Looking at the orthogonal matrices A and B in Example 5.7, you may notice that not only do their columns form orthonormal sets—so do their *rows*. In fact, every orthogonal matrix has this property, as the next theorem shows.

Theorem 5.7 If Q is an orthogonal matrix, then its rows form an orthonormal set.

Proof From Theorem 5.5, we know that $Q^{-1} = Q^T$. Therefore,

$$(Q^T)^{-1} = (Q^{-1})^{-1} = Q = (Q^T)^T$$

so Q^T is an orthogonal matrix. Thus, the columns of Q^T—which are just the rows of Q—form an orthonormal set.

The final theorem in this section lists some other properties of orthogonal matrices.

Theorem 5.8 Let Q be an orthogonal matrix.

a. Q^{-1} is orthogonal.
b. $\det Q = \pm 1$
c. If λ is an eigenvalue of Q, then $|\lambda| = 1$.
d. If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is Q_1Q_2.

Proof We will prove property (c) and leave the proofs of the remaining properties as exercises.

(c) Let λ be an eigenvalue of Q with corresponding eigenvector $\mathbf{v}$. Then $Q\mathbf{v} = \lambda\mathbf{v}$, and, using Theorem 5.6(b), we have

$$\|\mathbf{v}\| = \|Q\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|\,\|\mathbf{v}\|$$

Since $\|\mathbf{v}\| \neq 0$, this implies that $|\lambda| = 1$.

$a + bi$

Remark Property (c) holds even for complex eigenvalues. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal with eigenvalues i and $-i$, both of which have absolute value 1.

Exercises 5.1

In Exercises 1–6, determine which sets of vectors are orthogonal.

1. $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

4. $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

5. $\begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 2 \\ 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7–10, show that the given vectors form an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$. Then use Theorem 5.2 to express $\mathbf{w}$ *as a linear combination of these basis vectors. Give the coordinate vector* $[\mathbf{w}]_{\mathcal{B}}$ *of* $\mathbf{w}$ *with respect to the basis* $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ *of* $\mathbb{R}^2$ *or* $\mathcal{B} = \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *of* $\mathbb{R}^3$.

7. $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

8. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -6 \\ 2 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

9. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

10. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

In Exercises 11–15, determine whether the given orthogonal set of vectors is orthonormal. If it is not, normalize the vectors to form an orthonormal set.

11. $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

12. $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -\frac{5}{2} \end{bmatrix}$

14. $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix}$

15. $\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{6}/3 \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ -\sqrt{3}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

In Exercises 16–21, determine whether the given matrix is orthogonal. If it is, find its inverse.

16. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

18. $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \\ -\frac{1}{3} & 0 & \frac{2}{5} \end{bmatrix}$

19. $\begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

20. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 0 & 1/\sqrt{6} \\ 0 & 2/3 & 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/3 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/3 & 0 & 1/\sqrt{2} \end{bmatrix}$

22. Prove Theorem 5.8(a).

23. Prove Theorem 5.8(b).

24. Prove Theorem 5.8(d).

25. Prove that every permutation matrix is orthogonal.

26. If Q is an orthogonal matrix, prove that any matrix obtained by rearranging the rows of Q is also orthogonal.

27. Let Q be an orthogonal 2×2 matrix and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^2$. If θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, prove that the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ is also θ. (This proves that the linear transformations defined by orthogonal matrices are *angle-preserving* in $\mathbb{R}^2$, a fact that is true in general.)

28. **(a)** Prove that an orthogonal 2×2 matrix must have the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector.

(b) Using part (a), show that every orthogonal 2×2 matrix is of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0 \le \theta < 2\pi$.

(c) Show that every orthogonal 2×2 matrix corresponds to either a rotation or a reflection in $\mathbb{R}^2$.

(d) Show that an orthogonal 2×2 matrix Q corresponds to a rotation in $\mathbb{R}^2$ if $\det Q = 1$ and a reflection in $\mathbb{R}^2$ if $\det Q = -1$.

In Exercises 29–32, use Exercise 28 to determine whether the given orthogonal matrix represents a rotation or a reflection. If it is a rotation, give the angle of rotation; if it is a reflection, give the line of reflection.

29. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

30. $\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

31. $\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

32. $\begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$

33. Let A and B be $n \times n$ orthogonal matrices.

(a) Prove that $A(A^T + B^T)B = A + B$.

(b) Use part (a) to prove that, if $\det A + \det B = 0$, then $A + B$ is not invertible.

34. Let $\mathbf{x}$ be a unit vector in $\mathbb{R}^n$. Partition $\mathbf{x}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \hdashline x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathbf{y} \end{bmatrix}$$

Let

$$Q = \left[\begin{array}{c:c} x_1 & \mathbf{y}^T \\ \hdashline \mathbf{y} & I - \left(\dfrac{1}{1 - x_1}\right)\mathbf{y}\mathbf{y}^T \end{array}\right]$$

Prove that Q is orthogonal. (This procedure gives a quick method for finding an orthonormal basis for $\mathbb{R}^n$ with a prescribed first vector $\mathbf{x}$, a construction that is frequently useful in applications.)

35. Prove that if an upper triangular matrix is orthogonal, then it must be a diagonal matrix.

36. Prove that if $n > m$, then there is no $m \times n$ matrix A such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

37. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$.

(a) Prove that, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$,

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{v}_1)(\mathbf{y} \cdot \mathbf{v}_1) + (\mathbf{x} \cdot \mathbf{v}_2)(\mathbf{y} \cdot \mathbf{v}_2) + \cdots + (\mathbf{x} \cdot \mathbf{v}_n)(\mathbf{y} \cdot \mathbf{v}_n)$$

(This identity is called ***Parseval's Identity.***)

(b) What does Parseval's Identity imply about the relationship between the dot products $\mathbf{x} \cdot \mathbf{y}$ and $[\mathbf{x}]_{\mathcal{B}} \cdot [\mathbf{y}]_{\mathcal{B}}$?

5.2 Orthogonal Complements and Orthogonal Projections

In this section, we generalize two concepts that we encountered in Chapter 1. The notion of a normal vector to a plane will be extended to orthogonal complements, and the projection of one vector onto another will give rise to the concept of orthogonal projection onto a subspace.

Orthogonal Complements

$W^\perp$ is pronounced "W perp."

A normal vector $\mathbf{n}$ to a plane is orthogonal to every vector in that plane. If the plane passes through the origin, then it is a subspace W of $\mathbb{R}^3$, as is span($\mathbf{n}$). Hence, we have two subspaces of $\mathbb{R}^3$ with the property that every vector of one is orthogonal to every vector of the other. This is the idea behind the following definition.

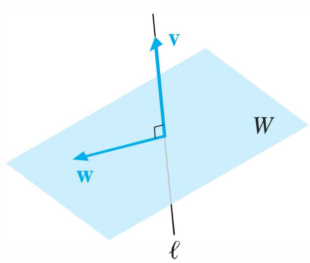


Figure 5.5
$\ell = W^\perp$ and $W = \ell^\perp$

Definition Let W be a subspace of $\mathbb{R}^n$. We say that a vector $\mathbf{v}$ in $\mathbb{R}^n$ is ***orthogonal to*** W if $\mathbf{v}$ is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the ***orthogonal complement of*** W, denoted $W^\perp$. That is,

$$W^\perp = \{\mathbf{v} \text{ in } \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all } \mathbf{w} \text{ in } W\}$$

Example 5.8

If W is a plane through the origin in $\mathbb{R}^3$ and ℓ is the line through the origin perpendicular to W (i.e., parallel to the normal vector to W), then every vector $\mathbf{v}$ on ℓ is orthogonal to every vector $\mathbf{w}$ in W; hence, $\ell = W^\perp$. Moreover, W consists *precisely* of those vectors $\mathbf{w}$ that are orthogonal to every $\mathbf{v}$ on ℓ; hence, we also have $W = \ell^\perp$. Figure 5.5 illustrates this situation.

In Example 5.8, the orthogonal complement of a subspace turned out to be another subspace. Also, the complement of the complement of a subspace was the original subspace. These properties are true in general and are proved as properties (a) and (b) of Theorem 5.9. Properties (c) and (d) will also be useful. (Recall that the *intersection* $A \cap B$ of sets A and B consists of their common elements. See Appendix A.)

Theorem 5.9

Let W be a subspace of $\mathbb{R}^n$.

a. $W^\perp$ is a subspace of $\mathbb{R}^n$.
b. $(W^\perp)^\perp = W$
c. $W \cap W^\perp = \{\mathbf{0}\}$
d. If $W = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then $\mathbf{v}$ is in $W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$.

Proof (a) Since $\mathbf{0} \cdot \mathbf{w} = 0$ for all $\mathbf{w}$ in W, $\mathbf{0}$ is in $W^\perp$. Let $\mathbf{u}$ and $\mathbf{v}$ be in $W^\perp$ and let c be a scalar. Then

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all } \mathbf{w} \text{ in } W$$

Therefore,

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = 0 + 0 = 0$$

so $\mathbf{u} + \mathbf{v}$ is in $W^\perp$.

We also have

$$(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w}) = c(0) = 0$$

from which we see that $c\mathbf{u}$ is in $W^\perp$. It follows that $W^\perp$ is a subspace of $\mathbb{R}^n$.

(b) We will prove this property as Corollary 5.12.

(c) You are asked to prove this property in Exercise 23.

(d) You are asked to prove this property in Exercise 24.

We can now express some fundamental relationships involving the subspaces associated with an $m \times n$ matrix.

Theorem 5.10

Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\operatorname{row}(A))^\perp = \operatorname{null}(A) \quad \text{and} \quad (\operatorname{col}(A))^\perp = \operatorname{null}(A^T)$$

Proof If $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then $\mathbf{x}$ is in $(\operatorname{row}(A))^\perp$ if and only if $\mathbf{x}$ is orthogonal to every row of A. But this is true if and only if $A\mathbf{x} = \mathbf{0}$, which is equivalent to $\mathbf{x}$ being in $\operatorname{null}(A)$, so we have established the first identity. To prove the second identity, we simply replace A by A^T and use the fact that $\operatorname{row}(A^T) = \operatorname{col}(A)$.

Thus, an $m \times n$ matrix has four subspaces: $\operatorname{row}(A)$, $\operatorname{null}(A)$, $\operatorname{col}(A)$, and $\operatorname{null}(A^T)$. The first two are orthogonal complements in $\mathbb{R}^n$, and the last two are orthogonal

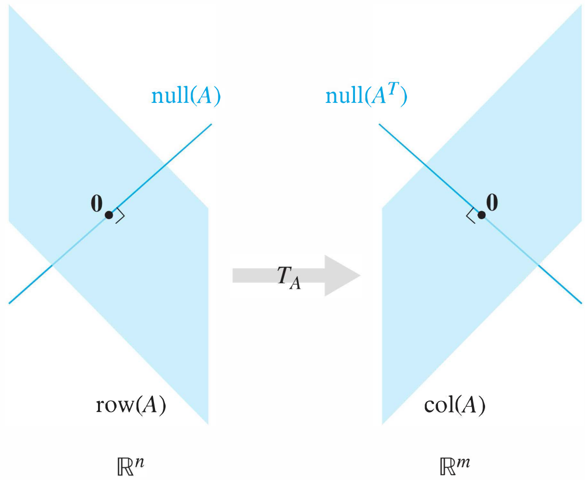


Figure 5.6
The four fundamental subspaces

complements in $\mathbb{R}^m$. The $m \times n$ matrix A defines a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ whose range is col(A). Moreover, this transformation sends null(A) to $\mathbf{0}$ in $\mathbb{R}^m$. Figure 5.6 illustrates these ideas schematically. These four subspaces are called the ***fundamental subspaces*** of the $m \times n$ matrix A.

Example 5.9

Find bases for the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

and verify Theorem 5.10.

Solution In Examples 3.45, 3.47, and 3.48, we computed bases for the row space, column space, and null space of A. We found that row(A) = span($\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$), where

$$\mathbf{u}_1 = [1 \quad 0 \quad 1 \quad 0 \quad -1], \quad \mathbf{u}_2 = [0 \quad 1 \quad 2 \quad 0 \quad 3], \quad \mathbf{u}_3 = [0 \quad 0 \quad 0 \quad 1 \quad 4]$$

Also, null(A) = span($\mathbf{x}_1, \mathbf{x}_2$), where

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

To show that $(\text{row}(A))^\perp = \text{null}(A)$, it is enough to show that every $\mathbf{u}_i$ is orthogonal to each $\mathbf{x}_j$, which is an easy exercise. (Why is this sufficient?)

The column space of A is $\text{col}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

We still need to compute the null space of A^T. Row reduction produces

$$[A^T \mid \mathbf{0}] = \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 4 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ 3 & 0 & 1 & 6 & 0 \\ 1 & 1 & -2 & 1 & 0 \\ 6 & -1 & 1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, if $\mathbf{y}$ is in the null space of A^T, then $y_1 = -y_4$, $y_2 = -6y_4$, and $y_3 = -3y_4$. It follows that

$$\text{null}(A^T) = \left\{ \begin{bmatrix} -y_4 \\ -6y_4 \\ -3y_4 \\ y_4 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ -6 \\ -3 \\ 1 \end{bmatrix} \right)$$

and it is easy to check that this vector is orthogonal to $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

The method of Example 5.9 is easily adapted to other situations.

Example 5.10

Let W be the subspace of $\mathbb{R}^5$ spanned by

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 5 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

Find a basis for $W^\perp$.

Solution The subspace W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ is the same as the column space of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & -1 \\ 5 & 2 & 4 \\ 0 & -2 & -1 \\ 5 & 3 & 5 \end{bmatrix}$$

Therefore, by Theorem 5.10, $W^{\perp} = (\text{col}(A))^{\perp} = \text{null}(A^T)$, and we may proceed as in the previous example. We compute

$$[A^T \mid \mathbf{0}] = \left[\begin{array}{rrrrr|r} 1 & -3 & 5 & 0 & 5 & 0 \\ -1 & 1 & 2 & -2 & 3 & 0 \\ 0 & -1 & 4 & -1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right]$$

Hence, $\mathbf{y}$ is in $W^{\perp}$ if and only if $y_1 = -3y_4 - 4y_5$, $y_2 = -y_4 - 3y_5$, and $y_3 = -2y_5$. It follows that

$$W^{\perp} = \left\{ \begin{bmatrix} -3y_4 - 4y_5 \\ -y_4 - 3y_5 \\ -2y_5 \\ y_4 \\ y_5 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

and these two vectors form a basis for $W^{\perp}$.

Orthogonal Projections

Recall that, in $\mathbb{R}^2$, the projection of a vector $\mathbf{v}$ onto a nonzero vector $\mathbf{u}$ is given by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$$

Furthermore, the vector $\text{perp}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})$ is orthogonal to $\text{proj}_{\mathbf{u}}(\mathbf{v})$, and we can decompose $\mathbf{v}$ as

$$\mathbf{v} = \text{proj}_{\mathbf{u}}(\mathbf{v}) + \text{perp}_{\mathbf{u}}(\mathbf{v})$$

as shown in Figure 5.7.

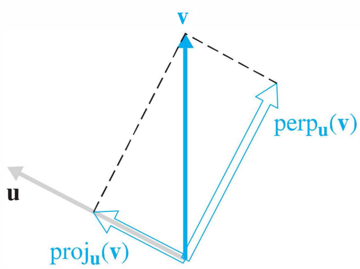


Figure 5.7
$\mathbf{v} = \text{proj}_{\mathbf{u}}(\mathbf{v}) + \text{perp}_{\mathbf{u}}(\mathbf{v})$

If we let $W = \text{span}(\mathbf{u})$, then $\mathbf{w} = \text{proj}_{\mathbf{u}}(\mathbf{v})$ is in W and $\mathbf{w}^{\perp} = \text{perp}_{\mathbf{u}}(\mathbf{v})$ is in $W^{\perp}$. We therefore have a way of "decomposing" $\mathbf{v}$ into the sum of two vectors, one from W and the other orthogonal to W—namely, $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$. We now generalize this idea to $\mathbb{R}^n$.

Definition Let W be a subspace of $\mathbb{R}^n$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal basis for W. For any vector $\mathbf{v}$ in $\mathbb{R}^n$, the ***orthogonal projection of* $\mathbf{v}$ *onto* W** is defined as

$$\text{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 + \cdots + \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k} \right) \mathbf{u}_k$$

The ***component of* $\mathbf{v}$ *orthogonal to* W** is the vector

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

Each summand in the definition of $\text{proj}_W(\mathbf{v})$ is also a projection onto a single vector (or, equivalently, the one-dimensional subspace spanned by it—in our previous sense). Therefore, with the notation of the preceding definition, we can write

$$\text{proj}_W(\mathbf{v}) = \text{proj}_{\mathbf{u}_1}(\mathbf{v}) + \cdots + \text{proj}_{\mathbf{u}_k}(\mathbf{v})$$

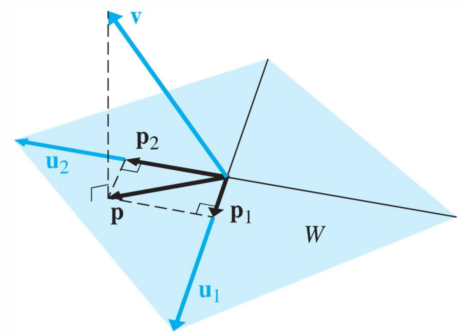


Figure 5.8
$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$

Since the vectors $\mathbf{u}_i$ are orthogonal, the orthogonal projection of $\mathbf{v}$ onto W is the sum of its projections onto one-dimensional subspaces that are mutually orthogonal. Figure 5.8 illustrates this situation with $W = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$, $\mathbf{p} = \text{proj}_W(\mathbf{v})$, $\mathbf{p}_1 = \text{proj}_{\mathbf{u}_1}(\mathbf{v})$, and $\mathbf{p}_2 = \text{proj}_{\mathbf{u}_2}(\mathbf{v})$.

As a special case of the definition of $\text{proj}_W(\mathbf{v})$, we now also have a nice geometric interpretation of Theorem 5.2. In terms of our present notation and terminology, that theorem states that if $\mathbf{w}$ is in the subspace W of $\mathbb{R}^n$, which has orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, then

$$\begin{aligned}\mathbf{w} &= \left(\frac{\mathbf{w}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \cdots + \left(\frac{\mathbf{w}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k}\right)\mathbf{v}_k \\ &= \text{proj}_{\mathbf{v}_1}(\mathbf{w}) + \cdots + \text{proj}_{\mathbf{v}_k}(\mathbf{w})\end{aligned}$$

Thus, $\mathbf{w}$ is decomposed into a sum of orthogonal projections onto mutually orthogonal one-dimensional subspaces of W.

The definition above seems to depend on the choice of orthogonal basis; that is, a different basis $\{\mathbf{u}'_1, \dots, \mathbf{u}'_k\}$ for W would appear to give a "different" $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$. Fortunately, this is not the case, as we will soon prove. For now, let's be content with an example.

Example 5.11

Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$, and let $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{v}$ onto W and the component of $\mathbf{v}$ orthogonal to W.

Solution In Example 5.3, we found an orthogonal basis for W. Taking

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} \mathbf{u}_1\cdot\mathbf{v} &= 2 & \mathbf{u}_2\cdot\mathbf{v} &= -2 \\ \mathbf{u}_1\cdot\mathbf{u}_1 &= 2 & \mathbf{u}_2\cdot\mathbf{u}_2 &= 3 \end{aligned}$$

Therefore,

$$\operatorname{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2$$
$$= \tfrac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

and $$\operatorname{perp}_W(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{8}{3} \end{bmatrix}$$

It is easy to see that $\operatorname{proj}_W(\mathbf{v})$ is in W, since it satisfies the equation of the plane. It is equally easy to see that $\operatorname{perp}_W(\mathbf{v})$ is orthogonal to W, since it is a scalar multiple of the normal vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ to W. (See Figure 5.9.)

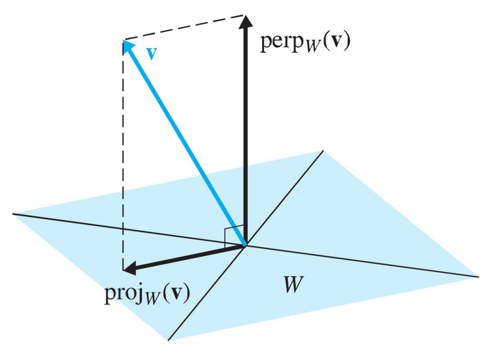


Figure 5.9
$\mathbf{v} = \operatorname{proj}_W(\mathbf{v}) + \operatorname{perp}_W(\mathbf{v})$

The next theorem shows that we can always find a decomposition of a vector with respect to a subspace and its orthogonal complement.

Theorem 5.11 **The Orthogonal Decomposition Theorem**

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Then there are unique vectors $\mathbf{w}$ in W and $\mathbf{w}^\perp$ in $W^\perp$ such that

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

Proof We need to show two things: that such a decomposition *exists* and that it is *unique*.

To show existence, we choose an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for W. Let $\mathbf{w} = \operatorname{proj}_W(\mathbf{v})$ and let $\mathbf{w}^\perp = \operatorname{perp}_W(\mathbf{v})$. Then

$$\mathbf{w} + \mathbf{w}^\perp = \operatorname{proj}_W(\mathbf{v}) + \operatorname{perp}_W(\mathbf{v}) = \operatorname{proj}_W(\mathbf{v}) + (\mathbf{v} - \operatorname{proj}_W(\mathbf{v})) = \mathbf{v}$$

Clearly, $\mathbf{w} = \text{proj}_W(\mathbf{v})$ is in W, since it is a linear combination of the basis vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$. To show that $\mathbf{w}^\perp$ is in $W^\perp$, it is enough to show that $\mathbf{w}^\perp$ is orthogonal to each of the basis vectors $\mathbf{u}_i$, by Theorem 5.9(d). We compute

$$\begin{aligned}
\mathbf{u}_i \cdot \mathbf{w}^\perp &= u_i \cdot \text{perp}_W(\mathbf{v}) \\
&= \mathbf{u}_i \cdot (\mathbf{v} - \text{proj}_W(\mathbf{v})) \\
&= \mathbf{u}_i \cdot \left(\mathbf{v} - \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \cdots - \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)\mathbf{u}_k\right) \\
&= \mathbf{u}_i \cdot \mathbf{v} - \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)(\mathbf{u}_i \cdot \mathbf{u}_1) - \cdots - \left(\frac{\mathbf{u}_i \cdot \mathbf{v}}{\mathbf{u}_i \cdot \mathbf{u}_i}\right)(\mathbf{u}_i \cdot \mathbf{u}_i) - \cdots \\
&\quad - \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)(\mathbf{u}_i \cdot \mathbf{u}_k) \\
&= \mathbf{u}_i \cdot \mathbf{v} - 0 - \cdots - \left(\frac{\mathbf{u}_i \cdot \mathbf{v}}{\mathbf{u}_i \cdot \mathbf{u}_i}\right)(\mathbf{u}_i \cdot \mathbf{u}_i) - \cdots - 0 \\
&= \mathbf{u}_i \cdot \mathbf{v} - \mathbf{u}_i \cdot \mathbf{v} = 0
\end{aligned}$$

since $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for $j \neq i$. This proves that $\mathbf{w}^\perp$ is in $W^\perp$ and completes the existence part of the proof.

To show the uniqueness of this decomposition, let's suppose we have another decomposition $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_1^\perp$, where $\mathbf{w}_1$ is in W and $\mathbf{w}_1^\perp$ is in $W^\perp$. Then $\mathbf{w} + \mathbf{w}^\perp = \mathbf{w}_1 + \mathbf{w}_1^\perp$, so

$$\mathbf{w} - \mathbf{w}_1 = \mathbf{w}_1^\perp - \mathbf{w}^\perp$$

But since $\mathbf{w} - \mathbf{w}_1$ is in W and $\mathbf{w}_1^\perp - \mathbf{w}^\perp$ is in $W^\perp$ (because these are subspaces), we know that this common vector is in $W \cap W^\perp = \{\mathbf{0}\}$ [using Theorem 5.9(c)]. Thus,

$$\mathbf{w} - \mathbf{w}_1 = \mathbf{w}_1^\perp - \mathbf{w}^\perp = 0$$

so $\mathbf{w}_1 = \mathbf{w}$ and $\mathbf{w}_1^\perp = \mathbf{w}_1$.

Example 5.11 illustrated the Orthogonal Decomposition Theorem. When W is the subspace of $\mathbb{R}^3$ given by the plane with equation $x - y + 2z = 0$, the orthogonal decomposition of $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ with respect to W is $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$, where

$$\mathbf{w} = \text{proj}_W(\mathbf{v}) = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \quad \text{and} \quad \mathbf{w}^\perp = \text{perp}_W(\mathbf{v}) = \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{8}{3} \end{bmatrix}$$

The uniqueness of the orthogonal decomposition guarantees that the definitions of $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$ do not depend on the choice of orthogonal basis. The Orthogonal Decomposition Theorem also allows us to prove property (b) of Theorem 5.9. We state that property here as a corollary to the Orthogonal Decomposition Theorem.

Corollary 5.12 If W is a subspace of $\mathbb{R}^n$, then

$$(W^\perp)^\perp = W$$

Proof If $\mathbf{w}$ is in W and $\mathbf{x}$ is in $W^{\perp}$, then $\mathbf{w} \cdot \mathbf{x} = 0$. But this now implies that $\mathbf{w}$ is in $(W^{\perp})^{\perp}$. Hence, $W \subseteq (W^{\perp})^{\perp}$. Now let $\mathbf{v}$ be in $(W^{\perp})^{\perp}$. By Theorem 5.11, we can write $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ for (unique) vectors $\mathbf{w}$ in W and $\mathbf{w}^{\perp}$ in $W^{\perp}$. But now

$$0 = \mathbf{v} \cdot \mathbf{w}^{\perp} = (\mathbf{w} + \mathbf{w}^{\perp}) \cdot \mathbf{w}^{\perp} = \mathbf{w} \cdot \mathbf{w}^{\perp} + \mathbf{w}^{\perp} \cdot \mathbf{w}^{\perp} = 0 + \mathbf{w}^{\perp} \cdot \mathbf{w}^{\perp} = \mathbf{w}^{\perp} \cdot \mathbf{w}^{\perp}$$

so $\mathbf{w}^{\perp} = \mathbf{0}$. Therefore, $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp} = \mathbf{w}$, and thus $\mathbf{v}$ is in W. This shows that $(W^{\perp})^{\perp} \subseteq W$ and, since the reverse inclusion is also true, we conclude that $(W^{\perp})^{\perp} = W$, as required.

There is also a nice relationship between the dimensions of W and $W^{\perp}$, expressed in Theorem 5.13.

Theorem 5.13 If W is a subspace of $\mathbb{R}^n$, then

$$\dim W + \dim W^{\perp} = n$$

Proof Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal basis for W and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ be an orthogonal basis for $W^{\perp}$. Then $\dim W = k$ and $\dim W^{\perp} = l$. Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_l\}$. We claim that $\mathcal{B}$ is an orthogonal basis for $\mathbb{R}^n$.

We first note that, since each $\mathbf{u}_i$ is in W and each $\mathbf{v}_j$ is in $W^{\perp}$,

$$\mathbf{u}_i \cdot \mathbf{v}_j = 0 \quad \text{for } i = 1, \ldots, k \text{ and } j = 1, \ldots, l$$

Thus, $\mathcal{B}$ is an orthogonal set and, hence, is linearly independent, by Theorem 5.1. Next, if $\mathbf{v}$ is a vector in $\mathbb{R}^n$, the Orthogonal Decomposition Theorem tells us that $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ for some $\mathbf{w}$ in W and $\mathbf{w}^{\perp}$ in $W^{\perp}$. Since $\mathbf{w}$ can be written as a linear combination of the vectors $\mathbf{u}_i$ and $\mathbf{w}^{\perp}$ can be written as a linear combination of the vectors $\mathbf{v}_j$, $\mathbf{v}$ can be written as a linear combination of the vectors in $\mathcal{B}$. Therefore, $\mathcal{B}$ spans $\mathbb{R}^n$ also and so is a basis for $\mathbb{R}^n$. It follows that $k + l = \dim \mathbb{R}^n$, or

$$\dim W + \dim W^{\perp} = n$$

As a lovely bonus, when we apply this result to the fundamental subspaces of a matrix, we get a quick proof of the Rank Theorem (Theorem 3.26), restated here as Corollary 5.14.

Corollary 5.14 **The Rank Theorem**

If A is an $m \times n$ matrix, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

Proof In Theorem 5.13, take $W = \operatorname{row}(A)$. Then $W^{\perp} = \operatorname{null}(A)$, by Theorem 5.10, so $\dim W = \operatorname{rank}(A)$ and $\dim W^{\perp} = \operatorname{nullity}(A)$. The result follows.

Note that we get a counterpart identity by taking $W = \operatorname{col}(A)$ [and therefore $W^{\perp} = \operatorname{null}(A^T)$]:

$$\operatorname{rank}(A) + \operatorname{nullity}(A^T) = m$$

Sections 5.1 and 5.2 have illustrated some of the advantages of working with orthogonal bases. However, we have not established that every subspace *has* an orthogonal basis, nor have we given a method for constructing such a basis (except in particular examples, such as Example 5.3). These issues are the subject of the next section.

Exercises 5.2

In Exercises 1–6, find the orthogonal complement $W^\perp$ of W and give a basis for $W^\perp$.

1. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x - y = 0 \right\}$

2. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 3x + 4y = 0 \right\}$

3. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$

4. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y + 3z = 0 \right\}$

5. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = -t, z = 3t \right\}$

6. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = \frac{1}{2}t, y = -\frac{1}{2}t, z = 2t \right\}$

In Exercises 7 and 8, find bases for the row space and null space of A. Verify that every vector in row(A) is orthogonal to every vector in null(A).

7. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 & -1 & 0 & 2 \\ -2 & 0 & 2 & 4 & 4 \\ 2 & 2 & -2 & 0 & 1 \\ -3 & -1 & 3 & 4 & 5 \end{bmatrix}$

In Exercises 9 and 10, find bases for the column space of A and the null space of A^T for the given exercise. Verify that every vector in col(A) is orthogonal to every vector in null(A^T).

9. Exercise 7

10. Exercise 8

In Exercises 11–14, let W be the subspace spanned by the given vectors. Find a basis for $W^\perp$.

11. $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

12. $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

13. $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 6 \\ 3 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ -3 \\ -2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 1 \end{bmatrix}$

14. $\mathbf{w}_1 = \begin{bmatrix} 4 \\ 6 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -3 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$

In Exercises 15–18, find the orthogonal projection of $\mathbf{v}$ onto the subspace W spanned by the vectors $\mathbf{u}_i$. (You may assume that the vectors $\mathbf{u}_i$ are orthogonal.)

15. $\mathbf{v} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

16. $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

17. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

18. $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 19–22, find the orthogonal decomposition of $\mathbf{v}$ *with respect to W.*

19. $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$

20. $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

21. $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$

22. $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right)$

23. Prove Theorem 5.9(c).

24. Prove Theorem 5.9(d).

25. Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ a vector in $\mathbb{R}^n$. Suppose that $\mathbf{w}$ and $\mathbf{w}'$ are orthogonal vectors with $\mathbf{w}$ in W and that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$. Is it necessarily true that $\mathbf{w}'$ is in $W^\perp$? Either prove that it is true or find a counterexample.

26. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and let $W = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Is it necessarily true that $W^\perp = \operatorname{span}(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$? Either prove that it is true or find a counterexample.

In Exercises 27–29, let W be a subspace of $\mathbb{R}^n$, *and let* $\mathbf{x}$ *be a vector in* $\mathbb{R}^n$.

27. Prove that $\mathbf{x}$ is in W if and only if $\operatorname{proj}_W(\mathbf{x}) = \mathbf{x}$.

28. Prove that $\mathbf{x}$ is orthogonal to W if and only if $\operatorname{proj}_W(\mathbf{x}) = \mathbf{0}$.

29. Prove that $\operatorname{proj}_W(\operatorname{proj}_W(\mathbf{x})) = \operatorname{proj}_W(\mathbf{x})$.

30. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthonormal set in $\mathbb{R}^n$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$.

(a) Prove that

$$\|\mathbf{x}\|^2 \geq |\mathbf{x}\cdot\mathbf{v}_1|^2 + |\mathbf{x}\cdot\mathbf{v}_2|^2 + \cdots + |\mathbf{x}\cdot\mathbf{v}_k|^2$$

(This inequality is called ***Bessel's Inequality.***)

(b) Prove that Bessel's Inequality is an equality if and only if $\mathbf{x}$ is in $\operatorname{span}(S)$.

5.3 The Gram-Schmidt Process and the *QR* Factorization

In this section, we present a simple method for constructing an orthogonal (or orthonormal) basis for any subspace of $\mathbb{R}^n$. This method will then lead us to one of the most useful of all matrix factorizations.

The Gram-Schmidt Process

We would like to be able to find an orthogonal basis for a subspace W of $\mathbb{R}^n$. The idea is to begin with an arbitrary basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for W and to "orthogonalize" it one vector at a time. We will illustrate the basic construction with the subspace W from Example 5.3.

Example 5.12

Let $W = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Construct an orthogonal basis for W.

Solution Starting with $\mathbf{x}_1$, we get a second vector that is orthogonal to it by taking the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$ (Figure 5.10).

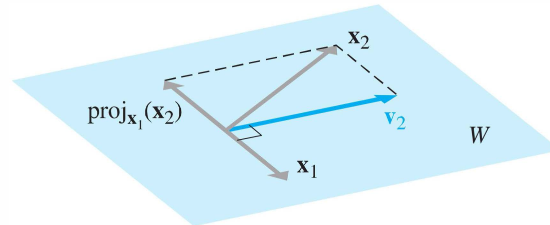


Figure 5.10
Constructing $\mathbf{v}_2$ orthogonal to $\mathbf{x}_1$

Algebraically, we set $\mathbf{v}_1 = \mathbf{x}_1$, so

$$\begin{aligned}\mathbf{v}_2 &= \text{perp}_{\mathbf{x}_1}(\mathbf{x}_2) = \mathbf{x}_2 - \text{proj}_{\mathbf{x}_1}(\mathbf{x}_2)\\ &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_1\cdot\mathbf{x}_2}{\mathbf{x}_1\cdot\mathbf{x}_1}\right)\mathbf{x}_1\\ &= \begin{bmatrix}-2\\0\\1\end{bmatrix} - \left(\frac{-2}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set of vectors in W. Hence, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set and therefore a basis for W, since $\dim W = 2$.

Remark Observe that this method depends on the *order* of the original basis vectors. In Example 5.12, if we had taken $\mathbf{x}_1 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, we would have obtained a different orthogonal basis for W. (Verify this.)

The generalization of this method to more than two vectors begins as in Example 5.12. Then the process is to iteratively construct the components of subsequent vectors orthogonal to all of the vectors that have already been constructed. The method is known as the ***Gram-Schmidt Process.***

Theorem 5.15 **The Gram-Schmidt Process**

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, & W_1 &= \text{span}(\mathbf{x}_1)\\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1\cdot\mathbf{x}_2}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1, & W_2 &= \text{span}(\mathbf{x}_1, \mathbf{x}_2)\\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1\cdot\mathbf{x}_3}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2\cdot\mathbf{x}_3}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2, & W_3 &= \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\\
&\vdots\\
\mathbf{v}_k &= \mathbf{x}_k - \left(\frac{\mathbf{v}_1\cdot\mathbf{x}_k}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2\cdot\mathbf{x}_k}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 - \cdots\\
&\quad - \left(\frac{\mathbf{v}_{k-1}\cdot\mathbf{x}_k}{\mathbf{v}_{k-1}\cdot\mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, & W_k &= \text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)
\end{aligned}$$

Then for each $i = 1, \ldots, k$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W.

Jörgen Pedersen Gram (1850–1916) was a Danish actuary (insurance statistician) who was interested in the science of measurement. He first published the process that bears his name in an 1883 paper on least squares. Erhard Schmidt (1876–1959) was a German mathematician who studied under the great David Hilbert and is considered one of the founders of the branch of mathematics known as functional analysis. His contribution to the Gram-Schmidt Process came in a 1907 paper on integral equations, in which he wrote out the details of the method more explicitly than Gram had done.

Stated succinctly, Theorem 5.15 says that every subspace of $\mathbb{R}^n$ has an orthogonal basis, and it gives an algorithm for constructing such a basis.

Proof We will prove by induction that, for each $i = 1, \dots, k$, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i.

Since $\mathbf{v}_1 = \mathbf{x}_1$, clearly $\{\mathbf{v}_1\}$ is an (orthogonal) basis for $W_1 = \operatorname{span}(\mathbf{x}_1)$. Now assume that, for some $i < k$, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. Then

$$\mathbf{v}_{i+1} = \mathbf{x}_{i+1} - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_{i+1}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_{i+1}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_i \cdot \mathbf{x}_{i+1}}{\mathbf{v}_i \cdot \mathbf{v}_i}\right)\mathbf{v}_i$$

By the induction hypothesis, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for $\operatorname{span}(\mathbf{x}_1, \dots, \mathbf{x}_i) = W_i$. Hence,

$$\mathbf{v}_{i+1} = \mathbf{x}_{i+1} - \operatorname{proj}_{W_i}(\mathbf{x}_{i+1}) = \operatorname{perp}_{W_i}(\mathbf{x}_{i+1})$$

So, by the Orthogonal Decomposition Theorem, $\mathbf{v}_{i+1}$ is orthogonal to W_i. By definition, $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linear combinations of $\mathbf{x}_1, \dots, \mathbf{x}_i$ and, hence, are in W_i. Therefore, $\{\mathbf{v}_1, \dots, \mathbf{v}_{i+1}\}$ is an orthogonal set of vectors in W_{i+1}.

Moreover, $\mathbf{v}_{i+1} \neq \mathbf{0}$, since otherwise $\mathbf{x}_{i+1} = \operatorname{proj}_{W_i}(\mathbf{x}_{i+1})$, which in turn implies that $\mathbf{x}_{i+1}$ is in W_i. But this is impossible, since $W_i = \operatorname{span}(\mathbf{x}_1, \dots, \mathbf{x}_i)$ and $\{\mathbf{x}_1, \dots, \mathbf{x}_{i+1}\}$ is linearly independent. (Why?) We conclude that $\{\mathbf{v}_1, \dots, \mathbf{v}_{i+1}\}$ is a set of $i + 1$ linearly independent vectors in W_{i+1}. Consequently, $\{\mathbf{v}_1, \dots, \mathbf{v}_{i+1}\}$ is a basis for W_{i+1}, since $\dim W_{i+1} = i + 1$. This completes the proof.

If we require an orthonormal basis for W, we simply need to normalize the orthogonal vectors produced by the Gram-Schmidt Process. That is, for each i, we replace $\mathbf{v}_i$ by the unit vector $\mathbf{q}_i = (1/\|\mathbf{v}_i\|)\mathbf{v}_i$.

Example 5.13

Apply the Gram-Schmidt Process to construct an orthonormal basis for the subspace $W = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of $\mathbb{R}^4$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution First we note that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set, so it forms a basis for W. We begin by setting $\mathbf{v}_1 = \mathbf{x}_1$. Next, we compute the component of $\mathbf{x}_2$ orthogonal to $W_1 = \operatorname{span}(\mathbf{v}_1)$:

$$\begin{aligned}
\mathbf{v}_2 = \operatorname{perp}_{W_1}(\mathbf{x}_2) &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
&= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\tfrac{2}{4}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\end{aligned}$$

For hand calculations, it is a good idea to "scale" $\mathbf{v}_2$ at this point to eliminate fractions. When we are finished, we can rescale the orthogonal set we are constructing to obtain an orthonormal set; thus, we can replace each $\mathbf{v}_i$ by any convenient scalar multiple without affecting the final result. Accordingly, we replace $\mathbf{v}_2$ by

$$\mathbf{v}_2' = 2\mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

We now find the component of $\mathbf{x}_3$ orthogonal to

$$W_2 = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2')$$

using the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$:

$$\begin{aligned}\mathbf{v}_3 = \operatorname{perp}_{W_2}(\mathbf{x}_3) &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2' \cdot \mathbf{x}_3}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\right)\mathbf{v}_2' \\ &= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \left(\tfrac{1}{4}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \left(\tfrac{15}{20}\right)\begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}\end{aligned}$$

Again, we rescale and use $\mathbf{v}_3' = 2\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

We now have an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\}$ for W. (Check to make sure that these vectors are orthogonal.) To obtain an orthonormal basis, we normalize each vector:

$$\mathbf{q}_1 = \left(\frac{1}{\|\mathbf{v}_1\|}\right)\mathbf{v}_1 = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\mathbf{q}_2 = \left(\frac{1}{\|\mathbf{v}_2'\|}\right)\mathbf{v}_2' = \left(\frac{1}{2\sqrt{5}}\right)\begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}$$

$$\mathbf{q}_3 = \left(\frac{1}{\|\mathbf{v}_3'\|}\right)\mathbf{v}_3' = \left(\frac{1}{\sqrt{6}}\right)\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

Then $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for W.

One of the important uses of the Gram-Schmidt Process is to construct an orthogonal basis that contains a specified vector. The next example illustrates this application.

Example 5.14

Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution We first find *any* basis for $\mathbb{R}^3$ containing $\mathbf{v}_1$. If we take

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then $\{\mathbf{v}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly a basis for $\mathbb{R}^3$. (Why?) We now apply the Gram-Schmidt Process to this basis to obtain

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\tfrac{2}{14}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ \frac{5}{7} \\ -\frac{3}{7} \end{bmatrix}, \quad \mathbf{v}_2' = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

and finally

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2' \cdot \mathbf{x}_3}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\right)\mathbf{v}_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\tfrac{3}{14}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\tfrac{-3}{35}\right)\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{-3}{10} \\ 0 \\ \frac{1}{10} \end{bmatrix},$$

$$\mathbf{v}_3' = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\}$ is an orthogonal basis for $\mathbb{R}^3$ that contains $\mathbf{v}_1$.

Similarly, given a unit vector, we can find an orthonormal basis that contains it by using the preceding method and then normalizing the resulting orthogonal vectors.

Remark When the Gram-Schmidt Process is implemented on a computer, there is almost always some roundoff error, leading to a loss of orthogonality in the vectors $\mathbf{q}_i$. To avoid this loss of orthogonality, some modifications are usually made. The vectors $\mathbf{v}_i$ are normalized as soon as they are computed, rather than at the end, to give the vectors $\mathbf{q}_i$, and as each $\mathbf{q}_i$ is computed, the remaining vectors $\mathbf{x}_j$ are modified to be orthogonal to $\mathbf{q}_i$. This procedure is known as the ***Modified Gram-Schmidt Process.*** In practice, however, a version of the QR factorization is used to compute orthonormal bases.

The *QR* Factorization

If A is an $m \times n$ matrix with linearly independent columns (requiring that $m \geq n$), then applying the Gram-Schmidt Process to these columns yields a very useful factorization of A into the product of a matrix Q with orthonormal columns and an upper triangular matrix R. This is the ***QR factorization,*** and it has applications to the numerical approximation of eigenvalues, which we explore at the end of this section, and to the problem of least squares approximation, which we discuss in Chapter 7.

To see how the QR factorization arises, let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the (linearly independent) columns of A and let $\mathbf{q}_1, \ldots, \mathbf{q}_n$ be the orthonormal vectors obtained by applying the Gram-Schmidt Process to A with normalizations. From Theorem 5.15, we know that, for each $i = 1, \ldots, n$,

$$W_i = \operatorname{span}(\mathbf{a}_1, \ldots, \mathbf{a}_i) = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_i)$$

Therefore, there are scalars $r_{1i}, r_{2i}, \ldots, r_{ii}$ such that

$$\mathbf{a}_i = r_{1i}\mathbf{q}_1 + r_{2i}\mathbf{q}_2 + \cdots + r_{ii}\mathbf{q}_i \quad \text{for } i = 1, \ldots, n$$

That is,

$$\begin{aligned} \mathbf{a}_1 &= r_{11}\mathbf{q}_1 \\ \mathbf{a}_2 &= r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2 \\ &\vdots \\ \mathbf{a}_n &= r_{1n}\mathbf{q}_1 + r_{2n}\mathbf{q}_2 + \cdots + r_{nn}\mathbf{q}_n \end{aligned}$$

which can be written in matrix form as

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n]\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix} = QR$$

Clearly, the matrix Q has orthonormal columns. It is also the case that the diagonal entries of R are all nonzero. To see this, observe that if $r_{ii} = 0$, then $\mathbf{a}_i$ is a linear combination of $\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}$ and, hence, is in W_{i-1}. But then $\mathbf{a}_i$ would be a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}$, which is impossible, since $\mathbf{a}_1, \ldots, \mathbf{a}_i$ are linearly independent. We conclude that $r_{ii} \neq 0$ for $i = 1, \ldots, n$. Since R is upper triangular, it follows that it must be invertible. (See Exercise 23.)

We have proved the following theorem.

Theorem 5.16 **The QR Factorization**

Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix.

Remarks

- We can also arrange for the diagonal entries of R to be *positive*. If any $r_{ii} < 0$, simply replace $\mathbf{q}_i$ by $-\mathbf{q}_i$ and r_{ii} by $-r_{ii}$.
- The requirement that A have linearly independent columns is a necessary one. To prove this, suppose that A is an $m \times n$ matrix that has a QR factorization, as in Theorem 5.16. Then, since R is invertible, we have $Q = AR^{-1}$. Hence, $\operatorname{rank}(Q) = \operatorname{rank}(A)$, by Exercise 61 in Section 3.5. But $\operatorname{rank}(Q) = n$, since its columns are orthonormal and, therefore, linearly independent. So $\operatorname{rank}(A) = n$ too, and consequently the columns of A are linearly independent, by the Fundamental Theorem.

- The QR factorization can be extended to arbitrary matrices in a slightly modified form. If A is $m \times n$, it is possible to find a sequence of orthogonal matrices $Q_1, \dots, Q_{m-1}$ such that $Q_{m-1} \cdots Q_2Q_1A$ is an upper triangular $m \times n$ matrix R. Then $A = QR$, where $Q = (Q_{m-1} \cdots Q_2Q_1)^{-1}$ is an orthogonal matrix. We will examine this approach in Exploration: The Modified QR Factorization.

Example 5.15

Find a QR factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution The columns of A are just the vectors from Example 5.13. The orthonormal basis for col(A) produced by the Gram-Schmidt Process was

$$\mathbf{q}_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

so

$$Q = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3] = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix}$$

From Theorem 5.16, $A = QR$ for some upper triangular matrix R. To find R, we use the fact that Q has orthonormal columns and, hence, $Q^TQ = I$. Therefore,

$$Q^TA = Q^TQR = IR = R$$

We compute

$$R = Q^TA = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

Exercises 5.3

In Exercises 1–4, the given vectors form a basis for $\mathbb{R}^2$ or $\mathbb{R}^3$. Apply the Gram-Schmidt Process to obtain an orthogonal basis. Then normalize this basis to obtain an orthonormal basis.

1. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. $\mathbf{x}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

4. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 5 and 6, the given vectors form a basis for a subspace W of $\mathbb{R}^3$ or $\mathbb{R}^4$. Apply the Gram-Schmidt Process to obtain an orthogonal basis for W.

5. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$

6. $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7 and 8, find the orthogonal decomposition of $\mathbf{v}$ *with respect to the subspace W.*

7. $\mathbf{v} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}$, W as in Exercise 5

8. $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$, W as in Exercise 6

Use the Gram-Schmidt Process to find an orthogonal basis for the column spaces of the matrices in Exercises 9 and 10.

9. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$

11. Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$.

12. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

In Exercises 13 and 14, fill in the missing entries of Q to make Q an orthogonal matrix.

13. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & * \\ 0 & 1/\sqrt{3} & * \\ -1/\sqrt{2} & 1/\sqrt{3} & * \end{bmatrix}$

14. $Q = \begin{bmatrix} 1/2 & 2/\sqrt{14} & * & * \\ 1/2 & 1/\sqrt{14} & * & * \\ 1/2 & 0 & * & * \\ 1/2 & -3/\sqrt{14} & * & * \end{bmatrix}$

In Exercises 15 and 16, find a QR factorization of the matrix in the given exercise.

15. Exercise 9

16. Exercise 10

In Exercises 17 and 18, the columns of Q were obtained by applying the Gram-Schmidt Process to the columns of A. Find the upper triangular matrix R such that $A = QR$.

17. $A = \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}$

19. If A is an orthogonal matrix, find a QR factorization of A.

20. Prove that A is invertible if and only if $A = QR$, where Q is orthogonal and R is upper triangular with nonzero entries on its diagonal.

In Exercises 21 and 22, use the method suggested by Exercise 20 to compute A^{-1} for the matrix A in the given exercise.

21. Exercise 9

22. Exercise 15

23. Let A be an $m \times n$ matrix with linearly independent columns. Give an alternative proof that the upper triangular matrix R in a QR factorization of A must be invertible, using property (c) of the Fundamental Theorem.

24. Let A be an $m \times n$ matrix with linearly independent columns and let $A = QR$ be a QR factorization of A. Show that A and Q have the same column space.

Explorations

The Modified *QR* Factorization

When the matrix A does not have linearly independent columns, the Gram-Schmidt Process as we have stated it does not work and so cannot be used to develop a generalized QR factorization of A. There is a modification of the Gram-Schmidt Process that can be used, but instead we will explore a method that converts A into upper triangular form one column at a time, using a sequence of orthogonal matrices. The method is analogous to that of LU factorization, in which the matrix L is formed using a sequence of elementary matrices.

The first thing we need is the "orthogonal analogue" of an elementary matrix; that is, we need to know how to construct an orthogonal matrix Q that will transform a given column of A—call it $\mathbf{x}$—into the corresponding column of R—call it $\mathbf{y}$. By Theorem 5.6, it will be necessary that $\|\mathbf{x}\| = \|Q\mathbf{x}\| = \|\mathbf{y}\|$. Figure 5.11 suggests a way to proceed: We can reflect $\mathbf{x}$ in a line perpendicular to $\mathbf{x} - \mathbf{y}$. If

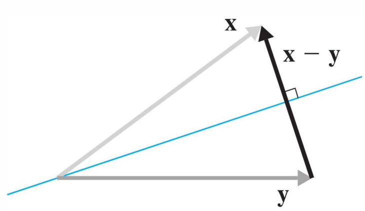


Figure 5.11

$$\mathbf{u} = \left(\frac{1}{\|\mathbf{x} - \mathbf{y}\|}\right)(\mathbf{x} - \mathbf{y}) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

is the unit vector in the direction of $\mathbf{x} - \mathbf{y}$, then $\mathbf{u}^{\perp} = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$ is orthogonal to $\mathbf{u}$, and we can use Exercise 26 in Section 3.6 to find the standard matrix Q of the reflection in the line through the origin in the direction of $\mathbf{u}^{\perp}$.

1. Show that $Q = \begin{bmatrix} 1 - 2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1 - 2d_2^2 \end{bmatrix} = I - 2\mathbf{u}\mathbf{u}^T$.

2. Compute Q for

$$\text{(a)}\ \mathbf{u} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \qquad \text{(b)}\ \mathbf{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

We can generalize the definition of Q as follows. If $\mathbf{u}$ is any unit vector in $\mathbb{R}^n$, we define an $n \times n$ matrix Q as

$$Q = I - 2\mathbf{u}\mathbf{u}^T$$

Courtesy Oak Ridge National Lab

Alston Householder (1904–1993) was one of the pioneers in the field of numerical linear algebra. He was the first to present a systematic treatment of algorithms for solving problems involving linear systems. In addition to introducing the widely used Householder transformations that bear his name, he was one of the first to advocate the systematic use of norms in linear algebra. His 1964 book *The Theory of Matrices in Numerical Analysis* is considered a classic.

Such a matrix is called a ***Householder matrix*** (or an ***elementary reflector***).

3. Prove that every Householder matrix Q satisfies the following properties:
(a) Q is symmetric. (b) Q is orthogonal. (c) $Q^2 = I$

4. Prove that if Q is a Householder matrix corresponding to the unit vector $\mathbf{u}$, then

$$Q\mathbf{v} = \begin{cases} -\mathbf{v} & \text{if } \mathbf{v} \text{ is in span}(\mathbf{u}) \\ \mathbf{v} & \text{if } \mathbf{v}\cdot\mathbf{u} = \mathbf{0} \end{cases}$$

5. Compute Q for $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and verify Problems 3 and 4.

6. Let $\mathbf{x} \neq \mathbf{y}$ with $\|\mathbf{x}\| = \|\mathbf{y}\|$ and set $\mathbf{u} = (1/\|\mathbf{x} - \mathbf{y}\|)(\mathbf{x} - \mathbf{y})$. Prove that the corresponding Householder matrix Q satisfies $Q\mathbf{x} = \mathbf{y}$. [*Hint:* Apply Exercise 57 in Section 1.2 to the result in Problem 4.]

7. Find Q and verify Problem 6 for

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

We are now ready to perform the triangularization of an $m \times n$ matrix A, column by column.

8. Let $\mathbf{x}$ be the first column of A and let

$$\mathbf{y} = \begin{bmatrix} \|\mathbf{x}\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Show that if Q_1 is the Householder matrix given by Problem 6, then Q_1A is a matrix with the block form

$$Q_1A = \begin{bmatrix} * & * \\ \mathbf{0} & A_1 \end{bmatrix}$$

where A_1 is $(m-1)\times(n-1)$.

If we repeat Problem 8 on the matrix A_1, we use a Householder matrix P_2 such that

$$P_2A_1 = \begin{bmatrix} * & * \\ \mathbf{0} & A_2 \end{bmatrix}$$

where A_2 is $(m-2)\times(n-2)$.

9. Set $Q_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix}$. Show that Q_2 is an orthogonal matrix and that

$$Q_2Q_1A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ \mathbf{0} & \mathbf{0} & A_2 \end{bmatrix}$$

10. Show that we can continue in this fashion to find a sequence of orthogonal matrices $Q_1, \ldots, Q_{m-1}$ such that $Q_{m-1} \cdots Q_2 Q_1 A = R$ is an upper triangular $m \times n$ matrix (i.e., $r_{ij} = 0$ if $i > j$).

11. Deduce that $A = QR$ with $Q = Q_1 Q_2 \cdots Q_{m-1}$ orthogonal.

12. Use the method of this exploration to find a QR factorization of

$$\text{(a) } A = \begin{bmatrix} 3 & 9 & 1 \\ -4 & 3 & 2 \end{bmatrix} \quad \text{(b) } A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & -4 & 1 & 1 \\ 2 & -5 & -1 & -2 \end{bmatrix}$$

Approximating Eigenvalues with the *QR* Algorithm

See G. H. Golub and C. F. Van Loan, *Matrix Computations* (Baltimore: Johns Hopkins University Press, 1983).

One of the best (and most widely used) methods for numerically approximating the eigenvalues of a matrix makes use of the QR factorization. The purpose of this exploration is to introduce this method, the ***QR* algorithm,** and to show it at work in a few examples. For a more complete treatment of this topic, consult any good text on numerical linear algebra. (You will find it helpful to use a CAS to perform the calculations in the problems below.)

Given a square matrix A, the first step is to factor it as $A = QR$ (using whichever method is appropriate). Then we define $A_1 = RQ$.

1. First prove that A_1 is similar to A. Then prove that A_1 has the same eigenvalues as A.

2. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$, find A_1 and verify that it has the same eigenvalues as A.

Continuing the algorithm, we factor A_1 as $A_1 = Q_1 R_1$ and set $A_2 = R_1 Q_1$. Then we factor $A_2 = Q_2 R_2$ and set $A_3 = R_2 Q_2$, and so on. That is, for $k \geq 1$, we compute $A_k = Q_k R_k$ and then set $A_{k+1} = R_k Q_k$.

3. Prove that A_k is similar to A for all $k \geq 1$.

4. Continuing Problem 2, compute A_2, A_3, A_4, and A_5, using two-decimal-place accuracy. What do you notice?

It can be shown that if the eigenvalues of A are all real and have distinct absolute values, then the matrices A_k approach an upper triangular matrix U.

5. What will be true of the diagonal entries of this matrix U?

6. Approximate the eigenvalues of the following matrices by applying the QR algorithm. Use two-decimal-place accuracy and perform at least five iterations.

$$\text{(a) } \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \qquad \text{(b) } \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\text{(c) } \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -4 & 0 & 1 \end{bmatrix} \qquad \text{(d) } \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -2 & 4 & 2 \end{bmatrix}$$

7. Apply the QR algorithm to the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$. What happens? Why?

8. Shift the eigenvalues of the matrix in Problem 7 by replacing A with $B = A + 0.9I$. Apply the QR algorithm to B and then shift back by subtracting 0.9 from the (approximate) eigenvalues of B. Verify that this method approximates the eigenvalues of A.

9. Let $Q_0 = Q$ and $R_0 = R$. First show that

$$Q_0Q_1 \cdots Q_{k-1}A_k = AQ_0Q_1 \cdots Q_{k-1}$$

for all $k \geq 1$. Then show that

$$(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$$

[*Hint:* Repeatedly use the same approach used for the first equation, working from the "inside out."] Finally, deduce that $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0)$ is the QR factorization of A^{k+1}.

5.4 Orthogonal Diagonalization of Symmetric Matrices

We saw in Chapter 4 that a square matrix with real entries will not necessarily have real eigenvalues. Indeed, the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has complex eigenvalues i and $-i$. We also discovered that not all square matrices are diagonalizable. The situation changes dramatically if we restrict our attention to real *symmetric* matrices. As we will show in this section, all of the eigenvalues of a real symmetric matrix are real, and such a matrix is always diagonalizable.

Recall that a symmetric matrix is one that equals its own transpose. Let's begin by studying the diagonalization process for a symmetric 2×2 matrix.

Example 5.16

If possible, diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution The characteristic polynomial of A is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$, from which we see that A has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$. Solving for the corresponding eigenvectors, we find

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

respectively. So A is diagonalizable, and if we set $P = [\mathbf{v}_1 \ \ \mathbf{v}_2]$, then we know that $P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = D$.

However, we can do better. Observe that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. So, if we normalize them to get the unit eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

and then take

$$Q = [\mathbf{u}_1 \ \ \mathbf{u}_2] = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

we have $Q^{-1}AQ = D$ also. But now Q is an *orthogonal* matrix, since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set of vectors. Therefore, $Q^{-1} = Q^T$, and we have $Q^TAQ = D$. (Note that checking is easy, since computing Q^{-1} only involves taking a transpose!)

The situation in Example 5.16 is the one that interests us. It is important enough to warrant a new definition.

Definition A square matrix A is ***orthogonally diagonalizable*** if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

We are interested in finding conditions under which a matrix is orthogonally diagonalizable. Theorem 5.17 shows us where to look.

Theorem 5.17 If A is orthogonally diagonalizable, then A is symmetric.

Proof If A is orthogonally diagonalizable, then there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$. Since $Q^{-1} = Q^T$, we have $Q^TQ = I = QQ^T$, so

$$QDQ^T = QQ^TAQQ^T = IAI = A$$

But then

$$A^T = (QDQ^T)^T = (Q^T)^TD^TQ^T = QDQ^T = A$$

since every diagonal matrix is symmetric. Hence, A is symmetric.

Remark Theorem 5.17 shows that the orthogonally diagonalizable matrices are all to be found *among* the symmetric matrices. It does *not* say that every symmetric matrix must be orthogonally diagonalizable. However, it is a remarkable fact that this indeed is true! Finding a proof for this amazing result will occupy us for much of the rest of this section.

a + bi

We next prove that we don't need to worry about *complex* eigenvalues when working with symmetric matrices with *real* entries.

Theorem 5.18 If A is a real symmetric matrix, then the eigenvalues of A are real.

Recall that the *complex conjugate* of a complex number $z = a + bi$ is the number $\bar{z} = a - bi$ (see Appendix C). To show that z is real, we need to show that $b = 0$. One way to do this is to show that $z = \bar{z}$, for then $bi = -bi$ (or $2bi = 0$), from which it follows that $b = 0$.

We can also extend the notion of complex conjugate to vectors and matrices by, for example, defining $\overline{A}$ to be the matrix whose entries are the complex conjugates of the entries of A; that is, if $A = [a_{ij}]$, then $\overline{A} = [\overline{a}_{ij}]$. The rules for complex conjugation extend easily to matrices; in particular, we have $\overline{AB} = \overline{A}\,\overline{B}$ for compatible matrices A and B.

Proof Suppose that λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then $A\mathbf{v} = \lambda\mathbf{v}$, and, taking complex conjugates, we have $\overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}}$. But then

$$A\overline{\mathbf{v}} = \overline{A}\,\overline{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}$$

since A is real. Taking transposes and using the fact that A is symmetric, we have

$$\overline{\mathbf{v}}^TA = \overline{\mathbf{v}}^TA^T = (A\overline{\mathbf{v}})^T = (\overline{\lambda}\,\overline{\mathbf{v}})^T = \overline{\lambda}\,\overline{\mathbf{v}}^T$$

Therefore,

$$\lambda(\overline{\mathbf{v}}^T\mathbf{v}) = \overline{\mathbf{v}}^T(\lambda\mathbf{v}) = \overline{\mathbf{v}}^T(A\mathbf{v}) = (\overline{\mathbf{v}}^TA)\mathbf{v} = (\overline{\lambda}\,\overline{\mathbf{v}}^T)\mathbf{v} = \overline{\lambda}(\overline{\mathbf{v}}^T\mathbf{v})$$

or $(\lambda - \overline{\lambda})(\overline{\mathbf{v}}^T\mathbf{v}) = 0$.

Now if $\mathbf{v} = \begin{bmatrix} a_1 + b_1i \\ \vdots \\ a_n + b_ni \end{bmatrix}$, then $\overline{\mathbf{v}} = \begin{bmatrix} a_1 - b_1i \\ \vdots \\ a_n - b_ni \end{bmatrix}$, so

$$\overline{\mathbf{v}}^T\mathbf{v} = (a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2) \neq 0$$

since $\mathbf{v} \neq \mathbf{0}$ (because it is an eigenvector). We conclude that $\lambda - \overline{\lambda} = 0$, or $\lambda = \overline{\lambda}$. Hence, λ is real.

Theorem 4.20 showed that, for any square matrix, eigenvectors corresponding to distinct eigenvalues are linearly independent. For symmetric matrices, something stronger is true: Such eigenvectors are *orthogonal*.

Theorem 5.19 If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Proof Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors corresponding to the distinct eigenvalues $\lambda_1 \neq \lambda_2$ so that $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. Using $A^T = A$ and the fact that $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}$ for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we have

$$\begin{aligned}\lambda_1(\mathbf{v}_1\cdot\mathbf{v}_2) = (\lambda_1\mathbf{v}_1)\cdot\mathbf{v}_2 = A\mathbf{v}_1\cdot\mathbf{v}_2 &= (A\mathbf{v}_1)^T\mathbf{v}_2 \\ = (\mathbf{v}_1^TA^T)\mathbf{v}_2 = (\mathbf{v}_1^TA)\mathbf{v}_2 &= \mathbf{v}_1^T(A\mathbf{v}_2) \\ = \mathbf{v}_1^T(\lambda_2\mathbf{v}_2) = \lambda_2(\mathbf{v}_1^T\mathbf{v}_2) &= \lambda_2(\mathbf{v}_1\cdot\mathbf{v}_2)\end{aligned}$$

Hence, $(\lambda_1 - \lambda_2)(\mathbf{v}_1\cdot\mathbf{v}_2) = 0$. But $\lambda_1 - \lambda_2 \neq 0$, so $\mathbf{v}_1\cdot\mathbf{v}_2 = 0$, as we wished to show.

Example 5.17

Verify the result of Theorem 5.19 for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution The characteristic polynomial of A is $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda - 4)\cdot(\lambda - 1)^2$, from which it follows that the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 1$. The corresponding eigenspaces are

$$E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \quad\text{and}\quad E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

(Check this.) We easily verify that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0 \quad\text{and}\quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

from which it follows that every vector in E_4 is orthogonal to every vector in E_1. (Why?)

Remark Note that $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1$. Thus, eigenvectors corresponding to the *same* eigenvalue need not be orthogonal.

We can now prove the main result of this section. It is called the Spectral Theorem, since the set of eigenvalues of a matrix is sometimes called the ***spectrum*** of the matrix. (Technically, we should call Theorem 5.20 the *Real* Spectral Theorem, since there is a corresponding result for matrices with complex entries.)

Theorem 5.20 **The Spectral Theorem**

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

Spectrum is a Latin word meaning "image." When atoms vibrate, they emit light. And when light passes through a prism, it spreads out into a spectrum—a band of rainbow colors. Vibration frequencies correspond to the eigenvalues of a certain operator and are visible as bright lines in the spectrum of light that is emitted from a prism. Thus, we can literally see the eigenvalues of the atom in its spectrum, and for this reason, it is appropriate that the word *spectrum* has come to be applied to the set of all eigenvalues of a matrix (or operator).

Proof We have already proved the "if" part as Theorem 5.17. To prove the "only if" implication, we proceed by induction on n. For $n = 1$, there is nothing to do, since a 1×1 matrix is already in diagonal form. Now assume that every $k \times k$ real symmetric matrix with real eigenvalues is orthogonally diagonalizable. Let $n = k + 1$ and let A be an $n \times n$ real symmetric matrix with real eigenvalues.

Let λ_1 be one of the eigenvalues of A and let $\mathbf{v}_1$ be a corresponding eigenvector. Then $\mathbf{v}_1$ is a *real* vector (why?) and we can assume that $\mathbf{v}_1$ is a unit vector, since otherwise we can normalize it and we will still have an eigenvector corresponding to λ_1. Using the Gram-Schmidt Process, we can extend $\mathbf{v}_1$ to an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$. Now we form the matrix

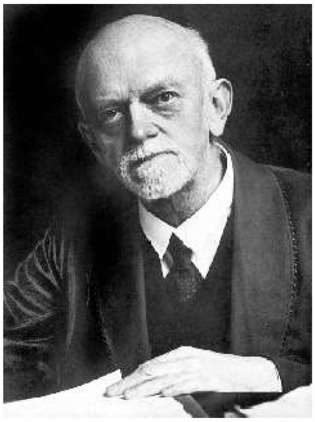
$$Q_1 = [\mathbf{v}_1 \quad \mathbf{v}_2 \cdots \mathbf{v}_n]$$

Then Q_1 is orthogonal, and

$$Q_1^TAQ_1 = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} A\,[\mathbf{v}_1 \quad \mathbf{v}_2 \cdots \mathbf{v}_n] = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} [A\mathbf{v}_1 \quad A\mathbf{v}_2 \cdots A\mathbf{v}_n]$$

$$= \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} [\lambda_1\mathbf{v}_1 \quad A\mathbf{v}_2 \cdots A\mathbf{v}_n]$$

$$= \left[\begin{array}{c|c} \lambda_1 & * \\ \hline \mathbf{0} & A_1 \end{array}\right] = B$$



Baldwin H. Ward & Kathryn C. Ward/CORBIS

In a lecture he delivered at the University of Göttingen in 1905, the German mathematician David Hilbert (1862–1943) considered linear operators acting on certain infinite-dimensional vector spaces. Out of this lecture arose the notion of a quadratic form in infinitely many variables, and it was in this context that Hilbert first used the term *spectrum* to mean a complete set of eigenvalues. The spaces in question are now called *Hilbert spaces.*

Hilbert made major contributions to many areas of mathematics, among them integral equations, number theory, geometry, and the foundations of mathematics. In 1900, at the Second International Congress of Mathematicians in Paris, Hilbert gave an address entitled "The Problems of Mathematics." In it, he challenged mathematicians to solve 23 problems of fundamental importance during the coming century. Many of the problems have been solved—some were proved true, others false—and some may never be solved. Nevertheless, Hilbert's speech energized the mathematical community and is often regarded as the most influential speech ever given about mathematics.

since $\mathbf{v}_1^T(\lambda_1\mathbf{v}_1) = \lambda_1(\mathbf{v}_1^T\mathbf{v}_1) = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_1) = \lambda_1$ and $\mathbf{v}_i^T(\lambda_1\mathbf{v}_1) = \lambda_1(\mathbf{v}_i^T\mathbf{v}_1) = \lambda_1(\mathbf{v}_i \cdot \mathbf{v}_1) = 0$ for $i \neq 1$, because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal set.

But

$$B^T = (Q_1^TAQ_1)^T = Q_1^TA^T(Q_1^T)^T = Q_1^TAQ_1 = B$$

so B is symmetric. Therefore, B has the block form

$$B = \left[\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & A_1 \end{array}\right]$$

and A_1 is symmetric. Furthermore, B is similar to A (why?), so the characteristic polynomial of B is equal to the characteristic polynomial of A, by Theorem 4.22. By Exercise 39 in Section 4.3, the characteristic polynomial of A_1 divides the characteristic polynomial of A. It follows that the eigenvalues of A_1 are also eigenvalues of A and, hence, are real. We also see that A_1 has real entries. (Why?) Thus, A_1 is a $k \times k$ real symmetric matrix with real eigenvalues, so the induction hypothesis applies to it. Hence, there is an orthogonal matrix P_2 such that $P_2^TA_1P_2$ is a diagonal matrix—say, D_1. Now let

$$Q_2 = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & P_2 \end{array}\right]$$

Then Q_2 is an orthogonal $(k+1)\times(k+1)$ matrix, and therefore so is $Q = Q_1Q_2$. Consequently,

$$\begin{aligned} Q^TAQ &= (Q_1Q_2)^TA(Q_1Q_2) = (Q_2^TQ_1^T)A(Q_1Q_2) = Q_2^T(Q_1^TAQ_1)Q_2 = Q_2^TBQ_2 \\ &= \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & P_2^T \end{array}\right]\left[\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & A_1 \end{array}\right]\left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & P_2 \end{array}\right] \\ &= \left[\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & P_2^TA_1P_2 \end{array}\right] \\ &= \left[\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & D_1 \end{array}\right] \end{aligned}$$

which is a diagonal matrix. This completes the induction step, and we conclude that, for all $n \geq 1$, an $n \times n$ real symmetric matrix with real eigenvalues is orthogonally diagonalizable.

Example 5.18

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution This is the matrix from Example 5.17. We have already found that the eigenspaces of A are

$$E_4 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad E_1 = \operatorname{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

We need three orthonormal eigenvectors. First, we apply the Gram-Schmidt Process to

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

The new vector, which has been constructed to be orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, is still in E_1 (why?) and so is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus, we have three mutually orthogonal vectors, and all we need to do is normalize them and construct a matrix Q with these vectors as its columns. We find that

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

and it is straightforward to verify that

$$Q^TAQ = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Spectral Theorem allows us to write a real symmetric matrix A in the form $A = QDQ^T$, where Q is orthogonal and D is diagonal. The diagonal entries of D are just the eigenvalues of A, and if the columns of Q are the orthonormal vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$, then, using the column-row representation of the product, we have

$$\begin{aligned} A = QDQ^T &= [\mathbf{q}_1 \ \cdots \ \mathbf{q}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \\ &= [\lambda_1\mathbf{q}_1 \ \cdots \ \lambda_n\mathbf{q}_n] \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \\ &= \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \cdots + \lambda_n\mathbf{q}_n\mathbf{q}_n^T \end{aligned}$$

This is called the ***spectral decomposition*** of A. Each of the terms $\lambda_i\mathbf{q}_i\mathbf{q}_i^T$ is a rank 1 matrix, by Exercise 62 in Section 3.5, and $\mathbf{q}_i\mathbf{q}_i^T$ is actually the matrix of the projection onto the subspace spanned by $\mathbf{q}_i$. (See Exercise 25.) For this reason, the spectral decomposition

$$A = \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \cdots + \lambda_n\mathbf{q}_n\mathbf{q}_n^T$$

is sometimes referred to as the ***projection form of the Spectral Theorem.***

Example 5.19

Find the spectral decomposition of the matrix A from Example 5.18.

Solution From Example 5.18, we have:

$$\lambda_1 = 4, \qquad \lambda_2 = 1, \qquad \lambda_3 = 1$$

$$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Therefore,

$$\mathbf{q}_1\mathbf{q}_1^T = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} [1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}] = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\mathbf{q}_2\mathbf{q}_2^T = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} [-1/\sqrt{2} \quad 0 \quad 1/\sqrt{2}] = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$\mathbf{q}_3\mathbf{q}_3^T = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} [-1/\sqrt{6} \quad 2/\sqrt{6} \quad -1/\sqrt{6}] = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

so

$$\begin{aligned} A &= \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \lambda_3\mathbf{q}_3\mathbf{q}_3^T \\ &= 4\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \end{aligned}$$

which can be easily verified.

In this example, $\lambda_2 = \lambda_3$, so we could combine the last two terms $\lambda_2\mathbf{q}_2\mathbf{q}_2^T + \lambda_3\mathbf{q}_3\mathbf{q}_3^T$ to get

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The rank 2 matrix $\mathbf{q}_2\mathbf{q}_2^T + \mathbf{q}_3\mathbf{q}_3^T$ is the matrix of a projection onto the two-dimensional subspace (i.e., the plane) spanned by $\mathbf{q}_2$ and $\mathbf{q}_3$. (See Exercise 26.)

Observe that the spectral decomposition expresses a symmetric matrix A explicitly in terms of its eigenvalues and eigenvectors. This gives us a way of constructing a matrix with given eigenvalues and (orthonormal) eigenvectors.

Example 5.20

Find a 2×2 matrix with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$ and corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Solution We begin by normalizing the vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$, with

$$\mathbf{q}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

Now, we compute the matrix A whose spectral decomposition is

$$\begin{aligned} A &= \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T \\ &= 3\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} - 2\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= 3\begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix} - 2\begin{bmatrix} \frac{16}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{6}{5} \end{bmatrix} \end{aligned}$$

It is easy to check that A has the desired properties. (Do this.)

Exercises 5.4

Orthogonally diagonalize the matrices in Exercises 1–10 by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

1. $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ **2.** $A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$ **4.** $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

5. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$ **6.** $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **8.** $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ **10.** $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

11. If $b \neq 0$, orthogonally diagonalize $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

12. If $b \neq 0$, orthogonally diagonalize $A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix}$.

13. Let A and B be orthogonally diagonalizable $n \times n$ matrices and let c be a scalar. Use the Spectral Theorem to prove that the following matrices are orthogonally diagonalizable:

(a) $A + B$ **(b)** cA **(c)** A^2

14. If A is an invertible matrix that is orthogonally diagonalizable, show that A^{-1} is orthogonally diagonalizable.

15. If A and B are orthogonally diagonalizable and $AB = BA$, show that AB is orthogonally diagonalizable.

16. If A is a symmetric matrix, show that every eigenvalue of A is nonnegative if and only if $A = B^2$ for some symmetric matrix B.

In Exercises 17–20, find a spectral decomposition of the matrix in the given exercise.

17. Exercise 1

18. Exercise 2

19. Exercise 5

20. Exercise 8

In Exercises 21 and 22, find a symmetric 2×2 matrix with eigenvalues λ_1 and λ_2 and corresponding orthogonal eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

21. $\lambda_1 = -1, \lambda_2 = 2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

22. $\lambda_1 = 3, \lambda_2 = -3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

In Exercises 23 and 24, find a symmetric 3×3 matrix with eigenvalues λ_1, λ_2, and λ_3 and corresponding orthogonal eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

23. $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$

$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

24. $\lambda_1 = 1, \lambda_2 = -4, \lambda_3 = -4, \mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$

$\mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

25. Let $\mathbf{q}$ be a unit vector in $\mathbb{R}^n$ and let W be the subspace spanned by $\mathbf{q}$. Show that the orthogonal projection of a vector $\mathbf{v}$ onto W (as defined in Sections 1.2 and 5.2) is given by

$$\text{proj}_W(\mathbf{v}) = (\mathbf{q}\mathbf{q}^T)\mathbf{v}$$

and that the matrix of this projection is thus $\mathbf{q}\mathbf{q}^T$. [*Hint:* Remember that, for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}$.]

26. Let $\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$ and let W be the subspace spanned by this set.

(a) Show that the matrix of the orthogonal projection onto W is given by

$$P = \mathbf{q}_1\mathbf{q}_1^T + \cdots + \mathbf{q}_k\mathbf{q}_k^T$$

(b) Show that the projection matrix P in part (a) is symmetric and satisfies $P^2 = P$.

(c) Let $Q = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_k]$ be the $n \times k$ matrix whose columns are the orthonormal basis vectors of W. Show that $P = QQ^T$ and deduce that $\text{rank}(P) = k$.

27. Let A be an $n \times n$ real matrix, all of whose eigenvalues are real. Prove that there exist an orthogonal matrix Q and an upper triangular matrix T such that $Q^TAQ = T$. This very useful result is known as ***Schur's Triangularization Theorem.*** [*Hint:* Adapt the proof of the Spectral Theorem.]

28. Let A be a nilpotent matrix (see Exercise 56 in Section 4.2). Prove that there is an orthogonal matrix Q such that $Q^T AQ$ is upper triangular with zeros on its diagonal. [*Hint:* Use Exercise 27.]

5.5 Applications

Quadratic Forms

An expression of the form

$$ax^2 + by^2 + cxy$$

is called a ***quadratic form*** in x and y. Similarly,

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

is a quadratic form in x, y, and z. In words, a quadratic form is a sum of terms, each of which has total degree *two* in the variables. Therefore, $5x^2 - 3y^2 + 2xy$ is a quadratic form, but $x^2 + y^2 + x$ is not.

We can represent quadratic forms using matrices as follows:

$$ax^2 + by^2 + cxy = [x \quad y]\begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = [x \quad y \quad z]\begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(Verify these.) Each has the form $\mathbf{x}^T A\mathbf{x}$, where the matrix A is symmetric. This observation leads us to the following general definition.

Definition A ***quadratic form*** in n variables is a function $f: \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

where A is a symmetric $n \times n$ matrix and $\mathbf{x}$ is in $\mathbb{R}^n$. We refer to A as the ***matrix associated with f***.

Example 5.21

What is the quadratic form with associated matrix $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$?

Solution If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 5x_2^2 - 6x_1x_2$$

Observe that the *off-diagonal* entries $a_{12} = a_{21} = -3$ of A are *combined* to give the coefficient -6 of x_1x_2. This is true generally. We can expand a quadratic form in n variables $\mathbf{x}^T A\mathbf{x}$ as follows:

$$\mathbf{x}^T A\mathbf{x} = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + \sum_{i<j} 2a_{ij}x_ix_j$$

Thus, if $i \neq j$, the coefficient of x_ix_j is $2a_{ij}$.

Example 5.22

Find the matrix associated with the quadratic form

$$f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2 - 3x_1x_3$$

Solution The coefficients of the squared terms x_i^2 go on the diagonal as a_{ii}, and the coefficients of the cross-product terms x_ix_j are split between a_{ij} and a_{ji}. This gives

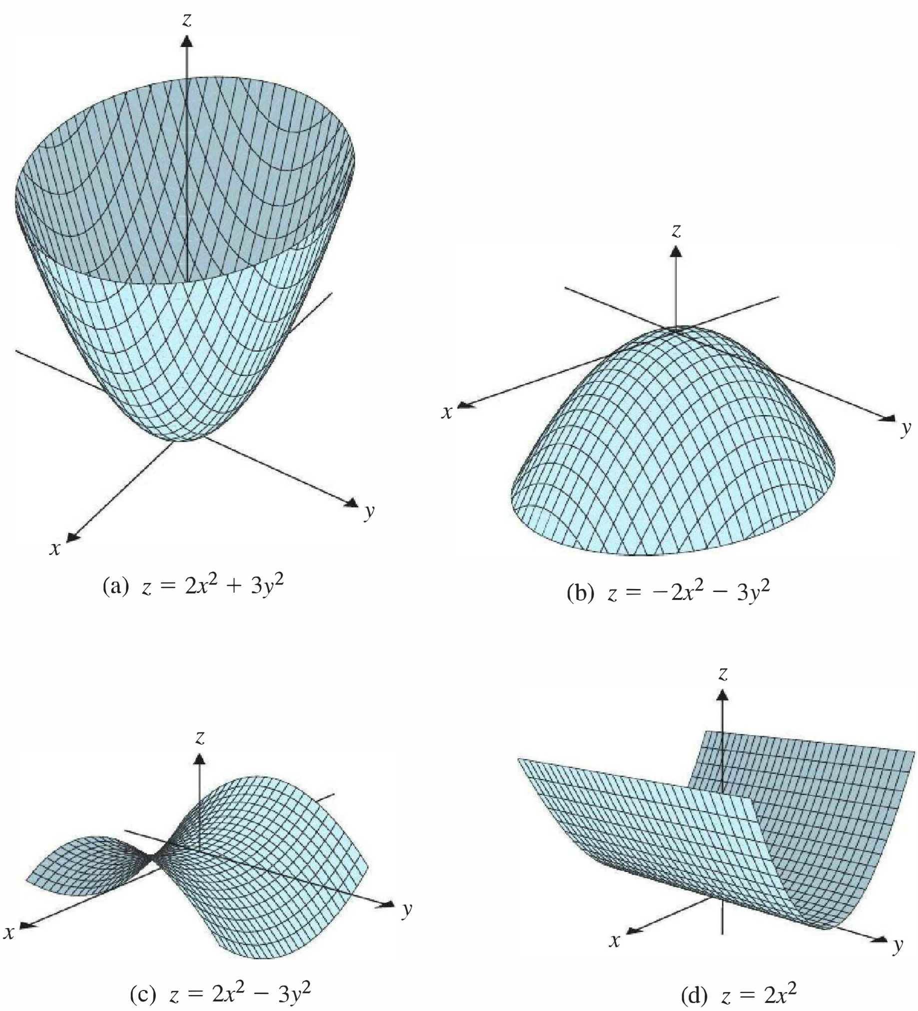
$$A = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & -1 & 0 \\ -\frac{3}{2} & 0 & 5 \end{bmatrix}$$

so
$$f(x_1, x_2, x_3) = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & -1 & 0 \\ -\frac{3}{2} & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

as you can easily check.

In the case of a quadratic form $f(x, y)$ in two variables, the graph of $z = f(x, y)$ is a surface in $\mathbb{R}^3$. Some examples are shown in Figure 5.12.

Observe that the effect of holding x or y constant is to take a cross section of the graph parallel to the yz or xz planes, respectively. For the graphs in Figure 5.12, all of these cross sections are easy to identify. For example, in Figure 5.12(a), the cross sections we get by holding x or y constant are all parabolas opening upward, so $f(x, y) \geq 0$ for all values of x and y. In Figure 5.12(c), holding x constant gives parabolas opening downward and holding y constant gives parabolas opening upward, producing a *saddle point*.



(a) $z = 2x^2 + 3y^2$

(b) $z = -2x^2 - 3y^2$

(c) $z = 2x^2 - 3y^2$

(d) $z = 2x^2$

Figure 5.12

Graphs of quadratic forms $f(x, y)$

What makes this type of analysis quite easy is the fact that these quadratic forms have no cross-product terms. The matrix associated with such a quadratic form is a diagonal matrix. For example,

$$2x^2 - 3y^2 = [x \quad y]\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

In general, the matrix of a quadratic form is a symmetric matrix, and we saw in Section 5.4 that such matrices can always be diagonalized. We will now use this fact to show that, for *every* quadratic form, we can eliminate the cross-product terms by means of a suitable change of variable.

Let $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form in n variables, with A a symmetric $n \times n$ matrix. By the Spectral Theorem, there is an orthogonal matrix Q that diagonalizes A; that is, $Q^T AQ = D$, where D is a diagonal matrix displaying the eigenvalues of A. We now set

$$\mathbf{x} = Q\mathbf{y} \quad \text{or, equivalently,} \quad \mathbf{y} = Q^{-1}\mathbf{x} = Q^T\mathbf{x}$$

Substitution into the quadratic form yields

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= (Q\mathbf{y})^T A(Q\mathbf{y}) \\ &= \mathbf{y}^T Q^T AQ\mathbf{y} \\ &= \mathbf{y}^T D\mathbf{y} \end{aligned}$$

which is a quadratic form without cross-product terms, since D is diagonal. Furthermore, if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then Q can be chosen so that

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

If $\mathbf{y} = [y_1 \quad \cdots \quad y_n]^T$, then, with respect to these new variables, the quadratic form becomes

$$\mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

This process is called ***diagonalizing a quadratic form.*** We have just proved the following theorem, known as the ***Principal Axes Theorem.*** (The reason for this name will become clear in the next subsection.)

Theorem 5.21 **The Principal Axes Theorem**

Every quadratic form can be diagonalized. Specifically, if A is the $n \times n$ symmetric matrix associated with the quadratic form $\mathbf{x}^T A\mathbf{x}$ and if Q is an orthogonal matrix such that $Q^T AQ = D$ is a diagonal matrix, then the change of variable $\mathbf{x} = Q\mathbf{y}$ transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into the quadratic form $\mathbf{y}^T D\mathbf{y}$, which has no cross-product terms. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and $\mathbf{y} = [y_1 \quad \cdots \quad y_n]^T$, then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

Example 5.23

Find a change of variable that transforms the quadratic form

$$f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$$

into one with no cross-product terms.

Solution The matrix of f is

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$. Corresponding unit eigenvectors are

$$\mathbf{q}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

(Check this.) If we set

$$Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

then $Q^TAQ = D$. The change of variable $\mathbf{x} = Q\mathbf{y}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

converts f into

$$f(\mathbf{y}) = f(y_1, y_2) = [y_1 \quad y_2]\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 6y_1^2 + y_2^2$$

The original quadratic form $\mathbf{x}^TA\mathbf{x}$ and the new one $\mathbf{y}^TD\mathbf{y}$ (referred to in the Principal Axes Theorem) are *equal* in the following sense. In Example 5.23, suppose we want to evaluate $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$ at $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. We have

$$f(-1, 3) = 5(-1)^2 + 4(-1)(3) + 2(3)^2 = 11$$

In terms of the new variables,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{y} = Q^T\mathbf{x} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -7/\sqrt{5} \end{bmatrix}$$

so

$$f(y_1, y_2) = 6y_1^2 + y_2^2 = 6(1/\sqrt{5})^2 + (-7/\sqrt{5})^2 = 55/5 = 11$$

exactly as before.

The Principal Axes Theorem has some interesting and important consequences. We will consider two of these. The first relates to the possible *values* that a quadratic form can take on.

Definition A quadratic form $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$ is classified as one of the following:

1. ***positive definite*** if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$
2. ***positive semidefinite*** if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$
3. ***negative definite*** if $f(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$
4. ***negative semidefinite*** if $f(\mathbf{x}) \leq 0$ for all $\mathbf{x}$
5. ***indefinite*** if $f(\mathbf{x})$ takes on both positive and negative values

A symmetric matrix A is called ***positive definite, positive semidefinite, negative definite, negative semidefinite,*** or ***indefinite*** if the associated quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ has the corresponding property.

The quadratic forms in parts (a), (b), (c), and (d) of Figure 5.12 are positive definite, negative definite, indefinite, and positive semidefinite, respectively. The Principal Axes Theorem makes it easy to tell if a quadratic form has one of these properties.

Theorem 5.22 Let A be an $n \times n$ symmetric matrix. The quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is

a. positive definite if and only if all of the eigenvalues of A are positive.
b. positive semidefinite if and only if all of the eigenvalues of A are nonnegative.
c. negative definite if and only if all of the eigenvalues of A are negative.
d. negative semidefinite if and only if all of the eigenvalues of A are nonpositive.
e. indefinite if and only if A has both positive and negative eigenvalues.

You are asked to prove Theorem 5.22 in Exercise 27.

Example 5.24 Classify $f(x, y, z) = 3x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz$ as positive definite, negative definite, indefinite, or none of these.

