Higher-Order Derivatives, Taylor's Theorem, and Convexity

Outline

Definition of Higher-Order Derivatives

First Derivative: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Second Derivative: The derivative of the first derivative

$$f''(x) = \frac{d}{dx}[f'(x)]$$

Third Derivative: $f'''(x) = \frac{d}{dx}[f''(x)]$

n**-th Derivative:** $f^{(n)}(x)$ denotes the n-th derivative

Notation:

$$f'(x), \quad f''(x), \quad f'''(x), \quad f^{(4)}(x), \ldots, \quad f^{(n)}(x)$$

or

$$\frac{df}{dx}, \quad \frac{d^2f}{dx^2}, \quad \frac{d^3f}{dx^3}, \quad \ldots, \quad \frac{d^nf}{dx^n}$$

Examples of Higher-Order Derivatives

Example 1: $f(x) = x^4$

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$
$$f'''(x) = 24x$$
$$f^{(4)}(x) = 24$$
$$f^{(5)}(x) = 0$$

Example 2: $f(x) = e^x$

$$f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

Examples of Higher-Order Derivatives (Continued)

Example 3: $f(x) = \sin(x)$

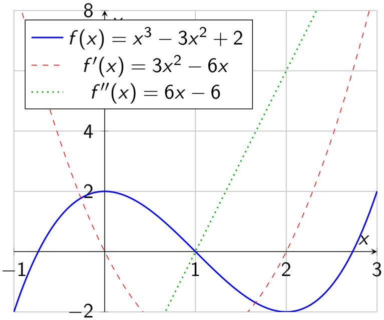
$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) \end{aligned}$$

The pattern repeats every 4 derivatives.

Example 4: $f(x) = \ln(x)$

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \end{aligned}$$

Visualizing Higher-Order Derivatives



The second derivative $f''(x)$ tells us about the curvature of $f(x)$.

Physical Interpretation

If $s(t)$ represents position at time t:

First Derivative: $s'(t) = v(t)$ is *velocity*

$$v(t) = \frac{ds}{dt}$$

Second Derivative: $s''(t) = v'(t) = a(t)$ is *acceleration*

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Third Derivative: $s'''(t) = a'(t)$ is *jerk* (rate of change of acceleration)

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Motivation for Taylor Series

Question: Can we approximate a complicated function with a polynomial?

Linear Approximation: Near $x = a$:

$$f(x) \approx f(a) + f'(a)(x - a)$$

This is the tangent line approximation.

Quadratic Approximation: Include curvature:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Taylor's Theorem extends this idea to higher-order approximations.

Taylor's Theorem

Taylor Polynomial of degree n at $x = a$:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or more compactly:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor Series: If the limit exists as $n \to \infty$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Maclaurin Series: Special case when $a = 0$:

$$\sum^{\infty} \frac{f^{(k)}(0)}{}$$

Common Taylor Series (Maclaurin Series)

Exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Sine:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Cosine:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Natural Logarithm:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad |x| < 1$$

Example: Taylor Series for e^x

Find the Maclaurin series for $f(x) = e^x$:

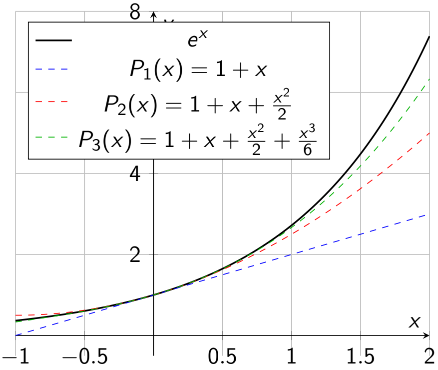
All derivatives of e^x equal e^x, so:

$$
\begin{aligned}
f(0) &= e^0 = 1 \\
f'(0) &= e^0 = 1 \\
f''(0) &= e^0 = 1 \\
&\vdots \\
f^{(n)}(0) &= 1
\end{aligned}
$$

Therefore:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Visualizing Taylor Approximations



Higher-degree polynomials provide better approximations near $x = 0$.

Taylor's Theorem with Remainder

Taylor's Theorem: If f has $n+1$ continuous derivatives, then:

$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x)$ is the Taylor polynomial and $R_n(x)$ is the remainder term.

Lagrange Form of Remainder:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

This tells us the *error* in approximating $f(x)$ by $P_n(x)$.

Definition of Convexity

A function f is **convex** on an interval I if for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

Geometric Interpretation: The line segment connecting any two points on the graph lies *above* the graph.

