Optimization with Lagrange Multipliers

Unconstrained basics + Equality and Inequality Constraints (KKT)

Plan

1. **Unconstrained Optimization**: gradients, Hessians, minima
2. **Equality Constraints**: Lagrange multipliers and intuition
3. **Inequality Constraints**: KKT conditions and active constraints
4. **Worked Minimization Examples**: step-by-step solutions

Learning Objectives

By the end, you should be able to:

- State and use first- and second-order optimality conditions (unconstrained)
- Form and solve the *Lagrangian* for equality-constrained problems
- State KKT conditions for inequality constraints and interpret them
- Solve standard minimization problems (including checking feasibility and activity)

Optimization Problem (Unconstrained)

Goal:

$$\min_{x \in \mathbb{R}^n} f(x)$$

Definitions:

- **Objective function** $f : \mathbb{R}^n \to \mathbb{R}$: what we want to minimize.
- **Global minimizer** $x^\star$: $f(x^\star) \leq f(x)$ for all x.
- **Local minimizer** $x^\star$: $f(x^\star) \leq f(x)$ for all x near $x^\star$.

Gradient and Stationary Points

Assume f is differentiable.

Gradient:

$$\nabla f(x) = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

First-order necessary condition (FONC): If $x^\star$ is a local minimizer (and f is differentiable), then

$$\nabla f(x^\star) = 0.$$

Stationary point: any x with $\nabla f(x) = 0$.

Hessian and Second-Order Conditions

If f is twice differentiable, the **Hessian** is

$$\nabla^2 f(x) = \left[\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right]_{i,j=1}^{n}.$$

Second-order test (common version):

- If $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star) \succ 0$ (positive definite), then $x^\star$ is a *strict local minimizer*.
- If $\nabla^2 f(x^\star) \prec 0$, then $x^\star$ is a local maximizer.
- If $\nabla^2 f(x^\star)$ is indefinite, then $x^\star$ is a saddle point.

Example (Unconstrained Minimization)

Problem:

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) = (x-1)^2 + (y+2)^2.$$

Step 1: Compute the gradient

$$\nabla f(x,y) = \begin{bmatrix} 2(x-1) \\ 2(y+2) \end{bmatrix}.$$

Step 2: Set $\nabla f = 0$

$$2(x-1) = 0 \Rightarrow x = 1, \qquad 2(y+2) = 0 \Rightarrow y = -2.$$

Conclusion: Unique minimizer is $(1,-2)$ with minimum value $f(1,-2) = 0$.

Equality-Constrained Optimization

Problem:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) = 0,$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

Feasible set:

$$\mathcal{F} = \{x \in \mathbb{R}^n : h(x) = 0\}.$$

Interpretation: We optimize f but we are only allowed to move along the constraint surface $h(x) = 0$.

Lagrangian and Lagrange Multipliers

Lagrangian:

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^\top h(x),$$

where $\lambda \in \mathbb{R}^m$ are **Lagrange multipliers**.
Lagrange conditions (first-order, equality case):

$$\nabla_x \mathcal{L}(x^\star, \lambda^\star) = \nabla f(x^\star) + J_h(x^\star)^\top \lambda^\star = 0,$$

$$h(x^\star) = 0.$$

Here $J_h(x)$ is the Jacobian matrix of h.

Geometric Intuition (Equality Constraints)

For a single equality constraint $h(x) = 0$ in $\mathbb{R}^n$:

- The feasible directions at x lie in the tangent space to the constraint surface.
- $\nabla h(x)$ is perpendicular (normal) to the constraint surface.
- At an optimum, the component of $\nabla f(x)$ along feasible directions must be zero.

Key idea:

$$\nabla f(x^\star) = -\lambda^\star \nabla h(x^\star)$$

So ∇f must be parallel to ∇h (in the 1-constraint case).

Worked Example (Equality Constraint)

Problem:

$$\min_{x,y\in\mathbb{R}} \; x^2 + y^2 \quad \text{s.t.} \quad x + y = 1.$$

Step 1: Build the Lagrangian

$$\mathcal{L}(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 1).$$

Step 2: Stationarity

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} = 2y + \lambda = 0.$$

Worked Example (Equality Constraint) — continued

From stationarity:

$$2x + \lambda = 0,\ 2y + \lambda = 0 \Rightarrow x = y.$$

Step 3: Enforce feasibility

$$x + y = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2},\ y = \frac{1}{2}.$$

Step 4: Value at optimum

$$f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}.$$

Conclusion: Minimizer is $(x^\star, y^\star) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Multiplier Interpretation (Sensitivity)

Consider equality constraint $h(x) = b$.
Under regularity conditions, the optimal multiplier $\lambda^\star$ approximates how the optimal value changes when we slightly change b:

$$\frac{d}{db} f^\star(b) \approx -\lambda^\star.$$

Meaning:

- Large $|\lambda^\star|$: the constraint is "expensive" (tight/important).
- Small $|\lambda^\star|$: the constraint has little effect near optimum.