Solution The matrix associated with f is

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

which has eigenvalues 1, 4, and 4. (Verify this.) Since all of these eigenvalues are positive, f is a positive definite quadratic form.

If a quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is positive definite, then, since $f(\mathbf{0}) = 0$, the *minimum* value of $f(\mathbf{x})$ is 0 and it occurs at the origin. Similarly, a negative definite quadratic form has a maximum at the origin. Thus, Theorem 5.22 allows us to solve certain types of maxima/minima problems easily, without resorting to calculus. A type of problem that falls into this category is the ***constrained optimization problem.***

It is often important to know the maximum or minimum values of a quadratic form subject to certain constraints. (Such problems arise not only in mathematics but also in statistics, physics, engineering, and economics.) We will be interested in finding the extreme values of $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ subject to the constraint that $\|\mathbf{x}\| = 1$. In the case of a quadratic form in two variables, we can visualize what the problem means. The graph of $z = f(x, y)$ is a surface in $\mathbb{R}^3$, and the constraint $\|\mathbf{x}\| = 1$ restricts the point (x, y) to the unit circle in the xy-plane. Thus, we are considering those points that lie simultaneously on the surface and on the unit cylinder perpendicular to the xy plane. These points form a curve lying on the surface, and we want the highest and lowest points on this curve. Figure 5.13 shows this situation for the quadratic form and corresponding surface in Figure 5.12(c).

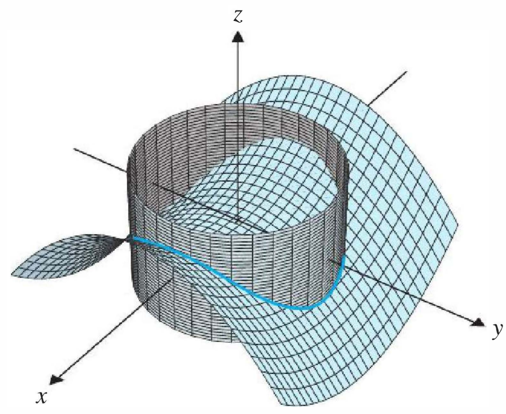


Figure 5.13
The intersection of $z = 2x^2 - 3y^2$ with the cylinder $x^2 + y^2 = 1$

In this case, the maximum and minimum values of $f(x, y) = 2x^2 - 3y^2$ (the highest and lowest points on the curve of intersection) are 2 and -3, respectively, which are just the eigenvalues of the associated matrix. Theorem 5.23 shows that this is always the case.

Theorem 5.23 Let $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form with associated $n \times n$ symmetric matrix A. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the following are true, subject to the constraint $\|\mathbf{x}\| = 1$:

a. $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$
b. The maximum value of $f(\mathbf{x})$ is λ_1, and it occurs when $\mathbf{x}$ is a unit eigenvector corresponding to λ_1.
c. The minimum value of $f(\mathbf{x})$ is λ_n, and it occurs when $\mathbf{x}$ is a unit eigenvector corresponding to λ_n.

Proof As usual, we begin by orthogonally diagonalizing A. Accordingly, let Q be an orthogonal matrix such that $Q^T AQ$ is the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Then, by the Principal Axes Theorem, the change of variable $\mathbf{x} = Q\mathbf{y}$ gives $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y}$. Now note that $\mathbf{y} = Q^T\mathbf{x}$ implies that

$$\mathbf{y}^T\mathbf{y} = (Q^T\mathbf{x})^T(Q^T\mathbf{x}) = \mathbf{x}^T(Q^T)^T Q^T\mathbf{x} = \mathbf{x}^T QQ^T\mathbf{x} = \mathbf{x}^T\mathbf{x}$$

since $Q^T = Q^{-1}$. Hence, using $\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x}$, we see that $\|\mathbf{y}\| = \sqrt{\mathbf{y}^T\mathbf{y}} = \sqrt{\mathbf{x}^T\mathbf{x}} = \|\mathbf{x}\| = 1$. Thus, if $\mathbf{x}$ is a unit vector, so is the corresponding $\mathbf{y}$, and the values of $\mathbf{x}^T A\mathbf{x}$ and $\mathbf{y}^T D\mathbf{y}$ are the same.

(a) To prove property (a), we observe that if $\mathbf{y} = [y_1 \quad \cdots \quad y_n]^T$, then

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\,\mathbf{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \\ &\le \lambda_1 y_1^2 + \lambda_1 y_2^2 + \cdots + \lambda_1 y_n^2 \\ &= \lambda_1 (y_1^2 + y_2^2 + \cdots + y_n^2) \\ &= \lambda_1 \|\mathbf{y}\|^2 \\ &= \lambda_1 \end{aligned}$$

Thus, $f(\mathbf{x}) \le \lambda_1$ for all $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$. The proof that $f(\mathbf{x}) \ge \lambda_n$ is similar. (See Exercise 37.)

(b) If $\mathbf{q}_1$ is a unit eigenvector corresponding to λ_1, then $A\mathbf{q}_1 = \lambda_1 \mathbf{q}_1$ and

$$f(\mathbf{q}_1) = \mathbf{q}_1^T A\mathbf{q}_1 = \mathbf{q}_1^T \lambda_1 \mathbf{q}_1 = \lambda_1(\mathbf{q}_1^T \mathbf{q}_1) = \lambda_1$$

This shows that the quadratic form actually takes on the value λ_1, and so, by property (a), it is the maximum value of $f(\mathbf{x})$ and it occurs when $\mathbf{x} = \mathbf{q}_1$.

(c) You are asked to prove this property in Exercise 38.

Example 5.25

Find the maximum and minimum values of the quadratic form $f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$ subject to the constraint $x_1^2 + x_2^2 = 1$, and determine values of x_1 and x_2 for which each of these occurs.

Solution In Example 5.23, we found that f has the associated eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$, with corresponding unit eigenvectors

$$\mathbf{q}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Therefore, the maximum value of f is 6 when $x_1 = 2/\sqrt{5}$ and $x_2 = 1/\sqrt{5}$. The minimum value of f is 1 when $x_1 = 1/\sqrt{5}$ and $x_2 = -2/\sqrt{5}$. (Observe that these extreme values occur twice—in opposite directions—since $-\mathbf{q}_1$ and $-\mathbf{q}_2$ are also unit eigenvectors for λ_1 and λ_2, respectively.)

Graphing Quadratic Equations

The general form of a quadratic equation in two variables x and y is

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

where at least one of a, b, and c is nonzero. The graphs of such quadratic equations are called ***conic sections*** (or ***conics***), since they can be obtained by taking cross sections of a (double) cone (i.e., slicing it with a plane). The most important of the conic sections are the ellipses (with circles as a special case), hyperbolas, and parabolas. These are called the ***nondegenerate*** conics. Figure 5.14 shows how they arise.

It is also possible for a cross section of a cone to result in a single point, a straight line, or a pair of lines. These are called ***degenerate*** conics. (See Exercises 59–64.)

The graph of a nondegenerate conic is said to be in ***standard position*** relative to the coordinate axes if its equation can be expressed in one of the forms in Figure 5.15.

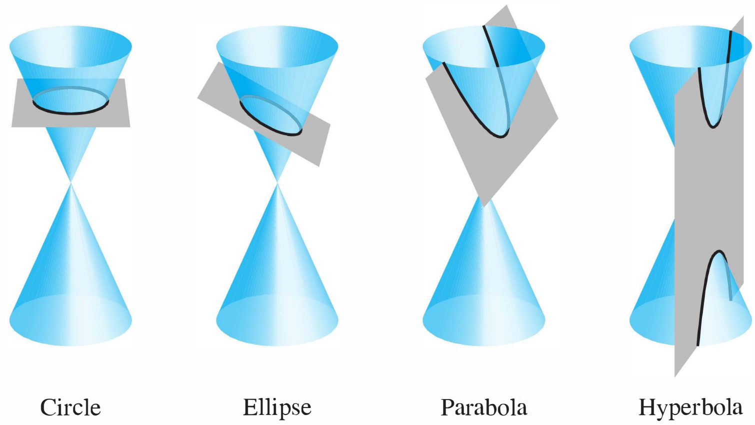


Figure 5.14
The nondegenerate conics

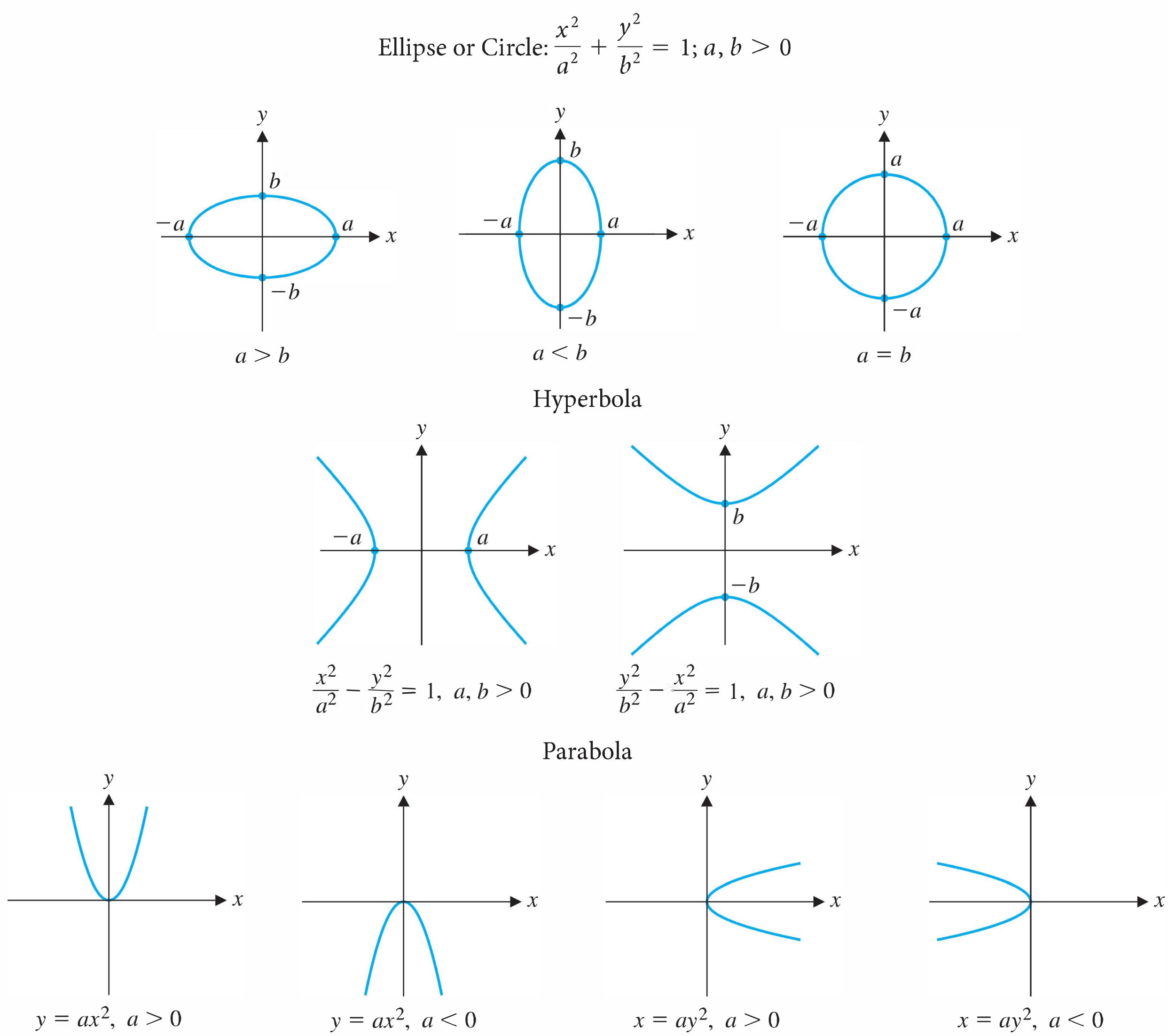


Figure 5.15
Nondegenerate conics in standard position

Example 5.26

If possible, write each of the following quadratic equations in the form of a conic in standard position and identify the resulting graph.

(a) $4x^2 + 9y^2 = 36$ (b) $4x^2 - 9y^2 + 1 = 0$ (c) $4x^2 - 9y = 0$

Solution (a) The equation $4x^2 + 9y^2 = 36$ can be written in the form

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

so its graph is an ellipse intersecting the x-axis at $(\pm 3, 0)$ and the y-axis at $(0, \pm 2)$.

(b) The equation $4x^2 - 9y^2 + 1 = 0$ can be written in the form

$$\frac{y^2}{\frac{1}{9}} - \frac{x^2}{\frac{1}{4}} = 1$$

so its graph is a hyperbola, opening up and down, intersecting the y-axis at $(0, \pm\frac{1}{3})$.

(c) The equation $4x^2 - 9y = 0$ can be written in the form

$$y = \frac{4}{9}x^2$$

so its graph is a parabola opening upward.

If a quadratic equation contains too many terms to be written in one of the forms in Figure 5.15, then its graph is not in standard position. When there are additional terms but no xy term, the graph of the conic has been *translated* out of standard position.

Example 5.27

Identify and graph the conic whose equation is

$$x^2 + 2y^2 - 6x + 8y + 9 = 0$$

Solution We begin by grouping the x and y terms separately to get

$$(x^2 - 6x) + (2y^2 + 8y) = -9$$

or

$$(x^2 - 6x) + 2(y^2 + 4y) = -9$$

Next, we complete the squares on the two expressions in parentheses to obtain

$$(x^2 - 6x + 9) + 2(y^2 + 4y + 4) = -9 + 9 + 8$$

or

$$(x - 3)^2 + 2(y + 2)^2 = 8$$

We now make the substitutions $x' = x - 3$ and $y' = y + 2$, turning the above equation into

$$(x')^2 + 2(y')^2 = 8 \quad \text{or} \quad \frac{(x')^2}{8} + \frac{(y')^2}{4} = 1$$

This is the equation of an ellipse in standard position in the $x'y'$ coordinate system, intersecting the x'-axis at $(\pm 2\sqrt{2}, 0)$ and the y'-axis at $(0, \pm 2)$. The origin in the $x'y'$ coordinate system is at $x = 3$, $y = -2$, so the ellipse has been translated out of standard position 3 units to the right and 2 units down. Its graph is shown in Figure 5.16.

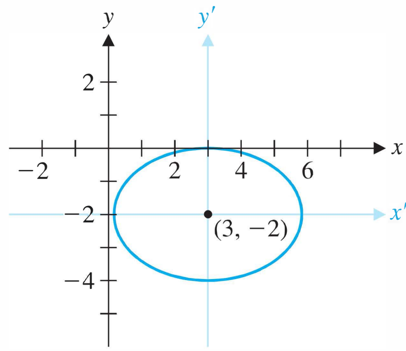


Figure 5.16
A translated ellipse

If a quadratic equation contains a cross-product term, then it represents a conic that has been *rotated*.

Example 5.28

Identify and graph the conic whose equation is

$$5x^2 + 4xy + 2y^2 = 6$$

Solution The left-hand side of the equation is a quadratic form, so we can write it in matrix form as $\mathbf{x}^T A\mathbf{x} = 6$, where

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

In Example 5.23, we found that the eigenvalues of A are 6 and 1, and a matrix Q that orthogonally diagonalizes A is

$$Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

Observe that $\det Q = -1$. In this example, we will interchange the columns of this matrix to make the determinant equal to $+1$. Then Q will be the matrix of a *rotation*, by Exercise 28 in Section 5.1. It is always possible to rearrange the columns of an orthogonal matrix Q to make its determinant equal to $+1$. (Why?) We set

$$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

instead, so that

$$Q^T A Q = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = D$$

The change of variable $\mathbf{x} = Q\mathbf{x}'$ converts the given equation into the form $(\mathbf{x}')^T D\mathbf{x}' = 6$ by means of a rotation. If $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, then this equation is just

$$(x')^2 + 6(y')^2 = 6 \quad \text{or} \quad \frac{(x')^2}{6} + (y')^2 = 1$$

which represents an ellipse in the $x'y'$ coordinate system.

To graph this ellipse, we need to know which vectors play the roles of $\mathbf{e}_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the new coordinate system. (These two vectors locate the positions of the x' and y' axes.) But, from $\mathbf{x} = Q\mathbf{x}'$, we have

$$Q\mathbf{e}_1' = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

and

$$Q\mathbf{e}_2' = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

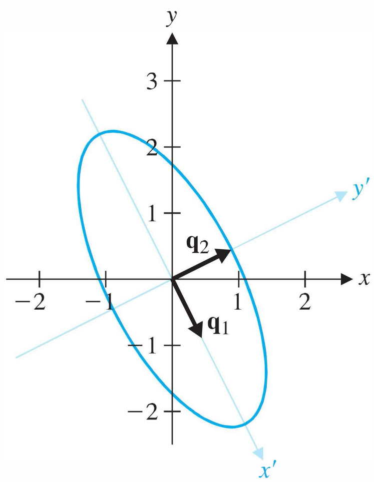


Figure 5.17
A rotated ellipse

These are just the columns $\mathbf{q}_1$ and $\mathbf{q}_2$ of Q, which are the eigenvectors of A! The fact that these are orthonormal vectors agrees perfectly with the fact that the change of variable is just a rotation. The graph is shown in Figure 5.17.

You can now see why the Principal Axes Theorem is so named. If a real symmetric matrix A arises as the coefficient matrix of a quadratic equation, the eigenvectors of A give the directions of the principal axes of the corresponding graph.

It is possible for the graph of a conic to be both rotated and translated out of standard position, as illustrated in Example 5.29.

Example 5.29

Identify and graph the conic whose equation is

$$5x^2 + 4xy + 2y^2 - \frac{28}{\sqrt{5}}x - \frac{4}{\sqrt{5}}y + 4 = 0$$

Solution The strategy is to eliminate the cross-product term first. In matrix form, the equation is $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + 4 = 0$, where

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{28}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \end{bmatrix}$$

The cross-product term comes from the quadratic form $\mathbf{x}^T A\mathbf{x}$, which we diagonalize as in Example 5.28 by setting $\mathbf{x} = Q\mathbf{x}'$, where

$$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Then, as in Example 5.28,

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = (x')^2 + 6(y')^2$$

But now we also have

$$B\mathbf{x} = BQ\mathbf{x}' = \begin{bmatrix} -\frac{28}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = -4x' - 12y'$$

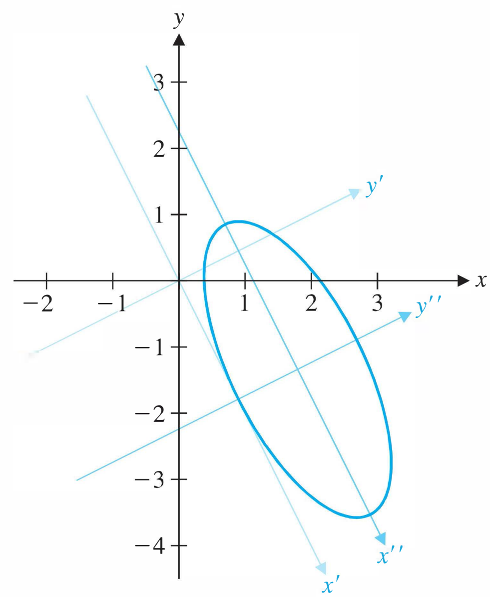


Figure 5.18

Thus, in terms of x' and y', the given equation becomes

$$(x')^2 + 6(y')^2 - 4x' - 12y' + 4 = 0$$

To bring the conic represented by this equation into standard position, we need to *translate* the $x'y'$ axes. We do so by completing the squares, as in Example 5.27. We have

$$((x')^2 - 4x' + 4) + 6((y')^2 - 2y' + 1) = -4 + 4 + 6 = 6$$

or

$$(x' - 2)^2 + 6(y' - 1)^2 = 6$$

This gives us the translation equations

$$x'' = x' - 2 \quad \text{and} \quad y'' = y' - 1$$

In the $x''y''$ coordinate system, the equation is simply

$$(x'')^2 + 6(y'')^2 = 6$$

which is the equation of an ellipse (as in Example 5.28). We can sketch this ellipse by first rotating and then translating. The resulting graph is shown in Figure 5.18.

The general form of a quadratic equation in three variables x, y, and z is

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$$

where at least one of $a, b, \ldots, f$ is nonzero. The graph of such a quadratic equation is called a ***quadric surface*** (or ***quadric***). Once again, to recognize a quadric we need

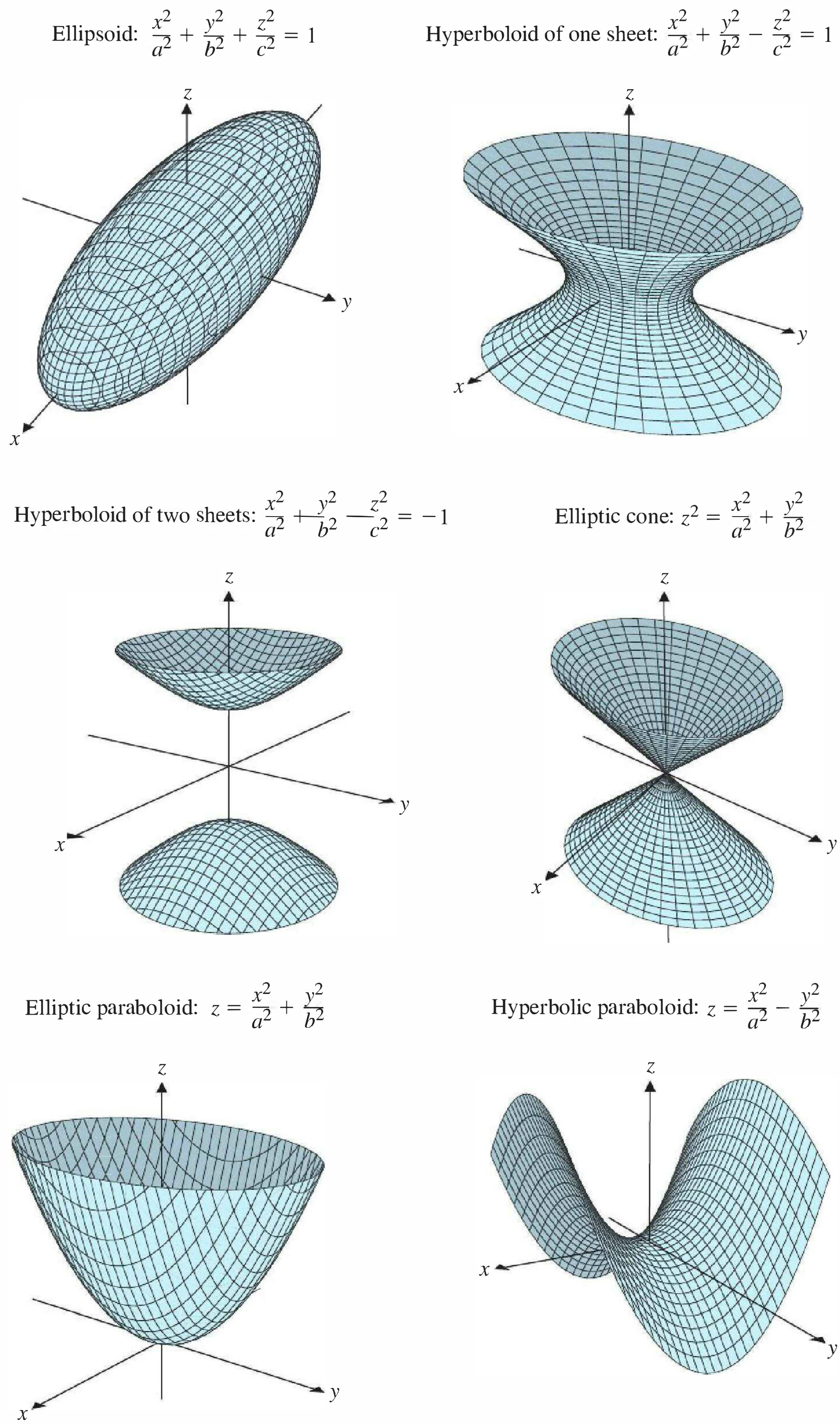


Figure 5.19
Quadric surfaces

to put it into standard position. Some quadrics in standard position are shown in Figure 5.19; others are obtained by permuting the variables.

Example 5.30

Identify the quadric surface whose equation is

$$5x^2 + 11y^2 + 2z^2 + 16xy + 20xz - 4yz = 36$$

Solution The equation can be written in matrix form as $\mathbf{x}^T A\mathbf{x} = 36$, where

$$A = \begin{bmatrix} 5 & 8 & 10 \\ 8 & 11 & -2 \\ 10 & -2 & 2 \end{bmatrix}$$

We find the eigenvalues of A to be 18, 9, and -9, with corresponding orthogonal eigenvectors

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

respectively. We normalize them to obtain

$$\mathbf{q}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \text{and} \quad \mathbf{q}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

and form the orthogonal matrix

$$Q = [\mathbf{q}_1 \ \ \mathbf{q}_2 \ \ \mathbf{q}_3] = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Note that in order for Q to be the matrix of a rotation, we require $\det Q = 1$, which is true in this case. (Otherwise, $\det Q = -1$, and swapping two columns changes the sign of the determinant.) Therefore,

$$Q^T AQ = D = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

and, with the change of variable $\mathbf{x} = Q\mathbf{x}'$, we get $\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')D\mathbf{x}' = 36$, so

$$18(x')^2 + 9(y')^2 - 9(z')^2 = 36 \quad \text{or} \quad \frac{(x')^2}{2} + \frac{(y')^2}{4} - \frac{(z')^2}{4} = 1$$

From Figure 5.19, we recognize this equation as the equation of a hyperboloid of one sheet. The x', y', and z' axes are in the directions of the eigenvectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, respectively. The graph is shown in Figure 5.20.

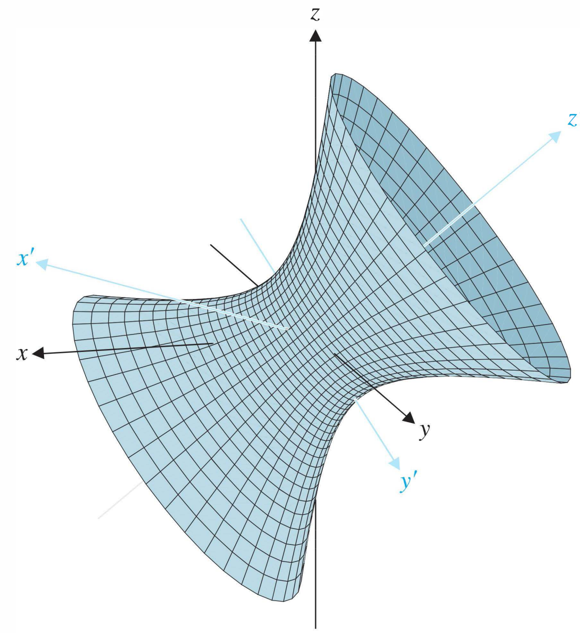


Figure 5.20
A hyperboloid of one sheet in nonstandard position

We can also identify and graph quadrics that have been translated out of standard position using the "complete-the-squares method" of Examples 5.27 and 5.29. You will be asked to do so in the exercises.

Exercises 5.5

Quadratic Forms

In Exercises 1–6, evaluate the quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for the given A and $\mathbf{x}$.

1. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

2. $A = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

In Exercises 7–12, find the symmetric matrix A associated with the given quadratic form.

7. $x_1^2 + 2x_2^2 + 6x_1x_2$

8. x_1x_2

9. $3x^2 - 3xy - y^2$

10. $x_1^2 - x_3^2 + 8x_1x_2 - 6x_2x_3$

11. $5x_1^2 - x_2^2 + 2x_3^2 + 2x_1x_2 - 4x_1x_3 + 4x_2x_3$

12. $2x^2 - 3y^2 + z^2 - 4xz$

Diagonalize the quadratic forms in Exercises 13–18 by finding an orthogonal matrix Q such that the change of variable $\mathbf{x} = Q\mathbf{y}$ transforms the given form into one with no cross-product terms. Give Q and the new quadratic form.

13. $2x_1^2 + 5x_2^2 - 4x_1x_2$

14. $x^2 + 8xy + y^2$

15. $7x_1^2 + x_2^2 + x_3^2 + 8x_1x_2 + 8x_1x_3 - 16x_2x_3$

16. $x_1^2 + x_2^2 + 3x_3^2 - 4x_1x_2$

17. $x^2 + z^2 - 2xy + 2yz$

18. $2xy + 2xz + 2yz$

Classify each of the quadratic forms in Exercises 19–26 as positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

19. $x_1^2 + 2x_2^2$ **20.** $x_1^2 + x_2^2 - 2x_1x_2$

21. $-2x^2 - 2y^2 + 2xy$ **22.** $x^2 + y^2 + 4xy$

23. $2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

24. $x_1^2 + x_2^2 + x_3^2 + 2x_1x_3$ **25.** $x_1^2 + x_2^2 - x_3^2 + 4x_1x_2$

26. $-x^2 - y^2 - z^2 - 2xy - 2xz - 2yz$

27. Prove Theorem 5.22.

28. Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ be a symmetric 2×2 matrix. Prove that A is positive definite if and only if $a > 0$ and $\det A > 0$. [*Hint:* $ax^2 + 2bxy + dy^2 = a\left(x + \frac{b}{a}y\right)^2 + \left(d - \frac{b^2}{a}\right)y^2$.]

29. Let B be an invertible matrix. Show that $A = B^TB$ is positive definite.

30. Let A be a positive definite symmetric matrix. Show that there exists an invertible matrix B such that $A = B^TB$. [*Hint:* Use the Spectral Theorem to write $A = QDQ^T$. Then show that D can be factored as C^TC for some invertible matrix C.]

31. Let A and B be positive definite symmetric $n \times n$ matrices and let c be a positive scalar. Show that the following matrices are positive definite.

(a) cA **(b)** A^2 **(c)** $A + B$

(d) A^{-1} (First show that A is necessarily invertible.)

32. Let A be a positive definite symmetric matrix. Show that there is a positive definite symmetric matrix B such that $A = B^2$. (Such a matrix B is called a ***square root*** of A.)

In Exercises 33–36, find the maximum and minimum values of the quadratic form $f(\mathbf{x})$ in the given exercise, subject to the constraint $\|\mathbf{x}\| = 1$, and determine the values of $\mathbf{x}$ for which these occur.

33. Exercise 20 **34.** Exercise 22

35. Exercise 23 **36.** Exercise 24

37. Finish the proof of Theorem 5.23(a).

38. Prove Theorem 5.23(c).

Graphing Quadratic Equations

In Exercises 39–44, identify the graph of the given equation.

39. $x^2 + 5y^2 = 25$ **40.** $x^2 - y^2 - 4 = 0$

41. $x^2 - y - 1 = 0$ **42.** $2x^2 + y^2 - 8 = 0$

43. $3x^2 = y^2 - 1$ **44.** $x = -2y^2$

In Exercises 45–50, use a translation of axes to put the conic in standard position. Identify the graph, give its equation in the translated coordinate system, and sketch the curve.

45. $x^2 + y^2 - 4x - 4y + 4 = 0$

46. $4x^2 + 2y^2 - 8x + 12y + 6 = 0$

47. $9x^2 - 4y^2 - 4y = 37$ **48.** $x^2 + 10x - 3y = -13$

49. $2y^2 + 4x + 8y = 0$

50. $2y^2 - 3x^2 - 18x - 20y + 11 = 0$

In Exercises 51–54, use a rotation of axes to put the conic in standard position. Identify the graph, give its equation in the rotated coordinate system, and sketch the curve.

51. $x^2 + xy + y^2 = 6$ **52.** $4x^2 + 10xy + 4y^2 = 9$

53. $4x^2 + 6xy - 4y^2 = 5$ **54.** $3x^2 - 2xy + 3y^2 = 8$

In Exercises 55–58, identify the conic with the given equation and give its equation in standard form.

55. $3x^2 - 4xy + 3y^2 - 28\sqrt{2}x + 22\sqrt{2}y + 84 = 0$

56. $6x^2 - 4xy + 9y^2 - 20x - 10y - 5 = 0$

57. $2xy + 2\sqrt{2}x - 1 = 0$

58. $x^2 - 2xy + y^2 + 4\sqrt{2}x - 4 = 0$

Sometimes the graph of a quadratic equation is a straight line, a pair of straight lines, or a single point. We refer to such a graph as a ***degenerate conic.*** *It is also possible that the equation is not satisfied for any values of the variables, in which case there is no graph at all and we refer to the conic as an* ***imaginary conic.*** *In Exercises 59–64, identify the conic with the given equation as either degenerate or imaginary and, where possible, sketch the graph.*

59. $x^2 - y^2 = 0$ **60.** $x^2 + 2y^2 + 2 = 0$

61. $3x^2 + y^2 = 0$ **62.** $x^2 + 2xy + y^2 = 0$

63. $x^2 - 2xy + y^2 + 2\sqrt{2}x - 2\sqrt{2}y = 0$

64. $2x^2 + 2xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y + 6 = 0$

65. Let A be a symmetric 2×2 matrix and let k be a scalar. Prove that the graph of the quadratic equation $\mathbf{x}^TA\mathbf{x} = k$ is

(a) a hyperbola if $k \neq 0$ and $\det A < 0$

(b) an ellipse, circle, or imaginary conic if $k \neq 0$ and $\det A > 0$

(c) a pair of straight lines or an imaginary conic if $k \neq 0$ and $\det A = 0$

(d) a pair of straight lines or a single point if $k = 0$ and $\det A \neq 0$

(e) a straight line if $k = 0$ and $\det A = 0$

[*Hint:* Use the Principal Axes Theorem.]

In Exercises 66–73, identify the quadric with the given equation and give its equation in standard form.

66. $4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz = 8$

67. $x^2 + y^2 + z^2 - 4yz = 1$

68. $-x^2 - y^2 - z^2 + 4xy + 4xz + 4yz = 12$

69. $2xy + z = 0$

70. $16x^2 + 100y^2 + 9z^2 - 24xz - 60x - 80z = 0$

71. $x^2 + y^2 - 2z^2 + 4xy - 2xz + 2yz - x + y + z = 0$

72. $10x^2 + 25y^2 + 10z^2 - 40xz + 20\sqrt{2}x + 50y + 20\sqrt{2}z = 15$

73. $11x^2 + 11y^2 + 14z^2 + 2xy + 8xz - 8yz - 12x + 12y + 12z = 6$

74. Let A be a real 2×2 matrix with complex eigenvalues $\lambda = a \pm bi$ such that $b \neq 0$ and $|\lambda| = 1$. Prove that every trajectory of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ lies on an ellipse. [*Hint:* Theorem 4.43 shows that if $\mathbf{v}$ is an eigenvector corresponding to $\lambda = a - bi$, then the matrix $P = [\operatorname{Re}\mathbf{v} \quad \operatorname{Im}\mathbf{v}]$ is invertible and $A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}$. Set $B = (PP^T)^{-1}$. Show that the quadratic $\mathbf{x}^TB\mathbf{x} = k$ defines an ellipse for all $k > 0$, and prove that if $\mathbf{x}$ lies on this ellipse, so does $A\mathbf{x}$.]

Chapter Review

Key Definitions and Concepts

Review Questions

1. Mark each of the following statements true or false:

(a) Every orthonormal set of vectors is linearly independent.

(b) Every nonzero subspace of $\mathbb{R}^n$ has an orthogonal basis.

(c) If A is a square matrix with orthonormal rows, then A is an orthogonal matrix.

(d) Every orthogonal matrix is invertible.

(e) If A is a matrix with $\det A = 1$, then A is an orthogonal matrix.

(f) If A is an $m \times n$ matrix such that $(\operatorname{row}(A))^\perp = \mathbb{R}^n$, then A must be the zero matrix.

(g) If W is a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $\operatorname{proj}_W(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v}$ must be the zero vector.

(h) If A is a symmetric, orthogonal matrix, then $A^2 = I$.

(i) Every orthogonally diagonalizable matrix is invertible.

(j) Given any n real numbers $\lambda_1, \ldots, \lambda_n$, there exists a symmetric $n \times n$ matrix with $\lambda_1, \ldots, \lambda_n$ as its eigenvalues.

2. Find all values of a and b such that

$$\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 3 \end{bmatrix}\right\} \text{ is an orthogonal set of vectors.}$$

3. Find the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$ with respect to the orthogonal basis

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\} \text{ of } \mathbb{R}^3$$

4. The coordinate vector of a vector **v** with respect to an orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$ is $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1/2 \end{bmatrix}$. If $\mathbf{v}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, find all possible vectors **v**.

5. Show that $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 4/7\sqrt{5} & -15/7\sqrt{5} & 2/7\sqrt{5} \end{bmatrix}$ is an orthogonal matrix.

6. If $\begin{bmatrix} 1/2 & a \\ b & c \end{bmatrix}$ is an orthogonal matrix, find all possible values of a, b, and c.

7. If Q is an orthogonal $n \times n$ matrix and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthonormal set in $\mathbb{R}^n$, prove that $\{Q\mathbf{v}_1, \dots, Q\mathbf{v}_k\}$ is an orthonormal set.

8. If Q is an $n \times n$ matrix such that the angles $\angle(Q\mathbf{x}, Q\mathbf{y})$ and $\angle(\mathbf{x}, \mathbf{y})$ are equal for all vectors **x** and **y** in $\mathbb{R}^n$, prove that Q is an orthogonal matrix.

In Questions 9–12, find a basis for $W^{\perp}$.

9. W is the line in $\mathbb{R}^2$ with general equation
$2x - 5y = 0$

10. W is the line in $\mathbb{R}^3$ with parametric equations
$x = t$
$y = 2t$
$z = -t$

11. $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$

12. $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

13. Find bases for each of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ -1 & 2 & -2 & 1 & -2 \\ 2 & 1 & 4 & 8 & 9 \\ 3 & -5 & 6 & -1 & 7 \end{bmatrix}$$

14. Find the orthogonal decomposition of

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

with respect to

$$W = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

15. (a) Apply the Gram-Schmidt Process to

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

to find an orthogonal basis for $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

(b) Use the result of part (a) to find a QR factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

16. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

17. Find an orthogonal basis for the subspace

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\} \text{ of } \mathbb{R}^4$$

18. Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

(a) Orthogonally diagonalize A.
(b) Give the spectral decomposition of A.

19. Find a symmetric matrix with eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -2$ and eigenspaces

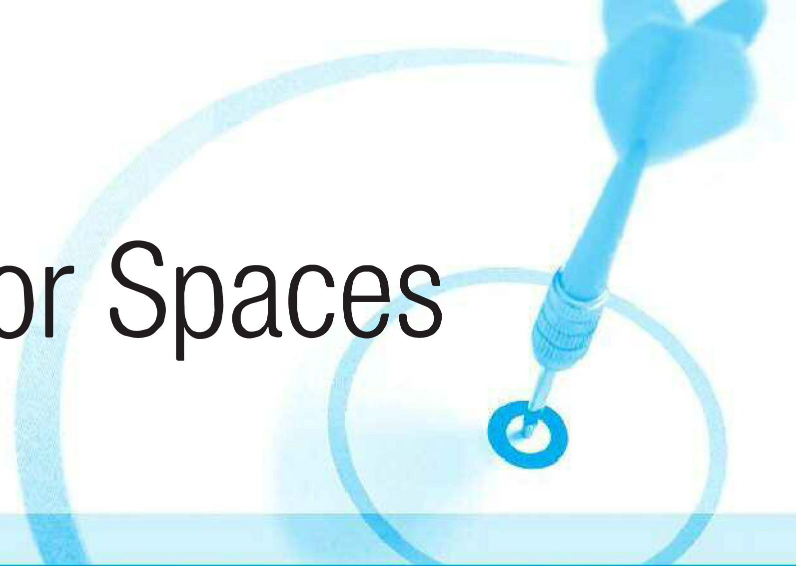
$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right), E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

20. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and

$$A = c_1\mathbf{v}_1\mathbf{v}_1^T + c_2\mathbf{v}_2\mathbf{v}_2^T + \cdots + c_n\mathbf{v}_n\mathbf{v}_n^T$$

prove that A is a symmetric matrix with eigenvalues $c_1, c_2, \dots, c_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

6 Vector Spaces



6.0 Introduction: Fibonacci in (Vector) Space

Algebra is generous; she often gives more than is asked of her.
—Jean le Rond d'Alembert (1717–1783)
In Carl B. Boyer
A History of Mathematics
Wiley, 1968, p. 481

The Fibonacci sequence was introduced in Section 4.6. It is the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

of nonnegative integers with the property that after the first two terms, each term is the sum of the two terms preceding it. Thus $0 + 1 = 1$, $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on.

If we denote the terms of the Fibonacci sequence by $f_0, f_1, f_2, \ldots$, then the entire sequence is completely determined by specifying that

$$f_0 = 0, f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

By analogy with vector notation, let's write a sequence $x_0, x_1, x_2, x_3, \ldots$ as

$$\mathbf{x} = [x_0, x_1, x_2, x_3, \ldots)$$

The Fibonacci sequence then becomes

$$\mathbf{f} = [f_0, f_1, f_2, f_3, \ldots) = [0, 1, 1, 2, \ldots)$$

We now generalize this notion.

Definition A ***Fibonacci-type sequence*** is *any* sequence $\mathbf{x} = [x_0, x_1, x_2, x_3, \ldots)$ such that x_0 and x_1 are real numbers and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$.

For example, $[1, \sqrt{2}, 1 + \sqrt{2}, 1 + 2\sqrt{2}, 2 + 3\sqrt{2}, \ldots)$ is a Fibonacci-type sequence.

Problem 1 Write down the first five terms of three more Fibonacci-type sequences.

By analogy with vectors again, let's define the *sum* of two sequences $\mathbf{x} = [x_0, x_1, x_2, \ldots)$ and $\mathbf{y} = [y_0, y_1, y_2, \ldots)$ to be the sequence

$$\mathbf{x} + \mathbf{y} = [x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$$

If c is a scalar, we can likewise define the scalar multiple of a sequence by

$$c\mathbf{x} = [cx_0, cx_1, cx_2, \ldots)$$

Problem 2 (a) Using your examples from Problem 1 or other examples, compute the sums of various pairs of Fibonacci-type sequences. Do the resulting sequences appear to be Fibonacci-type?

(b) Compute various scalar multiples of your Fibonacci-type sequences from Problem 1. Do the resulting sequences appear to be Fibonacci-type?

Problem 3 (a) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are Fibonacci-type sequences, then so is $\mathbf{x} + \mathbf{y}$.

(b) Prove that if $\mathbf{x}$ is a Fibonacci-type sequence and c is a scalar, then $c\mathbf{x}$ is also a Fibonacci-type sequence.

Let's denote the set of all Fibonacci-type sequences by *Fib*. Problem 3 shows that, like $\mathbb{R}^n$, *Fib* is closed under addition and scalar multiplication. The next exercises show that *Fib* has much more in common with $\mathbb{R}^n$.

Problem 4 Review the algebraic properties of vectors in Theorem 1.1. Does *Fib* satisfy all of these properties? What Fibonacci-type sequence plays the role of $\mathbf{0}$? For a Fibonacci-type sequence $\mathbf{x}$, what is $-\mathbf{x}$? Is $-\mathbf{x}$ also a Fibonacci-type sequence?

Problem 5 In $\mathbb{R}^n$, we have the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The Fibonacci sequence $\mathbf{f} = [0, 1, 1, 2, \ldots)$ can be thought of as the analogue of $\mathbf{e}_2$ because its first two terms are 0 and 1. What sequence $\mathbf{e}$ in *Fib* plays the role of $\mathbf{e}_1$?

What about $\mathbf{e}_3, \mathbf{e}_4, \ldots$? Do these vectors have analogues in *Fib*?

Problem 6 Let $\mathbf{x} = [x_0, x_1, x_2, \ldots)$ be a Fibonacci-type sequence. Show that $\mathbf{x}$ is a linear combination of $\mathbf{e}$ and $\mathbf{f}$.

Problem 7 Show that $\mathbf{e}$ and $\mathbf{f}$ are linearly independent. (That is, show that if $c\mathbf{e} + d\mathbf{f} = \mathbf{0}$, then $c = d = 0$.)

Problem 8 Given your answers to Problems 6 and 7, what would be a sensible value to assign to the "dimension" of *Fib*? Why?

Problem 9 Are there any geometric sequences in *Fib*? That is, if

$$[1, r, r^2, r^3, \ldots)$$

is a Fibonacci-type sequence, what are the possible values of r?

Problem 10 Find a "basis" for *Fib* consisting of geometric Fibonacci-type sequences.

Problem 11 Using your answer to Problem 10, give an alternative derivation of *Binet's formula* [formula (5) in Section 4.6]:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for the terms of the Fibonacci sequence $\mathbf{f} = [f_0, f_1, f_2, \ldots)$. [*Hint:* Express $\mathbf{f}$ in terms of the basis from Problem 10.]

The Lucas sequence is named after Edouard Lucas (see page 336).

The ***Lucas sequence*** is the Fibonacci-type sequence

$$\mathbf{l} = [l_0, l_1, l_2, l_3, \ldots) = [2, 1, 3, 4, \ldots)$$

Problem 12 Use the basis from Problem 10 to find an analogue of Binet's formula for the nth term l_n of the Lucas sequence.

Problem 13 Prove that the Fibonacci and Lucas sequences are related by the identity

$$f_{n-1} + f_{n+1} = l_n \quad \text{for } n \geq 1$$

[*Hint:* The Fibonacci-type sequences $\mathbf{f}^- = [1, 1, 2, 3, \ldots)$ and $\mathbf{f}^+ = [1, 0, 1, 1, \ldots)$ form a basis for *Fib*. (Why?)]

In this Introduction, we have seen that the collection *Fib* of all Fibonacci-type sequences behaves in many respects like $\mathbb{R}^2$, even though the "vectors" are actually infinite sequences. This useful analogy leads to the general notion of a *vector space* that is the subject of this chapter.

6.1 Vector Spaces and Subspaces

In Chapters 1 and 3, we saw that the algebra of vectors and the algebra of matrices are similar in many respects. In particular, we can add both vectors and matrices, and we can multiply both by scalars. The properties that result from these two operations (Theorem 1.1 and Theorem 3.2) are identical in both settings. In this section, we use these properties to define generalized "vectors" that arise in a wide variety of examples. By proving general theorems about these "vectors," we will therefore simultaneously be proving results about all of these examples. This is the real power of algebra: its ability to take properties from a concrete setting, like $\mathbb{R}^n$, and *abstract* them into a general setting.

Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associativity
4. There exists an element $\mathbf{0}$ in V, called a ***zero vector,*** such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. — Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

The German mathematician Hermann Grassmann (1809–1877) is generally credited with first introducing the idea of a vector space (although he did not call it that) in 1844. Unfortunately, his work was very difficult to read and did not receive the attention it deserved. One person who did study it was the Italian mathematician Giuseppe Peano (1858–1932). In his 1888 book *Calcolo Geometrico*, Peano clarified Grassmann's earlier work and laid down the axioms for a vector space as we know them today. Peano's book is also remarkable for introducing operations on sets. His notations ∪, ∩, and ∈ (for "union," "intersection," and "is an element of") are the ones we still use, although they were not immediately accepted by other mathematicians. Peano's axiomatic definition of a vector space also had very little influence for many years. Acceptance came in 1918, after Hermann Weyl (1885–1955) repeated it in his book *Space, Time, Matter*, an introduction to Einstein's general theory of relativity.

Remarks

- By "scalars" we will usually mean the real numbers. Accordingly, we should refer to V as a *real vector space* (or a *vector space over the real numbers*). It is also possible for scalars to be complex numbers or to belong to $\mathbb{Z}_p$, where p is prime. In these cases, V is called a *complex vector space* or a *vector space over* $\mathbb{Z}_p$, respectively. Most of our examples will be real vector spaces, so we will usually omit the adjective "real." If something is referred to as a "vector space," assume that we are working over the real number system.

  In fact, the scalars can be chosen from any number system in which, roughly speaking, we can add, subtract, multiply, and divide according to the usual laws of arithmetic. In abstract algebra, such a number system is called a ***field***.
- The definition of a vector space does not specify what the set V consists of. Neither does it specify what the operations called "addition" and "scalar multiplication" look like. Often, they will be familiar, but they need not be. See Example 6.6 and Exercises 5–7.

We will now look at several examples of vector spaces. In each case, we need to specify the set V and the operations of addition and scalar multiplication and to verify axioms 1 through 10. We need to pay particular attention to axioms 1 and 6 (closure), axiom 4 (the existence of a zero vector in V), and axiom 5 (each vector in V must have a negative in V).

Example 6.1

For any $n \geq 1$, $\mathbb{R}^n$ is a vector space with the usual operations of addition and scalar multiplication. Axioms 1 and 6 follow from the definitions of these operations, and the remaining axioms follow from Theorem 1.1.

Example 6.2

The set of all 2×3 matrices is a vector space with the usual operations of matrix addition and matrix scalar multiplication. Here the "vectors" are actually matrices. We know that the sum of two 2×3 matrices is also a 2×3 matrix and that multiplying a 2×3 matrix by a scalar gives another 2×3 matrix; hence, we have closure. The remaining axioms follow from Theorem 3.2. In particular, the zero vector $\mathbf{0}$ is the 2×3 zero matrix, and the negative of a 2×3 matrix A is just the 2×3 matrix $-A$.

There is nothing special about 2×3 matrices. For any positive integers m and n, the set of all $m \times n$ matrices forms a vector space with the usual operations of matrix addition and matrix scalar multiplication. This vector space is denoted M_{mn}.

Example 6.3

Let $\mathscr{P}_2$ denote the set of all polynomials of degree 2 or less with real coefficients. Define addition and scalar multiplication in the usual way. (See Appendix D.) If

$$p(x) = a_0 + a_1x + a_2x^2 \quad \text{and} \quad q(x) = b_0 + b_1x + b_2x^2$$

are in $\mathscr{P}_2$, then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

has degree at most 2 and so is in $\mathscr{P}_2$. If c is a scalar, then

$$cp(x) = ca_0 + ca_1x + ca_2x^2$$

is also in $\mathscr{P}_2$. This verifies axioms 1 and 6.

The zero vector $\mathbf{0}$ is the zero polynomial—that is, the polynomial all of whose coefficients are zero. The negative of a polynomial $p(x) = a_0 + a_1x + a_2x^2$ is the polynomial $-p(x) = -a_0 - a_1x - a_2x^2$. It is now easy to verify the remaining axioms. We will check axiom 2 and leave the others for Exercise 12. With $p(x)$ and $q(x)$ as above, we have

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \\ &= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 \\ &= (b_0 + b_1x + b_2x^2) + (a_0 + a_1x + a_2x^2) \\ &= q(x) + p(x) \end{aligned}$$

where the third equality follows from the fact that addition of real numbers is commutative.

In general, for any fixed $n \geq 0$, the set $\mathscr{P}_n$ of all polynomials of degree less than or equal to n is a vector space, as is the set $\mathscr{P}$ of *all* polynomials.

Example 6.4

Let $\mathscr{F}$ denote the set of all real-valued functions defined on the real line. If f and g are two such functions and c is a scalar, then $f + g$ and cf are defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$

In other words, the *value* of $f + g$ at x is obtained by adding together the values of f and g at x [Figure 6.1(a)]. Similarly, the value of cf at x is just the value of f at x multiplied by the scalar c [Figure 6.1(b)]. The zero vector in $\mathscr{F}$ is the constant function f_0 that is identically zero; that is, $f_0(x) = 0$ for all x. The negative of a function f is the function $-f$ defined by $(-f)(x) = -f(x)$ [Figure 6.1(c)].

Axioms 1 and 6 are obviously true. Verification of the remaining axioms is left as Exercise 13. Thus, $\mathscr{F}$ is a vector space.

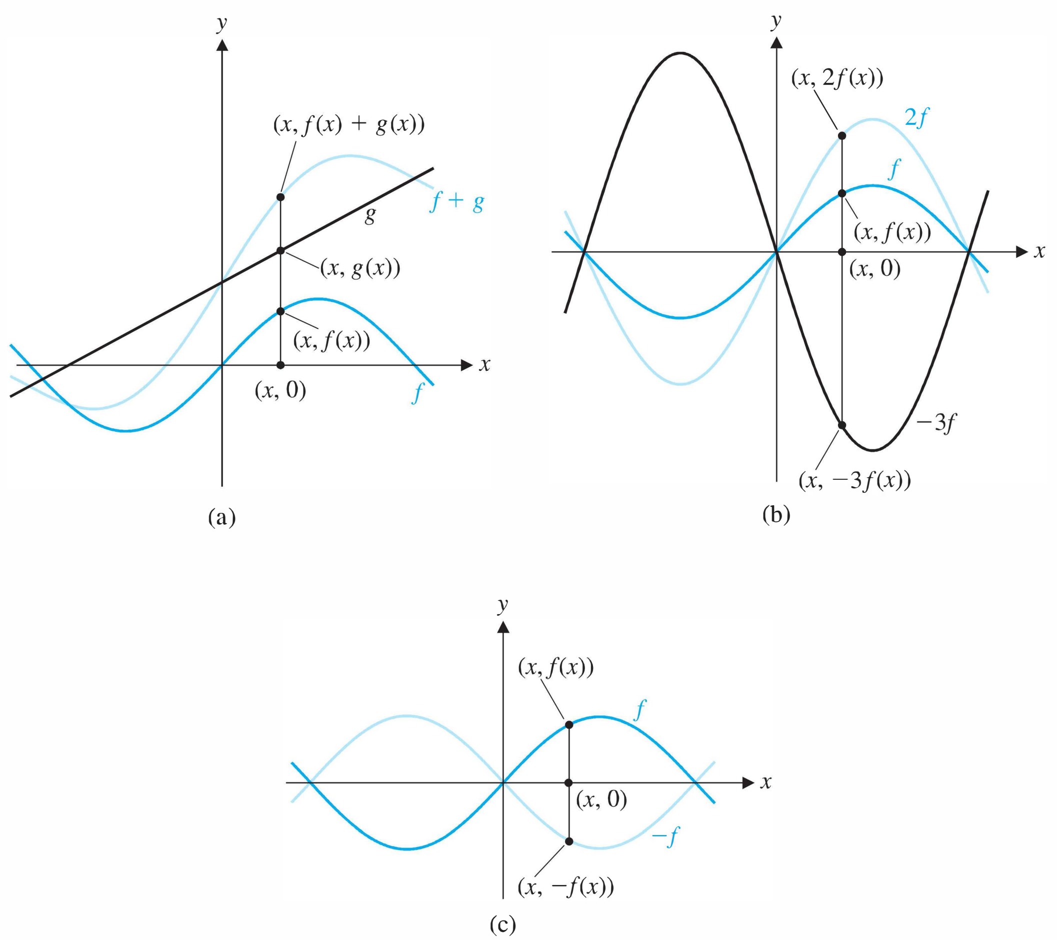


Figure 6.1
The graphs of (a) f, g, and $f + g$, (b) f, $2f$, and $-3f$, and (c) f and $-f$

In Example 6.4, we could also have considered only those functions defined on some *closed interval* $[a, b]$ of the real line. This approach also produces a vector space, denoted by $\mathscr{F}[a, b]$.

Example 6.5

The set $\mathbb{Z}$ of integers with the usual operations is *not* a vector space. To demonstrate this, it is enough to find that *one* of the ten axioms fails and to give a specific instance in which it fails (a *counterexample*). In this case, we find that we do not have closure under scalar multiplication. For example, the multiple of the integer 2 by the scalar $\frac{1}{3}$ is $(\frac{1}{3})(2) = \frac{2}{3}$, which is not an integer. Thus, it is not true that cx is in $\mathbb{Z}$ for *every* x in $\mathbb{Z}$ and *every* scalar c (i.e., axiom 6 fails).

Example 6.6

Let $V = \mathbb{R}^2$ with the usual definition of addition but the following definition of scalar multiplication:

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix}$$

Then, for example,
$$1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so axiom 10 fails. [In fact, the other nine axioms are all true (check this), but we do not need to look into them, because V has already failed to be a vector space. This example shows the value of looking ahead, rather than working through the list of axioms in the order in which they have been given.]

$a + bi$

Example 6.7

Let $\mathbb{C}^2$ denote the set of all ordered pairs of complex numbers. Define addition and scalar multiplication as in $\mathbb{R}^2$, except here the scalars are complex numbers. For example,

$$\begin{bmatrix} 1 + i \\ 2 - 3i \end{bmatrix} + \begin{bmatrix} -3 + 2i \\ 4 \end{bmatrix} = \begin{bmatrix} -2 + 3i \\ 6 - 3i \end{bmatrix}$$

and
$$(1 - i)\begin{bmatrix} 1 + i \\ 2 - 3i \end{bmatrix} = \begin{bmatrix} (1 - i)(1 + i) \\ (1 - i)(2 - 3i) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 - 5i \end{bmatrix}$$

Using properties of the complex numbers, it is straightforward to check that all ten axioms hold. Therefore, $\mathbb{C}^2$ is a complex vector space.

In general, $\mathbb{C}^n$ is a complex vector space for all $n \geq 1$.

Example 6.8

If p is prime, the set $\mathbb{Z}_p^n$ (with the usual definitions of addition and multiplication by scalars from $\mathbb{Z}_p$) is a vector space over $\mathbb{Z}_p$ for all $n \geq 1$.

Before we consider further examples, we state a theorem that contains some useful properties of vector spaces. It is important to note that, by proving this theorem for vector spaces in *general*, we are actually proving it for *every specific* vector space.

Theorem 6.1 Let V be a vector space, $\mathbf{u}$ a vector in V, and c a scalar.

a. $0\mathbf{u} = \mathbf{0}$
b. $c\mathbf{0} = \mathbf{0}$
c. $(-1)\mathbf{u} = -\mathbf{u}$
d. If $c\mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof We prove properties (b) and (d) and leave the proofs of the remaining properties as exercises.

(b) We have

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$$

by vector space axioms 4 and 7. Adding the negative of $c\mathbf{0}$ to both sides produces

$$c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$$

which implies

$$\begin{aligned} \mathbf{0} &= c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0})) && \text{by axioms 5 and 3} \\ &= c\mathbf{0} + \mathbf{0} && \text{by axiom 5} \\ &= c\mathbf{0} && \text{by axiom 4} \end{aligned}$$

(d) Suppose $c\mathbf{u} = \mathbf{0}$. To show that either $c = 0$ or $\mathbf{u} = \mathbf{0}$, let's assume that $c \neq 0$. (If $c = 0$, there is nothing to prove.) Then, since $c \neq 0$, its reciprocal $1/c$ is defined, and

$$\begin{aligned} \mathbf{u} &= 1\mathbf{u} && \text{by axiom 10} \\ &= \left(\frac{1}{c}c\right)\mathbf{u} \\ &= \frac{1}{c}(c\mathbf{u}) && \text{by axiom 9} \\ &= \frac{1}{c}\mathbf{0} \\ &= \mathbf{0} && \text{by property (b)} \end{aligned}$$

We will write $\mathbf{u} - \mathbf{v}$ for $\mathbf{u} + (-\mathbf{v})$, thereby defining ***subtraction*** of vectors. We will also exploit the associativity property of addition to unambiguously write $\mathbf{u} + \mathbf{v} + \mathbf{w}$ for the sum of three vectors and, more generally,

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$$

for a ***linear combination*** of vectors.

Subspaces

We have seen that, in $\mathbb{R}^n$, it is possible for one vector space to sit inside another one, giving rise to the notion of a subspace. For example, a plane through the origin is a subspace of $\mathbb{R}^3$. We now extend this concept to general vector spaces.

Definition A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V.

As in $\mathbb{R}^n$, checking to see whether a subset W of a vector space V is a subspace of V involves testing only two of the ten vector space axioms. We prove this observation as a theorem.

Theorem 6.2 Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

a. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
b. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Proof Assume that W is a subspace of V. Then W satisfies vector space axioms 1 to 10. In particular, axiom 1 is condition (a) and axiom 6 is condition (b).

Conversely, assume that W is a subset of a vector space V, satisfying conditions (a) and (b). By hypothesis, axioms 1 and 6 hold. Axioms 2, 3, 7, 8, 9, and 10 hold in W because they are true for *all* vectors in V and thus are true in particular for those vectors in W. (We say that W *inherits* these properties from V.) This leaves axioms 4 and 5 to be checked.

Since W is nonempty, it contains at least one vector $\mathbf{u}$. Then condition (b) and Theorem 6.1(a) imply that $0\mathbf{u} = \mathbf{0}$ is also in W. This is axiom 4.

If $\mathbf{u}$ is in V, then, by taking $c = -1$ in condition (b), we have that $-\mathbf{u} = (-1)\mathbf{u}$ is also in W, using Theorem 6.1(c).

Remark Since Theorem 6.2 generalizes the notion of a subspace from the context of $\mathbb{R}^n$ to general vector spaces, all of the subspaces of $\mathbb{R}^n$ that we encountered in Chapter 3 are subspaces of $\mathbb{R}^n$ in the current context. In particular, lines and planes through the origin are subspaces of $\mathbb{R}^3$.

Example 6.9 We have already shown that the set $\mathscr{P}_n$ of all polynomials with degree at most n is a vector space. Hence, $\mathscr{P}_n$ is a subspace of the vector space $\mathscr{P}$ of *all* polynomials.

Example 6.10 Let W be the set of symmetric $n \times n$ matrices. Show that W is a subspace of M_{nn}.

Solution Clearly, W is nonempty, so we need only check conditions (a) and (b) in Theorem 6.2. Let A and B be in W and let c be a scalar. Then $A^T = A$ and $B^T = B$, from which it follows that

$$(A + B)^T = A^T + B^T = A + B$$

Therefore, $A + B$ is symmetric and, hence, is in W. Similarly,

$$(cA)^T = cA^T = cA$$

so cA is symmetric and, thus, is in W. We have shown that W is closed under addition and scalar multiplication. Therefore, it is a subspace of M_{nn}, by Theorem 6.2.

Example 6.11

Let $\mathscr{C}$ be the set of all continuous real-valued functions defined on $\mathbb{R}$ and let $\mathscr{D}$ be the set of all differentiable real-valued functions defined on $\mathbb{R}$. Show that $\mathscr{C}$ and $\mathscr{D}$ are subspaces of $\mathscr{F}$, the vector space of all real-valued functions defined on $\mathbb{R}$.

Solution From calculus, we know that if f and g are continuous functions and c is a scalar, then $f + g$ and cf are also continuous. Hence, $\mathscr{C}$ is closed under addition and scalar multiplication and so is a subspace of $\mathscr{F}$. If f and g are differentiable, then so are $f + g$ and cf. Indeed,

$$(f + g)' = f' + g' \quad \text{and} \quad (cf)' = c(f')$$

So $\mathscr{D}$ is also closed under addition and scalar multiplication, making it a subspace of $\mathscr{F}$.

It is a theorem of calculus that every differentiable function is continuous. Consequently, $\mathscr{D}$ is contained in $\mathscr{C}$ (denoted by $\mathscr{D} \subset \mathscr{C}$), making $\mathscr{D}$ a subspace of $\mathscr{C}$. It is also the case that every polynomial function is differentiable, so $\mathscr{P} \subset \mathscr{D}$, and thus $\mathscr{P}$ is a subspace of $\mathscr{D}$. We therefore have a *hierarchy* of subspaces of $\mathscr{F}$, one inside the other:

$$\mathscr{P} \subset \mathscr{D} \subset \mathscr{C} \subset \mathscr{F}$$

This hierarchy is depicted in Figure 6.2.

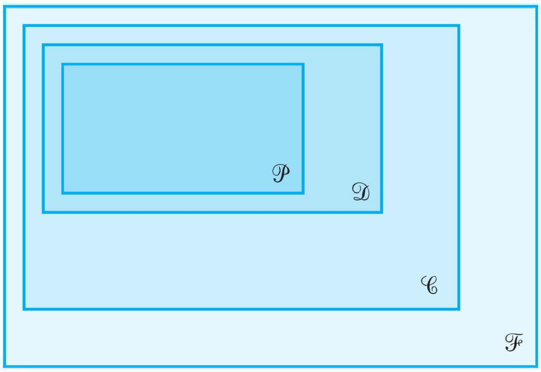


Figure 6.2
The hierarchy of subspaces of $\mathscr{F}$

There are other subspaces of $\mathscr{F}$ that can be placed into this hierarchy. Some of these are explored in the exercises.

In the preceding discussion, we could have restricted our attention to functions defined on a closed interval $[a, b]$. Then the corresponding subspaces of $\mathscr{F}[a, b]$ would be $\mathscr{C}[a, b]$, $\mathscr{D}[a, b]$, and $\mathscr{P}[a, b]$.

Example 6.12

Let S be the set of all functions that satisfy the differential equation

$$f'' + f = 0 \tag{1}$$

That is, S is the solution set of Equation (1). Show that S is a subspace of $\mathscr{F}$.

Solution S is nonempty, since the zero function clearly satisfies Equation (1). Let f and g be in S and let c be a scalar. Then

$$\begin{aligned}(f+g)'' + (f+g) &= (f''+g'') + (f+g)\\ &= (f''+f) + (g''+g)\\ &= 0+0\\ &= 0\end{aligned}$$

which shows that $f+g$ is in S. Similarly,

$$\begin{aligned}(cf)'' + cf &= cf'' + cf\\ &= c(f''+f)\\ &= c0\\ &= 0\end{aligned}$$

so cf is also in S.

Therefore, S is closed under addition and scalar multiplication and is a subspace of $\mathscr{F}$.

The differential Equation (1) is an example of a ***homogeneous linear differential equation.*** The solution sets of such equations are always subspaces of $\mathscr{F}$. Note that in Example 6.12 we did not actually solve Equation (1) (i.e., we did not find any specific solutions, other than the zero function). We will discuss techniques for finding solutions to this type of equation in Section 6.7.

As you gain experience working with vector spaces and subspaces, you will notice that certain examples tend to resemble one another. For example, consider the vector spaces $\mathbb{R}^4$, $\mathscr{P}_3$, and M_{22}. Typical elements of these vector spaces are, respectively,

$$\mathbf{u} = \begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix}, \quad p(x) = a + bx + cx^2 + dx^3, \quad \text{and} \quad A = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

In the words of Yogi Berra, "It's déjà vu all over again."

Any calculations involving the vector space operations of addition and scalar multiplication are essentially the same in all three settings. To highlight the similarities, in the next example we will perform the necessary steps in the three vector spaces side by side.

Example 6.13

(a) Show that the set W of all vectors of the form

$$\begin{bmatrix} a\\ b\\ -b\\ a\end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

(b) Show that the set W of all polynomials of the form $a + bx - bx^2 + ax^3$ is a subspace of $\mathscr{P}_3$.

(c) Show that the set W of all matrices of the form $\begin{bmatrix} a & b\\ -b & a\end{bmatrix}$ is a subspace of M_{22}.

Solution

(a) W is nonempty because it contains the zero vector $\mathbf{0}$. (Take $a = b = 0$.) Let $\mathbf{u}$ and $\mathbf{v}$ be in W—say,

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + c \\ b + d \\ -b - d \\ a + c \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \\ -(b + d) \\ a + c \end{bmatrix}$$

so $\mathbf{u} + \mathbf{v}$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$k\mathbf{u} = \begin{bmatrix} ka \\ kb \\ -kb \\ ka \end{bmatrix}$$

so $k\mathbf{u}$ is in W.

Thus, W is a nonempty subset of $\mathbb{R}^4$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 6.2.

(b) W is nonempty because it contains the zero polynomial. (Take $a = b = 0$.) Let $p(x)$ and $q(x)$ be in W—say,

$$p(x) = a + bx - bx^2 + ax^3$$

and

$$q(x) = c + dx - dx^2 + cx^3$$

Then

$$\begin{aligned} p(x) + q(x) = (a + c) &+ (b + d)x \\ &- (b + d)x^2 \\ &+ (a + c)x^3 \end{aligned}$$

so $p(x) + q(x)$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kp(x) = ka + kbx - kbx^2 + kax^3$$

so $kp(x)$ is in W.

Thus, W is a nonempty subset of $\mathscr{P}_3$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathscr{P}_3$ by Theorem 6.2.

(c) W is nonempty because it contains the zero matrix O. (Take $a = b = 0$.) Let A and B be in W—say,

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and

$$B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a + c & b + d \\ -(b + d) & a + c \end{bmatrix}$$

so $A + B$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kA = \begin{bmatrix} ka & kb \\ -kb & ka \end{bmatrix}$$

so kA is in W.

Thus, W is a nonempty subset of M_{22} that is closed under addition and scalar multiplication. Therefore, W is a subspace of M_{22}, by Theorem 6.2.

Example 6.13 shows that it is often possible to relate examples that, on the surface, appear to have nothing in common. Consequently, we can apply our knowledge of $\mathbb{R}^n$ to polynomials, matrices, and other examples. We will encounter this idea several times in this chapter and will make it precise in Section 6.5.

Example 6.14

If V is a vector space, then V is clearly a subspace of itself. The set $\{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace of V, called the ***zero subspace.*** To show this, we simply note that the two closure conditions of Theorem 6.2 are satisfied:

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad c\mathbf{0} = \mathbf{0} \quad \text{for any scalar } c$$

The subspaces $\{\mathbf{0}\}$ and V are called the ***trivial subspaces*** of V.

An examination of the proof of Theorem 6.2 reveals the following useful fact:

If W is a subspace of a vector space V, then W contains the zero vector $\mathbf{0}$ of V.

This fact is consistent with, and analogous to, the fact that lines and planes are subspaces of $\mathbb{R}^3$ if and only if they contain the origin. The requirement that every subspace must contain $\mathbf{0}$ is sometimes useful in showing that a set is *not* a subspace.

Example 6.15

Let W be the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix}$$

Is W a subspace of M_{22}?

Solution Each matrix in W has the property that its (1, 2) entry is one more than its (1, 1) entry. Since the zero matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

does not have this property, it is not in W. Hence, W is not a subspace of M_{22}.

Example 6.16

Let W be the set of all 2×2 matrices with determinant equal to 0. Is W a subspace of M_{22}? (Since $\det O = 0$, the zero matrix is in W, so the method of Example 6.15 is of no use to us.)

Solution Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $\det A = \det B = 0$, so A and B are in W. But

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\det(A + B) = 1 \neq 0$, and therefore $A + B$ is not in W. Thus, W is not closed under addition and so is not a subspace of M_{22}.

Spanning Sets

The notion of a spanning set of vectors carries over easily from $\mathbb{R}^n$ to general vector spaces.

Definition If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the ***span*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\text{span}(S)$. If $V = \text{span}(S)$, then S is called a ***spanning set*** for V and V is said to be ***spanned*** by S.

Example 6.17

Show that the polynomials 1, x, and x^2 span $\mathscr{P}_2$.

Solution By its very definition, a polynomial $p(x) = a + bx + cx^2$ is a linear combination of 1, x, and x^2. Therefore, $\mathscr{P}_2 = \text{span}(1, x, x^2)$.

Example 6.17 can clearly be generalized to show that $\mathscr{P}_n = \text{span}(1, x, x^2, \ldots, x^n)$. However, no finite set of polynomials can possibly span $\mathscr{P}$, the vector space of all polynomials. (See Exercise 44 in Section 6.2.) But, if we allow a spanning set to be infinite, then clearly the set of *all* nonnegative powers of x will do. That is, $\mathscr{P} = \text{span}(1, x, x^2, \ldots)$.

Example 6.18

Show that $M_{23} = \text{span}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$, where

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(That is, E_{ij} is the matrix with a 1 in row i, column j and zeros elsewhere.)

Solution We need only observe that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23}$$

Extending this example, we see that, in general, M_{mn} is spanned by the mn matrices E_{ij}, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Example 6.19

In $\mathscr{P}_2$, determine whether $r(x) = 1 - 4x + 6x^2$ is in $\text{span}(p(x), q(x))$, where

$$p(x) = 1 - x + x^2 \quad \text{and} \quad q(x) = 2 + x - 3x^2$$

Solution We are looking for scalars c and d such that $cp(x) + dq(x) = r(x)$. This means that

$$c(1 - x + x^2) + d(2 + x - 3x^2) = 1 - 4x + 6x^2$$

Regrouping according powers of x, we have

$$(c + 2d) + (-c + d)x + (c - 3d)x^2 = 1 - 4x + 6x^2$$

Equating the coefficients of like powers of x gives

$$\begin{aligned} c + 2d &= 1 \\ -c + d &= -4 \\ c - 3d &= 6 \end{aligned}$$

which is easily solved to give $c = 3$ and $d = -1$. Therefore, $r(x) = 3p(x) - q(x)$, so $r(x)$ is in span$(p(x), q(x))$. (Check this.)

Example 6.20

In $\mathscr{F}$, determine whether $\sin 2x$ is in span$(\sin x, \cos x)$.

Solution We set $c \sin x + d \cos x = \sin 2x$ and try to determine c and d so that this equation is true. Since these are functions, the equation must be true for *all* values of x. Setting $x = 0$, we have

$$c \sin 0 + d \cos 0 = \sin 0 \quad \text{or} \quad c(0) + d(1) = 0$$

from which we see that $d = 0$. Setting $x = \pi/2$, we get

$$c \sin(\pi/2) + d \cos(\pi/2) = \sin(\pi) \quad \text{or} \quad c(1) + d(0) = 0$$

giving $c = 0$. But this implies that $\sin 2x = 0(\sin x) + 0(\cos x) = 0$ for all x, which is absurd, since $\sin 2x$ is not the zero function. We conclude that $\sin 2x$ is not in span$(\sin x, \cos x)$.

Remark It *is* true that $\sin 2x$ can be written in terms of $\sin x$ and $\cos x$. For example, we have the double angle formula $\sin 2x = 2 \sin x \cos x$. However, this is not a *linear* combination.

Example 6.21

In M_{22}, describe the span of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution Every linear combination of A, B, and C is of the form

$$\begin{aligned} cA + dB + eC &= c\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c + d & c + e \\ c + e & d \end{bmatrix} \end{aligned}$$

This matrix is symmetric, so span(A, B, C) is contained within the subspace of symmetric 2×2 matrices. In fact, we have equality; that is, *every* symmetric 2×2 matrix is in span(A, B, C). To show this, we let $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$ be a symmetric 2×2 matrix. Setting

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} c + d & c + e \\ c + e & d \end{bmatrix}$$

and solving for c and d, we find that $c = x - z$, $d = z$, and $e = -x + y + z$. Therefore,

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = (x - z)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-x + y + z)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(Check this.) It follows that span(A, B, C) is the subspace of symmetric 2×2 matrices.

As was the case in $\mathbb{R}^n$, the span of a set of vectors is always a subspace of the vector space that contains them. The next theorem makes this result precise. It generalizes Theorem 3.19.

Theorem 6.3 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V.

a. $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of V.
b. $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is the smallest subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Proof (a) The proof of property (a) is identical to the proof of Theorem 3.19, with $\mathbb{R}^n$ replaced by V.

(b) To establish property (b), we need to show that any subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ also contains $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$. Accordingly, let W be a subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then, since W is closed under addition and scalar multiplication, it contains every linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Therefore, $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is contained in W.

Exercises 6.1

In Exercises 1–11, determine whether the given set, together with the specified operations of addition and scalar multiplication, is a vector space. If it is not, list all of the axioms that fail to hold.

1. The set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} x \\ x \end{bmatrix}$, with the usual vector addition and scalar multiplication
2. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 0, y \geq 0$ (i.e., the first quadrant), with the usual vector addition and scalar multiplication
3. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $xy \geq 0$ (i.e., the union of the first and third quadrants), with the usual vector addition and scalar multiplication
4. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq y$, with the usual vector addition and scalar multiplication
5. $\mathbb{R}^2$, with the usual addition but scalar multiplication defined by
$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ y \end{bmatrix}$$
6. $\mathbb{R}^2$, with the usual scalar multiplication but addition defined by
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \end{bmatrix}$$
7. The set of all positive real numbers, with addition $\oplus$ defined by $x \oplus y = xy$ and scalar multiplication $\odot$ defined by $c \odot x = x^c$
8. The set of all rational numbers, with the usual addition and multiplication
9. The set of all upper triangular 2×2 matrices, with the usual matrix addition and scalar multiplication
10. The set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad = 0$, with the usual matrix addition and scalar multiplication
11. The set of all skew-symmetric $n \times n$ matrices, with the usual matrix addition and scalar multiplication (see page 162).
12. Finish verifying that $\mathscr{P}_2$ is a vector space (see Example 6.3).
13. Finish verifying that $\mathscr{F}$ is a vector space (see Example 6.4).

In Exercises 14–17, determine whether the given set, together with the specified operations of addition and scalar multiplication, is a complex vector space. If it is not, list all of the axioms that fail to hold.

14. The set of all vectors in $\mathbb{C}^2$ of the form $\begin{bmatrix} z \\ \bar{z} \end{bmatrix}$, with the usual vector addition and scalar multiplication

15. The set $M_{mn}(\mathbb{C})$ of all $m \times n$ complex matrices, with the usual matrix addition and scalar multiplication

16. The set $\mathbb{C}^2$, with the usual vector addition but scalar multiplication defined by $c\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{c}z_1 \\ \bar{c}z_2 \end{bmatrix}$

17. $\mathbb{R}^n$, with the usual vector addition and scalar multiplication

In Exercises 18–21, determine whether the given set, together with the specified operations of addition and scalar multiplication, is a vector space over the indicated $\mathbb{Z}_p$. If it is not, list all of the axioms that fail to hold.

18. The set of all vectors in $\mathbb{Z}_2^n$ with an even number of 1s, over $\mathbb{Z}_2$ with the usual vector addition and scalar multiplication

19. The set of all vectors in $\mathbb{Z}_2^n$ with an odd number of 1s, over $\mathbb{Z}_2$ with the usual vector addition and scalar multiplication

20. The set $M_{mn}(\mathbb{Z}_p)$ of all $m \times n$ matrices with entries from $\mathbb{Z}_p$, over $\mathbb{Z}_p$ with the usual matrix addition and scalar multiplication

21. $\mathbb{Z}_6$, over $\mathbb{Z}_3$ with the usual addition and multiplication (Think this one through carefully!)

22. Prove Theorem 6.1(a).

23. Prove Theorem 6.1(c).

In Exercises 24–45, use Theorem 6.2 to determine whether W is a subspace of V.

24. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ 0 \\ a \end{bmatrix}\right\}$ **25.** $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ -a \\ 2a \end{bmatrix}\right\}$

26. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix}\right\}$

27. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ b \\ |a| \end{bmatrix}\right\}$

28. $V = M_{22}$, $W = \left\{\begin{bmatrix} a & b \\ b & 2a \end{bmatrix}\right\}$

29. $V = M_{22}$, $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad \geq bc\right\}$

30. $V = M_{nn}$, $W = \{A \text{ in } M_{nn} : \det A = 1\}$

31. $V = M_{nn}$, W is the set of diagonal $n \times n$ matrices

32. $V = M_{nn}$, W is the set of idempotent $n \times n$ matrices

33. $V = M_{nn}$, $W = \{A \text{ in } M_{nn} : AB = BA\}$, where B is a given (fixed) matrix

34. $V = \mathscr{P}_2$, $W = \{bx + cx^2\}$

35. $V = \mathscr{P}_2$, $W = \{a + bx + cx^2 : a + b + c = 0\}$

36. $V = \mathscr{P}_2$, $W = \{a + bx + cx^2 : abc = 0\}$

37. $V = \mathscr{P}$, W is the set of all polynomials of degree 3

38. $V = \mathscr{F}$, $W = \{f \text{ in } \mathscr{F} : f(-x) = f(x)\}$

39. $V = \mathscr{F}$, $W = \{f \text{ in } \mathscr{F} : f(-x) = -f(x)\}$

40. $V = \mathscr{F}$, $W = \{f \text{ in } \mathscr{F} : f(0) = 1\}$

41. $V = \mathscr{F}$, $W = \{f \text{ in } \mathscr{F} : f(0) = 0\}$

42. $V = \mathscr{F}$, W is the set of all integrable functions

43. $V = \mathscr{D}$, $W = \{f \text{ in } \mathscr{D} : f'(x) \geq 0 \text{ for all } x\}$

44. $V = \mathscr{F}$, $W = \mathscr{C}^{(2)}$, the set of all functions with continuous second derivatives

45. $V = \mathscr{F}$, $W = \{f \text{ in } \mathscr{F} : = \lim_{x \to 0} f(x) = \infty\}$

46. Let V be a vector space with subspaces U and W. Prove that $U \cap W$ is a subspace of V.

47. Let V be a vector space with subspaces U and W. Give an example with $V = \mathbb{R}^2$ to show that $U \cup W$ need not be a subspace of V.

48. Let V be a vector space with subspaces U and W. Define the ***sum of U and W*** to be

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \text{ is in } U, \mathbf{w} \text{ is in } W\}$$

(a) If $V = \mathbb{R}^3$, U is the x-axis, and W is the y-axis, what is $U + W$?

(b) If U and W are subspaces of a vector space V, prove that $U + W$ is a subspace of V.

49. If U and V are vector spaces, define the ***Cartesian product*** of U and V to be

$$U \times V = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \text{ is in } U \text{ and } \mathbf{v} \text{ is in } V\}$$

Prove that $U \times V$ is a vector space.

50. Let W be a subspace of a vector space V. Prove that $\Delta = \{(\mathbf{w}, \mathbf{w}) : \mathbf{w} \text{ is in } W\}$ is a subspace of $V \times V$.

In Exercises 51 and 52, let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Determine whether C is in span(A, B).

51. $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **52.** $C = \begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix}$

In Exercises 53 and 54, let $p(x) = 1 - 2x$, $q(x) = x - x^2$, *and* $r(x) = -2 + 3x + x^2$. *Determine whether* $s(x)$ *is in span*$(p(x), q(x), r(x))$.

53. $s(x) = 3 - 5x - x^2$

54. $s(x) = 1 + x + x^2$

In Exercises 55–58, let $f(x) = \sin^2 x$ *and* $g(x) = \cos^2 x$. *Determine whether* $h(x)$ *is in span*$(f(x), g(x))$.

55. $h(x) = 1$

56. $h(x) = \cos 2x$

57. $h(x) = \sin 2x$

58. $h(x) = \sin x$

59. Is M_{22} spanned by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$?

60. Is M_{22} spanned by $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$?

61. Is $\mathscr{P}_2$ spanned by $1 + x, x + x^2, 1 + x^2$?

62. Is $\mathscr{P}_2$ spanned by $1 + x + 2x^2, 2 + x + 2x^2, -1 + x + 2x^2$?

63. Prove that every vector space has a unique zero vector.

64. Prove that for every vector $\mathbf{v}$ in a vector space V, there is a unique $\mathbf{v}'$ in V such that $\mathbf{v} + \mathbf{v}' = \mathbf{0}$.

Writing Project

The Rise of Vector Spaces

As noted in the sidebar on page 429, in the late 19th century, the mathematicians Hermann Grassmann and Giuseppe Peano were instrumental in introducing the idea of a vector space and the vector space axioms that we use today. Grassmann's work had its origins in barycentric coordinates, a technique invented in 1827 by August Ferdinand Möbius (of Möbius strip fame). However, widespread acceptance of the vector space concept did not come until the early 20th century.

Write a report on the history of vector spaces. Discuss the origins of the notion of a vector space and the contributions of Grassmann and Peano. Why was the mathematical community slow to adopt these ideas, and how did acceptance come about?

1. Carl B. Boyer and Uta C. Merzbach, *A History of Mathematics* (Third Edition) (Hoboken, NJ: Wiley, 2011).
2. Jean-Luc, Dorier (1995), A General Outline of the Genesis of Vector Space Theory, *Historia Mathematica* 22 (1995), pp. 227–261.
3. Victor J. Katz, *A History of Mathematics: An Introduction* (Third Edition) (Reading, MA: Addison Wesley Longman, 2008).

6.2 Linear Independence, Basis, and Dimension

In this section, we extend the notions of linear independence, basis, and dimension to general vector spaces, generalizing the results of Sections 2.3 and 3.5. In most cases, the proofs of the theorems carry over; we simply replace $\mathbb{R}^n$ by the vector space V.

Linear Independence

Definition A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_k$, *at least one of which is not zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

A set of vectors that is not linearly dependent is said to be ***linearly independent.***

As in $\mathbb{R}^n$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent in a vector space V if and only if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \quad \text{implies} \quad c_1 = 0, c_2 = 0, \ldots, c_k = 0$$

We also have the following useful alternative formulation of linear dependence.

Theorem 6.4 A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

Proof The proof is identical to that of Theorem 2.5.

As a special case of Theorem 6.4, note that a set of *two* vectors is linearly dependent if and only if one is a scalar multiple of the other.

Example 6.22 In $\mathscr{P}_2$, the set $\{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$ is linearly dependent, since

$$2(1 + x + x^2) - (1 - x + 3x^2) = 1 + 3x - x^2$$

Example 6.23 In M_{22}, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Then $A + B = C$, so the set $\{A, B, C\}$ is linearly dependent.

Example 6.24 In $\mathscr{F}$, the set $\{\sin^2 x, \cos^2 x, \cos 2x\}$ is linearly dependent, since

$$\cos 2x = \cos^2 x - \sin^2 x$$

Example 6.25 Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathscr{P}_n$.

Solution 1 Suppose that $c_0, c_1, \ldots, c_n$ are scalars such that

$$c_0 \cdot 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$$

Then the polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ is zero for all values of x. But a polynomial of degree at most n cannot have more than n zeros (see Appendix D). So $p(x)$ must be the zero polynomial, meaning that $c_0 = c_1 = c_2 = \cdots = c_n = 0$. Therefore, $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Solution 2 We begin, as in the first solution, by assuming that

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$$

Since this is true for all x, we can substitute $x = 0$ to obtain $c_0 = 0$. This leaves

$$c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$$

Taking derivatives, we obtain

$$c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} = 0$$

and setting $x = 0$, we see that $c_1 = 0$. Differentiating $2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} = 0$ and setting $x = 0$, we find that $2c_2 = 0$, so $c_2 = 0$. Continuing in this fashion, we find that $k!c_k = 0$ for $k = 0, \ldots, n$. Therefore, $c_0 = c_1 = c_2 = \cdots = c_n = 0$, and $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Example 6.26

In $\mathscr{P}_2$, determine whether the set $\{1 + x, x + x^2, 1 + x^2\}$ is linearly independent.

Solution Let c_1, c_2, and c_3 be scalars such that

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = 0$$

Then

$$(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = 0$$

This implies that

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

the solution to which is $c_1 = c_2 = c_3 = 0$. It follows that $\{1 + x, x + x^2, 1 + x^2\}$ is linearly independent.

Remark Compare Example 6.26 with Example 2.23(b). The system of equations that arises is exactly the same. This is because of the correspondence between $\mathscr{P}_2$ and $\mathbb{R}^3$ that relates

$$1 + x \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x + x^2 \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad 1 + x^2 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and produces the columns of the coefficient matrix of the linear system that we have to solve. Thus, showing that $\{1 + x, x + x^2, 1 + x^2\}$ is linearly independent is equivalent to showing that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent. This can be done simply by establishing that the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has rank 3, by the Fundamental Theorem of Invertible Matrices.

Example 6.27

In $\mathscr{F}$, determine whether the set $\{\sin x, \cos x\}$ is linearly independent.

Solution The functions $f(x) = \sin x$ and $g(x) = \cos x$ are linearly *dependent* if and only if one of them is a scalar multiple of the other. But it is clear from their graphs that this is not the case, since, for example, any nonzero multiple of $f(x) = \sin x$ has the same zeros, none of which are zeros of $g(x) = \cos x$.

This approach may not always be appropriate to use, so we offer the following direct, more computational method. Suppose c and d are scalars such that

$$c \sin x + d \cos x = 0$$

Setting $x = 0$, we obtain $d = 0$, and setting $x = \pi/2$, we obtain $c = 0$. Therefore, the set $\{\sin x, \cos x\}$ is linearly independent.

