g(a) = 0$ (can extend by continuity)
2. By Cauchy's Mean Value Theorem: For x near a, $\exists c$ between a and x such that:
$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$
3. Since $f(a) = g(a) = 0$:
$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$
4. As $x \to a$, we have $c \to a$ (since c is between a and x)
5. If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then:
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{c\to a} \frac{f'(c)}{g'(c)} = L$$

$\square$

Example 4: Basic L'Hôpital Application

Problem: Evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$

Solution:

1. Check the form: $\lim_{x \to 0} \sin(x) = 0$ and $\lim_{x \to 0} x = 0$
2. This is $\frac{0}{0}$ form, so apply L'Hôpital's Rule
3. Take derivatives:
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos(x)}{1}$$
4. Evaluate:
$$\lim_{x \to 0} \cos(x) = \cos(0) = 1$$

Answer: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Example 5: Multiple Applications

Problem: Evaluate $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

Solution:

1. Check: $e^0 - 1 - 0 = 0$ and $0^2 = 0$, so this is $\frac{0}{0}$
2. Apply L'Hôpital's Rule:
$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}$$
3. Still $\frac{0}{0}$! Apply again:
$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2}$$
4. Evaluate:
$$\lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

Answer: $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$

Example 6: Infinity/Infinity Form

Problem: Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$

Solution:

1. As $x \to \infty$: $x^2 \to \infty$ and $e^x \to \infty$, so this is $\frac{\infty}{\infty}$
2. Apply L'Hôpital's Rule:
$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$$
3. Still $\frac{\infty}{\infty}$! Apply again:
$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x}$$
4. Now evaluate:
$$\lim_{x\to\infty} \frac{2}{e^x} = 0$$

Conclusion: Exponentials grow faster than polynomials!

Example 7: $0 \cdot \infty$ Form

Problem: Evaluate $\lim_{x \to 0^+} x \ln(x)$

Solution:

1. As $x \to 0^+$: $x \to 0$ and $\ln(x) \to -\infty$, so this is $0 \cdot \infty$
2. Rewrite to get $\frac{\infty}{\infty}$ form:
$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x}$$
3. As $x \to 0^+$: numerator $\to -\infty$, denominator $\to +\infty$, so $\frac{-\infty}{\infty}$
4. Apply L'Hôpital's Rule:

$$\lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{x} = \lim_{x \to 0^+} (-x) = 0$$

Answer: $\lim_{x \to 0^+} x \ln(x) = 0$

Example 8: 1^∞ Form

Problem: Evaluate $\lim_{x\to 0^+}(1+x)^{1/x}$

Solution: This is 1^∞ form. Use logarithms!

1. Let $L = \lim_{x\to 0^+}(1+x)^{1/x}$
2. Take logarithm:
$$\ln L = \lim_{x\to 0^+} \ln\left[(1+x)^{1/x}\right] = \lim_{x\to 0^+} \frac{\ln(1+x)}{x}$$
3. This is $\frac{0}{0}$ form. Apply L'Hôpital:
$$\ln L = \lim_{x\to 0^+} \frac{1/(1+x)}{1} = \lim_{x\to 0^+} \frac{1}{1+x} = 1$$
4. Thus $\ln L = 1$, so:
$$L = e^1 = e$$

Answer: $\lim_{x\to 0^+}(1+x)^{1/x} = e$ (Definition of e!)

Common Mistakes with L'Hôpital's Rule

Warning: Do NOT do these!

1. × Using L'Hôpital when NOT indeterminate:

$$\lim_{x\to 0} \frac{x+1}{x+2} = \frac{0+1}{0+2} = \frac{1}{2}$$

   No need for L'Hôpital! (Not $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

2. × Using quotient rule instead of taking derivatives separately:

$$\lim_{x\to 0} \frac{\sin x}{x} \neq \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2}$$

3. × Forgetting to check if the derivative limit exists
4. × Applying infinitely many times without checking convergence

Always: Verify the form, differentiate correctly, and check your answer!

Summary: Three Key Theorems

Theorem	**Key Idea**
IVT	Continuous functions on $[a, b]$ take all intermediate values between $f(a)$ and $f(b)$. Used for proving existence of roots.
Sequences	Convergence means terms get arbitrarily close to limit. Monotone Convergence Theorem guarantees convergence for bounded monotone sequences.
L'Hôpital	For indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, can evaluate limit of ratio by taking derivatives: $\lim \frac{f}{g} = \lim \frac{f'}{g'}$

Common Thread: All use fundamental properties of real numbers (completeness, continuity, differentiability)

Applications

Intermediate Value Theorem:

- Root finding algorithms (bisection method)
- Fixed point theorems
- Proving existence of solutions

Sequences:

- Iterative algorithms (convergence analysis)
- Numerical methods (Newton's method, gradient descent)
- Series convergence (prerequisite)

L'Hôpital's Rule:

- Computing difficult limits
- Taylor series analysis
- Asymptotic analysis
- Machine learning (analyzing loss function behavior)

Practice Problems

IVT:

1. Prove $x^5 + x - 1 = 0$ has a root in $[0, 1]$
2. Show $\tan(x) = x$ has infinitely many solutions

Sequences:

1. Prove $\lim_{n\to\infty} \frac{3n^2+2n}{n^2+1} = 3$
2. Determine if $a_n = \frac{(-1)^n}{n}$ converges

L'Hôpital:

1. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$
2. $\lim_{x\to\infty} x^{1/x}$
3. $\lim_{x\to 0^+} x^x$

Thank You!

Questions?

These three concepts are fundamental tools for analysis and will appear throughout calculus and beyond!