Inequality-Constrained Optimization

We often solve

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_i(x) \leq 0 \; (i = 1, \dots, m), \quad h_j(x) = 0 \; (j = 1, \dots, p).$$

New concepts:

- **Active constraint**: $g_i(x^\star) = 0$ (tight/binding).
- **Inactive constraint**: $g_i(x^\star) < 0$ (slack).

KKT: The Lagrangian for Inequalities

Define the Lagrangian

$$\mathcal{L}(x,\lambda,\nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \nu_j h_j(x),$$

with multipliers

$$\lambda_i \geq 0 \text{ (for inequalities)}, \qquad \nu_j \in \mathbb{R} \text{ (for equalities)}.$$

KKT Conditions (Statement)

Under suitable regularity conditions (constraint qualifications), a local optimum $x^\star$ satisfies:

1) Primal feasibility

$$g_i(x^\star) \le 0, \quad h_j(x^\star) = 0.$$

2) Dual feasibility

$$\lambda_i^\star \ge 0 \quad \forall i.$$

3) Complementary slackness

$$\lambda_i^\star \, g_i(x^\star) = 0 \quad \forall i.$$

4) Stationarity

$$\nabla f(x^\star) + \sum_{i=1}^{m} \lambda_i^\star \, \nabla g_i(x^\star) + \sum_{j=1}^{p} \nu_j^\star \, \nabla h_j(x^\star) = 0.$$

Complementary Slackness (What it Means)

For each inequality constraint $g_i(x) \leq 0$:

- If the constraint is **inactive** ($g_i(x^\star) < 0$), then complementary slackness forces $\lambda_i^\star = 0$.
- If the constraint is **active** ($g_i(x^\star) = 0$), then $\lambda_i^\star$ may be positive.

Interpretation: Only the constraints that are tight at the optimum influence the stationarity equation.

When Are KKT Conditions Also Sufficient?

KKT conditions are *necessary* for many problems.
They are also *sufficient* for global optimality in the important **convex** case:

- f is convex
- each g_i is convex
- each h_j is affine (linear + constant)
- a regularity condition holds (e.g. Slater's condition)

Then any point satisfying KKT is a **global minimizer**.

Worked Example (Inequality, Active Constraint)

Problem:

$$\min_{x\in\mathbb{R}} \; x^2 \quad \text{s.t.} \quad x \geq 1.$$

Rewrite as $g(x) \leq 0$ with

$$g(x) = 1 - x \leq 0.$$

Lagrangian:

$$\mathcal{L}(x,\lambda) = x^2 + \lambda(1-x), \quad \lambda \geq 0.$$

Stationarity: $\frac{d\mathcal{L}}{dx} = 2x - \lambda = 0 \Rightarrow \lambda = 2x.$

Worked Example (Inequality, Active Constraint) — continued

Primal feasibility: $x \geq 1$.
Complementary slackness: $\lambda(1 - x) = 0$.
Case analysis:

- If $x > 1$ then $1 - x < 0$ so complementary slackness forces $\lambda = 0$, but then stationarity gives $2x = 0$ (impossible).
- Therefore the constraint must be **active**: $x = 1$.

Then stationarity gives $\lambda = 2x = 2 \geq 0$ (dual feasible).
Conclusion: Minimizer is $x^\star = 1$ with value $f(x^\star) = 1$.

Worked Example (Inequality, Inactive Constraint)

Problem:

$$\min_{x \in \mathbb{R}} (x-2)^2 \quad \text{s.t.} \quad x \geq 0.$$

Rewrite $g(x) = -x \leq 0$.
Unconstrained minimizer: $x = 2$.
Check feasibility: $2 \geq 0$ so the constraint is **inactive** at the solution.
KKT perspective:

- Since $g(2) = -2 < 0$, complementary slackness forces $\lambda^\star = 0$.
- Stationarity becomes derivative of f equals zero at $x = 2$.

Conclusion: Minimizer remains $x^\star = 2$.