Although the definitions of linear dependence and independence are phrased in terms of *finite* sets of vectors, we can extend the concepts to *infinite* sets as follows:

A set S of vectors in a vector space V is ***linearly dependent*** if it contains finitely many linearly dependent vectors. A set of vectors that is not linearly dependent is said to be ***linearly independent.***

Note that for finite sets of vectors, this is just the original definition. Following is an example of an infinite set of linearly independent vectors.

Example 6.28

In $\mathscr{P}$, show that $S = \{1, x, x^2, \ldots\}$ is linearly independent.

Solution Suppose there is a finite subset T of S that is linearly dependent. Let x^m be the highest power of x in T and let x^n be the lowest power of x in T. Then there are scalars $c_n, c_{n+1}, \ldots, c_m$, not all zero, such that

$$c_n x^n + c_{n+1} x^{n+1} + \cdots + c_m x^m = 0$$

But, by an argument similar to that used in Example 6.25, this implies that $c_n = c_{n+1} = \cdots = c_m = 0$, which is a contradiction. Hence, S cannot contain finitely many linearly dependent vectors, so it is linearly independent.

Bases

The important concept of a basis now can be extended easily to arbitrary vector spaces.

Definition A subset $\mathcal{B}$ of a vector space V is a ***basis*** for V if

1. $\mathcal{B}$ spans V and
2. $\mathcal{B}$ is linearly independent.

Example 6.29

If $\mathbf{e}_i$ is the ith column of the $n \times n$ identity matrix, then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, called the ***standard basis*** for $\mathbb{R}^n$.

Example 6.30

$\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathscr{P}_n$, called the ***standard basis*** for $\mathscr{P}_n$.

Example 6.31

The set $\mathcal{E} = \{E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, E_{m1}, \ldots, E_{mn}\}$ is a basis for M_{mn}, where the matrices E_{ij} are as defined in Example 6.18. $\mathcal{E}$ is called the ***standard basis*** for M_{mn}.

We have already seen that $\mathcal{E}$ spans M_{mn}. It is easy to show that $\mathcal{E}$ is linearly independent. (Verify this!) Hence, $\mathcal{E}$ is a basis for M_{mn}.

Example 6.32

Show that $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}$ is a basis for $\mathscr{P}_2$.

Solution We have already shown that $\mathcal{B}$ is linearly independent, in Example 6.26. To show that $\mathcal{B}$ spans $\mathscr{P}_2$, let $a + bx + cx^2$ be an arbitrary polynomial in $\mathscr{P}_2$. We must show that there are scalars c_1, c_2, and c_3 such that

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = a + bx + cx^2$$

or, equivalently,

$$(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = a + bx + cx^2$$

Equating coefficients of like powers of x, we obtain the linear system

$$\begin{aligned} c_1 + c_3 &= a \\ c_1 + c_2 &= b \\ c_2 + c_3 &= c \end{aligned}$$

which has a solution, since the coefficient matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has rank 3 and, hence, is invertible. (We do not need to know *what* the solution is; we only need to know that it exists.) Therefore, $\mathcal{B}$ is a basis for $\mathscr{P}_2$.

Remark Observe that the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is the key to Example 6.32. We can immediately obtain it using the correspondence between $\mathscr{P}_2$ and $\mathbb{R}^3$, as indicated in the Remark following Example 6.26.

Example 6.33 Show that $\mathcal{B} = \{1, x, x^2, \ldots\}$ is a basis for $\mathcal{P}$.

Solution In Example 6.28, we saw that $\mathcal{B}$ is linearly independent. It also spans $\mathcal{P}$, since clearly every polynomial is a linear combination of (finitely many) powers of x.

Example 6.34 Find bases for the three vector spaces in Example 6.13:

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \right\}$ (b) $W_2 = \{a + bx - bx^2 + ax^3\}$ (c) $W_3 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\}$

Solution Once again, we will work the three examples side by side to highlight the similarities among them. In a strong sense, they are all the *same* example, but it will take us until Section 6.5 to make this idea perfectly precise.

(a) Since

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

we have $W_1 = \text{span}(\mathbf{u}, \mathbf{v})$, where

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Since $\{\mathbf{u}, \mathbf{v}\}$ is clearly linearly independent, it is also a basis for W_1.

(b) Since

$$\begin{aligned} a + bx - bx^2 + ax^3 \\ = a(1 + x^3) + b(x - x^2) \end{aligned}$$

we have $W_2 = \text{span}(u(x), v(x))$, where

$$u(x) = 1 + x^3$$

and

$$v(x) = x - x^2$$

Since $\{u(x), v(x)\}$ is clearly linearly independent, it is also a basis for W_2.

(c) Since

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

we have $W_3 = \text{span}(U, V)$, where

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since $\{U, V\}$ is clearly linearly independent, it is also a basis for W_3.

Coordinates

Section 3.5 introduced the idea of the coordinates of a vector with respect to a basis for subspaces of $\mathbb{R}^n$. We now extend this concept to arbitrary vector spaces.

Theorem 6.5 Let V be a vector space and let $\mathcal{B}$ be a basis for V. For every vector $\mathbf{v}$ in V, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in $\mathcal{B}$.

Proof The proof is the same as the proof of Theorem 3.29. It works even if the basis $\mathcal{B}$ is infinite, since linear combinations are, by definition, finite.

The converse of Theorem 6.5 is also true. That is, if $\mathcal{B}$ is a set of vectors in a vector space V with the property that every vector in V can be written uniquely as a linear combination of the vectors in $\mathcal{B}$, then $\mathcal{B}$ is a basis for V (see Exercise 30). In this sense, the *unique representation property* characterizes a basis.

Since representation of a vector with respect to a basis is unique, the next definition makes sense.

Definition Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{v}$ be a vector in V, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Then $c_1, c_2, \ldots, c_n$ are called the ***coordinates of* $\mathbf{v}$ *with respect to* $\mathcal{B}$**, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the ***coordinate vector of* $\mathbf{v}$ *with respect to* $\mathcal{B}$**.

Observe that if the basis $\mathcal{B}$ of V has n vectors, then $[\mathbf{v}]_{\mathcal{B}}$ is a (column) vector in $\mathbb{R}^n$.

Example 6.35

Find the coordinate vector $[p(x)]_{\mathcal{B}}$ of $p(x) = 2 - 3x + 5x^2$ with respect to the standard basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$.

Solution The polynomial $p(x)$ is already a linear combination of 1, x, and x^2, so

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

This is the correspondence between $\mathcal{P}_2$ and $\mathbb{R}^3$ that we remarked on after Example 6.26, and it can easily be generalized to show that the coordinate vector of a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad \text{in } \mathcal{P}_n$$

with respect to the standard basis $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$ is just the vector

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{in } \mathbb{R}^{n+1}$$

Remark The *order* in which the basis vectors appear in $\mathcal{B}$ affects the order of the entries in a coordinate vector. For example, in Example 6.35, assume that the standard basis vectors are ordered as $\mathcal{B}' = \{x^2, x, 1\}$. Then the coordinate vector of $p(x) = 2 - 3x + 5x^2$ with respect to $\mathcal{B}'$ is

$$[p(x)]_{\mathcal{B}'} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$$

Example 6.36

Find the coordinate vector $[A]_{\mathcal{B}}$ of $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ with respect to the standard basis $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_{22}.

Solution Since

$$\begin{aligned} A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} &= 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 2E_{11} - E_{12} + 4E_{21} + 3E_{22} \end{aligned}$$

we have

$$[A]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

This is the correspondence between M_{22} and $\mathbb{R}^4$ that we noted before the introduction to Example 6.13. It too can easily be generalized to give a correspondence between M_{mn} and $\mathbb{R}^{mn}$.

Example 6.37

Find the coordinate vector $[p(x)]_{\mathcal{B}}$ of $p(x) = 1 + 2x - x^2$ with respect to the basis $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathscr{P}_2$.

Solution We need to find c_1, c_2, and c_3 such that

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = 1 + 2x - x^2$$

or, equivalently,

$$(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = 1 + 2x - x^2$$

As in Example 6.32, this means we need to solve the system

$$\begin{aligned} c_1 + \phantom{c_2 +{}} c_3 &= 1 \\ c_1 + c_2 \phantom{{}+ c_3} &= 2 \\ c_2 + c_3 &= -1 \end{aligned}$$

whose solution is found to be $c_1 = 2$, $c_2 = 0$, $c_3 = -1$. Therefore,

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

[Since this result says that $p(x) = 2(1 + x) - (1 + x^2)$, it is easy to check that it is correct.]

The next theorem shows that the process of forming coordinate vectors is compatible with the vector space operations of addition and scalar multiplication.

Theorem 6.6 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and let c be a scalar. Then

a. $[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$
b. $[c\mathbf{u}]_{\mathcal{B}} = c[\mathbf{u}]_{\mathcal{B}}$

Proof We begin by writing $\mathbf{u}$ and $\mathbf{v}$ in terms of the basis vectors—say, as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$$

Then, using vector space properties, we have

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_n + d_n)\mathbf{v}_n$$

and

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_n)\mathbf{v}_n$$

so

$$[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$$

and

$$[c\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{bmatrix} = c\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c[\mathbf{u}]_{\mathcal{B}}$$

An easy corollary to Theorem 6.6 states that coordinate vectors preserve linear combinations:

$$[c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_k[\mathbf{u}_k]_{\mathcal{B}} \tag{1}$$

You are asked to prove this corollary in Exercise 31.

The most useful aspect of coordinate vectors is that they allow us to transfer information from a general vector space to $\mathbb{R}^n$, where we have the tools of Chapters 1 to 3 at our disposal. We will explore this idea in some detail in Sections 6.3 and 6.6. For now, we have the following useful theorem.

Theorem 6.7 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in V. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_k]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.

Proof Assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent in V and let

$$c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_k[\mathbf{u}_k]_{\mathcal{B}} = \mathbf{0}$$

in $\mathbb{R}^n$. But then we have

$$[c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k]_{\mathcal{B}} = \mathbf{0}$$

using Equation (1), so the coordinates of the vector $c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$ with respect to $\mathcal{B}$ are all zero. That is,

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

The linear independence of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ now forces $c_1 = c_2 = \cdots = c_k = 0$, so $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_k]_{\mathcal{B}}\}$ is linearly independent.

The converse implication, which uses similar ideas, is left as Exercise 32.

Observe that, in the special case where $\mathbf{u}_i = \mathbf{v}_i$, we have

$$\mathbf{v}_i = 0 \cdot \mathbf{v}_1 + \cdots + 1 \cdot \mathbf{v}_i + \cdots + 0 \cdot \mathbf{v}_n$$

so $[\mathbf{v}_i]_{\mathcal{B}} = \mathbf{e}_i$ and $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_n]_{\mathcal{B}}\} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis in $\mathbb{R}^n$.

Dimension

The definition of dimension is the same for a vector space as for a subspace of $\mathbb{R}^n$—the number of vectors in a basis for the space. Since a vector space can have more than one basis, we need to show that this definition makes sense; that is, we need to establish that different bases for the same vector space contain the same number of vectors.

Part (a) of the next theorem generalizes Theorem 2.8.

Theorem 6.8 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.

a. Any set of more than n vectors in V must be linearly dependent.
b. Any set of fewer than n vectors in V cannot span V.

Proof (a) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a set of vectors in V, with $m > n$. Then $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_m]_{\mathcal{B}}\}$ is a set of more than n vectors in $\mathbb{R}^n$ and, hence, is linearly dependent, by Theorem 2.8. This means that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is linearly dependent as well, by Theorem 6.7.

(b) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a set of vectors in V, with $m < n$. Then $S = \{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_m]_{\mathcal{B}}\}$ is a set of fewer than n vectors in $\mathbb{R}^n$. Now $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) = V$ if and only if $\text{span}(S) = \mathbb{R}^n$ (see Exercise 33). But $\text{span}(S)$ is just the column space of the $n \times m$ matrix

$$A = [\,[\mathbf{u}_1]_{\mathcal{B}} \cdots [\mathbf{u}_m]_{\mathcal{B}}\,]$$

so $\dim(\text{span}(S)) = \dim(\text{col}(A)) \le m < n$. Hence, S cannot span $\mathbb{R}^n$, so $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ does not span V.

Now we extend Theorem 3.23.

Theorem 6.9 **The Basis Theorem**

If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.

The proof of Theorem 3.23 also works here, virtually word for word. However, it is easier to make use of Theorem 6.8.

Proof Let $\mathcal{B}$ be a basis for V with n vectors and let $\mathcal{B}'$ be another basis for V with m vectors. By Theorem 6.8, $m \leq n$; otherwise, $\mathcal{B}'$ would be linearly dependent.

Now use Theorem 6.8 with the roles of $\mathcal{B}$ and $\mathcal{B}'$ interchanged. Since $\mathcal{B}'$ is a basis of V with m vectors, Theorem 6.8 implies that any set of more than m vectors in V is linearly dependent. Hence, $n \leq m$, since $\mathcal{B}$ is a basis and is, therefore, linearly independent.

Since $n \leq m$ and $m \leq n$, we must have $n = m$, as required.

The following definition now makes sense, since the number of vectors in a (finite) basis does not depend on the choice of basis.

Definition A vector space V is called ***finite-dimensional*** if it has a basis consisting of finitely many vectors. The ***dimension*** of V, denoted by dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. A vector space that has no finite basis is called ***infinite-dimensional.***

Example 6.38

Since the standard basis for $\mathbb{R}^n$ has n vectors, dim $\mathbb{R}^n = n$. In the case of $\mathbb{R}^3$, a one-dimensional subspace is just the span of a single nonzero vector and thus is a line through the origin. A two-dimensional subspace is spanned by its basis of two linearly independent (i.e., nonparallel) vectors and therefore is a plane through the origin. Any three linearly independent vectors must span $\mathbb{R}^3$, by the Fundamental Theorem. The subspaces of $\mathbb{R}^3$ are now completely classified according to dimension, as shown in Table 6.1.

Table 6.1

dim V	V
3	$\mathbb{R}^3$
2	Plane through the origin
1	Line through the origin
0	$\{\mathbf{0}\}$

Example 6.39

The standard basis for $\mathscr{P}_n$ contains $n + 1$ vectors (see Example 6.30), so dim $\mathscr{P}_n = n + 1$.

Example 6.40

The standard basis for M_{mn} contains mn vectors (see Example 6.31), so $\dim M_{mn} = mn$.

Example 6.41

Both $\mathscr{P}$ and $\mathscr{F}$ are infinite-dimensional, since they each contain the infinite linearly independent set $\{1, x, x^2, \ldots\}$ (see Exercise 44).

Example 6.42

Find the dimension of the vector space W of symmetric 2×2 matrices (see Example 6.10).

Solution A symmetric 2×2 matrix is of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so W is spanned by the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

If S is linearly independent, then it will be a basis for W. Setting

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we obtain

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which it immediately follows that $a = b = c = 0$. Hence, S is linearly independent and is, therefore, a basis for W. We conclude that $\dim W = 3$.

The dimension of a vector space is its "magic number." Knowing the dimension of a vector space V provides us with much information about V and can greatly simplify the work needed in certain types of calculations, as the next few theorems and examples illustrate.

Theorem 6.10

Let V be a vector space with $\dim V = n$. Then:

a. Any linearly independent set in V contains at most n vectors.
b. Any spanning set for V contains at least n vectors.
c. Any linearly independent set of exactly n vectors in V is a basis for V.
d. Any spanning set for V consisting of exactly n vectors is a basis for V.
e. Any linearly independent set in V can be extended to a basis for V.
f. Any spanning set for V can be reduced to a basis for V.

Proof The proofs of properties (a) and (b) follow from parts (a) and (b) of Theorem 6.8, respectively.

(c) Let S be a linearly independent set of exactly n vectors in V. If S does not span V, then there is some vector $\mathbf{v}$ in V that is not a linear combination of the vectors in S. Inserting $\mathbf{v}$ into S produces a set S' with $n + 1$ vectors that is still linearly independent (see Exercise 54). But this is impossible, by Theorem 6.8(a). We conclude that S must span V and therefore be a basis for V.

(d) Let S be a spanning set for V consisting of exactly n vectors. If S is linearly dependent, then some vector $\mathbf{v}$ in S is a linear combination of the others. Throwing $\mathbf{v}$ away leaves a set S' with $n - 1$ vectors that still spans V (see Exercise 55). But this is impossible, by Theorem 6.8(b). We conclude that S must be linearly independent and therefore be a basis for V.

(e) Let S be a linearly independent set of vectors in V. If S spans V, it is a basis for V and so consists of exactly n vectors, by the Basis Theorem. If S does not span V, then, as in the proof of property (c), there is some vector $\mathbf{v}$ in V that is not a linear combination of the vectors in S. Inserting $\mathbf{v}$ into S produces a set S' that is still linearly independent. If S' still does not span V, we can repeat the process and expand it into a larger, linearly independent set. Eventually, this process must stop, since no linearly independent set in V can contain more than n vectors, by Theorem 6.8(a). When the process stops, we have a linearly independent set S^* that contains S and also spans V. Therefore, S^* is a basis for V that extends S.

(f) You are asked to prove this property in Exercise 56.

You should view Theorem 6.10 as, in part, a labor-saving device. In many instances, it can dramatically decrease the amount of work needed to check that a set of vectors is linearly independent, a spanning set, or a basis.

Example 6.43

In each case, determine whether S is a basis for V.

(a) $V = \mathscr{P}_2$, $S = \{1 + x, 2 - x + x^2, 3x - 2x^2, -1 + 3x + x^2\}$

(b) $V = M_{22}$, $S = \left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right\}$

(c) $V = \mathscr{P}_2$, $S = \{1 + x, x + x^2, 1 + x^2\}$

Solution (a) Since $\dim(\mathscr{P}_2) = 3$ and S contains four vectors, S is linearly dependent, by Theorem 6.10(a). Hence, S is not a basis for $\mathscr{P}_2$.

(b) Since $\dim(M_{22}) = 4$ and S contains three vectors, S cannot span M_{22}, by Theorem 6.10(b). Hence, S is not a basis for M_{22}.

(c) Since $\dim(\mathscr{P}_2) = 3$ and S contains three vectors, S will be a basis for $\mathscr{P}_2$ if it is linearly independent or if it spans $\mathscr{P}_2$, by Theorem 6.10(c) or (d). It is easier to show that S is linearly independent; we did this in Example 6.26. Therefore, S is a basis for $\mathscr{P}_2$. (This is the same problem as in Example 6.32—but see how much easier it becomes using Theorem 6.10!)

Example 6.44

Extend $\{1 + x, 1 - x\}$ to a basis for $\mathscr{P}_2$.

Solution First note that $\{1 + x, 1 - x\}$ is linearly independent. (Why?) Since $\dim(\mathscr{P}_2) = 3$, we need a third vector—one that is not linearly dependent on the first two.

We could proceed, as in the proof of Theorem 6.10(e), to find such a vector using trial and error. However, it is easier in practice to proceed in a different way.

We enlarge the given set of vectors by throwing in the *entire* standard basis for $\mathscr{P}_2$. This gives

$$S = \{1 + x, 1 - x, 1, x, x^2\}$$

Now S is linearly dependent, by Theorem 6.10(a), so we need to throw away some vectors—in this case, two. Which ones? We use Theorem 6.10(f), starting with the first vector that was added, 1. Since $1 = \frac{1}{2}(1 + x) + \frac{1}{2}(1 - x)$, the set $\{1 + x, 1 - x, 1\}$ is linearly dependent, so we throw away 1. Similarly, $x = \frac{1}{2}(1 + x) - \frac{1}{2}(1 - x)$, so $\{1 + x, 1 - x, x\}$ is linearly dependent also. Finally, we check that $\{1 + x, 1 - x, x^2\}$ is linearly independent. (Can you see a quick way to tell this?) Therefore, $\{1 + x, 1 - x, x^2\}$ is a basis for $\mathscr{P}_2$ that extends $\{1 + x, 1 - x\}$.

In Example 6.42, the vector space W of symmetric 2×2 matrices is a subspace of the vector space M_{22} of all 2×2 matrices. As we showed, $\dim W = 3 \leq 4 = \dim M_{22}$. This is an example of a general result, as the final theorem of this section shows.

Theorem 6.11 Let W be a subspace of a finite-dimensional vector space V. Then:

a. W is finite-dimensional and $\dim W \leq \dim V$.
b. $\dim W = \dim V$ if and only if $W = V$.

Proof (a) Let $\dim V = n$. If $W = \{\mathbf{0}\}$, then $\dim(W) = 0 \leq n = \dim V$. If W is nonzero, then any basis $\mathcal{B}$ for V (containing n vectors) certainly spans W, since W is contained in V. But $\mathcal{B}$ can be reduced to a basis $\mathcal{B}'$ for W (containing at most n vectors), by Theorem 6.10(f). Hence, W is finite-dimensional and $\dim(W) \leq n = \dim V$.

(b) If $W = V$, then certainly $\dim W = \dim V$. On the other hand, if $\dim W = \dim V = n$, then any basis $\mathcal{B}$ for W consists of exactly n vectors. But these are then n linearly independent vectors in V and, hence, a basis for V, by Theorem 6.10(c). Therefore, $V = \operatorname{span}(\mathcal{B}) = W$.

Exercises 6.2

In Exercises 1–4, test the sets of matrices for linear independence in M_{22}. For those that are linearly dependent, express one of the matrices as a linear combination of the others.

1. $\left\{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\right\}$

2. $\left\{\begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}\right\}$

3. $\left\{\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix}\right\}$

4. $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

In Exercises 5–9, test the sets of polynomials for linear independence. For those that are linearly dependent, express one of the polynomials as a linear combination of the others.

5. $\{x, 1 + x\}$ in $\mathscr{P}_1$

6. $\{1 + x, 1 + x^2, 1 - x + x^2\}$ in $\mathscr{P}_2$

7. $\{x, 2x - x^2, 3x + 2x^2\}$ in $\mathscr{P}_2$

8. $\{2x, x - x^2, 1 + x^3, 2 - x^2 + x^3\}$ in $\mathscr{P}_3$

9. $\{1 - 2x, 3x + x^2 - x^3, 1 + x^2 + 2x^3, 3 + 2x + 3x^3\}$ in $\mathscr{P}_3$

In Exercises 10–14, test the sets of functions for linear independence in $\mathscr{F}$. For those that are linearly dependent, express one of the functions as a linear combination of the others.

10. $\{1, \sin x, \cos x\}$

11. $\{1, \sin^2 x, \cos^2 x\}$

12. $\{e^x, e^{-x}\}$

13. $\{1, \ln(2x), \ln(x^2)\}$

14. $\{\sin x, \sin 2x, \sin 3x\}$

15. If f and g are in $\mathscr{C}^{(1)}$, the vector space of all functions with continuous derivatives, then the determinant

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

is called the ***Wronskian*** of f and g [named after the Polish-French mathematician Jósef Maria Hoëné-Wronski (1776–1853), who worked on the theory of determinants and the philosophy of mathematics]. Show that f and g are linearly independent if their Wronskian is not identically zero (that is, if there is some x such that $W(x) \neq 0$).

16. In general, the Wronskian of $f_1, \dots, f_n$ in $\mathscr{C}^{(n-1)}$ is the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

and $f_1, \dots, f_n$ are linearly independent, provided $W(x)$ is not identically zero. Repeat Exercises 10–14 using the Wronskian test.

17. Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a linearly independent set of vectors in a vector space V.
   (a) Is $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}\}$ linearly independent? Either prove that it is or give a counterexample to show that it is not.
   (b) Is $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{w}\}$ linearly independent? Either prove that it is or give a counterexample to show that it is not.

In Exercises 18–25, determine whether the set $\mathcal{B}$ is a basis for the vector space V.

18. $V = M_{22}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

19. $V = M_{22}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$

20. $V = M_{22}$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

21. $V = M_{22}$,
$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right\}$

22. $V = \mathscr{P}_2$, $\mathcal{B} = \{x, 1 + x, x - x^2\}$

23. $V = \mathscr{P}_2$, $\mathcal{B} = \{1 - x, 1 - x^2, x - x^2\}$

24. $V = \mathscr{P}_2$, $\mathcal{B} = \{1, 1 + 2x + 3x^2\}$

25. $V = \mathscr{P}_2$, $\mathcal{B} = \{1, 2 - x, 3 - x^2, x + 2x^2\}$

26. Find the coordinate vector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{E_{22}, E_{21}, E_{12}, E_{11}\}$ of M_{22}.

27. Find the coordinate vector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ of M_{22}.

28. Find the coordinate vector of $p(x) = 1 + 2x + 3x^2$ with respect to the basis $\mathcal{B} = \{1 + x, 1 - x, x^2\}$ of $\mathscr{P}_2$.

29. Find the coordinate vector of $p(x) = 2 - x + 3x^2$ with respect to the basis $\mathcal{B} = \{1, 1 + x, -1 + x^2\}$ of $\mathscr{P}_2$.

30. Let $\mathcal{B}$ be a set of vectors in a vector space V with the property that every vector in V can be written uniquely as a linear combination of the vectors in $\mathcal{B}$. Prove that $\mathcal{B}$ is a basis for V.

31. Let $\mathcal{B}$ be a basis for a vector space V, let $\mathbf{u}_1, \dots, \mathbf{u}_k$ be vectors in V, and let $c_1, \dots, c_k$ be scalars. Show that $[c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_k[\mathbf{u}_k]_{\mathcal{B}}$.

32. Finish the proof of Theorem 6.7 by showing that if $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_k]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$ then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent in V.

33. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a set of vectors in an n-dimensional vector space V and let $\mathcal{B}$ be a basis for V. Let $S = \{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_m]_{\mathcal{B}}\}$ be the set of coordinate vectors of $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ with respect to $\mathcal{B}$. Prove that $\text{span}(\mathbf{u}_1, \dots, \mathbf{u}_m) = V$ if and only if $\text{span}(S) = \mathbb{R}^n$.

In Exercises 34–39, find the dimension of the vector space V and give a basis for V.

34. $V = \{p(x) \text{ in } \mathscr{P}_2 : p(0) = 0\}$

35. $V = \{p(x) \text{ in } \mathscr{P}_2 : p(1) = 0\}$

36. $V = \{p(x) \text{ in } \mathscr{P}_2 : xp'(x) = p(x)\}$

37. $V = \{A \text{ in } M_{22} : A \text{ is upper triangular}\}$

38. $V = \{A \text{ in } M_{22} : A \text{ is skew-symmetric}\}$

39. $V = \{A \text{ in } M_{22} : AB = BA\}$, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

40. Find a formula for the dimension of the vector space of symmetric $n \times n$ matrices.

41. Find a formula for the dimension of the vector space of skew-symmetric $n \times n$ matrices.

42. Let U and W be subspaces of a finite-dimensional vector space V. Prove ***Grassmann's Identity:***

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

[*Hint:* The subspace $U + W$ is defined in Exercise 48 of Section 6.1. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $U \cap W$. Extend $\mathcal{B}$ to a basis $\mathcal{C}$ of U and a basis $\mathcal{D}$ of W. Prove that $\mathcal{C} \cup \mathcal{D}$ is a basis for $U + W$.]

43. Let U and V be finite-dimensional vector spaces.

(a) Find a formula for $\dim(U \times V)$ in terms of $\dim U$ and $\dim V$. (See Exercise 49 in Section 6.1.)

(b) If W is a subspace of V, show that $\dim \Delta = \dim W$, where $\Delta = \{(\mathbf{w}, \mathbf{w}) : \mathbf{w} \text{ is in } W\}$.

44. Prove that the vector space $\mathscr{P}$ is infinite-dimensional. [*Hint:* Suppose it has a finite basis. Show that there is some polynomial that is not a linear combination of this basis.]

45. Extend $\{1 + x, 1 + x + x^2\}$ to a basis for $\mathscr{P}_2$.

46. Extend $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ to a basis for M_{22}.

47. Extend $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ to a basis for M_{22}.

48. Extend $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ to a basis for the vector space of symmetric 2×2 matrices.

49. Find a basis for $\operatorname{span}(1, 1 + x, 2x)$ in $\mathscr{P}_1$.

50. Find a basis for $\operatorname{span}(1 - 2x, 2x - x^2, 1 - x^2, 1 + x^2)$ in $\mathscr{P}_2$.

51. Find a basis for $\operatorname{span}(1 - x, x - x^2, 1 - x^2, 1 - 2x + x^2)$ in $\mathscr{P}_2$.

52. Find a basis for $\operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right)$ in M_{22}.

53. Find a basis for $\operatorname{span}(\sin^2 x, \cos^2 x, \cos 2x)$ in $\mathscr{F}$.

54. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set in a vector space V. Show that if $\mathbf{v}$ is a vector in V that is not in $\operatorname{span}(S)$, then $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is still linearly independent.

55. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for a vector space V. Show that if $\mathbf{v}_n$ is in $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$, then $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ is still a spanning set for V.

56. Prove Theorem 6.10(f).

57. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $c_1, \ldots, c_n$ be nonzero scalars. Prove that $\{c_1\mathbf{v}_1, \ldots, c_n\mathbf{v}_n\}$ is also a basis for V.

58. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Prove that

$$\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \cdots + \mathbf{v}_n\}$$

is also a basis for V.

Let $a_0, a_1, \ldots, a_n$ be $n + 1$ distinct real numbers. Define polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ by

$$p_i(x) = \frac{(x - a_0)\cdots(x - a_{i-1})(x - a_{i+1})\cdots(x - a_n)}{(a_i - a_0)\cdots(a_i - a_{i-1})(a_i - a_{i+1})\cdots(a_i - a_n)}$$

*These are called the **Lagrange polynomials** associated with $a_0, a_1, \ldots, a_n$. [Joseph-Louis Lagrange (1736–1813) was born in Italy but spent most of his life in Germany and France. He made important contributions to such fields as number theory, algebra, astronomy, mechanics, and the calculus of variations. In 1773, Lagrange was the first to give the volume interpretation of a determinant (see Chapter 4).]*

59. **(a)** Compute the Lagrange polynomials associated with $a_0 = 1, a_1 = 2, a_2 = 3$.

(b) Show, in general, that

$$p_i(a_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

60. **(a)** Prove that the set $\mathcal{B} = \{p_0(x), p_1(x), \ldots, p_n(x)\}$ of Lagrange polynomials is linearly independent in $\mathscr{P}_n$. [*Hint:* Set $c_0p_0(x) + \cdots + c_np_n(x) = 0$ and use Exercise 59(b).]

(b) Deduce that $\mathcal{B}$ is a basis for $\mathscr{P}_n$.

61. If $q(x)$ is an arbitrary polynomial in $\mathscr{P}_n$, it follows from Exercise 60(b) that

$$q(x) = c_0p_0(x) + \cdots + c_np_n(x) \qquad (1)$$

for some scalars $c_0, \ldots, c_n$.

(a) Show that $c_i = q(a_i)$ for $i = 0, \ldots, n$, and deduce that $q(x) = q(a_0)p_0(x) + \cdots + q(a_n)p_n(x)$ is the unique representation of $q(x)$ with respect to the basis $\mathcal{B}$.

(b) Show that for any $n + 1$ points $(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n)$ with distinct first components, the function $q(x)$ defined by Equation (1) is the unique polynomial of degree at most n that passes through all of the points. This formula is known as the ***Lagrange interpolation formula.*** (Compare this formula with Problem 19 in Exploration: Geometric Applications of Determinants in Chapter 4.)

(c) Use the Lagrange interpolation formula to find the polynomial of degree at most 2 that passes through the points

(i) $(1, 6)$, $(2, -1)$, and $(3, -2)$

(ii) $(-1, 10)$, $(0, 5)$, and $(3, 2)$

62. Use the Lagrange interpolation formula to show that if a polynomial in $\mathscr{P}_n$ has $n + 1$ zeros, then it must be the zero polynomial.

63. Find a formula for the number of invertible matrices in $M_{nn}(\mathbb{Z}_p)$. [*Hint:* This is the same as determining the number of different bases for $\mathbb{Z}_p^n$. (Why?) Count the number of ways to construct a basis for $\mathbb{Z}_p^n$, one vector at a time.]

Exploration

Magic Squares

The engraving shown on page 461 is Albrecht Dürer's *Melancholia I* (1514). Among the many mathematical artifacts in this engraving is the chart of numbers that hangs on the wall in the upper right-hand corner. (It is enlarged in the detail shown.) Such an array of numbers is known as a *magic square*. We can think of it as a 4×4 matrix

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

Observe that the numbers in each row, in each column, and in both diagonals have the same sum: 34. Observe further that the entries are the integers $1, 2, \ldots, 16$. (Note that Dürer cleverly placed the 15 and 14 adjacent to each other in the last row, giving the date of the engraving.) These observations lead to the following definition.

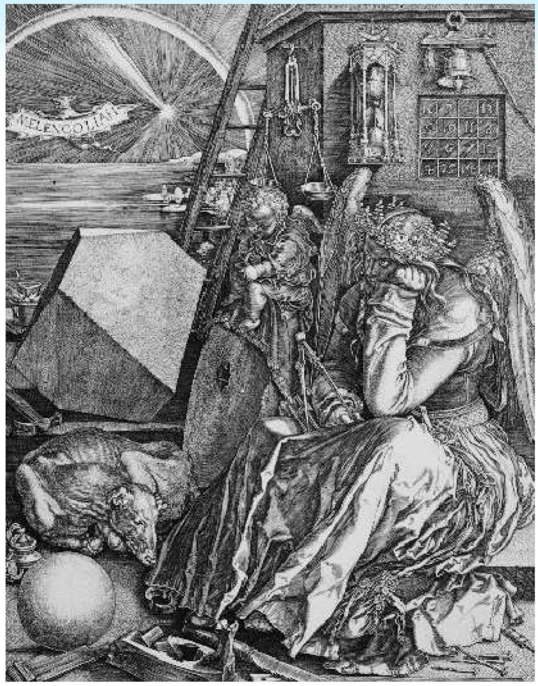
Definition An $n \times n$ matrix M is called a ***magic square*** if the sum of the entries is the same in each row, each column, and both diagonals. This common sum is called the ***weight*** of M, denoted $\text{wt}(M)$. If M is an $n \times n$ magic square that contains each of the entries $1, 2, \ldots, n^2$ exactly once, then M is called a ***classical magic square.***

1. If M is a classical $n \times n$ magic square, show that

$$\text{wt}(M) = \frac{n(n^2 + 1)}{2}$$

[*Hint:* Use Exercise 51 in Section 2.4.]

2. Find a classical 3×3 magic square. Find a different one. Are your two examples related in any way?



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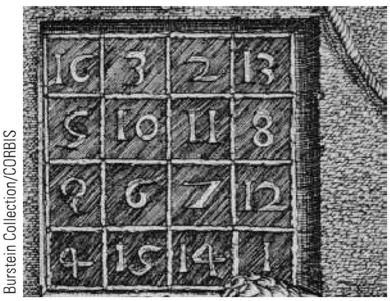

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3. Clearly, the 3×3 matrix with all entries equal to $\frac{1}{3}$ is a magic square with weight 1. Using your answer to Problem 2, find a 3×3 magic square with weight 1, *all of whose entries are different.* Describe a method for constructing a 3×3 magic square with distinct entries and weight w for any real number w.

Let Mag_n denote the set of all $n \times n$ magic squares, and let Mag_n^0 denote the set of all $n \times n$ magic squares of weight 0.

4. (a) Prove that Mag_3 is a subspace of M_{33}.
   (b) Prove that Mag_3^0 is a subspace of Mag_3.

5. Use Problems 3 and 4 to show that if M is a 3×3 magic square with weight w, then we can write M as

$$M = M_0 + kJ$$

where M_0 is a 3×3 magic square of weight 0, J is the 3×3 matrix consisting entirely of ones, and k is a scalar. What must k be? [*Hint:* Show that $M - kJ$ is in Mag_3^0 for an appropriate value of k.]

Let's try to find a way of describing *all* 3×3 magic squares. Let

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

be a magic square with weight 0. The conditions on the rows, columns, and diagonals give rise to a system of eight homogeneous linear equations in the variables $a, b, \ldots, i$.

6. Write out this system of equations and solve it. [*Note:* Using a CAS will facilitate the calculations.]

7. Find the dimension of Mag_3^0. *Hint:* By doing a substitution, if necessary, use your solution to Problem 6 to show that M can be written in the form

$$M = \begin{bmatrix} s & -s-t & t \\ -s+t & 0 & s-t \\ -t & s+t & -s \end{bmatrix}$$

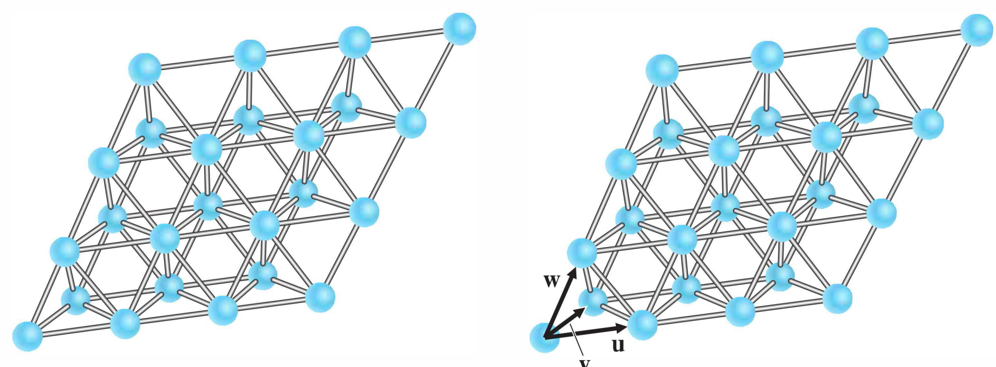
8. Find the dimension of Mag_3. [*Hint:* Combine the results of Problems 5 and 7.]

9. Can you find a direct way of showing that the (2, 2) entry of a 3×3 magic square with weight w must be $w/3$? [*Hint:* Add and subtract certain rows, columns, and diagonals to leave a multiple of the central entry.]

10. Let M be a 3×3 magic square of weight 0, obtained from a classical 3×3 magic square as in Problem 5. If M has the form given in Problem 7, write out an equation for the sum of the squares of the entries of M. Show that this is the equation of a circle in the variables s and t, and carefully plot it. Show that there are exactly eight points (s, t) on this circle with both s and t integers. Using Problem 8, show that these eight points give rise to eight classical 3×3 magic squares. How are these magic squares related to one another?

6.3 Change of Basis

In many applications, a problem described using one coordinate system may be solved more easily by switching to a new coordinate system. This switch is usually accomplished by performing a change of variables, a process that you have probably encountered in other mathematics courses. In linear algebra, a basis provides us with a coordinate system for a vector space, via the notion of coordinate vectors. Choosing the right basis will often greatly simplify a particular problem. For example, consider the molecular structure of zinc, shown in Figure 6.3(a). A scientist studying zinc might wish to measure the lengths of the bonds between the atoms, the angles between these bonds, and so on. Such an analysis will be greatly facilitated by introducing coordinates and making use of the tools of linear algebra. The standard basis and the associated standard xyz coordinate axes are not always the best choice. As Figure 6.3(b) shows, in this case $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is probably a better choice of basis for $\mathbb{R}^3$ than the standard basis, since these vectors align nicely with the bonds between the atoms of zinc.



(a) (b)

Figure 6.3

Change-of-Basis Matrices

Figure 6.4 shows two different coordinate systems for $\mathbb{R}^2$, each arising from a different basis. Figure 6.4(a) shows the coordinate system related to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, while Figure 6.4(b) arises from the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The same vector $\mathbf{x}$ is shown relative to each coordinate system. It is clear from the diagrams that the coordinate vectors of $\mathbf{x}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ are

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

respectively. It turns out that there is a direct connection between the two coordinate vectors. One way to find the relationship is to use $[\mathbf{x}]_{\mathcal{B}}$ to calculate

$$\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

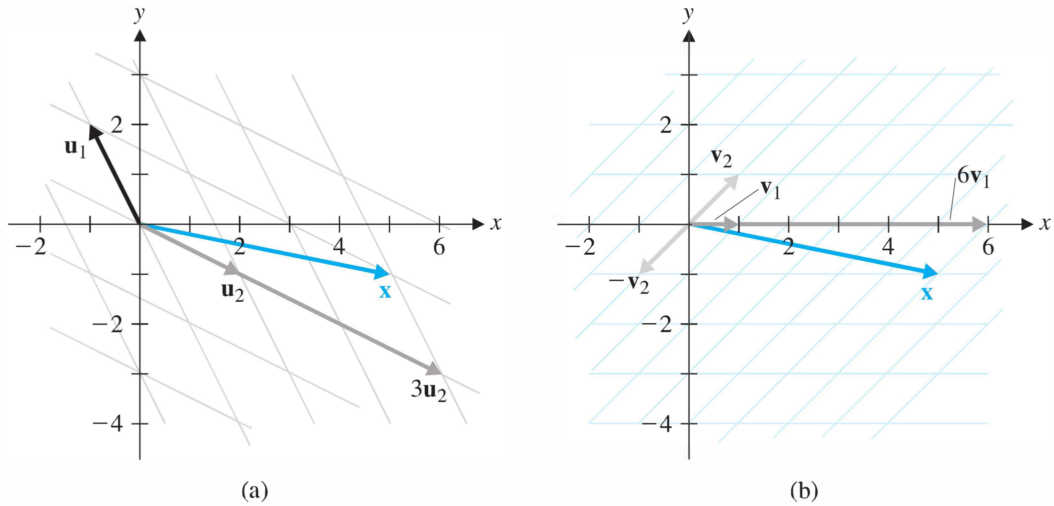


Figure 6.4

Then we can find $[\mathbf{x}]_{\mathcal{C}}$ by writing $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. However, there is a better way to proceed—one that will provide us with a general mechanism for such problems. We illustrate this approach in the next example.

Example 6.45 Using the bases $\mathcal{B}$ and $\mathcal{C}$ above, find $[\mathbf{x}]_{\mathcal{C}}$, given that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Solution Since $\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2$, writing $\mathbf{u}_1$ and $\mathbf{u}_2$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$ will give us the required coordinates of $\mathbf{x}$ with respect to $\mathcal{C}$. We find that

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -3\mathbf{v}_1 + 2\mathbf{v}_2$$

and

$$\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{v}_1 - \mathbf{v}_2$$

so

$$\begin{aligned} \mathbf{x} &= \mathbf{u}_1 + 3\mathbf{u}_2 \\ &= (-3\mathbf{v}_1 + 2\mathbf{v}_2) + 3(3\mathbf{v}_1 - \mathbf{v}_2) \\ &= 6\mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

This gives

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

in agreement with Figure 6.4(b).

This method may not look any easier than the one suggested prior to Example 6.45, but it has one big advantage: We can now find $[\mathbf{y}]_{\mathcal{C}}$ from $[\mathbf{y}]_{\mathcal{B}}$ for *any* vector $\mathbf{y}$ in $\mathbb{R}^2$

with very little additional work. Let's look at the calculations in Example 6.45 from a different point of view. From $\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2$, we have

$$[\mathbf{x}]_{\mathcal{C}} = [\mathbf{u}_1 + 3\mathbf{u}_2]_{\mathcal{C}} = [\mathbf{u}_1]_{\mathcal{C}} + 3[\mathbf{u}_2]_{\mathcal{C}}$$

by Theorem 6.6. Thus,

$$\begin{aligned}[\mathbf{x}]_{\mathcal{C}} &= \big[[\mathbf{u}_1]_{\mathcal{C}}\,[\mathbf{u}_2]_{\mathcal{C}}\big]\begin{bmatrix}1\\3\end{bmatrix}\\ &= \begin{bmatrix}-3 & 3\\ 2 & -1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\\ &= P[\mathbf{x}]_{\mathcal{B}}\end{aligned}$$

where P is the matrix whose columns are $[\mathbf{u}_1]_{\mathcal{C}}$ and $[\mathbf{u}_2]_{\mathcal{C}}$. This procedure generalizes very nicely.

Definition Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \ldots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ with respect to $\mathcal{C}$ is denoted by $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and is called the ***change-of-basis matrix*** from $\mathcal{B}$ to $\mathcal{C}$. That is,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[[\mathbf{u}_1]_{\mathcal{C}}\,[\mathbf{u}_2]_{\mathcal{C}}\ \cdots\ [\mathbf{u}_n]_{\mathcal{C}}\big]$$

Think of $\mathcal{B}$ as the "old" basis and $\mathcal{C}$ as the "new" basis. Then the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are just the coordinate vectors obtained by writing the old basis vectors in terms of the new ones. Theorem 6.12 shows that Example 6.45 is a special case of a general result.

Theorem 6.12 Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

a. $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
b. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
c. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

Proof (a) Let $\mathbf{x}$ be in V and let

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}$$

That is, $\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. Then

$$\begin{aligned}[\mathbf{x}]_{\mathcal{C}} &= [c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n]_{\mathcal{C}}\\ &= c_1[\mathbf{u}_1]_{\mathcal{C}} + \cdots + c_n[\mathbf{u}_n]_{\mathcal{C}}\\ &= \big[[\mathbf{u}_1]_{\mathcal{C}}\ \cdots\ [\mathbf{u}_n]_{\mathcal{C}}\big]\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\\ &= P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}\end{aligned}$$

(b) Suppose that P is an $n \times n$ matrix with the property that $P[\mathbf{x}]_\mathcal{B} = [\mathbf{x}]_\mathcal{C}$ for all $\mathbf{x}$ in V. Taking $\mathbf{x} = \mathbf{u}_i$, the ith basis vector in $\mathcal{B}$, we see that $[\mathbf{x}]_\mathcal{B} = [\mathbf{u}_i]_\mathcal{B} = \mathbf{e}_i$, so the ith column of P is

$$\mathbf{p}_i = P\mathbf{e}_i = P[\mathbf{u}_i]_\mathcal{B} = [\mathbf{u}_i]_\mathcal{C}$$

which is the ith column of $P_{\mathcal{C}\leftarrow\mathcal{B}}$, by definition. It follows that $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$.

(c) Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly independent in V, the set $\{[\mathbf{u}_1]_\mathcal{C}, \ldots, [\mathbf{u}_n]_\mathcal{C}\}$ is linearly independent in $\mathbb{R}^n$, by Theorem 6.7. Hence, $P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{u}_1]_\mathcal{C} \;\cdots\; [\mathbf{u}]_\mathcal{C}$ is invertible, by the Fundamental Theorem.

For all $\mathbf{x}$ in V, we have $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_\mathcal{B} = [\mathbf{x}]_\mathcal{C}$. Solving for $[\mathbf{x}]_\mathcal{B}$, we find that

$$[\mathbf{x}]_\mathcal{B} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_\mathcal{C}$$

for all $\mathbf{x}$ in V. Therefore, $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ is a matrix that changes bases from $\mathcal{C}$ to $\mathcal{B}$. Thus, by the uniqueness property (b), we must have $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}$.

Remarks

- You may find it helpful to think of change of basis as a transformation (indeed, it is a linear transformation) from $\mathbb{R}^n$ to itself that simply switches from one coordinate system to another. The transformation corresponding to $P_{\mathcal{C}\leftarrow\mathcal{B}}$ accepts $[\mathbf{x}]_\mathcal{B}$ as input and returns $[\mathbf{x}]_\mathcal{C}$ as output; $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}$ does just the opposite. Figure 6.5 gives a schematic representation of the process.

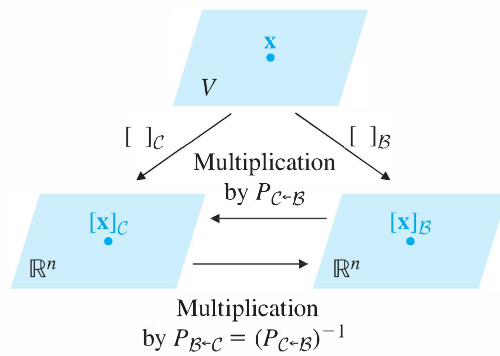


Figure 6.5
Change of basis

- The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the coordinate vectors of one basis with respect to the other basis. To remember which basis is which, think of the notation $\mathcal{C} \leftarrow \mathcal{B}$ as saying "$\mathcal{B}$ in terms of $\mathcal{C}$." It is also helpful to remember that $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_\mathcal{B}$ is a linear combination of the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$. But since the result of this combination is $[\mathbf{x}]_\mathcal{C}$, the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ must themselves be coordinate vectors with respect to $\mathcal{C}$.

Example 6.46

Find the change-of-basis matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ for the bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathcal{P}_2$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ with respect to $\mathcal{C}$.

Solution Changing *to* a standard basis is easy, so we find $P_{\mathcal{B}\leftarrow\mathcal{C}}$ first. Observe that the coordinate vectors for $\mathcal{C}$ in terms of $\mathcal{B}$ are

$$[1 + x]_\mathcal{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [x + x^2]_\mathcal{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [1 + x^2]_\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(Look back at the Remark following Example 6.26.) It follows that

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

To find $P_{\mathcal{C}\leftarrow\mathcal{B}}$, we could express each vector in $\mathcal{B}$ as a linear combination of the vectors in $\mathcal{C}$ (do this), but it is much easier to use the fact that $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$, by Theorem 6.12(c). We find that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It now follows that

$$\begin{aligned} [p(x)]_{\mathcal{C}} &= P_{\mathcal{C}\leftarrow\mathcal{B}}[p(x)]_{\mathcal{B}} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

which agrees with Example 6.37.

Remark If we do not need $P_{\mathcal{C}\leftarrow\mathcal{B}}$ explicitly, we can find $[p(x)]_{\mathcal{C}}$ from $[p(x)]_{\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ using Gaussian elimination. Row reduction produces

$$[P_{\mathcal{B}\leftarrow\mathcal{C}} \,|\, [p(x)]_{\mathcal{B}}] \longrightarrow [I \,|\, (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}[p(x)]_{\mathcal{B}}] = [I \,|\, P_{\mathcal{C}\leftarrow\mathcal{B}}[p(x)]_{\mathcal{B}}] = [I \,|\, [p(x)]_{\mathcal{C}}]$$

(See the next section on using Gauss-Jordan elimination.)

It is worth repeating the observation in Example 6.46: Changing *to* a standard basis is easy. If $\mathcal{E}$ is the standard basis for a vector space V and $\mathcal{B}$ is any other basis, then the columns of $P_{\mathcal{E}\leftarrow\mathcal{B}}$ are the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{E}$, and these are usually "visible." We make use of this observation again in the next example.

Example 6.47

In M_{22}, let $\mathcal{B}$ be the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let $\mathcal{C}$ be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and verify that $[X]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution 1 To solve this problem directly, we must find the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{C}$. This involves solving four linear combination problems of the form $X = aA + bB + cC + dD$, where X is in $\mathcal{B}$ and we must find a, b, c, and d. However, here we are lucky, since we can find the required coefficients by inspection.

Clearly, $E_{11} = A$, $E_{21} = -B + C$, $E_{12} = -A + B$, and $E_{22} = -C + D$. Thus,

$$[E_{11}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [E_{21}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad [E_{12}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [E_{22}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[E_{11}]_{\mathcal{C}} \;\; [E_{21}]_{\mathcal{C}} \;\; [E_{12}]_{\mathcal{C}} \;\; [E_{22}]_{\mathcal{C}}] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$[X]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

and

$$P_{\mathcal{C}\leftarrow\mathcal{B}}[X]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

This is the coordinate vector with respect to $\mathcal{C}$ of the matrix

$$\begin{aligned} -A - B - C + 4D &= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = X \end{aligned}$$

as it should be.

Solution 2 We can compute $P_{\mathcal{C}\leftarrow\mathcal{B}}$ in a different way, as follows. As you will be asked to prove in Exercise 21, if $\mathcal{E}$ is another basis for M_{22}, then $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}}P_{\mathcal{E}\leftarrow\mathcal{B}} = (P_{\mathcal{E}\leftarrow\mathcal{C}})^{-1}P_{\mathcal{E}\leftarrow\mathcal{B}}$. If $\mathcal{E}$ is the standard basis, then $P_{\mathcal{E}\leftarrow\mathcal{B}}$ and $P_{\mathcal{E}\leftarrow\mathcal{C}}$ can be found by inspection. We have

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Do you see why?) Therefore,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{E}\leftarrow\mathcal{C}})^{-1}P_{\mathcal{E}\leftarrow\mathcal{B}}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which agrees with the first solution.

Remark The second method has the advantage of not requiring the computation of any linear combinations. It has the disadvantage of requiring that we find a matrix inverse. However, using a CAS will facilitate finding a matrix inverse, so in general the second method is preferable to the first. For certain problems, though, the first method may be just as easy to use. In any event, we are about to describe yet a third approach, which you may find best of all.

The Gauss-Jordan Method for Computing a Change-of-Basis Matrix

Finding the change-of-basis matrix to a standard basis is easy and can be done by inspection. Finding the change-of-basis matrix from a standard basis is almost as easy, but requires the calculation of a matrix inverse, as in Example 6.46. If we do it by hand, then (except for the 2×2 case) we will usually find the necessary inverse by Gauss-Jordan elimination. We now look at a modification of the Gauss-Jordan method that can be used to find the change-of-basis matrix between two nonstandard bases, as in Example 6.47.

Suppose $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are bases for a vector space V and $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. The ith column of P is

$$[\mathbf{u}_i]_{\mathcal{C}} = \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{bmatrix}$$

so $\mathbf{u}_i = p_{1i}\mathbf{v}_1 + \cdots + p_{ni}\mathbf{v}_n$. If $\mathcal{E}$ is any basis for V, then

$$[\mathbf{u}_i]_{\mathcal{E}} = [p_{1i}\mathbf{v}_1 + \cdots + p_{ni}\mathbf{v}_n]_{\mathcal{E}} = p_{1i}[\mathbf{v}_1]_{\mathcal{E}} + \cdots + p_{ni}[\mathbf{v}_n]_{\mathcal{E}}$$

This can be rewritten in matrix form as

$$\left[\, [\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}} \,\right] \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{bmatrix} = [\mathbf{u}_i]_{\mathcal{E}}$$

which we can solve by applying Gauss-Jordan elimination to the augmented matrix

$$[[\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}} \mid [\mathbf{u}_i]_{\mathcal{E}}]$$

There are n such systems of equations to be solved, one for each column of $P_{\mathcal{C}\leftarrow\mathcal{B}}$, but *the coefficient matrix* $[[\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}}]$ *is the same in each case*. Hence, we can solve all the systems simultaneously by row reducing the $n \times 2n$ augmented matrix

$$[[\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}} \mid [\mathbf{u}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{u}_n]_{\mathcal{E}}] = [C \mid B]$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, so is $\{[\mathbf{v}_1]_{\mathcal{E}}, \ldots, [\mathbf{v}_n]_{\mathcal{E}}\}$, by Theorem 6.7. Therefore, the matrix C whose columns are $[\mathbf{v}_1]_{\mathcal{E}}, \ldots, [\mathbf{v}_n]_{\mathcal{E}}$ has the $n \times n$ identity matrix I for its reduced row echelon form, by the Fundamental Theorem. It follows that Gauss-Jordan elimination will necessarily produce

$$[C \mid B] \to [I \mid P]$$

where $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$.

We have proved the following theorem.

Theorem 6.13

Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. Let $B = [[\mathbf{u}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{u}_n]_{\mathcal{E}}]$ and $C = [[\mathbf{v}_1]_{\mathcal{E}} \ldots [\mathbf{v}_n]_{\mathcal{E}}]$, where $\mathcal{E}$ is any basis for V. Then row reduction applied to the $n \times 2n$ augmented matrix $[C \mid B]$ produces

$$[C \mid B] \to [I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}]$$

If $\mathcal{E}$ is a standard basis, this method is particularly easy to use, since in that case $B = P_{\mathcal{E}\leftarrow\mathcal{B}}$ and $C = P_{\mathcal{E}\leftarrow\mathcal{C}}$. We illustrate this method by reworking the problem in Example 6.47.

Example 6.48

Rework Example 6.47 using the Gauss-Jordan method.

Solution Taking $\mathcal{E}$ to be the standard basis for M_{22}, we see that

$$B = P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row reduction produces

$$[C \mid B] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

(Verify this row reduction.) It follows that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

as we found before.

Exercises 6.3

In Exercises 1–4:
(a) Find the coordinate vectors $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ of $\mathbf{x}$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, respectively.
(b) Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.
(c) Use your answer to part (b) to compute $[\mathbf{x}]_{\mathcal{C}}$, and compare your answer with the one found in part (a).
(d) Find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{B}$.
(e) Use your answers to parts (c) and (d) to compute $[\mathbf{x}]_{\mathcal{B}}$, and compare your answer with the one found in part (a).

1. $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ in $\mathbb{R}^2$

2. $\mathbf{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$ in $\mathbb{R}^2$

3. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^3$

4. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^3$

In Exercises 5–8, follow the instructions for Exercises 1–4 using $p(x)$ instead of $\mathbf{x}$.

5. $p(x) = 2 - x, \mathcal{B} = \{1, x\}, \mathcal{C} = \{x, 1 + x\}$ in $\mathcal{P}_1$

6. $p(x) = 1 + 3x, \mathcal{B} = \{1 + x, 1 - x\},$
$\mathcal{C} = \{2x, 4\}$ in $\mathcal{P}_1$

7. $p(x) = 1 + x^2, \mathcal{B} = \{1 + x + x^2, x + x^2, x^2\},$
$\mathcal{C} = \{1, x, x^2\}$ in $\mathcal{P}_2$

8. $p(x) = 4 - 2x - x^2, \mathcal{B} = \{x, 1 + x^2, x + x^2\},$
$\mathcal{C} = \{1, 1 + x, x^2\}$ in $\mathcal{P}_2$

In Exercises 9 and 10, follow the instructions for Exercises 1–4 using A instead of $\mathbf{x}$.

9. $A = \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix}$, $\mathcal{B} =$ the standard basis,
$\mathcal{C} = \left\{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ in M_{22}

10. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$
$\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ in M_{22}

In Exercises 11 and 12, follow the instructions for Exercises 1–4 using $f(x)$ instead of $\mathbf{x}$.

11. $f(x) = 2\sin x - 3\cos x, \mathcal{B} = \{\sin x + \cos x, \cos x\},$
$\mathcal{C} = \{\sin x + \cos x, \sin x - \cos x\}$ in $\operatorname{span}(\sin x, \cos x)$

12. $f(x) = \sin x, \mathcal{B} = \{\sin x + \cos x, \cos x\},$
$\mathcal{C} = \{\cos x - \sin x, \sin x + \cos x\}$ in $\operatorname{span}(\sin x, \cos x)$

13. Rotate the xy-axes in the plane counterclockwise through an angle $\theta = 60°$ to obtain new $x'y'$-axes. Use the methods of this section to find (a) the $x'y'$-coordinates of the point whose xy-coordinates are $(3, 2)$ and (b) the xy-coordinates of the point whose $x'y'$-coordinates are $(4, -4)$.

14. Repeat Exercise 13 with $\theta = 135°$.

15. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathbb{R}^2$. If $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

find $\mathcal{B}$.

16. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathcal{P}_2$. If $\mathcal{B} = \{x, 1 + x, 1 - x + x^2\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

find $\mathcal{C}$.

In calculus, you learn that a ***Taylor polynomial of degree n about a*** *is a polynomial of the form*

$$p(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n$$

where $a_n \neq 0$. *In other words, it is a polynomial that has been expanded in terms of powers of* $x - a$ *instead of powers of* x. *Taylor polynomials are very useful for approximating functions that are "well behaved" near* $x = a$.

The set $\mathcal{B} = \{1, x - a, (x - a)^2, \ldots, (x - a)^n\}$ *is a basis for* $\mathcal{P}_n$ *for any real number* a. *(Do you see a quick way to show this? Try using Theorem 6.7.) This fact allows us to use the techniques of this section to rewrite a polynomial as a Taylor polynomial about a given* a.

17. Express $p(x) = 1 + 2x - 5x^2$ as a Taylor polynomial about $a = 1$.

18. Express $p(x) = 1 + 2x - 5x^2$ as a Taylor polynomial about $a = -2$.

19. Express $p(x) = x^3$ as a Taylor polynomial about $a = -1$.

20. Express $p(x) = x^3$ as a Taylor polynomial about $a = \frac{1}{2}$.

21. Let $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be bases for a finite-dimensional vector space V. Prove that

$$P_{\mathcal{D}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{B}}$$

22. Let V be an n-dimensional vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let P be an invertible $n \times n$ matrix and set

$$\mathbf{u}_i = p_{1i}\mathbf{v}_1 + \cdots + p_{ni}\mathbf{v}_n$$

for $i = 1, \ldots, n$. Prove that $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for V and show that $P = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

6.4 Linear Transformations

We encountered linear transformations in Section 3.6 in the context of matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section, we extend this concept to linear transformations between arbitrary vector spaces.

Definition A ***linear transformation*** from a vector space V to a vector space W is a mapping $T : V \to W$ such that, for all $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars c,

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

It is straightforward to show that this definition is equivalent to the requirement that T preserve all linear combinations. That is,

$T : V \to W$ is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$$

for all $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V and scalars $c_1, \ldots, c_k$.

Example 6.49

Every matrix transformation is a linear transformation. That is, if A is an $m \times n$ matrix, then the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n$$

is a linear transformation. This is a restatement of Theorem 3.30.

Example 6.50

Define $T : M_{nn} \to M_{nn}$ by $T(A) = A^T$. Show that T is a linear transformation.

Solution We check that, for A and B in M_{nn} and scalars c,

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B)$$

and

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T is a linear transformation.

Example 6.51

Let D be the ***differential operator*** $D : \mathscr{D} \to \mathscr{F}$ defined by $D(f) = f'$. Show that D is a linear transformation.

Solution Let f and g be differentiable functions and let c be a scalar. Then, from calculus, we know that

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$$

and

$$D(cf) = (cf)' = cf' = cD(f)$$

Hence, D is a linear transformation.

In calculus, you learn that every continuous function on $[a, b]$ is integrable. The next example shows that integration is a linear transformation.

Example 6.52

Define $S : \mathscr{C}[a, b] \to \mathbb{R}$ by $S(f) = \int_a^b f(x)\,dx$. Show that S is a linear transformation.

Solution Let f and g be in $\mathscr{C}[a, b]$. Then

$$\begin{aligned} S(f + g) &= \int_a^b (f + g)(x)\,dx \\ &= \int_a^b (f(x) + g(x))\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\ &= S(f) + S(g) \end{aligned}$$

and

$$\begin{aligned} S(cf) &= \int_a^b (cf)(x)\,dx \\ &= \int_a^b cf(x)\,dx \\ &= c\int_a^b f(x)\,dx \\ &= cS(f) \end{aligned}$$

It follows that S is linear.

Example 6.53

Show that none of the following transformations is linear:

(a) $T: M_{22} \to \mathbb{R}$ defined by $T(A) = \det A$

(b) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2^x$

(c) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$

Solution In each case, we give a specific counterexample to show that one of the properties of a linear transformation fails to hold.

(a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so

$$T(A + B) = \det(A + B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

But

$$T(A) + T(B) = \det A + \det B = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 + 0 = 0$$

so $T(A + B) \neq T(A) + T(B)$ and T is not linear.

(b) Let $x = 1$ and $y = 2$. Then

$$T(x + y) = T(3) = 2^3 = 8 \neq 6 = 2^1 + 2^2 = T(x) + T(y)$$

so T is not linear.

(c) Let $x = 1$ and $y = 2$. Then

$$T(x + y) = T(3) = 3 + 1 = 4 \neq 5 = (1 + 1) + (2 + 1) = T(x) + T(y)$$

Therefore, T is not linear.

Remark Example 6.53(c) shows that you need to be careful when you encounter the word "linear." As a *function*, $T(x) = x + 1$ is linear, since its graph is a straight line. However, it is not a *linear transformation* from the vector space $\mathbb{R}$ to itself, since it fails to satisfy the definition. (Which linear functions from $\mathbb{R}$ to $\mathbb{R}$ will also be linear transformations?)

There are two special linear transformations that deserve to be singled out.

Example 6.54

(a) For any vector spaces V and W, the transformation $T_0 : V \to W$ that maps every vector in V to the zero vector in W is called the ***zero transformation.*** That is,

$$T_0(\mathbf{v}) = \mathbf{0} \quad \text{for all } \mathbf{v} \text{ in } V$$

(b) For any vector space V, the transformation $I : V \to V$ that maps every vector in V to itself is called the ***identity transformation.*** That is,

$$I(\mathbf{v}) = \mathbf{v} \quad \text{for all } \mathbf{v} \text{ in } V$$

(If it is important to identify the vector space V, we may write I_V for clarity.) The proofs that the zero and identity transformations are linear are left as easy exercises.

Properties of Linear Transformations

In Chapter 3, all linear transformations were matrix transformations, and their properties were directly related to properties of the matrices involved. The following theorem is easy to prove for matrix transformations. (Do it!) The full proof for linear transformations in general takes a bit more care, but it is still straightforward.

Theorem 6.14 Let $T : V \to W$ be a linear transformation. Then:

a. $T(\mathbf{0}) = \mathbf{0}$
b. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.
c. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.

Proof We prove properties (a) and (c) and leave the proof of property (b) for Exercise 21.

(a) Let $\mathbf{v}$ be any vector in V. Then $T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$, as required. (Can you give a reason for each step?)

(c) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u} + (-1)\mathbf{v}) = T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

Remark Property (a) can be useful in showing that certain transformations are *not* linear. As an illustration, consider Example 6.53(b). If $T(x) = 2^x$, then $T(0) = 2^0 = 1 \neq 0$, so T is not linear, by Theorem 6.14(a). Be warned, however, that there are lots of transformations that *do* map the zero vector to the zero vector but that are still *not* linear. Example 6.53(a) is a case in point: The zero vector is the 2×2 zero matrix O, so $T(O) = \det O = 0$, but we have seen that $T(A) = \det A$ is not linear.

The most important property of a linear transformation $T : V \to W$ is that T is completely determined by its effect on a basis for V. The next example shows what this means.

Example 6.55 Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathscr{P}_2$ such that

$$T\begin{bmatrix}1\\1\end{bmatrix} = 2 - 3x + x^2 \quad \text{and} \quad T\begin{bmatrix}2\\3\end{bmatrix} = 1 - x^2$$

Find $T\begin{bmatrix}-1\\2\end{bmatrix}$ and $T\begin{bmatrix}a\\b\end{bmatrix}$.

Solution Since $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ (why?), every vector in $\mathbb{R}^2$ is in span$(\mathcal{B})$. Solving

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

we find that $c_1 = -7$ and $c_2 = 3$. Therefore,

$$\begin{aligned} T\begin{bmatrix} -1 \\ 2 \end{bmatrix} &= T\left(-7\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) \\ &= -7T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3T\begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= -7(2 - 3x + x^2) + 3(1 - x^2) \\ &= -11 + 21x - 10x^2 \end{aligned}$$

Similarly, we discover that

$$\begin{bmatrix} a \\ b \end{bmatrix} = (3a - 2b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so

$$\begin{aligned} T\begin{bmatrix} a \\ b \end{bmatrix} &= T\left((3a - 2b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a)\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) \\ &= (3a - 2b)T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a)T\begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= (3a - 2b)(2 - 3x + x^2) + (b - a)(1 - x^2) \\ &= (5a - 3b) + (-9a + 6b)x + (4a - 3b)x^2 \end{aligned}$$

(Note that by setting $a = -1$ and $b = 2$, we recover the solution $T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -11 + 21x - 10x^2$.)

The proof of the general theorem is quite straightforward.

Theorem 6.15 Let $T : V \to W$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for V. Then $T(\mathcal{B}) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans the range of T.

Proof The range of T is the set of all vectors in W that are of the form $T(\mathbf{v})$, where $\mathbf{v}$ is in V. Let $T(\mathbf{v})$ be in the range of T. Since $\mathcal{B}$ spans V, there are scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

Applying T and using the fact that it is a linear transformation, we see that

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n)$$

In other words, $T(\mathbf{v})$ is in $\text{span}(T(\mathcal{B}))$, as required.

Theorem 6.15 applies, in particular, when $\mathcal{B}$ is a basis for V. You might guess that, in this case, $T(\mathcal{B})$ would then be a basis for the range of T. Unfortunately, this is not always the case. We will address this issue in Section 6.5.

Composition of Linear Transformations

In Section 3.6, we defined the composition of matrix transformations. The definition extends to general linear transformations in an obvious way.

$S \circ T$ is read "S of T."

Definition If $T : U \to V$ and $S : V \to W$ are linear transformations, then the ***composition of S with T*** is the mapping $S \circ T$, defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

where $\mathbf{u}$ is in U.

Observe that $S \circ T$ is a mapping from U to W (see Figure 6.6). Notice also that for the definition to make sense, the range of T must be contained in the domain of S.

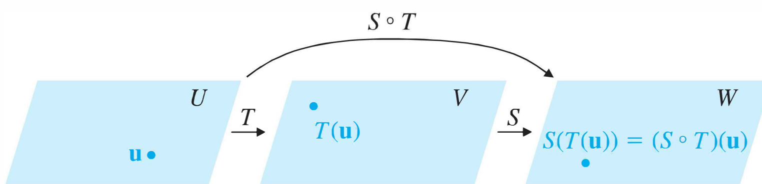


Figure 6.6
Composition of linear transformations

Example 6.56

Let $T : \mathbb{R}^2 \to \mathscr{P}_1$ and $S : \mathscr{P}_1 \to \mathscr{P}_2$ be the linear transformations defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x \quad \text{and} \quad S(p(x)) = xp(x)$$

Find $(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}$.

Solution We compute

$$(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix} = S\left(T\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = S(3 + (3-2)x) = S(3+x) = x(3+x)$$
$$= 3x + x^2$$

and

$$(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = S\left(T\begin{bmatrix} a \\ b \end{bmatrix}\right) = S(a + (a+b)x) = x(a + (a+b)x)$$
$$= ax + (a+b)x^2$$

Chapter 3 showed that the composition of two matrix transformations was another matrix transformation. In general, we have the following theorem.

Theorem 6.16 If $T : U \to V$ and $S : V \to W$ are linear transformations, then $S \circ T : U \to W$ is a linear transformation.

Proof Let $\mathbf{u}$ and $\mathbf{v}$ be in U and let c be a scalar. Then

$$\begin{aligned}(S \circ T)(\mathbf{u} + \mathbf{v}) &= S(T(\mathbf{u} + \mathbf{v})) \\ &= S(T(\mathbf{u}) + T(\mathbf{v})) && \text{since } T \text{ is linear} \\ &= S(T(\mathbf{u})) + S(T(\mathbf{v})) && \text{since } S \text{ is linear} \\ &= (S \circ T)(\mathbf{u}) + (S \circ T)(\mathbf{v})\end{aligned}$$

and

$$\begin{aligned}(S \circ T)(c\mathbf{u}) &= S(T(c\mathbf{u})) \\ &= S(cT(\mathbf{u})) && \text{since } T \text{ is linear} \\ &= cS(T(\mathbf{u})) && \text{since } S \text{ is linear} \\ &= c(S \circ T)(\mathbf{u})\end{aligned}$$

Therefore, $S \circ T$ is a linear transformation.

The algebraic properties of linear transformations mirror those of matrix transformations, which, in turn, are related to the algebraic properties of matrices. For example, composition of linear transformations is associative. That is, if R, S, and T are linear transformations, then

$$R \circ (S \circ T) = (R \circ S) \circ T$$

provided these compositions make sense. The proof of this property is identical to that given in Section 3.6.

The next example gives another useful (but not surprising) property of linear transformations.

Example 6.57

Let $S: U \to V$ and $T: V \to W$ be linear transformations and let $I: V \to V$ be the identity transformation. Then for every $\mathbf{v}$ in V, we have

$$(T \circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$

Since $T \circ I$ and T have the same value at every $\mathbf{v}$ in their domain, it follows that $T \circ I = T$. Similarly, $I \circ S = S$.

Remark The method of Example 6.57 is worth noting. Suppose we want to show that two linear transformations T_1 and T_2 (both from V to W) are equal. It suffices to show that $T_1(\mathbf{v}) = T_2(\mathbf{v})$ for every $\mathbf{v}$ in V.

Further properties of linear transformations are explored in the exercises.

Inverses of Linear Transformations

Definition A linear transformation $T: V \to W$ is ***invertible*** if there is a linear transformation $T': W \to V$ such that

$$T' \circ T = I_V \quad \text{and} \quad T \circ T' = I_W$$

In this case, T' is called an ***inverse*** for T.

Remarks

- The domain V and codomain W of T do not have to be the same, as they do in the case of invertible matrix transformations. However, we will see in the next section that V and W must be very closely related.
- The requirement that T' be linear could have been omitted from this definition. For, as we will see in Theorem 6.24, if T' is *any* mapping from W to V such that $T' \circ T = I_V$ and $T \circ T' = I_W$, then T' is forced to be linear as well.
- If T' is an inverse for T, then the definition implies that T is an inverse for T'. Hence, T' is invertible too.

Example 6.58

Verify that the mappings $T : \mathbb{R}^2 \to \mathscr{P}_1$ and $T' : \mathscr{P}_1 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x \quad \text{and} \quad T'(c + dx) = \begin{bmatrix} c \\ d - c \end{bmatrix}$$

are inverses.

Solution We compute

$$(T' \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = T'\left(T\begin{bmatrix} a \\ b \end{bmatrix}\right) = T'(a + (a + b)x) = \begin{bmatrix} a \\ (a + b) - a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$(T \circ T')(c + dx) = T(T'(c + dx)) = T\begin{bmatrix} c \\ d - c \end{bmatrix} = c + (c + (d - c))x = c + dx$$

Hence, $T' \circ T = I_{\mathbb{R}^2}$ and $T \circ T' = I_{\mathscr{P}_1}$. Therefore, T and T' are inverses of each other.

As was the case for invertible matrices, inverses of linear transformations are unique if they exist. The following theorem is the analogue of Theorem 3.6.

Theorem 6.17 If T is an invertible linear transformation, then its inverse is unique.

Proof The proof is the same as that of Theorem 3.6, with products of matrices replaced by compositions of linear transformations. (You are asked to complete this proof in Exercise 31.)

Thanks to Theorem 6.17, if T is invertible, we can refer to *the* inverse of T. It will be denoted by T^{-1} (pronounced "T inverse"). In the next two sections, we will address the issue of determining when a given linear transformation is invertible and finding its inverse when it exists.

Exercises 6.4

In Exercises 1–12, determine whether T is a linear transformation.

1. $T : M_{22} \to M_{22}$ defined by
$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix}$$

2. $T : M_{22} \to M_{22}$ defined by
$$T\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & w-z \\ x-y & 1 \end{bmatrix}$$

3. $T : M_{nn} \to M_{nn}$ defined by $T(A) = AB$, where B is a fixed $n \times n$ matrix

4. $T : M_{nn} \to M_{nn}$ defined by $T(A) = AB - BA$, where B is a fixed $n \times n$ matrix

5. $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = \operatorname{tr}(A)$

6. $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = a_{11}a_{22}\cdots a_{nn}$

7. $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = \operatorname{rank}(A)$

8. $T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(a + bx + cx^2) = (a + 1) + (b + 1)x + (c + 1)x^2$

9. $T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(a + bx + cx^2) = a + b(x + 1) + b(x + 1)^2$

10. $T : \mathscr{F} \to \mathscr{F}$ defined by $T(f) = f(x^2)$

11. $T : \mathscr{F} \to \mathscr{F}$ defined by $T(f) = (f(x))^2$

12. $T : \mathscr{F} \to \mathbb{R}$ defined by $T(f) = f(c)$, where c is a fixed scalar

13. Show that the transformations S and T in Example 6.56 are both linear.

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation for which
$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad\text{and}\quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$
Find $T\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $T\begin{bmatrix} a \\ b \end{bmatrix}$.

15. Let $T : \mathbb{R}^2 \to \mathscr{P}_2$ be a linear transformation for which
$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 - 2x \quad\text{and}\quad T\begin{bmatrix} 3 \\ -1 \end{bmatrix} = x + 2x^2$$
Find $T\begin{bmatrix} -7 \\ 9 \end{bmatrix}$ and $T\begin{bmatrix} a \\ b \end{bmatrix}$.

16. Let $T : \mathscr{P}_2 \to \mathscr{P}_2$ be a linear transformation for which
$$T(1) = 3 - 2x,\ T(x) = 4x - x^2, \text{ and } T(x^2) = 2 + 2x^2$$
Find $T(6 + x - 4x^2)$ and $T(a + bx + cx^2)$.

17. Let $T : \mathscr{P}_2 \to \mathscr{P}_2$ be a linear transformation for which
$$T(1 + x) = 1 + x^2, \quad T(x + x^2) = x - x^2,$$
$$T(1 + x^2) = 1 + x + x^2$$
Find $T(4 - x + 3x^2)$ and $T(a + bx + cx^2)$.

18. Let $T : M_{22} \to \mathbb{R}$ be a linear transformation for which
$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1, \quad T\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 2,$$
$$T\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = 3, \quad T\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 4$$
Find $T\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $T\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

19. Let $T : M_{22} \to \mathbb{R}$ be a linear transformation. Show that there are scalars a, b, c, and d such that
$$T\begin{bmatrix} w & x \\ y & z \end{bmatrix} = aw + bx + cy + dz$$
for all $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ in M_{22}.

20. Show that there is no linear transformation $T : \mathbb{R}^3 \to \mathscr{P}_2$ such that
$$T\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 1 + x, \quad T\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = 2 - x + x^2,$$
$$T\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix} = -2 + 2x^2$$

21. Prove Theorem 6.14(b).

22. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $T : V \to V$ be a linear transformation. Prove that if $T(\mathbf{v}_1) = \mathbf{v}_1, T(\mathbf{v}_2) = \mathbf{v}_2 \ldots, T(\mathbf{v}_n) = \mathbf{v}_n$, then T is the identity transformation on V.

23. Let $T : \mathscr{P}_n \to \mathscr{P}_n$ be a linear transformation such that $T(x^k) = kx^{k-1}$ for $k = 0, 1, \ldots, n$. Show that T must be the differential operator D.

24. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V and let $T: V \to W$ be a linear transformation.

(a) If $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in W, show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent in V.

(b) Show that the converse of part (a) is false. That is, it is not necessarily true that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent in V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in W. Illustrate this with an example $T: \mathbb{R}^2 \to \mathbb{R}^2$.

25. Define linear transformations $S: \mathbb{R}^2 \to M_{22}$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$S\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b & b \\ 0 & a-b \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d \\ -d \end{bmatrix}$$

Compute $(S \circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix}$. Can you compute $(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix}$? If so, compute it.

26. Define linear transformations $S: \mathscr{P}_1 \to \mathscr{P}_2$ and $T: \mathscr{P}_2 \to \mathscr{P}_1$ by

$$S(a + bx) = a + (a + b)x + 2bx^2$$

and $$T(a + bx + cx^2) = b + 2cx$$

Compute $(S \circ T)(3 + 2x - x^2)$ and $(S \circ T)(a + bx + cx^2)$. Can you compute $(T \circ S)(a + bx)$? If so, compute it.

27. Define linear transformations $S: \mathscr{P}_n \to \mathscr{P}_n$ and $T: \mathscr{P}_n \to \mathscr{P}_n$ by

$$S(p(x)) = p(x + 1) \quad \text{and} \quad T(p(x)) = p'(x)$$

Find $(S \circ T)(p(x))$ and $(T \circ S)(p(x))$. [*Hint:* Remember the Chain Rule.]

28. Define linear transformations $S: \mathscr{P}_n \to \mathscr{P}_n$ and $T: \mathscr{P}_n \to \mathscr{P}_n$ by

$$S(p(x)) = p(x + 1) \quad \text{and} \quad T(p(x)) = xp'(x)$$

Find $(S \circ T)(p(x))$ and $(T \circ S)(p(x))$.

In Exercises 29 and 30, verify that S and T are inverses.

29. $S: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x + y \\ 3x + y \end{bmatrix}$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ -3x + 4y \end{bmatrix}$

30. $S: \mathscr{P}_1 \to \mathscr{P}_1$ defined by $S(a + bx) = (-4a + b) + 2ax$ and $T: \mathscr{P}_1 \to \mathscr{P}_1$ defined by

$$T(a + bx) = b/2 + (a + 2b)x$$

31. Prove Theorem 6.17.

32. Let $T: V \to V$ be a linear transformation such that $T \circ T = I$.

(a) Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \pm\mathbf{v}$.

(b) Give an example of such a linear transformation with $V = \mathbb{R}^2$.

33. Let $T: V \to V$ be a linear transformation such that $T \circ T = T$.

(a) Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \mathbf{v}$ or $T(\mathbf{v}) = \mathbf{0}$.

(b) Give an example of such a linear transformation with $V = \mathbb{R}^2$.

The set of all linear transformations from a vector space V to a vector space W is denoted by $\mathscr{L}(V, W)$. If S and T are in $\mathscr{L}(V, W)$, we can define the **sum** *$S + T$ of S and T by*

$$(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$$

for all $\mathbf{v}$ in V. If c is a scalar, we define the **scalar multiple** *cT of T by c to be*

$$(cT)(\mathbf{v}) = cT(\mathbf{v})$$

for all $\mathbf{v}$ in V. Then $S + T$ and cT are both transformations from V to W.

34. Prove that $S + T$ and cT are linear transformations.

35. Prove that $\mathscr{L}(V, W)$ is a vector space with this addition and scalar multiplication.

36. Let R, S, and T be linear transformations such that the following operations make sense. Prove that:

(a) $R \circ (S + T) = R \circ S + R \circ T$

(b) $c(R \circ S) = (cR) \circ S = R \circ (cS)$ for any scalar c

6.5 The Kernel and Range of a Linear Transformation

The null space and column space are two of the fundamental subspaces associated with a matrix. In this section, we extend these notions to the kernel and range of a linear transformation.

The word *kernel* is derived from the Old English word *cyrnel*, a form of the word *corn*, meaning "seed" or "grain." Like a kernel of corn, the kernel of a linear transformation is its "core" or "seed" in the sense that it carries information about many of the important properties of the transformation.

Definition Let $T: V \to W$ be a linear transformation. The ***kernel*** of T, denoted $\ker(T)$, is the set of all vectors in V that are mapped by T to $\mathbf{0}$ in W. That is,

$$\ker(T) = \{\mathbf{v} \text{ in } V: T(\mathbf{v}) = \mathbf{0}\}$$

The ***range*** of T, denoted $\operatorname{range}(T)$, is the set of all vectors in W that are images of vectors in V under T. That is,

$$\begin{aligned}\operatorname{range}(T) &= \{T(\mathbf{v}) : \mathbf{v} \text{ in } V\}\\ &= \{\mathbf{w} \text{ in } W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}\end{aligned}$$

Example 6.59

Let A be an $m \times n$ matrix and let $T = T_A$ be the corresponding matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$. Then, as we saw in Chapter 3, the range of T is the column space of A.

The kernel of T is

$$\begin{aligned}\ker(T) &= \{\mathbf{v} \text{ in } \mathbb{R}^n : T(\mathbf{v}) = \mathbf{0}\}\\ &= \{\mathbf{v} \text{ in } \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}\\ &= \operatorname{null}(A)\end{aligned}$$

In words, the kernel of a matrix transformation is just the null space of the corresponding matrix.

Example 6.60

Find the kernel and range of the differential operator $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution Since $D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$, we have

$$\begin{aligned}\ker(D) &= \{a + bx + cx^2 + dx^3 : D(a + bx + cx^2 + dx^3) = 0\}\\ &= \{a + bx + cx^2 + dx^3 : b + 2cx + 3dx^2 = 0\}\end{aligned}$$

But $b + 2cx + 3dx^2 = 0$ if and only if $b = 2c = 3d = 0$, which implies that $b = c = d = 0$. Therefore,

$$\begin{aligned}\ker(D) &= \{a + bx + cx^2 + dx^3 : b = c = d = 0\}\\ &= \{a : a \text{ in } \mathbb{R}\}\end{aligned}$$

In other words, the kernel of D is the set of constant polynomials.

The range of D is all of $\mathscr{P}_2$, since *every* polynomial in $\mathscr{P}_2$ is the image under D (i.e., the derivative) of *some* polynomial in $\mathscr{P}_3$. To be specific, if $a + bx + cx^2$ is in $\mathscr{P}_2$, then

$$a + bx + cx^2 = D\left(ax + \left(\frac{b}{2}\right)x^2 + \left(\frac{c}{3}\right)x^3\right)$$

Example 6.61

Let $S : \mathscr{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$S(p(x)) = \int_0^1 p(x)\,dx$$

Find the kernel and range of S.

Solution In detail, we have

$$\begin{aligned} S(a + bx) &= \int_0^1 (a + bx)\,dx \\ &= \left[ax + \frac{b}{2}x^2\right]_0^1 \\ &= \left(a + \frac{b}{2}\right) - 0 = a + \frac{b}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \ker(S) &= \{a + bx : S(a + bx) = 0\} \\ &= \left\{a + bx : a + \frac{b}{2} = 0\right\} \\ &= \left\{a + bx : a = -\frac{b}{2}\right\} \\ &= \left\{-\frac{b}{2} + bx\right\} \end{aligned}$$

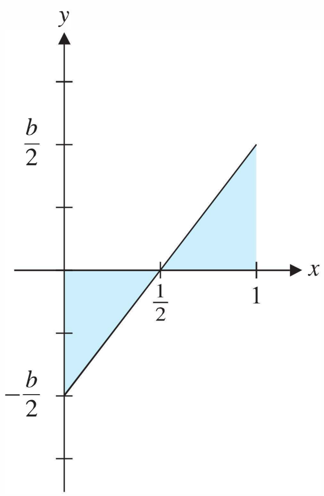


Figure 6.7

If $y = -\frac{b}{2} + bx$, then $\int_0^1 y\,dx = 0$

Geometrically, $\ker(S)$ consists of all those linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis on the interval $[0, 1]$ (see Figure 6.7).

The range of S is $\mathbb{R}$, since every real number can be obtained as the image under S of some polynomial in $\mathscr{P}_1$. For example, if a is an arbitrary real number, then

$$\int_0^1 a\,dx = [ax]_0^1 = a - 0 = a$$

so $a = S(a)$.

Example 6.62

Let $T : M_{22} \to M_{22}$ be the linear transformation defined by taking transposes: $T(A) = A^T$. Find the kernel and range of T.

Solution We see that

$$\begin{aligned} \ker(T) &= \{A \text{ in } M_{22} : T(A) = O\} \\ &= \{A \text{ in } M_{22} : A^T = O\} \end{aligned}$$

But if $A^T = O$, then $A = (A^T)^T = O^T = O$. It follows that $\ker(T) = \{O\}$.

Since, for any matrix A in M_{22}, we have $A = (A^T)^T = T(A^T)$ (and A^T is in M_{22}), we deduce that $\text{range}(T) = M_{22}$.

In all of these examples, the kernel and range of a linear transformation are subspaces of the domain and codomain, respectively, of the transformation. Since we are generalizing the null space and column space of a matrix, this is perhaps not surprising. Nevertheless, we should not take anything for granted, so we need to prove that it is not a coincidence.

Theorem 6.18 Let $T: V \to W$ be a linear transformation. Then:

a. The kernel of T is a subspace of V.
b. The range of T is a subspace of W.

Proof (a) Since $T(\mathbf{0}) = \mathbf{0}$, the zero vector of V is in $\ker(T)$, so $\ker(T)$ is nonempty. Let $\mathbf{u}$ and $\mathbf{v}$ be in $\ker(T)$ and let c be a scalar. Then $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, so

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and
$$T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$$

Therefore, $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are in $\ker(T)$, and $\ker(T)$ is a subspace of V.