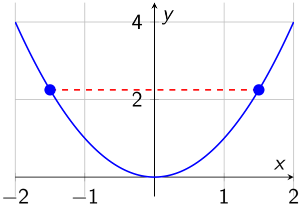
A function f is **concave** if $-f$ is convex, or equivalently:

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$$

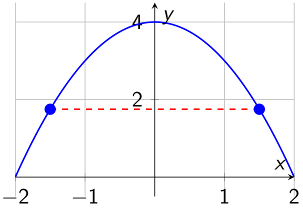
The line segment lies *below* the graph.

Visualizing Convex and Concave Functions

Convex: $f(x) = x^2$



Concave: $f(x) = -x^2 + 4$



Second Derivative Test for Convexity

Theorem: Let f be twice differentiable on an interval I.

- If $f''(x) \geq 0$ for all $x \in I$, then f is **convex** on I
- If $f''(x) \leq 0$ for all $x \in I$, then f is **concave** on I
- If $f''(x) > 0$ for all $x \in I$, then f is **strictly convex** on I
- If $f''(x) < 0$ for all $x \in I$, then f is **strictly concave** on I

Intuition:

- $f''(x) > 0$: function curves upward (convex, "holds water")
- $f''(x) < 0$: function curves downward (concave, "spills water")

Examples of Convex and Concave Functions

Convex Functions:

- $f(x) = x^2 \quad (f''(x) = 2 > 0)$
- $f(x) = e^x \quad (f''(x) = e^x > 0)$
- $f(x) = |x| \quad$ (convex but not differentiable at $x = 0$)
- $f(x) = -\ln(x)$ for $x > 0 \quad (f''(x) = \frac{1}{x^2} > 0)$

Concave Functions:

- $f(x) = -x^2 \quad (f''(x) = -2 < 0)$
- $f(x) = \ln(x)$ for $x > 0 \quad (f''(x) = -\frac{1}{x^2} < 0)$
- $f(x) = \sqrt{x}$ for $x \geq 0 \quad (f''(x) = -\frac{1}{4x^{3/2}} < 0$ for $x > 0)$

Inflection Points

Definition: A point $(c, f(c))$ is an **inflection point** if:

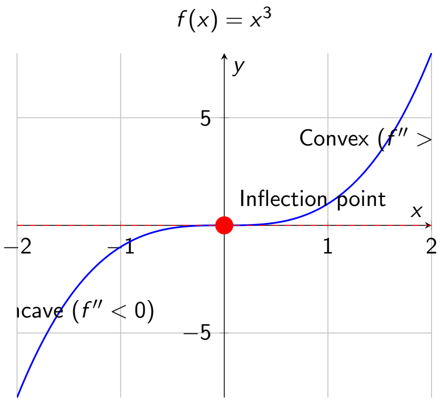
- $f''(c) = 0$ or $f''(c)$ does not exist, AND
- f'' changes sign at $x = c$

At an inflection point, the function changes from convex to concave (or vice versa).

Example: $f(x) = x^3$

- $f'(x) = 3x^2$
- $f''(x) = 6x$
- $f''(0) = 0$
- $f''(x) < 0$ for $x < 0$ (concave)
- $f''(x) > 0$ for $x > 0$ (convex)
- Inflection point at $(0, 0)$

Visualizing Inflection Points



Critical Points

Definition: A point $x = c$ is a **critical point** of f if:

- $f'(c) = 0$, OR
- $f'(c)$ does not exist

Why are critical points important?

- Local maxima and minima occur at critical points
- Not all critical points are local extrema (e.g., inflection points)

Example: $f(x) = x^3 - 3x^2 + 2$

- $f'(x) = 3x^2 - 6x = 3x(x - 2)$
- Critical points: $x = 0$ and $x = 2$

First Derivative Test

First Derivative Test: Let c be a critical point of f.

Local Maximum at $x = c$:

- $f'(x) > 0$ for $x < c$ (increasing)
- $f'(x) < 0$ for $x > c$ (decreasing)

Local Minimum at $x = c$:

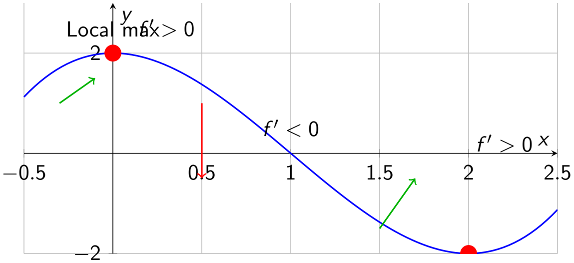
- $f'(x) < 0$ for $x < c$ (decreasing)
- $f'(x) > 0$ for $x > c$ (increasing)

Neither (possible inflection point):

- f' does not change sign at $x = c$

Summary: Look at the *sign change* of $f'(x)$ around $x = c$.

