Derivatives: Rules and Applications

Mathematics Lecture Notes

1 Introduction

The derivative measures the rate of change of a function. In this lecture, we'll cover the fundamental rules for computing derivatives and explore derivatives of exponential and logarithmic functions.

1.1 Definition of the Derivative

For a function $f(x)$, the derivative at point x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2 Properties of Exponents and Logarithms

Before diving into derivatives, let's review the essential properties of exponentials and logarithms that we'll use throughout this lecture.

2.1 Exponent Properties

For any real numbers $a, b > 0$ and any real numbers m, n:

1. **Product rule:** $a^m \cdot a^n = a^{m+n}$
   Example: $2^3 \cdot 2^5 = 2^8 = 256$

2. **Quotient rule:** $\frac{a^m}{a^n} = a^{m-n}$
   Example: $\frac{x^5}{x^2} = x^{5-2} = x^3$

3. **Power of a power:** $(a^m)^n = a^{mn}$
   Example: $(x^2)^3 = x^6$

4. **Power of a product:** $(ab)^n = a^n b^n$
   Example: $(2x)^3 = 2^3 x^3 = 8x^3$

5. **Power of a quotient:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
   Example: $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$

6. **Negative exponent:** $a^{-n} = \frac{1}{a^n}$
   Example: $x^{-2} = \frac{1}{x^2}$

7. **Zero exponent:** $a^0 = 1$ (for $a \neq 0$)

8. **Fractional exponent:** $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Example: $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

2.2 Logarithm Properties

The logarithm $\log_a x$ is the inverse of the exponential function: if $y = a^x$, then $x = \log_a y$.

The natural logarithm uses base e: $\ln x = \log_e x$.

Key Identity: $a^{\log_a x} = x$ and $\log_a(a^x) = x$

Fundamental Properties:

For $a, b > 0$, $a \neq 1$, and $x, y > 0$:

1. **Product rule:** $\log_a(xy) = \log_a x + \log_a y$

   Example: $\ln(5 \cdot 3) = \ln 5 + \ln 3$

   This is why logarithms are useful: they turn multiplication into addition!

2. **Quotient rule:** $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

   Example: $\log_2\left(\frac{8}{4}\right) = \log_2 8 - \log_2 4 = 3 - 2 = 1$

3. **Power rule:** $\log_a(x^r) = r \log_a x$

   Example: $\ln(x^3) = 3 \ln x$

   This property is particularly useful in calculus!

4. **Change of base formula:** $\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$

   Example: $\log_2 10 = \frac{\ln 10}{\ln 2} \approx \frac{2.303}{0.693} \approx 3.32$

5. **Special values:**

   - $\log_a 1 = 0$ (because $a^0 = 1$)
   - $\log_a a = 1$ (because $a^1 = a$)
   - $\ln e = 1$

2.3 Common Simplifications

Example 1: Simplify $\ln(e^{3x})$

$$\ln(e^{3x}) = 3x \ln e = 3x \cdot 1 = 3x$$

Example 2: Simplify $e^{\ln(x^2)}$

$$e^{\ln(x^2)} = x^2$$

Example 3: Expand $\ln\left(\frac{x^2\sqrt{y}}{z^3}\right)$

$$\ln\left(\frac{x^2\sqrt{y}}{z^3}\right) = \ln(x^2\sqrt{y}) - \ln(z^3)$$
$$= \ln(x^2) + \ln(\sqrt{y}) - \ln(z^3)$$
$$= 2\ln x + \frac{1}{2}\ln y - 3\ln z$$

Example 4: Condense $3\ln x - 2\ln y + \ln z$

$$3\ln x - 2\ln y + \ln z = \ln(x^3) - \ln(y^2) + \ln z = \ln\left(\frac{x^3 z}{y^2}\right)$$

3 Differentiation Rules

3.1 Sum Rule

The derivative of a sum is the sum of the derivatives.

Theorem 3.1 (Sum Rule). *If $f(x)$ and $g(x)$ are differentiable, then:*

$$(f + g)' = f' + g'$$

Or more explicitly:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Example:

$$\frac{d}{dx}(x^3 + 5x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(5x^2) = 3x^2 + 10x$$

Extension: This extends to any finite number of functions:

$$\frac{d}{dx}[f_1(x) + f_2(x) + \cdots + f_n(x)] = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$$

3.2 Product Rule

The derivative of a product is NOT simply the product of derivatives.