(b) Since $\mathbf{0} = T(\mathbf{0})$, the zero vector of W is in range(T), so range(T) is nonempty. Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be in the range of T and let c be a scalar. Then $T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v})$ is the image of the vector $\mathbf{u} + \mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are in V, so is $\mathbf{u} + \mathbf{v}$, and hence $T(\mathbf{u}) + T(\mathbf{v})$ is in range (T). Similarly, $cT(\mathbf{u}) = T(c\mathbf{u})$. Since $\mathbf{u}$ is in V, so is $c\mathbf{u}$, and hence $cT(\mathbf{u})$ is in range(T). Therefore, range(T) is a nonempty subset of W that is closed under addition and scalar multiplication, and thus it is a subspace of W.

Figure 6.8 gives a schematic representation of the kernel and range of a linear transformation.

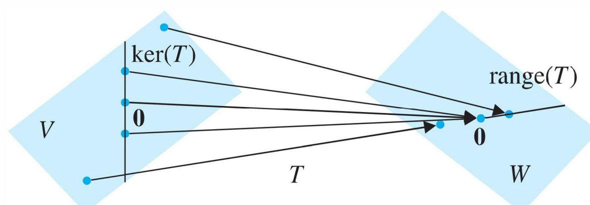


Figure 6.8
The kernel and range of $T: V \to W$

In Chapter 3, we defined the rank of a matrix to be the dimension of its column space and the nullity of a matrix to be the dimension of its null space. We now extend these definitions to linear transformations.

Definition Let $T: V \to W$ be a linear transformation. The ***rank*** of T is the dimension of the range of T and is denoted by rank(T). The ***nullity*** of T is the dimension of the kernel of T and is denoted by nullity(T).

Example 6.63 If A is a matrix and $T = T_A$ is the matrix transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, then the range and kernel of T are the column space and the null space of A, respectively, by Example 6.59. Hence, from Section 3.5, we have

$$\text{rank}(T) = \text{rank}(A) \quad \text{and} \quad \text{nullity}(T) = \text{nullity}(A)$$

Example 6.64

Find the rank and the nullity of the linear transformation $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution In Example 6.60, we computed range $(D) = \mathscr{P}_2$, so

$$\operatorname{rank}(D) = \dim \mathscr{P}_2 = 3$$

The kernel of D is the set of all constant polynomials: $\ker(D) = \{a : a \text{ in } \mathbb{R}\} = \{a \cdot 1 : a \text{ in } \mathbb{R}\}$. Hence, $\{1\}$ is a basis for $\ker(D)$, so

$$\operatorname{nullity}(D) = \dim(\ker(D)) = 1$$

Example 6.65

Find the rank and the nullity of the linear transformation $S : \mathscr{P}_1 \to \mathbb{R}$ defined by

$$S(p(x)) = \int_0^1 p(x)\, dx$$

Solution From Example 6.61, $\operatorname{range}(S) = \mathbb{R}$ and $\operatorname{rank}(S) = \dim \mathbb{R} = 1$. Also,

$$\begin{aligned}\ker(S) &= \left\{-\frac{b}{2} + bx : b \text{ in } \mathbb{R}\right\}\\ &= \{b(-\tfrac{1}{2} + x) : b \text{ in } \mathbb{R}\}\\ &= \operatorname{span}(-\tfrac{1}{2} + x)\end{aligned}$$

so $\{-\frac{1}{2} + x\}$ is a basis for $\ker(S)$. Therefore, $\operatorname{nullity}(S) = \dim(\ker(S)) = 1$.

Example 6.66

Find the rank and the nullity of the linear transformation $T : M_{22} \to M_{22}$ defined by $T(A) = A^T$.

Solution In Example 6.62, we found that $\operatorname{range}(T) = M_{22}$ and $\ker(T) = \{O\}$. Hence,

$$\operatorname{rank}(T) = \dim M_{22} = 4 \quad \text{and} \quad \operatorname{nullity}(T) = \dim\{O\} = 0$$

In Chapter 3, we saw that the rank and nullity of an $m \times n$ matrix A are related by the formula $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$. This is the Rank Theorem (Theorem 3.26). Since the matrix transformation $T = T_A$ has $\mathbb{R}^n$ as its domain, we could rewrite the relationship as

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \dim \mathbb{R}^n$$

This version of the Rank Theorem extends very nicely to general linear transformations, as you can see from the last three examples:

$$\begin{aligned}\operatorname{rank}(D) + \operatorname{nullity}(D) &= 3 + 1 = 4 = \dim \mathscr{P}_3 && \text{Example 6.64}\\ \operatorname{rank}(S) + \operatorname{nullity}(S) &= 1 + 1 = 2 = \dim \mathscr{P}_1 && \text{Example 6.65}\\ \operatorname{rank}(T) + \operatorname{nullity}(T) &= 4 + 0 = 4 = \dim M_{22} && \text{Example 6.66}\end{aligned}$$

Theorem 6.19 **The Rank Theorem**

Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

In the next section, you will see how to adapt the proof of Theorem 3.26 to prove this version of the result. For now, we give an alternative proof that does not use matrices.

Proof Let $\dim V = n$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $\ker(T)$[so that $\text{nullity}(T) = \dim(\ker(T)) = k$]. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set, it can be extended to a basis for V, by Theorem 6.28. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ be such a basis. If we can show that the set $\mathcal{C} = \{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is a basis for $\text{range}(T)$, then we will have $\text{rank}(T) = \dim(\text{range}(T)) = n - k$ and thus

$$\text{rank}(T) + \text{nullity}(T) = k + (n - k) = n = \dim V$$

as required.

Certainly $\mathcal{C}$ is contained in the range of T. To show that $\mathcal{C}$ spans the range of T, let $T(\mathbf{v})$ be a vector in the range of T. Then $\mathbf{v}$ is in V, and since $\mathcal{B}$ is a basis for V, we can find scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are in the kernel of T, we have $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_k) = \mathbf{0}$, so

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k) + c_{k+1}T(\mathbf{v}_{k+1}) + \cdots + c_nT(\mathbf{v}_n) \\ &= c_{k+1}T(\mathbf{v}_{k+1}) + \cdots + c_nT(\mathbf{v}_n) \end{aligned}$$

This shows that the range of T is spanned by $\mathcal{C}$.

To show that $\mathcal{C}$ is linearly independent, suppose that there are scalars $c_{k+1}, \ldots, c_n$ such that

$$c_{k+1}T(\mathbf{v}_{k+1}) + \cdots + c_nT(\mathbf{v}_n) = \mathbf{0}$$

Then $T(c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n) = \mathbf{0}$, which means that $c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n$ is in the kernel of T and is, hence, expressible as a linear combination of the basis vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of $\ker(T)$—say,

$$c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

But now $$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k - c_{k+1}\mathbf{v}_{k+1} - \cdots - c_n\mathbf{v}_n = \mathbf{0}$$

and the linear independence of $\mathcal{B}$ forces $c_1 = \cdots = c_n = 0$. In particular, $c_{k+1} = \cdots = c_n = 0$, which means $\mathcal{C}$ is linearly independent.

We have shown that $\mathcal{C}$ is a basis for the range of T, so, by our comments above, the proof is complete.

We have verified the Rank Theorem for Examples 6.64, 6.65, and 6.66. In practice, this theorem allows us to find the rank and nullity of a linear transformation with only half the work. The following examples illustrate the process.

Example 6.67

Find the rank and nullity of the linear transformation $T : \mathscr{P}_2 \to \mathscr{P}_3$ defined by $T(p(x)) = xp(x)$. (Check that T really is linear.)

Solution In detail, we have

$$T(a + bx + cx^2) = ax + bx^2 + cx^3$$

It follows that

$$\begin{aligned} \ker(T) &= \{a + bx + cx^2 : T(a + bx + cx^2) = 0\} \\ &= \{a + bx + cx^2 : ax + bx^2 + cx^3 = 0\} \\ &= \{a + bx + cx^2 : a = b = c = 0\} \\ &= \{0\} \end{aligned}$$

so we have $\text{nullity}(T) = \dim(\ker(T)) = 0$. The Rank Theorem implies that

$$\text{rank}(T) = \dim \mathscr{P}_2 - \text{nullity}(T) = 3 - 0 = 3$$

Remark In Example 6.67, it would be just as easy to find the rank of T first, since $\{x, x^2, x^3\}$ is easily seen to be a basis for the range of T. Usually, though, one of the two (the rank or the nullity of a linear transformation) will be easier to compute; the Rank Theorem can then be used to find the other. With practice, you will become better at knowing which way to proceed.

Example 6.68

Let W be the vector space of all symmetric 2×2 matrices. Define a linear transformation $T : W \to \mathscr{P}_2$ by

$$T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a - b) + (b - c)x + (c - a)x^2$$

(Check that T is linear.) Find the rank and nullity of T.

Solution The nullity of T is easier to compute directly than the rank, so we proceed as follows:

$$\begin{aligned} \ker(T) &= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a - b) + (b - c)x + (c - a)x^2 = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a - b) = (b - c) = (c - a) = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a = b = c \right\} \\ &= \left\{ \begin{bmatrix} c & c \\ c & c \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \end{aligned}$$

Therefore, $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for the kernel of T, so $\text{nullity}(T) = \dim(\ker(T)) = 1$. The Rank Theorem and Example 6.42 tell us that $\text{rank}(T) = \dim W - \text{nullity}(T) = 3 - 1 = 2$.

One-to-One and Onto Linear Transformations

We now investigate criteria for a linear transformation to be invertible. The keys to the discussion are the very important properties one-to-one and onto.

Definition A linear transformation $T : V \to W$ is called ***one-to-one*** if T maps distinct vectors in V to distinct vectors in W. If range$(T) = W$, then T is called ***onto.***

Remarks

- The definition of one-to-one may be written more formally as follows:

$T : V \to W$ is one-to-one if, for all $\mathbf{u}$ and $\mathbf{v}$ in V,

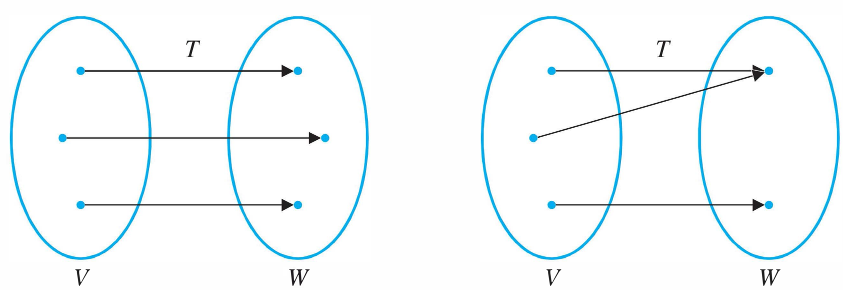
$$\mathbf{u} \neq \mathbf{v} \text{ implies that } T(\mathbf{u}) \neq T(\mathbf{v})$$

The above statement is equivalent to the following:

$T : V \to W$ is one-to-one if, for all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$T(\mathbf{u}) = T(\mathbf{v}) \text{ implies that } \mathbf{u} = \mathbf{v}$$

Figure 6.9 illustrates these two statements.



(a) T is one-to-one (b) T is not one-to-one

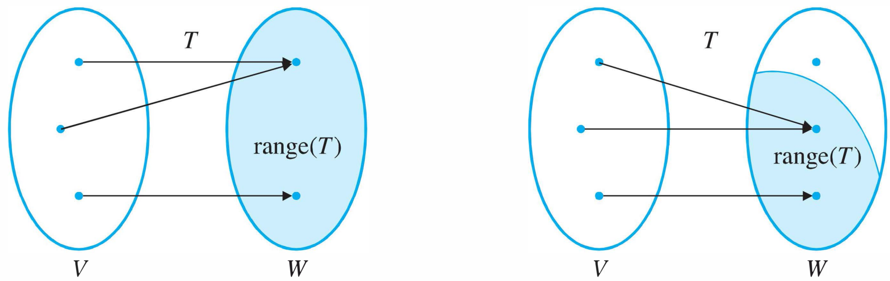
Figure 6.9

- Another way to write the definition of onto is as follows:

$T : V \to W$ is onto if, for all $\mathbf{w}$ in W, there is at least one $\mathbf{v}$ in V such that

$$\mathbf{w} = T(\mathbf{v})$$

In other words, *given* $\mathbf{w}$ in W, does there exist some $\mathbf{v}$ in V such that $\mathbf{w} = T(\mathbf{v})$? If, for an arbitrary $\mathbf{w}$, we can solve this equation for $\mathbf{v}$, then T is onto (see Figure 6.10).



(a) T is onto (b) T is not onto

Figure 6.10

Example 6.69

Which of the following linear transformations are one-to-one? onto?

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x - y \\ 0 \end{bmatrix}$

(b) $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$

(c) $T : M_{22} \to M_{22}$ defined by $T(A) = A^T$

Solution (a) Let $T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$\begin{bmatrix} 2x_1 \\ x_1 - y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 - y_2 \\ 0 \end{bmatrix}$$

so $2x_1 = 2x_2$ and $x_1 - y_1 = x_2 - y_2$. Solving these equations, we see that $x_1 = x_2$ and $y_1 = y_2$. Hence, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, so T is one-to-one.

T is not onto, since its range is not all of $\mathbb{R}^3$. To be specific, there is no vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (Why not?)

(b) In Example 6.60, we showed that range$(D) = \mathscr{P}_2$, so D is onto. D is not one-to-one, since distinct polynomials in $\mathscr{P}_3$ can have the same derivative. For example, $x^3 \neq x^3 + 1$, but $D(x^3) = 3x^2 = D(x^3 + 1)$.

(c) Let A and B be in M_{22}, with $T(A) = T(B)$. Then $A^T = B^T$, so $A = (A^T)^T = (B^T)^T = B$. Hence, T is one-to-one. In Example 6.62, we showed that range$(T) = M_{22}$. Hence, T is onto.

It turns out that there is a very simple criterion for determining whether a linear transformation is one-to-one.

Theorem 6.20 A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Proof Assume that T is one-to-one. If $\mathbf{v}$ is in the kernel of T, then $T(\mathbf{v}) = \mathbf{0}$. But we also know that $T(\mathbf{0}) = \mathbf{0}$, so $T(\mathbf{v}) = T(\mathbf{0})$. Since T is one-to-one, this implies that $\mathbf{v} = \mathbf{0}$, so the only vector in the kernel of T is the zero vector.

Conversely, assume that $\ker(T) = \{\mathbf{0}\}$. To show that T is one-to-one, let $\mathbf{u}$ and $\mathbf{v}$ be in V with $T(\mathbf{u}) = T(\mathbf{v})$. Then $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$, which implies that $\mathbf{u} - \mathbf{v}$ is in the kernel of T. But $\ker(T) = \{\mathbf{0}\}$, so we must have $\mathbf{u} - \mathbf{v} = \mathbf{0}$ or, equivalently, $\mathbf{u} = \mathbf{v}$. This proves that T is one-to-one.

Example 6.70

Show that the linear transformation $T : \mathbb{R}^2 \to \mathscr{P}_1$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

is one-to-one and onto.

Solution If $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the kernel of T, then

$$0 = T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

It follows that $a = 0$ and $a + b = 0$. Hence, $b = 0$, and therefore $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Consequently, $\ker(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, and T is one-to-one, by Theorem 6.20.

By the Rank Theorem,

$$\operatorname{rank}(T) = \dim \mathbb{R}^2 - \operatorname{nullity}(T) = 2 - 0 = 2$$

Therefore, the range of T is a two-dimensional subspace of $\mathbb{R}^2$, and hence $\operatorname{range}(T) = \mathbb{R}^2$. It follows that T is onto.

For linear transformations between two n-dimensional vector spaces, the properties of one-to-one and onto are closely related. Observe first that for a linear transformation $T : V \to W$, $\ker(T) = \{\mathbf{0}\}$ if and only if $\operatorname{nullity}(T) = 0$, and T is onto if and only if $\operatorname{rank}(T) = \dim W$. (Why?) The proof of the next theorem essentially uses the method of Example 6.70.

Theorem 6.21

Let $\dim V = \dim W = n$. Then a linear transformation $T : V \to W$ is one-to-one if and only if it is onto.

Proof Assume that T is one-to-one. Then $\operatorname{nullity}(T) = 0$ by Theorem 6.20 and the remark preceding Theorem 6.21. The Rank Theorem implies that

$$\operatorname{rank}(T) = \dim V - \operatorname{nullity}(T) = n - 0 = n$$

Therefore, T is onto.

Conversely, assume that T is onto. Then $\operatorname{rank}(T) = \dim W = n$. By the Rank Theorem,

$$\operatorname{nullity}(T) = \dim V - \operatorname{rank}(T) = n - n = 0$$

Hence, $\ker(T) = \{\mathbf{0}\}$, and T is one-to-one.

In Section 6.4, we pointed out that if $T : V \to W$ is a linear transformation, then the image of a basis for V under T need not be a basis for the range of T. We can now give a condition that ensures that a basis for V will be mapped by T to a basis for W.

Theorem 6.22

Let $T : V \to W$ be a one-to-one linear transformation. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set in V, then $T(S) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is a linearly independent set in W.

Proof Let $c_1, \ldots, c_k$ be scalars such that

$$c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k) = \mathbf{0}$$

Then $T(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = \mathbf{0}$, which implies that $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is in the kernel of T. But, since T is one-to-one, $\ker(T) = \{\mathbf{0}\}$, by Theorem 6.20. Hence,

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

But, since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, all of the scalars c_i must be 0. Therefore, $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.

Corollary 6.23

Let $\dim V = \dim W = n$. Then a one-to-one linear transformation $T : V \to W$ maps a basis for V to a basis for W.

Proof Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. By Theorem 6.22, $T(\mathcal{B}) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent set in W, so we need only show that $T(\mathcal{B})$ spans W. But, by Theorem 6.15, $T(\mathcal{B})$ spans the range of T. Moreover, T is onto, by Theorem 6.21, so $\operatorname{range}(T) = W$. Therefore, $T(\mathcal{B})$ spans W, which completes the proof.

Example 6.71

Let $T : \mathbb{R}^2 \to \mathscr{P}_1$ be the linear transformation from Example 6.70, defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

Then, by Corollary 6.23, the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$ is mapped to a basis $T(\mathcal{E}) = \{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ of $\mathscr{P}_1$. We find that

$$T(\mathbf{e}_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + x \quad \text{and} \quad T(\mathbf{e}_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$

It follows that $\{1 + x, x\}$ is a basis for $\mathscr{P}_1$.

We can now determine which linear transformations $T : V \to W$ are invertible.

Theorem 6.24

A linear transformation $T : V \to W$ is invertible if and only if it is one-to-one and onto.

Proof Assume that T is invertible. Then there exists a linear transformation $T^{-1}: W \to V$ such that

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \circ T^{-1} = I_W$$

To show that T is one-to-one, let $\mathbf{v}$ be in the kernel of T. Then $T(\mathbf{v}) = \mathbf{0}$. Therefore,

$$\begin{aligned} T^{-1}(T(\mathbf{v})) = T^{-1}(\mathbf{0}) &\Rightarrow (T^{-1} \circ T)(\mathbf{v}) = \mathbf{0} \\ &\Rightarrow I(\mathbf{v}) = \mathbf{0} \\ &\Rightarrow \mathbf{v} = \mathbf{0} \end{aligned}$$

which establishes that $\ker(T) = \{\mathbf{0}\}$. Therefore, T is one-to-one, by Theorem 6.20.

To show that T is onto, let $\mathbf{w}$ be in W and let $\mathbf{v} = T^{-1}(\mathbf{w})$. Then

$$\begin{aligned} T(\mathbf{v}) &= T(T^{-1}(\mathbf{w})) \\ &= (T \circ T^{-1})(\mathbf{w}) \\ &= I(\mathbf{w}) \\ &= \mathbf{w} \end{aligned}$$

which shows that $\mathbf{w}$ is the image of $\mathbf{v}$ under T. Since $\mathbf{v}$ is in V, this shows that T is onto.

Conversely, assume that T is one-to-one and onto. This means that $\text{nullity}(T) = 0$ and $\text{rank}(T) = \dim W$. We need to show that there exists a linear transformation $T': W \to V$ such that $T' \circ T = I_V$ and $T \circ T' = I_W$.

Let $\mathbf{w}$ be in W. Since T is onto, there exists some vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. There is only one such vector $\mathbf{v}$, since, if $\mathbf{v}'$ is another vector in V such that $T(\mathbf{v}') = \mathbf{w}$, then $T(\mathbf{v}) = T(\mathbf{v}')$; the fact that T is one-to-one then implies that $\mathbf{v} = \mathbf{v}'$. It therefore makes sense to define a mapping $T': W \to V$ by setting $T'(\mathbf{w}) = \mathbf{v}$.

It follows that

$$(T' \circ T)(\mathbf{v}) = T'(T(\mathbf{v})) = T'(\mathbf{w}) = \mathbf{v}$$

and

$$(T \circ T')(\mathbf{w}) = T(T'(\mathbf{w})) = T(\mathbf{v}) = \mathbf{w}$$

It then follows that $T' \circ T = I_V$ and $T \circ T' = I_W$. Now we must show that T' is a *linear* transformation.

To this end, let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in W and let c_1 and c_2 be scalars. As above, let $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Then $\mathbf{v}_1 = T'(\mathbf{w}_1)$ and $\mathbf{v}_2 = T'(\mathbf{w}_2)$ and

$$\begin{aligned} T'(c_1\mathbf{w}_1 + c_2\mathbf{w}_2) &= T'(c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)) \\ &= T'(T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)) \\ &= I(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \\ &= c_1T'(\mathbf{w}_1) + c_2T'(\mathbf{w}_2) \end{aligned}$$

Consequently, T' is linear, so, by Theorem 6.17, $T' = T^{-1}$.

The words *isomorphism* and *isomorphic* are derived from the Greek words *isos,* meaning "equal," and *morph,* meaning "shape." Thus, figuratively speaking, isomorphic vector spaces have "equal shapes."

Isomorphisms of Vector Spaces

We now are in a position to describe, in concrete terms, what it means for two vector spaces to be "essentially the same."

Definition A linear transformation $T: V \to W$ is called an ***isomorphism*** if it is one-to-one and onto. If V and W are two vector spaces such that there is an isomorphism from V to W, then we say that V is ***isomorphic*** to W and write $V \cong W$.

Example 6.72

Show that $\mathscr{P}_{n-1}$ and $\mathbb{R}^n$ are isomorphic.

Solution The process of forming the coordinate vector of a polynomial provides us with one possible isomorphism (as we observed already in Section 6.2, although we did not use the term *isomorphism* there). Specifically, define $T: \mathscr{P}_{n-1} \to \mathbb{R}^n$ by $T(p(x)) = [p(x)]_{\mathcal{E}}$, where $\mathcal{E} = \{1, x, \ldots, x^{n-1}\}$ is the standard basis for $\mathscr{P}_{n-1}$. That is,

$$T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Theorem 6.6 shows that T is a linear transformation. If $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ is in the kernel of T, then

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix} = T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence, $a_0 = a_1 = \cdots = a_{n-1} = 0$, so $p(x) = 0$. Therefore, $\ker(T) = \{0\}$, and T is one-to-one. Since $\dim \mathscr{P}_{n-1} = \dim \mathbb{R}^n = n$, T is also onto, by Theorem 6.21. Thus, T is an isomorphism, and $\mathscr{P}_{n-1} \cong \mathbb{R}^n$.

Example 6.73

Show that M_{mn} and $\mathbb{R}^{mn}$ are isomorphic.

Solution Once again, the coordinate mapping from M_{mn} to $\mathbb{R}^{mn}$ (as in Example 6.36) is an isomorphism. The details of the proof are left as an exercise.

In fact, the easiest way to tell if two vector spaces are isomorphic is simply to check their dimensions, as the next theorem shows.

Theorem 6.25 Let V and W be two finite-dimensional vector spaces (over the same field of scalars). Then V is isomorphic to W if and only if $\dim V = \dim W$.

Proof Let $n = \dim V$. If V is isomorphic to W, then there is an isomorphism $T: V \to W$. Since T is one-to-one, $\operatorname{nullity}(T) = 0$. The Rank Theorem then implies that

$$\operatorname{rank}(T) = \dim V - \operatorname{nullity}(T) = n - 0 = n$$

Therefore, the range of T is an n-dimensional subspace of W. But, since T is onto, $W = \operatorname{range}(T)$, so $\dim W = n$, as we wished to show.

Conversely, assume that V and W have the same dimension, n. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V and let $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be a basis for W. We will define a linear transformation $T: V \to W$ and then show that T is one-to-one and onto. An arbitrary vector $\mathbf{v}$ in V can be written uniquely as a linear combination of the vectors in the basis $\mathcal{B}$—say,

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

We define T by

$$T(\mathbf{v}) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$$

It is straightforward to check that T is linear. (Do so.) To see that T is one-to-one, suppose $\mathbf{v}$ is in the kernel of T. Then

$$c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n = T(\mathbf{v}) = \mathbf{0}$$

and the linear independence of $\mathcal{C}$ forces $c_1 = \cdots = c_n = 0$. But then

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

so $\ker(T) = \{\mathbf{0}\}$, meaning that T is one-to-one. Since $\dim V = \dim W$, T is also onto, by Theorem 6.21. Therefore, T is an isomorphism, and $V \cong W$.

Example 6.74 Show that $\mathbb{R}^n$ and $\mathcal{P}_n$ are not isomorphic.

Solution Since $\dim \mathbb{R}^n = n \neq n + 1 = \dim \mathcal{P}_n$, $\mathbb{R}^n$ and $\mathcal{P}_n$ are not isomorphic, by Theorem 6.25.

Example 6.75 Let W be the vector space of all symmetric 2×2 matrices. Show that W is isomorphic to $\mathbb{R}^3$.

Solution In Example 6.42, we showed that $\dim W = 3$. Hence, $\dim W = \dim \mathbb{R}^3$, so $W \cong \mathbb{R}^3$, by Theorem 6.25. (There is an obvious candidate for an isomorphism $T: W \to \mathbb{R}^3$. What is it?)

Remark Our examples have all been *real* vector spaces, but the theorems we have proved are true for vector spaces over the complex numbers $\mathbb{C}$ or $\mathbb{Z}_p$, where p is prime. For example, the vector space $M_{22}(\mathbb{Z}_2)$ of all 2×2 matrices with entries from $\mathbb{Z}_2$ has dimension 4 as a vector space over $\mathbb{Z}_2$, and hence $M_{22}(\mathbb{Z}_2) \cong \mathbb{Z}_2^4$.

Exercises 6.5

1. Let $T: M_{22} \to M_{22}$ be the linear transformation defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

(a) Which, if any, of the following matrices are in $\ker(T)$?

(i) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

(b) Which, if any, of the matrices in part (a) are in $\text{range}(T)$?

(c) Describe $\ker(T)$ and $\text{range}(T)$.

2. Let $T: M_{22} \to \mathbb{R}$ be the linear transformation defined by $T(A) = \text{tr}(A)$.

(a) Which, if any, of the following matrices are in $\ker(T)$?

(i) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$

(b) Which, if any, of the following scalars are in $\text{range}(T)$?

(i) 0 **(ii)** 2 **(iii)** $\sqrt{2}/2$

(c) Describe $\ker(T)$ and $\text{range}(T)$.

3. Let $T: \mathcal{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - b \\ b + c \end{bmatrix}$$

(a) Which, if any, of the following polynomials are in $\ker(T)$?

(i) $1 + x$ **(ii)** $x - x^2$ **(iii)** $1 + x - x^2$

(b) Which, if any, of the following vectors are in $\text{range}(T)$?

(i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) Describe $\ker(T)$ and $\text{range}(T)$.

4. Let $T: \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by $T(p(x)) = xp'(x)$.

(a) Which, if any, of the following polynomials are in $\ker(T)$?

(i) 1 **(ii)** x **(iii)** x^2

(b) Which, if any, of the polynomials in part (a) are in $\text{range}(T)$?

(c) Describe $\ker(T)$ and $\text{range}(T)$.

In Exercises 5–8, find bases for the kernel and range of the linear transformations T in the indicated exercises. In each case, state the nullity and rank of T and verify the Rank Theorem.

5. Exercise 1

6. Exercise 2

7. Exercise 3

8. Exercise 4

In Exercises 9–14, find either the nullity or the rank of T and then use the Rank Theorem to find the other.

9. $T: M_{22} \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b \\ c - d \end{bmatrix}$

10. $T: \mathcal{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

11. $T: M_{22} \to M_{22}$ defined by $T(A) = AB$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

12. $T: M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

13. $T: \mathcal{P}_2 \to \mathbb{R}$ defined by $T(p(x)) = p'(0)$

14. $T: M_{33} \to M_{33}$ defined by $T(A) = A - A^T$

In Exercises 15–20, determine whether the linear transformation T is (a) one-to-one and (b) onto.

15. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix}$

16. $T : \mathbb{R}^2 \to \mathcal{P}_2$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = (a - 2b) + (3a + b)x + (a + b)x^2$$

17. $T : \mathcal{P}_2 \to \mathbb{R}^3$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} 2a - b \\ a + b - 3c \\ c - a \end{bmatrix}$$

18. $T : \mathcal{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

19. $T : \mathbb{R}^3 \to \mathbb{M}_{22}$ defined by $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a - b & b - c \\ a + b & b + c \end{bmatrix}$

20. $T : \mathbb{R}^3 \to W$ defined by $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} =$

$\begin{bmatrix} a + b + c & b - 2c \\ b - 2c & a - c \end{bmatrix}$, where W is the vector space of all symmetric 2×2 matrices

In Exercises 21–26, determine whether V and W are isomorphic. If they are, give an explicit isomorphism $T : V \to W$.

21. $V = D_3$ (diagonal 3×3 matrices), $W = \mathbb{R}^3$

22. $V = S_3$ (symmetric 3×3 matrices), $W = U_3$ (upper triangular 3×3 matrices)

23. $V = S_3$ (symmetric 3×3 matrices), $W = S_3'$ (skew-symmetric 3×3 matrices)

24. $V = \mathcal{P}_2$, $W = \{p(x) \text{ in } \mathcal{P}_3 : p(0) = 0\}$

25. $V = \mathbb{C}$, $W = \mathbb{R}^2$

26. $V = \{A \text{ in } M_{22} : \operatorname{tr}(A) = 0\}$, $W = \mathbb{R}^2$

27. Show that $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by $T(p(x)) = p(x) + p'(x)$ is an isomorphism.

28. Show that $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by $T(p(x)) = p(x - 2)$ is an isomorphism.

29. Show that $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by $T(p(x)) = x^n p\left(\frac{1}{x}\right)$ is an isomorphism.

30. **(a)** Show that $\mathscr{C}[0, 1] \cong \mathscr{C}[2, 3]$. [*Hint:* Define $T : \mathscr{C}[0, 1] \to \mathscr{C}[2, 3]$ by letting $T(f)$ be the function whose value at x is $(T(f))(x) = f(x - 2)$ for x in $[2, 3]$.]

(b) Show that $\mathscr{C}[0, 1] \cong \mathscr{C}[a, a + 1]$ for all a.

31. Show that $\mathscr{C}[0, 1] \cong \mathscr{C}[0, 2]$.

32. Show that $\mathscr{C}[a, b] \cong \mathscr{C}[c, d]$ for all $a < b$ and $c < d$.

33. Let $S : V \to W$ and $T : U \to V$ be linear transformations.

(a) Prove that if S and T are both one-to-one, so is $S \circ T$.

(b) Prove that if S and T are both onto, so is $S \circ T$.

34. Let $S : V \to W$ and $T : U \to V$ be linear transformations.

(a) Prove that if $S \circ T$ is one-to-one, so is T.

(b) Prove that if $S \circ T$ is onto, so is S.

35. Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces.

(a) Prove that if $\dim V < \dim W$, then T cannot be onto.

(b) Prove that if $\dim V > \dim W$, then T cannot be one-to-one.

36. Let $a_0, a_1, \ldots, a_n$ be $n + 1$ distinct real numbers. Define $T : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by

$$T(p(x)) = \begin{bmatrix} p(a_0) \\ p(a_1) \\ \vdots \\ p(a_n) \end{bmatrix}$$

Prove that T is an isomorphism.

37. If V is a finite-dimensional vector space and $T : V \to V$ is a linear transformation such that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, prove that $\operatorname{range}(T) \cap \ker(T) = \{\mathbf{0}\}$. [*Hint:* T^2 denotes $T \circ T$. Use the Rank Theorem to help show that the kernels of T and T^2 are the same.]

38. Let U and W be subspaces of a finite-dimensional vector space V. Define $T : U \times W \to V$ by $T(\mathbf{u}, \mathbf{w}) = \mathbf{u} - \mathbf{w}$.

(a) Prove that T is a linear transformation.

(b) Show that $\operatorname{range}(T) = U + W$.

(c) Show that $\ker(T) \cong U \cap W$. [*Hint:* See Exercise 50 in Section 6.1.]

(d) Prove ***Grassmann's Identity:***

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

[*Hint:* Apply the Rank Theorem, using results (a) and (b) and Exercise 43(b) in Section 6.2.]

6.6 The Matrix of a Linear Transformation

Theorem 6.15 showed that a linear transformation $T : V \to W$ is completely determined by its effect on a spanning set for V. In particular, if we know how T acts on a basis for V, then we can compute $T(\mathbf{v})$ for any vector $\mathbf{v}$ in V. Example 6.55 illustrated the process. We implicitly used this important property of linear transformations in Theorem 3.31 to help us compute the standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. In this section, we will show that every linear transformation between finite-dimensional vector spaces can be represented as a matrix transformation.

Suppose that V is an n-dimensional vector space, W is an m-dimensional vector space, and $T : V \to W$ is a linear transformation. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Then the coordinate vector mapping $R(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ defines an isomorphism $R : V \to \mathbb{R}^n$. At the same time, we have an isomorphism $S : W \to \mathbb{R}^m$ given by $S(\mathbf{w}) = [\mathbf{w}]_{\mathcal{C}}$, which allows us to associate the image $T(\mathbf{v})$ with the vector $[T(\mathbf{v})]_{\mathcal{C}}$ in $\mathbb{R}^m$. Figure 6.11 illustrates the relationships.

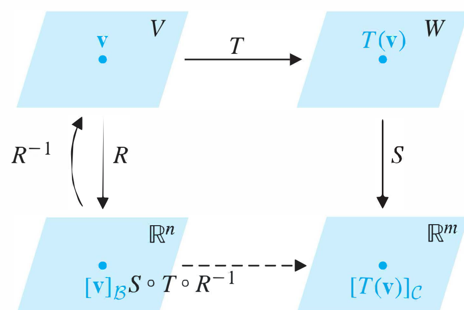


Figure 6.11

Since R is an isomorphism, it is invertible, so we may form the composite mapping

$$S \circ T \circ R^{-1} : \mathbb{R}^n \to \mathbb{R}^m$$

which maps $[\mathbf{v}]_{\mathcal{B}}$ to $[T(\mathbf{v})]_{\mathcal{C}}$. Since this mapping goes from $\mathbb{R}^n$ to $\mathbb{R}^m$, we know from Chapter 3 that it is a matrix transformation. What, then, is the standard matrix of $S \circ T \circ R^{-1}$? We would like to find the $m \times n$ matrix A such that $A[\mathbf{v}]_{\mathcal{B}} = (S \circ T \circ R^{-1})([\mathbf{v}]_{\mathcal{B}})$. Or, since $(S \circ T \circ R^{-1})([\mathbf{v}]_{\mathcal{B}}) = [T(\mathbf{v})]_{\mathcal{C}}$, we require

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

It turns out to be surprisingly easy to find. The basic idea is that of Theorem 3.31. The columns of A are the images of the standard basis vectors for $\mathbb{R}^n$ under $S \circ T \circ R^{-1}$. But, if $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V, then

$$\begin{aligned} R(\mathbf{v}_i) &= [\mathbf{v}_i]_{\mathcal{B}} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \leftarrow i\text{th entry} \\ &= \mathbf{e}_i \end{aligned}$$

so $R^{-1}(\mathbf{e}_i) = \mathbf{v}_i$. Therefore, the ith column of the matrix A we seek is given by

$$\begin{aligned}(S \circ T \circ R^{-1})(\mathbf{e}_i) &= S(T(R^{-1}(\mathbf{e}_i))) \\ &= S(T(\mathbf{v}_i)) \\ &= [T(\mathbf{v}_i)]_{\mathcal{C}}\end{aligned}$$

which is the coordinate vector of $T(\mathbf{v}_i)$ with respect to the basis $\mathcal{C}$ of W.

We summarize this discussion as a theorem.

Theorem 6.26 Let V and W be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively, where $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $T: V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = \big[\,[T(\mathbf{v}_1)]_{\mathcal{C}} \mid [T(\mathbf{v}_2)]_{\mathcal{C}} \mid \cdots \mid [T(\mathbf{v}_n)]_{\mathcal{C}}\,\big]$$

satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every vector $\mathbf{v}$ in V.

The matrix A in Theorem 6.26 is called the ***matrix of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.*** The relationship is illustrated below. (Recall that T_A denotes multiplication by A.)

$$\begin{array}{ccc} \mathbf{v} & \xrightarrow{T} & T(\mathbf{v}) \\ \downarrow & & \downarrow \\ [\mathbf{v}]_{\mathcal{B}} & \xrightarrow{T_A} & A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}} \end{array}$$

Remarks

- The matrix of a linear transformation T with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is sometimes denoted by $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$. Note the direction of the arrow: right-to-left (not left-to-right, as for $T: V \to W$). With this notation, the final equation in Theorem 6.26 becomes

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

Observe that the $\mathcal{B}$s in the subscripts appear side by side and appear to "cancel" each other. In words, this equation says, "The matrix for T times the coordinate vector for $\mathbf{v}$ gives the coordinate vector for $T(\mathbf{v})$."

In the special case where $V = W$ and $\mathcal{B} = \mathcal{C}$, we write $[T]_{\mathcal{B}}$ (instead of $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$). Theorem 6.26 then states that

$$[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$$

- The matrix of a linear transformation with respect to given bases is unique. That is, for every vector $\mathbf{v}$ in V, there is only *one* matrix A with the property specified by Theorem 6.26—namely,

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

(You are asked to prove this in Exercise 39.)

- The diagram that follows Theorem 6.26 is sometimes called a *commutative diagram* because we can start in the upper left-hand corner with the vector **v** and get to $[T(\mathbf{v})]_{\mathcal{C}}$ in the lower right-hand corner in two different, but equivalent, ways. If, as before, we denote the coordinate mappings that map **v** to $[\mathbf{v}]_{\mathcal{B}}$ and **w** to $[\mathbf{w}]_{\mathcal{C}}$ by R and S, respectively, then we can summarize this "commutativity" by

$$S \circ T = T_A \circ R$$

The reason for the term *commutative* becomes clearer when $V = W$ and $\mathcal{B} = \mathcal{C}$, for then $R = S$ too, and we have

$$R \circ T = T_A \circ R$$

suggesting that the coordinate mapping R commutes with the linear transformation T (provided we use the matrix version of T—namely, $T_A = T_{[T]_{\mathcal{B}}}$—where it is required).

- The matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ depends on the *order* of the vectors in the bases $\mathcal{B}$ and $\mathcal{C}$. Rearranging the vectors within either basis will affect the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$. [See Example 6.77(b).]

Example 6.76

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

and let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathcal{C} = \{\mathbf{e}_2, \mathbf{e}_1\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ and verify Theorem 6.26 for $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

Solution First, we compute

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Next, we need their coordinate vectors with respect to $\mathcal{C}$. Since

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e}_2 + \mathbf{e}_1, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \mathbf{e}_2 - 2\mathbf{e}_1, \quad \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3\mathbf{e}_2 + 0\mathbf{e}_1$$

we have

$$[T(\mathbf{e}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad [T(\mathbf{e}_2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad [T(\mathbf{e}_3)]_{\mathcal{C}} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Therefore, the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is

$$\begin{aligned} A = [T]_{\mathcal{C}\leftarrow\mathcal{B}} &= [\,[T(\mathbf{e}_1)]_{\mathcal{C}} \;\; [T(\mathbf{e}_2)]_{\mathcal{C}} \;\; [T(\mathbf{e}_3)]_{\mathcal{C}}\,] \\ &= \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \end{aligned}$$

To verify Theorem 6.26 for **v**, we first compute

$$T(\mathbf{v}) = T\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

Then

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

and

$$[T(\mathbf{v})]_{\mathcal{C}} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

(Check these.)

Using all of these facts, we confirm that

$$A[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = [T(\mathbf{v})]_{\mathcal{C}}$$

Example 6.77

Let $D : \mathscr{P}_3 \to \mathscr{P}_2$ be the differential operator $D(p(x)) = p'(x)$. Let $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{C} = \{1, x, x^2\}$ be bases for $\mathscr{P}_3$ and $\mathscr{P}_2$, respectively.

(a) Find the matrix A of D with respect to $\mathcal{B}$ and $\mathcal{C}$.

(b) Find the matrix A' of D with respect to $\mathcal{B}'$ and $\mathcal{C}$, where $\mathcal{B}' = \{x^3, x^2, x, 1\}$.

(c) Using part (a), compute $D(5 - x + 2x^3)$ and $D(a + bx + cx^2 + dx^3)$ to verify Theorem 6.26.

Solution First note that $D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$. (See Example 6.60.)

(a) Since the images of the basis $\mathcal{B}$ under D are $D(1) = 0$, $D(x) = 1$, $D(x^2) = 2x$, and $D(x^3) = 3x^2$, their coordinate vectors with respect to $\mathcal{C}$ are

$$[D(1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [D(x^3)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Consequently,

$$\begin{aligned} A = [D]_{\mathcal{C}\leftarrow\mathcal{B}} &= [[D(1)]_{\mathcal{C}} \mid [D(x)]_{\mathcal{C}} \mid [D(x^2)]_{\mathcal{C}} \mid [D(x^3)]_{\mathcal{C}}] \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

(b) Since the basis $\mathcal{B}'$ is just $\mathcal{B}$ in the *reverse* order, we see that

$$\begin{aligned} A' = [D]_{\mathcal{C}\leftarrow\mathcal{B}'} &= [[D(x^3)]_{\mathcal{C}} \mid [D(x^2)]_{\mathcal{C}} \mid [D(x)]_{\mathcal{C}} \mid [D(1)]_{\mathcal{C}}] \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(This shows that the *order* of the vectors in the bases $\mathcal{B}$ and $\mathcal{C}$ affects the matrix of a transformation with respect to these bases.)

(c) First we compute $D(5 - x + 2x^3) = -1 + 6x^2$ directly, getting the coordinate vector

$$[D(5 - x + 2x^3)]_{\mathcal{C}} = [-1 + 6x^2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

On the other hand,

$$[5 - x + 2x^3]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

so

$$A[5 - x + 2x^3]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = [D(5 - x + 2x^3)]_{\mathcal{C}}$$

which agrees with Theorem 6.26. We leave proof of the general case as an exercise.

Since the linear transformation in Example 6.77 is easy to use directly, there is really no advantage to using the matrix of this transformation to do calculations. However, in other examples—especially large ones—the matrix approach may be simpler, as it is very well-suited to computer implementation. Example 6.78 illustrates the basic idea behind this indirect approach.

Example 6.78

Let $T : \mathscr{P}_2 \to \mathscr{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1)$$

(a) Find the matrix of T with respect to $\mathcal{E} = \{1, x, x^2\}$.

(b) Compute $T(3 + 2x - x^2)$ indirectly, using part (a).

Solution (a) We see that

$$T(1) = 1, \quad T(x) = 2x - 1, \quad T(x^2) = (2x - 1)^2 = 1 - 4x + 4x^2$$

so the coordinate vectors are

$$[T(1)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

Therefore,

$$[T]_{\mathcal{E}} = [[T(1)]_{\mathcal{E}} \mid [T(x)]_{\mathcal{E}} \mid [T(x^2)]_{\mathcal{E}}] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) We apply Theorem 6.26 as follows: The coordinate vector of $p(x) = 3 + 2x - x^2$ with respect to $\mathcal{E}$ is

$$[p(x)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Therefore, by Theorem 6.26,

$$\begin{aligned} [T(3 + 2x - x^2)]_{\mathcal{E}} &= [T(p(x))]_{\mathcal{E}} \\ &= [T]_{\mathcal{E}}[p(x)]_{\mathcal{E}} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix} \end{aligned}$$

It follows that $T(3 + 2x - x^2) = 0 \cdot 1 + 8 \cdot x - 4 \cdot x^2 = 8x - 4x^2$. [Verify this by computing $T(3 + 2x - x^2) = 3 + 2(2x - 1) - (2x - 1)^2$ directly.]

The matrix of a linear transformation can sometimes be used in surprising ways. Example 6.79 shows its application to a traditional calculus problem.

Example 6.79

Let $\mathscr{D}$ be the vector space of all differentiable functions. Consider the subspace W of $\mathscr{D}$ given by $W = \text{span}(e^{3x}, xe^{3x}, x^2e^{3x})$. Since the set $\mathcal{B} = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ is linearly independent (why?), it is a basis for W.

(a) Show that the differential operator D maps W into itself.

(b) Find the matrix of D with respect to $\mathcal{B}$.

(c) Compute the derivative of $5e^{3x} + 2xe^{3x} - x^2e^{3x}$ indirectly, using Theorem 6.26, and verify it using part (a).

Solution (a) Applying D to a general element of W, we see that

$$D(ae^{3x} + bxe^{3x} + cx^2e^{3x}) = (3a + b)e^{3x} + (3b + 2c)xe^{3x} + 3cx^2e^{3x}$$

(check this), which is again in W.

(b) Using the formula in part (a), we see that

$$D(e^{3x}) = 3e^{3x}, \quad D(xe^{3x}) = e^{3x} + 3xe^{3x}, \quad D(x^2e^{3x}) = 2xe^{3x} + 3x^2e^{3x}$$

so

$$[D(e^{3x})]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad [D(xe^{3x})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad [D(x^2e^{3x})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

It follows that

$$[D]_{\mathcal{B}} = [[D(e^{3x})]_{\mathcal{B}} \mid [D(xe^{3x})]_{\mathcal{B}} \mid [D(x^2e^{3x})]_{\mathcal{B}}] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

(c) For $f(x) = 5e^{3x} + 2xe^{3x} - x^2e^{3x}$, we see by inspection that

$$[f(x)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

Hence, by Theorem 6.26, we have

$$[D(f(x))]_{\mathcal{B}} = [D]_{\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ -3 \end{bmatrix}$$

which, in turn, implies that $f'(x) = D(f(x)) = 17e^{3x} + 4xe^{3x} - 3x^2e^{3x}$, in agreement with the formula in part (a).

Remark The point of Example 6.79 is not that this method is easier than direct differentiation. Indeed, once the formula in part (a) has been established, there is little to do. What is significant is that matrix methods can be used at all in what appears, on the surface, to be a calculus problem. We will explore this idea further in Example 6.83.

Example 6.80

Let V be an n-dimensional vector space and let I be the identity transformation on V. What is the matrix of I with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V if $\mathcal{B} = \mathcal{C}$ (including the order of the basis vectors)? What if $\mathcal{B} \neq \mathcal{C}$?

Solution Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then $I(\mathbf{v}_1) = \mathbf{v}_1, \ldots, I(\mathbf{v}_n) = \mathbf{v}_n$, so

$$[I(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1, \quad [I(\mathbf{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_2, \quad \ldots, \quad [I(\mathbf{v}_n)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{e}_n$$

and, if $\mathcal{B} = \mathcal{C}$,

$$\begin{aligned} [I]_{\mathcal{B}} &= [\,[I(\mathbf{v}_1)]_{\mathcal{B}} \mid [I(\mathbf{v}_2)]_{\mathcal{B}} \mid \cdots \mid [I(\mathbf{v}_n)]_{\mathcal{B}}\,] \\ &= [\,\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n\,] \\ &= I_n \end{aligned}$$

the $n \times n$ identity matrix. (This is what you expected, isn't it?)

In the case $\mathcal{B} \neq \mathcal{C}$, we have

$$[I(\mathbf{v}_1)]_{\mathcal{C}} = [\mathbf{v}_1]_{\mathcal{C}}, \quad \ldots, \quad [I(\mathbf{v}_n)]_{\mathcal{C}} = [\mathbf{v}_n]_{\mathcal{C}}$$

so

$$\begin{aligned} [I]_{\mathcal{C} \leftarrow \mathcal{B}} &= [\,[\mathbf{v}_1]_{\mathcal{C}} \mid \cdots \mid [\mathbf{v}_n]_{\mathcal{C}}\,] \\ &= P_{\mathcal{C} \leftarrow \mathcal{B}} \end{aligned}$$

the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Matrices of Composite and Inverse Linear Transformations

We now generalize Theorems 3.32 and 3.33 to get a theorem that will allow us to easily find the inverse of a linear transformation between finite-dimensional vector spaces (if it exists).

Theorem 6.27 Let U, V, and W be finite-dimensional vector spaces with bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, respectively. Let $T: U \to V$ and $S: V \to W$ be linear transformations. Then

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}$$

Remarks

- In words, this theorem says, "The matrix of the composite is the product of the matrices."
- Notice how the "inner subscripts" $\mathcal{C}$ must match and appear to cancel each other out, leaving the "outer subscripts" in the form $\mathcal{D} \leftarrow \mathcal{B}$.

Proof We will show that corresponding columns of the matrices $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}}$ and $[S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ are the same. Let $\mathbf{v}_i$ be the ith basis vector in $\mathcal{B}$. Then the ith column of $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}}$ is

$$\begin{aligned} [(S \circ T)(\mathbf{v}_i)]_{\mathcal{D}} &= [S(T(\mathbf{v}_i))]_{\mathcal{D}} \\ &= [S]_{\mathcal{D} \leftarrow \mathcal{C}}[T(\mathbf{v}_i)]_{\mathcal{C}} \\ &= [S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{v}_i]_{\mathcal{B}} \end{aligned}$$

by two applications of Theorem 6.26. But $[\mathbf{v}_i]_{\mathcal{B}} = \mathbf{e}_i$ (why?), so

$$[S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{v}_i]_{\mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}\mathbf{e}_i$$

is the ith column of the matrix $[S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}$. Therefore, the ith columns of $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}}$ and $[S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ are the same, as we wished to prove.

Example 6.81 Use matrix methods to compute $(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}$ for the linear transformations S and T of Example 6.56.

Solution Recall that $T: \mathbb{R}^2 \to \mathscr{P}_1$ and $S: \mathscr{P}_1 \to \mathscr{P}_2$ are defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x \quad \text{and} \quad S(a + bx) = ax + bx^2$$

Choosing the standard bases $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$ for $\mathbb{R}^2$, $\mathscr{P}_1$, and $\mathscr{P}_2$, respectively, we see that

$$[T]_{\mathcal{E}' \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad [S]_{\mathcal{E}'' \leftarrow \mathcal{E}'} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(Verify these.) By Theorem 6.27, the matrix of $S \circ T$ with respect to $\mathcal{E}$ and $\mathcal{E}''$ is

$$\begin{aligned} [(S \circ T)]_{\mathcal{E}'' \leftarrow \mathcal{E}} &= [S]_{\mathcal{E}'' \leftarrow \mathcal{E}'}[T]_{\mathcal{E}' \leftarrow \mathcal{E}} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Thus, by Theorem 6.26,

$$\left[(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}\right]_{\mathcal{E}''} = [(S \circ T)]_{\mathcal{E}'' \leftarrow \mathcal{E}} \begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{E}}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ a + b \end{bmatrix}$$

Consequently, $(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = ax + (a + b)x^2$, which agrees with the solution to Example 6.56.

In Theorem 6.24, we proved that a linear transformation is invertible if and only if it is one-to-one and onto (i.e., if it is an isomorphism). When the vector spaces involved are finite-dimensional, we can use the matrix methods we have developed to find the inverse of such a linear transformation.

Theorem 6.28 Let $T : V \to W$ be a linear transformation between n-dimensional vector spaces V and W and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Then T is invertible if and only if the matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible. In this case,

$$([T]_{\mathcal{C} \leftarrow \mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B} \leftarrow \mathcal{C}}$$

Proof Observe that the matrices of T and T^{-1} (if T is invertible) are $n \times n$. If T is invertible, then $T^{-1} \circ T = I_V$. Applying Theorem 6.27, we have

$$I_n = [I_V]_{\mathcal{B}} = [T^{-1} \circ T]_{\mathcal{B}}$$
$$= [T^{-1}]_{\mathcal{B} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}}$$

This shows that $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible and that $([T]_{\mathcal{C} \leftarrow \mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B} \leftarrow \mathcal{C}}$.

Conversely, assume that $A = [T]_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible. To show that T is invertible, it is enough to show that $\ker(T) = \{\mathbf{0}\}$. (Why?) To this end, let $\mathbf{v}$ be in the kernel of T. Then $T(\mathbf{v}) = \mathbf{0}$, so

$$A[\mathbf{v}]_{\mathcal{B}} = [T]_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}} = [\mathbf{0}]_{\mathcal{C}} = \mathbf{0}$$

which means that $[\mathbf{v}]_{\mathcal{B}}$ is in the null space of the invertible matrix A. By the Fundamental Theorem, this implies that $[\mathbf{v}]_{\mathcal{B}} = \mathbf{0}$, which, in turn, implies that $\mathbf{v} = \mathbf{0}$, as required.

Example 6.82 In Example 6.70, the linear transformation $T : \mathbb{R}^2 \to \mathcal{P}_1$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

was shown to be one-to-one and onto and hence invertible. Find T^{-1}.

Solution In Example 6.81, we found the matrix of T with respect to the standard bases $\mathcal{E}$ and $\mathcal{E}'$ for $\mathbb{R}^2$ and $\mathcal{P}_1$, respectively, to be

$$[T]_{\mathcal{E}' \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

By Theorem 6.28, it follows that the matrix of T^{-1} with respect to $\mathcal{E}'$ and $\mathcal{E}$ is

$$[T^{-1}]_{\mathcal{E} \leftarrow \mathcal{E}'} = ([T]_{\mathcal{E}' \leftarrow \mathcal{E}})^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

By Theorem 6.26,

$$\begin{aligned} [T^{-1}(a + bx)]_{\mathcal{E}} &= [T^{-1}]_{\mathcal{E} \leftarrow \mathcal{E}'}[a + bx]_{\mathcal{E}'} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a \\ b - a \end{bmatrix} \end{aligned}$$

This means that

$$T^{-1}(a + bx) = a\mathbf{e}_1 + (b - a)\mathbf{e}_2 = \begin{bmatrix} a \\ b - a \end{bmatrix}$$

(Note that the choice of the standard basis makes this last calculation virtually irrelevant.)

The next example, a continuation of Example 6.79, shows that matrices can be used in certain integration problems in calculus. The specific integral we consider is usually evaluated in a calculus course by means of two applications of integration by parts. Contrast this approach with our method.

Example 6.83

Show that the differential operator, restricted to the subspace $W = \text{span}(e^{3x}, xe^{3x}, x^2e^{3x})$ of $\mathcal{D}$, is invertible, and use this fact to find the integral

$$\int x^2 e^{3x}\, dx$$

Solution In Example 6.79, we found the matrix of D with respect to the basis $\mathcal{B} = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ of W to be

$$[D]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

By Theorem 6.28, therefore, D is invertible on W, and the matrix of D^{-1} is

$$[D^{-1}]_{\mathcal{B}} = ([D]_{\mathcal{B}})^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{9} & \frac{2}{27} \\ 0 & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Since integration is *antidifferentiation*, this is the matrix corresponding to integration on W. We want to integrate the function x^2e^{3x} whose coordinate vector is

$$[x^2e^{3x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Consequently, by Theorem 6.26,

$$\begin{aligned}\left[\int x^2e^{3x}\,dx\right]_{\mathcal{B}} &= [D^{-1}(x^2e^{3x})]_{\mathcal{B}} \\ &= [D^{-1}]_{\mathcal{B}}[x^2e^{3x}]_{\mathcal{B}} \\ &= \begin{bmatrix} \frac{1}{3} & -\frac{1}{9} & \frac{2}{27} \\ 0 & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{27} \\ -\frac{2}{9} \\ \frac{1}{3} \end{bmatrix}\end{aligned}$$

It follows that

$$\int x^2e^{3x}\,dx = \tfrac{2}{27}e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{1}{3}x^2e^{3x}$$

(To be fully correct, we need to add a constant of integration. It does not show up here because we are working with *linear* transformations, which must send zero vectors to zero vectors, forcing the constant of integration to be zero as well.)

Warning In general, differentiation is *not* an invertible transformation. (See Exercise 22.) What the preceding example shows is that, suitably restricted, it sometimes is. Exercises 27–30 explore this idea further.

Change of Basis and Similarity

Suppose $T: V \to V$ is a linear transformation and $\mathcal{B}$ and $\mathcal{C}$ are two different bases for V. It is natural to wonder how, if at all, the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are related. It turns out that the answer to this question is quite satisfying and relates to some questions we first considered in Chapter 4.

Figure 6.12 suggests one way to address this problem. Chasing the arrows around the diagram from the upper left-hand corner to the lower right-hand corner in two different, but equivalent, ways shows that $I \circ T = T \circ I$, something we already knew, since both are equal to T. However, if the "upper" version of T is with respect to the

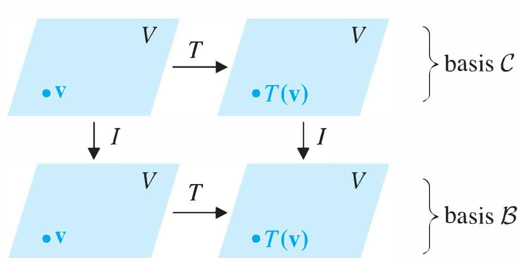


Figure 6.12
$I \circ T = T \circ I$

basis $\mathcal{C}$ and the "lower" version is with respect to $\mathcal{B}$, then $T = I \circ T = T \circ I$ is with respect to $\mathcal{C}$ in its domain and with respect to $\mathcal{B}$ in its codomain. Thus, the matrix of T in this case is $[T]_{\mathcal{B}\leftarrow\mathcal{C}}$. But

$$[T]_{\mathcal{B}\leftarrow\mathcal{C}} = [I \circ T]_{\mathcal{B}\leftarrow\mathcal{C}} = [I]_{\mathcal{B}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{C}}$$

and

$$[T]_{\mathcal{B}\leftarrow\mathcal{C}} = [T \circ I]_{\mathcal{B}\leftarrow\mathcal{C}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}}[I]_{\mathcal{B}\leftarrow\mathcal{C}}$$

Therefore, $[I]_{\mathcal{B}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{C}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}}[I]_{\mathcal{B}\leftarrow\mathcal{C}}$.

From Example 6.80, we know that $[I]_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$, the (invertible) change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$. If we denote this matrix by P, then we also have

$$P^{-1} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = P_{\mathcal{C}\leftarrow\mathcal{B}}$$

With this notation,

$$P[T]_{\mathcal{C}\leftarrow\mathcal{C}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}}P$$

so

$$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P^{-1}[T]_{\mathcal{B}\leftarrow\mathcal{B}}P \quad \text{or} \quad [T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$

Thus, the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar, in the terminology of Section 4.4.

We summarize the foregoing discussion as a theorem.

Theorem 6.29 Let V be a finite-dimensional vector space with bases $\mathcal{B}$ and $\mathcal{C}$ and let $T : V \to V$ be a linear transformation. Then

$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$

where P is the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

Remark As an aid in remembering that P must be the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$, and not $\mathcal{B}$ to $\mathcal{C}$, it is instructive to look at what Theorem 6.29 says when written in full detail. As shown below, the "inner subscripts" must be the same (all $\mathcal{B}$s) and must appear to cancel, leaving the "outer subscripts," which are both $\mathcal{C}$s.

$$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}$$

Same Same

Same

Theorem 6.29 is often used when we are trying to find a basis with respect to which the matrix of a linear transformation is particularly simple. For example, we can ask whether there is a basis $\mathcal{C}$ of V such that the matrix $[T]_{\mathcal{C}}$ of $T : V \to V$ is a diagonal matrix. Example 6.84 illustrates this application.

Example 6.84 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ 2x + 2y \end{bmatrix}$$

If possible, find a basis $\mathcal{C}$ for $\mathbb{R}^2$ such that the matrix of T with respect to $\mathcal{C}$ is diagonal.

Solution The matrix of T with respect to the standard basis $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

This matrix is diagonalizable, as we saw in Example 4.24. Indeed, if

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

then $P^{-1}[T]_{\mathcal{E}}P = D$. If we let $\mathcal{C}$ be the basis of $\mathbb{R}^2$ consisting of the columns of P, then P is the change-of-basis matrix $P_{\mathcal{E}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{E}$. By Theorem 6.29,

$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{E}}P = D$$

so the matrix of T with respect to the basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$ is diagonal.

Remarks

- It is easy to check that the solution above is correct by computing $[T]_{\mathcal{C}}$ directly. We find that

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Thus, the coordinate vectors that form the columns of $[T]_{\mathcal{C}}$ are

$$\left[T\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \text{and} \quad \left[T\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

in agreement with our solution above.

- The general procedure for a problem like Example 6.84 is to take the standard matrix $[T]_{\mathcal{E}}$ and determine whether it is diagonalizable by finding bases for its eigenspaces, as in Chapter 4. The solution then proceeds exactly as in the preceding example.

Example 6.84 motivates the following definition.

Definition Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear transformation. Then T is called ***diagonalizable*** if there is a basis $\mathcal{C}$ for V such that the matrix $[T]_{\mathcal{C}}$ is a diagonal matrix.

It is not hard to show that if $\mathcal{B}$ is *any* basis for V, then T is diagonalizable if and only if the matrix $[T]_{\mathcal{B}}$ is diagonalizable. This is essentially what we did, for a special case, in the last example. You are asked to prove this result in general in Exercise 42.

Sometimes it is easiest to write down the matrix of a linear transformation with respect to a "nonstandard" basis. We can then reverse the process of Example 6.84 to find the standard matrix. We illustrate this idea by revisiting Example 3.59.

Example 6.85

Let ℓ be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Find the standard matrix of the projection onto ℓ.

Solution Let T denote the projection. There is no harm in assuming that $\mathbf{d}$ is a unit vector (i.e., $d_1^2 + d_2^2 = 1$), since any nonzero multiple of $\mathbf{d}$ can serve as a direction vector for ℓ. Let $\mathbf{d}' = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$ so that $\mathbf{d}$ and $\mathbf{d}'$ are orthogonal. Since $\mathbf{d}'$ is also a unit vector, the set $\mathcal{D} = \{\mathbf{d}, \mathbf{d}'\}$ is an orthonormal basis for $\mathbb{R}^2$.

As Figure 6.13 shows, $T(\mathbf{d}) = \mathbf{d}$ and $T(\mathbf{d}') = \mathbf{0}$. Therefore,

$$[T(\mathbf{d})]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{d}')]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

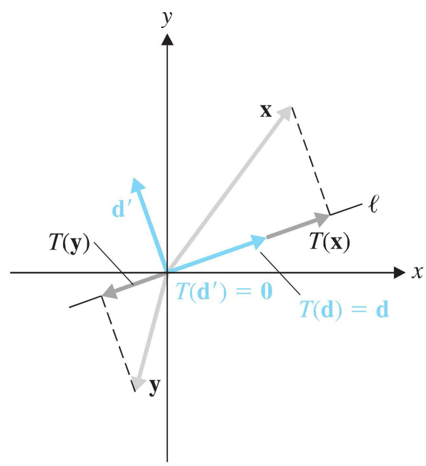


Figure 6.13
Projection onto ℓ

so
$$[T]_{\mathcal{D}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The change-of-basis matrix from $\mathcal{D}$ to the standard basis $\mathcal{E}$ is

$$P_{\mathcal{E}\leftarrow\mathcal{D}} = \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}$$

so the change-of-basis matrix from $\mathcal{E}$ to $\mathcal{D}$ is

$$P_{\mathcal{D}\leftarrow\mathcal{E}} = (P_{\mathcal{E}\leftarrow\mathcal{D}})^{-1} = \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}^{-1} = \begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix}$$

By Theorem 6.29, then, the standard matrix of T is

$$\begin{aligned} [T]_{\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{D}\leftarrow\mathcal{E}} \\ &= \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix} \\ &= \begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix} \end{aligned}$$

which agrees with part (b) of Example 3.59.

Example 6.86

Let $T: \mathscr{P}_2 \to \mathscr{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1)$$

(a) Find the matrix of T with respect to the basis $\mathcal{B} = \{1 + x, 1 - x, x^2\}$ of $\mathscr{P}_2$.

(b) Show that T is diagonalizable and find a basis $\mathcal{C}$ for $\mathscr{P}_2$ such that $[T]_{\mathcal{C}}$ is a diagonal matrix.

Solution (a) In Example 6.78, we found that the matrix of T with respect to the standard basis $\mathcal{E} = \{1, x, x^2\}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

The change-of-basis matrix from $\mathcal{B}$ to $\mathcal{E}$ is

$$P = P_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that the matrix of T with respect to $\mathcal{B}$ is

$$\begin{aligned} [T]_{\mathcal{B}} &= P^{-1}[T]_{\mathcal{E}}P \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ -1 & 2 & \frac{5}{2} \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

(Check this.)

(b) The eigenvalues of $[T]_{\mathcal{E}}$ are 1, 2, and 4 (why?), so we know from Theorem 4.25 that $[T]_{\mathcal{E}}$ is diagonalizable. Eigenvectors corresponding to these eigenvalues are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

respectively. Therefore, setting

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

we have $P^{-1}[T]_{\mathcal{E}}P = D$. Furthermore, P is the change-of-basis matrix from a basis $\mathcal{C}$ to $\mathcal{E}$, and the columns of P are thus the coordinate vectors of $\mathcal{C}$ in terms of $\mathcal{E}$. It follows that

$$\mathcal{C} = \{1, -1 + x, 1 - 2x + x^2\}$$

and $[T]_{\mathcal{C}} = D$.

The preceding ideas can be generalized to relate the matrices $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ and $[T]_{\mathcal{C}'\leftarrow\mathcal{B}'}$ of a linear transformation $T : V \to W$, where $\mathcal{B}$ and $\mathcal{B}'$ are bases for V and $\mathcal{C}$ and $\mathcal{C}'$ are bases for W. (See Exercise 44.)

We conclude this section by revisiting the Fundamental Theorem of Invertible Matrices and incorporating some results from this chapter.

Theorem 6.30 **The Fundamental Theorem of Invertible Matrices: Version 4**

Let A be an $n \times n$ matrix and let $T : V \to W$ be a linear transformation whose matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively, is A. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\operatorname{rank}(A) = n$
g. $\operatorname{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A.
p. T is invertible.
q. T is one-to-one.
r. T is onto.
s. $\ker(T) = \{\mathbf{0}\}$
t. $\operatorname{range}(T) = W$

Proof The equivalence (q) $\Leftrightarrow$ (s) is Theorem 6.20, and (r) $\Leftrightarrow$ (t) is the definition of onto. Since A is $n \times n$, we must have $\dim V = \dim W = n$. From Theorems 6.21 and 6.24, we get (p) $\Leftrightarrow$ (q) $\Leftrightarrow$ (r). Finally, we connect the last five statements to the others by Theorem 6.28, which implies that (a) $\Leftrightarrow$ (p).

Exercises 6.6

In Exercises 1–12, find the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ of the linear transformation $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

1. $T : \mathcal{P}_1 \to \mathcal{P}_1$ defined by $T(a + bx) = b - ax$, $\mathcal{B} = \mathcal{C} = \{1, x\}$, $\mathbf{v} = p(x) = 4 + 2x$

2. $T : \mathcal{P}_1 \to \mathcal{P}_1$ defined by $T(a + bx) = b - ax$, $\mathcal{B} = \{1 + x, 1 - x\}$, $\mathcal{C} = \{1, x\}$, $\mathbf{v} = p(x) = 4 + 2x$

3. $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(p(x)) = p(x + 2)$, $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{1, x + 2, (x + 2)^2\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

4. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x+2)$, $\mathcal{B} = \{1, x+2, (x+2)^2\}$, $\mathcal{C} = \{1, x, x^2\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

5. $T: \mathscr{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$, $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

6. $T: \mathscr{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$, $\mathcal{B} = \{x^2, x, 1\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

7. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + 2b \\ -a \\ b \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\},$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathbf{v} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

8. Repeat Exercise 7 with $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

9. $T: M_{22} \to M_{22}$ defined by $T(A) = A^T$, $\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

10. Repeat Exercise 9 with $\mathcal{B} = \{E_{22}, E_{21}, E_{12}, E_{11}\}$ and $\mathcal{C} = \{E_{12}, E_{21}, E_{22}, E_{11}\}$.

11. $T: M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

12. $T: M_{22} \to M_{22}$ defined by $T(A) = A - A^T$, $\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

13. Consider the subspace W of $\mathscr{D}$, given by $W = \operatorname{span}(\sin x, \cos x)$.

(a) Show that the differential operator D maps W into itself.

(b) Find the matrix of D with respect to $\mathcal{B} = \{\sin x, \cos x\}$.

(c) Compute the derivative of $f(x) = 3 \sin x - 5 \cos x$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

14. Consider the subspace W of $\mathscr{D}$, given by $W = \operatorname{span}(e^{2x}, e^{-2x})$.

(a) Show that the differential operator D maps W into itself.

(b) Find the matrix of D with respect to $\mathcal{B} = \{e^{2x}, e^{-2x}\}$.

(c) Compute the derivative of $f(x) = e^{2x} - 3e^{-2x}$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

15. Consider the subspace W of $\mathscr{D}$, given by $W = \operatorname{span}(e^{2x}, e^{2x} \cos x, e^{2x} \sin x)$.

(a) Find the matrix of D with respect to $\mathcal{B} = \{e^{2x}, e^{2x} \cos x, e^{2x} \sin x\}$.

(b) Compute the derivative of $f(x) = 3e^{2x} - e^{2x} \cos x + 2e^{2x} \sin x$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

16. Consider the subspace W of $\mathscr{D}$, given by $W = \operatorname{span}(\cos x, \sin x, x \cos x, x \sin x)$.

(a) Find the matrix of D with respect to $\mathcal{B} = \{\cos x, \sin x, x \cos x, x \sin x\}$.

(b) Compute the derivative of $f(x) = \cos x + 2x \cos x$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

In Exercises 17 and 18, $T: U \to V$ and $S: V \to W$ are linear transformations and $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ are bases for U, V, and W, respectively. Compute $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}}$ in two ways: (a) by finding $S \circ T$ directly and then computing its matrix and (b) by finding the matrices of S and T separately and using Theorem 6.27.

17. $T: \mathscr{P}_1 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$, $S: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - 2b \\ 2a - b \end{bmatrix}$, $\mathcal{B} = \{1, x\}$, $\mathcal{C} = \mathcal{D} = \{\mathbf{e}_1, \mathbf{e}_2\}$

18. $T: \mathscr{P}_1 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x+1)$, $S: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $S(p(x)) = p(x+1)$, $\mathcal{B} = \{1, x\}$, $\mathcal{C} = \mathcal{D} = \{1, x, x^2\}$

In Exercises 19–26, determine whether the linear transformation T is invertible by considering its matrix with respect to the standard bases. If T is invertible, use Theorem 6.28 and the method of Example 6.82 to find T^{-1}.

19. T in Exercise 1

20. T in Exercise 5

21. T in Exercise 3

22. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p'(x)$

23. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x) + p'(x)$

24. $T: M_{22} \to M_{22}$ defined by $T(A) = AB$, where

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

25. T in Exercise 11 **26.** T in Exercise 12

In Exercises 27–30, use the method of Example 6.83 to evaluate the given integral.

27. $\int (\sin x - 3\cos x)\, dx$. (See Exercise 13.)

28. $\int 5e^{-2x}\, dx$. (See Exercise 14.)

29. $\int (e^{2x}\cos x - 2e^{2x}\sin x)\, dx$. (See Exercise 15.)

30. $\int (x\cos x + x\sin x)\, dx$. (See Exercise 16.)

In Exercises 31–36, a linear transformation $T: V \to V$ is given. If possible, find a basis $\mathcal{C}$ for V such that the matrix $[T]_{\mathcal{C}}$ of T with respect to $\mathcal{C}$ is diagonal.

31. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4b \\ a + 5b \end{bmatrix}$

32. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$

33. $T: \mathscr{P}_1 \to \mathscr{P}_1$ defined by $T(a + bx) = (4a + 2b) + (a + 3b)x$

34. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x + 1)$

35. $T: \mathscr{P}_1 \to \mathscr{P}_1$ defined by $T(p(x)) = p(x) + xp'(x)$

36. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(3x + 2)$

37. Let ℓ be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Use the method of Example 6.85 to find the standard matrix of a reflection in ℓ.

38. Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$. Use the method of Example 6.85 to find the standard matrix of an orthogonal projection onto W. Verify that your answer is correct by using it to compute the orthogonal projection of $\mathbf{v}$ onto W, where

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Compare your answer with Example 5.11. [*Hint:* Find an orthogonal decomposition of $\mathbb{R}^3$ as $\mathbb{R}^3 = W + W^\perp$ using an orthogonal basis for W. See Example 5.3.]

39. Let $T: V \to W$ be a linear transformation between finite-dimensional vector spaces and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Show that the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is unique. That is, if A is a matrix such that $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for all $\mathbf{v}$ in V, then $A = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$. [*Hint:* Find values of $\mathbf{v}$ that will show this, one column at a time.]

In Exercises 40–45, let $T: V \to W$ be a linear transformation between finite-dimensional vector spaces V and W. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively, and let $A = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

40. Show that $\text{nullity}(T) = \text{nullity}(A)$.

41. Show that $\text{rank}(T) = \text{rank}(A)$.

42. If $V = W$ and $\mathcal{B} = \mathcal{C}$, show that T is diagonalizable if and only if A is diagonalizable.

43. Use the results of this section to give a matrix-based proof of the Rank Theorem (Theorem 6.19).

44. If $\mathcal{B}'$ and $\mathcal{C}'$ are also bases for V and W, respectively, what is the relationship between $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ and $[T]_{\mathcal{C}'\leftarrow\mathcal{B}'}$? Prove your assertion.

45. If $\dim V = n$ and $\dim W = m$, prove that $\mathscr{L}(V, W) \cong M_{mn}$. (See the exercises for Section 6.4.) [*Hint:* Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Show that the mapping $\varphi(T) = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$, for T in $\mathscr{L}(V, W)$, defines a linear transformation $\varphi: \mathscr{L}(V, W) \to M_{mn}$ that is an isomorphism.]

46. If V is a vector space, then the ***dual space*** of V is the vector space $V^* = \mathscr{L}(V, \mathbb{R})$. Prove that if V is finite-dimensional, then $V^* \cong V$.

Exploration

Tilings, Lattices, and the Crystallographic Restriction

Repeating patterns are frequently found in nature and in art. The molecular structure of crystals often exhibits repetition, as do the tilings and mosaics found in the artwork of many cultures. *Tiling* (or *tessellation*) is covering of a plane by shapes that do not overlap and leave no gaps. The Dutch artist M. C. Escher (1898–1972) produced many works in which he explored the possibility of tiling a plane using fanciful shapes (Figure 6.14).

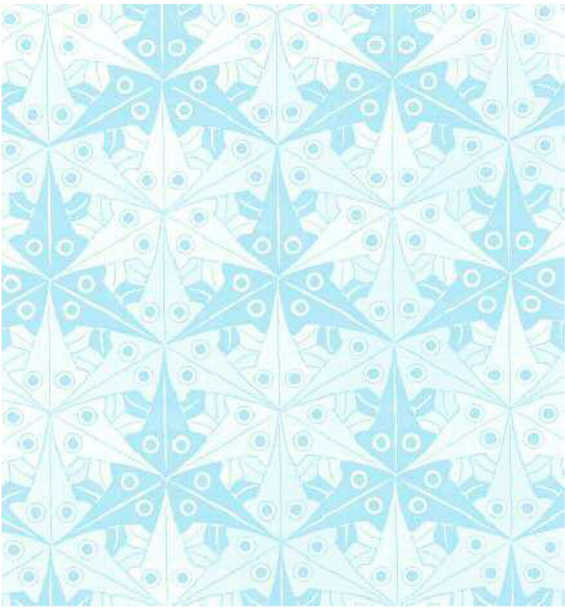


Figure 6.14
M. C. Escher's "Symmetry Drawing E103"

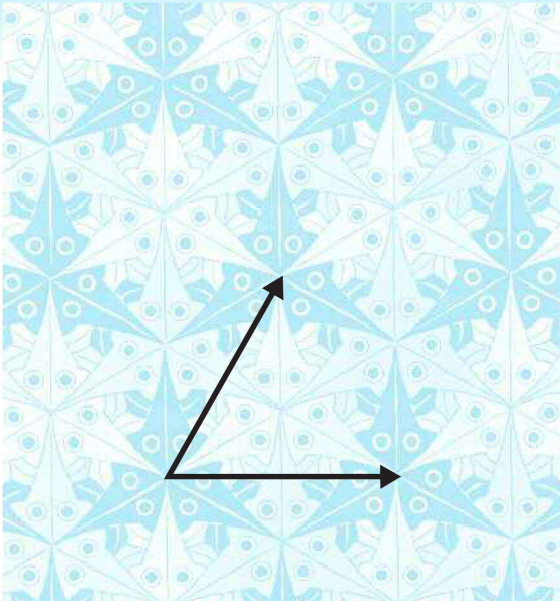


Figure 6.15
Invariance under translation
M. C. Escher's "Symmetry Drawing E103"

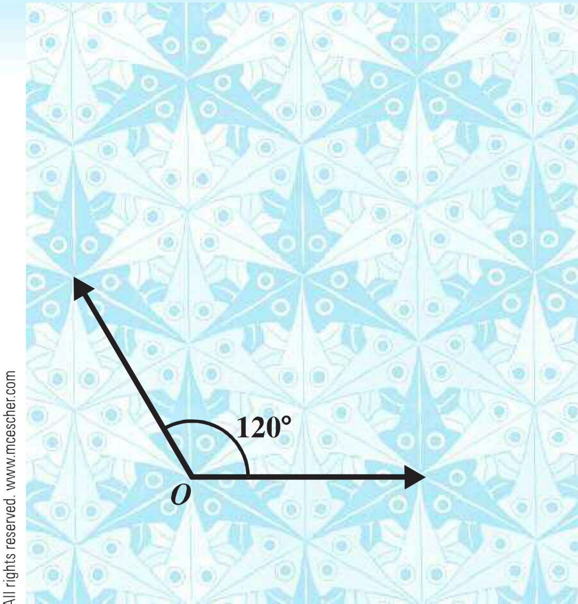


Figure 6.17
Rotational symmetry
M. C. Escher's "Symmetry Drawing E103"

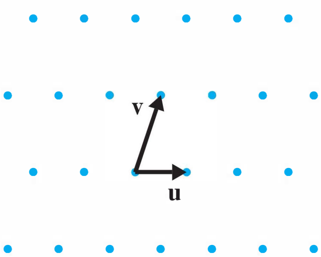


Figure 6.16
A lattice

In this exploration, we will be interested in patterns such as those in Figure 6.14, which we assume to be infinite and repeating in all directions of the plane. Such a pattern has the property that it can be shifted (or *translated*) in at least two directions (corresponding to two linearly independent vectors) so that it appears not to have been moved at all. We say that the pattern is *invariant* under translations and has ***translational symmetry*** in these directions. For example, the pattern in Figure 6.14 has translational symmetry in the directions shown in Figure 6.15.

If a pattern has translational symmetry in two directions, it has translational symmetry in infinitely many directions.

1. Let the two vectors shown in Figure 6.15 be denoted by $\mathbf{u}$ and $\mathbf{v}$. Show that the pattern in Figure 6.14 is invariant under translation by any *integer* linear combination of $\mathbf{u}$ and $\mathbf{v}$—that is, by any vector of the form $a\mathbf{u} + b\mathbf{v}$, where a and b are integers.

For any two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, the set of points determined by all integer linear combinations of $\mathbf{u}$ and $\mathbf{v}$ is called a ***lattice.*** Figure 6.16 shows an example of a lattice.

2. Draw the lattice corresponding to the vectors $\mathbf{u}$ and $\mathbf{v}$ of Figure 6.15.

Figure 6.14 also exhibits ***rotational symmetry.*** That is, it is possible to rotate the entire pattern about some point and have it appear unchanged. We say that it is *invariant* under such a rotation. For example, the pattern of Figure 6.14 is invariant under a rotation of 120° about the point O, as shown in Figure 6.17. We call O a ***center*** of rotational symmetry (or a ***rotation center***).

Note that if a pattern is based on an underlying lattice, then any symmetries of the pattern must also be possessed by the lattice.

3. Explain why, if a point O is a rotation center through an angle θ, then it is a rotation center through every integer multiple of θ. Deduce that if $0 < \theta \leq 360°$, then $360/\theta$ must be an integer. (If $360/\theta = n$, we say the pattern or lattice has ***n-fold*** rotational symmetry.)

4. What is the smallest positive angle of rotational symmetry for the lattice in Problem 2? Does the pattern in Figure 6.14 also have rotational symmetry through this angle?

5. Take various values of θ such that $0 < \theta \leq 360°$ and $360/\theta$ is an integer. Try to draw a lattice that has rotational symmetry through the angle θ. In particular, can you draw a lattice with eight-fold rotational symmetry?

We will show that values of θ that are possible angles of rotational symmetry for a lattice are severely restricted. The technique we will use is to consider rotation transformations in terms of different bases. Accordingly, let R_θ denote a rotation about the origin through an angle θ and let $\mathcal{E}$ be the standard basis for $\mathbb{R}^2$. Then the standard matrix of R_θ is

$$[R_\theta]_{\mathcal{E}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

6. Referring to Problems 2 and 4, take the origin to be at the tails of $\mathbf{u}$ and $\mathbf{v}$.

(a) What is the actual (i.e., numerical) value of $[R_\theta]_{\mathcal{E}}$ in this case?

(b) Let $\mathcal{B}$ be the basis $\{\mathbf{u}, \mathbf{v}\}$. Compute the matrix $[R_\theta]_{\mathcal{B}}$.

7. In general, let $\mathbf{u}$ and $\mathbf{v}$ be any two linearly independent vectors in $\mathbb{R}^2$ and suppose that the lattice determined by $\mathbf{u}$ and $\mathbf{v}$ is invariant under a rotation through an angle θ. If $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$, show that the matrix of R_θ with respect to $\mathcal{B}$ must have the form

$$[R_\theta]_{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, and d are *integers*.

8. In the terminology and notation of Problem 7, show that $2\cos\theta$ must be an integer. [*Hint:* Use Exercise 35 in Section 4.4 and Theorem 6.29.]

9. Using Problem 8, make a list of all possible values of θ, with $0 < \theta \leq 360°$, that can be angles of rotational symmetry of a lattice. Record the corresponding values of n, where $n = 360/\theta$, to show that a lattice can have n-fold rotational symmetry if and only if $n = 1, 2, 3, 4$, or 6. This result, known as the ***crystallographic restriction,*** was first proved by W. Barlow in 1894.

10. In the library or on the Internet, see whether you can find an Escher tiling for each of the five possible types of rotational symmetry—that is, where the *smallest* angle of rotational symmetry of the pattern is one of those specified by the crystallographic restriction.

6.7 Applications

Homogeneous Linear Differential Equations

In Exercises 69–72 in Section 4.6, we showed that if $y = y(t)$ is a twice-differentiable function that satisfies the differential equation

$$y'' + ay' + by = 0 \tag{1}$$

then y is of the form

$$y = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$$

if λ_1 and λ_2 are *distinct* roots of the associated characteristic equation $\lambda^2 + a\lambda + b = 0$. (The case where $\lambda_1 = \lambda_2$ was left unresolved.) Example 6.12 and Exercise 20 in this section show that the set of solutions to Equation (1) forms a subspace of $\mathscr{F}$, the vector space of functions. In this section, we pursue these ideas further, paying particular attention to the role played by vector spaces, bases, and dimension.

To set the stage, we consider a simpler class of examples. A differential equation of the form

$$y' + ay = 0 \tag{2}$$

is called a ***first-order, homogeneous, linear differential equation.*** ("First-order" refers to the fact that the highest derivative that is involved is a first derivative, and "homogeneous" means that the right-hand side is zero. Do you see why the equation is "linear"?) A ***solution*** to Equation (2) is a differentiable function $y = y(t)$ that satisfies Equation (2) for all values of t.

It is easy to check that one solution to Equation (2) is $y = e^{-at}$. (Do it.) However, we would like to describe *all* solutions—and this is where vector spaces come in. We have the following theorem.

Theorem 6.31 The set S of all solutions to $y' + ay = 0$ is a subspace of $\mathscr{F}$.

Proof Since the zero function certainly satisfies Equation (2), S is nonempty. Let x and y be two differentiable functions of t that are in S and let c be a scalar. Then

$$x' + ax = 0 \quad \text{and} \quad y' + ay = 0$$

so, using rules for differentiation, we have

$$(x + y)' + a(x + y) = x' + y' + ax + ay = (x' + ax) + (y' + ay) = 0 + 0 = 0$$

and

$$(cy)' + a(cy) = cy' + c(ay) = c(y' + ay) = c \cdot 0 = 0$$

Hence, $x + y$ and cy are also in S, so S is a subspace of $\mathscr{F}$.

Now we will show that S is a one-dimensional subspace of $\mathscr{F}$ and that $\{e^{-at}\}$ is a basis. To this end, let $x = x(t)$ be in S. Then, for all t,

$$x'(t) + ax(t) = 0 \quad \text{or} \quad x'(t) = -ax(t)$$

Define a new function $z(t) = x(t)e^{at}$. Then, by the Chain Rule for differentiation,

$$\begin{aligned} z'(t) &= x(t)ae^{at} + x'(t)e^{at} \\ &= ax(t)e^{at} - ax(t)e^{at} \\ &= 0 \end{aligned}$$

Since z' is identically zero, z must be a constant function—say, $z(t) = k$. But this means that

$$x(t)e^{at} = z(t) = k \quad \text{for all } t$$

so $x(t) = ke^{-at}$. Therefore, all solutions to Equation (2) are scalar multiples of the single solution $y = e^{-at}$. We have proved the following theorem.

Theorem 6.32 If S is the solution space of $y' + ay = 0$, then $\dim S = 1$ and $\{e^{-at}\}$ is a basis for S.

One model for population growth assumes that the growth rate of the population is proportional to the size of the population. This model works well if there are few restrictions (such as limited space, food, or the like) on growth. If the size of the population at time t is $p(t)$, then the growth rate, or rate of change of the population, is its derivative $p'(t)$. Our assumption that the growth rate of the population is proportional to its size can be written as

$$p'(t) = kp(t)$$

where k is the proportionality constant. Thus, p satisfies the differential equation $p' - kp = 0$, so, by Theorem 6.32,

$$p(t) = ce^{kt}$$

for some scalar c. The constants c and k are determined using experimental data.

Example 6.87 The bacterium *Escherichia coli* (or *E. coli,* for short) is commonly found in the intestines of humans and other mammals. It poses severe health risks if it escapes into the environment. Under laboratory conditions, each cell of the bacterium divides into two every 20 minutes. If we start with a single *E. coli* cell, how many will there be after 1 day?

E. coli is mentioned in Michael Crichton's novel *The Andromeda Strain* (New York: Dell, 1969), although the "villain" in that novel was supposedly an alien virus. In real life, *E. coli* contaminated the town water supply of Walkerton, Ontario, in 2000, resulting in seven deaths and causing hundreds of people to become seriously ill.

Solution We do not need to use differential equations to solve this problem, but we will, in order to illustrate the basic method.