Visualizing the First Derivative Test



Second Derivative Test

Second Derivative Test: Let c be a critical point with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a **local minimum** at $x = c$

  (function is convex near c, curves upward)

- If $f''(c) < 0$, then f has a **local maximum** at $x = c$

  (function is concave near c, curves downward)

- If $f''(c) = 0$, the test is **inconclusive**

  (use first derivative test or higher-order derivatives)

Advantage: Only need to evaluate f'' at the critical point, not in an interval.

Example: Finding Local Extrema

Find and classify the critical points of $f(x) = x^3 - 3x^2 + 2$

Step 1: Find critical points

$$f'(x) = 3x^2 - 6x = 3x(x-2) = 0$$

Critical points: $x = 0$ and $x = 2$

Step 2: Apply second derivative test

$$f''(x) = 6x - 6$$

At $x = 0$: $f''(0) = -6 < 0 \Rightarrow$ **local maximum**
$f(0) = 2$, so local maximum at $(0, 2)$

At $x = 2$: $f''(2) = 6 > 0 \Rightarrow$ **local minimum**
$f(2) = -2$, so local minimum at $(2, -2)$

Comparison: First vs. Second Derivative Test

First Derivative Test	Second Derivative Test
Requires checking sign of f' on both sides of critical point	Only requires evaluating f'' at the critical point
Always works (assuming f' exists)	May be inconclusive if $f''(c) = 0$
More information about behavior	Less information, but faster

When to use which test?

- Use **second derivative test** if it's easy to compute $f''(c)$ and $f''(c) \neq 0$
- Use **first derivative test** if second derivative test is inconclusive or complicated

Example Where Second Derivative Test Fails

Example: $f(x) = x^4$

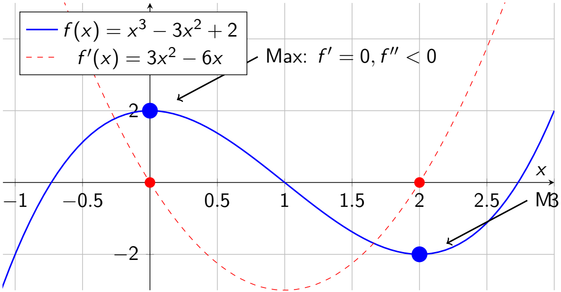
$f'(x) = 4x^3 = 0$ at $x = 0$ (critical point)
$f''(x) = 12x^2$, so $f''(0) = 0$ (inconclusive!)

Use first derivative test:

- For $x < 0$: $f'(x) = 4x^3 < 0$ (decreasing)
- For $x > 0$: $f'(x) = 4x^3 > 0$ (increasing)

Therefore, f has a **local minimum** at $x = 0$.

Alternative: Check $f'''(0) = 0$ and $f^{(4)}(0) = 24 > 0$
Higher-order derivative tests exist, but are rarely used.


$f(x) = x^3 - 3x^2 + 2$
$f'(x) = 3x^2 - 6x$
Max: $f' = 0, f'' < 0$
M3
x
-1
-0.5
0.5
1
1.5
2
2.5
2
-2

Summary

Higher-Order Derivatives:

- Provide information about curvature and behavior
- Physical interpretation: velocity, acceleration, jerk

Taylor's Theorem:

- Approximates functions with polynomials
- Better approximation with more terms

Convexity and Concavity:

- $f''(x) > 0 \Rightarrow$ convex (curves up)
- $f''(x) < 0 \Rightarrow$ concave (curves down)
- Inflection points where f'' changes sign

Derivative Tests:

- First derivative test: check sign change of f'
- Second derivative test: check sign of f'' at critical point

Practice Problems

Problem 1: Find the Taylor polynomial of degree 3 for $f(x) = \cos(x)$ at $x = 0$.

Problem 2: Determine where $f(x) = x^4 - 4x^3$ is convex and concave.

Problem 3: Find and classify all critical points of $f(x) = x^3 - 6x^2 + 9x + 1$ using both the first and second derivative tests.

Problem 4: Find the inflection points of $f(x) = x^4 - 6x^2 + 8x - 3$.