Theorem 3.2 (Product Rule). *If $f(x)$ and $g(x)$ are differentiable, then:*

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Or:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Mnemonic: "First times derivative of second, plus second times derivative of first."
Example 1:

$$\frac{d}{dx}(x^2 \sin x) = 2x \cdot \sin x + x^2 \cdot \cos x$$

Example 2:

$$\frac{d}{dx}[(3x + 1)(x^2 - 2)] = 3(x^2 - 2) + (3x + 1)(2x) = 3x^2 - 6 + 6x^2 + 2x = 9x^2 + 2x - 6$$

3.3 Quotient Rule

For the derivative of a quotient, we use the quotient rule.

Theorem 3.3 (Quotient Rule). *If $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$, then:*

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

Or:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Mnemonic: "Low dee-high minus high dee-low, over low-low" (denominator times derivative of numerator minus numerator times derivative of denominator, all over denominator squared).

Example 1:

$$\frac{d}{dx}\left(\frac{x^2}{x+1}\right) = \frac{2x(x+1) - x^2(1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$$

Example 2:

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x \cdot 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

3.4   Chain Rule

The chain rule is used to differentiate composite functions.

Theorem 3.4 (Chain Rule). *If $y = f(g(x))$, then:*

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Or using Leibniz notation: if $y = f(u)$ and $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Intuition: The rate of change of y with respect to x equals the rate of change of y with respect to u times the rate of change of u with respect to x.

Example 1: Find $\frac{d}{dx}(3x + 1)^5$

Let $u = 3x + 1$, then $y = u^5$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot 3 = 15(3x + 1)^4$$

Example 2: Find $\frac{d}{dx}\sin(x^2)$

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

Example 3: Find $\frac{d}{dx}e^{x^3 + 2x}$

$$\frac{d}{dx}e^{x^3 + 2x} = e^{x^3 + 2x} \cdot (3x^2 + 2)$$

3.5   Combining Rules

Often, we need to combine multiple rules.

Example: Find $\frac{d}{dx}\left[\frac{(x^2+1)^3}{x}\right]$

Using quotient rule combined with chain rule:

$$\frac{d}{dx}\left[\frac{(x^2 + 1)^3}{x}\right] = \frac{x \cdot 3(x^2 + 1)^2 \cdot 2x - (x^2 + 1)^3 \cdot 1}{x^2}$$

$$= \frac{6x^2(x^2 + 1)^2 - (x^2 + 1)^3}{x^2}$$

$$= \frac{(x^2 + 1)^2[6x^2 - (x^2 + 1)]}{x^2}$$

$$= \frac{(x^2 + 1)^2(5x^2 - 1)}{x^2}$$

4 Higher-Order Derivatives

The derivative of a derivative is called a higher-order derivative.

4.1 Notation

- **Second derivative:** $f''(x)$, $\frac{d^2 f}{dx^2}$, or $\frac{d^2 y}{dx^2}$

- **Third derivative:** $f'''(x)$ or $\frac{d^3 f}{dx^3}$

- **n-th derivative:** $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$

4.2 Examples

Example 1: Find all derivatives of $f(x) = x^4 - 3x^3 + 2x - 5$

$$f'(x) = 4x^3 - 9x^2 + 2$$
$$f''(x) = 12x^2 - 18x$$
$$f'''(x) = 24x - 18$$
$$f^{(4)}(x) = 24$$
$$f^{(5)}(x) = 0$$

All subsequent derivatives are zero.
Example 2: Find $f''(x)$ for $f(x) = \sin x$

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$

Interpretation:

- $f'(x)$: instantaneous rate of change (velocity if f is position)

- $f''(x)$: rate of change of the rate of change (acceleration if f is position)

- $f'''(x)$: jerk (rate of change of acceleration)

5 Exponential Functions

5.1 The Natural Exponential Function

The exponential function e^x has a remarkable property: it is its own derivative.

Theorem 5.1.
$$\frac{d}{dx}(e^x) = e^x$$

This is why e is called the "natural" base for exponential functions.
Examples:
1. $\frac{d}{dx}(e^x + x^2) = e^x + 2x$
2. Using the chain rule:
$$\frac{d}{dx}(e^{3x}) = e^{3x} \cdot 3 = 3e^{3x}$$

3. Using the product rule:

$$\frac{d}{dx}(xe^x) = 1 \cdot e^x + x \cdot e^x = e^x(1+x)$$

4. Using the chain rule:

$$\frac{d}{dx}(e^{x^2+1}) = e^{x^2+1} \cdot 2x = 2xe^{x^2+1}$$

5.2   General Exponential Functions

For any base $a > 0$, $a \neq 1$:

Theorem 5.2.