To determine c and k, we use the data given in the statement of the problem. If we take 1 unit of time to be 20 minutes, then we are given that $p(0) = 1$ and $p(1) = 2$. Therefore,

$$c = c \cdot 1 = ce^{k \cdot 0} = 1 \quad \text{and} \quad 2 = ce^{k \cdot 1} = e^k$$

It follows that $k = \ln 2$, so

$$p(t) = e^{t \ln 2} = e^{\ln 2^t} = 2^t$$

After 1 day, $t = 72$, so the number of bacteria cells will be $p(72) = 2^{72} \approx 4.72 \times 10^{21}$ (see Figure 6.18).

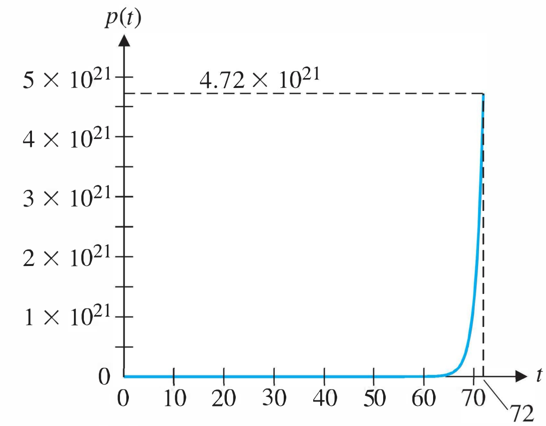


Figure 6.18
Exponential growth

Radioactive substances decay by emitting radiation. If $m(t)$ denotes the mass of the substance at time t, then the rate of decay is $m'(t)$. Physicists have found that the rate of decay of a substance is proportional to its mass; that is,

$$m'(t) = km(t) \quad \text{or} \quad m' - km = 0$$

where k is a negative constant. Applying Theorem 6.32, we have

$$m(t) = ce^{kt}$$

for some constant c. The time required for half of a radioactive substance to decay is called its ***half-life.***

Example 6.88

After 5.5 days, a 100 mg sample of radon-222 decayed to 37 mg.

(a) Find a formula for $m(t)$, the mass remaining after t days.

(b) What is the half-life of radon-222?

(c) When will only 10 mg remain?

Solution (a) From $m(t) = ce^{kt}$, we have

$$100 = m(0) = ce^{k \cdot 0} = c \cdot 1 = c$$

so

$$m(t) = 100e^{kt}$$

With time measured in days, we are given that $m(5.5) = 37$. Therefore,

$$100e^{5.5k} = 37$$

so

$$e^{5.5k} = 0.37$$

Solving for k, we find

$$5.5k = \ln(0.37)$$

so

$$k = \frac{\ln(0.37)}{5.5} \approx -0.18$$

Therefore, $m(t) = 100e^{-0.18t}$.

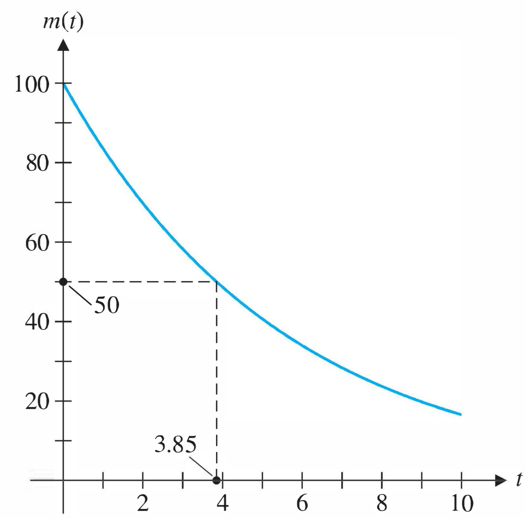


Figure 6.19
Radioactive decay

(b) To find the half-life of radon-222, we need the value of t for which $m(t) = 50$. Solving this equation, we find

$$100e^{-0.18t} = 50$$

so

$$e^{-0.18t} = 0.50$$

Hence,

$$-0.18t = \ln\left(\tfrac{1}{2}\right) = -\ln 2$$

and

$$t = \frac{\ln 2}{0.18} \approx 3.85$$

Thus, radon-222 has a half-life of approximately 3.85 days. (See Figure 6.19.)

(c) We need to determine the value of t such that $m(t) = 10$. That is, we must solve the equation

$$100e^{-0.18t} = 10 \quad \text{or} \quad e^{-0.18t} = 0.1$$

Taking the natural logarithm of both sides yields $-0.18t = \ln 0.1$. Thus,

$$t = \frac{\ln 0.1}{-0.18} \approx 12.79$$

so 10 mg of the sample will remain after approximately 12.79 days.

See *Linear Algebra* by S. H. Friedberg, A. J. Insel, and L. E. Spence (Englewood Cliffs, NJ: Prentice-Hall, 1979).

The solution set S of the second-order differential equation $y'' + ay' + by = 0$ is also a subspace of $\mathscr{F}$ (Exercise 20), and it turns out that the dimension of S is 2. Part (a) of Theorem 6.33, which extends Theorem 6.32, is implied by Theorem 4.40. Our approach here is to use the power of vector spaces; doing so allows us to obtain part (b) of Theorem 6.33 as well, a result that we could not obtain with our previous methods.

Theorem 6.33

Let S be the solution space of

$$y'' + ay' + by = 0$$

and let λ_1 and λ_2 be the roots of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

a. If $\lambda_1 \neq \lambda_2$, then $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ is a basis for S.
b. If $\lambda_1 = \lambda_2$, then $\{e^{\lambda_1 t}, te^{\lambda_1 t}\}$ is a basis for S.

Remarks

- Observe that what the theorem says, in other words, is that the solutions of $y'' + ay' + by = 0$ are of the form

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

in the first case and

$$y = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

in the second case.
- Compare Theorem 6.33 with Theorem 4.38. Linear differential equations and linear recurrence relations have much in common. Although the former belong to *continuous* mathematics and the latter to *discrete* mathematics, there are many parallels.

Proof (a) We first show that $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ is contained in S. Let λ be any root of the characteristic equation and let $f(t) = e^{\lambda t}$. Then

$$f'(t) = \lambda e^{\lambda t} \quad \text{and} \quad f''(t) = \lambda^2 e^{\lambda t}$$

from which it follows that

$$\begin{aligned} f'' + af' + bf &= \lambda^2 e^{\lambda t} + a\lambda e^{\lambda t} + be^{\lambda t} \\ &= (\lambda^2 + a\lambda + b)e^{\lambda t} \\ &= 0 \cdot e^{\lambda t} = 0 \end{aligned}$$

Therefore, f is in S. But, since λ_1 and λ_2 are roots of the characteristic equation, this means that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are in S.

The set $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ is also linearly independent, since if

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = 0$$

then, setting $t = 0$, we have

$$c_1 + c_2 = 0 \quad \text{or} \quad c_2 = -c_1$$

Next, we set $t = 1$ to obtain

$$c_1 e^{\lambda_1} - c_1 e^{\lambda_2} = 0 \quad \text{or} \quad c_1(e^{\lambda_1} - e^{\lambda_2}) = 0$$

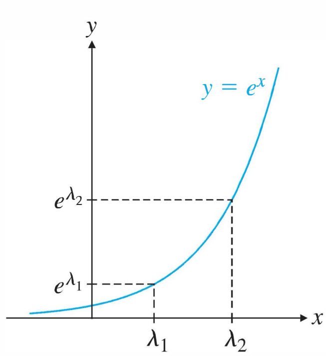


Figure 6.20

But $e^{\lambda_1} - e^{\lambda_2} \neq 0$, since $e^{\lambda_1} - e^{\lambda_2} = 0$ implies that $e^{\lambda_1} = e^{\lambda_2}$, which is clearly impossible if $\lambda_1 \neq \lambda_2$. (See Figure 6.20.) We deduce that $c_1 = 0$ and, hence, $c_2 = 0$, so $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ is linearly independent.

Since $\dim S = 2$, $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ must be a basis for S.

(b) You are asked to prove this property in Exercise 21.

Example 6.89

Find all solutions of $y'' - 5y' + 6y = 0$.

Solution The characteristic equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$. Thus, the roots are 2 and 3, so $\{e^{2t}, e^{3t}\}$ is a basis for the solution space. It follows that the solutions to the given equation are of the form

$$y = c_1e^{2t} + c_2e^{3t}$$

The constants c_1 and c_2 can be determined if additional equations, called ***boundary conditions,*** are specified.

Example 6.90

Find the solution of $y'' + 6y' + 9y = 0$ that satisfies $y(0) = 1$, $y'(0) = 0$.

Solution The characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$, so -3 is a repeated root. Therefore, $\{e^{-3t}, te^{-3t}\}$ is a basis for the solution space, and the general solution is of the form

$$y = c_1e^{-3t} + c_2te^{-3t}$$

The first boundary condition gives

$$1 = y(0) = c_1e^{-3\cdot 0} + 0 = c_1$$

so $y = e^{-3t} + c_2te^{-3t}$. Differentiating, we have

$$y' = -3e^{-3t} + c_2(-3te^{-3t} + e^{-3t})$$

so the second boundary condition gives

$$0 = y'(0) = -3e^{-3\cdot 0} + c_2(0 + e^{-3\cdot 0}) = -3 + c_2$$

or

$$c_2 = 3$$

Therefore, the required solution is

$$y = e^{-3t} + 3te^{-3t} = (1 + 3t)e^{-3t}$$

a + bi

Theorem 6.33 includes the case in which the roots of the characteristic equation are complex. If $\lambda = p + qi$ is a complex root of the equation $\lambda^2 + a\lambda + b = 0$, then so is its conjugate $\bar{\lambda} = p - qi$. (See Appendices C and D.) By Theorem 6.33(a), the solution space S of the differential equation $y'' + ay' + by = 0$ has $\{e^{\lambda t}, e^{\bar{\lambda} t}\}$ as a basis. Now

$$e^{\lambda t} = e^{(p+qi)t} = e^{pt}e^{i(qt)} = e^{pt}(\cos qt + i \sin qt)$$

and

$$e^{\bar{\lambda} t} = e^{(p-qi)t} = e^{pt}e^{i(-qt)} = e^{pt}(\cos qt - i \sin qt)$$

so

$$e^{pt}\cos qt = \frac{e^{\lambda t} + e^{\bar{\lambda} t}}{2} \quad\text{and}\quad e^{pt}\sin qt = \frac{e^{\lambda t} - e^{\bar{\lambda} t}}{2i}$$

It follows that $\{e^{pt}\cos qt, e^{pt}\sin qt\}$ is contained in $\operatorname{span}(e^{\lambda t}, e^{\bar{\lambda} t}) = S$. Since $e^{pt}\cos qt$ and $e^{pt}\sin qt$ are linearly independent (see Exercise 22) and $\dim S = 2$, $\{e^{pt}\cos qt, e^{pt}\sin qt\}$ is also a basis for S. Thus, when its characteristic equation has a complex root $p + qi$, the differential equation $y'' + ay' + by = 0$ has solutions of the form

$$y = c_1e^{pt}\cos qt + c_2e^{pt}\sin qt$$

$a + bi$ **Example 6.91**

Find all solutions of $y'' - 2y' + 4 = 0$.

Solution The characteristic equation is $\lambda^2 - 2\lambda + 4 = 0$ with roots $1 \pm i\sqrt{3}$. The foregoing discussion tells us that the general solution to the given differential equation is

$$y = c_1 e^t \cos \sqrt{3}t + c_2 e^t \sin \sqrt{3}t$$

$a + bi$ **Example 6.92**

A mass is attached to the end of a vertical spring (Figure 6.21). If the mass is pulled downward and released, it will oscillate up and down. Two laws of physics govern this situation. The first, ***Hooke's law,*** states that if the spring is stretched (or compressed) x units, the force F needed to restore it to its original position is proportional to x:

$$F = -kx$$

where k is a positive constant (called the spring constant). ***Newton's Second Law of Motion*** states that force equals mass times acceleration. Since $x = x(t)$ represents distance, or displacement, of the spring at time t, x' gives its velocity and x'' its acceleration. Thus, we have

$$mx'' = -kx \quad \text{or} \quad x'' + \left(\frac{k}{m}\right)x = 0$$

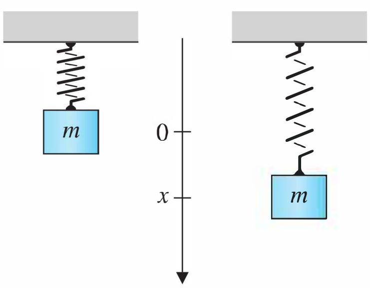


Figure 6.21

Since both k and m are positive, so is $K = k/m$, and our differential equation has the form $x'' + Kx = 0$, where K is positive.

The characteristic equation is $\lambda^2 + K = 0$ with roots $\pm i\sqrt{K}$. Therefore, the general solution to the differential equation of the oscillating spring is

$$x = c_1 \cos \sqrt{K}t + c_2 \sin \sqrt{K}t$$

Suppose the spring is at rest ($x = 0$) at time $t = 0$ seconds and is stretched as far as possible, to a length of 20 cm, before it is released. Then

$$0 = x(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

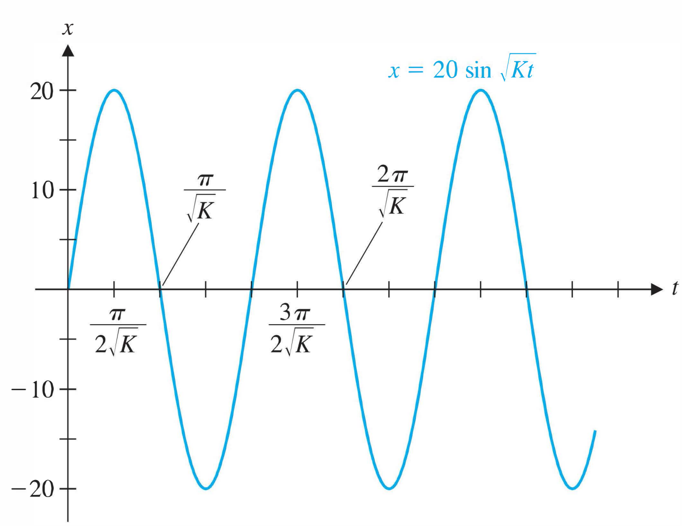


Figure 6.22

so $x = c_2 \sin \sqrt{K}t$. Since the maximum value of the sine function is 1, we must have $c_2 = 20$ (occurring for the first time when $t = \pi/2\sqrt{K}$), giving us the solution

$$x = 20 \sin \sqrt{K}t$$

(See Figure 6.22.)

Of course, this is an idealized solution, since it neglects any form of resistance and predicts that the spring will oscillate forever. It is possible to take damping effects (such as friction) into account, but this simple model has served to introduce an important application of differential equations and the techniques we have developed.

Exercises 6.7

Homogeneous Linear Differential Equations

In Exercises 1–12, find the solution of the differential equation that satisfies the given boundary condition(s).

1. $y' - 3y = 0, y(1) = 2$

2. $x' + x = 0, x(1) = 1$

3. $y'' - 7y' + 12y = 0, y(0) = y(1) = 1$

4. $x'' + x' - 12x = 0, x(0) = 0, x'(0) = 1$

5. $f'' - f' - f = 0, f(0) = 0, f(1) = 1$

6. $g'' - 2g = 0, g(0) = 1, g(1) = 0$

7. $y'' - 2y' + y = 0, y(0) = y(1) = 1$

8. $x'' + 4x' + 4x = 0, x(0) = 1, x'(0) = 1$

9. $y'' - k^2y = 0, k \neq 0, y(0) = y'(0) = 1$

10. $y'' - 2ky' + k^2y = 0, k \neq 0, y(0) = 1, y(1) = 0$

11. $f'' - 2f' + 5f = 0, f(0) = 1, f(\pi/4) = 0$

12. $h'' - 4h' + 5h = 0, h(0) = 0, h'(0) = -1$

13. A strain of bacteria has a growth rate that is proportional to the size of the population. Initially, there are 100 bacteria; after 3 hours, there are 1600.

(a) If $p(t)$ denotes the number of bacteria after t hours, find a formula for $p(t)$.

(b) How long does it take for the population to double?

(c) When will the population reach one million?

CAS **14.** Table 6.2 gives the population of the United States at 10-year intervals for the years 1900–2000.

(a) Assuming an exponential growth model, use the data for 1900 and 1910 to find a formula for $p(t)$, the population in year t. [*Hint:* Let $t = 0$ be 1900 and let $t = 1$ be 1910.] How accurately does your formula calculate the U.S. population in 2000?

(b) Repeat part (a), but use the data for the years 1970 and 1980 to solve for $p(t)$. Does this approach give a better approximation for the year 2000?

(c) What can you conclude about U.S. population growth?

Table 6.2

Year	Population (in millions)
1900	76
1910	92
1920	106
1930	123
1940	131
1950	150
1960	179
1970	203
1980	227
1990	250
2000	281

Source: U.S. Bureau of the Census

15. The half-life of radium-226 is 1590 years. Suppose we start with a sample of radium-226 whose mass is 50 mg.

(a) Find a formula for the mass $m(t)$ remaining after t years and use this formula to predict the mass remaining after 1000 years.

(b) When will only 10 mg remain?

16. *Radiocarbon dating* is a method used by scientists to estimate the age of ancient objects that were once living matter, such as bone, leather, wood, or paper.

All of these contain carbon, a proportion of which is carbon-14, a radioactive isotope that is continuously being formed in the upper atmosphere. Since living organisms take up radioactive carbon along with other carbon atoms, the ratio between the two forms remains constant. However, when an organism dies, the carbon-14 in its cells decays and is not replaced. Carbon-14 has a known half-life of 5730 years, so by measuring the concentration of carbon-14 in an object, scientists can determine its approximate age.

One of the most successful applications of radiocarbon dating has been to determine the age of the Stonehenge monument in England (Figure 6.23). Samples taken from the remains of wooden posts were found to have a concentration of carbon-14 that was 45% of that found in living material. What is the estimated age of these posts?

Figure 6.23
Stonehenge

17. A mass is attached to a spring, as in Example 6.92. At time $t = 0$ second, the spring is stretched to a length of 10 cm below its position at rest. The spring is released, and its length 10 seconds later is observed to be 5 cm. Find a formula for the length of the spring at time t seconds.

18. A 50 g mass is attached to a spring, as in Example 6.92. If the period of oscillation is 10 seconds, find the spring constant.

19. A pendulum consists of a mass, called a *bob*, that is affixed to the end of a string of length L (see Figure 6.24). When the bob is moved from its rest position and released, it swings back and forth. The time it takes the pendulum to swing from its farthest right position to its farthest left position and back to its next farthest right position is called the *period* of the pendulum.

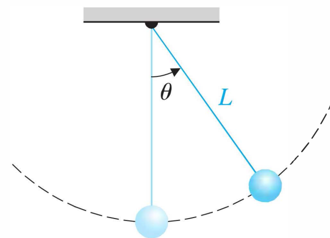


Figure 6.24

Let $\theta = \theta(t)$ be the angle of the pendulum from the vertical. It can be shown that if there is no resistance, then when θ is small it satisfies the differential equation

$$\theta'' + \frac{g}{L}\theta = 0$$

where g is the constant of acceleration due to gravity, approximately 9.7 m/s^2. Suppose that $L = 1$ m and that the pendulum is at rest (i.e., $\theta = 0$) at time $t = 0$ second. The bob is then drawn to the right at an angle of θ_1 radians and released.

(a) Find the period of the pendulum.

(b) Does the period depend on the angle θ_1 at which the pendulum is released? This question was posed and answered by Galileo in 1638. [Galileo Galilei (1564–1642) studied medicine as a student at the University of Pisa, but his real interest was always mathematics. In 1592, Galileo was appointed professor of mathematics at the University of Padua in Venice, where he taught primarily geometry and astronomy. He was the first to use a telescope to look at the stars and planets, and in so doing, he produced experimental data in support of the Copernican view that the planets revolve around the sun and not the earth. For this, Galileo was summoned before the Inquisition, placed under house arrest, and forbidden to publish his results. While under house arrest, he was able to write up his research on falling objects and pendulums. His notes were smuggled out of Italy and published as *Discourses on Two New Sciences* in 1638.]

20. Show that the solution set S of the second-order differential equation $y'' + ay' + by = 0$ is a subspace of $\mathscr{F}$.

21. Prove Theorem 6.33(b).

22. Show that $e^{pt}\cos qt$ and $e^{pt}\sin qt$ are linearly independent.

Chapter Review

Key Definitions and Concepts

Review Questions

1. Mark each of the following statements true or false:

(a) If $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then every spanning set for V contains at least n vectors.

(b) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors, then so is $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}\}$.

(c) M_{22} has a basis consisting of invertible matrices.

(d) M_{22} has a basis consisting of matrices whose trace is zero.

(e) The transformation $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T(\mathbf{x}) = \|\mathbf{x}\|$ is a linear transformation.

(f) If $T : V \to W$ is a linear transformation and $\dim V \neq \dim W$, then T cannot be both one-to-one and onto.

(g) If $T : V \to W$ is a linear transformation and $\ker(T) = V$, then $W = \{\mathbf{0}\}$.

(h) If $T : M_{33} \to \mathscr{P}_4$ is a linear transformation and $\text{nullity}(T) = 4$, then T is onto.

(i) The vector space $V = \{p(x) \text{ in } \mathscr{P}_4 : p(1) = 0\}$ is isomorphic to $\mathscr{P}_3$.

(j) If $I : V \to V$ is the identity transformation, then the matrix $[I]_{\mathcal{C} \leftarrow \mathcal{B}}$ is the identity matrix for any bases $\mathcal{B}$ and $\mathcal{C}$ of V.

In Questions 2–5, determine whether W is a subspace of V.

2. $V = \mathbb{R}^2$, $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + 3y^2 = 0 \right\}$

3. $V = M_{22}$, $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b = c + d = a + c = b + d \right\}$

4. $V = \mathscr{P}_3$, $W = \{p(x) \text{ in } \mathscr{P}_3 : x^3 p(1/x) = p(x)\}$

5. $V = \mathscr{F}$, $W = \{f \text{ in } \mathscr{F} : f(x + \pi) = f(x) \text{ for all } x\}$

6. Determine whether $\{1, \cos 2x, 3 \sin^2 x\}$ is linearly dependent or independent.

7. Let A and B be nonzero $n \times n$ matrices such that A is symmetric and B is skew-symmetric. Prove that $\{A, B\}$ is linearly independent.

In Questions 8 and 9, find a basis for W and state the dimension of W.

8. $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = b + c \right\}$

9. $W = \{p(x) \text{ in } \mathscr{P}_5 : p(-x) = p(x)\}$

10. Find the change-of-basis matrices $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and $P_{\mathcal{B} \leftarrow \mathcal{C}}$ with respect to the bases $\mathcal{B} = \{1, 1 + x, 1 + x + x^2\}$ and $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathscr{P}_2$.

In Questions 11–13, determine whether T is a linear transformation.

11. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{y}\mathbf{x}^T\mathbf{y}$, where $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

12. $T: M_{nn} \to M_{nn}$ defined by $T(A) = A^TA$

13. $T: \mathscr{P}_n \to \mathscr{P}_n$ defined by $T(p(x)) = p(2x - 1)$

14. If $T: \mathscr{P}_2 \to M_{22}$ is a linear transformation such that $T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T(1 + x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T(1 + x + x^2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find $T(5 - 3x + 2x^2)$.

15. Find the nullity of the linear transformation $T: M_{nn} \to \mathbb{R}$ defined by $T(A) = \operatorname{tr}(A)$.

16. Let W be the vector space of upper triangular 2×2 matrices.

(a) Find a linear transformation $T: M_{22} \to M_{22}$ such that $\ker(T) = W$.

(b) Find a linear transformation $T: M_{22} \to M_{22}$ such that $\operatorname{range}(T) = W$.

17. Find the matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ of the linear transformation T in Question 14 with respect to the standard bases $\mathcal{B} = \{1, x, x^2\}$ of $\mathscr{P}_2$ and $\mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_{22}.

18. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors in a vector space V with the property that every vector in V can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in exactly one way. Prove that S is a basis for V.

19. If $T: U \to V$ and $S: V \to W$ are linear transformations such that $\operatorname{range}(T) \subseteq \ker(S)$, what can be deduced about $S \circ T$?

20. Let $T: V \to V$ be a linear transformation, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V such that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is also a basis for V. Prove that T is invertible.

7 Distance and Approximation

A straight line may be the shortest distance between two points, but it is by no means the most interesting.

—Doctor Who
In "The Time Monster"
By Robert Sloman
BBC, 1972

Although this may seem a paradox, all exact science is dominated by the idea of approximation.

—Bertrand Russell
In W. H. Auden and
L. Kronenberger, eds.
The Viking Book of Aphorisms
Viking, 1962, p. 263

7.0 Introduction: Taxicab Geometry

We live in a three-dimensional Euclidean world, and, therefore, concepts from Euclidean geometry govern our way of looking at the world. In particular, imagine stopping people on the street and asking them to fill in the blank in the following sentence: "The shortest distance between two points is a ______." They will almost certainly respond with "straight line." There are, however, other equally sensible and intuitive notions of distance. By allowing ourselves to think of "distance" in a more flexible way, we will open the door to the possibility of having a "distance" between polynomials, functions, matrices, and many other objects that arise in linear algebra.

In this section, you will discover a type of "distance" that is every bit as real as the straight-line distance you are used to from Euclidean geometry (the one that is a consequence of Pythagoras' Theorem). As you'll see, this new type of "distance" still behaves in some familiar ways.

Suppose you are standing at an intersection in a city, trying to get to a restaurant at another intersection. If you ask someone how far it is to the restaurant, that person is unlikely to measure distance "as the crow flies" (i.e., using the Euclidean version of distance). Instead, the response will be something like "It's five blocks away." Since this is the way taxicab drivers measure distance, we will refer to this notion of "distance" as ***taxicab distance.***

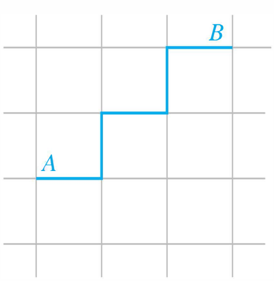


Figure 7.1
Taxicab distance

Figure 7.1 shows an example of taxicab distance. The shortest path from A to B requires traversing the sides of five city blocks. Notice that although there is more than one route from A to B, all shortest routes require three horizontal moves and two vertical moves, where a "move" corresponds to the side of one city block. (How many shortest routes are there from A to B?) Therefore, the taxicab distance from A to B is 5.

Idealizing this situation, we will assume that all blocks are unit squares, and we will use the notation $d_t(A, B)$ for the taxicab distance from A to B.

Problem 1 Find the taxicab distance between the following pairs of points:

(a) $(1, 2)$ and $(5, 5)$ (b) $(2, 4)$ and $(3, -2)$

(c) $(0, 0)$ and $(-4, -3)$ (d) $(-2, 3)$ and $(1, 3)$

(e) $(1, \frac{1}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$ (f) $(2.5, 4.6)$ and $(3.1, 1.5)$

Problem 2 Which of the following is the correct formula for the taxicab distance $d_t(A, B)$ between $A = (a_1, a_2)$ and $B = (b_1, b_2)$?

(a) $d_t(A, B) = (a_1 - b_1) + (a_2 - b_2)$

(b) $d_t(A, B) = (|a_1| - |b_1|) + (|a_2| - |b_2|)$

(c) $d_t(A, B) = |a_1 - b_1| + |a_2 - b_2|$

We can define the ***taxicab norm*** of a vector $\mathbf{v}$ as

$$\|\mathbf{v}\|_t = d_t(\mathbf{v}, \mathbf{0})$$

Problem 3 Find $\|\mathbf{v}\|_t$ for the following vectors:

(a) $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ (b) $\mathbf{v} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

(c) $\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ (d) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Problem 4 Show that Theorem 1.3 is true for the taxicab norm.

Problem 5 Verify the Triangle Inequality (Theorem 1.5), using the taxicab norm and the following pairs of vectors:

(a) $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Problem 6 Show that the Triangle Inequality is true, in general, for the taxicab norm.

In Euclidean geometry, we can define a circle of radius r, centered at the origin, as the set of all $\mathbf{x}$ such that $\|\mathbf{x}\| = r$. Analogously, we can define a ***taxicab circle*** of radius r, centered at the origin, as the set of all $\mathbf{x}$ such that $\|\mathbf{x}\|_t = r$.

Problem 7 Draw taxicab circles centered at the origin with the following radii:

(a) $r = 3$ (b) $r = 4$ (c) $r = 1$

Problem 8 In Euclidean geometry, the value of π is half the circumference of a unit circle (a circle of radius 1). Let's define ***taxicab pi*** to be the number π_t that is half the circumference of a taxicab unit circle. What is the value of π_t?

In Euclidean geometry, the perpendicular bisector of a line segment $\overline{AB}$ can be defined as the set of all points that are equidistant from A and B. If we use taxicab distance instead of Euclidean distance, it is reasonable to ask what the perpendicular bisector of a line segment now looks like. To be precise, the ***taxicab perpendicular bisector*** of $\overline{AB}$ is the set of all points X such that

$$d_t(X, A) = d_t(X, B)$$

Problem 9 Draw the taxicab perpendicular bisector of $\overline{AB}$ for the following pairs of points:

(a) $A = (2, 1), B = (4, 1)$ (b) $A = (-1, 3), B = (-1, -2)$

(c) $A = (1, 1), B = (5, 3)$ (d) $A = (1, 1), B = (5, 5)$

As these problems illustrate, taxicab geometry shares some properties with Euclidean geometry, but it also differs in some striking ways. In this chapter, we will encounter several other types of distances and norms, each of which is useful in its own way. We will try to discover what they have in common and use these common properties to our advantage. We will also explore a variety of approximation problems in which the notion of "distance" plays an important role.

7.1 Inner Product Spaces

In Chapter 1, we defined the dot product $\mathbf{u} \cdot \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and we have made repeated use of this operation throughout this book. In this section, we will use the properties of the dot product as a means of defining the general notion of an *inner product*. In the next section, we will show that inner products can be used to define analogues of "length" and "distance" in vector spaces other than $\mathbb{R}^n$.

The following definition is our starting point; it is based on the properties of the dot product proved in Theorem 1.2.

Definition An ***inner product*** on a vector space V is an operation that assigns to every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an ***inner product space.***

Remark Technically, this definition defines a *real* inner product space, since it assumes that V is a real vector space and since the inner product of two vectors is a real number. There are *complex* inner product spaces too, but their definition is somewhat different. (See Exploration: Vectors and Matrices with Complex Entries at the end of this section.)

Example 7.1

$\mathbb{R}^n$ is an inner product space with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$. Properties (1) through (4) were verified as Theorem 1.2.

The dot product is not the only inner product that can be defined on $\mathbb{R}^n$.

Example 7.2

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

defines an inner product.

Solution We must verify properties (1) through (4). Property (1) holds because

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 = 2v_1u_1 + 3v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$$

Next, let $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. We check that

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= 2u_1(v_1 + w_1) + 3u_2(v_2 + w_2) \\ &= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2 \\ &= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2) \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \end{aligned}$$

which proves property (2).

If c is a scalar, then

$$\begin{aligned} \langle c\mathbf{u}, \mathbf{v} \rangle &= 2(cu_1)v_1 + 3(cu_2)v_2 \\ &= c(2u_1v_1 + 3u_2v_2) \\ &= c\langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

which verifies property (3).

Finally,

$$\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1u_1 + 3u_2u_2 = 2u_1^2 + 3u_2^2 \geq 0$$

and it is clear that $\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1^2 + 3u_2^2 = 0$ if and only if $u_1 = u_2 = 0$ (that is, if and only if $\mathbf{u} = \mathbf{0}$). This verifies property (4), completing the proof that $\langle \mathbf{u}, \mathbf{v} \rangle$, as defined, is an inner product.

Example 7.2 can be generalized to show that if $w_1, \ldots, w_n$ are *positive* scalars and

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + \cdots + w_nu_nv_n \tag{1}$$

defines an inner product on $\mathbb{R}^n$, called a ***weighted dot product.*** If any of the weights w_i is negative or zero, then Equation (1) does not define an inner product. (See Exercises 13 and 14.)

Recall that the dot product can be expressed as $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$. Observe that we can write the weighted dot product in Equation (1) as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T W \mathbf{v}$$

where W is the $n \times n$ diagonal matrix

$$W = \begin{bmatrix} w_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n \end{bmatrix}$$

The next example further generalizes this type of inner product.

Example 7.3

Let A be a symmetric, positive definite $n \times n$ matrix (see Section 5.5) and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v}$$

defines an inner product.

Solution We check that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= u^T A\mathbf{v} = \mathbf{u} \cdot A\mathbf{v} = A\mathbf{v} \cdot \mathbf{u} \\ &= A^T\mathbf{v} \cdot \mathbf{u} = (\mathbf{v}^T A)^T \cdot \mathbf{u} = \mathbf{v}^T A\mathbf{u} = \langle \mathbf{v}, \mathbf{u} \rangle\end{aligned}$$

Also,

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \mathbf{u}^T A(\mathbf{v} + \mathbf{w}) = \mathbf{u}^T A\mathbf{v} + \mathbf{u}^T A\mathbf{w} = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

and

$$\langle c\mathbf{u}, \mathbf{v} \rangle = (c\mathbf{u})^T A\mathbf{v} = c(\mathbf{u}^T A\mathbf{v}) = c\langle \mathbf{u}, \mathbf{v} \rangle$$

Finally, since A is positive definite, $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^T A\mathbf{u} > 0$ for all $\mathbf{u} \neq \mathbf{0}$, so $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^T A\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$. This establishes the last property.

To illustrate Example 7.3, let $A = \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$. Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v} = [u_1 \; u_2] \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 7u_2v_2$$

The matrix A is positive definite, by Theorem 5.24, since its eigenvalues are 3 and 8. Hence, $\langle \mathbf{u}, \mathbf{v} \rangle$ defines an inner product on $\mathbb{R}^2$.

We now define some inner products on vector spaces other than $\mathbb{R}^n$.

Example 7.4

In $\mathcal{P}_2$, let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$. Show that

$$\langle p(x), q(x) \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

defines an inner product on $\mathcal{P}_2$. (For example, if $p(x) = 1 - 5x + 3x^2$ and $q(x) = 6 + 2x - x^2$, then $\langle p(x), q(x) \rangle = 1 \cdot 6 + (-5) \cdot 2 + 3 \cdot (-1) = -7$.)

Solution Since $\mathcal{P}_2$ is isomorphic to $\mathbb{R}^3$, we need only show that the dot product in $\mathbb{R}^3$ is an inner product, which we have already established.

Example 7.5

Let f and g be in $\mathscr{C}[a, b]$, the vector space of all continuous functions on the closed interval $[a, b]$. Show that

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

defines an inner product on $\mathscr{C}[a, b]$.

Solution We have

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g, f\rangle$$

Also, if h is in $\mathscr{C}[a, b]$, then

$$\begin{aligned}\langle f, g + h\rangle &= \int_a^b f(x)(g(x) + h(x))\,dx \\ &= \int_a^b (f(x)g(x) + f(x)h(x))\,dx \\ &= \int_a^b f(x)g(x)\,dx + \int_a^b f(x)h(x)\,dx \\ &= \langle f, g\rangle + \langle f, h\rangle\end{aligned}$$

If c is a scalar, then

$$\begin{aligned}\langle cf, g\rangle &= \int_a^b cf(x)g(x)\,dx \\ &= c\int_a^b f(x)g(x)\,dx \\ &= c\langle f, g\rangle\end{aligned}$$

Finally, $\langle f, f\rangle = \int_a^b (f(x))^2\,dx \geq 0$, and it follows from a theorem of calculus that, since f is continuous, $\langle f, f\rangle = \int_a^b (f(x))^2\,dx = 0$ if and only if f is the zero function. Therefore, $\langle f, g\rangle$ is an inner product on $\mathscr{C}[a, b]$.

Example 7.5 also defines an inner product on any *subspace* of $\mathscr{C}[a, b]$. For example, we could restrict our attention to polynomials defined on the interval $[a, b]$. Suppose we consider $\mathscr{P}[0, 1]$, the vector space of all polynomials on the interval $[0, 1]$. Then, using the inner product of Example 7.5, we have

$$\begin{aligned}\langle x^2, 1 + x\rangle &= \int_0^1 x^2(1 + x)\,dx = \int_0^1 (x^2 + x^3)\,dx \\ &= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}\end{aligned}$$

Properties of Inner Products

The following theorem summarizes some additional properties that follow from the definition of inner product.

Theorem 7.1 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space V and let c be a scalar.

a. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
b. $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
c. $\langle \mathbf{u}, \mathbf{0} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle = 0$

Proof We prove property (a), leaving the proofs of properties (b) and (c) as Exercises 23 and 24. Referring to the definition of inner product, we have

$$\begin{aligned}
\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= \langle \mathbf{w}, \mathbf{u} + \mathbf{v} \rangle && \text{by (1)} \\
&= \langle \mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle && \text{by (2)} \\
&= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle && \text{by (1)}
\end{aligned}$$

Length, Distance, and Orthogonality

In an inner product space, we can define the length of a vector, distance between vectors, and orthogonal vectors, just as we did in Section 1.2. We simply have to replace every use of the dot product $\mathbf{u} \cdot \mathbf{v}$ by the more general inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.

Definition Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. The ***length*** (or ***norm***) of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
2. The ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
3. $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Note that $\|\mathbf{v}\|$ is always defined, since $\langle \mathbf{v}, \mathbf{v} \rangle \ge 0$ by the definition of inner product, so we can take the square root of this nonnegative quantity. As in $\mathbb{R}^n$, a vector of length 1 is called a ***unit vector.*** The ***unit sphere*** in V is the set S of all unit vectors in V.

Example 7.6 Consider the inner product on $\mathscr{C}[0, 1]$ given in Example 7.5. If $f(x) = x$ and $g(x) = 3x - 2$, find

(a) $\|f\|$ (b) $d(f, g)$ (c) $\langle f, g \rangle$

Solution (a) We find that

$$\langle f, f \rangle = \int_0^1 f^2(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$$

so $\|f\| = \sqrt{\langle f, f \rangle} = 1/\sqrt{3}$.

(b) Since $d(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle}$ and

$$f(x) - g(x) = x - (3x - 2) = 2 - 2x = 2(1 - x)$$

we have
$$\langle f - g, f - g \rangle = \int_0^1 (f(x) - g(x))^2 \, dx = \int_0^1 4(1 - 2x + x^2) \, dx$$
$$= 4\left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{4}{3}$$

Combining these facts, we see that $d(f, g) = \sqrt{4/3} = 2/\sqrt{3}$.

(c) We compute

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx = \int_0^1 x(3x - 2) \, dx = \int_0^1 (3x^2 - 2x) \, dx = [x^3 - x^2]_0^1 = 0$$

Thus, f and g are orthogonal.

It is important to remember that the "distance" between f and g in Example 7.6 does *not* refer to any measurement related to the graphs of these functions. Neither does the fact that f and g are orthogonal mean that their graphs intersect at right angles. We are simply applying the definition of a particular inner product. However, in doing so, we should be guided by the corresponding notions in $\mathbb{R}^2$ and $\mathbb{R}^3$, where the inner product is the dot product. The geometry of Euclidean space can still guide us here, even though we cannot visualize things in the same way.

Example 7.7

Using the inner product on $\mathbb{R}^2$ defined in Example 7.2, draw a sketch of the unit sphere (circle).

Solution If $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $\langle \mathbf{x}, \mathbf{x} \rangle = 2x^2 + 3y^2$. Since the unit sphere (circle) consists of all $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$, we have

$$1 = \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{2x^2 + 3y^2} \quad \text{or} \quad 2x^2 + 3y^2 = 1$$

This is the equation of an ellipse, and its graph is shown in Figure 7.2.

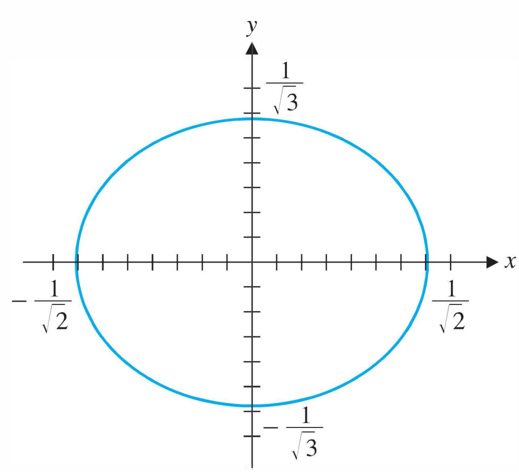


Figure 7.2
A unit circle that is an ellipse

We will discuss properties of length, distance, and orthogonality in the next section and in the exercises. One result that we will need in this section is the generalized version of Pythagoras' Theorem, which extends Theorem 1.6.

Theorem 7.2 **Pythagoras' Theorem**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V. Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof As you will be asked to prove in Exercise 32, we have

$$\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$$

It follows immediately that $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Orthogonal Projections and the Gram-Schmidt Process

In Chapter 5, we discussed orthogonality in $\mathbb{R}^n$. Most of this material generalizes nicely to general inner product spaces. For example, an ***orthogonal set*** of vectors in an inner product space V is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors from V such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $\mathbf{v}_i \neq \mathbf{v}_j$. An ***orthonormal set*** of vectors is then an orthogonal set of *unit* vectors. An ***orthogonal basis*** for a subspace W of V is just a basis for W that is an orthogonal set; similarly, an ***orthonormal basis*** for a subspace W of V is a basis for W that is an orthonormal set.

In $\mathbb{R}^n$, the Gram-Schmidt Process (Theorem 5.15) shows that every subspace has an orthogonal basis. We can mimic the construction of the Gram-Schmidt Process to show that every finite-dimensional subspace of an inner product space has an orthogonal basis—all we need to do is replace the dot product by the more general inner product. We illustrate this approach with an example. (Compare the steps here with those in Example 5.13.)

Example 7.8

Construct an orthogonal basis for $\mathscr{P}_2$ with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Solution Let $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, and $\mathbf{x}_3 = x^2$. We begin by setting $\mathbf{v}_1 = \mathbf{x}_1 = 1$. Next we compute

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} dx = x\Big]_{-1}^{1} = 2 \quad \text{and} \quad \langle \mathbf{v}_1, \mathbf{x}_2 \rangle = \int_{-1}^{1} x\, dx = \frac{x^2}{2}\Big]_{-1}^{1} = 0$$

Adrien Marie Legendre (1752–1833) was a French mathematician who worked in astronomy, number theory, and elliptic functions. He was involved in several heated disputes with Gauss. Legendre gave the first published statement of the law of quadratic reciprocity in number theory in 1765. Gauss, however, gave the first rigorous proof of this result in 1801 and claimed credit for the result, prompting understandable outrage from Legendre. Then in 1806, Legendre gave the first published application of the method of least squares in a book on the orbits of comets. Gauss published on the same topic in 1809 but claimed he had been using the method since 1795, once again infuriating Legendre.

Therefore,

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{v}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = x - \frac{0}{2}(1) = x$$

To find $\mathbf{v}_3$, we first compute

$$\langle \mathbf{v}_1, \mathbf{x}_3 \rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\bigg]_{-1}^{1} = \frac{2}{3}, \qquad \langle \mathbf{v}_2, \mathbf{x}_3 \rangle = \int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\bigg]_{-1}^{1} = 0,$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{v}_1, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 = x^2 - \frac{\frac{2}{3}}{2}(1) - \frac{0}{\frac{2}{3}}x = x^2 - \frac{1}{3}$$

It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathscr{P}_2$ on the interval $[-1, 1]$. The polynomials

$$1, \quad x, \quad x^2 - \tfrac{1}{3}$$

are the first three ***Legendre polynomials.*** If we divide each of these polynomials by its length relative to the same inner product, we obtain ***normalized Legendre polynomials*** (see Exercise 41).

Just as we did in Section 5.2, we can define the ***orthogonal projection*** $\text{proj}_W(\mathbf{v})$ of a vector $\mathbf{v}$ onto a subspace W of an inner product space. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis for W, then

$$\text{proj}_W(\mathbf{v}) = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 + \cdots + \frac{\langle \mathbf{u}_k, \mathbf{v} \rangle}{\langle \mathbf{u}_k, \mathbf{u}_k \rangle}\mathbf{u}_k$$

Then the ***component of* v *orthogonal to W*** is the vector

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

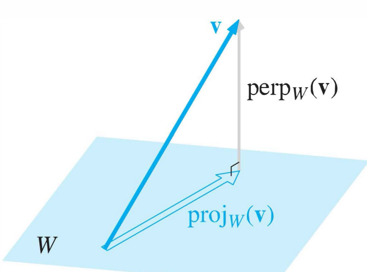


Figure 7.3

As in the Orthogonal Decomposition Theorem (Theorem 5.11), $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$ are orthogonal (see Exercise 43), and so, schematically, we have the situation illustrated in Figure 7.3.

We will make use of these formulas in Sections 7.3 and 7.5 when we consider approximation problems—in particular, the problem of how best to approximate a given function by "nice" functions. Consequently, we will defer any examples until then, when they will make more sense. Our immediate use of orthogonal projection will be to prove an inequality that we first encountered in Chapter 1.

The Cauchy-Schwarz and Triangle Inequalities

The proofs of identities and inequalities involving the dot product in $\mathbb{R}^n$ are easily adapted to give corresponding results in general inner product spaces. Some of these are given in Exercises 31–36. In Section 1.2, we first encountered the Cauchy-Schwarz Inequality, which is important in many branches of mathematics. We now give a proof of this result for inner product spaces.

Theorem 7.3 **The Cauchy-Schwarz Inequality**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V. Then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$$

with equality holding if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

Proof If $\mathbf{u} = \mathbf{0}$, then the inequality is actually an equality, since

$$|\langle \mathbf{0}, \mathbf{v}\rangle| = 0 = \|\mathbf{0}\|\,\|\mathbf{v}\|$$

If $\mathbf{u} \neq \mathbf{0}$, then let W be the subspace of V spanned by $\mathbf{u}$. Since $\text{proj}_W(\mathbf{v}) = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$ and $\text{perp}_W\,\mathbf{v} = \mathbf{v} - \text{proj}_W(\mathbf{v})$ are orthogonal, we can apply Pythagoras' Theorem to obtain

$$\begin{aligned}\|\mathbf{v}\|^2 &= \|\text{proj}_W(\mathbf{v}) + (\mathbf{v} - \text{proj}_W(\mathbf{v}))\|^2 = \|\text{proj}_W(\mathbf{v}) + \text{perp}_W(\mathbf{v})\|^2 \\ &= \|\text{proj}_W(\mathbf{v})\|^2 + \|\text{perp}_W(\mathbf{v})\|^2\end{aligned} \tag{2}$$

It follows that $\|\text{proj}_W(\mathbf{v})\|^2 \leq \|\mathbf{v}\|^2$. Now

$$\|\text{proj}_W(\mathbf{v})\|^2 = \left\langle \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} \right\rangle = \left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\right)^2 \langle \mathbf{u}, \mathbf{u}\rangle = \frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\langle \mathbf{u}, \mathbf{u}\rangle} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\|\mathbf{u}\|^2}$$

so we have

$$\frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\|\mathbf{u}\|^2} \leq \|\mathbf{v}\|^2 \quad \text{or, equivalently,} \quad \langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2$$

Taking square roots, we obtain $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$.

Clearly this last inequality is an equality if and only if $\|\text{proj}_W(\mathbf{v})\|^2 = \|\mathbf{v}\|^2$. By Equation (2) this is true if and only if $\text{perp}_W(\mathbf{v}) = \mathbf{0}$ or, equivalently,

$$\mathbf{v} = \text{proj}_W(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$$

This inequality was discovered by several different mathematicians, in several different contexts. It is no surprise that the name of the prolific Cauchy is attached to it. The second name associated with this result is that of Karl Herman Amandus Schwarz (1843–1921), a German mathematician who taught at the University of Berlin. His version of the inequality that bears his name was published in 1885 in a paper that used integral equations to study surfaces of minimal area. A third name also associated with this important result is that of the Russian mathematician Viktor Yakovlevitch Bunyakovsky (1804–1889). Bunyakovsky published the inequality in 1859, a full quarter-century before Schwarz's work on the same subject. Hence, it is more proper to refer to the result as the Cauchy-Bunyakovsky-Schwarz Inequality.

If this is so, then $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$. Conversely, if $\mathbf{v} = c\mathbf{u}$, then

$$\operatorname{perp}_W(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = c\mathbf{u} - \frac{\langle \mathbf{u}, c\mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = c\mathbf{u} - \frac{c\langle \mathbf{u}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = \mathbf{0}$$

so equality holds in the Cauchy-Schwarz Inequality.

For an alternative proof of this inequality, see Exercise 44. We will investigate some interesting consequences of the Cauchy-Schwarz Inequality and related inequalities in Exploration: Geometric Inequalities and Optimization Problems, which follows this section. For the moment, we use it to prove a generalized version of the Triangle Inequality (Theorem 1.5).

Theorem 7.4 **The Triangle Inequality**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V. Then

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof Starting with the equality you will be asked to prove in Exercise 32, we have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v}\rangle| + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \qquad \text{by Cauchy-Schwarz} \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

Taking square roots yields the result.

Exercises 7.1

In Exercises 1–4, let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

1. $\langle \mathbf{u}, \mathbf{v}\rangle$ is the inner product of Example 7.2. Compute

(a) $\langle \mathbf{u}, \mathbf{v}\rangle$ **(b)** $\|\mathbf{u}\|$ **(c)** $d(\mathbf{u}, \mathbf{v})$

2. $\langle \mathbf{u}, \mathbf{v}\rangle$ is the inner product of Example 7.3 with $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$. Compute

(a) $\langle \mathbf{u}, \mathbf{v}\rangle$ **(b)** $\|\mathbf{u}\|$ **(c)** $d(\mathbf{u}, \mathbf{v})$

3. In Exercise 1, find a nonzero vector orthogonal to $\mathbf{u}$.

4. In Exercise 2, find a nonzero vector orthogonal to $\mathbf{u}$.

In Exercises 5–8, let $p(x) = 3 - 2x$ and $q(x) = 1 + x + x^2$.

5. $\langle p(x), q(x)\rangle$ is the inner product of Example 7.4. Compute

(a) $\langle p(x), q(x)\rangle$ **(b)** $\|p(x)\|$ **(c)** $d(p(x), q(x))$

6. $\langle p(x), q(x)\rangle$ is the inner product of Example 7.5 on the vector space $\mathscr{P}_2[0, 1]$. Compute

(a) $\langle p(x), q(x)\rangle$ **(b)** $\|p(x)\|$ **(c)** $d(p(x), q(x))$

7. In Exercise 5, find a nonzero vector orthogonal to $p(x)$.

8. In Exercise 6, find a nonzero vector orthogonal to $p(x)$.

In Exercises 9 and 10, let $f(x) = \sin x$ *and* $g(x) = \sin x + \cos x$ *in the vector space* $\mathscr{C}[0, 2\pi]$ *with the inner product defined by Example 7.5.*

9. Compute

(a) $\langle f, g\rangle$ **(b)** $\|f\|$ **(c)** $d(f, g)$

10. Find a nonzero vector orthogonal to f.

11. Let a, b, and c be distinct real numbers. Show that

$$\langle p(x), q(x)\rangle = p(a)q(a) + p(b)q(b) + p(c)q(c)$$

defines an inner product on $\mathscr{P}_2$. [*Hint:* You will need the fact that a polynomial of degree n has at most n zeros. See Appendix D.]

12. Repeat Exercise 5 using the inner product of Exercise 11 with $a = 0$, $b = 1$, $c = 2$.

In Exercises 13–18, determine which of the four inner product axioms do not hold. Give a specific example in each case.

13. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$. Define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1$.

14. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$. Define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2$.

15. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$. Define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_2 + u_2 v_1$.

16. In $\mathscr{P}_2$, define $\langle p(x), q(x) \rangle = p(0)q(0)$.

17. In $\mathscr{P}_2$, define $\langle p(x), q(x) \rangle = p(1)q(1)$.

18. In M_{22}, define $\langle A, B \rangle = \det(AB)$.

In Exercises 19 and 20, $\langle \mathbf{u}, \mathbf{v} \rangle$ defines an inner product on $\mathbb{R}^2$, where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Find a symmetric matrix A such that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$.

19. $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + u_1v_2 + u_2v_1 + 4u_2v_2$

20. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_1v_2 + 2u_2v_1 + 5u_2v_2$

In Exercises 21 and 22, sketch the unit circle in $\mathbb{R}^2$ for the given inner product, where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

21. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + \frac{1}{4}u_2v_2$

22. $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + u_1v_2 + u_2v_1 + 4u_2v_2$

23. Prove Theorem 7.1(b).

24. Prove Theorem 7.1(c).

In Exercises 25–29, suppose that **u**, **v**, *and* **w** *are vectors in an inner product space such that*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1, \quad \langle \mathbf{u}, \mathbf{w} \rangle = 5, \quad \langle \mathbf{v}, \mathbf{w} \rangle = 0$$
$$\|\mathbf{u}\| = 1, \quad \|\mathbf{v}\| = \sqrt{3}, \quad \|\mathbf{w}\| = 2$$

Evaluate the expressions in Exercises 25–28.

25. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle$

26. $\langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle$

27. $\|\mathbf{u} + \mathbf{v}\|$

28. $\|2\mathbf{u} - 3\mathbf{v} + \mathbf{w}\|$

29. Show that $\mathbf{u} + \mathbf{v} = \mathbf{w}$. [*Hint:* How can you use the properties of inner product to verify that $\mathbf{u} + \mathbf{v} - \mathbf{w} = \mathbf{0}$?]

30. Show that, in an inner product space, there cannot be unit vectors **u** and **v** with $\langle \mathbf{u}, \mathbf{v} \rangle < -1$.

In Exercises 31–36, $\langle \mathbf{u}, \mathbf{v} \rangle$ is an inner product. In Exercises 31–34, prove that the given statement is an identity.

31. $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

32. $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$

33. $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \frac{1}{2}\|\mathbf{u} + \mathbf{v}\|^2 + \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|^2$

34. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$

35. Prove that $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ if and only if **u** and **v** are orthogonal.

36. Prove that $d(\mathbf{u}, \mathbf{v}) = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}$ if and only if **u** and **v** are orthogonal.

In Exercises 37–40, apply the Gram-Schmidt Process to the basis $\mathcal{B}$ to obtain an orthogonal basis for the inner product space V relative to the given inner product.

37. $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, with the inner product in Example 7.2

38. $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, with the inner product immediately following Example 7.3

39. $V = \mathscr{P}_2$, $\mathcal{B} = \{1, 1 + x, 1 + x + x^2\}$, with the inner product in Example 7.4

40. $V = \mathscr{P}_2[0, 1]$, $\mathcal{B} = \{1, 1 + x, 1 + x + x^2\}$, with the inner product in Example 7.5

41. **(a)** Compute the first three normalized Legendre polynomials. (See Example 7.8.)

(b) Use the Gram-Schmidt Process to find the fourth normalized Legendre polynomial.

42. If we multiply the Legendre polynomial of degree n by an appropriate scalar we can obtain a polynomial $L_n(x)$ such that $L_n(1) = 1$.

(a) Find $L_0(x)$, $L_1(x)$, $L_2(x)$, and $L_3(x)$.

(b) It can be shown that $L_n(x)$ satisfies the recurrence relation

$$L_n(x) = \frac{2n - 1}{n} x L_{n-1}(x) - \frac{n - 1}{n} L_{n-2}(x)$$

for all $n \geq 2$. Verify this recurrence for $L_2(x)$ and $L_3(x)$. Then use it to compute $L_4(x)$ and $L_5(x)$.

43. Verify that if W is a subspace of an inner product space V and $\mathbf{v}$ is in V, then $\operatorname{perp}_W(\mathbf{v})$ is orthogonal to all $\mathbf{w}$ in W.

44. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V. Prove the Cauchy-Schwarz Inequality for $\mathbf{u} \neq \mathbf{0}$ as follows:

(a) Let t be a real scalar. Then $\langle t\mathbf{u} + \mathbf{v}, t\mathbf{u} + \mathbf{v} \rangle \geq 0$ for all values of t. Expand this inequality to obtain a quadratic inequality of the form

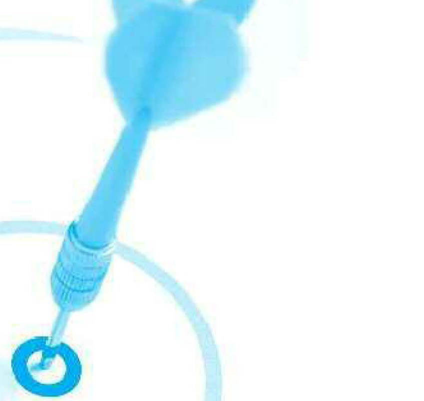
$$at^2 + bt + c \geq 0$$

What are a, b, and c in terms of $\mathbf{u}$ and $\mathbf{v}$?

(b) Use your knowledge of quadratic equations and their graphs to obtain a condition on a, b, and c for which the inequality in part (a) is true.

(c) Show that, in terms of $\mathbf{u}$ and $\mathbf{v}$, your condition in part (b) is equivalent to the Cauchy-Schwarz Inequality.

Explorations



Vectors and Matrices with Complex Entries

In this book, we have developed the theory and applications of real vector spaces, the most basic example of which is $\mathbb{R}^n$. We have also explored the finite vector spaces $\mathbb{Z}_p^n$ and their applications. The set $\mathbb{C}^n$ of n-tuples of complex numbers is also a vector space, with the complex numbers $\mathbb{C}$ as scalars. The vector space axioms (Section 6.1) all hold for $\mathbb{C}^n$, and concepts such as linear independence, basis, and dimension carry over from $\mathbb{R}^n$ without difficulty.

The first notable difference between $\mathbb{R}^n$ and $\mathbb{C}^n$ is in the definition of dot product. If we define the dot product in $\mathbb{C}^n$ as in $\mathbb{R}^n$, then for the nonzero vector $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ we have

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{i^2 + 1^2} = \sqrt{-1+1} = \sqrt{0} = 0$$

This is clearly an undesirable situation (a nonzero vector whose length is zero) and violates Theorems 1.2(d) and 1.3. We now generalize the real dot product to $\mathbb{C}^n$ in a way that avoids this type of difficulty.

Definition If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{C}^n$, then the ***complex dot product*** of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u}\cdot\mathbf{v} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n$$

The norm (or length) of a complex vector $\mathbf{v}$ is defined as in the real case: $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$. Likewise, the distance between two complex vectors $\mathbf{u}$ and $\mathbf{v}$ is still defined as $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$.

1. Show that, for $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$, $\|\mathbf{v}\| = \sqrt{|v_1^2| + |v_2^2| + \cdots + |v_n^2|}$.

2. Let $\mathbf{u} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 - 3i \\ 1 + 5i \end{bmatrix}$. Find:

(a) $\mathbf{u} \cdot \mathbf{v}$ (b) $\|\mathbf{u}\|$ (c) $\|\mathbf{v}\|$ (d) $d(\mathbf{u}, \mathbf{v})$ (e) a nonzero vector orthogonal to $\mathbf{u}$
(f) a nonzero vector orthogonal to $\mathbf{v}$

The complex dot product is an example of the more general notion of a complex inner product, which satisfies the same conditions as a real inner product with two exceptions. Problem 3 provides a summary.

3. Prove that the complex dot product satisfies the following properties for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{C}^n$ and all complex scalars.

(a) $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$
(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
(c) $(c\mathbf{u}) \cdot \mathbf{v} = \overline{c}\,(\mathbf{u} \cdot \mathbf{v})$ and $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
(d) $\mathbf{u} \cdot \mathbf{u} \geqslant 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

For matrices with complex entries, addition, multiplication by complex scalars, transpose, and matrix multiplication are all defined exactly as we did for real matrices in Section 3.1, and the algebraic properties of these operations still hold. (See Section 3.2.) Likewise, we have the notion of the inverse and determinant of a square complex matrix just as in the real case, and the techniques and properties all carry over to the complex case. (See Sections 3.3 and 4.2.)

The notion of transpose is, however, less useful in the complex case than in the real case. The following definition provides an alternative.

Definition If A is a complex matrix, then the ***conjugate transpose*** of A is the matrix A^* defined by

$$A^* = \overline{A}^T$$

In the preceding definition, $\overline{A}$ refers to the matrix whose entries are the complex conjugates of the corresponding entries of A; that is, if $A = [a_{ij}]$, then $\overline{A} = [\overline{a}_{ij}]$.

4. Find the conjugate transpose A^* of the given matrix:

(a) $A = \begin{bmatrix} i & 2i \\ -i & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 5 - 2i \\ 5 + 2i & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 - i & 1 + 3i & -2 \\ 4 & 0 & 3 - 4i \end{bmatrix}$ (d) $A = \begin{bmatrix} 3i & 0 & 1 + i \\ 1 - i & 4 & i \\ 1 + i & 0 & -i \end{bmatrix}$

Properties of the complex conjugate (Appendix C) extend to matrices, as the next problem shows.

5. Let A and B be complex matrices, and let c be a complex scalar. Prove the following properties:

(a) $\overline{\overline{A}} = A$ (b) $\overline{A + B} = \overline{A} + \overline{B}$
(c) $\overline{cA} = \overline{c}\,\overline{A}$ (d) $\overline{AB} = \overline{A}\,\overline{B}$
(e) $(\overline{A})^T = (\overline{A^T})$

The properties in Problem 5 can be used to establish the following properties of the conjugate transpose, which are analogous to the properties of the transpose for real matrices (Theorem 3.4).

6. Let A and B be complex matrices, and let c be a complex scalar. Prove the following properties:

(a) $(A^*)^* = A$ (b) $(A + B)^* = A^* + B^*$
(c) $(cA)^* = \bar{c}A^*$ (d) $(AB)^* = B^*A^*$

7. Show that for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{C}^n$, the complex dot product satisfies $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^*\mathbf{v}$. (This result is why we defined the complex dot product as we did. It gives us the analogue of the formula $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$ for vectors in $\mathbb{R}^n$.)

For real matrices, we have seen the importance of symmetric matrices, especially in our study of diagonalization. Recall that a real matrix A is symmetric if $A^T = A$. For complex matrices, the following definition is the correct generalization.

Hermitian matrices are named after the French mathematician Charles Hermite (1822–1901). Hermite is best known for his proof that the number e is transcendental, but he also was the first to use the term *orthogonal matrices*, and he proved that symmetric (and Hermitian) matrices have real eigenvalues.

Definition A square complex matrix A is called ***Hermitian*** if $A^* = A$—that is, if it is equal to its own conjugate transpose.

8. Prove that the diagonal entries of a Hermitian matrix must be real.

9. Which of the following matrices are Hermitian?

(a) $A = \begin{bmatrix} 2 & 1+i \\ 1-i & i \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 2-3i \\ 2-3i & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} -3 & -1+5i \\ 1-5i & 3 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 1+4i & 3-i \\ 1-4i & 2 & i \\ 3+i & -i & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ (f) $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 5 \end{bmatrix}$

10. Prove that the eigenvalues of a Hermitian matrix are real numbers. [*Hint:* The proof of Theorem 5.18 can be adapted by making use of the conjugate transpose operation.]

11. Prove that if A is a Hermitian matrix, then eigenvectors corresponding to distinct eigenvalues of A are orthogonal. [*Hint:* Adapt the proof of Theorem 5.19 using $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^*\mathbf{v}$ instead of $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$.]

Recall that a square real matrix Q is orthogonal if $Q^{-1} = Q^T$. The next definition provides the complex analogue.

Definition A square complex matrix U is called ***unitary*** if $U^{-1} = U^*$.

Just as for orthogonal matrices, in practice it is not necessary to compute U^{-1} directly. You need only show that $U^*U = I$ to verify that U is unitary.

12. Which of the following matrices are unitary? For those that are unitary, give their inverses.

(a) $\begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$ (b) $\begin{bmatrix} 1+i & 1+i \\ 1-i & -1+i \end{bmatrix}$

(c) $\begin{bmatrix} 3/5 & -4/5 \\ 4i/5 & 3i/5 \end{bmatrix}$ (d) $\begin{bmatrix} (1+i)/\sqrt{6} & 0 & 2/\sqrt{6} \\ 0 & 1 & 0 \\ (-1-i)/\sqrt{3} & 0 & 1/\sqrt{3} \end{bmatrix}$

Unitary matrices behave in most respects like orthogonal matrices. The following problem gives some alternative characterizations of unitary matrices.

13. Prove that the following statements are equivalent for a square complex matrix U:

(a) U is unitary.
(b) The columns of U form an orthonormal set in $\mathbb{C}^n$ with respect to the complex dot product.
(c) The rows of U form an orthonormal set in $\mathbb{C}^n$ with respect to the complex dot product.
(d) $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x}$ in $\mathbb{C}^n$.
(e) $U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^n$.

[*Hint:* Adapt the proofs of Theorems 5.4–5.7.]

14. Repeat Problem 12, this time by applying the criterion in part (b) or part (c) of Problem 13.

The next definition is the natural generalization of orthogonal diagonalizability to complex matrices.

Definition A square complex matrix A is called ***unitarily diagonalizable*** if there exists a unitary matrix U and a diagonal matrix D such that

$$U^*AU = D$$

The process for diagonalizing a unitarily diagonalizable $n \times n$ matrix A mimics the real case. The columns of U must form an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A. Therefore, we (1) compute the eigenvalues of A, (2) find a basis for each eigenspace, (3) ensure that each eigenspace basis consists of orthonormal vectors (using the Gram-Schmidt Process, with the complex dot product, if necessary), (4) form the matrix U whose columns are the orthonormal eigenvectors just found. Then U^*AU will be a diagonal matrix D whose diagonal entries are the eigenvalues of A, arranged in the same order as the corresponding eigenvectors in the columns of U.

15. In each of the following, find a unitary matrix U and a diagonal matrix D such that $U^*AU = D$.

(a) $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} -1 & 1+i \\ 1-i & 0 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1-i \\ 0 & 1+i & 3 \end{bmatrix}$

The matrices in (a), (c), and (d) of the preceding problem are all Hermitian. It turns out that every Hermitian matrix is unitarily diagonalizable. (This is the *Complex Spectral Theorem,* which can be proved by adapting the proof of Theorem 5.20.) At this point you probably suspect that the converse of this result must also be true—namely, that every unitarily diagonalizable matrix must be Hermitian. But unfortunately this is *false*! (Can you see where the complex analogue of the proof of Theorem 5.17 breaks down?)

For a specific counterexample, take the matrix in part (b) of Problem 15. It is not Hermitian, but it *is* unitarily diagonalizable.

See *Linear Algebra with Applications* by S. J. Leon (Upper Saddle River, NJ: Prentice-Hall, 2002).

It turns out that the correct characterization of unitary diagonalizability is the following theorem, the proof of which can be found in more advanced textbooks.

A square complex matrix A is ***unitarily diagonalizable*** if and only if

$$A^*A = AA^*$$

A matrix A for which $A^*A = AA^*$ is called ***normal.***

16. Show that every Hermitian matrix, every unitary matrix, and every *skew-Hermitian* matrix ($A^* = -A$) is normal. (Note that in the real case, this result refers to symmetric, orthogonal, and skew-symmetric matrices, respectively.)

17. Prove that if a square complex matrix is unitarily diagonalizable, then it must be normal.

Geometric Inequalities and Optimization Problems

This exploration will introduce some powerful (and perhaps surprising) applications of various inequalities, such as the Cauchy-Schwarz Inequality. As you will see, certain maximization/minimization problems (*optimization problems*) that typically arise in a calculus course can be solved without using calculus at all!

Recall that the Cauchy-Schwarz Inequality in $\mathbb{R}^n$ states that for all vectors $\mathbf{u}$ and $\mathbf{v}$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$$

with equality if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other. If $\mathbf{u} = [x_1 \ \cdots \ x_n]^T$ and $\mathbf{v} = [y_1 \ \cdots \ y_n]^T$, the above inequality is equivalent to

$$|x_1y_1 + \cdots + x_ny_n| \leq \sqrt{x_1^2 + \cdots + x_n^2}\sqrt{y_1^2 + \cdots + y_n^2}$$

Squaring both sides and using summation notation, we have

$$\left(\sum_{i=1}^{n} x_iy_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Equality holds if and only if there is some scalar k such that $y_i = kx_i$ for $i = 1, \ldots, n$.

Let's begin by using Cauchy-Schwarz to derive a special case of one of the most useful of all inequalities.

1. Let x and y be nonnegative real numbers. Apply the Cauchy-Schwarz Inequality to $\mathbf{u} = \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \sqrt{y} \\ \sqrt{x} \end{bmatrix}$ to show that

$$\sqrt{xy} \le \frac{x + y}{2} \tag{1}$$

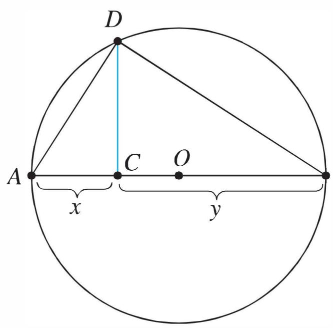


Figure 7.4

with equality if and only if $x = y$.

2. (a) Prove inequality (1) directly. [*Hint:* Square both sides.] (b) Figure 7.4 shows a circle with center O and diameter $AB = AC + CB = x + y$. The segment $\overline{CD}$ is perpendicular to $\overline{AB}$. Prove that $CD = \sqrt{xy}$ and use this result to deduce inequality (1). [*Hint:* Use similar triangles.]

The right-hand side of inequality (1) is the familiar ***arithmetic mean*** (or *average*) of the numbers x and y. The left-hand side shows the less familiar ***geometric mean*** of x and y. Accordingly, inequality (1) is known as the ***Arithmetic Mean–Geometric Mean Inequality (AMGM)***. It holds more generally; for n nonnegative variables $x_1, \ldots, x_n$, it states

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

In words, the AMGM Inequality says that the geometric mean of a set of nonnegative numbers is always less than or equal to their arithmetic mean, and the two are the same precisely when all of the numbers are the same. (For the general proof, see Appendix B.)

We now explore how such an inequality can be applied to optimization problems. Here is a typical calculus problem.

Example 7.9

Prove that among all rectangles whose perimeter is 100 units, the square has the largest area.

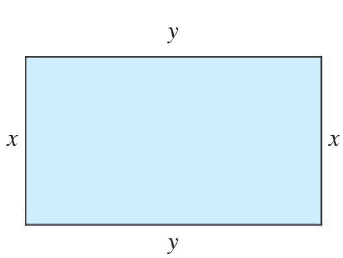


Figure 7.5

Solution If we let x and y be the dimensions of the rectangle (see Figure 7.5), then the area we want to maximize is given by

$$A = xy$$

We are given that the perimeter satisfies

$$2x + 2y = 100$$

which is the same as $x + y = 50$. We can relate xy and $x + y$ using the AMGM Inequality:

$$\sqrt{xy} \leq \frac{x + y}{2} \quad \text{or, equivalently,} \quad xy \leq \tfrac{1}{4}(x + y)^2$$

Since $x + y = 50$ is a *constant* (and this is the key), we see that the maximum value of $A = xy$ is $50^2/4 = 625$ and it occurs when $x = y = 25$.

Not a derivative in sight! Isn't that impressive? Notice that in this maximization problem, the crucial step was showing that the right-hand side of the AMGM Inequality was *constant*. In a similar fashion, we may be able to apply the inequality to a *minimization* problem if we can arrange for the left-hand side to be constant.

Example 7.10

Prove that among all rectangular prisms with volume 8 m^3, the cube has the minimum surface area.

Solution As shown in Figure 7.6, if the dimensions of such a prism are x, y, and z, then its volume is given by

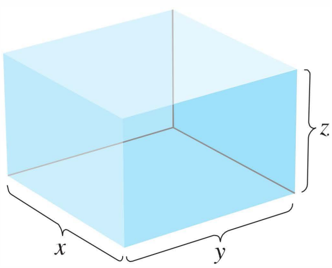


Figure 7.6

$$V = xyz$$

Thus, we are given that $xyz = 8$. The surface area to be minimized is

$$S = 2xy + 2yz + 2zx$$

Since this is a three-variable problem, the obvious thing to try is the version of the AMGM Inequality for $n = 3$—namely,

$$\sqrt[3]{xyz} \leq \frac{x + y + z}{3}$$

Unfortunately, the expression for S does not appear here. However, the AMGM Inequality also implies that

$$\begin{aligned}\frac{S}{3} &= \frac{2xy + 2yz + 2zx}{3} \\ &\geq \sqrt[3]{(2xy)(2yz)(2zx)} \\ &= 2\sqrt[3]{(xyz)^2} \\ &= 2\sqrt[3]{64} = 8\end{aligned}$$

which is equivalent to $S \geq 24$. Therefore, the minimum value of S is 24, and it occurs when

$$2xy = 2yz = 2zx$$

(Why?) This implies that $x = y = z = 2$ (i.e., the rectangular prism is a cube).

3. Prove that among all rectangles with area 100 square units, the square has the smallest perimeter.

4. What is the minimum value of $f(x) = x + \dfrac{1}{x}$ for $x > 0$?

5. A cardboard box with a square base and an open top is to be constructed from a square of cardboard 10 cm on a side by cutting out four squares at the corners and folding up the sides. What should the dimensions of the box be in order to make the enclosed volume as large as possible?

6. Find the minimum value of $f(x, y, z) = (x + y)(y + z)(z + x)$ if x, y, and z are positive real numbers such that $xyz = 1$.

7. For $x > y > 0$, find the minimum value of $x + \dfrac{8}{y(x - y)}$. [*Hint:* A substitution might help.]

The Cauchy-Schwarz Inequality itself can be applied to similar problems, as the next example illustrates.

Example 7.11

Find the maximum value of the function $f(x, y, z) = 3x + y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$. Where does the maximum value occur?

Solution This sort of problem is usually handled by techniques covered in a multivariable calculus course. Here's how to use the Cauchy-Schwarz Inequality. The function $3x + y + 2z$ has the form of a dot product, so we let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then the componentwise form of the Cauchy-Schwarz Inequality gives

$$(3x + y + 2z)^2 \le (3^2 + 1^2 + 2^2)(x^2 + y^2 + z^2) = 14$$

Thus, the maximum value of our function is $\sqrt{14}$, and it occurs when

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Therefore, $x = 3k$, $y = k$, and $z = 2k$, so $3(3k) + k + 2(2k) = \sqrt{14}$. It follows that $k = 1/\sqrt{14}$, and hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/\sqrt{14} \\ 1/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}$$

8. Find the maximum value of $f(x, y, z) = x + 2y + 4z$ subject to $x^2 + 2y^2 + z^2 = 1$.

9. Find the minimum value of $f(x, y, z) = x^2 + y^2 + \dfrac{z^2}{2}$ subject to $x + y + z = 10$.

10. Find the maximum value of $\sin\theta + \cos\theta$.

11. Find the point on the line $x + 2y = 5$ that is closest to the origin.

There are many other inequalities that can be used to solve optimization problems. The ***quadratic mean*** of the numbers $x_1, \ldots, x_n$ is defined as

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

If $x_1, \ldots, x_n$ are nonzero, their ***harmonic mean*** is given by

$$\frac{n}{1/x_1 + 1/x_2 + \cdots + 1/x_n}$$

It turns out that the quadratic, arithmetic, geometric, and harmonic means are all related.

12. Let x and y be positive real numbers. Show that

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2} \geq \sqrt{xy} \geq \frac{2}{1/x + 1/y}$$

with equality if and only if $x = y$. (The middle inequality is just AMGM, so you need only establish the first and third inequalities.)

13. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r (Figure 7.7).

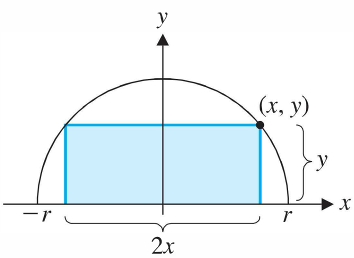


Figure 7.7

14. Find the minimum value of the function

$$f(x, y) = \frac{(x + y)^2}{xy}$$

for $x, y > 0$. [*Hint:* $(x + y)^2/xy = (x + y)(1/x + 1/y)$.]

15. Let x and y be positive real numbers with $x + y = 1$. Show that the minimum value of

$$f(x, y) = \left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2$$

is $\frac{25}{2}$, and determine the values of x and y for which it occurs.

7.2 Norms and Distance Functions

In the last section, you saw that it is possible to define length and distance in an inner product space. As you will see shortly, there are also some versions of these two concepts that are not defined in terms of an inner product.

To begin, we need to specify the properties that we want a "length function" to have. The following definition does this, using as its basis Theorem 1.3 and the Triangle Inequality.

Definition A ***norm*** on a vector space V is a mapping that associates with each vector $\mathbf{v}$ a real number $\|\mathbf{v}\|$, called the ***norm*** of $\mathbf{v}$, such that the following properties are satisfied for all vectors $\mathbf{u}$ and $\mathbf{v}$ and all scalars c:

1. $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

A vector space with a norm is called a ***normed linear space***.

Example 7.12 Show that in an inner product space, $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ defines a norm.

Solution Clearly, $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \geq 0$. Moreover,

$$\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0 \Leftrightarrow \langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$$

by the definition of inner product. This proves property (1).

For property (2), we only need to note that

$$\|c\mathbf{v}\| = \sqrt{\langle c\mathbf{v}, c\mathbf{v} \rangle} = \sqrt{c^2 \langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{c^2}\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = |c|\|\mathbf{v}\|$$

Property (3) is just the Triangle Inequality, which we verified in Theorem 7.4.

We now look at some examples of norms that are not defined in terms of an inner product. Example 7.13 is the mathematical generalization to $\mathbb{R}^n$ of the taxicab norm that we explored in the Introduction to this chapter.

Example 7.13 The ***sum norm*** $\|\mathbf{v}\|_s$ of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the sum of the absolute values of its components. That is, if $\mathbf{v} = [v_1 \ \cdots \ v_n]^T$, then

$$\|\mathbf{v}\|_s = |v_1| + \cdots + |v_n|$$

Show that the sum norm is a norm.

Solution Clearly, $\|\mathbf{v}\|_s = |v_1| + \cdots + |v_n| \geq 0$, and the only way to achieve equality is if $|v_1| = \cdots = |v_n| = 0$. But this is so if and only if $v_1 = \cdots = v_n = 0$ or, equivalently, $\mathbf{v} = \mathbf{0}$, proving property (1). For property (2), we see that $c\mathbf{v} = [cv_1 \ \cdots \ cv_n]^T$, so

$$\|c\mathbf{v}\|_s = |cv_1| + \cdots + |cv_n| = |c|(|v_1| + \cdots + |v_n|) = |c|\|\mathbf{v}\|_s$$

Finally, the Triangle Inequality holds, because if $\mathbf{u} = [u_1 \ \cdots \ u_n]^T$, then

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_s &= |u_1+v_1| + \cdots + |u_n+v_n| \\ &\le (|u_1|+|v_1|) + \cdots + (|u_n|+|v_n|) \\ &= (|u_1| + \cdots + |u_n|) + (|v_1| + \cdots + |v_n|) = \|\mathbf{u}\|_s + \|\mathbf{v}\|_s\end{aligned}$$

The sum norm is also known as the **1-*norm*** and is often denoted by $\|\mathbf{v}\|_1$. On $\mathbb{R}^2$, it is the same as the taxicab norm. As Example 7.13 shows, it is possible to have several norms on the same vector space. Example 7.14 illustrates another norm on $\mathbb{R}^n$.

Example 7.14

The ***max norm*** $\|\mathbf{v}\|_m$ of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the largest number among the absolute values of its components. That is, if $\mathbf{v} = [v_1 \ \cdots \ v_n]^T$, then

$$\|\mathbf{v}\|_m = \max\{|v_1|, \ldots, |v_n|\}$$

Show that the max norm is a norm.

Solution Again, it is clear that $\|\mathbf{v}\|_m \ge 0$. If $\|\mathbf{v}\|_m = 0$, then the largest of $|v_1|, \ldots, |v_n|$ is zero, and so they all are. Hence, $v_1 = \cdots = v_n = 0$, so $\mathbf{v} = \mathbf{0}$. This verifies property (1). Next, we observe that for any scalar c,

$$\|c\mathbf{v}\|_m = \max\{|cv_1|, \ldots, |cv_n|\} = |c|\max\{|v_1|, \ldots, |v_n|\} = |c|\,\|\mathbf{v}\|_m$$

Finally, for $\mathbf{u} = [u_1 \ \cdots \ u_n]^T$, we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_m &= \max\{|u_1+v_1|, \ldots, |u_n+v_n|\} \\ &\le \max\{|u_1|+|v_1|, \ldots, |u_n|+|v_n|\} \\ &\le \max\{|u_1|, \ldots, |u_n|\} + \max\{|v_1|, \ldots, |v_n|\} = \|\mathbf{u}\|_m + \|\mathbf{v}\|_m\end{aligned}$$

(Why is the second inequality true?) This verifies the Triangle Inequality.

The max norm is also known as the ∞-***norm*** or ***uniform norm*** and is often denoted by $\|\mathbf{v}\|_\infty$. In general, it is possible to define a norm $\|\mathbf{v}\|_p$ on $\mathbb{R}^n$ by

$$\|\mathbf{v}\|_p = (|v_1|^p + \cdots + |v_n|^p)^{1/p}$$

for any real number $p \ge 1$. For $p = 1$, $\|\mathbf{v}\|_1 = \|\mathbf{v}\|_s$, justifying the term 1-norm. For $p = 2$,

$$\|\mathbf{v}\|_2 = (|v_1|^2 + \cdots + |v_n|^2)^{1/2} = \sqrt{v_1^2 + \cdots + v_n^2}$$

which is just the familiar norm on $\mathbb{R}^n$ obtained from the dot product. Called the **2-*norm*** or ***Euclidean norm,*** it is often denoted by $\|\mathbf{v}\|_E$. As p gets large, it can be shown using calculus that $\|\mathbf{v}\|_p$ approaches the max norm $\|\mathbf{v}\|_m$. This justifies the use of the alternative notation $\|\mathbf{v}\|_\infty$ for this norm.

Example 7.15

For a vector $\mathbf{v}$ in $\mathbb{Z}_2^n$, define $\|\mathbf{v}\|_H$ to be $w(\mathbf{v})$, the weight of $\mathbf{v}$. Show that it defines a norm.

Solution Certainly, $\|\mathbf{v}\|_H = w(\mathbf{v}) \geq 0$, and the only vector whose weight is zero is the zero vector. Therefore, property (1) is true. Since the only candidates for a scalar c are 0 and 1, property (2) is immediate.

To verify the Triangle Inequality, first observe that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{Z}_2^n$, then $w(\mathbf{u}+\mathbf{v})$ counts the number of places in which $\mathbf{u}$ and $\mathbf{v}$ differ. [For example, if

$$\mathbf{u} = [1 \quad 1 \quad 0 \quad 1 \quad 0]^T \quad \text{and} \quad \mathbf{v} = [0 \quad 1 \quad 1 \quad 1 \quad 1]^T$$

then $\mathbf{u}+\mathbf{v} = [1 \quad 0 \quad 1 \quad 0 \quad 1]^T$, so $w(\mathbf{u}+\mathbf{v}) = 3$, in agreement with the fact that $\mathbf{u}$ and $\mathbf{v}$ differ in exactly three positions.] Suppose that both $\mathbf{u}$ and $\mathbf{v}$ have zeros in n_0 positions and 1s in n_1 positions, $\mathbf{u}$ has a 0 and $\mathbf{v}$ has a 1 in n_{01} positions, and $\mathbf{u}$ has a 1 and $\mathbf{v}$ has a 0 in n_{10} positions. (In the example above, $n_0 = 0$, $n_1 = 2$, $n_{01} = 2$, and $n_{10} = 1$.) Now

$$w(\mathbf{u}) = n_1 + n_{10}, \; w(\mathbf{v}) = n_1 + n_{01}, \quad \text{and} \quad w(\mathbf{u}+\mathbf{v}) = n_{10} + n_{01}$$

Therefore,

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_H = w(\mathbf{u}+\mathbf{v}) &= n_{10} + n_{01}\\ &= (n_1 + n_{10}) + (n_1 + n_{01}) - 2n_1\\ &\leq (n_1 + n_{10}) + (n_1 + n_{01})\\ &= w(\mathbf{u}) + w(\mathbf{v}) = \|\mathbf{u}\|_H + \|\mathbf{v}\|_H\end{aligned}$$

The norm $\|\mathbf{v}\|_H$ is called the ***Hamming norm***.

Distance Functions

For any norm, we can define a distance function just as we did in the last section—namely,

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Example 7.16

Let $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Compute $d(\mathbf{u}, \mathbf{v})$ relative to (a) the Euclidean norm, (b) the sum norm, and (c) the max norm.

Solution Each calculation requires knowing that $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

(a) As is by now quite familiar,

$$d_E(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_E = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$$

(b) $d_s(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_s = |4| + |-3| = 7$

(c) $d_m(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_m = \max\{|4|, |-3|\} = 4$

The distance function on $\mathbb{Z}_2^n$ determined by the Hamming norm is called the ***Hamming distance.*** We will explore its use in error-correcting codes in Section 8.5. Example 7.17 provides an illustration of the Hamming distance.

Example 7.17

Find the Hamming distance between

$$\mathbf{u} = [1 \quad 1 \quad 0 \quad 1 \quad 0]^T \quad \text{and} \quad \mathbf{v} = [0 \quad 1 \quad 1 \quad 1 \quad 1]^T$$

Solution Since we are working over $\mathbb{Z}_2$, $\mathbf{u} - \mathbf{v} = \mathbf{u} + \mathbf{v}$. But

$$d_H(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} + \mathbf{v}\|_H = w(\mathbf{u} + \mathbf{v})$$

As we noted in Example 7.15, this is just the number of positions in which $\mathbf{u}$ and $\mathbf{v}$ differ. The given vectors are the same ones used in that example; the calculation is therefore exactly the same. Hence, $d_H(\mathbf{u}, \mathbf{v}) = 3$.

Theorem 7.5 summarizes the most important properties of a distance function.

Theorem 7.5

Let d be a distance function defined on a normed linear space V. The following properties hold for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V:

a. $d(\mathbf{u}, \mathbf{v}) \geq 0$, and $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.
b. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
c. $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$

Proof (a) Using property (1) from the definition of a norm, it is easy to check that $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \geq 0$, with equality holding if and only if $\mathbf{u} - \mathbf{v} = \mathbf{0}$ or, equivalently, $\mathbf{u} = \mathbf{v}$.

(b) You are asked to prove property (b) in Exercise 19.

(c) We apply the Triangle Inequality to obtain

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}) &= \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \\ &\geq \|(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w})\| \\ &= \|\mathbf{u} - \mathbf{w}\| = d(\mathbf{u}, \mathbf{w}) \end{aligned}$$

A function d satisfying the three properties of Theorem 7.5 is also called a ***metric,*** and a vector space that possesses such a function is called a ***metric space.*** These are very important in many branches of mathematics and are studied in detail in more advanced courses.

Matrix Norms

We can define norms for matrices exactly as we defined norms for vectors in $\mathbb{R}^n$. After all, the vector space M_{mn} of all $m \times n$ matrices is isomorphic to $\mathbb{R}^{mn}$, so this is not difficult to do. Of course, properties (1), (2), and (3) of a norm will also hold in the setting of matrices. It turns out that, for matrices, the norms that are most useful satisfy an additional property. (We will restrict our attention to square matrices, but it is possible to generalize everything to arbitrary matrices.)

Definition A ***matrix norm*** on M_{nn} is a mapping that associates with each $n \times n$ matrix A a real number $\|A\|$, called the ***norm*** of A, such that the following properties are satisfied for all $n \times n$ matrices A and B and all scalars c.

1. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = O$.
2. $\|cA\| = |c|\|A\|$
3. $\|A + B\| \leq \|A\| + \|B\|$
4. $\|AB\| \leq \|A\|\|B\|$

A matrix norm on M_{nn} is said to be ***compatible*** with a vector norm $\|\mathbf{x}\|$ on $\mathbb{R}^n$ if, for all $n \times n$ matrices A and all vectors $\mathbf{x}$ in $\mathbb{R}^n$, we have

$$\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$$

Example 7.18

The ***Frobenius norm*** $\|A\|_F$ of a matrix A is obtained by stringing out the entries of the matrix into a vector and then taking the Euclidean norm. In other words, $\|A\|_F$ is just the square root of the sum of the squares of the entries of A. So, if $A = [a_{ij}]$, then

$$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}$$

(a) Find the Frobenius norm of

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

(b) Show that the Frobenius norm is compatible with the Euclidean norm.

(c) Show that the Frobenius norm is a matrix norm.

Solution (a) $\|A\|_F = \sqrt{3^2 + (-1)^2 + 2^2 + 4^2} = \sqrt{30}$

Before we continue, observe that if $\mathbf{A}_1 = [3 \quad -1]$ and $\mathbf{A}_2 = [2 \quad 4]$ are the row vectors of A, then $\|\mathbf{A}_1\|_E = \sqrt{3^2 + (-1)^2}$ and $\|\mathbf{A}_2\|_E = \sqrt{2^2 + 4^2}$. Thus,

$$\|A\|_F = \sqrt{\|\mathbf{A}_1\|_E^2 + \|\mathbf{A}_2\|_E^2}$$

Similarly, if $\mathbf{a}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ are the column vectors of A, then

$$\|A\|_F = \sqrt{\|\mathbf{a}_1\|_E^2 + \|\mathbf{a}_2\|_E^2}$$

It is easy to see that these facts extend to $n \times n$ matrices in general. We will use these observations to solve parts (b) and (c).

(b) Write $$A = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$$

Then

$$\begin{aligned}\|A\mathbf{x}\|_E &= \left\| \begin{bmatrix} A_1\mathbf{x} \\ \vdots \\ A_n\mathbf{x} \end{bmatrix} \right\|_E \\ &= \sqrt{\|\mathbf{A}_1\mathbf{x}\|_E^2 + \cdots + \|\mathbf{A}_n\mathbf{x}\|_E^2} \\ &\le \sqrt{\|\mathbf{A}_1\|_E^2\|\mathbf{x}\|_E^2 + \cdots + \mathbf{A}_n\|_E^2\|\mathbf{x}\|_E^2} \\ &= (\sqrt{\|\mathbf{A}_1\|_E^2 + \cdots + \|\mathbf{A}_n\|_E^2})\|\mathbf{x}\|_E \\ &= \|\mathbf{A}\|_F\|\mathbf{x}\|_E\end{aligned}$$

where the inequality arises from the Cauchy-Schwarz Inequality applied to the dot products of the row vectors $\mathbf{A}_i$ with the column vector $\mathbf{x}$. (Do you see how Cauchy-Schwarz has been applied?) Hence, the Frobenius norm is compatible with the Euclidean norm.

(c) Let $\mathbf{b}_i$ denote the ith column of B. Using the matrix-column representation of the product AB, we have

$$\begin{aligned}\|AB\|_F &= \|[A\mathbf{b}_1 \cdots A\mathbf{b}_n]\|_F \\ &= \sqrt{\|A\mathbf{b}_1\|_E^2 + \cdots + \|A\mathbf{b}_n\|_E^2} \\ &\le \sqrt{\|A\|_F^2\|\mathbf{b}_1\|_E^2 + \cdots + \|A\|_F^2\|\mathbf{b}_n\|_E^2} && \text{by part (b)} \\ &= \|A\|_F\sqrt{\|\mathbf{b}_1\|_E^2 + \cdots + \|\mathbf{b}_n\|_E^2} \\ &= \|A\|_F\|B\|_F\end{aligned}$$

which proves property (4) of the definition of a matrix norm. Properties (1) through (3) are true, since the Frobenius norm is derived from the Euclidean norm, which satisfies these properties. Therefore, the Frobenius norm is a matrix norm.

For many applications, the Frobenius matrix norm is not the best (or the easiest) one to use. The most useful types of matrix norms arise from considering the effect of the matrix transformation corresponding to the square matrix A. This transformation maps a vector $\mathbf{x}$ into $A\mathbf{x}$. One way to measure the "size" of A is to compare $\|\mathbf{x}\|$ and $\|A\mathbf{x}\|$ using any convenient (vector) norm. Let's think ahead. Whatever definition of $\|A\|$ we arrive at, we know we are going to want it to be compatible with the vector norm we are using; that is, we will need

$$\|A\mathbf{x}\| \le \|A\|\,\|\mathbf{x}\| \quad \text{or} \quad \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \le \|A\| \quad \text{for } \mathbf{x} \ne \mathbf{0}$$

The expression $\dfrac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$ measures the "stretching capability" of A. If we normalize each nonzero vector $\mathbf{x}$ by dividing it by its norm, we get unit vectors $\hat{\mathbf{x}} = \dfrac{1}{\|\mathbf{x}\|}\mathbf{x}$ and thus

$$\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{1}{\|\mathbf{x}\|}\|A\mathbf{x}\| = \left\|\frac{1}{\|\mathbf{x}\|}(A\mathbf{x})\right\| = \left\|A\left(\frac{1}{\|\mathbf{x}\|}\mathbf{x}\right)\right\| = \|A\hat{\mathbf{x}}\|$$

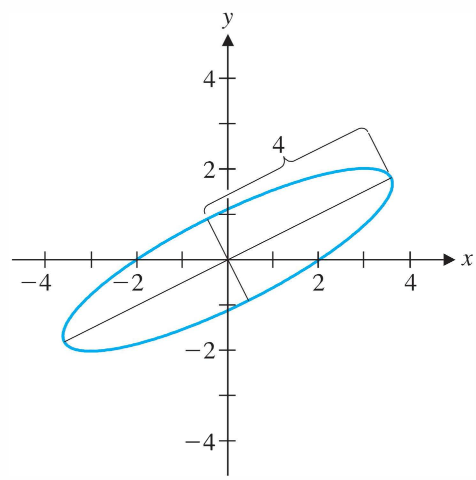


Figure 7.8

If $\mathbf{x}$ ranges over all nonzero vectors in $\mathbb{R}^n$, then $\hat{\mathbf{x}}$ ranges over all *unit* vectors (i.e., the unit sphere) and the set of all vectors $A\hat{\mathbf{x}}$ determines some curve in $\mathbb{R}^n$. For example, Figure 7.8 shows how the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ affects the unit circle in $\mathbb{R}^2$—it maps it into an ellipse. With the Euclidean norm, the maximum value of $\|A\hat{\mathbf{x}}\|$ is clearly just half the length of the principal axis—in this case, 4 units. We express this by writing $\max_{\|\hat{\mathbf{x}}\|=1} \|A\hat{\mathbf{x}}\| = 4$.

In Section 7.4, we will see that this is not an isolated phenomenon. That is,

$$\max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\hat{\mathbf{x}}\|=1} \|A\hat{\mathbf{x}}\|$$

always exists, and there is a particular unit vector $\mathbf{y}$ for which $\|A\mathbf{y}\|$ *is* maximum. Now we prove that $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ defines a matrix norm.

Theorem 7.6 If $\|\mathbf{x}\|$ is a vector norm on $\mathbb{R}^n$, then $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ defines a matrix norm on M_{nn} that is compatible with the vector norm that induces it.

Proof (1) Certainly, $\|A\mathbf{x}\| \geq 0$ for all vectors $\mathbf{x}$, so, in particular, this inequality is true if $\|\mathbf{x}\| = 1$. Hence, $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \geq 0$ also. If $\|A\| = 0$, then we must have $\|A\mathbf{x}\| = 0$—and, hence, $A\mathbf{x} = \mathbf{0}$—for all $\mathbf{x}$ with $\|\mathbf{x}\| = 1$. In particular, $A\mathbf{e}_i = \mathbf{0}$ for each of the standard basis vectors $\mathbf{e}_i$ in $\mathbb{R}^n$. But $A\mathbf{e}_i$ is just the ith column of A, so we must have $A = O$. Conversely, if $A = O$, it is clear that $\|A\| = 0$. (Why?)

(2) Let c be a scalar. Then

$$\|cA\| = \max_{\|\mathbf{x}\|=1} \|cA\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} |c|\,\|A\mathbf{x}\| = |c| \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = |c|\,\|A\|$$

(3) Let B be an $n \times n$ matrix and let $\mathbf{y}$ be a unit vector for which

$$\|A + B\| = \max_{\|\mathbf{x}\|=1} \|(A + B)\mathbf{x}\| = \|(A + B)\mathbf{y}\|$$

Then

$$\begin{aligned} \|A + B\| &= \|(A + B)\mathbf{y}\| \\ &= \|A\mathbf{y} + B\mathbf{y}\| \\ &\le \|A\mathbf{y}\| + \|B\mathbf{y}\| \\ &\le \|A\| + \|B\| \end{aligned}$$

(Where does the second inequality come from?) Next, we show that our definition is compatible with the vector norm [property (5)] and then use this fact to complete the proof that we have a matrix norm.

(5) If $\mathbf{x} = \mathbf{0}$, then the inequality $\|A\mathbf{x}\| \le \|A\| \, \|\mathbf{x}\|$ is true, since both sides are zero. If $\mathbf{x} \ne \mathbf{0}$, then from the comments preceding this theorem,

$$\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \le \max_{\mathbf{x} \ne \mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \|A\|$$

Hence, $\|A\mathbf{x}\| \le \|A\| \, \|\mathbf{x}\|$.

(4) Let $\mathbf{z}$ be a unit vector such that $\|AB\| = \max_{\|\mathbf{x}\|=1} \|(AB)\mathbf{x}\| = \|AB\mathbf{z}\|$. Then

$$\begin{aligned} \|AB\| &= \|AB\mathbf{z}\| \\ &= \|A(B\mathbf{z})\| \\ &\le \|A\| \, \|B\mathbf{z}\| && \text{by property (5)} \\ &\le \|A\| \, \|B\| \, \|\mathbf{z}\| && \text{by property (5)} \\ &= \|A\| \, \|B\| \end{aligned}$$

This completes the proof that $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ defines a matrix norm on M_{nn} that is compatible with the vector norm that induces it.

Definition The matrix norm $\|A\|$ in Theorem 7.6 is called the ***operator norm*** induced by the vector norm $\|\mathbf{x}\|$.

The term *operator norm* reflects the fact that a matrix transformation arising from a square matrix is also called a *linear operator*. This norm is therefore a measure of the stretching capability of a linear operator.

The three most commonly used operator norms are those induced by the sum norm, the Euclidean norm, and the max norm—namely,

$$\|A\|_1 = \max_{\|\mathbf{x}\|_s=1} \|A\mathbf{x}\|_s, \quad \|A\|_2 = \max_{\|\mathbf{x}\|_E=1} \|A\mathbf{x}\|_E, \quad \|A\|_\infty = \max_{\|\mathbf{x}\|_m=1} \|A\mathbf{x}\|_m$$

respectively. The first and last of these turn out to have especially nice formulas that make them very easy to compute.

Theorem 7.7 Let A be an $n \times n$ matrix with column vectors $\mathbf{a}_j$ and row vectors $\mathbf{A}_i$ for $i = 1, \ldots, n$.

a. $\|A\|_1 = \max_{j=1,\ldots,n} \{\|\mathbf{a}_j\|_s\} = \max_{j=1,\ldots,n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}$

b. $\|A\|_\infty = \max_{i=1,\ldots,n} \{\|\mathbf{A}_i\|_s\} = \max_{i=1,\ldots,n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$

In other words, $\|A\|_1$ is the largest absolute column sum, and $\|A\|_\infty$ is the largest absolute row sum. Before we prove the theorem, let's look at an example to see how easy it is to use.

Example 7.19 Let

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix}$$

Find $\|A\|_1$ and $\|A\|_\infty$.

Solution Clearly, the largest absolute column sum is in the first column, so

$$\|A\|_1 = \|\mathbf{a}_1\|_s = |1| + |4| + |-5| = 10$$

The third row has the largest absolute row sum, so

$$\|A\|_\infty = \|\mathbf{A}_3\|_s = |-5| + |1| + |3| = 9$$

With reference to the definition $\|A\|_1 = \max_{\|\mathbf{x}\|_s=1} \|A\mathbf{x}\|_s$, we see that the maximum value of 10 is actually achieved when we take $\mathbf{x} = \mathbf{e}_1$, for then

$$\|A\mathbf{e}_1\|_s = \|\mathbf{a}_1\|_s = 10 = \|A\|_1$$

For $\|A\|_\infty = \max_{\|\mathbf{x}\|_m=1} \|A\mathbf{x}\|_m$, if we take

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

we obtain

$$\|A\mathbf{x}\|_m = \left\| \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\|_m = \left\| \begin{bmatrix} -2 \\ -7 \\ 9 \end{bmatrix} \right\|_m$$
$$= \max\{|-2|, |-7|, |9|\} = 9 = \|A\|_\infty$$

We will use these observations in proving Theorem 7.7.

Proof of Theorem 7.7 The strategy is the same in the case of both the column sum and the row sum. If M represents the maximum value, we show that $\|\mathbf{Ax}\| \le M$ for all unit vectors $\mathbf{x}$. Then we find a specific unit vector $\mathbf{x}$ for which equality occurs. It is important to remember that for property (a) the vector norm is the sum norm whereas for property (b) it is the max norm.

(a) To prove (a), let $M = \max_{j=1,\ldots,n} \{\|\mathbf{a}_j\|_s\}$, the maximum absolute column sum, and let $\|\mathbf{x}\|_s = 1$. Then $|x_1| + \cdots + |x_n| = 1$, so

$$\begin{aligned}\|\mathbf{Ax}\|_s &= \|x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n\|_s \\ &\le |x_1|\,\|\mathbf{a}_1\|_s + \cdots + |x_n|\,\|\mathbf{a}_n\|_s \\ &\le |x_1|M + \cdots + |x_n|M \\ &= (|x_1| + \cdots + |x_n|)M = 1 \cdot M = M\end{aligned}$$

If the maximum absolute column sum occurs in column k, then with $\mathbf{x} = \mathbf{e}_k$ we obtain

$$\|\mathbf{Ae}_k\|_s = \|\mathbf{a}_k\|_s = M$$

Therefore, $\|A\|_1 = \max_{\|\mathbf{x}\|_s=1} \|A\mathbf{x}\|_s = M = \max_{j=1,\ldots,n} \{\|\mathbf{a}_j\|_s\}$, as required.

(b) The proof of property (b) is left as Exercise 32.

In Section 7.4, we will discover a formula for the operator norm $\|A\|_2$, although it is not as computationally feasible as the formula for $\|A\|_1$ or $\|A\|_\infty$.

The Condition Number of a Matrix

In Exploration: Lies My Computer Told Me in Chapter 2, we encountered the notion of an *ill-conditioned* system of linear equations. Here is the definition as it applies to matrices.

Definition A matrix A is ***ill-conditioned*** if small changes in its entries can produce large changes in the solutions to $A\mathbf{x} = \mathbf{b}$. If small changes in the entries of A produce only small changes in the solutions to $A\mathbf{x} = \mathbf{b}$, then A is called ***well-conditioned.***

Although the definition applies to arbitrary matrices, we will restrict our attention to square matrices.

Example 7.20

Show that $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0005 \end{bmatrix}$ is ill-conditioned.

Solution If we take $\mathbf{b} = \begin{bmatrix} 3 \\ 3.0010 \end{bmatrix}$, then the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. However, if A changes to

$$A' = \begin{bmatrix} 1 & 1 \\ 1 & 1.0010 \end{bmatrix}$$

then the solution changes to $\mathbf{x}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (Check these assertions.) Therefore, a relative change of $0.0005/1.0005 \approx 0.0005$, or about 0.05%, causes a change of $(2 - 1)/1 = 1$, or 100%, in x_1 and $(1 - 2)/2 = -0.5$, or -50%, in x_2. Hence, A is ill-conditioned.

We can use matrix norms to give a more precise way of determining when a matrix is ill-conditioned. Think of the change from A to A' as an error ΔA that, in turn, introduces an error $\Delta\mathbf{x}$ in the solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. Then $A' = A + \Delta A$ and $\mathbf{x}' = \mathbf{x} + \Delta\mathbf{x}$. In Example 7.20,

$$\Delta A = \begin{bmatrix} 0 & 0 \\ 0 & 0.0005 \end{bmatrix} \quad \text{and} \quad \Delta\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then, since $A\mathbf{x} = \mathbf{b}$ and $A'\mathbf{x}' = \mathbf{b}$, we have $(A + \Delta A)(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b}$. Expanding and canceling off $A\mathbf{x} = \mathbf{b}$, we obtain

$$A(\Delta\mathbf{x}) + (\Delta A)\mathbf{x} + (\Delta A)(\Delta\mathbf{x}) = 0 \quad \text{or} \quad A(\Delta\mathbf{x}) = -\Delta A(\mathbf{x} + \Delta\mathbf{x})$$

Since we are assuming that $A\mathbf{x} = \mathbf{b}$ has a solution, A must be invertible. Therefore, we can rewrite the last equation as

$$\Delta\mathbf{x} = -A^{-1}(\Delta A)(\mathbf{x} + \Delta\mathbf{x}) = -A^{-1}(\Delta A)\mathbf{x}'$$

Taking norms of both sides (using a matrix norm that is compatible with a vector norm), we have

$$\begin{aligned} \|\Delta\mathbf{x}\| &= \|-A^{-1}(\Delta A)\mathbf{x}'\| = \|A^{-1}(\Delta A)\mathbf{x}'\| \\ &\le \|A^{-1}(\Delta A)\|\,\|\mathbf{x}'\| \\ &\le \|A^{-1}\|\,\|\Delta A\|\,\|\mathbf{x}'\| \end{aligned}$$

(What is the justification for each step?) Therefore,

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}'\|} \le \|A^{-1}\|\,\|\Delta A\| = (\|A^{-1}\|\,\|A\|)\frac{\|\Delta A\|}{\|A\|}$$

The expression $\|A^{-1}\|\,\|A\|$ is called the ***condition number*** of A and is denoted by $\operatorname{cond}(A)$. If A is not invertible, we define $\operatorname{cond}(A) = \infty$.

What are we to make of the inequality just above? The ratio $\|\Delta A\|/\|A\|$ is a measure of the *relative change* in the matrix A, which we are assuming to be small. Similarly, $\|\Delta\mathbf{x}\|/\|\mathbf{x}'\|$ is a measure of the relative error created in the solution to $A\mathbf{x} = \mathbf{b}$ (although, in this case, the error is measured relative to the *new* solution, $\mathbf{x}'$, not the original one, $\mathbf{x}$). Thus, the inequality

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}'\|} \le \operatorname{cond}(A)\frac{\|\Delta A\|}{\|A\|} \tag{1}$$

gives an upper bound on how large the relative error in the solution can be in terms of the relative error in the coefficient matrix. The larger the condition number, the more ill-conditioned the matrix, since there is more "room" for the error to be large relative to the solution.

Remarks

- The condition number of a matrix depends on the choice of norm. The most commonly used norms are the operator norms $\|A\|_1$ and $\|A\|_\infty$.
- For any norm, $\text{cond}(A) \geq 1$. (See Exercise 45.)

Example 7.21

Find the condition number of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0005 \end{bmatrix}$ relative to the ∞-norm.

Solution We first compute

$$A^{-1} = \begin{bmatrix} 2001 & -2000 \\ -2000 & 2000 \end{bmatrix}$$

Therefore, in the ∞-norm (maximum absolute row sum),

$$\|A\|_\infty = 1 + 1.0005 = 2.0005 \quad \text{and} \quad \|A^{-1}\|_\infty = 2001 + |-2000| = 4001$$

so $\text{cond}_\infty(A) = \|A^{-1}\|_\infty \|A\|_\infty = 4001(2.0005) \approx 8004$.

It turns out that if the condition number is large relative to one compatible matrix norm, it will be large relative to *any* compatible matrix norm. For example, it can be shown that for matrix A in Examples 7.20 and 7.21, $\text{cond}_1(A) \approx 8004$, $\text{cond}_2(A) \approx 8002$ (relative to the 2-norm), and $\text{cond}_F(A) \approx 8002$ (relative to the Frobenius norm).

The Convergence of Iterative Methods

In Section 2.5, we explored two iterative methods for solving a system of linear equations: Jacobi's method and the Gauss-Seidel method. In Theorem 2.9, we stated without proof that if A is a strictly diagonally dominant $n \times n$ matrix, then both of these methods converge to the solution of $A\mathbf{x} = \mathbf{b}$. We are now in a position to prove this theorem. Indeed, one of the important uses of matrix norms is to establish the convergence properties of various iterative methods.

We will deal only with Jacobi's method here. (The Gauss-Seidel method can be handled using similar techniques, but it requires a bit more care.) The key is to rewrite the iterative process in terms of matrices. Let's revisit Example 2.37 with this in mind. The system of equations is

$$\begin{aligned} 7x_1 - x_2 &= 5 \\ 3x_1 - 5x_2 &= -7 \end{aligned} \tag{2}$$

so $$A = \begin{bmatrix} 7 & -1 \\ 3 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

We rewrote Equation (2) as

$$\begin{aligned} x_1 &= \frac{5 + x_2}{7} \\ x_2 &= \frac{7 + 3x_1}{5} \end{aligned} \tag{3}$$

which is equivalent to

$$\begin{aligned} 7x_1 &= \phantom{-3x_1 +{}} x_2 + 5 \\ -5x_2 &= -3x_1 - 7 \end{aligned} \tag{4}$$

or, in terms of matrices,

$$\begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \end{bmatrix} \tag{5}$$

Study Equation (5) carefully: The matrix on the left-hand side contains the diagonal entries of A, while on the right-hand side we see the *negative* of the off-diagonal entries of A and the vector $\mathbf{b}$. So, if we decompose A as

$$A = \begin{bmatrix} 7 & -1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = L + D + U$$

then Equation (5) can be written as

$$D\mathbf{x} = -(L + U)\mathbf{x} + \mathbf{b}$$

or, equivalently,

$$\mathbf{x} = -D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b} \tag{6}$$

since the matrix D is invertible. Equation (6) is the matrix version of Equation (3). It is easy to see that we can do this in general: An $n \times n$ matrix A can be written as $A = L + D + U$, where D is the diagonal part of A and L and U are, respectively, the portions of A below and above the diagonal. The system $A\mathbf{x} = \mathbf{b}$ can then be written in the form of Equation (6), provided D is invertible—which it is if A is strictly diagonally dominant. (Why?) To simplify the notation, let's let $M = -D^{-1}(L + U)$ and $\mathbf{c} = D^{-1}\mathbf{b}$ so that Equation (6) becomes

$$\mathbf{x} = M\mathbf{x} + \mathbf{c} \tag{7}$$

Recall how we use this equation in Jacobi's method. We start with an initial vector $\mathbf{x}_0$ and plug it into the right-hand side of Equation (7) to get the first iterate $\mathbf{x}_1$—that is, $\mathbf{x}_1 = M\mathbf{x}_0 + \mathbf{c}$. Then we plug $\mathbf{x}_1$ into the right-hand side of Equation (7) to get the second iterate $\mathbf{x}_2 = M\mathbf{x}_1 + \mathbf{c}$. In general, we have

$$\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{c} \tag{8}$$

for $k \geq 0$. For Example 2.37, we have

$$M = -D^{-1}(L + U) = -\begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}^{-1}\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{3}{5} & 0 \end{bmatrix}$$

and

$$\mathbf{c} = D^{-1}\mathbf{b} = \begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{7}{5} \end{bmatrix}$$

so
$$\mathbf{x}_1 = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{3}{5} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5}{7} \\ \frac{7}{5} \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{7}{5} \end{bmatrix} \approx \begin{bmatrix} 0.714 \\ 1.400 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{3}{5} & 0 \end{bmatrix}\begin{bmatrix} 0.714 \\ 1.400 \end{bmatrix} + \begin{bmatrix} \frac{5}{7} \\ \frac{7}{5} \end{bmatrix} \approx \begin{bmatrix} 0.914 \\ 1.829 \end{bmatrix}$$

and so on. (These are exactly the same calculations we did in Example 2.37, but written in matrix form.)

To show that Jacobi's method will converge, we need to show that the iterates $\mathbf{x}_k$ approach the actual solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. It is enough to show that the ***error vectors*** $\mathbf{x}_k - \mathbf{x}$ approach the zero vector. From our calculations above, $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = M\mathbf{x} + \mathbf{c}$. Using Equation (8), we then have

$$\begin{aligned} \mathbf{x}_{k+1} - \mathbf{x} &= M\mathbf{x}_k + \mathbf{c} - (M\mathbf{x} + \mathbf{c}) \\ &= M(\mathbf{x}_k - \mathbf{x}) \end{aligned}$$

Now we take the norm of both sides of this equation. (At this point, it is not important which norm we use as long as we choose a matrix norm that is compatible with a vector norm.) We have

$$\|\mathbf{x}_{k+1} - \mathbf{x}\| = \|M(\mathbf{x}_k - \mathbf{x})\| \le \|M\|\,\|\mathbf{x}_k - \mathbf{x}\| \tag{9}$$

If we can show that $\|M\| < 1$, then we will have $\|\mathbf{x}_{k+1} - \mathbf{x}\| < \|\mathbf{x}_k - \mathbf{x}\|$ for all $k \ge 0$, and it follows that $\|\mathbf{x}_k - \mathbf{x}\|$ approaches zero, so the error vectors $\mathbf{x}_k - \mathbf{x}$ approach the zero vector.

The fact that strict diagonal dominance is defined in terms of the absolute values of the entries in the *rows* of a matrix suggests that the ∞-norm of a matrix (the operator norm induced by the max norm) is the one to choose. If $A = [a_{ij}]$, then

$$M = \begin{bmatrix} 0 & -a_{12}/a_{11} & \cdots & -a_{1n}/a_{11} \\ -a_{21}/a_{22} & 0 & \cdots & -a_{2n}/a_{22} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & \cdots & 0 \end{bmatrix}$$

(verify this), so, by Theorem 7.7, $\|M\|_\infty$ is the maximum absolute row sum of M. Suppose it occurs in the kth row. Then

$$\begin{aligned} \|M\|_\infty &= \left|\frac{-a_{k1}}{a_{kk}}\right| + \cdots + \left|\frac{-a_{k,k-1}}{a_{kk}}\right| + \left|\frac{-a_{k,k+1}}{a_{kk}}\right| + \cdots + \left|\frac{-a_{kn}}{a_{kk}}\right| \\ &= \frac{|a_{k1}| + \cdots + |a_{k,k-1}| + |a_{k,k+1}| + \cdots + |a_{kn}|}{|a_{kk}|} < 1 \end{aligned}$$

since A is strictly diagonally dominant. Thus, $\|M\|_\infty < 1$, so $\|\mathbf{x}_k - \mathbf{x}\| \to 0$, as we wished to show.

Example 7.22

Compute $\|M\|_\infty$ in Example 2.37 and use this value to find the number of iterations required to approximate the solution to three-decimal-place accuracy (after rounding) if the initial vector is $\mathbf{x}_0 = \mathbf{0}$.

Solution We have already computed $M = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{3}{5} & 0 \end{bmatrix}$, so $\|M\|_\infty = \frac{3}{5} = 0.6 < 1$ (implying that Jacobi's method converges in Example 2.37, as we saw). The approximate solution $\mathbf{x}_k$ will be accurate to three decimal places if the error vector $\mathbf{x}_k - \mathbf{x}$ has the property that each of its components is less than 0.0005 in absolute value. (Why?) Thus, we need only guarantee that the *maximum* absolute component of $\mathbf{x}_k - \mathbf{x}$ is less than 0.0005. In other words, we need to find the smallest value of k such that

$$\|\mathbf{x}_k - \mathbf{x}\|_m < 0.0005$$

Using Equation (9) above, we see that

$$\|\mathbf{x}_k - \mathbf{x}\|_m \le \|M\|_\infty \|\mathbf{x}_{k-1} - \mathbf{x}\|_m \le \|M\|_\infty^2 \|\mathbf{x}_{k-2} - \mathbf{x}\|_m \le \cdots \le \|M\|_\infty^k \|\mathbf{x}_0 - \mathbf{x}\|_m$$

Now $\|M\|_\infty = 0.6$ and $\|\mathbf{x}_0 - \mathbf{x}\|_m \approx \|\mathbf{x}_0 - \mathbf{x}_1\|_m = \|\mathbf{x}_1\|_m = \left\| \begin{bmatrix} 0.714 \\ 1.400 \end{bmatrix} \right\|_m = 1.4,$ so

$$\|M\|_\infty^k \|\mathbf{x}_0 - \mathbf{x}\|_m \approx (0.6)^k(1.4)$$

(If we knew the exact solution in advance, we could use it instead of $\mathbf{x}_1$. In practice, this is not the case, so we use an approximation to the solution, as we have done here.) Therefore, we need to find k such that

$$(0.6)^k(1.4) < 0.0005$$

We can solve this inequality by taking logarithms (base 10) of both sides. We have

$$\begin{aligned} \log_{10}((0.6)^k(1.4)) < \log_{10}(5 \times 10^{-4}) &\Rightarrow k\log_{10}(0.6) + \log_{10}(1.4) < \log_{10}5 - 4 \\ &\Rightarrow -0.222k + 0.146 < -3.301 \\ &\Rightarrow k > 15.5 \end{aligned}$$

Since k must be an integer, we can therefore conclude that $k = 16$ will work and that 16 iterations of Jacobi's method will give us three-decimal-place accuracy in this example. (In fact, it appears from our calculations in Example 2.37 that we get this degree of accuracy sooner, but our goal here was only to come up with an estimate.)

Exercises 7.2

In Exercises 1–3, let $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$.

1. Compute the Euclidean norm, the sum norm, and the max norm of $\mathbf{u}$.

2. Compute the Euclidean norm, the sum norm, and the max norm of $\mathbf{v}$.

3. Compute $d(\mathbf{u}, \mathbf{v})$ relative to the Euclidean norm, the sum norm, and the max norm.

4. (a) What does $d_s(\mathbf{u}, \mathbf{v})$ measure?
(b) What does $d_m(\mathbf{u}, \mathbf{v})$ measure?

In Exercises 5 and 6, let $\mathbf{u} = [1 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 1]^T$ *and* $\mathbf{v} = [0 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 1]^T$.

5. Compute the Hamming norms of $\mathbf{u}$ and $\mathbf{v}$.

6. Compute the Hamming distance between $\mathbf{u}$ and $\mathbf{v}$.

7. (a) For which vectors $\mathbf{v}$ is $\|\mathbf{v}\|_E = \|\mathbf{v}\|_m$? Explain your answer.

(b) For which vectors $\mathbf{v}$ is $\|\mathbf{v}\|_s = \|\mathbf{v}\|_m$? Explain your answer.

(c) For which vectors $\mathbf{v}$ is $\|\mathbf{v}\|_s = \|\mathbf{v}\|_m = \|\mathbf{v}\|_E$? Explain your answer.

8. (a) Under what conditions on $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} + \mathbf{v}\|_E = \|\mathbf{u}\|_E + \|\mathbf{v}\|_E$? Explain your answer.

(b) Under what conditions on $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} + \mathbf{v}\|_s = \|\mathbf{u}\|_s + \|\mathbf{v}\|_s$? Explain your answer.

(c) Under what conditions on $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} + \mathbf{v}\|_m = \|\mathbf{u}\|_m + \|\mathbf{v}\|_m$? Explain your answer.

9. Show that for all $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{v}\|_m \le \|\mathbf{v}\|_E$.

10. Show that for all $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{v}\|_E \le \|\mathbf{v}\|_s$.

11. Show that for all $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{v}\|_s \le n\|\mathbf{v}\|_m$.

12. Show that for all $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{v}\|_E \le \sqrt{n}\|\mathbf{v}\|_m$.

13. Draw the unit circles in $\mathbb{R}^2$ relative to the sum norm and the max norm.

14. By showing that the identity of Exercise 33 in Section 7.1 fails, show that the sum norm does not arise from any inner product.

In Exercises 15–18, prove that $\| \ \|$ defines a norm on the vector space V.

15. $V = \mathbb{R}^2, \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\| = \max\{|2a|, |3b|\}$

16. $V = M_{mn}, \|A\| = \max\limits_{i,j}\{|a_{ij}|\}$

17. $V = \mathscr{C}[0, 1], \|f\| = \displaystyle\int_0^1 |f(x)|\,dx$

18. $\|f\| = \max\limits_{0 \le x \le 1} |f(x)|$

19. Prove Theorem 7.5(b).

In Exercises 20–25, compute $\|A\|_F$, $\|A\|_1$, and $\|A\|_\infty$.

20. $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ **21.** $A = \begin{bmatrix} 0 & -1 \\ -3 & 3 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix}$ **23.** $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & -5 & 2 \\ 3 & 1 & -3 \\ -4 & -4 & 3 \end{bmatrix}$ **25.** $A = \begin{bmatrix} 4 & -2 & -1 \\ 0 & -1 & 2 \\ 3 & -3 & 0 \end{bmatrix}$

In Exercises 26–31, find vectors $\mathbf{x}$ and $\mathbf{y}$ with $\|\mathbf{x}\|_s = 1$ and $\|\mathbf{y}\|_m = 1$ such that $\|A\|_1 = \|A\mathbf{x}\|_s$ and $\|A\|_\infty = \|A\mathbf{y}\|_m$, where A is the matrix in the given exercise.

26. Exercise 20 **27.** Exercise 21 **28.** Exercise 22

29. Exercise 23 **30.** Exercise 24 **31.** Exercise 25

32. Prove Theorem 7.7(b).

33. (a) If $\|A\|$ is an operator norm, prove that $\|I\| = 1$, where I is an identity matrix.

(b) Is there a vector norm that induces the Frobenius norm as an operator norm? Why or why not?

34. Let $\|A\|$ be a matrix norm that is compatible with a vector norm $\|\mathbf{x}\|$. Prove that $\|A\| \ge |\lambda|$ for every eigenvalue λ of A.

In Exercises 35–40, find $\operatorname{cond}_1(A)$ and $\operatorname{cond}_\infty(A)$. State whether the given matrix is ill-conditioned.

35. $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ **36.** $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$

37. $A = \begin{bmatrix} 1 & 0.99 \\ 1 & 1 \end{bmatrix}$ **38.** $A = \begin{bmatrix} 150 & 200 \\ 3001 & 4002 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 6 \\ 1 & 0 & 0 \end{bmatrix}$ **40.** $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

41. Let $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$.

(a) Find a formula for $\operatorname{cond}_\infty(A)$ in terms of k.

(b) What happens to $\operatorname{cond}_\infty(A)$ as k approaches 1?

42. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where A is invertible. Suppose an error $\Delta\mathbf{b}$ changes $\mathbf{b}$ to $\mathbf{b}' = \mathbf{b} + \Delta\mathbf{b}$. Let $\mathbf{x}'$ be the solution to the new system; that is, $A\mathbf{x}' = \mathbf{b}'$. Let $\mathbf{x}' = \mathbf{x} + \Delta\mathbf{x}$ so that $\Delta\mathbf{x}$ represents the resulting error in the solution of the system. Show that

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \operatorname{cond}(A)\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

for any compatible matrix norm.

43. Let $A = \begin{bmatrix} 10 & 10 \\ 10 & 9 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 100 \\ 99 \end{bmatrix}$.

(a) Compute $\operatorname{cond}_\infty(A)$.

(b) Suppose A is changed to $A' = \begin{bmatrix} 10 & 10 \\ 10 & 11 \end{bmatrix}$. How large a relative change can this change produce in the solution to $A\mathbf{x} = \mathbf{b}$? [*Hint:* Use inequality (1) from this section.]

(c) Solve the systems using A and A' and determine the actual relative error.

(d) Suppose $\mathbf{b}$ is changed to $\mathbf{b}' = \begin{bmatrix} 100 \\ 101 \end{bmatrix}$. How large a relative change can this change produce in the solution to $A\mathbf{x} = \mathbf{b}$? [*Hint:* Use Exercise 42.]

(e) Solve the systems using $\mathbf{b}$ and $\mathbf{b}'$ and determine the actual relative error.

44. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Compute $\text{cond}_1(A)$.

(b) Suppose A is changed to $A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & -1 & 2 \end{bmatrix}$. How large a relative change can this change produce in the solution to $A\mathbf{x} = \mathbf{b}$? [*Hint:* Use inequality (1) from this section.]

(c) Solve the systems using A and A' and determine the actual relative error.

(d) Suppose $\mathbf{b}$ is changed to $\mathbf{b}' = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. How large a relative change can this change produce in the solution to $A\mathbf{x} = \mathbf{b}$? [*Hint:* Use Exercise 42.]

(e) Solve the systems using $\mathbf{b}$ and $\mathbf{b}'$ and determine the actual relative error.

45. Show that if A is an invertible matrix, then $\text{cond}(A) \geq 1$ with respect to any matrix norm.

46. Show that if A and B are invertible matrices, then $\text{cond}(AB) \leq \text{cond}(A)\,\text{cond}(B)$ with respect to any matrix norm.

47. Let A be an invertible matrix and let λ_1 and λ_n be the eigenvalues with the largest and smallest absolute values, respectively. Show that

$$\text{cond}\,(A) \geq \frac{|\lambda_1|}{|\lambda_n|}$$

[*Hint:* See Exercise 34 and Theorem 4.18(b) in Section 4.3.]

CAS *In Exercises 48–51, write the given system in the form of Equation (7). Then use the method of Example 7.22 to estimate the number of iterations of Jacobi's method that will be needed to approximate the solution to three-decimal-place accuracy. (Use $\mathbf{x}_0 = \mathbf{0}$.) Compare your answer with the solution computed in the given exercise from Section 2.5.*

48. Exercise 1, Section 2.5

49. Exercise 3, Section 2.5

50. Exercise 4, Section 2.5

51. Exercise 5, Section 2.5

Exercise 52(c) refers to the Leontief model of an open economy, as discussed in Sections 2.4 and 3.7.

52. Let A be an $n \times n$ matrix such that $\|A\| < 1$, where the norm is either the sum norm or the max norm.

(a) Prove that $A^n \to O$ as $n \to \infty$.

(b) Deduce from (a) that $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

[*Hint*: See the proof of Theorem 3.34.]

(c) Show that (b) can be used to prove Corollaries 3.35 and 3.36.

7.3 Least Squares Approximation

In many branches of science, experimental data are used to infer a mathematical relationship among the variables being measured. For example, we might measure the height of a tree at various points in time and try to deduce a function that expresses the tree's height h in terms of time t. Or, we might measure the size p of a population over time and try to find a rule that relates p to t. Relationships between variables are also of interest in business; for example, a company producing widgets may be interested in knowing the relationship between its total costs c and the number n of widgets produced.

In each of these examples, the data come in the form of two measurements: one for the independent variable and one for the (supposedly) dependent variable. Thus, we have a set of *data points* (x_i, y_i), and we are looking for a function that best approximates the relationship between the independent variable x and the dependent variable y. Figure 7.9 shows examples in which experimental data points are plotted, along with a curve that approximately "fits" the data.

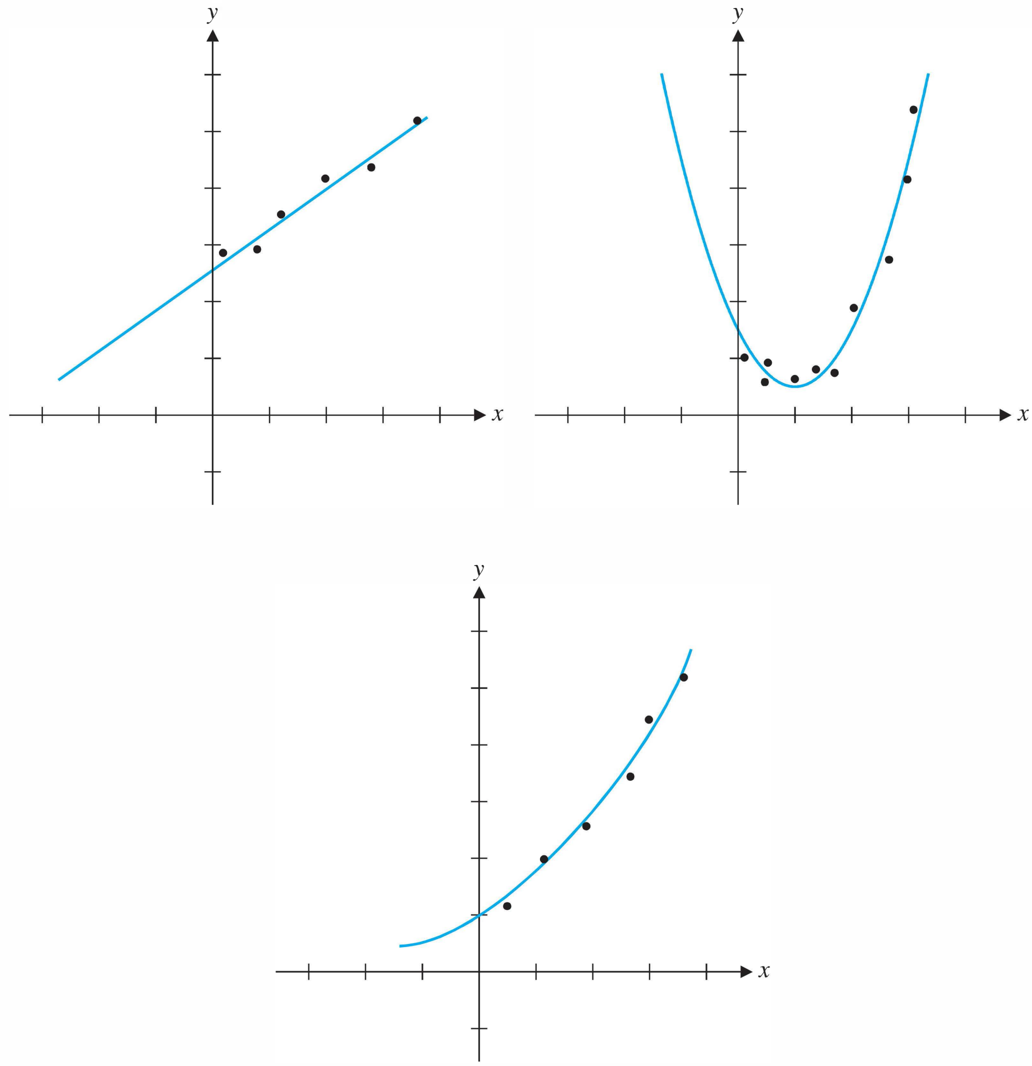


Figure 7.9
Curves of "best fit"

The method of least squares, which we are about to consider, is attributed to Gauss. A new asteroid, Ceres, was discovered on New Year's Day, 1801, but it disappeared behind the sun shortly after it was observed. Astronomers predicted when and where Ceres would reappear, but their calculations differed greatly from those done, independently, by Gauss. Ceres reappeared on December 7, 1801, almost exactly where Gauss had predicted it would be. Although he did not disclose his methods at the time, Gauss had used his least squares approximation method, which he described in a paper in 1809. The same method was actually known earlier; Cotes anticipated the method in the early 18th century, and Legendre published a paper on it in 1806. Nevertheless, Gauss is generally given credit for the method of least squares approximation.

Roger Cotes (1682–1716) was an English mathematician who, while a fellow at Cambridge, edited the second edition of Newton's *Principia*. Although he published little, he made important discoveries in the theory of logarithms, integral calculus, and numerical methods.

We begin our exploration of approximation with a more general result.

The Best Approximation Theorem

In the sciences, there are many problems that can be phrased generally as "What is the best approximation to X of type Y?" X might be a set of data points, a function, a vector, or many other things, while Y might be a particular type of function, a vector belonging to a certain vector space, etc. A typical example of such a problem is finding the vector $\mathbf{w}$ in a subspace W of a vector space V that best approximates (i.e., is closest to) a given vector $\mathbf{v}$ in V. This problem gives rise to the following definition.

Definition If W is a subspace of a normed linear space V and if $\mathbf{v}$ is a vector in V, then the ***best approximation to* v *in* W** is the vector $\bar{\mathbf{v}}$ in W such that

$$\|\mathbf{v} - \bar{\mathbf{v}}\| < \|\mathbf{v} - \mathbf{w}\|$$

for every vector $\mathbf{w}$ in W different from $\bar{\mathbf{v}}$.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, we are used to thinking of "shortest distance" as corresponding to "perpendicular distance." In algebraic terminology, "shortest distance" relates to the notion of orthogonal projection: If W is a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ is a vector in $\mathbb{R}^n$, then we expect $\mathrm{proj}_W(\mathbf{v})$ to be the vector in W that is closest to $\mathbf{v}$ (Figure 7.10).

Since orthogonal projection can be defined in any inner product space, we have the following theorem.

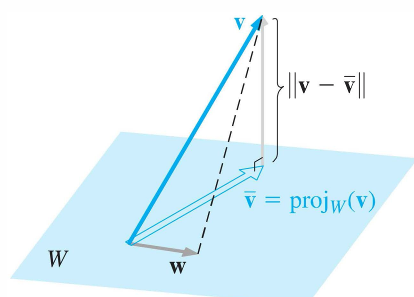


Figure 7.10

If $\bar{\mathbf{v}} = \mathrm{proj}_W(\mathbf{v})$, then $\|\mathbf{v} - \bar{\mathbf{v}}\| < \|\mathbf{v} - \mathbf{w}\|$ for all $\mathbf{w} \neq \bar{\mathbf{v}}$

Theorem 7.8 **The Best Approximation Theorem**

If W is a finite-dimensional subspace of an inner product space V and if $\mathbf{v}$ is a vector in V, then $\mathrm{proj}_W(\mathbf{v})$ is the best approximation to $\mathbf{v}$ in W.

Proof Let $\mathbf{w}$ be a vector in W different from $\mathrm{proj}_W(\mathbf{v})$. Then $\mathrm{proj}_W(\mathbf{v}) - \mathbf{w}$ is also in W, so $\mathbf{v} - \mathrm{proj}_W(\mathbf{v}) = \mathrm{perp}_W(\mathbf{v})$ is orthogonal to $\mathrm{proj}_W(\mathbf{v}) - \mathbf{w}$, by Exercise 43 in Section 7.1. Pythagoras' Theorem now implies that

$$\begin{aligned}\|\mathbf{v} - \mathrm{proj}_W(\mathbf{v})\|^2 + \|\mathrm{proj}_W(\mathbf{v}) - \mathbf{w}\|^2 &= \|(\mathbf{v} - \mathrm{proj}_W(\mathbf{v})) + (\mathrm{proj}_W(\mathbf{v}) - \mathbf{w})\|^2 \\ &= \|\mathbf{v} - \mathbf{w}\|^2\end{aligned}$$

as Figure 7.10 illustrates. However, $\|\text{proj}_W(\mathbf{v}) - \mathbf{w}\|^2 > 0$, since $\mathbf{w} \neq \text{proj}_W(\mathbf{v})$, so

$$\|\mathbf{v} - \text{proj}_W(\mathbf{v})\|^2 < \|\mathbf{v} - \text{proj}_W(\mathbf{v})\|^2 + \|\text{proj}_W(\mathbf{v}) - \mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$$

or, equivalently,

$$\|\mathbf{v} - \text{proj}_W(\mathbf{v})\| < \|\mathbf{v} - \mathbf{w}\|$$

Example 7.23

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Find the best approximation to $\mathbf{v}$ in the plane $W = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$ and find the Euclidean distance from $\mathbf{v}$ to W.

Solution The vector in W that best approximates $\mathbf{v}$ is $\text{proj}_W(\mathbf{v})$. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal,

$$\text{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2$$

$$= \tfrac{2}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \tfrac{16}{30}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

The distance from $\mathbf{v}$ to W is the distance from $\mathbf{v}$ to the point in W closest to $\mathbf{v}$. But this distance is just $\|\text{perp}_W(\mathbf{v})\| = \|\mathbf{v} - \text{proj}_W(\mathbf{v})\|$. We compute

$$\mathbf{v} - \text{proj}_W(\mathbf{v}) = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{12}{5} \\ \frac{24}{5} \end{bmatrix}$$

so $$\|\mathbf{v} - \text{proj}_W(\mathbf{v})\| = \sqrt{0^2 + (\tfrac{12}{5})^2 + (\tfrac{24}{5})^2} = \sqrt{\tfrac{720}{25}} = 12\sqrt{5}/5$$

which is the distance from $\mathbf{v}$ to W.

In Section 7.5, we will look at other examples of the Best Approximation Theorem when we explore the problem of approximating functions.

Remark The orthogonal projection of a vector $\mathbf{v}$ onto a subspace W is defined in terms of an orthogonal basis for W. The Best Approximation Theorem gives us an alternative proof that $\text{proj}_W(\mathbf{v})$ does not depend on the choice of this basis, since there can be only one vector in W that is closest to $\mathbf{v}$—namely, $\text{proj}_W(\mathbf{v})$.

Least Squares Approximation

We now turn to the problem of finding a curve that "best fits" a set of data points. Before we can proceed, however, we need to define what we mean by "best fit." Suppose the data points (1, 2), (2, 2), and (3, 4) have arisen from measurements taken during some experiment. Also suppose we have reason to believe that the x and y values are related by a linear function; that is, we expect the points to lie on some line with equation $y = a + bx$. If our measurements were accurate, all three points would satisfy this equation and we would have

$$2 = a + b \cdot 1 \qquad 2 = a + b \cdot 2 \qquad 4 = a + b \cdot 3$$

This is a system of three linear equations in two variables:

$$\begin{aligned} a + b &= 2 \\ a + 2b &= 2 \\ a + 3b &= 4 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Unfortunately, this system is inconsistent (since the three points do not lie on a straight line). So we will settle for a line that comes "as close as possible" to passing through our points. For any line, we will measure the vertical distance from each data point to the line (representing the *errors* in the y-direction), and then we will try to choose the line that minimizes the *total error*. Figure 7.11 illustrates the situation.

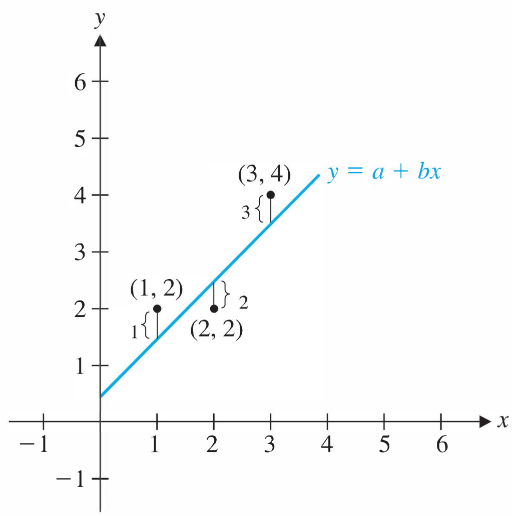


Figure 7.11
Finding the line that minimizes $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$

If the errors are denoted by ε_1, ε_2, and ε_3, then we can form the ***error vector***

$$\mathbf{e} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

We want $\mathbf{e}$ to be as small as possible, so $\|\mathbf{e}\|$ must be as close to zero as possible. Which norm should we use? It turns out that the familiar Euclidean norm is the best choice. (The sum norm would also be a sensible choice, since $\|\mathbf{e}\|_s = |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3|$ is the actual sum of the errors in Figure 7.11. However, the absolute value signs are hard to work with, and, as you will soon see, the choice of the Euclidean norm leads to some very nice formulas.) So we are going to minimize

$$\|\mathbf{e}\| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2} \quad \text{or, equivalently,} \quad \|\mathbf{e}\|^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$$

This is where the term "least squares" comes from: We need to find the smallest sum of squares, in the sense of the foregoing equation. The number $\|\mathbf{e}\|$ is called the ***least squares error*** of the approximation.

From Figure 7.11, we also obtain the following formulas for ε_1, ε_2, and ε_3 in our example:

$$\varepsilon_1 = 2 - (a + b \cdot 1) \quad \varepsilon_2 = 2 - (a + b \cdot 2) \quad \varepsilon_3 = 4 - (a + b \cdot 3)$$

Example 7.24

Which of the following lines gives the smallest least squares error for the data points (1, 2), (2, 2), and (3, 4)?

(a) $y = 1 + x$

(b) $y = -2 + 2x$

(c) $y = \frac{2}{3} + x$

Solution Table 7.1 shows the necessary calculations.

Table 7.1

	$y = 1 + x$	$y = -2 + 2x$	$y = \frac{2}{3} + x$
ε_1	$2 - (1 + 1) = 0$	$2 - (-2 + 2) = 2$	$2 - (\frac{2}{3} + 1) = \frac{1}{3}$
ε_2	$2 - (1 + 2) = -1$	$2 - (-2 + 4) = 0$	$2 - (\frac{2}{3} + 2) = -\frac{2}{3}$
ε_3	$4 - (1 + 3) = 0$	$4 - (-2 + 6) = 0$	$4 - (\frac{2}{3} + 3) = \frac{1}{3}$
$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$	$0^2 + (-1)^2 + 0^2 = 1$	$2^2 + 0^2 + 0^2 = 4$	$(\frac{1}{3})^2 + (-\frac{2}{3})^2 + (\frac{1}{3})^2 = \frac{2}{3}$
$\lVert \mathbf{e} \rVert$	1	2	$\sqrt{\frac{2}{3}} \approx 0.816$

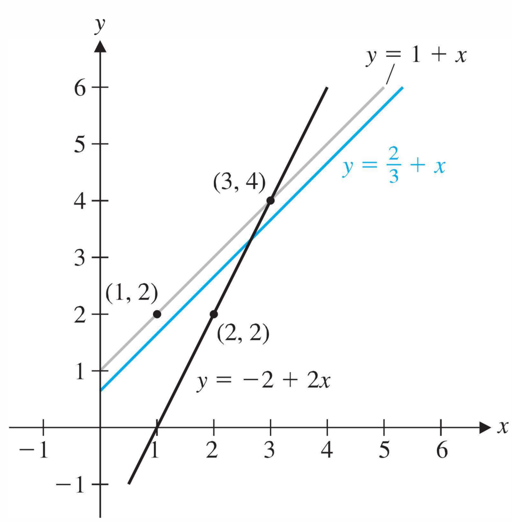


Figure 7.12

We see that the line $y = \frac{2}{3} + x$ produces the smallest least squares error among these three lines. Figure 7.12 shows the data points and all three lines.

It turns out that the line $y = \frac{2}{3} + x$ in Example 7.24 gives the smallest least squares error of *any* line, even though it passes through *none* of the given points. The rest of this section is devoted to illustrating why this is so.

In general, suppose we have n data points $(x_1, y_1), \dots, (x_n, y_n)$ and a line $y = a + bx$. Our error vector is

$$\mathbf{e} = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

where $\varepsilon_i = y_i - (a + bx_i)$. The line $y = a + bx$ that minimizes $\varepsilon_1^2 + \cdots + \varepsilon_n^2$ is called the ***least squares approximating line*** (or the ***line of best fit***) for the points $(x_1, y_1), \dots, (x_n, y_n)$. As noted prior to Example 7.24, we can express this problem in matrix form. If the given points were actually on the line $y = a + bx$, then the n linear equations

$$\begin{aligned} a + bx_1 &= y_1 \\ &\vdots \\ a + bx_n &= y_n \end{aligned}$$

would all be true (i.e., the system would be consistent). Our interest is in the case where the points are *not* collinear, in which case the system is *inconsistent*. In matrix form, we have

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

which is of the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The error vector $\mathbf{e}$ is just $\mathbf{b} - A\mathbf{x}$ (check this), and we want to minimize $\|\mathbf{e}\|^2$ or, equivalently, $\|\mathbf{e}\|$. We can therefore rephrase our problem in terms of matrices as follows.

Definition If A is an $m \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^m$, a ***least squares solution*** of $A\mathbf{x} = \mathbf{b}$ is a vector $\overline{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$\|\mathbf{b} - A\overline{\mathbf{x}}\| \le \|\mathbf{b} - A\mathbf{x}\|$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.

Solution of the Least Squares Problem

Any vector of the form $A\mathbf{x}$ is in the column space of A, and as $\mathbf{x}$ varies over all vectors in $\mathbb{R}^n$, $A\mathbf{x}$ varies over all vectors in col(A). A least squares solution of $A\mathbf{x} = \mathbf{b}$ is therefore equivalent to a vector $\overline{\mathbf{y}}$ in col(A) such that

$$\|\mathbf{b} - \overline{\mathbf{y}}\| \le \|\mathbf{b} - \mathbf{y}\|$$

for all $\mathbf{y}$ in col(A). In other words, we need the closest vector in col(A) to $\mathbf{b}$. By the Best Approximation Theorem, the vector we want is the orthogonal projection of $\mathbf{b}$ onto col(A). Thus, if $\overline{\mathbf{x}}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$, we have

$$A\overline{\mathbf{x}} = \mathrm{proj}_{\mathrm{col}(A)}(\mathbf{b}) \tag{1}$$

In order to find $\overline{\mathbf{x}}$, it would appear that we need to first compute $\mathrm{proj}_{\mathrm{col}(A)}(\mathbf{b})$ and then solve the system (1). However, there is a better way to proceed.

We know that

$$\mathbf{b} - A\overline{\mathbf{x}} = \mathbf{b} - \mathrm{proj}_{\mathrm{col}(A)}(\mathbf{b}) = \mathrm{perp}_{\mathrm{col}(A)}(\mathbf{b})$$

is orthogonal to col(A). So $\mathbf{b} - A\overline{\mathbf{x}}$ is in $(\mathrm{col}(A))^\perp = \mathrm{null}(A^T)$. Therefore $A^T(\mathbf{b} - A\overline{\mathbf{x}}) = \mathbf{0}$, which, in turn, is equivalent to $A^T\mathbf{b} - A^TA\overline{\mathbf{x}} = \mathbf{0}$ or

$$A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$$

This represents a system of equations known as the ***normal equations*** for $\overline{\mathbf{x}}$.

We have just established that the solutions of the normal equations for $\overline{\mathbf{x}}$ are precisely the least squares solutions of $A\mathbf{x} = \mathbf{b}$. This proves the first part of the following theorem.

Theorem 7.9 **The Least Squares Theorem**

Let A be an $m \times n$ matrix and let $\mathbf{b}$ be in $\mathbb{R}^m$. Then $A\mathbf{x} = \mathbf{b}$ always has at least one least squares solution $\overline{\mathbf{x}}$. Moreover:

a. $\overline{\mathbf{x}}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\overline{\mathbf{x}}$ is a solution of the normal equations $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$.
b. A has linearly independent columns if and only if A^TA is invertible. In this case, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is unique and is given by

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$$

Proof We have already established property (a). For property (b), we note that the n columns of A are linearly independent if and only if rank(A) $= n$. But this is true if and only if A^TA is invertible, by Theorem 3.28. If A^TA is invertible, then the unique solution of $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$ is clearly $\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$.

Example 7.25 Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 5 \\ 2 & -2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Solution We compute

$$A^TA = \begin{bmatrix} 1 & 2 & -1 \\ 5 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 30 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & -1 \\ 5 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \end{bmatrix}$$

The normal equations $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$ are just

$$\begin{bmatrix} 6 & 0 \\ 0 & 30 \end{bmatrix}\overline{\mathbf{x}} = \begin{bmatrix} 2 \\ 16 \end{bmatrix}$$

which yield $\overline{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} \\ \frac{8}{15} \end{bmatrix}$. The fact that this solution is unique was guaranteed by Theorem 7.9(b), since the columns of A are clearly linearly independent.

Remark We could have phrased Example 7.25 as follows: Find the best approximation to $\mathbf{b}$ in the column space of A. The resulting equations give the system $A\mathbf{x} = \mathbf{b}$ whose least squares solution we just found. (Verify this.) In this case, the components of $\overline{\mathbf{x}}$ are the *coefficients* of that linear combination of the columns of A that produces the best approximation to $\mathbf{b}$—namely,

$$\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{8}{15}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

This is exactly the result of Example 7.23. Compare the two approaches.

Example 7.26 Find the least squares approximating line for the data points (1, 2), (2, 2), and (3, 4) from Example 7.24.

Solution We have already seen that the corresponding system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

where $y = a + bx$ is the line we seek. Since the columns of A are clearly linearly independent, there will be a unique least squares solution, by part (b) of the Least Squares Theorem. We compute

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \quad \text{and} \quad A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

Hence, we can solve the normal equations $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$, using Gaussian elimination to obtain

$$[A^TA \mid A^T\mathbf{b}] = \left[\begin{array}{cc|c} 3 & 6 & 8 \\ 6 & 14 & 18 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & 1 \end{array}\right]$$

So $\overline{\mathbf{x}} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$, from which we see that $a = \frac{2}{3}$, $b = 1$ are the coefficients of the least squares approximating line: $y = \frac{2}{3} + x$.

The line we just found is the line in Example 7.24(c), so we have justified our claim that this line produces the smallest least squares error for the data points (1, 2), (2, 2), and (3, 4). Notice that if $\overline{\mathbf{x}}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$, we may compute the least squares error as

$$\|\mathbf{e}\| = \|\mathbf{b} - A\overline{\mathbf{x}}\|$$

Since $A\overline{\mathbf{x}} = \text{proj}_{\text{col}(A)}(\mathbf{b})$, this is just the length of $\text{perp}_{\text{col}(A)}(\mathbf{b})$—that is, the distance from $\mathbf{b}$ to the column space of A. In Example 7.26, we had

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

so, as in Example 7.24(c), we have a least squares error of $\|\mathbf{e}\| = \sqrt{\frac{2}{3}} \approx 0.816$.

Remark Note that the columns of A in Example 7.26 are linearly independent, so $(A^TA)^{-1}$ exists, and we could calculate $\overline{\mathbf{x}}$ as $\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$. However, it is almost always easier to solve the normal equations using Gaussian elimination (or to let your CAS do it for you!).

It is interesting to look at Example 7.26 from two different geometric points of view. On the one hand, we have the least squares approximating line $y = \frac{2}{3} + x$, with corresponding errors $\varepsilon_1 = \frac{1}{3}$, $\varepsilon_2 = -\frac{2}{3}$, and $\varepsilon_3 = \frac{1}{3}$, as shown in Figure 7.13(a). Equivalently, we have the projection of $\mathbf{b}$ onto the column space of A, as shown in Figure 7.13(b). Here,

$$\mathbf{p} = \text{proj}_{\text{col}(A)}(\mathbf{b}) = A\overline{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{8}{3} \\ \frac{11}{3} \end{bmatrix}$$

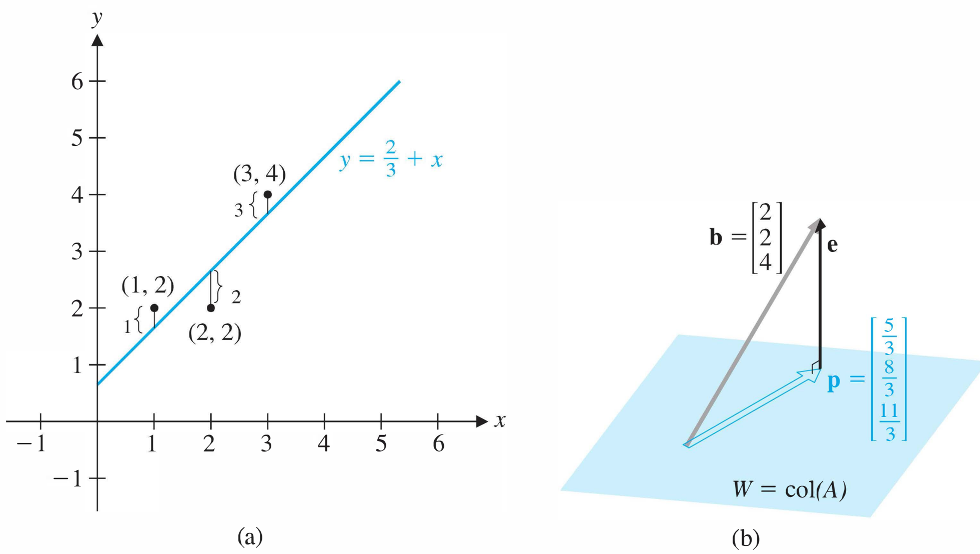


Figure 7.13

and the least squares error vector is $\mathbf{e} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$. [What would Figure 7.13(b) look like if the data points *were* collinear?]

Example 7.27

Find the least squares approximating line and the least squares error for the points (1, 1), (2, 2), (3, 2), and (4, 3).

Solution Let $y = a + bx$ be the equation of the line we seek. Then, substituting the four points into this equation, we obtain

$$\begin{aligned} a + b &= 1 \\ a + 2b &= 2 \\ a + 3b &= 2 \\ a + 4b &= 3 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

So we want the least squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

Since the columns of A are linearly independent, the solution we want is

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{5} \end{bmatrix}$$

(Check this calculation.) Therefore, we take $a = \frac{1}{2}$ and $b = \frac{3}{5}$, producing the least squares approximating line $y = \frac{1}{2} + \frac{3}{5}x$, as shown in Figure 7.14.

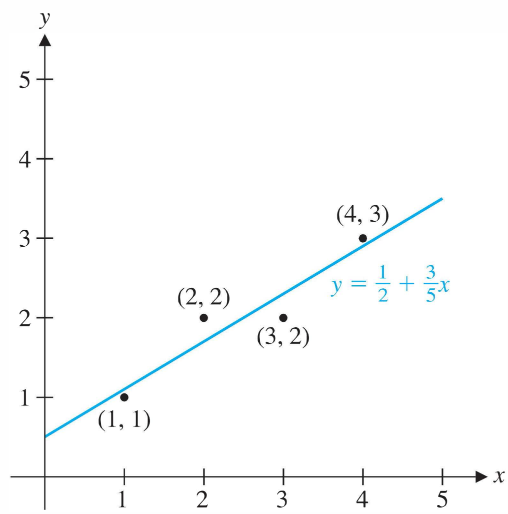


Figure 7.14

Since

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ \frac{3}{10} \\ -\frac{3}{10} \\ \frac{1}{10} \end{bmatrix}$$

the least squares error is $\|\mathbf{e}\| = \sqrt{5}/5 \approx 0.447$.

We can use the method of least squares to approximate data points by curves other than straight lines.

Example 7.28

Find the parabola that gives the best least squares approximation to the points $(-1, 1)$, $(0, -1)$, $(1, 0)$, and $(2, 2)$.

Solution The equation of a parabola is a quadratic $y = a + bx + cx^2$. Substituting the given points into this quadratic, we obtain the linear system

$$\begin{aligned} a - b + c &= 1 \\ a &= -1 \\ a + b + c &= 0 \\ a + 2b + 4c &= 2 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

Thus, we want the least squares approximation of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

We compute

$$A^TA = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \quad \text{and} \quad A^T\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$$

so the normal equations are given by

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$$

whose solution is

$$\overline{\mathbf{x}} = \begin{bmatrix} -\frac{7}{10} \\ -\frac{3}{5} \\ 1 \end{bmatrix}$$

Thus, the least squares approximating parabola has the equation

$$y = -\tfrac{7}{10} - \tfrac{3}{5}x + x^2$$

as shown in Figure 7.15.

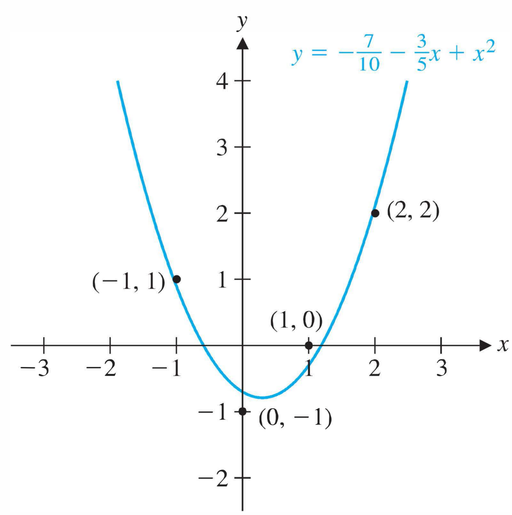


Figure 7.15
A least squares approximating parabola

One of the important uses of least squares approximation is to estimate constants associated with various processes. The next example illustrates this application in the context of population growth. Recall from Section 6.7 that a population that is growing (or decaying) exponentially satisfies an equation of the form $p(t) = ce^{kt}$, where $p(t)$ is the size of the population at time t and c and k are constants. Clearly, $c = p(0)$, but k is not so easy to determine. It is easy to see that

$$k = \frac{p'(t)}{p(t)}$$

which explains why k is sometimes referred to as the *relative growth rate* of the population: It is the ratio of the growth rate $p'(t)$ to the size of the population $p(t)$.

CAS **Example 7.29**

Table 7.2 gives the population of the world at 10-year intervals for the second half of the 20th century. Assuming an exponential growth model, find the relative growth rate and predict the world's population in 2010.

Table 7.2

Year	Population (in billions)
1950	2.56
1960	3.04
1970	3.71
1980	4.46
1990	5.28
2000	6.08

Source: U.S. Bureau of the Census, International Data Base

Solution Let's agree to measure time t in 10-year intervals so that $t = 0$ is 1950, $t = 1$ is 1960, and so on. Since $c = p(0) = 2.56$, the equation for the growth rate of the population is

$$p = 2.56e^{kt}$$

How can we use the method of least squares on this equation? If we take the natural logarithm of both sides, we convert the equation into a linear one:

$$\begin{aligned} \ln p &= \ln(2.56e^{kt}) \\ &= \ln 2.56 + \ln(e^{kt}) \\ &\approx 0.94 + kt \end{aligned}$$

Plugging in the values of t and p from Table 7.2 yields the following system (where we have rounded calculations to three decimal places):

$$\begin{aligned} 0.94 &= 0.94 \\ k &= 0.172 \\ 2k &= 0.371 \\ 3k &= 0.555 \\ 4k &= 0.724 \\ 5k &= 0.865 \end{aligned}$$

We can ignore the first equation (it just corresponds to the initial condition $c = p(0) = 2.56$). The remaining equations correspond to a system $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0.172 \\ 0.371 \\ 0.555 \\ 0.724 \\ 0.865 \end{bmatrix}$$

Since $A^TA = 55$ and $A^T\mathbf{b} = 9.80$, the corresponding normal equations are just the single equation

$$55\overline{\mathbf{x}} = 9.80$$

Therefore, $k = \overline{\mathbf{x}} = 9.80/55 \approx 0.178$. Consequently, the least squares solution has the form $p = 2.56e^{0.178t}$ (see Figure 7.16).

The world's population in 2010 corresponds to $t = 6$, from which we obtain

$$p(6) = 2.56e^{0.178(6)} \approx 7.448$$

Thus, if our model is accurate, there will be approximately 7.45 billion people on Earth in the year 2010. (The U.S. Census Bureau estimates that the global population will be "only" 6.82 billion in 2010. Why do you think our estimate is higher?)

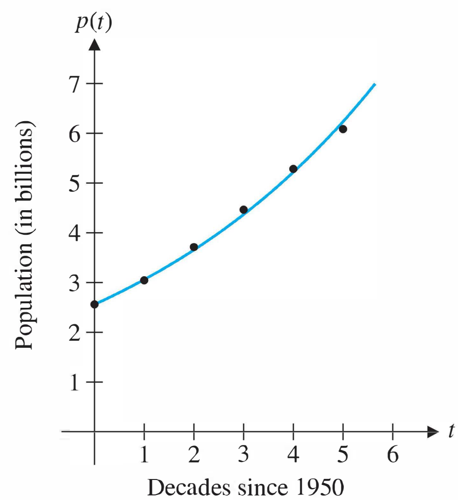


Figure 7.16

Least Squares via the QR Factorization

It is often the case that the normal equations for a least squares problem are ill-conditioned. Therefore, a small numerical error in performing Gaussian elimination will result in a large error in the least squares solution. Consequently, in practice, other methods are usually used to compute least squares approximations.

It turns out that the QR factorization of A yields a more reliable way of computing the least squares approximation of $A\mathbf{x} = \mathbf{b}$.

Theorem 7.10 Let A be an $m \times n$ matrix with linearly independent columns and let $\mathbf{b}$ be in $\mathbb{R}^m$. If $A = QR$ is a QR factorization of A, then the unique least squares solution $\overline{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ is

$$\overline{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$$

Proof Recall from Theorem 5.16 that the QR factorization $A = QR$ involves an $m \times n$ matrix Q with orthonormal columns and an invertible upper triangular matrix R. From the Least Squares Theorem, we have

$$\begin{aligned} A^TA\overline{\mathbf{x}} &= A^T\mathbf{b} \\ \Rightarrow (QR)^TQR\overline{\mathbf{x}} &= (QR)^T\mathbf{b} \\ \Rightarrow R^TQ^TQR\overline{\mathbf{x}} &= R^TQ^T\mathbf{b} \\ \Rightarrow R^TR\overline{\mathbf{x}} &= R^TQ^T\mathbf{b} \end{aligned}$$

since $Q^TQ = I$. (Why?)

Since R is invertible, so is R^T, and hence we have

$$R\overline{\mathbf{x}} = Q^T\mathbf{b} \quad \text{or, equivalently,} \quad \overline{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$$

Remark Since R is upper triangular, in practice it is easier to solve $R\overline{\mathbf{x}} = Q^T\mathbf{b}$ directly than to invert R and compute $R^{-1}Q^T\mathbf{b}$.

Example 7.30 Use the QR factorization to find a least squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ -2 \\ 0 \end{bmatrix}$$

Solution From Example 5.15,

$$A = QR = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

We have

$$Q^T\mathbf{b} = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -\sqrt{5}/2 \\ -2\sqrt{6}/3 \end{bmatrix}$$

so we require the solution to $R\overline{\mathbf{x}} = Q^T\mathbf{b}$, or

$$\begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 7/2 \\ -\sqrt{5}/2 \\ -2\sqrt{6}/3 \end{bmatrix}$$

Back substitution quickly yields

$$\overline{\mathbf{x}} = \begin{bmatrix} 4/3 \\ 3/2 \\ -4/3 \end{bmatrix}$$

Orthogonal Projection Revisited

One of the nice byproducts of the least squares method is a new formula for the orthogonal projection of a vector onto a subspace of $\mathbb{R}^m$.

Theorem 7.11 Let W be a subspace of $\mathbb{R}^m$ and let A be an $m \times n$ matrix whose columns form a basis for W. If $\mathbf{v}$ is any vector in $\mathbb{R}^m$, then the orthogonal projection of $\mathbf{v}$ onto W is the vector

$$\text{proj}_W(\mathbf{v}) = A(A^TA)^{-1}A^T\mathbf{v}$$

The linear transformation $P : \mathbb{R}^m \to \mathbb{R}^m$ that projects $\mathbb{R}^m$ onto W has $A(A^TA)^{-1}A^T$ as its standard matrix.

Proof Given the way we have constructed A, its column space is W. Since the columns of A are linearly independent, the Least Squares Theorem guarantees that there is a unique least squares solution to $A\mathbf{x} = \mathbf{v}$ given by

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{v}$$

By Equation (1),

$$A\bar{\mathbf{x}} = \text{proj}_{\text{col}(A)}(\mathbf{v}) = \text{proj}_W(\mathbf{v})$$

Therefore, $$\text{proj}_W(\mathbf{v}) = A((A^TA)^{-1}A^T\mathbf{v}) = (A(A^TA)^{-1}A^T)\mathbf{v}$$

as required. Since this equation holds for all $\mathbf{v}$ in $\mathbb{R}^m$, the last statement of the theorem follows immediately.

We will illustrate Theorem 7.11 by revisiting Example 5.11.

Example 7.31

Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto the plane W in $\mathbb{R}^3$ with equation $x - y + 2z = 0$, and give the standard matrix of the orthogonal projection transformation onto W.

Solution As in Example 5.11, we will take as a basis for W the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We form the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

with these basis vectors as its columns. Then

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

so $$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

By Theorem 7.11, the standard matrix of the orthogonal projection transformation onto W is

$$A(A^TA)^{-1}A^T = A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

so the orthogonal projection of $\mathbf{v}$ onto W is

$$\text{proj}_W(\mathbf{v}) = A(A^TA)^{-1}A^T\mathbf{v} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

which agrees with our solution to Example 5.11.

Remark Since the projection of a vector onto a subspace W is unique, the standard matrix of this linear transformation (as given by Theorem 7.11) cannot depend on the choice of basis for W. In other words, with a different basis for W, we have a different matrix A, but the matrix $A(A^TA)^{-1}A^T$ will be the same! (You are asked to verify this in Exercise 43.)

The Pseudoinverse of a Matrix

If A is an $n \times n$ matrix with linearly independent columns, then it is invertible, and the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. If $m > n$ and A is $m \times n$ with linearly independent columns, then $A\mathbf{x} = \mathbf{b}$ has no exact solution, but the best approximation is given by the unique least squares solution $\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$. The matrix $(A^TA)^{-1}A^T$ therefore plays the role of an "inverse of A" in this situation.

Definition If A is a matrix with linearly independent columns, then the ***pseudoinverse*** of A is the matrix A^+ defined by

$$A^+ = (A^TA)^{-1}A^T$$

Observe that if A is $m \times n$, then A^+ is $n \times m$.

Example 7.32

Find the pseudoinverse of $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution We have already done most of the calculations in Example 7.26. Using our previous work, we have

$$A^+ = (A^TA)^{-1}A^T = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

The pseudoinverse is a convenient shorthand notation for some of the concepts we have been exploring. For example, if A is $m \times n$ with linearly independent columns, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\overline{\mathbf{x}} = A^+\mathbf{b}$$

and the standard matrix of the orthogonal projection P from $\mathbb{R}^m$ onto col(A) is

$$[P] = AA^+$$

If A is actually a square matrix, then it is easy to show that $A^+ = A^{-1}$ (see Exercise 53). In this case, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is the *exact* solution, since

$$\overline{\mathbf{x}} = A^+\mathbf{b} = A^{-1}\mathbf{b} = \mathbf{x}$$

The projection matrix becomes $[P] = AA^+ = AA^{-1} = I$. (What is the geometric interpretation of this equality?)

Theorem 7.12 summarizes the key properties of the pseudoinverse of a matrix. (Before reading the proof of this theorem, verify these properties for the matrix in Example 7.32.)

Theorem 7.12 Let A be a matrix with linearly independent columns. Then the pseudoinverse A^+ of A satisfies the following properties, called the ***Penrose conditions*** for A:

a. $AA^+A = A$
b. $A^+AA^+ = A^+$
c. AA^+ and A^+A are symmetric.

Proof We prove condition (a) and half of condition (c) and leave the proofs of the remaining conditions as Exercises 54 and 55.

(a) We compute

$$\begin{aligned} AA^+A &= A((A^TA)^{-1}A^T)A \\ &= A(A^TA)^{-1}(A^TA) \\ &= AI = A \end{aligned}$$

(c) By Theorem 3.4, A^TA is symmetric. Therefore, $(A^TA)^{-1}$ is also symmetric, by Exercise 46 in Section 3.3. Taking the transpose of AA^+, we have

$$\begin{aligned} (AA^+)^T &= (A(A^TA)^{-1}A^T)^T \\ &= (A^T)^T((A^TA)^{-1})^TA^T \\ &= A(A^TA)^{-1}A^T \\ &= AA^+ \end{aligned}$$

Exercise 56 explores further properties of the pseudoinverse. In the next section, we will see how to extend the definition of A^+ to handle *all* matrices, whether or not the columns of A are linearly independent.

Exercises 7.3

CAS

In Exercises 1–3, consider the data points (1, 0), (2, 1), *and* (3, 5). *Compute the least squares error for the given line. In each case, plot the points and the line.*

1. $y = -2 + 2x$ **2.** $y = x$ **3.** $y = -3 + \frac{5}{2}x$

In Exercises 4–6, consider the data points (−5, 3), (0, 3), (5, 2), *and* (10, 0). *Compute the least squares error for the given line. In each case, plot the points and the line.*

4. $y = 3 - \frac{1}{3}x$ **5.** $y = \frac{5}{2}$ **6.** $y = 2 - \frac{1}{5}x$

In Exercises 7–14, find the least squares approximating line for the given points and compute the corresponding least squares error.

7. (1, 0), (2, 1), (3, 5)

8. (1, 6), (2, 3), (3, 1)

9. (0, 4), (1, 1), (2, 0)

10. (0, 3), (1, 3), (2, 5)

11. (−5, −1), (0, 1), (5, 2), (10, 4)

12. $(-5, 3), (0, 3), (5, 2), (10, 0)$

13. $(1, 1), (2, 3), (3, 4), (4, 5), (5, 7)$

14. $(1, 10), (2, 8), (3, 5), (4, 3), (5, 0)$

In Exercises 15–18, find the least squares approximating parabola for the given points.

15. $(1, 1), (2, -2), (3, 3), (4, 4)$

16. $(1, 6), (2, 0), (3, 0), (4, 2)$

17. $(-2, 4), (-1, 7), (0, 3), (1, 0), (2, -1)$

18. $(-2, 0), (-1, -11), (0, -10), (1, -9), (2, 8)$

In Exercises 19–22, find a least squares solution of $A\mathbf{x} = \mathbf{b}$ by constructing and solving the normal equations.

19. $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

21. $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 4 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 2 \end{bmatrix}$

In Exercises 23 and 24, show that the least squares solution of $A\mathbf{x} = \mathbf{b}$ is not unique and solve the normal equations to find all the least squares solutions.

23. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

In Exercises 25 and 26, find the best approximation to a solution of the given system of equations.

25.
$$\begin{aligned} x + y - z &= 2 \\ -y + 2z &= 6 \\ 3x + 2y - z &= 11 \\ -x + z &= 0 \end{aligned}$$

26.
$$\begin{aligned} 2x + 3y + z &= 21 \\ x + y + z &= 7 \\ -x + y - z &= 14 \\ 2y + z &= 0 \end{aligned}$$

In Exercises 27 and 28, a QR factorization of A is given. Use it to find a least squares solution of $A\mathbf{x} = \mathbf{b}$.

27. $A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, R = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix},$

$R = \begin{bmatrix} \sqrt{6} & -\sqrt{6}/2 \\ 0 & 1/\sqrt{2} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

29. A tennis ball is dropped from various heights, and the height of the ball on the first bounce is measured. Use the data in Table 7.3 to find the least squares approximating line for bounce height b as a linear function of initial height h.

Table 7.3

***h* (cm)**	20	40	48	60	80	100
***b* (cm)**	14.5	31	36	45.5	59	73.5

30. Hooke's Law states that the length L of a spring is a linear function of the force F applied to it. (See Figure 7.17 and Example 6.92.) Accordingly, there are constants a and b such that

$$L = a + bF$$

Table 7.4 shows the results of attaching various weights to a spring.

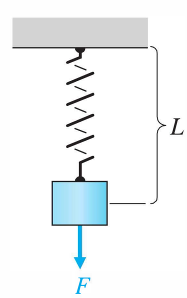


Figure 7.17

Table 7.4

***F* (oz)**	2	4	6	8
***L* (in.)**	7.4	9.6	11.5	13.6

Table 7.5

Year of Birth	1920	1930	1940	1950	1960	1970	1980	1990
Life Expectancy (years)	54.1	59.7	62.9	68.2	69.7	70.8	73.7	75.4

Source: World Almanac and Book of Facts. New York: World Almanac Books, 1999

(a) Determine the constants a and b by finding the least squares approximating line for these data. What does a represent?

(b) Estimate the length of the spring when a weight of 5 ounces is attached.

31. Table 7.5 gives life expectancies for people born in the United States in the given years.

(a) Determine the least squares approximating line for these data and use it to predict the life expectancy of someone born in 2000.

(b) How good is this model? Explain.

32. When an object is thrown straight up into the air, Newton's Second Law of Motion states that its height $s(t)$ at time t is given by

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}gt^2$$

where v_0 is its initial velocity and g is the constant of acceleration due to gravity. Suppose we take the measurements shown in Table 7.6.

Table 7.6

Time (s)	0.5	1	1.5	2	3
Height (m)	11	17	21	23	18

(a) Find the least squares approximating quadratic for these data.

(b) Estimate the height at which the object was released (in m), its initial velocity (in m/s), and its acceleration due to gravity (in m/s^2).

(c) Approximately when will the object hit the ground?

33. Table 7.7 gives the population of the United States at 10-year intervals for the years 1950–2000.

(a) Assuming an exponential growth model of the form $p(t) = ce^{kt}$, where $p(t)$ is the population at time t, use least squares to find the equation for the growth rate of the population. [*Hint:* Let $t = 0$ be 1950.]

(b) Use the equation to estimate the U.S. population in 2010.

Table 7.7

Year	Population (in millions)
1950	150
1960	179
1970	203
1980	227
1990	250
2000	281

Source: U.S. Bureau of the Census

34. Table 7.8 shows average major league baseball salaries for the years 1970–2005.

(a) Find the least squares approximating quadratic for these data.

(b) Find the least squares approximating exponential for these data.

(c) Which equation gives the better approximation? Why?

(d) What do you estimate the average major league baseball salary will be in 2010 and 2015?

Table 7.8

Year	Average Salary (thousands of dollars)
1970	29.3
1975	44.7
1980	143.8
1985	371.6
1990	597.5
1995	1110.8
2000	1895.6
2005	2476.6

Source: Major League Baseball Players Association

35. A 200 mg sample of radioactive polonium-210 is observed as it decays. Table 7.9 shows the mass remaining at various times.

Assuming an exponential decay model, use least squares to find the half-life of polonium-210. (See Section 6.7.)

Table 7.9

Time (days)	0	30	60	90
Mass (mg)	200	172	148	128

36. Find the plane $z = a + bx + cy$ that best fits the data points $(0, -4, 0)$, $(5, 0, 0)$, $(4, -1, 1)$, $(1, -3, 1)$, and $(-1, -5, -2)$.

In Exercises 37–42, find the standard matrix of the orthogonal projection onto the subspace W. Then use this matrix to find the orthogonal projection of **v** *onto W.*

37. $W = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right), \mathbf{v} = \begin{bmatrix}3\\4\end{bmatrix}$

38. $W = \text{span}\left(\begin{bmatrix}1\\-2\end{bmatrix}\right), \mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix}$

39. $W = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right), \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

40. $W = \text{span}\left(\begin{bmatrix}2\\2\\-1\end{bmatrix}\right), \mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

41. $W = \text{span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right), \mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

42. $W = \text{span}\left(\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right), \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

43. Verify that the standard matrix of the projection onto W in Example 7.31 (as constructed by Theorem 7.11) does not depend on the choice of basis. Take

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right\}$$

as a basis for W and repeat the calculations to show that the resulting projection matrix is the same.

44. Let A be a matrix with linearly independent columns and let $P = A(A^TA)^{-1}A^T$ be the matrix of orthogonal projection onto col(A).

(a) Show that P is symmetric.
(b) Show that P is idempotent.

In Exercises 45–52, compute the pseudoinverse of A.

45. $A = \begin{bmatrix}1\\2\end{bmatrix}$

46. $A = \begin{bmatrix}1\\-1\\2\end{bmatrix}$

47. $A = \begin{bmatrix}1 & 3\\-1 & 1\\0 & 2\end{bmatrix}$

48. $A = \begin{bmatrix}1 & 3\\3 & 1\\2 & 2\end{bmatrix}$

49. $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

50. $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

51. $A = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

52. $A = \begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\\1 & 1 & -2\\0 & 0 & 2\end{bmatrix}$

53. (a) Show that if A is a square matrix with linearly independent columns, then $A^+ = A^{-1}$.
(b) If A is an $m \times n$ matrix with orthonormal columns, what is A^+?