Worked Example (2D Inequalities): Geometry First

Problem:

$$\min_{x,y\in\mathbb{R}} \; x^2 + y^2 \quad \text{s.t.} \quad x \geq 1, \; y \geq 2.$$

Interpretation:

- $x^2 + y^2$ is the squared distance from (x, y) to the origin.
- The feasible set is the quadrant *shifted away* from the origin: $\{(x, y) : x \geq 1, y \geq 2\}$.
- The closest feasible point to the origin is the corner $(1, 2)$.

So we already expect the minimizer to be $(1, 2)$, and both constraints to be **active**.

Worked Example (2D Inequalities): KKT Setup

Write constraints as $g_1(x, y) = 1 - x \leq 0$ and $g_2(x, y) = 2 - y \leq 0$.
Lagrangian:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = x^2 + y^2 + \lambda_1(1 - x) + \lambda_2(2 - y), \quad \lambda_1, \lambda_2 \geq 0.$$

Stationarity:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda_1 = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda_2 = 0.$$

Complementary slackness:

$$\lambda_1(1 - x) = 0, \qquad \lambda_2(2 - y) = 0.$$

Worked Example (2D Inequalities): Solve + Explain

Unconstrained minimizer is $(0, 0)$, but it is infeasible.
Key KKT insight: If $x > 1$ then $(1 - x) < 0$ so complementary slackness forces $\lambda_1 = 0$, and stationarity would give $2x = 0$ (impossible). Therefore $x = 1$.
Similarly, if $y > 2$ then $\lambda_2 = 0$ and stationarity forces $2y = 0$ (impossible). Therefore $y = 2$.
Then multipliers come from stationarity:

$$\lambda_1 = 2x = 2 \geq 0, \qquad \lambda_2 = 2y = 4 \geq 0.$$

Conclusion: Minimizer is $(1, 2)$ with value 5.

Utility Maximization Problem

Utility:

$$u(x_1, x_2) = \ln x_1 + 2\ln x_2.$$

Constraints:

$$p_1x_1 + p_2x_2 \le b, \qquad x_1 \ge 0,\ x_2 \ge 0, \qquad p_1, p_2 > 0.$$

Important domain note: $\ln x$ is defined only for $x > 0$, so effectively we optimize over $x_1 > 0,\ x_2 > 0$. We assume $b > 0$ so feasible positive bundles exist.

(a) Lagrangian and Stationary Points

Write the constraints in KKT form:

$$g_0(x) = p_1 x_1 + p_2 x_2 - b \leq 0, \qquad g_1(x) = -x_1 \leq 0, \qquad g_2(x) = -x_2 \leq 0.$$

Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda, \mu_1, \mu_2) = \ln x_1 + 2 \ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - b) - \mu_1 x_1 - \mu_2 x_2,$$

with $\lambda, \mu_1, \mu_2 \geq 0$.
Stationarity (where defined):

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 - \mu_1 = 0, \qquad \frac{\partial \mathcal{L}}{\partial x_2} = \frac{2}{x_2} - \lambda p_2 - \mu_2 = 0.$$

(a) Stationary Point: Why It Must Be Interior

Recall $u(x_1, x_2) = \ln x_1 + 2\ln x_2$.
Because u is **strictly increasing** in each good on $(0,\infty)^2$:

$$\frac{\partial u}{\partial x_1} = \frac{1}{x_1} > 0, \qquad \frac{\partial u}{\partial x_2} = \frac{2}{x_2} > 0,$$

any maximizer must satisfy:

- **Budget binds**: $p_1x_1 + p_2x_2 = b$ (otherwise increase some x_i).
- **Nonnegativity is slack**: $x_1 > 0$, $x_2 > 0$ so $\mu_1 = \mu_2 = 0$.

So stationarity simplifies to

$$\frac{1}{x_1} = \lambda p_1, \qquad \frac{2}{x_2} = \lambda p_2.$$

(a) Solve the Stationary Point

From stationarity:

$$x_1 = \frac{1}{\lambda p_1}, \qquad x_2 = \frac{2}{\lambda p_2}.$$

Use the binding budget constraint:

$$p_1 x_1 + p_2 x_2 = b \Rightarrow p_1 \frac{1}{\lambda p_1} + p_2 \frac{2}{\lambda p_2} = b \Rightarrow \frac{1}{\lambda} + \frac{2}{\lambda} = b \Rightarrow \lambda = \frac{3}{b}.$$

Therefore the (candidate) stationary point is

$$x_1^\star = \frac{b}{3p_1}, \qquad x_2^\star = \frac{2b}{3p_2}.$$

(b) Global Maximum: Argue, Then Conclude

Why this is the global maximum (not just a critical point):

- $u(x_1, x_2) = \ln x_1 + 2 \ln x_2$ is **strictly concave** on $(0, \infty)^2$.
- The feasible set $\{(x_1, x_2) : p_1 x_1 + p_2 x_2 \leq b,\ x_1, x_2 \geq 0\}$ is convex.
- Maximizing a strictly concave function over a convex set gives a **unique global maximizer**.