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Note: When $a = e$, we get $\ln e = 1$, so $\frac{d}{dx}(e^x) = e^x \cdot 1 = e^x$.

Examples:
1. $\frac{d}{dx}(2^x) = 2^x \ln 2$
2. $\frac{d}{dx}(10^x) = 10^x \ln 10$
3. With chain rule:

$$\frac{d}{dx}(3^{2x}) = 3^{2x} \ln 3 \cdot 2 = 2 \ln 3 \cdot 3^{2x}$$

6   Logarithmic Functions

6.1   The Natural Logarithm

The natural logarithm $\ln x$ is the inverse of e^x.

Theorem 6.1.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$

Examples:
1. $\frac{d}{dx}(\ln x + x^3) = \frac{1}{x} + 3x^2$
2. Using the chain rule:

$$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

3. Using the chain rule:

$$\frac{d}{dx}[\ln(3x)] = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

Note: $\ln(3x) = \ln 3 + \ln x$, so the derivative is indeed $\frac{1}{x}$.
4. Product rule with logarithm:

$$\frac{d}{dx}[x \ln x] = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

6.2 General Logarithms

For logarithm with base $a > 0$, $a \neq 1$:

Theorem 6.2.
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \quad x > 0$$

Note: When $a = e$, we get $\ln e = 1$, so $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Examples:
1. $\frac{d}{dx}(\log_{10} x) = \frac{1}{x \ln 10}$
2. $\frac{d}{dx}(\log_2 x) = \frac{1}{x \ln 2}$

6.3 Logarithmic Differentiation

For complicated products, quotients, or powers, logarithmic differentiation can simplify calculations.

Method:

1. Take ln of both sides

2. Differentiate implicitly

3. Solve for $\frac{dy}{dx}$

Example: Find $\frac{dy}{dx}$ if $y = x^x$

$$\ln y = \ln(x^x) = x \ln x$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$
$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

7 Common Derivatives Reference

7.1 Power Functions

$$\frac{d}{dx}(c) = 0 \quad \text{(constant)}$$
$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$
$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

7.2 Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

7.3 Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

7.4 Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$$

8 Practice Problems

8.1 Problem Set

1. Find $\frac{d}{dx}(x^3 + 5x^2 - 3x + 7)$

   2. Find $\frac{d}{dx}[(2x + 1)(x^2 - 3)]$

   3. Find $\frac{d}{dx}\left(\frac{x^2+1}{x-2}\right)$

   4. Find $\frac{d}{dx}[(x^2 + 3x)^{10}]$

   5. Find $\frac{d}{dx}[e^{x^2}]$

   6. Find $\frac{d}{dx}[\ln(x^3 + 2x)]$

   7. Find $\frac{d^2}{dx^2}[x^4 - 2x^3 + x]$

   8. Find $\frac{d}{dx}[x^2 e^x]$

8.2 Solutions

1. $3x^2 + 10x - 3$

    2. $2(x^2 - 3) + (2x + 1)(2x) = 2x^2 - 6 + 4x^2 + 2x = 6x^2 + 2x - 6$

    3. $\frac{2x(x-2) - (x^2+1)(1)}{(x-2)^2} = \frac{2x^2 - 4x - x^2 - 1}{(x-2)^2} = \frac{x^2 - 4x - 1}{(x-2)^2}$

    4. $10(x^2 + 3x)^9 \cdot (2x + 3)$

    5. $e^{x^2} \cdot 2x = 2xe^{x^2}$

    6. $\frac{1}{x^3 + 2x} \cdot (3x^2 + 2) = \frac{3x^2 + 2}{x^3 + 2x}$

    7. First derivative: $4x^3 - 6x^2 + 1$; Second derivative: $12x^2 - 12x$

    8. $2x \cdot e^x + x^2 \cdot e^x = e^x(2x + x^2) = xe^x(2 + x)$

9 Summary

- **Sum Rule:** $(f + g)' = f' + g'$

- **Product Rule:** $(fg)' = f'g + fg'$

- **Quotient Rule:** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

- **Chain Rule:** $(f \circ g)' = f'(g(x)) \cdot g'(x)$

- **Exponential:** $(e^x)' = e^x$ and $(a^x)' = a^x \ln a$

- **Logarithm:** $(\ln x)' = \frac{1}{x}$ and $(\log_a x)' = \frac{1}{x \ln a}$

- Higher-order derivatives represent rates of change of rates of change

    These rules form the foundation for differential calculus and are essential for optimization, physics, and machine learning applications.