54. Prove Theorem 7.12(b).

55. Prove the remaining part of Theorem 7.12(c).

56. Let A be a matrix with linearly independent columns. Prove the following:

(a) $(cA)^+ = (1/c)A^+$ for all scalars $c \neq 0$.
(b) $(A^+)^+ = A$ if A is a square matrix.
(c) $(A^T)^+ = (A^+)^T$ if A is a square matrix.

57. Let n data points $(x_1, y_1), \ldots, (x_n, y_n)$ be given. Show that if the points do not all lie on the same vertical line, then they have a unique least squares approximating line.

58. Let n data points $(x_1, y_1), \ldots, (x_n, y_n)$ be given. Generalize Exercise 57 to show that if at least $k + 1$ of $x_1, \ldots, x_n$ are distinct, then the given points have a unique least squares approximating polynomial of degree at most k.

7.4 The Singular Value Decomposition

In Chapter 5, we saw that every symmetric matrix A can be factored as $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix displaying the eigenvalues of A. If A is not symmetric, such a factorization is not possible, but as we learned in Chapter 4, we may still be able to factor a square matrix A as $A = PDP^{-1}$, where D is as before but P is now simply an invertible matrix. However, not every matrix is diagonalizable, so it may surprise you that we will now show that *every* matrix (symmetric or not, square or not) has a factorization of the form $A = PDQ^T$, where P and Q are orthogonal and D is a diagonal matrix! This remarkable result is the *singular value decomposition* (SVD), and it is one of the most important of all matrix factorizations.

In this section, we will show how to compute the SVD of a matrix and then consider some of its many applications. Along the way, we will tie up some loose ends by answering a few questions that were left open in previous sections.

The Singular Values of a Matrix

For any $m \times n$ matrix A, the $n \times n$ matrix A^TA is symmetric and hence can be orthogonally diagonalized, by the Spectral Theorem. Not only are the eigenvalues of A^TA all real (Theorem 5.18), they are all *nonnegative*. To show this, let λ be an eigenvalue of A^TA with corresponding unit eigenvector $\mathbf{v}$. Then

$$\begin{aligned} 0 \le \|A\mathbf{v}\|^2 &= (A\mathbf{v})\cdot(A\mathbf{v}) = (A\mathbf{v})^TA\mathbf{v} = \mathbf{v}^TA^TA\mathbf{v} \\ &= \mathbf{v}^T\lambda\mathbf{v} = \lambda(\mathbf{v}\cdot\mathbf{v}) = \lambda\|\mathbf{v}\|^2 = \lambda \end{aligned}$$

It therefore makes sense to take (positive) square roots of these eigenvalues.

Definition If A is an $m \times n$ matrix, the ***singular values*** of A are the square roots of the eigenvalues of A^TA and are denoted by $\sigma_1, \ldots, \sigma_n$. It is conventional to arrange the singular values so that $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$.

Example 7.33

Find the singular values of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution The matrix

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$. Consequently, the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$ and $\sigma_2 = \sqrt{\lambda_2} = 1$.

To understand the significance of the singular values of an $m \times n$ matrix A, consider the eigenvectors of A^TA. Since A^TA is symmetric, we know that there is an *orthonormal* basis for $\mathbb{R}^n$ that consists of eigenvectors of A^TA. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be such a basis corresponding to the eigenvalues of A^TA, ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. From our calculations just before the definition,

$$\lambda_i = \|A\mathbf{v}_i\|^2$$

Therefore,

$$\sigma_i = \sqrt{\lambda_i} = \|A\mathbf{v}_i\|$$

In other words, the singular values of A are the lengths of the vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$.

Geometrically, this result has a nice interpretation. Consider Example 7.33 again. If $\mathbf{x}$ lies on the unit circle in $\mathbb{R}^2$ (i.e., $\|\mathbf{x}\| = 1$), then

$$\|A\mathbf{x}\|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x}$$

$$= [x_1 \quad x_2]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 2x_1x_2 + 2x_2^2$$

which we recognize is a quadratic form. By Theorem 5.25, the maximum and minimum values of this quadratic form, subject to the constraint $\|\mathbf{x}\| = 1$, are $\lambda_1 = 3$ and $\lambda_2 = 1$, respectively, and they occur at the corresponding eigenvectors of A^TA—that is, when $\mathbf{x} = \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{x} = \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, respectively. Since

$$\|A\mathbf{v}_i\|^2 = \mathbf{v}_i^TA^TA\mathbf{v}_i = \lambda_i$$

for $i = 1, 2$, we see that $\sigma_1 = \|A\mathbf{v}_1\| = \sqrt{3}$ and $\sigma_2 = \|A\mathbf{v}_2\| = 1$ are the maximum and minimum values of the lengths $\|A\mathbf{x}\|$ as $\mathbf{x}$ traverses the unit circle in $\mathbb{R}^2$.

Now, the linear transformation corresponding to A maps $\mathbb{R}^2$ onto the plane in $\mathbb{R}^3$ with equation $x - y - z = 0$ (verify this), and the image of the unit circle under this transformation is an ellipse that lies in this plane. (We will verify this fact in general shortly; see Figure 7.18.) So σ_1 and σ_2 are the lengths of half of the major and minor axes of this ellipse, as shown in Figure 7.19.

We can now describe the singular value decomposition of a matrix.

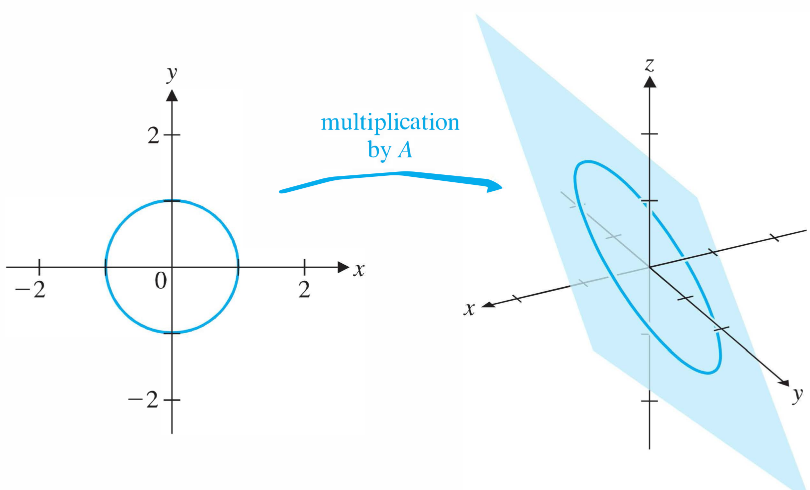


Figure 7.18
The matrix A transforms the unit circle in $\mathbb{R}^2$ into an ellipse in $\mathbb{R}^3$

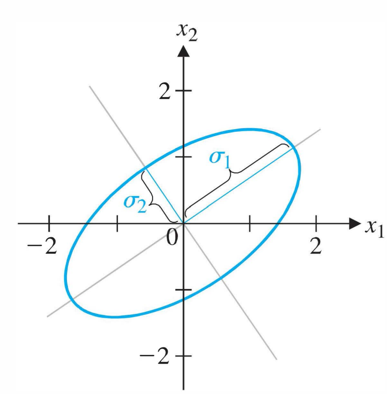


Figure 7.19

The Singular Value Decomposition

We want to show that an $m \times n$ matrix A can be factored as

$$A = U\Sigma V^T$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is an $m \times n$ "diagonal" matrix. If the *nonzero* singular values of A are

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$, then Σ will have the block form

$$\Sigma = \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \left[\begin{array}{c|c} D & O \\ \hline O & O \end{array}\right] \end{array} \begin{array}{l} \\ \}\ r \\ \}\ m-r \end{array}, \quad \text{where} \quad D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} \tag{1}$$

and each matrix O is a zero matrix of the appropriate size. (If $r = m$ or $r = n$, some of these will not appear.) Some examples of such a matrix Σ with $r = 2$ are

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(What is D in each case?)

To construct the orthogonal matrix V, we first find an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of the $n \times n$ symmetric matrix A^TA. Then

$$V = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n]$$

is an orthogonal $n \times n$ matrix.

For the orthogonal matrix U, we first note that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is an orthogonal set of vectors in $\mathbb{R}^m$. To see this, suppose that $\mathbf{v}_i$ is the eigenvector of A^TA corresponding to the eigenvalue λ_i. Then, for $i \neq j$, we have

$$\begin{aligned} (A\mathbf{v}_i) \cdot (A\mathbf{v}_j) &= (A\mathbf{v}_i)^T A\mathbf{v}_j \\ &= \mathbf{v}_i^T A^T A\mathbf{v}_j \\ &= \mathbf{v}_i^T \lambda_j \mathbf{v}_j \\ &= \lambda_j(\mathbf{v}_i \cdot \mathbf{v}_j) = 0 \end{aligned}$$

since the eigenvectors $\mathbf{v}_i$ are orthogonal. Now recall that the singular values satisfy $\sigma_i = \|A\mathbf{v}_i\|$ and that the first r of these are nonzero. Therefore, we can normalize $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ by setting

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i \quad \text{for } i = 1, \ldots, r$$

This guarantees that $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthonormal set in $\mathbb{R}^m$, but if $r < m$ it will not be a basis for $\mathbb{R}^m$. In this case, we extend the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ for $\mathbb{R}^m$. (This is the only tricky part of the construction; we will describe techniques for carrying it out in the examples below and in the exercises.) Then we set

$$U = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_m]$$

All that remains to be shown is that this works; that is, we need to verify that with U, V, and Σ as described, we have $A = U\Sigma V^T$. Since $V^T = V^{-1}$, this is equivalent to showing that

$$AV = U\Sigma$$

We know that $$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \quad \text{for } i = 1, \ldots, r$$

and $\|A\mathbf{v}_i\| = \sigma_i = 0$ for $i = r + 1, \ldots, n$. Hence,

$$A\mathbf{v}_i = \mathbf{0} \quad \text{for } i = r + 1, \ldots, n$$

Therefore,
$$\begin{aligned} AV &= A[\mathbf{v}_1 \ \cdots \ \mathbf{v}_n] \\ &= [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_n] \\ &= [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r \ \ \mathbf{0} \ \cdots \ \mathbf{0}] \\ &= [\sigma_1\mathbf{u}_1 \ \cdots \ \sigma_r\mathbf{u}_r \ \ \mathbf{0} \ \cdots \ \mathbf{0}] \\ &= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \left[\begin{array}{ccc:c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & O \\ 0 & \cdots & \sigma_r & \\ \hdashline & O & & O \end{array}\right] \\ &= U\Sigma \end{aligned}$$

as required.

We have just proved the following extremely important theorem.

Theorem 7.13 **The Singular Value Decomposition**

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ matrix Σ of the form shown in Equation (1) such that

$$A = U\Sigma V^T$$

A factorization of A as in Theorem 7.13 is called a ***singular value decomposition (SVD)*** of A. The columns of U are called ***left singular vectors*** of A, and the columns of V are called ***right singular vectors*** of A. The matrices U and V are not uniquely determined by A, but Σ *must* contain the singular values of A, as in Equation (1). (See Exercise 25.)

Example 7.34

Find a singular value decomposition for the following matrices:

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution (a) We compute

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and find that its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

(Verify this.) These vectors are orthogonal, so we normalize them to obtain

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

The singular values of A are $\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{1} = 1$, and $\sigma_3 = \sqrt{0} = 0$. Thus,

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find U, we compute

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{1}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These vectors already form an orthonormal basis (the standard basis) for $\mathbb{R}^2$, so we have

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This yields the SVD

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = U\Sigma V^T$$

which can be easily checked. (Note that V had to be transposed. Also note that the singular value σ_3 does not appear in Σ.)

(b) This is the matrix in Example 7.33, so we already know that the singular values are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$, corresponding to $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

For U, we compute

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

This time, we need to extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis for $\mathbb{R}^3$. There are several ways to proceed; one method is to use the Gram-Schmidt Process, as in Example 5.14. We first need to find a linearly independent set of three vectors that contains $\mathbf{u}_1$ and $\mathbf{u}_2$. If $\mathbf{e}_3$ is the third standard basis vector in $\mathbb{R}^3$, it is clear that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_3\}$ is linearly independent. (Here, you should be able to determine this by inspection, but a reliable method to use in general is to row reduce the matrix with these vectors as its columns and use the Fundamental Theorem.) Applying Gram-Schmidt (with normalization) to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_3\}$ (only the last step is needed), we find

$$\mathbf{u}_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

so

$$U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

and we have the SVD

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\Sigma V^T$$

There is another form of the singular value decomposition, analogous to the spectral decomposition of a symmetric matrix. It is obtained from the SVD by an outer product expansion and is very useful in applications. We can obtain this version of the SVD by imitating what we did to obtain the spectral decomposition.

Accordingly, we have

$$
\begin{aligned}
A = U\Sigma V^T &= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m]\left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & O \\ 0 & \cdots & \sigma_r & \\ \hline & O & & O \end{array}\right]\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\
&= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r \,\vdots\, \mathbf{u}_{r+1} \ \cdots \ \mathbf{u}_m]\left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & O \\ 0 & \cdots & \sigma_r & \\ \hline & O & & O \end{array}\right]\left[\begin{array}{c} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \\ \hline \mathbf{v}_{r+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{array}\right] \\
&= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r]\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} + [\mathbf{u}_{r+1} \ \cdots \ \mathbf{u}_m][O]\begin{bmatrix} \mathbf{v}_{r+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\
&= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r]\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} \\
&= [\sigma_1\mathbf{u}_1 \ \cdots \ \sigma_r\mathbf{u}_r]\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} \\
&= \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T
\end{aligned}
$$

using block multiplication and the column-row representation of the product. The following theorem summarizes the process for obtaining this ***outer product form of the SVD.***

Theorem 7.14 **The Outer Product Form of the SVD**

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be left singular vectors and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be right singular vectors of A corresponding to these singular values. Then

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T$$

Remark If A is a positive definite, symmetric matrix, then Theorems 7.13 and 7.14 both reduce to results that we already know. In this case, it is not hard to show that the SVD generalizes the Spectral Theorem and that Theorem 7.14 generalizes the spectral decomposition. (See Exercise 27.)

The SVD of a matrix A contains much important information about A, as outlined in the crucial Theorem 7.15.

Theorem 7.15 Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix A. Let $\sigma_1, \dots, \sigma_r$ be all the nonzero singular values of A. Then:

a. The rank of A is r.
b. $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal basis for col(A).
c. $\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$ is an orthonormal basis for null(A^T).
d. $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is an orthonormal basis for row(A).
e. $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal basis for null(A).

Proof (a) By Exercise 61 in Section 3.5, we have

$$\begin{aligned} \operatorname{rank}(A) &= \operatorname{rank}(U\Sigma V^T) \\ &= \operatorname{rank}(\Sigma V^T) \\ &= \operatorname{rank}(\Sigma) = r \end{aligned}$$

(b) We already know that $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal set. Therefore, it is linearly independent, by Theorem 5.1. Since $\mathbf{u}_i = (1/\sigma_i)A\mathbf{v}_i$ for $i = 1, \dots, r$, each $\mathbf{u}_i$ is in the column space of A. (Why?) Furthermore,

$$r = \operatorname{rank}(A) = \dim(\operatorname{col}(A))$$

Therefore, $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal basis for col(A), by Theorem 6.10(c).

(c) Since $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is an orthonormal basis for $\mathbb{R}^m$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is a basis for col(A), by property (b), it follows that $\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$ is an orthonormal basis for the orthogonal complement of col(A). But $(\operatorname{col}(A)) = \operatorname{null}(A^T)$, by Theorem 5.10.

(e) Since

$$A\mathbf{v}_{r+1} = \cdots = A\mathbf{v}_n = \mathbf{0}$$

the set $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal set contained in the null space of A. Therefore, $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is a linearly independent set of $n - r$ vectors in null(A). But

$$\dim(\operatorname{null}(A)) = n - r$$

by the Rank Theorem, so $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal basis for null(A), by Theorem 6.10(c).

(d) Property (d) follows from property (e) and Theorem 5.10. (You are asked to prove this in Exercise 32.)

The SVD provides new geometric insight into the effect of matrix transformations. We have noted several times (without proof) that an $m \times n$ matrix transforms the unit sphere in $\mathbb{R}^n$ into an ellipsoid in $\mathbb{R}^m$. This point arose, for example, in our discussions of Perron's Theorem and of operator norms, as well as in the introduction to singular values in this section. We now prove this result.

Theorem 7.16 Let A be an $m \times n$ matrix with rank r. Then the image of the unit sphere in $\mathbb{R}^n$ under the matrix transformation that maps $\mathbf{x}$ to $A\mathbf{x}$ is

a. the surface of an ellipsoid in $\mathbb{R}^m$ if $r = n$.
b. a solid ellipsoid in $\mathbb{R}^m$ if $r < n$.

Proof Let $A = U\Sigma V^T$ be a singular value decomposition of the $m \times n$ matrix A. Let the left and right singular vectors of A be $\mathbf{u}_1, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$, respectively. Since $\operatorname{rank}(A) = r$, the singular values of A satisfy

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0 \quad \text{and} \quad \sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$$

by Theorem 7.15(a). Let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a unit vector in $\mathbb{R}^n$. Now, since V is an orthogonal matrix, so is V^T, and hence $V^T\mathbf{x}$ is a unit vector, by Theorem 5.6. Now

$$V^T\mathbf{x} = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{v}_1^T\mathbf{x} \\ \vdots \\ \mathbf{v}_n^T\mathbf{x} \end{bmatrix}$$

so $(\mathbf{v}_1^T\mathbf{x})^2 + \cdots + (\mathbf{v}_n^T\mathbf{x})^2 = 1$.

By the outer product form of the SVD, we have $A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T$. Therefore,

$$\begin{aligned} A\mathbf{x} &= \sigma_1\mathbf{u}_1\mathbf{v}_1^T\mathbf{x} + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T\mathbf{x} \\ &= (\sigma_1\mathbf{v}_1^T\mathbf{x})\mathbf{u}_1 + \cdots + (\sigma_r\mathbf{v}_r^T\mathbf{x})\mathbf{u}_r \\ &= y_1\mathbf{u}_1 + \cdots + y_r\mathbf{u}_r \end{aligned}$$

where we are letting y_i denote the scalar $\sigma_i\mathbf{v}_i^T\mathbf{x}$.

(a) If $r = n$, then we must have $n \le m$ and

$$\begin{aligned} A\mathbf{x} &= y_1\mathbf{u}_1 + \cdots + y_n\mathbf{u}_n \\ &= U\mathbf{y} \end{aligned}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Therefore, again by Theorem 5.6, $\|A\mathbf{x}\| = \|U\mathbf{y}\| = \|\mathbf{y}\|$, since U is orthogonal. But

$$\left(\frac{y_1}{\sigma_1}\right)^2 + \cdots + \left(\frac{y_n}{\sigma_n}\right)^2 = (\mathbf{v}_1^T\mathbf{x})^2 + \cdots + (\mathbf{v}_n^T\mathbf{x})^2 = 1$$

which shows that the vectors $A\mathbf{x}$ form the surface of an ellipsoid in $\mathbb{R}^m$. (Why?)

(b) If $r < n$, the only difference in the above steps is that the equation becomes

$$\left(\frac{y_1}{\sigma_1}\right)^2 + \cdots + \left(\frac{y_r}{\sigma_r}\right)^2 \le 1$$

since we are missing some terms. This inequality corresponds to a solid ellipsoid in $\mathbb{R}^m$.

Example 7.35

Describe the image of the unit sphere in $\mathbb{R}^3$ under the action of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution In Example 7.34(a), we found the following SVD of A:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

Since $r = \operatorname{rank}(A) = 2 < 3 = n$, the second part of Theorem 7.16 applies. The image of the unit sphere will satisfy the inequality

$$\left(\frac{y_1}{\sqrt{2}}\right)^2 + \left(\frac{y_2}{1}\right)^2 \le 1 \quad \text{or} \quad \frac{y_1^2}{2} + y_2^2 \le 1$$

relative to y_1y_2 coordinate axes in $\mathbb{R}^2$ (corresponding to the left singular vectors $\mathbf{u}_1$ and $\mathbf{u}_2$). Since $\mathbf{u}_1 = \mathbf{e}_1$ and $\mathbf{u}_2 = \mathbf{e}_2$, the image is as shown in Figure 7.20.

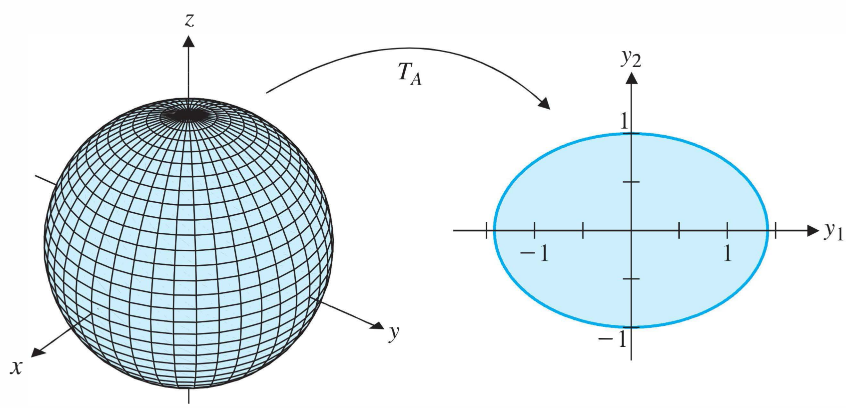


Figure 7.20

In general, we can describe the effect of an $m \times n$ matrix A on the unit sphere in $\mathbb{R}^n$ in terms of the effect of each factor in its SVD, $A = U\Sigma V^T$, from right to left. Since V^T is an orthogonal matrix, it maps the unit sphere to itself. The $m \times n$ matrix Σ does two things: The diagonal entries $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$ collapse $n - r$ of the dimensions of the unit sphere, leaving an r-dimensional unit sphere, which the nonzero diagonal entries $\sigma_1, \ldots, \sigma_r$ then distort into an ellipsoid. The orthogonal matrix U then aligns the axes of this ellipsoid with the orthonormal basis vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ in $\mathbb{R}^m$. (See Figure 7.21.)

Applications of the SVD

The singular value decomposition is an extremely useful tool, both practically and theoretically. We will look at just a few of its many applications.

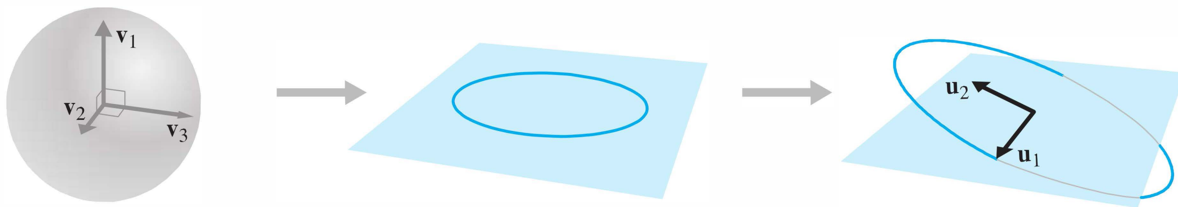


Figure 7.21

Rank Until now, we have not worried about calculating the rank of a matrix from a computational point of view. We compute the rank of a matrix by row reducing it to echelon form and counting the number of nonzero rows. However, as we have seen, roundoff errors can affect this process, especially if the matrix is ill-conditioned. Entries that should be zero may end up as very small nonzero numbers, affecting our ability to accurately determine the rank and other quantities associated with the matrix. In practice, the SVD is often used to find the rank of a matrix, since it is much more reliable when roundoff errors are present. The basic idea behind this approach is that the orthogonal matrices U and V in the SVD preserve lengths and thus do not introduce additional errors; any errors that occur will tend to show up in the matrix Σ.

CAS **Example 7.36**

Let

$$A = \begin{bmatrix} 8.1650 & -0.0041 & -0.0041 \\ 4.0825 & -3.9960 & 4.0042 \\ 4.0825 & 4.0042 & -3.9960 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8.17 & 0 & 0 \\ 4.08 & -4 & 4 \\ 4.08 & 4 & -4 \end{bmatrix}$$

The matrix B has been obtained by rounding off the entries in A to two decimal places. If we compute the ranks of these two approximately equal matrices, we find that $\operatorname{rank}(A) = 3$ but $\operatorname{rank}(B) = 2$. By the Fundamental Theorem, this implies, among other things, that A is invertible but B is not.

The explanation for this critical difference between two matrices that are approximately equal lies in their SVDs. The singular values of A are 10, 8, and 0.01, so A has rank 3. The singular values of B are 10, 8, and 0, so B has rank 2.

In practical applications, it is often assumed that if a singular value is computed to be close to zero, then roundoff error has crept in and the actual value should be zero. In this way, "noise" can be filtered out. In this example, if we compute $A = U\Sigma V^T$ and replace

$$\Sigma = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \quad \text{by} \quad \Sigma' = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then $U\Sigma' V^T = B$. (Try it!)

Matrix Norms and the Condition Number The SVD can provide simple formulas for certain expressions involving matrix norms. Consider, for example, the Frobenius norm of a matrix. The following theorem shows that it is completely determined by the singular values of the matrix.

Theorem 7.17 Let A be an $m \times n$ matrix and let $\sigma_1, \ldots, \sigma_r$ be all the nonzero singular values of A. Then

$$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$$

The proof of this result depends on the following analogue of Theorem 5.6:

If A is an $m \times n$ matrix and Q is an $m \times m$ orthogonal matrix, then

$$\|QA\|_F = \|A\|_F \tag{2}$$

To show that this is true, we compute

$$\begin{aligned}
\|QA\|_F^2 &= \|[Q\mathbf{a}_1 \quad \cdots \quad Q\mathbf{a}_n]\|_F^2 \\
&= \|Q\mathbf{a}_1\|_E^2 + \cdots + \|Q\mathbf{a}_n\|_E^2 \\
&= \|\mathbf{a}_1\|_E^2 + \cdots + \|\mathbf{a}_n\|_E^2 \\
&= \|A\|_F^2
\end{aligned}$$

Proof of Theorem 7.17 Let $A = U\Sigma V^T$ be a singular value decomposition of A. Then, using Equation (2) twice, we have

$$\begin{aligned}
\|A\|_F^2 &= \|U\Sigma V^T\|_F^2 \\
&= \|\Sigma V^T\|_F^2 = \|(\Sigma V^T)^T\|_F^2 \\
&= \|V\Sigma^T\|_F^2 = \|\Sigma^T\|_F^2 = \sigma_1^2 + \cdots + \sigma_r^2
\end{aligned}$$

which establishes the result.

CAS **Example 7.37** Verify Theorem 7.17 for the matrix A in Example 7.18.

Solution The matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ has singular values 4.5150 and 3.1008. We check that

$$\sqrt{4.5150^2 + 3.1008^2} = \sqrt{30} = \|A\|_F$$

which agrees with Example 7.18.

In Section 7.2, we commented that there is no easy formula for the operator 2-norm of a matrix A. Although that is true, the SVD of A provides us with a very nice expression for $\|A\|_2$. Recall that

$$\|A\|_2 = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where the vector norm is the ordinary Euclidean norm. By Theorem 7.16, for $\|\mathbf{x}\| = 1$, the set of vectors $\|A\mathbf{x}\|$ lies on or inside an ellipsoid whose semi-axes have lengths equal to the nonzero singular values of A. It follows immediately that the largest of these is σ_1, so

$$\|A\|_2 = \sigma_1$$

This provides us with a neat way to express the condition number of a (square) matrix with respect to the operator 2-norm. Recall that the condition number (with respect to the operator 2-norm) of an invertible matrix A is defined as

$$\operatorname{cond}_2(A) = \|A^{-1}\|_2\|A\|_2$$

As you will be asked to show in Exercise 28, if $A = U\Sigma V^T$, then $A^{-1} = V\Sigma^{-1}U^T$. Therefore, the singular values of A^{-1} are $1/\sigma_1, \ldots, 1/\sigma_n$ (why?), and

$$1/\sigma_n \geq \cdots \geq 1/\sigma_1$$

It follows that $\|A^{-1}\|_2 = 1/\sigma_n$, so

$$\operatorname{cond}_2(A) = \frac{\sigma_1}{\sigma_n}$$

Example 7.38

Find the 2-condition number of the matrix A in Example 7.36.

Solution Since $\sigma_1 = 10$ and $\sigma_3 = 0.01$,

$$\operatorname{cond}_2(A) = \frac{\sigma_1}{\sigma_3} = \frac{10}{0.01} = 1000$$

This value is large enough to suggest that A may be ill-conditioned and we should be wary of the effect of roundoff errors.

The Pseudoinverse and Least Squares Approximation In Section 7.3, we produced the formula $A^+ = (A^TA)^{-1}A^T$ for the pseudoinverse of a matrix A. Clearly, this formula is valid only if A^TA is invertible, as we noted at the time. Equipped with the SVD, we can now define the pseudoinverse of *any* matrix, generalizing our previous formula.

E. H. Moore (1862–1932) was an American mathematician who worked in group theory, number theory, and geometry. He was the first head of the mathematics department at the University of Chicago when it opened in 1892. In 1920, he introduced a generalized matrix inverse that included rectangular matrices. His work did not receive much attention because of his obscure writing style.

Definition Let $A = U\Sigma V^T$ be an SVD for an $m \times n$ matrix A, where $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$ and D is an $r \times r$ diagonal matrix containing the nonzero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ of A. The ***pseudoinverse*** (or ***Moore-Penrose inverse***) of A is the $n \times m$ matrix A^+ defined by

$$A^+ = V\Sigma^+U^T$$

where Σ^+ is the $n \times m$ matrix

$$\Sigma^+ = \begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix}$$

Example 7.39 Find the pseudoinverses of the matrices in Example 7.34.

Solution (a) From the SVD

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = U\Sigma V^T$$

we form

$$\Sigma^+ = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) We have the SVD

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\Sigma V^T$$

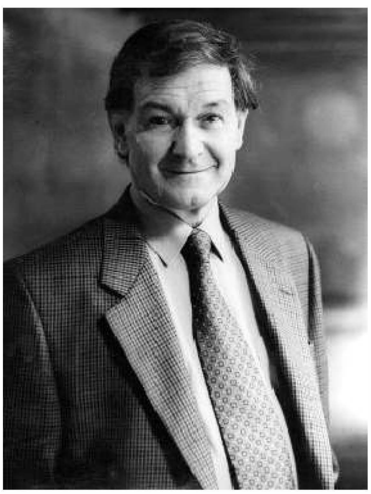
so

$$\Sigma^+ = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}$$



Jerry Bauer

One of those who was unaware of Moore's work on matrix inverses was Roger Penrose (b.1931), who introduced his own notion of a generalized matrix inverse in 1955. Penrose has made many contributions to geometry and theoretical physics. He is also the inventor of a type of *nonperiodic tiling* that covers the plane with only two different shapes of tile, yet has no repeating pattern. He has received many awards, including the 1988 Wolf Prize in Physics, which he shared with Stephen Hawking. In 1994, he was knighted for services to science. Sir Roger Penrose is currently the Emeritus Rouse Ball Professor of Mathematics at the University of Oxford.

It is straightforward to check that this new definition of the pseudoinverse generalizes the old one, for if the $m \times n$ matrix $A = U\Sigma V^T$ has linearly independent columns, then direct substitution shows that $(A^TA)^{-1}A^T = V\Sigma^+U^T$. (You are asked to verify this in Exercise 50.) Other properties of the pseudoinverse are explored in the exercises.

We have seen that when A has linearly independent columns, there is a unique least squares solution $\bar{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$; that is, the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$ have the unique solution

$$\bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = A^+\mathbf{b}$$

When the columns of A are linearly dependent, then A^TA is not invertible, so the normal equations have infinitely many solutions. In this case, we will ask for the solution $\bar{\mathbf{x}}$ of *minimum length* (i.e., the one closest to the origin). It turns out that this time we simply use the general version of the pseudoinverse.

Theorem 7.18 The least squares problem $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution $\bar{\mathbf{x}}$ of minimal length that is given by

$$\bar{\mathbf{x}} = A^{+}\mathbf{b}$$

Proof Let A be an $m \times n$ matrix of rank r with SVD $A = U\Sigma V^T$ (so that $A^+ = V\Sigma^+ U^T$). Let $\mathbf{y} = V^T\mathbf{x}$ and let $\mathbf{c} = U^T\mathbf{b}$. Write $\mathbf{y}$ and $\mathbf{c}$ in block form as

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}$$

where $\mathbf{y}_1$ and $\mathbf{c}_1$ are in $\mathbb{R}^r$.

We wish to minimize $\|\mathbf{b} - A\mathbf{x}\|$ or, equivalently, $\|\mathbf{b} - A\mathbf{x}\|^2$. Using Theorem 5.6 and the fact that U^T is orthogonal (because U is), we have

$$\begin{aligned} \|\mathbf{b} - A\mathbf{x}\|^2 &= \|U^T(\mathbf{b} - A\mathbf{x})\|^2 = \|U^T(\mathbf{b} - U\Sigma V^T\mathbf{x})\|^2 = \|U^T\mathbf{b} - U^TU\Sigma V^T\mathbf{x}\|^2 \\ &= \|\mathbf{c} - \Sigma\mathbf{y}\|^2 = \left\| \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} - \begin{bmatrix} D & O \\ O & O \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \mathbf{c}_1 - D\mathbf{y}_1 \\ \mathbf{c}_2 \end{bmatrix} \right\|^2 \end{aligned}$$

The only part of this expression that we have any control over is $\mathbf{y}_1$, so the minimum value occurs when $\mathbf{c}_1 - D\mathbf{y}_1 = \mathbf{0}$ or, equivalently, when $\mathbf{y}_1 = D^{-1}\mathbf{c}_1$. So all least squares solutions $\mathbf{x}$ are of the form

$$\mathbf{x} = V\mathbf{y} = V\begin{bmatrix} D^{-1}\mathbf{c}_1 \\ \mathbf{y}_2 \end{bmatrix}$$

Set

$$\bar{\mathbf{x}} = V\bar{\mathbf{y}} = V\begin{bmatrix} D^{-1}\mathbf{c}_1 \\ \mathbf{0} \end{bmatrix}$$

We claim that this $\bar{\mathbf{x}}$ is the least squares solution of minimal length. To show this, let's suppose that

$$\mathbf{x}' = V\mathbf{y}' = V\begin{bmatrix} D^{-1}\mathbf{c}_1 \\ \mathbf{y}_2 \end{bmatrix}$$

is a different least squares solution (hence, $\mathbf{y}_2 \neq \mathbf{0}$). Then

$$\|\bar{\mathbf{x}}\| = \|V\bar{\mathbf{y}}\| = \|\bar{\mathbf{y}}\| < \|\mathbf{y}'\| = \|V\mathbf{y}'\| = \|\mathbf{x}'\|$$

as claimed.

We still must show that $\bar{\mathbf{x}}$ is equal to $A^+\mathbf{b}$. To do so, we simply compute

$$\begin{aligned} \bar{\mathbf{x}} = V\bar{\mathbf{y}} &= V\begin{bmatrix} D^{-1}\mathbf{c}_1 \\ \mathbf{0} \end{bmatrix} = V\begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix}\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} \\ &= V\Sigma^+\mathbf{c} = V\Sigma^+U^T\mathbf{b} = A^+\mathbf{b} \end{aligned}$$

Example 7.40 Find the minimum length least squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution The corresponding equations

$$x + y = 0$$
$$x + y = 1$$

are clearly inconsistent, so a least squares solution is our only hope. Moreover, the columns of A are linearly dependent, so there will be infinitely many least squares solutions—among which we want the one with minimal length.

An SVD of A is given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = U\Sigma V^T$$

(Verify this.) It follows that

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

so

$$\bar{\mathbf{x}} = A^+\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

You can see that the minimum least squares solution in Example 7.40 satisfies $x + y = \frac{1}{2}$. In a sense, this is a compromise between the two equations we started with. In Exercise 49, you are asked to solve the normal equations for this problem directly and to verify that this solution really is the one closest to the origin.

The Fundamental Theorem of Invertible Matrices It is appropriate to conclude by revisiting the Fundamental Theorem of Invertible Matrices one more time. Not surprisingly, the singular values of a square matrix tell us when the matrix is invertible.

Theorem 7.19 **The Fundamental Theorem of Invertible Matrices: Final Version**

Let A be an $n \times n$ matrix and let $T: V \to W$ be a linear transformation whose matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively, is A. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\operatorname{rank}(A) = n$
g. $\operatorname{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.

m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A.
p. T is invertible.
q. T is one-to-one.
r. T is onto.
s. $\ker(T) = \{\mathbf{0}\}$
t. $\text{range}(T) = W$
u. 0 is not a singular value of A.

Proof First note that, by the definition of singular values, 0 is a singular value of A if and only if 0 is an eigenvalue of A^TA.

(a) $\Rightarrow$ (u) If A is invertible, so is A^T, and hence A^TA is as well. Therefore, property (o) implies that 0 is not an eigenvalue of A^TA, so 0 is not a singular value of A.

(u) $\Rightarrow$ (a) If 0 is not a singular value of A, then 0 is not an eigenvalue of A^TA. Therefore, A^TA is invertible, by the equivalence of properties (a) and (o). But then $\text{rank}(A) = n$, by Theorem 3.28, so A is invertible, by the equivalence of properties (a) and (f).

Vignette

Digital Image Compression

Among the many applications of the SVD, one of the most impressive is its use in compressing digital images so that they can be efficiently transmitted electronically (by satellite, fax, Internet, or the like). We have already discussed the problem of detecting and correcting errors in such transmissions. The problem we now wish to consider has to do with reducing the amount of information that has to be transmitted, without losing any essential information.

In the case of digital images, let's suppose we have a grayscale picture that is 340×280 pixels in size. Each pixel is one of 256 shades of gray, which we can represent by a number between 0 and 255. We can store this information in a 340×280 matrix A, but transmitting and manipulating these 95,200 numbers is very expensive. The idea behind image compression is that some parts of the picture are less interesting than others. For example, in a photograph of someone standing outside, there may be a lot of sky in the background, while the person's face contains a lot of detail. We can probably get away with transmitting every second or third pixel in the background, but we would like to keep all the pixels in the region of the face.

It turns out that the small singular values in the SVD of the matrix A come from the "boring" parts of the image, and we can ignore many of them. Suppose, then, that we have the SVD of A in outer product form

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

Let $k \leq r$ and define

$$A_k = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T$$

Then A_k is an approximation to A that corresponds to keeping only the first k singular values and the corresponding singular vectors. For our 340×280 example, we may discover that it is enough to transmit only the data corresponding to the first 20 singular values. Then, instead of transmitting 95,200 numbers, we need only send 20 singular values plus the 20 vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{20}$ in $\mathbb{R}^{340}$ and the 20 vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{20}$ in $\mathbb{R}^{280}$, for a total of

$$20 + 20 \cdot 340 + 20 \cdot 280 = 12{,}420$$

numbers. This represents a substantial saving!

Bettmann/Corbis

Figure 7.22

The picture of the mathematician Gauss in Figure 7.22 is a 340×280 pixel image. It has 256 shades of gray, so the corresponding matrix A is 340×280, with entries between 0 and 255.

It turns out that the matrix A has rank 280. If we approximate A by A_k, as described above, we get an image that corresponds to the first k singular values of A. Figure 7.23 shows several of these images for values of k from 2 to 256. At first, the image is very blurry, but fairly quickly it takes shape. Notice that A_{32} already gives a pretty good approximation to the actual image (which comes from $A = A_{280}$, as shown in the upper left-hand corner of Figure 7.23).

Some of the singular values of A are $\sigma_1 = 49{,}096$, $\sigma_{16} = 22{,}589$, $\sigma_{32} = 10{,}187$, $\sigma_{64} = 484$, $\sigma_{128} = 182$, $\sigma_{256} = 5$, and $\sigma_{280} = 0.5$. The smaller singular values contribute very little to the image, which is why the approximations quickly look so close to the original.

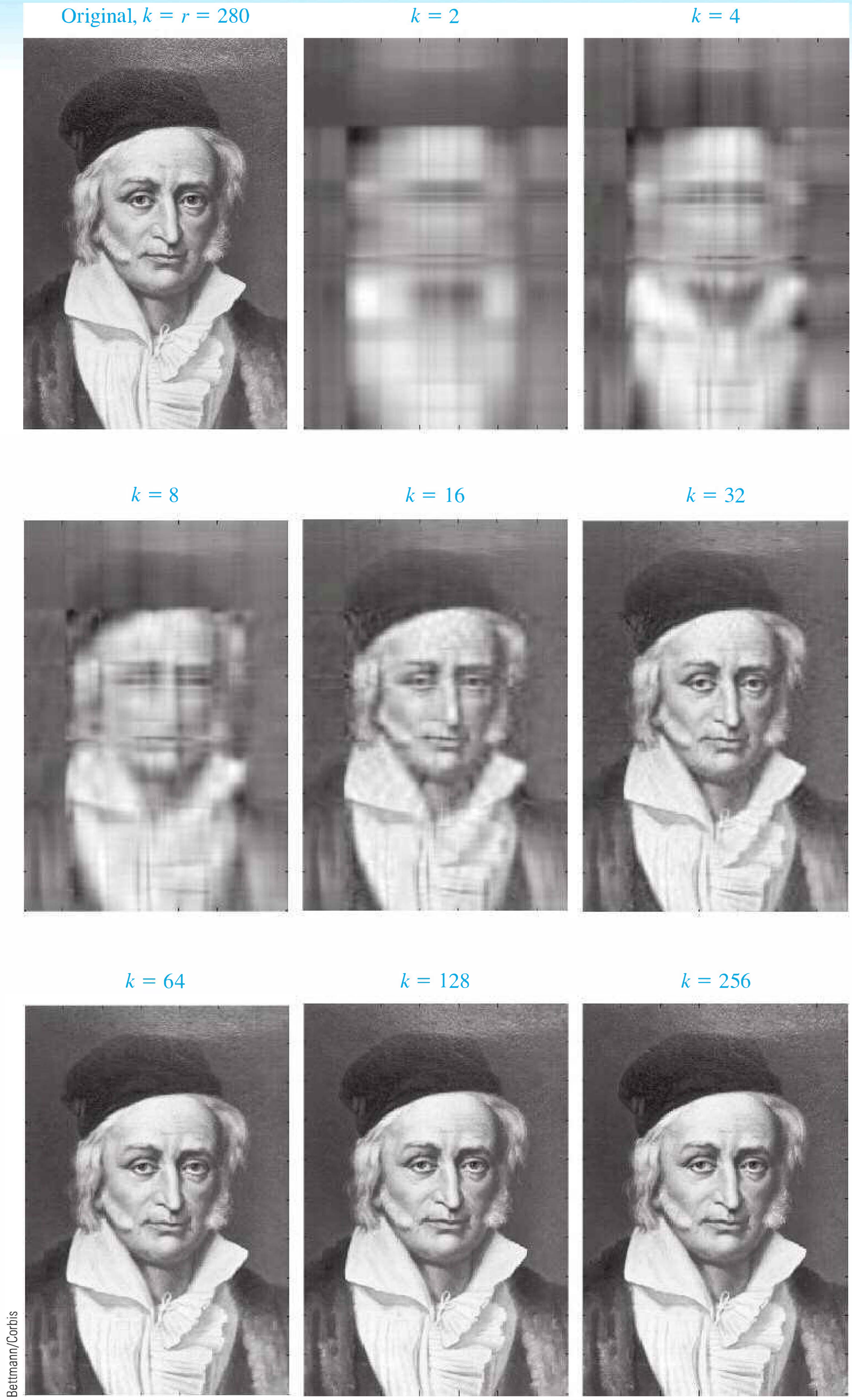

Original, $k = r = 280$
$k = 2$
$k = 4$
$k = 8$
$k = 16$
$k = 32$
$k = 64$
$k = 128$
$k = 256$
Bettmann/Corbis

Figure 7.23

Exercises 7.4

In Exercises 1–10, find the singular values of the given matrix.

1. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **4.** $A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$

5. $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **6.** $A = \begin{bmatrix} 3 & 4 \end{bmatrix}$

7. $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}$ **8.** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -2 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ **10.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

In Exercises 11–20, find an SVD of the indicated matrix.

11. A in Exercise 3 **12.** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

15. A in Exercise 5 **16.** A in Exercise 6

17. A in Exercise 7 **18.** A in Exercise 8

19. A in Exercise 9 **20.** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

In Exercises 21–24, find the outer product form of the SVD for the matrix in the given exercises.

21. Exercises 3 and 11 **22.** Exercise 14

23. Exercises 7 and 17 **24.** Exercises 9 and 19

25. Show that the matrices U and V in the SVD are not uniquely determined. [*Hint:* Find an example in which it would be possible to make different choices in the construction of these matrices.]

26. Let A be a symmetric matrix. Show that the singular values of A are:
(a) the absolute values of the eigenvalues of A.
(b) the eigenvalues of A if A is positive definite.

27. (a) Show that, for a positive definite, symmetric matrix A, Theorem 7.13 gives the orthogonal diagonalization of A, as guaranteed by the Spectral Theorem.
(b) Show that, for a positive definite, symmetric matrix A, Theorem 7.14 gives the spectral decomposition of A.

28. If A is an invertible matrix with SVD $A = U\Sigma V^T$, show that Σ is invertible and that $A^{-1} = V\Sigma^{-1}U^T$ is an SVD of A^{-1}.

29. Show that if $A = U\Sigma V^T$ is an SVD of A, then the left singular vectors are eigenvectors of AA^T.

30. Show that A and A^T have the same singular values.

31. Let Q be an orthogonal matrix such that QA makes sense. Show that A and QA have the same singular values.

32. Prove Theorem 7.15(d).

33. What is the image of the unit circle in $\mathbb{R}^2$ under the action of the matrix in Exercise 3?

34. What is the image of the unit circle in $\mathbb{R}^2$ under the action of the matrix in Exercise 7?

35. What is the image of the unit sphere in $\mathbb{R}^3$ under the action of the matrix in Exercise 9?

36. What is the image of the unit sphere in $\mathbb{R}^3$ under the action of the matrix in Exercise 10?

In Exercises 37–40, compute (a) $\|A\|_2$ and (b) cond$_2(A)$ for the indicated matrix.

37. A in Exercise 3 **38.** A in Exercise 8

39. $A = \begin{bmatrix} 1 & 0.9 \\ 1 & 1 \end{bmatrix}$ **40.** $A = \begin{bmatrix} 10 & 10 & 0 \\ 100 & 100 & 1 \end{bmatrix}$

In Exercises 41–44, compute the pseudoinverse A^+ of A in the given exercise.

41. Exercise 3 **42.** Exercise 8

43. Exercise 9 **44.** Exercise 10

In Exercises 45–48, find A^+ and use it to compute the minimal length least squares solution to $A\mathbf{x} = \mathbf{b}$.

45. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

46. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

49. (a) Set up and solve the normal equations for the system of equations in Example 7.40.
(b) Find a parametric expression for the length of a solution vector in part (a).
(c) Find the solution vector of minimal length and verify that it is the one produced by the method of Example 7.40. [*Hint:* Recall how to find the coordinates of the vertex of a parabola.]

50. Verify that when A has linearly independent columns, the definitions of pseudoinverse in this section and in Section 7.3 are the same.

51. Verify that the pseudoinverse (as defined in this section) satisfies the Penrose conditions for A (Theorem 7.12 in Section 7.3).

52. Show that A^+ is the *only* matrix that satisfies the Penrose conditions for A. To do this, assume that A' is a matrix satisfying the Penrose conditions: (a) $AA'A = A$, (b) $A'AA' = A'$, and (c) AA' and $A'A$ are symmetric. Prove that $A' = A^+$. [*Hint:* Use the Penrose conditions for A^+ and A' to show that $A^+ = A'AA^+$ and $A' = A'AA^+$. It is helpful to note that condition (c) can be written as $AA' = (A')^TA^T$ and $A'A = A^T(A')^T$, with similar versions for A^+.]

53. Show that $(A^+)^+ = A$. [*Hint:* Show that A satisfies the Penrose conditions for A^+. By Exercise 52, A must therefore be $(A^+)^+$.]

54. Show that $(A^+)^T = (A^T)^+$. [*Hint:* Show that $(A^+)^T$ satisfies the Penrose conditions for A^T. By Exercise 52, $(A^+)^T$ must therefore be $(A^T)^+$.]

55. Show that if A is a symmetric, idempotent matrix, then $A^+ = A$.

56. Let Q be an orthogonal matrix such that QA makes sense. Show that $(QA)^+ = A^+Q^T$.

57. Prove that if A is a positive definite matrix with SVD $A = U\Sigma V^T$, then $U = V$.

58. Prove that for a diagonal matrix, the 1-, 2-, and ∞-norms are the same.

59. Prove that for any square matrix A, $\|A\|_2^2 \le \|A\|_1\|A\|_\infty$. [*Hint:* $\|A\|_2^2$ is the square of the largest singular value of A and hence is equal to the largest eigenvalue of A^TA. Now use Exercise 34 in Section 7.2.]

Every complex number can be written in polar form as $z = re^{i\theta}$, where $r = |z|$ is a nonnegative real number and θ is its argument, with $|e^{i\theta}| = 1$. (See Appendix C.) Thus, z has been decomposed into a stretching factor r and a rotation factor $e^{i\theta}$. There is an analogous decomposition $A = RQ$ for square matrices, called the ***polar decomposition.***

60. Show that every square matrix A can be factored as $A = RQ$, where R is symmetric, positive semidefinite and Q is orthogonal. [*Hint:* Show that the SVD can be rewritten to give

$$A = U\Sigma V^T = U\Sigma(U^TU)V^T = (U\Sigma U^T)(UV^T)$$

Then show that $R = U\Sigma U^T$ and $Q = UV^T$ have the right properties.]

Find a polar decomposition of the matrices in Exercises 61–64.

61. A in Exercise 3

62. A in Exercise 14

63. $A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$

64. $A = \begin{bmatrix} 4 & 2 & -3 \\ -2 & 2 & 6 \\ 4 & -1 & 6 \end{bmatrix}$

7.5 Applications

Approximation of Functions

In many applications, it is necessary to approximate a given function by a "nicer" function. For example, we might want to approximate $f(x) = e^x$ by a linear function $g(x) = c + dx$ on some interval $[a, b]$. In this case, we have a continuous function f, and we want to approximate it as closely as possible on the interval $[a, b]$

by a function g in the subspace $\mathscr{P}_1$. The general problem can be phrased as follows:

> Given a continuous function f on an interval $[a, b]$ and a subspace W of $\mathscr{C}[a, b]$, find the function "closest" to f in W.

The problem is analogous to the least squares fitting of data points, except now we have infinitely many data points—namely, the points on the graph of the function f. What should "approximate" mean in this context? Once again, the Best Approximation Theorem holds the answer.

The given function f lives in the vector space $\mathscr{C}[a, b]$ of continuous functions on the interval $[a, b]$. This is an inner product space, with inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

If W is a finite-dimensional subspace of $\mathscr{C}[a, b]$, then the best approximation to f in W is given by the projection of f onto W, by Theorem 7.8. Furthermore, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis for W, then

$$\operatorname{proj}_W(f) = \frac{\langle \mathbf{u}_1, f\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \cdots + \frac{\langle \mathbf{u}_k, f\rangle}{\langle \mathbf{u}_k, \mathbf{u}_k\rangle}\mathbf{u}_k$$

Example 7.41

Find the best linear approximation to $f(x) = e^x$ on the interval $[-1, 1]$.

Solution Linear functions are polynomials of degree 1, so we use the subspace $W = \mathscr{P}_1[-1, 1]$ of $\mathscr{C}[-1, 1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\,dx$$

A basis for $\mathscr{P}_1[-1, 1]$ is given by $\{1, x\}$. Since

$$\langle 1, x\rangle = \int_{-1}^1 x\,dx = 0$$

this is an orthogonal basis, so the best approximation to f in W is

$$\begin{aligned} g(x) = \operatorname{proj}_W(e^x) &= \frac{\langle 1, e^x\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x, e^x\rangle}{\langle x, x\rangle}x \\ &= \frac{\int_{-1}^1 (1\cdot e^x)\,dx}{\int_{-1}^1 (1\cdot 1)\,dx} + \frac{\int_{-1}^1 xe^x\,dx}{\int_{-1}^1 x^2\,dx}x \\ &= \frac{e - e^{-1}}{2} + \frac{2e^{-1}}{\frac{2}{3}}x \\ &= \tfrac{1}{2}(e - e^{-1}) + 3e^{-1}x \approx 1.18 + 1.10x \end{aligned}$$

where we have used integration by parts to evaluate $\int_{-1}^{1} xe^x\,dx$. (Check these calculations.) See Figure 7.24.

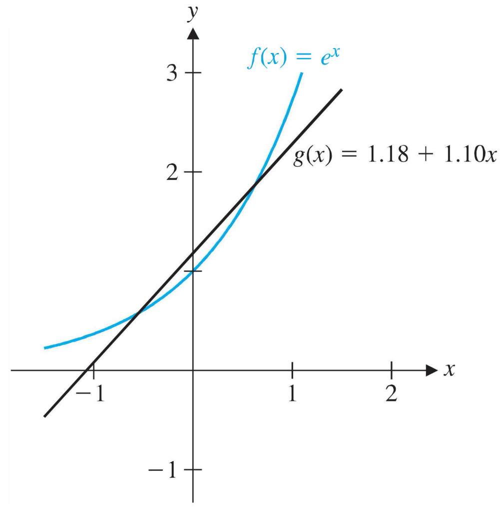


Figure 7.24

The error in approximating f by g is the one specified by the Best Approximation Theorem: the distance $\|f - g\|$ between f and g relative to the inner product on $\mathscr{C}[-1, 1]$. This error is just

$$\|f - g\| = \sqrt{\int_{-1}^{1} (f(x) - g(x))^2\,dx}$$

and is often called the ***root mean square error.*** With the aid of a CAS, we find that the root mean square error in Example 7.41 is

$$\|e^x - (\tfrac{1}{2}(e - e^{-1}) + 3e^{-1}x)\| = \sqrt{\int_{-1}^{1} (e^x - \tfrac{1}{2}(e - e^{-1}) - 3e^{-1}x)^2\,dx} \approx 0.23$$

Remark The root mean square error can be thought of as analogous to the area between the graphs of f and g on the specified interval. Recall that the area between the graphs of f and g on the interval $[a, b]$ is given by

$$\int_a^b |f(x) - g(x)|\,dx$$

(See Figure 7.25.)

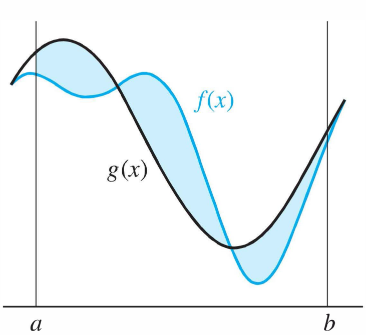


Figure 7.25

Although the equation in the above Remark is a sensible measure of the "error" between f and g, the absolute value sign makes it hard to work with. The root mean square error is easier to use and therefore preferable. The square root is necessary to "compensate" for the squaring and to keep the unit of measurement the same as it would be for the area between the curves. For comparison purposes, the area between the graphs of f and g in Example 7.41 is

$$\int_{-1}^{1} |e^x - \tfrac{1}{2}(e - e^{-1}) - 3e^{-1}x|\,dx \approx 0.28$$

Example 4.30

Find the best quadratic approximation to $f(x) = e^x$ on the interval $[-1, 1]$.

Solution A quadratic function is a polynomial of the form $g(x) = a + bx + cx^2$ in $W = \mathcal{P}_2[-1, 1]$. This time, the standard basis $\{1, x, x^2\}$ is not orthogonal. However, we can construct an orthogonal basis using the Gram-Schmidt Process, as we did in Example 7.8. The result is the set of Legendre polynomials

$$\{1, x, x^2 - \tfrac{1}{3}\}$$

Using this set as our basis, we compute the best approximation to f in W as $g(x) = \mathrm{proj}_W(e^x)$. The linear terms in this calculation are exactly as in Example 7.41, so we only require the additional calculations

$$\langle x^2 - \tfrac{1}{3}, e^x\rangle = \int_{-1}^{1} (x^2 - \tfrac{1}{3})e^x\,dx = \int_{-1}^{1} x^2e^x\,dx - \tfrac{1}{3}\int_{-1}^{1} e^x\,dx = \tfrac{2}{3}(e - 7e^{-1})$$

and $$\langle x^2 - \tfrac{1}{3}, x^2 - \tfrac{1}{3}\rangle = \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2\,dx = \int_{-1}^{1} (x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9})\,dx = \tfrac{8}{45}$$

Then the best quadratic approximation to $f(x) = e^x$ on the interval $[-1, 1]$ is

$$\begin{aligned} g(x) = \mathrm{proj}_W(e^x) &= \frac{\langle 1, e^x\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x, e^x\rangle}{\langle x, x\rangle}x + \frac{\langle x^2 - \frac{1}{3}, e^x\rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle}(x^2 - \tfrac{1}{3}) \\ &= \tfrac{1}{2}(e - e^{-1}) + 3e^{-1}x + \frac{\frac{2}{3}(e - 7e^{-1})}{\frac{8}{45}}(x^2 - \tfrac{1}{3}) \\ &= \frac{3(11e^{-1} - e)}{4} + 3e^{-1}x + \frac{15(e - 7e^{-1})}{4}x^2 \approx 1.00 + 1.10x + 0.54x^2 \end{aligned}$$

(See Figure 7.26.)

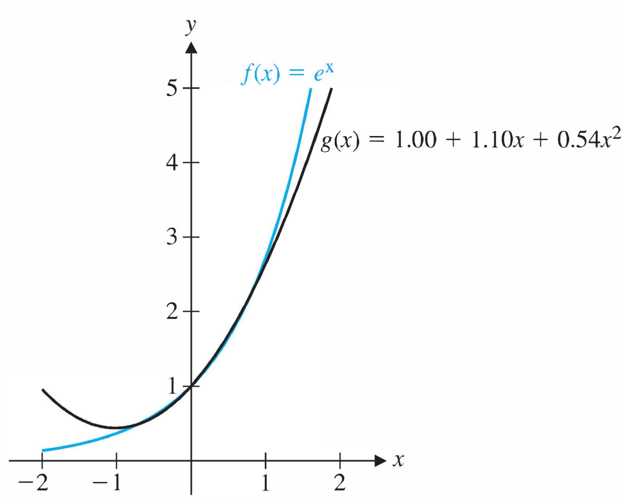


Figure 7.26

Notice how much better the quadratic approximation in Example 7.42 is than the linear approximation in Example 7.41. It turns out that, in the quadratic case, the root mean square error is

$$\|e^x - g(x)\| = \sqrt{\int_{-1}^{1} (e^x - g(x))^2\, dx} \approx 0.04$$

In general, the higher the degree of the approximating polynomial, the smaller the error and the better the approximation.

In many applications, functions are approximated by combinations of sine and cosine functions. This method is particularly useful if the function being approximated displays periodic or almost periodic behavior (such as that of a sound wave, an electrical impulse, or the motion of a vibrating system). A function of the form

$$\begin{aligned} p(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x \\ + b_2 \sin 2x + \cdots + b_n \sin nx \end{aligned} \tag{1}$$

is called a ***trigonometric polynomial;*** if a_n and b_n are not both zero, then $p(x)$ is said to have ***order n.*** For example,

$$p(x) = 3 - \cos x + \sin 2x + 4 \sin 3x$$

is a trigonometric polynomial of order 3.

Let's restrict our attention to the vector space $\mathscr{C}[-\pi, \pi]$ with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

The trigonometric polynomials of the form in Equation (1) are linear combinations of the set

$$\mathcal{B} = \{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\}$$

The best approximation to a function f in $\mathscr{C}[-\pi, \pi]$ by a trigonometric polynomial of order n will therefore be $\text{proj}_W(f)$, where $W = \text{span}(\mathcal{B})$. It turns out that $\mathcal{B}$ is an orthogonal set and, hence, a basis for W. Verification of this fact involves showing that any two distinct functions in $\mathcal{B}$ are orthogonal with respect to the given inner product. Example 7.43 presents some of the necessary calculations; you are asked to provide the remaining ones in Exercises 17–19.

Example 7.43

Show that $\sin jx$ is orthogonal to $\cos kx$ in $\mathscr{C}[-\pi, \pi]$ for $j, k \geq 1$.

Solution Using a trigonometric identity, we compute as follows: If $j \neq k$, then

$$\begin{aligned} \int_{-\pi}^{\pi} \sin jx \cos kx\, dx &= \tfrac{1}{2}\int_{-\pi}^{\pi} [\sin(j+k)x + \sin(j-k)x]\, dx \\ &= -\tfrac{1}{2}\left[\frac{\cos(j+k)x}{j+k} + \frac{\cos(j-k)x}{j-k}\right]_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

since the cosine function is periodic with period 2π.

If $j = k$, then

$$\int_{-\pi}^{\pi} \sin kx \cos kx \, dx = \frac{1}{2k}\left[\sin^2 kx\right]_{-\pi}^{\pi} = 0$$

since $\sin k\pi = 0$ for any integer k.

In order to find the orthogonal projection of a function f in $\mathscr{C}[-\pi, \pi]$ onto the subspace W spanned by the orthogonal basis $\mathcal{B}$, we need to know the squares of the norms of the basis vectors. For example, using a half-angle formula, we have

$$\begin{aligned}\langle \sin kx, \sin kx \rangle &= \int_{-\pi}^{\pi} \sin^2 kx \, dx \\ &= \tfrac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2kx)\, dx \\ &= \tfrac{1}{2}\left[x - \frac{\sin 2kx}{2k}\right]_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

In Exercise 20, you are asked to show that $\langle \cos kx, \cos kx \rangle = \pi$ and $\langle 1, 1 \rangle = 2\pi$.

We now have

$$\operatorname{proj}_W(f) = a_0 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx \qquad (2)$$

where

$$\begin{aligned} a_0 &= \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ a_k &= \frac{\langle \cos kx, f \rangle}{\langle \cos kx, \cos kx \rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx \\ b_k &= \frac{\langle \sin kx, f \rangle}{\langle \sin kx, \sin kx \rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx \end{aligned} \qquad (3)$$

for $k \geq 1$. The approximation to f given by Equations (2) and (3) is called the ***nth-order Fourier approximation*** to f on $[-\pi, \pi]$. The coefficients $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are called the ***Fourier coefficients*** of f.

Example 7.44

Find the fourth-order Fourier approximation to $f(x) = x$ on $[-\pi, \pi]$.

Solution Using formulas (3), we obtain

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \, dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

and for $k \geq 1$, integration by parts yields

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos kx \, dx = \frac{1}{\pi}\left[\frac{x}{k}\sin kx + \frac{1}{k^2}\cos kx\right]_{-\pi}^{\pi} = 0$$

Jean-Baptiste Joseph Fourier (1768–1830) was a French mathematician and physicist who gained prominence through his investigation into the theory of heat. In his landmark solution of the so-called heat equation, he introduced techniques related to what are now known as Fourier series, a tool widely used in many branches of mathematics, physics, and engineering. Fourier was a political activist during the French revolution and became a favorite of Napoleon, accompanying him on his Egyptian campaign in 1798. Later Napoleon appointed Fourier Prefect of Isère, where he oversaw many important engineering projects. In 1808, Fourier was made a baron. He is commemorated by a plaque on the Eiffel Tower.

and
$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx\, dx = \frac{1}{\pi}\left[-\frac{x}{k}\cos kx + \frac{1}{k^2}\sin kx\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-\pi\cos k\pi - \pi\cos(-k\pi)}{k}\right] \\
&= \begin{cases} -\dfrac{2}{k} & \text{if } k \text{ is even} \\ \dfrac{2}{k} & \text{if } k \text{ is odd} \end{cases} \\
&= \frac{2(-1)^{k+1}}{k}
\end{aligned}$$

It follows that the fourth-order Fourier approximation to $f(x) = x$ on $[-\pi, \pi]$ is

$$2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x)$$

Figure 7.27 shows the first four Fourier approximations to $f(x) = x$ on $[-\pi, \pi]$.

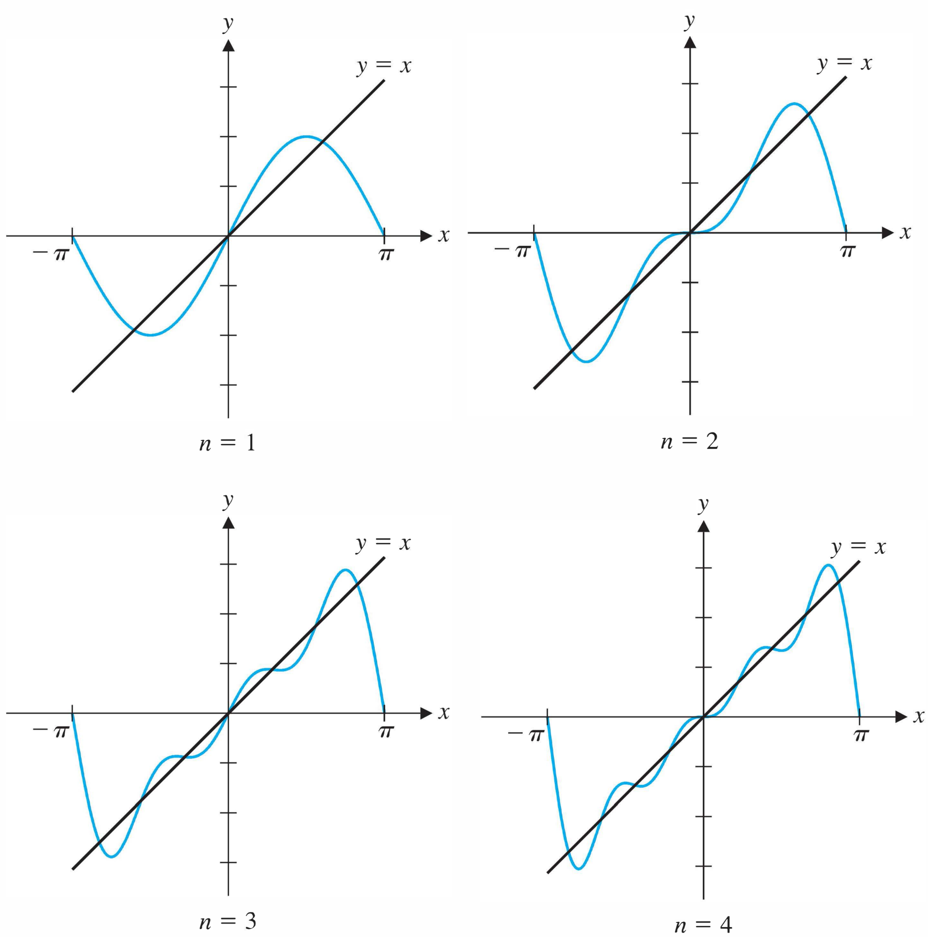


Figure 7.27

You can clearly see the approximations in Figure 7.27 improving, a fact that can be confirmed by computing the root mean square error in each case. As the order of the Fourier approximation increases, it can be shown that this error approaches zero. The trigonometric polynomial then becomes an *infinite series*, and we write

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

This is called the ***Fourier series*** of f on $[-\pi, \pi]$.

Mariner 9 used the Reed-Muller code R_5, whose minimum distance is $2^4 = 16$. By Theorem 1, this code can correct k errors, where $2k + 1 \leq 16$. The largest value of k for which this inequality is true is $k = 7$. Thus, R_5 not only contains exactly the right number of code vectors for transmitting 64 shades of gray but also is capable of correcting up to 7 errors, making it quite reliable. This explains why the images transmitted by *Mariner 9* were so sharp!

Exercises 7.5

Approximation of Functions

In Exercises 1–4, find the best linear approximation to f on the interval $[-1, 1]$.

1. $f(x) = x^2$ **2.** $f(x) = x^2 + 2x$

3. $f(x) = x^3$ **4.** $f(x) = \sin(\pi x/2)$

In Exercises 5 and 6, find the best quadratic approximation to f on the interval $[-1, 1]$.

5. $f(x) = |x|$ **6.** $f(x) = \cos(\pi x/2)$

7. Apply the Gram-Schmidt Process to the basis $\{1, x\}$ to construct an orthogonal basis for $\mathscr{P}_1[0, 1]$.

8. Apply the Gram-Schmidt Process to the basis $\{1, x, x^2\}$ to construct an orthogonal basis for $\mathscr{P}_2[0, 1]$.

In Exercises 9–12, find the best linear approximation to f on the interval $[0, 1]$.

9. $f(x) = x^2$ **10.** $f(x) = \sqrt{x}$

11. $f(x) = e^x$ **12.** $f(x) = \sin(\pi x/2)$

In Exercises 13–16, find the best quadratic approximation to f on the interval $[0, 1]$.

13. $f(x) = x^3$ **14.** $f(x) = \sqrt{x}$

15. $f(x) = e^x$ **16.** $f(x) = \sin(\pi x/2)$

17. Show that 1 is orthogonal to $\cos kx$ and $\sin kx$ in $\mathscr{C}[-\pi, \pi]$ for $k \geq 1$.

18. Show that $\cos jx$ is orthogonal to $\cos kx$ in $\mathscr{C}[-\pi, \pi]$ for $j \neq k$, $j, k \geq 1$.

19. Show that $\sin jx$ is orthogonal to $\sin kx$ in $\mathscr{C}[-\pi, \pi]$ for $j \neq k$, $j, k \geq 1$.

20. Show that $\|1\|^2 = 2\pi$ and $\|\cos kx\|^2 = \pi$ in $\mathscr{C}[-\pi, \pi]$.

In Exercises 21 and 22, find the third-order Fourier approximation to f on $[-\pi, \pi]$.

21. $f(x) = |x|$ **22.** $f(x) = x^2$

In Exercises 23–26, find the Fourier coefficients a_0, a_k, *and* b_k *of f on* $[-\pi, \pi]$.

23. $f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x \leq \pi \end{cases}$

24. $f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x \leq \pi \end{cases}$

25. $f(x) = \pi - x$ **26.** $f(x) = |x|$

Recall that a function f is an ***even function*** *if* $f(-x) = f(x)$ *for all x; f is called an* ***odd function*** *if* $f(-x) = -f(x)$ *for all x.*

27. (a) Prove that $\int_{-\pi}^{\pi} f(x)\,dx = 0$ if f is an odd function.

(b) Prove that the Fourier coefficients a_k are all zero if f is odd.

28. (a) Prove that $\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$ if f is an even function.

(b) Prove that the Fourier coefficients b_k are all zero if f is even.

Chapter Review

Key Definitions and Concepts

Review Questions

1. Mark each of the following statements true or false:

(a) If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + \pi u_2v_2$ defines an inner product on $\mathbb{R}^2$.

(b) If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 4u_2v_2$ defines an inner product on $\mathbb{R}^2$.

(c) $\langle A, B \rangle = \operatorname{tr}(A) + \operatorname{tr}(B)$ defines an inner product on M_{22}.

(d) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space with $\|\mathbf{u}\| = 4$, $\|\mathbf{v}\| = \sqrt{5}$, and $\langle \mathbf{u}, \mathbf{v} \rangle = 2$, then $\|\mathbf{u} + \mathbf{v}\| = 5$.

(e) The sum norm, max norm, and Euclidean norm on $\mathbb{R}^n$ are all equal to the absolute value function when $n = 1$.

(f) If a matrix A is well-conditioned, then $\operatorname{cond}(A)$ is small.

(g) If $\operatorname{cond}(A)$ is small, then the matrix A is well-conditioned.

(h) Every linear system has a unique least squares solution.

(i) If A is a matrix with orthonormal columns, then the standard matrix of an orthogonal projection onto the column space of A is $P = AA^T$.

(j) If A is a symmetric matrix, then the singular values of A are the same as the eigenvalues of A.

In Questions 2–4, determine whether the definition gives an inner product.

2. $\langle p(x), q(x) \rangle = p(0)q(1) + p(1)q(0)$ for $p(x), q(x)$ in $\mathscr{P}_1$

3. $\langle A, B \rangle = \operatorname{tr}(A^TB)$ for A, B in M_{22}

4. $\langle f, g \rangle = (\max_{0 \le x \le 1} f(x))(\max_{0 \le x \le 1} g(x))$ for f, g in $\mathscr{C}[0, 1]$

In Questions 5 and 6, compute the indicated quantity using the specified inner product.

5. $\|1 + x + x^2\|$ if $\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2 \rangle = a_0b_0 + a_1b_1 + a_2b_2$

6. $d(x, x^2)$ if $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)\,dx$

In Questions 7 and 8, construct an orthogonal set of vectors by applying the Gram-Schmidt Process to the given set of vectors using the specified inner product.

7. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v}$, where $A = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$

8. $\{1, x, x^2\}$ if $\langle p(x), q(x) \rangle = \displaystyle\int_0^1 p(x)q(x)\,dx$

In Questions 9 and 10, determine whether the definition gives a norm.

9. $\|\mathbf{v}\| = \mathbf{v}^T\mathbf{v}$ for $\mathbf{v}$ in $\mathbb{R}^n$

10. $\|p(x)\| = |p(0)| + |p(1) - p(0)|$ for $p(x)$ in $\mathscr{P}_1$

11. Show that the matrix $A = \begin{bmatrix} 1 & 0.1 & 0.11 \\ 0.1 & 0.11 & 0.111 \\ 0.11 & 0.111 & 0.1111 \end{bmatrix}$ is ill-conditioned.

12. Prove that if Q is an orthogonal $n \times n$ matrix, then its Frobenius norm is $\|Q\|_F = \sqrt{n}$.

13. Find the line of best fit through the points (1, 2), (2, 3), (3, 5), and (4, 7).

14. Find the least squares solution of

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

15. Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the column space of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

16. If $\mathbf{u}$ and $\mathbf{v}$ are orthonormal vectors, show that $P = \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T$ is the standard matrix of an orthogonal projection onto span $(\mathbf{u}, \mathbf{v})$. [*Hint:* Show that $P = A(A^TA)^{-1}A^T$ for some matrix A.]

In Questions 17 and 18, find (a) the singular values, (b) a singular value decomposition, and (c) the pseudoinverse of the matrix A.

17. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$ **18.** $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

19. If P and Q are orthogonal matrices for which PAQ is defined, prove that PAQ has the same singular values as A.

20. If A is a square matrix for which $A^2 = O$, prove that $(A^+)^2 = O$.

Appendix A*

Mathematical Notation and Methods of Proof

Please, sir, I want some more.
—Oliver
Charles Dickens, *Oliver Twist*

Anyone who understands algebraic notation reads at a glance in an equation results reached arithmetically only with great labour and pains.
—Augustin Cournot
Researches into the Mathematical Principles of the Theory of Wealth
Translated by Nathaniel T. Bacon
Macmillan, 1897, p. 4

In this book, an effort has been made to use "mathematical English" as much as possible, keeping mathematical notation to a minimum. However, mathematical notation is a convenient shorthand that can greatly simplify the amount of writing we have to do. Moreover, it is commonly used in every branch of mathematics, so the ability to read and write mathematical notation is an essential ingredient of mathematical understanding. Finally, there are some theorems whose proofs become "obvious" if the right notation is used.

Proving theorems in mathematics is as much an art as a science. For the beginner, it is often hard to know what approach to use in proving a theorem; there are many approaches, any one of which might turn out to be the best. To become proficient at proofs, it is important to study as many examples as possible and to get plenty of practice.

This appendix summarizes basic mathematical notation applied to sets. Summation notation, a useful shorthand for dealing with sums, is also discussed. Finally, some approaches to proofs are illustrated with generic examples.

Set Notation

A ***set*** is a collection of objects, called the ***elements*** (or ***members***) of the set. Examples of sets include the set of all words in this text, the set of all books in your college library, the set of positive integers, and the set of all vectors in the plane whose equation is $2x + 3y - z = 0$.

It is often possible to list the elements of a set, in which case it is conventional to enclose the list within braces. For example, we have

$$\{1, 2, 3\}, \quad \{a, t, x, z\}, \quad \{2, 4, 6, \ldots, 100\}, \quad \left\{\frac{\pi}{4}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{7}, \ldots, \frac{5\pi}{6}\right\}$$

Note that ellipses (. . .) denote elements omitted when a pattern is present. (What is the pattern in the last two examples?) Infinite sets are often expressed using ellipses. For example, the set of positive integers is usually denoted by $\mathbb{N}$ or $\mathbb{Z}^+$, so

$$\mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

The set of all integers is denoted by $\mathbb{Z}$, so

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Two sets are considered to be ***equal*** if they contain exactly the same elements. The *order* in which elements are listed does not matter, and repetitions are not counted. Thus,

*Exercises and selected odd-numbered answers for this appendix can be found on the student companion website.

$$\{1, 2, 3\} = \{2, 1, 3\} = \{1, 3, 2, 1\}$$

The symbol $\in$ means "is an element of" or "is in," and the symbol $\notin$ denotes the negation—that is, "is not an element of" or "is not in." For example,

$$5 \in \mathbb{Z}^+ \quad \text{but} \quad 0 \notin \mathbb{Z}^+$$

It is often more convenient to describe a set in terms of a rule satisfied by all of its elements. In such cases, ***set builder notation*** is appropriate. The format is

$$\{x : x \text{ satisfies } P\}$$

where P represents a property or a collection of properties that the element x must satisfy. The colon is pronounced "such that." For example,

$$\{n : n \in \mathbb{Z}, n > 0\}$$

is read as "the set of all n such that n is an integer and n is greater than zero." This is just another way of describing the positive integers $\mathbb{Z}^+$. (We could also write $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n > 0\}$.)

The ***empty set*** is the set with no elements. It is denoted by either $\varnothing$ or $\{\ \}$.

Example A.1

Describe in words the following sets:

(a) $A = \{n : n = 2k, k \in \mathbb{Z}\}$ (b) $B = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$

(c) $C = \{x \in \mathbb{R} : 4x^2 - 4x - 3 = 0\}$ (d) $D = \{x \in \mathbb{Z} : 4x^2 - 4x - 3 = 0\}$

Solution (a) A is the set of numbers n that are integer multiples of 2. Therefore, A is the set of all even integers.

(b) B is the set of all expressions of the form m/n, where m and n are integers and n is nonzero. This is the set of *rational numbers*, usually denoted by $\mathbb{Q}$. (Note that this way of describing $\mathbb{Q}$ produces many repetitions; however, our convention, as noted above, is that we include only one occurrence of each element. Thus, this expression precisely describes the set of all rational numbers.)

(c) C is the set of all real solutions of the equation $4x^2 - 4x - 3 = 0$. By factoring or using the quadratic formula, we find that the roots of this equation are $-\frac{1}{2}$ and $\frac{3}{2}$. (Verify this.) Therefore,

$$C = \{-\tfrac{1}{2}, \tfrac{3}{2}\}$$

(d) From the solution to (c) we see that there are *no* solutions to $4x^2 - 4x - 3 = 0$ in $\mathbb{R}$ that are integers. Therefore, D is the empty set, which we can express by writing $D = \varnothing$.

John Venn (1834–1923) was an English mathematician who studied at Cambridge University and later lectured there. He worked primarily in mathematical logic and is best known for inventing Venn diagrams.

If every element of a set A is also an element of a set B, then A is called a ***subset*** of B, denoted $A \subseteq B$. We can represent this situation schematically using a ***Venn diagram***, as shown in Figure A.1. (The rectangle represents the *universal set*, a set large enough to contain all of the other sets in question—in this case, A and B.)

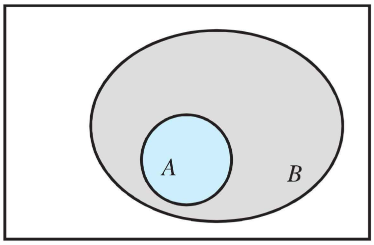


Figure A.1
$A \subseteq B$

Example A.2

(a) $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$

(b) $\mathbb{Z}^+ \subseteq \mathbb{Z} \subseteq \mathbb{R}$

(c) Let A be the set of all positive integers whose last two digits are 24 and let B be the set of all positive integers that are evenly divisible by 4. Then if n is in A, it is of the form

$$n = 100k + 24$$

for some integer k. (For example, $36{,}524 = 100 \cdot 365 + 24$.) But then

$$n = 100k + 24 = 4(25k + 6)$$

so $n/4 = 25k + 6$, which is an integer. Hence, n is evenly divisible by 4, so it is in B. Therefore, $A \subseteq B$.

We can show that two sets A and B are equal by showing that each is a subset of the other. This strategy is particularly useful if the sets are defined abstractly or if it is not easy to list and compare their elements.

Example A.3

Let A be the set of all positive integers whose last two digits form a number that is evenly divisible by 4. In the case of a one-digit number, we take its tens digit to be 0. Let B be the set of all positive integers that are evenly divisible by 4. Show that $A = B$.

Solution As in Example A.2(c), it is easy to see that $A \subseteq B$. If n is in A, then we can split off the number m formed by its last two digits by writing

$$n = 100k + m$$

for some integer k. But, since m is divisible by 4, we have $m = 4r$ for some integer r. Therefore,

$$n = 100k + m = 100k + 4r = 4(25k + r)$$

so n is also evenly divisible by 4. Hence, $A \subseteq B$.

To show that $B \subseteq A$, let n be in B. That is, n is evenly divisible by 4. Let's say that $n = 4s$, where s is an integer. If m is the number formed by the last two digits of n, then, as above, $n = 100k + m$ for some integer k. But now

$$m = n - 100k = 4s - 100k = 4(s - 25k)$$

which implies that m is evenly divisible by 4, since $s - 25k$ is an integer. Therefore, n is in A, and we have shown that $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, we must have $A = B$.

The ***intersection*** of sets A and B is denoted by $A \cap B$ and consists of the elements that A and B have in common. That is,

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}$$

Figure A.2 shows a Venn diagram of this case. The ***union*** of A and B is denoted by $A \cup B$ and consists of the elements that are in either A or B (or both). That is,

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

See Figure A.3.

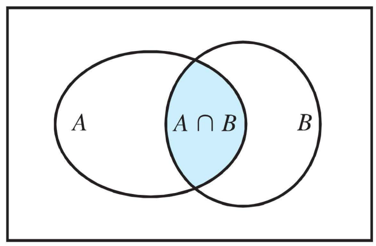


Figure A.2
$A \cap B$

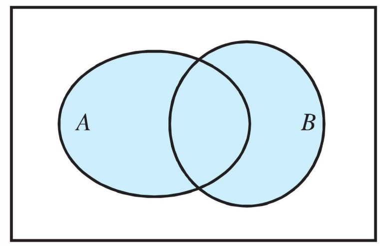


Figure A.3
$A \cup B$

Example A.4

Let $A = \{n^2 : n \in \mathbb{Z}^+, 1 \leq n \leq 4\}$ and let $B = \{n \in \mathbb{Z}^+ : n \leq 10 \text{ and } n \text{ is odd}\}$. Find $A \cap B$ and $A \cup B$.

Solution We see that

$$A = \{1^2, 2^2, 3^2, 4^2\} = \{1, 4, 9, 16\} \quad \text{and} \quad B = \{1, 3, 5, 7, 9\}$$

Therefore, $A \cap B = \{1, 9\}$ and $A \cup B = \{1, 3, 4, 5, 7, 9, 16\}$.

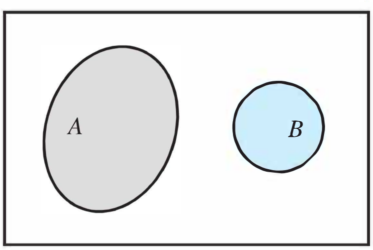


Figure A.4
Disjoint sets

If $A \cap B = \emptyset$, then A and B are called ***disjoint sets.*** (See Figure A.4.) For example, the set of even integers and the set of odd integers are disjoint.

Summation Notation

Summation notation is a convenient shorthand to use to write out a sum such as

$$1 + 2 + 3 + \cdots + 100$$

where we want to leave out all but a few terms. As in set notation, ellipses (. . .) convey that we have established a pattern and have simply left out some intermediate terms. In the above example, readers are expected to recognize that we are summing all of the positive integers from 1 to 100. However, ellipses can be ambiguous. For example, what would one make of the following sum?

$$1 + 2 + \cdots + 64$$

Is this the sum of all positive integers from 1 to 64 or just the powers of two, $1 + 2 + 4 + 8 + 16 + 32 + 64$? It is often clearer (and shorter) to use ***summation notation*** (or ***sigma notation***).

Σ is the capital Greek letter *sigma*, corresponding to S (for "sum"). Summation notation was introduced by Fourier in 1820 and was quickly adopted by the mathematical community.

We can abbreviate a sum of the form

$$a_1 + a_2 + \cdots + a_n \tag{1}$$

as

$$\sum_{k=1}^{n} a_k \tag{2}$$

which tells us to sum the terms a_k over all integers k ranging from 1 to n. An alternative version of this expression is

$$\sum_{1 \le k \le n} a_k$$

The subscript k is called the ***index of summation.*** It is a "dummy variable" in the sense that it does not appear in the actual sum in expression (1). Therefore, we can use any letter we like as the index of summation (as long as it doesn't already appear somewhere else in the expressions we are summing). Thus, expression (2) can also be written as

$$\sum_{i=1}^{n} a_i$$

The index of summation need not start at 1. The sum $a_3 + a_4 + \cdots + a_{99}$ becomes

$$\sum_{k=3}^{99} a_k$$

although we can arrange for the index to begin at 1 by rewriting the expression as

$$\sum_{k=1}^{97} a_{k+2}.$$

The key to using summation notation effectively is being able to recognize patterns.

Example A.5

Write the following sums using summation notation.

(a) $1 + 2 + 4 + \cdots + 64$ (b) $1 + 3 + 5 + \cdots + 99$ (c) $3 + 8 + 15 + \cdots + 99$

Solution (a) We recognize this expression as a sum of powers of 2:

$$1 + 2 + 4 + \cdots + 64 = 2^0 + 2^1 + 2^2 + \cdots + 2^6$$

Therefore, the index of summation appears as the exponent, and we have $\sum_{k=0}^{6} 2^k$.

(b) This expression is the sum of all the odd integers from 1 to 99. Every odd integer is of the form $2k + 1$, so the sum is

$$\begin{aligned} &1 + 3 + 5 + \cdots + 99 \\ &\quad = (2\cdot 0 + 1) + (2\cdot 1 + 1) + (2\cdot 2 + 1) + \cdots + (2\cdot 49 + 1) \\ &\quad = \sum_{k=0}^{49} (2k + 1) \end{aligned}$$

(c) The pattern here is less clear, but a little reflection reveals that each term is 1 less than a perfect square:

$$\begin{aligned} &3 + 8 + 15 + \cdots + 99 \\ &\quad = (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + \cdots + (10^2 - 1) \\ &\quad = \sum_{k=2}^{10} (k^2 - 1) \end{aligned}$$

Example A.6

Rewrite each of the sums in Example A.5 so that the index of summation starts at 1.

Solution (a) If we use the change of variable $i = k + 1$, then, as k goes from 0 to 6, i goes from 1 to 7. Since $k = i - 1$, we obtain

$$\sum_{k=0}^{6} 2^k = \sum_{i=1}^{7} 2^{i-1}$$

(b) Using the same substitution as in part (a), we get

$$\sum_{k=0}^{49} (2k + 1) = \sum_{i=1}^{50} (2(i - 1) + 1) = \sum_{i=1}^{50} (2i - 1)$$

(c) The substitution $i = k - 2$ will work (try it), but it is easier just to add a term corresponding to $k = 1$, since $1^2 - 1 = 0$. Therefore,

$$\sum_{k=2}^{10} (k^2 - 1) = \sum_{k=1}^{10} (k^2 - 1)$$

Multiple summations arise when there is more than one index of summation, as there is with a matrix. The notation

$$\sum_{i,j=1}^{n} a_{ij} \tag{3}$$

means to sum the terms a_{ij} as i and j each range independently from 1 to n. The sum in expression (3) is equivalent to either

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$$

where we sum first over j and then over i (we always work from the inside out), or

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}$$

where the order of summation is reversed.