So $\left(\frac{b}{3p_1}, \frac{2b}{3p_2}\right)$ is the unique global maximum.

Corresponding multipliers:

$$\lambda^\star = \frac{3}{b}, \qquad \mu_1^\star = \mu_2^\star = 0.$$

(b) Global Minimum and Other Local Extrema

There is no global minimum.
Reason: along feasible sequences approaching the boundary (still satisfying the budget), utility goes to $-\infty$. For example, fix any $\varepsilon > 0$ and take

$$x_1 = \varepsilon, \qquad x_2 = \frac{b - p_1\varepsilon}{p_2} > 0.$$

Then

$$u(x_1, x_2) = \ln \varepsilon + 2 \ln \left(\frac{b - p_1\varepsilon}{p_2} \right) \to -\infty \quad \text{as } \varepsilon \to 0^+.$$

So the infimum is $-\infty$ and it is not attained.
Other local extrema? No. Strict concavity implies the maximizer is unique and there are no other local maxima; and since the objective is unbounded below, there is no local minimum in the feasible interior.

Extreme Value Theorem (EVT)

Theorem (Weierstrass / Extreme Value Theorem): If f is **continuous** and $S \subset \mathbb{R}^n$ is **compact**, then there exist points $x_{\text{min}}, x_{\text{max}} \in S$ such that

$$f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}) \quad \text{for all } x \in S.$$

So: a continuous function attains both a global minimum and a global maximum on a compact set.
Key definition (in $\mathbb{R}^n$): compact $\iff$ **closed and bounded**.

Why EVT Matters in Optimization

EVT is an **existence theorem**.

- KKT / Lagrange multipliers help you find *candidates* for optima.
- EVT helps you answer: *Does an optimum exist at all?*

Typical workflow for constrained optimization:

1. Check whether the feasible set S is compact (closed + bounded).
2. Check whether the objective is continuous on S.
3. If yes, EVT guarantees global min and max exist.
4. Then use calculus/KKT to find them.

EVT Example (Optima Exist)

Example:

$$\min_{x\in[-1,2]} x^2, \qquad \max_{x\in[-1,2]} x^2.$$

Here $f(x) = x^2$ is continuous and $[-1, 2]$ is compact, so both extrema exist.
Compute:

- Minimum at $x = 0$ with value 0.
- Maximum at an endpoint: $\max\{(-1)^2, 2^2\} = 4$ achieved at $x = 2$.

EVT Counterexamples (Why Conditions Matter)

1) Set not closed: On $S = (0, 1)$, $f(x) = x$ is continuous but the minimum is not attained (infimum 0).

2) Set not bounded: On $S = \mathbb{R}$, $f(x) = x^2$ attains a minimum (at 0) but has no maximum.

3) Objective not continuous on the feasible set: Even if S is compact, discontinuities can break existence of extrema.

Connection to the Utility Example

The budget set $\{(x_1, x_2) : p_1x_1 + p_2x_2 \leq b,\ x_1, x_2 \geq 0\}$ is **compact** (closed + bounded).
But $u(x_1, x_2) = \ln x_1 + 2 \ln x_2$ is only defined (and continuous) on $x_1 > 0,\ x_2 > 0$, and it goes to $-\infty$ as $x_1 \to 0^+$ or $x_2 \to 0^+$.

Therefore:

- A maximum exists (and is unique) due to concavity and monotonicity.
- A minimum does not exist because the utility is unbounded below near the boundary.

Summary

- **Unconstrained:** local minima satisfy $\nabla f = 0$; Hessian helps classify.
- **Equality constraints:** solve $\nabla_x \mathcal{L} = 0$ with $h(x) = 0$.
- **Inequality constraints:** KKT adds $\lambda \geq 0$ and $\lambda_i g_i = 0$.
- **Active constraints matter:** only tight inequalities influence stationarity.

Quick Practice Problems

Try these after the lecture:

1. Minimize $x^2 + y^2$ subject to $x - 2y = 0$.
2. Minimize $(x-1)^2$ subject to $x \leq 0$.
3. Minimize $x^2 + y^2$ subject to $x + y \geq 1$.