Example A.7

Write out $\sum_{i,j=1}^{3} i^j$ using both possible orders of summation.

Solution

$$\begin{aligned}\sum_{i=1}^{3} \sum_{j=1}^{3} i^j &= \sum_{i=1}^{n} (i^1 + i^2 + i^3)\\ &= (1^1 + 1^2 + 1^3) + (2^1 + 2^2 + 2^3) + (3^1 + 3^2 + 3^3)\\ &= (1 + 1 + 1) + (2 + 4 + 8) + (3 + 9 + 27) = 56\end{aligned}$$

and

$$\sum_{j=1}^{3}\sum_{i=1}^{3} i^j = \sum_{j=1}^{n}(1^j + 2^j + 3^j)$$
$$= (1^1 + 2^1 + 3^1) + (1^2 + 2^2 + 3^2) + (1^3 + 2^3 + 3^3)$$
$$= (1 + 2 + 3) + (1 + 4 + 9) + (1 + 8 + 27) = 56$$

Remark Of course, the value of the sum in Example A.7 is the same no matter which order of summation we choose, because the sum is *finite*. It is also possible to consider *infinite sums* (known as *infinite series* in calculus), but such sums do not always have a value and great care must be taken when rearranging or manipulating their terms. For example, suppose we let

$$S = \sum_{k=0}^{\infty} 2^k$$

How to Solve It is the title of a book by the mathematician George Pólya (1887–1985). Since its publication in 1945, *How to Solve It* has sold over a million copies and has been translated into 17 languages. Pólya was born in Hungary, but because of the political situation in Europe, he moved to the United States in 1940. He subsequently taught at Brown and Stanford Universities, where he did mathematical research and developed a well-deserved reputation as an outstanding teacher. The Pólya Prize is awarded annually by the Society for Industrial and Applied Mathematics for major contributions to areas of mathematics close to those on which Pólya worked. The Mathematical Association of America annually awards Pólya Lectureships to mathematicians demonstrating the high-quality exposition for which Pólya was known.

Then

$$\begin{aligned} S &= 1 + 2 + 4 + 8 + \cdots \\ &= 1 + 2(1 + 2 + 4 + \cdots) \\ &= 1 + 2S \end{aligned}$$

from which it follows that $S = -1$. This is clearly nonsense, since S is a sum of *non-negative* terms! (Where is the error?)

Methods of Proof

The notion of proof is at the very heart of mathematics. It is one thing to know *what* is true; it is quite another to know *why* it is true and to be able to demonstrate its truth by means of a logically connected sequence of statements. The intention here is not to try to teach you how to do proofs; you will become better at doing proofs by studying examples and by practicing—something you should do often as you work through this text. The intention of this brief section is simply to provide a few elementary examples of some types of proofs. The proofs of theorems in the text will provide further illustrations of "how to solve it."

Roughly speaking, mathematical proofs fall into two categories: ***direct proofs*** and ***indirect proofs***. Many theorems have the structure "if P, then Q," where P (the *hypothesis*, or *premise*) and Q (the *conclusion*) are statements that are either true or false. We denote such an implication by $P \Rightarrow Q$. A direct proof proceeds by establishing a chain of implications

$$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_n \Rightarrow Q$$

leading directly from P to Q.

Example A.8 Prove that any two consecutive perfect squares differ by an odd number. This instruction can be rephrased as "Prove that if a and b are consecutive perfect squares, then $a - b$ is odd." Hence, it has the form $P \Rightarrow Q$, with P being "a and b are consecutive perfect squares" and Q being "$a - b$ is odd."

Solution Assume that a and b are consecutive perfect squares, with $a > b$. Then

$$a = (n + 1)^2 \quad \text{and} \quad b = n^2$$

for some integer n. But now

$$\begin{aligned} a - b &= (n + 1)^2 - n^2 \\ &= n^2 + 2n + 1 - n^2 \\ &= 2n + 1 \end{aligned}$$

so $a - b$ is odd.

There are two types of indirect proofs that can be used to establish a conditional statement of the form $P \Rightarrow Q$. A ***proof by contradiction*** assumes that the hypothesis P is true, just as in a direct proof, but then supposes that the conclusion Q is *false*. The strategy then is to show that this is not possible (i.e., to rule out the possibility that the conclusion is false) by finding a contradiction to the truth of P. It then follows that Q must be true.

Example A.9

Let n be a positive integer. Prove that if n^2 is even, so is n. (Take a few minutes to try to find a direct proof of this assertion; it will help you to appreciate the indirect proof that follows.)

Solution Assume that n is a positive integer such that n^2 is even. Now suppose that n is not even. Then n is odd, so

$$n = 2k + 1$$

for some integer k. But if so, we have

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$$

so n^2 is odd, since it is 1 more than the even number $4k^2 + 4k$. This contradicts our hypothesis that n^2 is even. We conclude that our supposition that n was *not* even must have been false; in other words, n must be even.

Closely related to the method of proof by contradiction is ***proof by contrapositive***. The *negative* of a statement P is the statement "it is not the case that P," abbreviated symbolically as $\neg P$ and pronounced "not P." For example, if P is "n is even," then $\neg P$ is "it is not the case that n is even"—in other words, "n is odd."

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$. A conditional statement $P \Rightarrow Q$ and its contrapositive $\neg Q \Rightarrow \neg P$ are logically equivalent in the sense that they are either both true or both false. (For example, if $P \Rightarrow Q$ is a theorem, then so is $\neg Q \Rightarrow \neg P$. To see this, note that if the hypothesis $\neg Q$ is true, then Q is false. The conclusion $\neg P$ cannot be false, for if it were, then P would be true and our known theorem $P \Rightarrow Q$ would imply the truth of Q, giving us a contradiction. It follows that $\neg P$ is true and we have proved $\neg Q \Rightarrow \neg P$.) Here is a contrapositive proof of the assertion in Example A.9.

Example A.10

Let n be a positive integer. Prove that if n^2 is even, so is n.

Solution The contrapositive of the given statement is

"If n is not even, then n^2 is not even" or "If n is odd, so is n^2"

To prove this contrapositive, assume that n is odd. Then $n = 2k + 1$ for some integer k. As before, this means that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ is odd, which completes the proof of the contrapositive. Since the contrapositive is true, so is the original statement.

Although we do not require a new method of proof to handle it, we will briefly consider how to prove an "if and only if" theorem. A statement of the form "P if and only if Q" signals a *double implication*, which we denote by $P \Leftrightarrow Q$. To prove such a statement, we must prove $P \Rightarrow Q$ and $Q \Rightarrow P$. To do so, we can use the techniques described earlier, where appropriate. It is important to notice that the "*if*" part of $P \Leftrightarrow Q$ is "P if Q," which is $Q \Rightarrow P$; the "only if" part of $P \Leftrightarrow Q$ is "P only if Q," meaning $P \Rightarrow Q$. The implication $P \Rightarrow Q$ is sometimes read as "P is sufficient for Q" or "Q is necessary for P"; $Q \Rightarrow P$ is read "Q is sufficient for P" or "P is necessary for Q." Taken together, they are $P \Leftrightarrow Q$, or "P is necessary and sufficient for Q" and vice versa.

Example A.11

A pawn is placed on a chessboard and is allowed to move one square at a time, either horizontally or vertically. A *pawn's tour* of a chessboard is a path taken by a pawn, moving as described, that visits each square exactly once, starting and ending on the same square. Prove that there is a pawn's tour of an $n \times n$ chessboard if and only if n is even.

Solution [$\Leftarrow$] ("if") Assume that n is even. It is easy to see that the strategy illustrated in Figure A.5 for a 6×6 chessboard will always give a pawn's tour.

[$\Rightarrow$] ("only if") Suppose that there is a pawn's tour of an $n \times n$ chessboard. We will give a proof by contradiction that n must be even. To this end, let's assume that n is odd. At each move, the pawn moves to a square of a different color. The total number of moves in its tour is n^2, which is also an odd number, according to the proof in Example A.10. Therefore, the pawn must end up on a square of the opposite color from that of the square on which it started. (Why?) This is impossible, since the pawn ends where it started, so we have a contradiction. It follows that n cannot be odd; hence, n is even and the proof is complete.

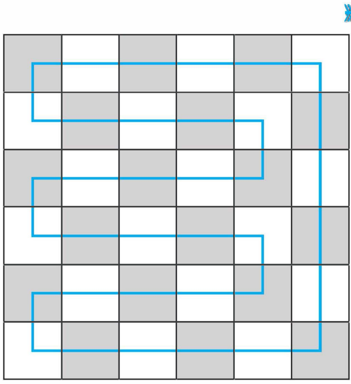


Figure A.5

Some theorems assert that several statements are *equivalent*. This means that each is true if and only if all of the others are true. Showing that n statements are equivalent requires $\binom{n}{2} = \dfrac{n!}{2!(n-2)!} = \dfrac{n^2 - n}{2}$ "if and only if" proofs. In practice, however, it is often easier to establish a "ring" of n implications that links all of the statements. The proof of the Fundamental Theorem of Invertible Matrices provides an excellent example of this approach.

Appendix B*

Mathematical Induction

The ability to spot patterns is one of the keys to success in mathematical problem solving. Consider the following pattern:

$$\begin{aligned} 1 &= 1 \\ 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ 1 + 3 + 5 + 7 &= 16 \\ 1 + 3 + 5 + 7 + 9 &= 25 \end{aligned}$$

Great fleas have little fleas
upon their backs to bite 'em,
And little fleas have lesser fleas,
and so ad infinitum.
—Augustus De Morgan
A Budget of Paradoxes
Longmans, Green, and Company, 1872, p. 377

The sums are all perfect squares: $1^2, 2^2, 3^2, 4^2, 5^2$. It seems reasonable to conjecture that this pattern will continue to hold; that is, the sum of consecutive odd numbers, starting at 1, will always be a perfect square. Let's try to be more precise. If the sum is n^2, then the last odd number in the sum is $2n - 1$. (Check this in the five cases above.) In symbols, our conjecture becomes

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{for all } n \geq 1 \tag{1}$$

Notice that Equation (1) is really an *infinite* collection of statements, one for each value of $n \geq 1$. Although our conjecture seems reasonable, we cannot assume that the pattern continues—we need to prove it. This is where ***mathematical induction*** comes in.

First Principle of Mathematical Induction

Let $S(n)$ be a statement about the positive integer n. If

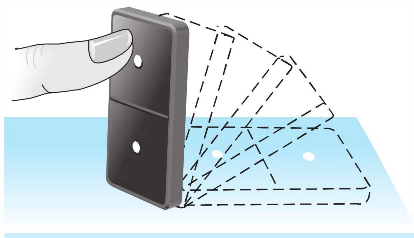
1. $S(1)$ is true and
2. for all $k \geq 1$, the truth of $S(k)$ implies the truth of $S(k + 1)$

then $S(n)$ is true for all $n \geq 1$.

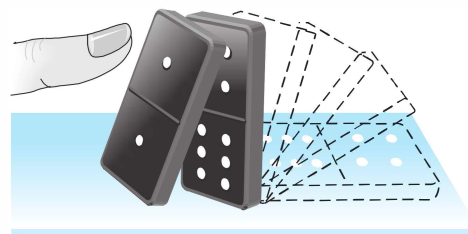
Verifying that $S(1)$ is true is called the ***basis step***. The assumption that $S(k)$ is true for some $k \geq 1$ is called the ***induction hypothesis.*** Using the induction hypothesis to prove that $S(k + 1)$ is then true is called the ***induction step.*** Mathematical induction has been referred to as the *domino principle* because it is analogous to showing that a line of dominoes will fall down if (1) the first domino can be knocked down (the basis step) and (2) knocking down any domino (the induction hypothesis) will knock over the next domino (the induction step). See Figure B.1.

We now use the principle of mathematical induction to prove Equation (1).

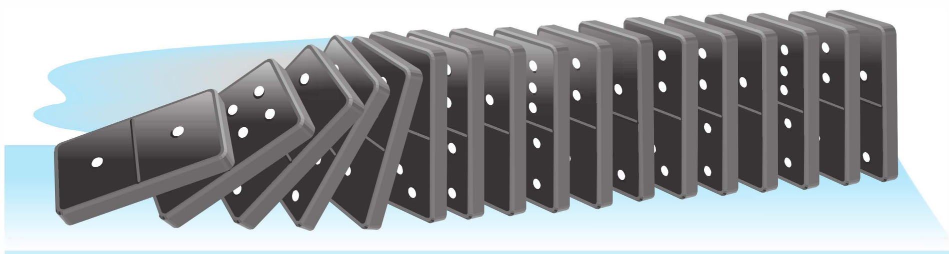
*Exercises and selected odd-numbered answers for this appendix can be found on the student companion website.



If the first domino falls, and . . .



each domino that falls knocks down the next one, . . .



then all the dominoes can be made to fall by pushing over the first one.

Figure B.1

Example B.1

Use mathematical induction to prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n \geq 1$.

Solution For $n = 1$, the sum on the left-hand side is just 1, while the right-hand side is 1^2. Since $1 = 1^2$, this completes the basis step.

Now assume that the formula is true for some integer $k \geq 1$. That is, assume that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

(This is the induction hypothesis.) The induction step consists of proving that the formula is true when $n = k + 1$. We see that when $n = k + 1$, the left-hand side of formula (1) is

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2(k+1) - 1) &= 1 + 3 + 5 + \cdots + (2k + 1) \\ &= \underbrace{1 + 3 + 5 + \cdots + (2k - 1)}_{k^2} + (2k + 1) \\ &= k^2 + 2k + 1 \quad \leftarrow \text{by the induction hypothesis} \\ &= (k + 1)^2 \end{aligned}$$

which is the right-hand side of Equation (1) when $n = k + 1$.

This completes the induction step, and we conclude that Equation (1) is true for all $n \geq 1$, by the principle of mathematical induction.

The next example gives a proof of a useful formula for the sum of the first n positive integers. The formula appears several times in the text; for example, see the solution to Exercise 51 in Section 2.4.

Example B.2

Prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

Solution The formula is true for $n = 1$, since

$$1 = \frac{1(1+1)}{2}$$

Assume that the formula is true for $n = k$; that is,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

We need to show that the formula is true when $n = k + 1$; that is, we must prove that

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

But we see that

$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad \text{by the induction hypothesis} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

which is what we needed to show.

This completes the induction step, and we conclude that the formula is true for all $n \geq 1$, by the principle of mathematical induction.

In a similar vein, we can prove that the sum of the squares of the first n positive integers satisfies the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$. (Verify this for yourself.)

The basis step need not be for $n = 1$, as the next two examples illustrate.

Example B.3

Prove that $n! > 2^n$ for all integers $n \geq 4$.

Solution The basis step here is when $n = 4$. The inequality is clearly true in this case, since

$$4! = 24 > 16 = 2^4$$

Assume that $k! > 2^k$ for some integer $k \geq 4$. Then

$$\begin{aligned}
(k + 1)! &= (k + 1)k! \\
&> (k + 1)2^k && \text{by the induction hypothesis} \\
&\geq 5 \cdot 2^k && \text{since } k \geq 4 \\
&> 2 \cdot 2^k = 2^{k+1}
\end{aligned}$$

which verifies the inequality for $n = k + 1$ and completes the induction step.

We conclude that $n! > 2^n$ for all integers $n \geq 4$, by the principle of mathematical induction.

If a is a nonzero real number and $n \geq 0$ is an integer, we can give a recursive definition of the power a^n that is compatible with mathematical induction. We define $a^0 = 1$ and, for $n \geq 0$,

$$a^{n+1} = a^n a$$

(This form avoids the ellipses used in the version $a^n = \overbrace{aa \cdots a}^{n \text{ times}}$.) We can now use mathematical induction to verify a familiar property of exponents.

Example B.4

Let a be a nonzero real number. Prove that $a^m a^n = a^{m+n}$ for all integers $m, n \geq 0$.

Solution At first glance, it is not clear how to proceed, since there are *two* variables, m and n. But we simply need to keep one of them fixed and perform our induction using the other. So, let $m \geq 0$ be a fixed integer. When $n = 0$, we have

$$a^m a^0 = a^m \cdot 1 = a^m = a^{m+0}$$

using the definition $a^0 = 1$. Hence, the basis step is true.

Now assume that the formula holds when $n = k$, where $k \geq 0$. Then $a^m a^k = a^{m+k}$. For $n = k + 1$, using our recursive definition and the fact that addition and multiplication are associative, we see that

$$\begin{aligned}
a^m a^{k+1} &= a^m(a^k a) && \text{by definition} \\
&= (a^m a^k)a \\
&= a^{m+k} a && \text{by the induction hypothesis} \\
&= a^{(m+k)+1} && \text{by definition} \\
&= a^{m+(k+1)}
\end{aligned}$$

Therefore, the formula is true for $n = k + 1$, and the induction step is complete.

We conclude that $a^m a^n = a^{m+n}$ for all integers $m, n \geq 0$, by the principle of mathematical induction.

In Examples B.1 through B.4, the use of the induction hypothesis during the induction step is relatively straightforward. However, this is not always the case. An alternative version of the principle of mathematical induction is often more useful.

Second Principle of Mathematical Induction

Let $S(n)$ be a statement about the positive integer n. If

1. $S(1)$ is true and
2. the truth of $S(1), S(2), \ldots, S(k)$ implies the truth of $S(k + 1)$

then $S(n)$ is true for all $n \geq 1$.

The only difference between the two principles of mathematical induction is in the induction hypothesis: The first version assumes that $S(k)$ is true, whereas the second version assumes that all of $S(1), S(2), \ldots, S(k)$ are true. This makes the second principle seem weaker than the first, since we need to assume more in order to prove $S(k + 1)$ (although, paradoxically, the second principle is sometimes called *strong* induction). In fact, however, the two principles are logically equivalent: Each one implies the other. (Can you see why?)

The next example presents an instance in which the second principle of mathematical induction is easier to use than the first. Recall that a prime number is a positive integer whose only positive integer factors are 1 and itself.

Example B.5

Prove that every positive integer $n \geq 2$ either is prime or can be factored into a product of primes.

Solution The result is clearly true when $n = 2$, since 2 is prime. Now assume that for all integers n between 2 and k, n either is prime or can be factored into a product of primes. Let $n = k + 1$. If $k + 1$ is prime, we are done. Otherwise, it must factor into a product of two smaller integers—say,

$$k + 1 = ab$$

Since $2 \leq a, b \leq k$ (why?), the induction hypothesis applies to a and b. Therefore,

$$a = p_1 \cdots p_r \quad \text{and} \quad b = q_1 \cdots q_s$$

where the p's and q's are all prime. Then

$$ab = p_1 \cdots p_r q_1 \cdots q_s$$

gives a factorization of ab into primes, completing the induction step.

We conclude that the result is true for all integers $n \geq 2$, by the second principle of mathematical induction.

Do you see why the first principle of mathematical induction would have been difficult to use here?

We conclude with a highly nontrivial example that involves a combination of induction and *backward* induction. The result is the Arithmetic Mean–Geometric Mean Inequality, discussed in Chapter 7 in Exploration: Geometric Inequalities and Optimization Problems. The clever proof in Example B.6 is due to Cauchy.

Example B.6

Let $x_1, \ldots, x_n$ be nonnegative real numbers. Prove that

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all integers $n \ge 2$.

Solution For $n = 2$, the inequality becomes $\sqrt{xy} \le (x + y)/2$. You are asked to verify this in Problems 1 and 2 of the Exploration mentioned above.

If $S(n)$ is the stated inequality, we will prove that $S(k)$ implies $S(2k)$. Assume that $S(k)$ is true; that is,

$$\sqrt[k]{x_1 x_2 \cdots x_k} \le \frac{x_1 + x_2 + \cdots + x_k}{k}$$

for all nonnegative real numbers $x_1, \ldots, x_k$. Let

$$x_1 = \frac{y_1 + y_2}{2}, \quad x_2 = \frac{y_3 + y_4}{2}, \quad \ldots, \quad x_k = \frac{y_{2k-1} + y_{2k}}{2}$$

Then

$$\begin{aligned}
\sqrt[2k]{y_1 \cdots y_{2k}} &= \sqrt[k]{\sqrt{y_1 \cdots y_{2k}}} = \sqrt[k]{\sqrt{y_1 y_2} \cdots \sqrt{y_{2k-1} y_{2k}}} \\
&\le \sqrt[k]{\left(\frac{y_1 + y_2}{2}\right) \cdots \left(\frac{y_{2k-1} + y_{2k}}{2}\right)} && \text{by } S(2) \\
&= \sqrt[k]{x_1 \cdots x_k} \\
&\le \frac{x_1 + x_2 + \cdots + x_k}{k} && \text{by } S(k) \\
&= \frac{\left(\frac{y_1 + y_2}{2}\right) + \cdots + \left(\frac{y_{2k-1} + y_{2k}}{2}\right)}{k} \\
&= \frac{y_1 + \cdots + y_{2k}}{2k}
\end{aligned}$$

which verifies $S(2k)$.

Thus, the Arithmetic Mean–Geometric Mean Inequality is true for $n = 2, 4, 8, \ldots$—the powers of 2. In order to complete the proof, we need to "fill in the gaps." We will use backward induction to prove that $S(k)$ implies $S(k - 1)$. Assuming $S(k)$ is true, let

$$x_k = \frac{x_1 + x_2 + \cdots + x_{k-1}}{k - 1}$$

Then

$$\sqrt[k]{x_1x_2\cdots x_{k-1}\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)} \le \frac{x_1+x_2+\cdots+\left(\dfrac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)}{k}$$

$$= \frac{kx_1+kx_2+\cdots+kx_{k-1}}{k(k-1)}$$

$$= \frac{x_1+x_2+\cdots+x_{k-1}}{k-1}$$

Equivalently,

$$x_1x_2\cdots x_{k-1}\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right) \le \left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)^k$$

or
$$x_1x_2\cdots x_{k-1} \le \left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)^{k-1}$$

Taking the $(k-1)$th root of both sides yields $S(k-1)$.

The two inductions, taken together, show that the Arithmetic Mean–Geometric Mean Inequality is true for all $n \ge 2$.

Remark Although mathematical induction is a powerful and indispensable tool, it cannot work miracles. That is, it cannot prove that a pattern or formula holds if it does not. Consider the diagrams in Figure B.2, which show the maximum number of regions $R(n)$ into which a circle can be subdivided by n straight lines.

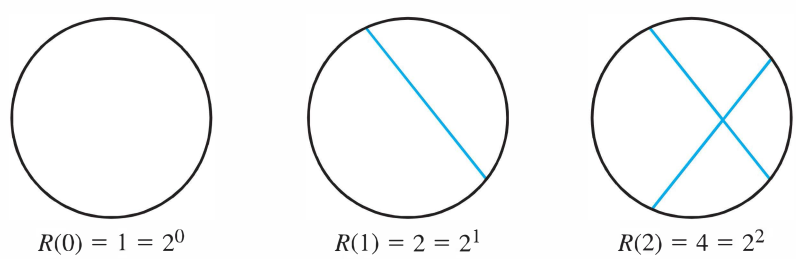


Figure B.2

Based on the evidence in Figure B.2, we might conjecture that $R(n) = 2^n$ for $n \ge 0$ and try to prove this conjecture using mathematical induction. We would not succeed, since this formula is not correct! If we had considered one more case, we would have discovered that $R(3) = 7 \ne 8 = 2^3$, thereby demolishing our conjecture. In fact, the correct formula turns out to be

$$R(n) = \frac{n^2+n+2}{2}$$

which *can* be verified by induction. (Can you do it?)

For other examples in which a pattern appears to be true, only to disappear when enough cases are considered, see Richard K. Guy's delightful article "The Strong Law of Small Numbers" in the *American Mathematical Monthly*, Vol. 95 (1988), pp. 697–712.

Appendix C*

Complex Numbers

[The] extension of the number concept to include the irrational, and we will at once add, the imaginary, is the greatest forward step which pure mathematics has ever taken.
—Hermann Hankel
Theorie der Complexen Zahlensysteme
Leipzig, 1867, p. 60

A ***complex number*** is a number of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The real number a is considered to be a special type of complex number, since $a = a + 0i$. If $z = a + bi$ is a complex number, then the ***real part*** of z, denoted by Re z, is a, and the ***imaginary part*** of z, denoted by Im z, is b. Two complex numbers $a + bi$ and $c + di$ are ***equal*** if their real parts are equal and their imaginary parts are equal—that is, if $a = c$ and $b = d$. A complex number $a + bi$ can be identified with the point (a, b) and plotted in the plane (called the ***complex plane,*** or the ***Argand plane***), as shown in Figure C.1. In the complex plane, the horizontal axis is called the ***real axis*** and the vertical axis is called the ***imaginary axis.***

There is nothing "imaginary" about complex numbers—they are just as "real" as the real numbers. The term *imaginary* arose from the study of polynomial equations such as $x^2 + 1 = 0$, whose solutions are not "real" (i.e., real numbers). It is worth remembering that at one time negative numbers were thought of as "imaginary" too.

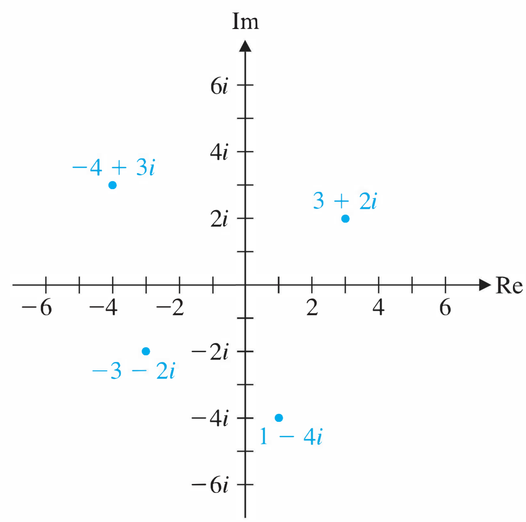


Figure C.1
The complex plane

Jean-Robert Argand (1768–1822) was a French accountant and amateur mathematician. His geometric interpretation of complex numbers appeared in 1806 in a book that he published privately. He was not, however, the first to give such an interpretation. The Norwegian-Danish surveyor Caspar Wessel (1745–1818) gave the same version of the complex plane in 1787, but his paper was not noticed by the mathematical community until after his death.

Operations on Complex Numbers

The ***sum*** of the complex numbers $a + bi$ and $c + di$ is defined as

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Notice that, with the identification of $a + bi$ with (a, b), $c + di$ with (c, d), and $(a + c) + (b + d)i$ with $(a + c, b + d)$, addition of complex numbers is the same as vector addition. The ***product*** of $a + bi$ and $c + di$ is

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di)\\ &= ac + adi + bci + bdi^2\end{aligned}$$

*Exercises and selected odd-numbered answers for this appendix can be found on the student companion website.

Since $i^2 = -1$, this expression simplifies to $(ac - bd) + (ad + bc)i$. Thus, we have

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Observe that, as a special case, $a(c + di) = ac + adi$, so the ***negative*** of $c + di$ is $-(c + di) = (-1)(c + di) = -c - di$. This fact allows us to compute the ***difference*** of $a + bi$ and $c + di$ as

$$\begin{aligned}(a + bi) - (c + di) &= (a + bi) + (-1)(c + di)\\ &= (a + (-c)) + (b + (-d))i\\ &= (a - c) + (b - d)i\end{aligned}$$

Example C.1

Find the sum, difference, and product of $3 - 4i$ and $-1 + 2i$.

Solution The sum is

$$(3 - 4i) + (-1 + 2i) = (3 - 1) + (-4 + 2)i = 2 - 2i$$

The difference is

$$(3 - 4i) - (-1 + 2i) = (3 - (-1)) + (-4 - 2)i = 4 - 6i$$

The product is

$$\begin{aligned}(3 - 4i)(-1 + 2i) &= -3 + 6i + 4i - 8i^2\\ &= -3 + 10i - 8(-1) = 5 + 10i\end{aligned}$$

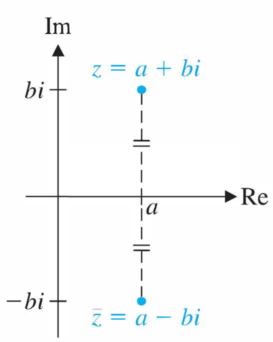


Figure C.2
Complex conjugates

The ***conjugate*** of $z = a + bi$ is the complex number

$$\bar{z} = a - bi$$

($\bar{z}$ is pronounced "z bar.") Figure C.2 gives the geometric interpretation of the conjugate.

To find the quotient of two complex numbers, we multiply the numerator and the denominator by the conjugate of the denominator.

Example C.2

Express $\dfrac{-1 + 2i}{3 + 4i}$ in the form $a + bi$.

Solution We multiply the numerator and denominator by $\overline{3 + 4i} = 3 - 4i$. Using Example C.1, we obtain

$$\frac{-1 + 2i}{3 + 4i} = \frac{-1 + 2i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{5 + 10i}{3^2 + 4^2} = \frac{5 + 10i}{25} = \frac{1}{5} + \frac{2}{5}i$$

On the following page is a summary of some of the properties of conjugates. The proofs follow from the definition of conjugate; you should verify them for yourself.

1. $\overline{\overline{z}} = z$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $\overline{zw} = \overline{z}\,\overline{w}$
4. If $z \neq 0$, then $\overline{(w/z)} = \overline{w}/\overline{z}$.
5. z is real if and only if $\overline{z} = z$.

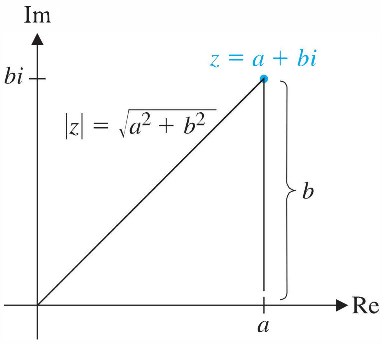


Figure C.3

The ***absolute value*** (or ***modulus***) $|z|$ of a complex number $z = a + bi$ is its distance from the origin. As Figure C.3 shows, Pythagoras' Theorem gives

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

Observe that

$$z\overline{z} = (a + bi)(a - bi) = a^2 - abi + bai - b^2i^2 = a^2 + b^2$$

Hence,

$$z\overline{z} = |z|^2$$

This gives us an alternative way of describing the division process for the quotient of two complex numbers. If w and $z \neq 0$ are two complex numbers, then

$$\frac{w}{z} = \frac{w}{z}\cdot\frac{\overline{z}}{\overline{z}} = \frac{w\overline{z}}{z\overline{z}} = \frac{w\overline{z}}{|z|^2}$$

Below is a summary of some of the properties of absolute value. (You should try to prove these using the definition of absolute value and other properties of complex numbers.)

1. $|z| = 0$ if and only if $z = 0$.
2. $|z| = |\overline{z}|$
3. $|zw| = |z||w|$
4. If $z \neq 0$, then $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.
5. $|z + w| \leq |z| + |w|$

Polar Form

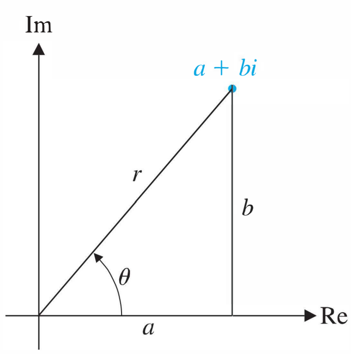


Figure C.4

As you have seen, the complex number $z = a + bi$ can be represented geometrically by the point (a, b). This point can also be expressed in terms of ***polar coordinates*** (r, θ), where $r \geq 0$, as shown in Figure C.4. We have

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

so

$$z = a + bi = r\cos\theta + (r\sin\theta)i$$

Thus, any complex number can be written in the ***polar form***

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan \theta = b/a$. The angle θ is called an ***argument*** of z and is denoted by arg z. Observe that arg z is not unique: Adding or subtracting any integer multiple of 2π gives another argument of z. However, there is only one argument θ that satisfies

$$-\pi < \theta \leq \pi$$

This is called the ***principal argument*** of z and is denoted by Arg z.

Example C.3

Write the following complex numbers in polar form using their principal arguments:

(a) $z = 1 + i$ (b) $w = 1 - \sqrt{3}i$

Solution (a) We compute

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{and} \quad \tan \theta = \frac{1}{1} = 1$$

Therefore, Arg $z = \theta = \dfrac{\pi}{4}$ (= 45°), and we have

$$z = \sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

as shown in Figure C.5.

(b) We have

$$r = |w| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2 \quad \text{and} \quad \tan \theta = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

Since w lies in the fourth quadrant, we must have Arg $z = \theta = -\dfrac{\pi}{3}$ (= −60°). Therefore,

$$w = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$$

See Figure C.5.

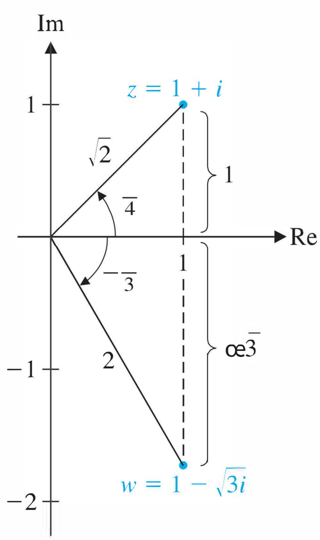


Figure C.5

The polar form of complex numbers can be used to give geometric interpretations of multiplication and division. Let

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Multiplying, we obtain

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \end{aligned}$$

Using the trigonometric identities

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \\ \sin(\theta_1 + \theta_2) &= \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \end{aligned}$$

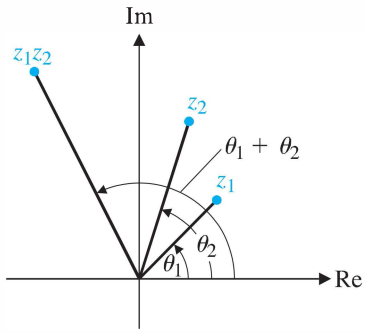


Figure C.6

we obtain

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \tag{1}$$

which is the polar form of a complex number with absolute value r_1r_2 and argument $\theta_1 + \theta_2$. This shows that

$$|z_1z_2| = |z_1||z_2| \quad \text{and} \quad \arg(z_1z_2) = \arg z_1 + \arg z_2$$

Equation (1) says that *to multiply two complex numbers, we multiply their absolute values and add their arguments*. See Figure C.6.

Similarly, using the subtraction identities for sine and cosine, we can show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad \text{if } z \neq 0$$

(Verify this.) Therefore,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

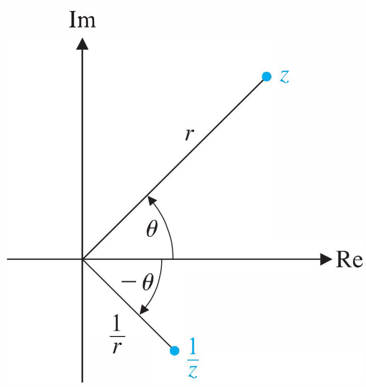


Figure C.7

and we see that *to divide two complex numbers, we divide their absolute values and subtract their arguments*.

As a special case of the last result, we obtain a formula for the reciprocal of a complex number in polar form. Setting $z_1 = 1$ (and therefore $\theta_1 = 0$) and $z_2 = z$ (and therefore $\theta_2 = \theta$), we obtain the following:

If $z = r(\cos\theta + i\sin\theta)$ is nonzero, then

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

See Figure C.7.

Example C.4

Find the product of $1 + i$ and $1 - \sqrt{3}i$ in polar form.

Solution From Example C.3, we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad 1 - \sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

Therefore,

$$\begin{aligned}(1 + i)(1 - \sqrt{3}i) &= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right] \\ &= 2\sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]\end{aligned}$$

See Figure C.8.

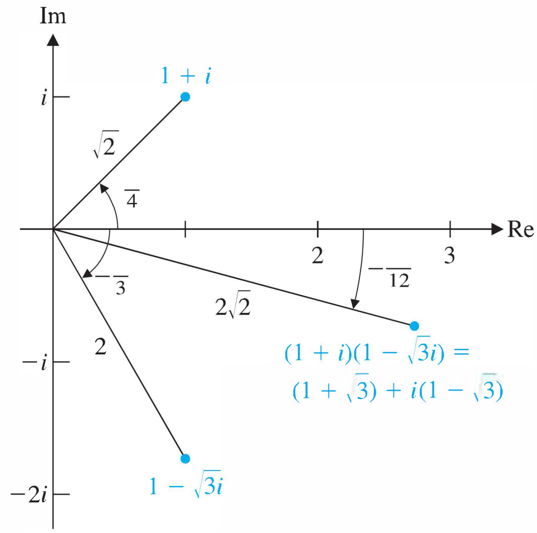


Figure C.8

Remark Since $(1 + i)(1 - \sqrt{3}i) = (1 + \sqrt{3}) + i(1 - \sqrt{3})$ (check this), we must have

$$1 + \sqrt{3} = 2\sqrt{2}\cos\left(-\frac{\pi}{12}\right) = -2\sqrt{2}\cos\left(\frac{\pi}{12}\right)$$

and
$$1 - \sqrt{3} = 2\sqrt{2}\sin\left(-\frac{\pi}{12}\right) = -2\sqrt{2}\sin\left(\frac{\pi}{12}\right)$$

(Why?) This implies that

$$\cos\left(\frac{\pi}{12}\right) = \frac{1 + \sqrt{3}}{2\sqrt{2}} \quad \text{and} \quad \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

We therefore have a method for finding the sine and cosine of an angle such as $\pi/12$ that is not a special angle but that can be obtained as a sum or difference of special angles.

De Moivre's Theorem

If n is a positive integer and $z = r(\cos\theta + i\sin\theta)$, then repeated use of Equation (1) yields formulas for the powers of z:

$$\begin{aligned} z^2 &= r^2(\cos 2\theta + i\sin 2\theta) \\ z^3 &= zz^2 = r^3(\cos 3\theta + i\sin 3\theta) \\ z^4 &= zz^3 = r^4(\cos 4\theta + i\sin 4\theta) \\ &\vdots \end{aligned}$$

In general, we have the following result, known as ***De Moivre's Theorem.***

Abraham De Moivre (1667–1754) was a French mathematician who made important contributions to trigonometry, analytic geometry, probability, and statistics.

Theorem C.1 **De Moivre's Theorem**

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Stated differently, we have

$$|z^n| = |z|^n \quad \text{and} \quad \arg(z^n) = n\arg z$$

In words, De Moivre's Theorem says that *to take the* nth *power of a complex number, we take the* nth *power of its absolute value and multiply its argument by* n.

Example C.5 Find $(1 + i)^6$.

Solution From Example C.3(a), we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Hence, De Moivre's Theorem gives

$$\begin{aligned}(1 + i)^6 &= (\sqrt{2})^6\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right)\\ &= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\\ &= 8(0 + i(-1)) = -8i\end{aligned}$$

See Figure C.9, which shows $1 + i, (1 + i)^2, (1 + i)^3, \ldots, (1 + i)^6$.

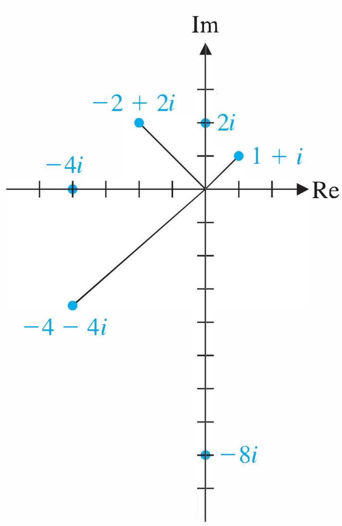


Figure C.9
Powers of $1 + i$

We can also use De Moivre's Theorem to find nth roots of complex numbers. An nth root of the complex number z is any complex number w such that

$$w^n = z$$

In polar form, we have

$$w = s(\cos\varphi + i\sin\varphi) \quad \text{and} \quad z = r(\cos\theta + i\sin\theta)$$

so, by De Moivre's Theorem,

$$s^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta)$$

Equating the absolute values, we see that

$$s^n = r \quad \text{or} \quad s = r^{1/n} = \sqrt[n]{r}$$

We must also have

$$\cos n\varphi = \cos\theta \quad \text{and} \quad \sin n\varphi = \sin\theta$$

(Why?) Since the sine and cosine functions each have period 2π, these equations imply that $n\varphi$ and θ differ by an integer multiple of 2π; that is,

$$n\varphi = \theta + 2k\pi \quad \text{or} \quad \varphi = \frac{\theta + 2k\pi}{n}$$

where k is an integer. Therefore,

$$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

describes the possible nth roots of z as k ranges over the integers. It is not hard to show that $k = 0, 1, 2, \ldots, n - 1$ produce distinct values of w, so there are exactly n different nth roots of $z = r(\cos\theta + i\sin\theta)$. We summarize this result as follows:

Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has exactly n distinct nth roots given by

$$r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \qquad (2)$$

for $k = 0, 1, 2, \ldots, n - 1$.

Example C.6

Find the three cube roots of -27.

Solution In polar form, $-27 = 27(\cos\pi + i\sin\pi)$. It follows that the cube roots of -27 are given by

$$(-27)^{1/3} = 27^{1/3}\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)\right] \quad \text{for } k = 0, 1, 2$$

Using formula (2) with $n = 3$, we obtain

$$27^{1/3}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$27^{1/3}\left[\cos\left(\frac{\pi + 2\pi}{3}\right) + i\sin\left(\frac{\pi + 2\pi}{3}\right)\right] = 3(\cos\pi + i\sin\pi) = -3$$

$$\begin{aligned}27^{1/3}\left[\cos\left(\frac{\pi + 4\pi}{3}\right) + i\sin\left(\frac{\pi + 4\pi}{3}\right)\right] &= 3\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\\ &= 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i\end{aligned}$$

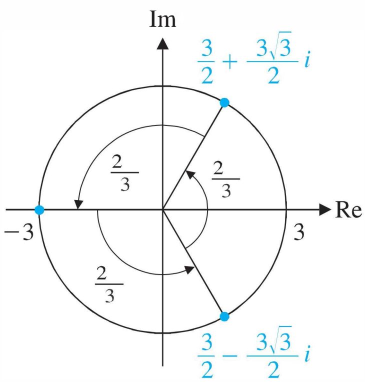
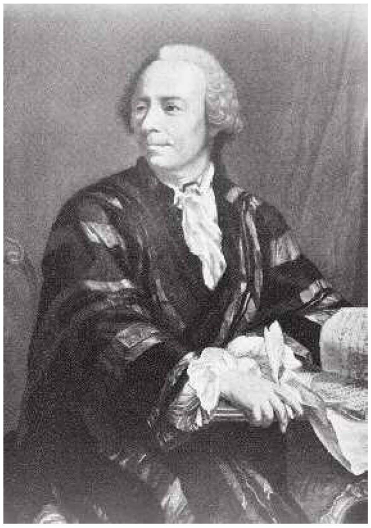


Figure C.10
The cube roots of -27

As Figure C.10 shows, the three cube roots of -27 are equally spaced $2\pi/3$ radians ($120°$) apart around a circle of radius 3 centered at the origin.

In general, formula (2) implies that the nth roots of $z = r(\cos\theta + i\sin\theta)$ will lie on a circle of radius $r^{1/n}$ centered at the origin. Moreover, they will be equally spaced $2\pi/n$ radians ($360/n°$) apart. (Verify this.) Thus, if we can find one nth root of z, the remaining nth roots of z can be obtained by rotating the first root through successive increments of $2\pi/n$ radians. Had we known this in Example C.6, we could have used the fact that the real cube root of -27 is -3 and then rotated it twice through an angle of $2\pi/3$ radians ($120°$) to get the other two cube roots.



Leonhard Euler (1707–1783) was the most prolific mathematician of all time. He has over 900 publications to his name, and his collected works fill over 70 volumes. There are so many results attributed to him that "Euler's formula" or "Euler's Theorem" can mean many different things, depending on the context.

Euler worked in so many areas of mathematics, it is difficult to list them all. His contributions to calculus and analysis, differential equations, number theory, geometry, topology, mechanics, and other areas of applied mathematics continue to be influential. He also introduced much of the notation we currently use, including π, e, i, Σ for summation, Δ for difference, and $f(x)$ for a function, and was the first to treat sine and cosine as functions.

Euler was born in Switzerland but spent most of his mathematical life in Russia and Germany. In 1727, he joined the St. Petersburg Academy of Sciences, which had been founded by Catherine I, the wife of Peter the Great. He went to Berlin in 1741 at the invitation of Frederick the Great, but returned in 1766 to St. Petersburg, where he remained until his death. When he was young, he lost the vision in one eye as the result of an illness, and by 1776 he had lost the vision in the other eye and was totally blind. Remarkably, his mathematical output did not diminish, and he continued to be productive until the day he died.

Euler's Formula

In calculus, you learn that the function e^z has a power series expansion

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

that converges for every real number z. It can be shown that this expansion also works when z is a complex number and that the complex exponential function e^z obeys the usual rules for exponents. The sine and cosine functions also have power series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots$$

If we let $z = ix$, where x is a real number, then we have

$$e^z = e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

Using the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on, repeating in a cycle of length 4, we see that

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + + - - \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

This remarkable result is known as ***Euler's formula.***

Theorem C.2 Euler's Formula

For any real number x,

$$e^{ix} = \cos x + i \sin x$$

Using Euler's formula, we see that the polar form of a complex number can be written more compactly as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

For example, from Example C.3(a), we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\pi/4}$$

We can also go in the other direction and convert a complex exponential back into polar or standard form.

Example C.7

Write the following in the form $a + bi$:

(a) $e^{i\pi}$ (b) $e^{2+i\pi/4}$

Solution (a) Using Euler's formula, we have

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0 = -1$$

(If we write this equation as $e^{i\pi} + 1 = 0$, we obtain what is surely one of the most remarkable equations in mathematics. It contains the fundamental operations of addition, multiplication, and exponentiation; the additive identity 0 and the multiplicative identity 1; the two most important transcendental numbers, π and e; and the complex unit i—all in one equation!)

(b) Using rules for exponents together with Euler's formula, we obtain

$$e^{2+i\pi/4} = e^2e^{i\pi/4} = e^2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = e^2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= \frac{e^2\sqrt{2}}{2} + \frac{e^2\sqrt{2}}{2}i$$

If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, then

$$\bar{z} = r(\cos\theta - i\sin\theta) \tag{3}$$

The trigonometric identities

$$\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta$$

allow us to rewrite Equation (3) as

$$\bar{z} = r(\cos(-\theta) + i\sin(-\theta)) = re^{i(-\theta)}$$

This gives the following useful formula for the conjugate:

If $z = re^{i\theta}$, then

$$\bar{z} = re^{-i\theta}$$

Note Euler's formula gives a quick, one-line proof of De Moivre's Theorem:

$$[r(\cos\theta + i\sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

Appendix D*

Polynomials

A ***polynomial*** is a function p of a single variable x that can be written in the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \tag{1}$$

Euler gave the most algebraic of the proofs of the existence of the roots of [a polynomial] equation. . . . I regard it as unjust to ascribe this proof exclusively to Gauss, who merely added the finishing touches.
—Georg Frobenius, 1907
Quoted on the MacTutor History of Mathematics archive, http://www-history.mcs.st-and.ac.uk/history/

where $a_0, a_1, \ldots, a_n$ are constants ($a_n \neq 0$), called the ***coefficients*** of p. With the convention that $x^0 = 1$, we can use summation notation to write p as

$$p(x) = \sum_{k=0}^{n} a_kx^k$$

The integer n is called the ***degree*** of p, which is denoted by writing $\deg p = n$. A polynomial of degree zero is called a ***constant polynomial.***

Example D.1

Which of the following are polynomials?

(a) $2 - \frac{1}{3}x + \sqrt{2}x^2$ (b) $2 - \dfrac{1}{3x^2}$ (c) $\sqrt{2x^2}$

(d) $\ln\left(\dfrac{2e^{5x^3}}{e^{3x}}\right)$ (e) $\dfrac{x^2 - 5x + 6}{x - 2}$ (f) $\sqrt{x}$

(g) $\cos(2\cos^{-1}x)$ (h) e^x

Solution (a) This is the only one that is obviously a polynomial.

(b) A polynomial of the form shown in Equation (1) cannot become infinite as x approaches a finite value $[\lim_{x \to c} p(x) \neq \pm\infty]$, whereas $2 - 1/3x^2$ approaches $-\infty$ as x approaches zero. Hence, it is not a polynomial.

(c) We have

$$\sqrt{2x^2} = \sqrt{2}\sqrt{x^2} = \sqrt{2}|x|$$

which is equal to $\sqrt{2}x$ when $x \geq 0$ and to $-\sqrt{2}x$ when $x < 0$. Therefore, this expression is formed by "splicing together" two polynomials (a *piecewise polynomial*), but it is not a polynomial itself.

*Exercises and selected odd-numbered answers for this appendix can be found on the student companion website.

(d) Using properties of exponents and logarithms, we have

$$\ln\left(\frac{2e^{5x^3}}{e^{3x}}\right) = \ln(2e^{5x^3 - 3x}) = \ln 2 + \ln(e^{5x^3 - 3x})$$
$$= \ln 2 + 5x^3 - 3x = \ln 2 - 3x + 5x^3$$

so this expression is a polynomial.

(e) The domain of this function consists of all real numbers $x \neq 2$. For these values of x, the function simplifies to

$$\frac{x^2 - 5x + 6}{x - 2} = \frac{(x-2)(x-3)}{x - 2} = x - 3$$

so we can say that it is a polynomial *on its domain*.

(f) We see that this function cannot be a polynomial (even on its domain $x \geq 0$), since repeated differentiation of a polynomial of the form shown in Equation (1) eventually results in zero and $\sqrt{x}$ does not have this property. (Verify this.)

(g) The domain of this expression is $-1 \leq x \leq 1$. Let $\theta = \cos^{-1} x$ so that $\cos\theta = x$. Using a trigonometric identity, we see that

$$\cos(2\cos^{-1} x) = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1$$

so this expression is a polynomial on its domain.

(h) Analyzing this expression as we did the one in (f), we conclude that it is not a polynomial.

Two polynomials are ***equal*** if the coefficients of corresponding powers of x are all equal. In particular, equal polynomials must have the same degree. The ***sum*** of two polynomials is obtained by adding together the coefficients of corresponding powers of x.

Example D.2

Find the sum of $2 - 4x + x^2$ and $1 + 2x - x^2 + 3x^3$.

Solution We compute

$$\begin{aligned}(2 - 4x + x^2) + (1 + 2x - x^2 + 3x^3) &= (2 + 1) + (-4 + 2)x \\ &\quad + (1 + (-1))x^2 + (0 + 3)x^3 \\ &= 3 - 2x + 3x^3\end{aligned}$$

where we have "padded" the first polynomial by giving it an x^3 coefficient of zero.

We define the ***difference*** of two polynomials analogously, subtracting coefficients instead of adding them. The ***product*** of two polynomials is obtained by repeatedly using the distributive law and then gathering together corresponding powers of x.

Example D.3 Find the product of $2 - 4x + x^2$ and $1 + 2x - x^2 + 3x^3$.

Solution We obtain

$$\begin{aligned}
&(2 - 4x + x^2)(1 + 2x - x^2 + 3x^3) \\
&\quad = 2(1 + 2x - x^2 + 3x^3) - 4x(1 + 2x - x^2 + 3x^3) \\
&\qquad + x^2(1 + 2x - x^2 + 3x^3) \\
&\quad = (2 + 4x - 2x^2 + 6x^3) + (-4x - 8x^2 + 4x^3 - 12x^4) \\
&\qquad + (x^2 + 2x^3 - x^4 + 3x^5) \\
&\quad = 2 + (4x - 4x) + (-2x^2 - 8x^2 + x^2) + (6x^3 + 4x^3 + 2x^3) \\
&\qquad + (-12x^4 - x^4) + 3x^5 \\
&\quad = 2 - 9x^2 + 12x^3 - 13x^4 + 3x^5
\end{aligned}$$

Observe that for two polynomials p and q, we have

$$\deg(pq) = \deg p + \deg q$$

If p and q are polynomials with $\deg q \leq \deg p$, we can divide q into p, using long division to obtain the quotient p/q. The next example illustrates the procedure, which is the same as for long division of one integer into another. Just as the quotient of two integers is not, in general, an integer, the quotient of two polynomials is not, in general, another polynomial.

Example D.4 Compute $\dfrac{1 + 2x - x^2 + 3x^3}{2 - 4x + x^2}$.

Solution We will perform long division. It is helpful to write each polynomial with *decreasing* powers of x. Accordingly, we have

$$x^2 - 4x + 2\overline{\smash{)}3x^3 - x^2 + 2x + 1}$$

We begin by dividing x^2 into $3x^3$ to obtain the partial quotient $3x$. We then multiply $3x$ by the divisor $x^2 - 4x + 2$, subtract the result, and bring down the next term from the dividend $(3x^3 - x^2 + 2x + 1)$:

$$\begin{array}{r@{}l}
 & \phantom{3x^3 - {}}\;3x \\
x^2 - 4x + 2\,\big) & \overline{3x^3 - x^2 + 2x + 1} \\
 & 3x^3 - 12x^2 + 6x \\
 & \overline{\phantom{3x^3 - {}}11x^2 - 4x + 1}
\end{array}$$

Then we repeat the process with $11x^2$, multiplying 11 by $x^2 - 4x + 2$ and subtracting the result from $11x^2 - 4x + 1$. We obtain

$$\begin{array}{r@{}l}
 & \phantom{3x^3 - 12x^2 + {}}3x + 11 \\
x^2 - 4x + 2\,\big) & \overline{3x^3 - x^2 + 2x + 1} \\
 & 3x^3 - 12x^2 + 6x \\
 & \overline{\phantom{3x^3 - {}}11x^2 - 4x + 1} \\
 & \phantom{3x^3 - {}}11x^2 - 44x + 22 \\
 & \overline{\phantom{3x^3 - 11x^2 - {}}40x - 21}
\end{array}$$

We now have a remainder $40x - 21$. Its degree is less than that of the divisor $x^2 - 4x + 2$, so the process stops, and we have found that

$$3x^3 - x^2 + 2x + 1 = (x^2 - 4x + 2)(3x + 11) + (40x - 21)$$

or

$$\frac{3x^3 - x^2 + 2x + 1}{x^2 - 4x + 2} = 3x + 11 + \frac{40x - 21}{x^2 - 4x + 2}$$

Example D.4 can be generalized to give the following result, known as the ***division algorithm.***

Theorem D.1 **The Division Algorithm**

If f and g are polynomials with $\deg g \le \deg f$, then there are polynomials q and r such that

$$f(x) = g(x)q(x) + r(x)$$

where either $r = 0$ or $\deg r < \deg g$.

In Example D.4,

$$f(x) = 3x^3 - x^2 + 2x + 1, \quad g(x) = x^2 - 4x + 2, \quad q(x) = 3x + 11,$$

$$\text{and} \quad r(x) = 40x - 21$$

In the division algorithm, if the remainder is zero, then

$$f(x) = g(x)q(x)$$

and we say that g is a ***factor*** of f. (Notice that q is also a factor of f.) There is a close connection between the factors of a polynomial and its zeros. A ***zero*** of a polynomial f is a number a such that $f(a) = 0$. [The number a is also called a ***root*** of the polynomial equation $f(x) = 0$.] The following result, known as the ***Factor Theorem,*** establishes the connection between factors of a polynomial and its zeros.

Theorem D.2 **The Factor Theorem**

Let f be a polynomial and let a be a constant. Then a is a zero of f if and only if $x - a$ is a factor of $f(x)$.

Proof By the division algorithm,

$$f(x) = (x - a)q(x) + r(x)$$

where either $r(x) = 0$ or $\deg r < \deg(x - a) = 1$. Thus, in either case, $r(x) = r$ is a constant. Now,

$$f(a) = (a - a)q(a) + r = r$$

so $f(a) = 0$ if and only if $r = 0$, which is equivalent to

$$f(x) = (x - a)q(x)$$

as we needed to prove.

There is no method that is guaranteed to find the zeros of a given polynomial. However, there are some guidelines that are useful in special cases. The case of a polynomial with *integer* coefficients is particularly interesting. The following result, known as the ***Rational Roots Theorem,*** gives criteria for a zero of such a polynomial to be a *rational* number.

Theorem D.3 The Rational Roots Theorem

Let

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

be a polynomial with integer coefficients and let a/b be a rational number written in lowest terms. If a/b is a zero of f, then a_0 is a multiple of a and a_n is a multiple of b.

Proof If a/b is a zero of f, then

$$a_0 + a_1\left(\frac{a}{b}\right) + \cdots + a_{n-1}\left(\frac{a}{b}\right)^{n-1} + a_n\left(\frac{a}{b}\right)^n = 0$$

Multiplying through by b^n, we have

$$a_0b^n + a_1ab^{n-1} + \cdots + a_{n-1}a^{n-1}b + a_na = 0 \tag{1}$$

which implies that

$$a_0b^n + a_1ab^{n-1} + \cdots + a_{n-1}a^{n-1}b = -a_na^n \tag{2}$$

The left-hand side of Equation (2) is a multiple of b, so a_na^n must be a multiple of b also. Since a/b is in lowest terms, a and b have no common factors greater than 1. Therefore, a_n must be a multiple of b.

We can also write Equation (1) as

$$-a_0b^n = a_1ab^{n-1} + \cdots + a_{n-1}a^{n-1}b + a_na^n$$

and a similar argument shows that a_0 must be a multiple of a. (Show this.)

Example D.5

Find all the rational roots of the equation

$$6x^3 + 13x^2 - 4 = 0 \tag{3}$$

Solution If a/b is a root of this equation, then 6 is a multiple of b and -4 is a multiple of a, by the Rational Roots Theorem. Therefore,

$$a \in \{\pm1, \pm2, \pm4\} \quad \text{and} \quad b \in \{\pm1, \pm2, \pm3, \pm6\}$$

Forming all possible rational numbers a/b with these choices of a and b, we see that the only possible rational roots of the given equation are

$$\pm 1, \pm 2, \pm 4, \pm \tfrac{1}{2}, \pm \tfrac{1}{3}, \pm \tfrac{2}{3}, \pm \tfrac{4}{3}, \pm \tfrac{1}{6}$$

Substituting these values into Equation (3) one at a time, we find that -2, $-\frac{2}{3}$, and $\frac{1}{2}$ are the only values from this list that are actually roots. (Check these.) As we will see shortly, a polynomial equation of degree 3 cannot have more than three roots, so these are not only all the *rational* roots of Equation (3) but also its *only* roots.

We can improve on the trial-and-error method of Example D.5 in various ways. For example, once we find one root a of a given polynomial equation $f(x) = 0$, we know that $x - a$ is a factor of $f(x)$—say, $f(x) = (x - a)g(x)$. We can therefore divide $f(x)$ by $x - a$ (using long division) to find $g(x)$. Since $\deg g < \deg f$, the roots of $g(x) = 0$ [which are also roots of $f(x) = 0$] may be easier to find. In particular, if $g(x)$ is a quadratic polynomial, we have access to the ***quadratic formula.***

Suppose

$$ax^2 + bx + c = 0$$

(We may assume that a is positive, since multiplying both sides by -1 would produce an equivalent equation otherwise.) Then, completing the square, we have

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = \frac{b^2}{4a} - c$$

(Verify this.) Equivalently,

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \quad \text{or} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Therefore,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's revisit the equation from Example D.5 with the quadratic formula in mind.

Example D.6

Find the roots of $6x^3 + 13x^2 - 4 = 0$.

Solution Let's suppose we use the Rational Roots Theorem to discover that $x = -2$ is a rational root of $6x^3 + 13x^2 - 4 = 0$. Then $x + 2$ is a factor of $6x^3 + 13x^2 - 4$, and long division gives

$$6x^3 + 13x^2 - 4 = (x + 2)(6x^2 + x - 2)$$

(Check this.) We can now apply the quadratic formula to the second factor to find that its zeros are

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4(6)(-2)}}{2 \cdot 6} \\ &= \frac{-1 \pm \sqrt{49}}{12} = \frac{-1 \pm 7}{12} \\ &= \tfrac{6}{12}, -\tfrac{8}{12} \end{aligned}$$

or, in lowest terms, $\frac{1}{2}$ and $-\frac{2}{3}$. Thus, the three roots of Equation (3) are -2, $\frac{1}{2}$, and $-\frac{2}{3}$, as we determined in Example D.5.

Remark The Factor Theorem establishes a connection between the zeros of a polynomial and its *linear* factors. However, a polynomial without linear factors may still have factors of higher degree. Furthermore, when asked to factor a polynomial, we need to know the number system to which the coefficients of the factors are supposed to belong.

For example, consider the polynomial

$$p(x) = x^4 + 1$$

Over the *rational numbers* $\mathbb{Q}$, the only possible zeros of p are 1 and -1, by the Rational Roots Theorem. A quick check shows that neither of these actually works, so $p(x)$ has no *linear* factors with rational coefficients, by the Factor Theorem. However, $p(x)$ may still factor into a product of two *quadratics*. We will check for quadratic factors using the method of ***undetermined coefficients.***

Suppose that

$$x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$$

Expanding the right-hand side and comparing coefficients, we obtain the equations

$$\begin{aligned} a + c &= 0 \\ b + ac + d &= 0 \\ bc + ad &= 0 \\ bd &= 1 \end{aligned}$$

If $a = 0$, then $c = 0$ and $d = -b$. This gives $-b^2 = 1$, which has no solutions in $\mathbb{Q}$. Hence, we may assume that $a \neq 0$. Then $c = -a$, and we obtain $d = b$. It now follows that $b^2 = 1$, so $b = 1$ or $b = -1$. This implies that $a^2 = 2$ or $a^2 = -2$, respectively, neither of which has solutions in $\mathbb{Q}$. It follows that $x^4 + 1$ cannot be factored over $\mathbb{Q}$. We say that it is ***irreducible*** over $\mathbb{Q}$.

However, over the *real numbers* $\mathbb{R}$, $x^4 + 1$ does factor. The calculations we have just done show that

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

(Why?) To see whether we can factor further, we apply the quadratic formula. We see that the first factor has zeros

$$x = \frac{-\sqrt{2} \pm \sqrt{(\sqrt{2})^2 - 4}}{2} = \frac{-\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(-1 \pm i) = -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$$

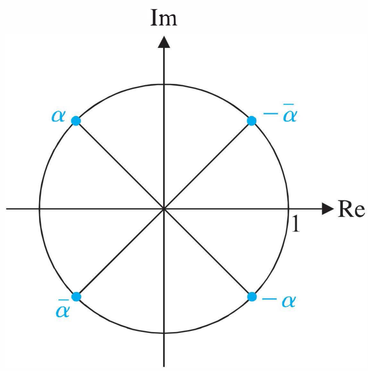


Figure D.1

which are in $\mathbb{C}$ but not in $\mathbb{R}$. Hence, $x^2 + \sqrt{2}x + 1$ cannot be factored into linear factors over $\mathbb{R}$. Similarly, $x^2 - \sqrt{2}x + 1$ cannot be factored into linear factors over $\mathbb{R}$.

Our calculations show that a complete factorization of $x^4 + 1$ is possible over the *complex numbers* $\mathbb{C}$. The four zeros of $x^4 + 1$ are

$$\alpha = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \quad \bar{\alpha} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \quad -\bar{\alpha} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$$
$$-\alpha = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

which, as Figure D.1 shows, all lie on the unit circle in the complex plane. Thus, the factorization of $x^4 + 1$ is

$$x^4 + 1 = (x - \alpha)(x - \bar{\alpha})(x + \bar{\alpha})(x + \alpha)$$

The preceding Remark illustrates several important properties of polynomials. Notice that the polynomial $p(x) = x^4 + 1$ satisfies $\deg p = 4$ and has exactly four zeros in $\mathbb{C}$. Furthermore, its complex zeros occur in *conjugate pairs*; that is, its complex zeros can be paired up as

$$\{\alpha, \bar{\alpha}\} \quad \text{and} \quad \{-\alpha, -\bar{\alpha}\}$$

These last two facts are true in general. The first is an instance of the ***Fundamental Theorem of Algebra (FTA)***, a result that was first proved by Gauss in 1797.

Theorem D.4 **The Fundamental Theorem of Algebra**

Every polynomial of degree n with real or complex coefficients has exactly n zeros (counting multiplicities) in $\mathbb{C}$.

This important theorem is sometimes stated as

"Every polynomial with real or complex coefficients has a zero in $\mathbb{C}$."

Let's call this statement FTA$'$. Certainly, FTA implies FTA$'$. Conversely, if FTA$'$ is true, then if we have a polynomial p of degree n, it has a zero α in $\mathbb{C}$. The Factor Theorem then tells us that $x - \alpha$ is a factor of $p(x)$, so

$$p(x) = (x - \alpha)q(x)$$

where q is a polynomial of degree $n - 1$ (also with real or complex coefficients). We can now apply FTA$'$ to q to get another zero, and so on, making FTA true. This argument can be made into a nice induction proof. (Try it.)

It is not possible to give a formula (along the lines of the quadratic formula) for the zeros of polynomials of degree 5 or more. (The work of Abel and Galois confirmed this; see page 311.) Consequently, other methods must be used to prove FTA. The proof that Gauss gave uses topological methods and can be found in more advanced mathematics courses.

Now suppose that

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

is a polynomial with real coefficients. Let α be a complex zero of p so that

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = p(\alpha) = 0$$

Then, using properties of conjugates, we have

$$\begin{aligned} p(\overline{\alpha}) &= a_0 + a_1\overline{\alpha} + \cdots + a_n\overline{\alpha}^n = \overline{a_0} + \overline{a_1}\,\overline{\alpha} + \cdots + \overline{a_n}\,\overline{\alpha}^n \\ &= \overline{a_0 + a_1\alpha + \cdots + a_n\alpha^n} \\ &= \overline{p(\alpha)} = \overline{0} = 0 \end{aligned}$$

Thus, $\overline{\alpha}$ is also a zero of p. This proves the following result:

The complex zeros of a polynomial with real coefficients occur in conjugate pairs.

In some situations, we do not need to know *what* the zeros of a polynomial are—we only need to know *where* they are located. For example, we might only need to know whether the zeros are positive or negative (as in Theorem 4.35). One theorem that is useful in this regard is ***Descartes' Rule of Signs.*** It allows us to make certain predictions about the number of positive zeros of a polynomial with real coefficients based on the signs of these coefficients.

Descartes' stated this rule in his 1637 book *La Géometrie,* but did not give a proof. Several mathematicians later furnished a proof, and Gauss provided a somewhat sharper version of the theorem in 1828.

Given a polynomial $a_0 + a_1x + \cdots + a_nx^n$, write its nonzero coefficients in order. Replace each positive coefficient by a plus sign and each negative coefficient by a minus sign. We will say that the polynomial has k ***sign changes*** if there are k places where the coefficients change sign. For example, the polynomial $2 - 3x + 4x^3 + x^4 - 7x^5$ has the sign pattern

$$\underbrace{+\ -}\!\!\underbrace{\ \ +}\ \underbrace{+\ -}$$

so it has three sign changes, as indicated.

Theorem D.5

Descartes' Rule of Signs

Let p be a polynomial with real coefficients that has k sign changes. Then the number of positive zeros of p (counting multiplicities) is at most k.

In words, Descartes' Rule of Signs says that a real polynomial cannot have more positive zeros than it has sign changes.

Example D.7

Show that the polynomial $p(x) = 4 + 2x^2 - 7x^4$ has exactly one positive zero.

Solution The coefficients of p have the sign pattern $+ + -$, which has only one sign change. So, by Descartes' Rule of Signs, p has at most one positive zero. But $p(0) = 4$ and $p(1) = -1$, so there is a zero somewhere in the interval (0, 1). Hence, this is the only positive zero of p.

We can also use Descartes' Rule of Signs to give a bound on the number of *negative* zeros of a polynomial with real coefficients. Let

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

and let b be a negative zero of p. Then $b = -c$ for $c > 0$, and we have

$$\begin{aligned} 0 = p(b) &= a_0 + a_1b + a_2b^2 + \cdots + a_nb^n \\ &= a_0 - a_1c + a_2c^2 - + \cdots + (-1)^n a_nc^n \end{aligned}$$

But

$$p(-x) = a_0 - a_1x + a_2x^2 - + \cdots + (-1)^n a_nx^n$$

so c is a positive zero of $p(-x)$. Therefore, $p(x)$ has exactly as many negative zeros as $p(-x)$ has positive zeros. Combined with Descartes' Rule of Signs, this observation yields the following:

> Let p be a polynomial with real coefficients. Then the number of negative zeros of p is at most the number of sign changes of $p(-x)$.

Example D.8

Show that the zeros of $p(x) = 1 + 3x + 2x^2 + x^5$ cannot all be real.

Solution The coefficients of $p(x)$ have no sign changes, so p has no positive zeros. Since $p(-x) = 1 - 3x + 2x^2 - x^5$ has three sign changes among its coefficients, p has at most three negative zeros. We note that 0 is not a zero of p either, so p has at most three real zeros. Therefore, it has at least two complex (nonreal) zeros.

Answers to Selected Odd-Numbered Exercises

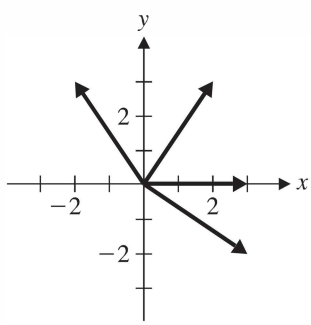
Answers are easy. It's asking the right questions [that's] hard.

—Doctor Who
"The Face of Evil,"
By Chris Boucher
BBC, 1977

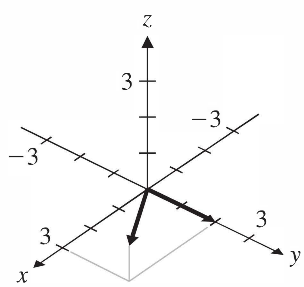
Chapter 1

Exercises 1.1

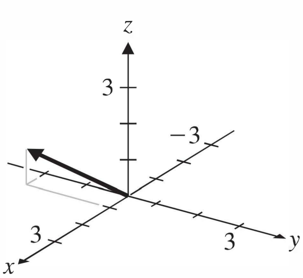
1.



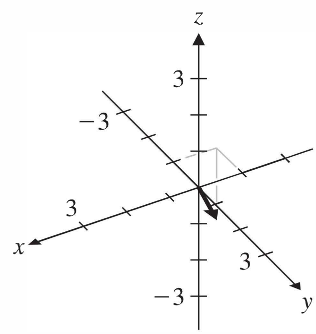
3. (a), (b)



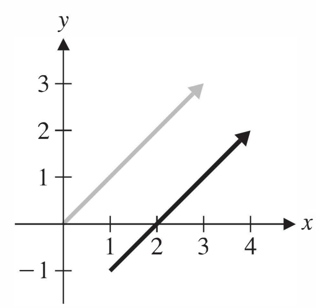
(c)



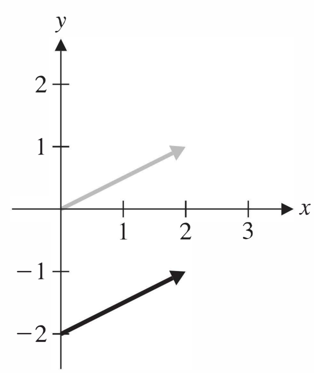
(d)



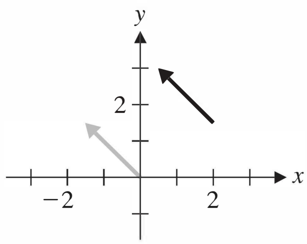
5. (a)



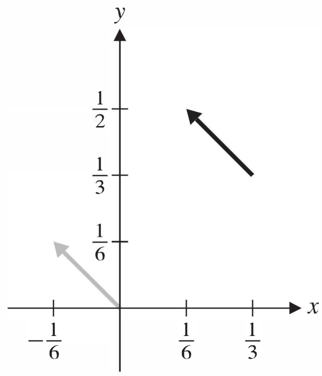
(b)



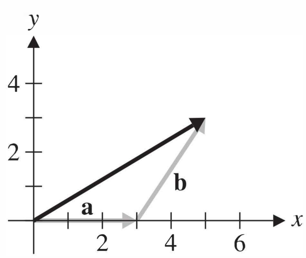
(c)



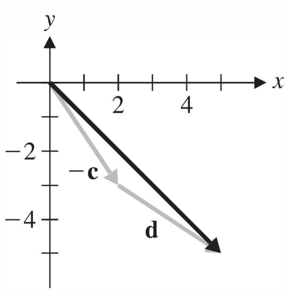
(d)



7. $\mathbf{a} + \mathbf{b} = [5, 3]$



9. $\mathbf{d} - \mathbf{c} = [5, -5]$



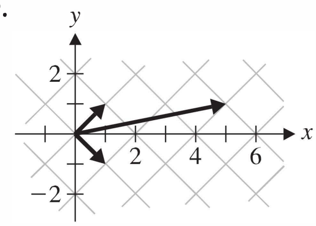
11. $[3, -2, 3]$

13. $\mathbf{u} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \mathbf{u} + \mathbf{v} = \begin{bmatrix} (1 - \sqrt{3})/2 \\ (\sqrt{3} - 1)/2 \end{bmatrix}, \mathbf{u} - \mathbf{v} = \begin{bmatrix} (1 + \sqrt{3})/2 \\ (1 + \sqrt{3})/2 \end{bmatrix}$

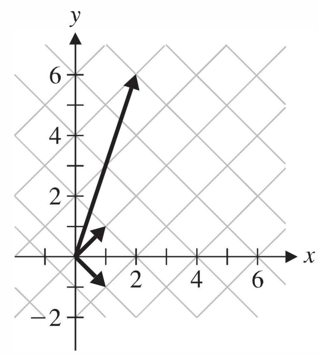
15. $5\mathbf{a}$

17. $\mathbf{x} = 3\mathbf{a}$

19.



21. $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$



25. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

27. $\mathbf{u} + \mathbf{v} = [0, 1, 0, 0]$

29.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

31. 0

33. 1

35. 0

37. 2, 0, 3

39. 5

41. [1, 1, 0]

43. [0, 0, 2, 2], [2, 3, 1, 1]

45. $x = 2$

47. No solution

49. $x = 3$

51. No solution

53. $x = 2$

55. $x = 1$, or $x = 5$

57. **(a)** All $a \neq 0$ **(b)** $a = 1, 5$
(c) a and m can have no common factors other than 1 [i.e., the greatest common divisor (gcd) of a and m is 1].

Exercises 1.2

1. -1

3. 11

5. 2

7. $\sqrt{5}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

9. $\sqrt{14}, \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$

11. $\sqrt{6}$, $[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}, 0]$

13. $\sqrt{17}$ **15.** $\sqrt{6}$

17. (a) $\mathbf{u} \cdot \mathbf{v}$ is a scalar, not a vector.

(c) $\mathbf{v} \cdot \mathbf{w}$ is a scalar and $\mathbf{u}$ is a vector.

19. Acute **21.** Acute **23.** Acute

25. 60° **27.** ≈88.10° **29.** ≈14.34°

31. Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = 0$, $\angle BAC$ is a right angle.

33. If we take the cube to be a unit cube (as in Figure 1.34), the four diagonals are given by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Since $\mathbf{d}_i \cdot \mathbf{d}_j \neq 0$ for all $i \neq j$ (six possibilities), no two diagonals are perpendicular.

35. $D = (-2, 1, 1)$

37. 5 mi/h at an angle of ≈ 53.13° to the bank

39. 60°

41. $\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ **43.** $\begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$ **45.** $\begin{bmatrix} -0.301 \\ 0.033 \\ -0.252 \end{bmatrix}$

47. $\mathcal{A} = \sqrt{45}/2$ **49.** $k = -2, 3$

51. $\mathbf{v}$ is of the form $k\begin{bmatrix} b \\ -a \end{bmatrix}$, where k is a scalar.

53. The Cauchy-Schwarz Inequality would be violated.

Exercises 1.3

1. (a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$ **(b)** $3x + 2y = 0$

3. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $\begin{aligned} x &= 1 - t \\ y &= 3t \end{aligned}$

5. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ **(b)** $\begin{aligned} x &= t \\ y &= -t \\ z &= 4t \end{aligned}$

7. (a) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2$ **(b)** $3x + 2y + z = 2$

9. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

(b) $\begin{aligned} x &= 2s - 3t \\ y &= s + 2t \\ z &= 2s + t \end{aligned}$

11. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

13. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$

15. (a) $\begin{aligned} x &= t \\ y &= -1 + 3t \end{aligned}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

17. Direction vectors for the two lines are given by $\mathbf{d}_1 = \begin{bmatrix} 1 \\ m_1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ m_2 \end{bmatrix}$. The lines are perpendicular if and only if $\mathbf{d}_1$ and $\mathbf{d}_2$ are orthogonal. But $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ if and only if $1 + m_1m_2 = 0$ or, equivalently, $m_1m_2 = -1$.

19. (a) Perpendicular **(b)** Parallel

(c) Perpendicular **(d)** Perpendicular

21. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

23. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

25. (a) $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

(b) $x - y = 0$ **(c)** $x + y - z = 0$

27. $3\sqrt{2}/2$ **29.** $2\sqrt{3}/3$ **31.** $(\frac{1}{2}, \frac{1}{2})$

33. $(\frac{4}{3}, \frac{4}{3}, \frac{8}{3})$ **35.** $18\sqrt{13}/13$ **37.** $\frac{5}{3}$

43. ≈78.9° **45.** ≈80.4°

Exercises 1.4

1. 13 N at approx N 67.38 E

3. $8\sqrt{3}$ N at an angle of 30° to $\mathbf{f}_1$

5. 4 N at an angle of 60° to $\mathbf{f}_2$

7. 5 N at an angle of 60° to the given force, $5\sqrt{3}$ N perpendicular to the 5 N force

9. $750\sqrt{2}$ N

11. 980 N

13. ≈ 117.6 N in the 15 cm wire, ≈ 88.2 N in the 20 cm wire

Review Questions

1. (a) T **(c)** F **(e)** T **(g)** F **(i)** T

3. $\mathbf{x} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ **5.** 120° **7.** $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

9. $2x + 3y - z = 7$ **11.** $\sqrt{6}/2$

13. The Cauchy-Schwarz Inequality would be violated.

15. $2\sqrt{6}/3$ **17.** $x = 2$ **19.** 3

Chapter 2

Exercises 2.1

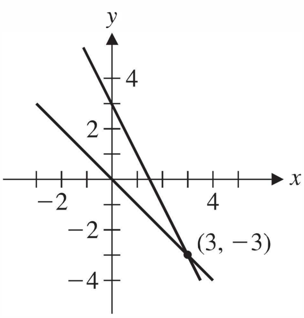
1. Linear **3.** Not linear because of the x^{-1} term

5. Not linear **7.** $2x + 4y = 7$

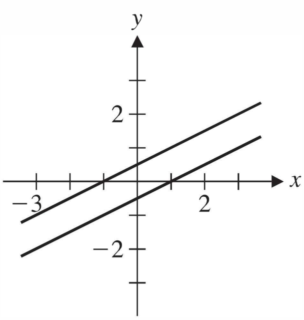
9. $x + y = 4(x, y \neq 0)$

11. $\left\{\begin{bmatrix} 2t \\ t \end{bmatrix}\right\}$ **13.** $\left\{\begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix}\right\}$

15. Unique solution, $x = 3, y = -3$



17. No solution



19. $[7, 3]$ **21.** $[\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}]$

23. $[5, -2, 1, 1]$ **25.** $[2, -7, -32]$

27. $\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$ **29.** $\left[\begin{array}{rr|r} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$

31.
$$\begin{aligned} y + z &= 1 \\ x - y \phantom{{}+z} &= 1 \\ 2x - y + z &= 1 \end{aligned}$$

33. $[1, 1]$

35. $[4, -1]$ **37.** No solution

39. (a) $\begin{aligned} 2x + y &= 3 \\ 4x + 2y &= 6 \end{aligned}$ **(b)** $\begin{aligned} x &= \frac{3}{2} - \frac{1}{2}s \\ y &= s \end{aligned}$

41. Let $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$. The solution is $x = \frac{1}{3}, y = -\frac{1}{2}$.

43. Let $u = \tan x$, $v = \sin y$, $w = \cos z$. One solution is $x = \pi/4$, $y = -\pi/6$, $z = \pi/3$. (There are infinitely many solutions.)

Exercises 2.2

1. No **3.** Reduced row echelon form

5. No **7.** No

9. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **11. (b)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

13. (b) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

15. Perform elementary row operations in the order $R_4 + 29R_3$, $8R_3$, $R_4 - 3R_2$, $R_2 \leftrightarrow R_3$, $R_4 - R_1$, $R_3 + 2R_1$, and, finally, $R_2 + 2R_1$.

17. One possibility is to perform elementary row operations on A in the order $R_2 - 3R_1$, $\frac{1}{2}R_2$, $R_1 + 2R_2$, $R_2 + 3R_1$, $R_1 \leftrightarrow R_2$.

19. *Hint:* Pick a random 2×2 matrix and try this—carefully!

21. This is really two elementary row operations combined: $3R_2$ and $R_2 - 2R_1$.

23. Exercise 1: 3; Exercise 3: 2; Exercise 5: 2; Exercise 7: 3

25. $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ **27.** $t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **29.** $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

31. $\begin{bmatrix} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

33. No solution

35. Unique solution

37. Infinitely many solutions

39. *Hint:* Show that if $ad - bc \neq 0$, the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2. (There are two cases: $a = 0$ and $a \neq 0$.) Use the Rank Theorem to deduce that the given system must have a unique solution.

41. **(a)** No solution if $k = -1$
(b) A unique solution if $k \neq \pm 1$
(c) Infinitely many solutions if $k = 1$

43. **(a)** No solution if $k = 1$
(b) A unique solution if $k \neq -2, 1$
(c) Infinitely many solutions if $k = -2$

45. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 9 \\ -10 \\ -7 \end{bmatrix}$

49. No intersection

51. The required vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are the solutions of the homogeneous system with augmented matrix

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right]$$

By Theorem 3, there are infinitely many solutions. If $u_1 \neq 0$ and $u_1v_2 - u_2v_1 \neq 0$, the solutions are given by

$$t\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

But a direct check shows that these are still solutions even if $u_1 = 0$ and/or $u_1v_2 - u_2v_1 = 0$.

53. $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ **55.** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Exercises 2.3

1. Yes **3.** No **5.** Yes **7.** Yes

9. We need to show that the vector equation $x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a solution for all values of a and b. This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right]$. Row reduction yields $\left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b - a \end{array}\right]$, from which we can see that there is a (unique) solution. [Further row operations yield $x = (a + b)/2$, $y = (a - b)/2$.] Hence, $\mathbb{R}^2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

11. We need to show that the vector equation $x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a solution for all values of a, b, and c. This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right]$. Row reduction yields $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & b + c - a \end{array}\right]$, from which we can see that there is a (unique) solution. [Further row operations yield $x = (a - b + c)/2$, $y = (a + b - c)/2$, $z = (-a + b + c)/2$.]

Hence, $\mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

13. **(a)** The line through the origin with direction vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
(b) The line with general equation $2x + y = 0$

15. **(a)** The plane through the origin with direction vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$
(b) The plane with general equation $2x - y + 4z = 0$

17. Substitution yields the linear system

$$\begin{aligned} a \qquad + 3c &= 0 \\ -a + b - 3c &= 0 \end{aligned}$$

whose solution is $t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. It follows that there are infinitely many solutions, the simplest perhaps being $a = -3, b = 0, c = 1$.

19. $\mathbf{u} = \mathbf{u} + 0(\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$
$\mathbf{v} = (-1)\mathbf{u} + (\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$
$\mathbf{w} = 0\mathbf{u} + (-1)(\mathbf{u} + \mathbf{v}) + (\mathbf{u} + \mathbf{v} + \mathbf{w})$

21. (c) We must show that span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) = span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$). We know that span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$) $\subseteq \mathbb{R}^3$ = span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$). From Exercise 19, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ all belong to span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$). Therefore, by Exercise 21(b), span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) = span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$).

23. Linearly independent

25. Linearly dependent, $-\begin{bmatrix}0\\1\\2\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix}$

27. Linearly dependent, since the set contains the zero vector

29. Linearly independent

31. Linearly dependent, $\begin{bmatrix}3\\-1\\1\\-1\end{bmatrix} + \begin{bmatrix}-1\\3\\1\\-1\end{bmatrix} + \begin{bmatrix}-1\\-1\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\3\\1\end{bmatrix}$

43. (a) Yes **(b)** No

Exercises 2.4

1. $x_1 = 160, x_2 = 120, x_3 = 160$

3. two small, three medium, four large

5. 65 bags of house blend, 30 bags of special blend, 45 bags of gourmet blend

7. $4FeS_2 + 11O_2 \longrightarrow 2Fe_2O_3 + 8SO_2$

9. $2C_4H_{10} + 13O_2 \longrightarrow 8CO_2 + 10H_2O$

11. $2C_5H_{11}OH + 15O_2 \longrightarrow 12H_2O + 10CO_2$

13. $Na_2CO_3 + 4C + N_2 \longrightarrow 2NaCN + 3CO$

15. (a) $\begin{aligned} f_1 &= 30 - t \\ f_2 &= -10 + t \\ f_3 &= t \end{aligned}$ **(b)** $f_1 = 15, f_3 = 15$

(c) $\begin{aligned} 0 &\le f_1 \le 20 \\ 0 &\le f_2 \le 20 \\ 10 &\le f_3 \le 30 \end{aligned}$

(d) Negative flow would mean that water was flowing backward, against the direction of the arrow.

17. (a) $\begin{aligned} f_1 &= -200 + s + t \\ f_2 &= 300 - s - t \\ f_3 &= s \\ f_4 &= 150 - t \\ f_5 &= t \end{aligned}$ **(b)** $200 \le f_3 \le 300$

(c) If $f_3 = s = 0$, then $f_5 = t \ge 200$ (from the f_1 equation), but $f_5 = t \le 150$ (from the f_4 equation). This is a contradiction.

(d) $50 \le f_3 \le 300$

19. $I_1 = 3$ amps, $I_2 = 5$ amps, $I_3 = 2$ amps

21. (a) $I = 10$ amps, $I_1 = I_5 = 6$ amps, $I_2 = I_4 = 4$ amps, $I_3 = 2$ amps

(b) $R_{eff} = \frac{7}{5}$ ohms

(c) Yes; change it to 4 ohms.

23. Farming : Manufacturing = 2 : 3

25. The painter charges \$39/hr, the plumber \$42/hr, the electrician \$54/hr.

27. (a) Coal should produce \$100 million and steel \$160 million.

(b) Coal should reduce production by ≈ \$4.2 million and steel should increase production by ≈ \$5.7 million.

29. (a) Yes; push switches 1, 2, and 3 or switches 3, 4, and 5.

(b) No

31. The states that can be obtained are represented by those vectors

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$$

in $\mathbb{Z}_2^5$ for which $x_1 + x_2 + x_4 + x_5 = 0$. (There are 16 such possibilities.)

33. If 0 = off, 1 = light blue, and 2 = dark blue, then the linear system that arises has augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

which reduces over $\mathbb{Z}_3$ to

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This yields the solutions

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}1\\1\\2\\2\\0\end{bmatrix} + t\begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix}$$

where t is in $\mathbb{Z}_3$. Hence, there are exactly three solutions:

$$\begin{bmatrix}1\\1\\2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\2\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\\0\\2\end{bmatrix}$$

where each entry indicates the number of times the corresponding switch should be pushed.

35. **(a)** Push squares 3 and 7.
(b) The 9×9 coefficient matrix A is row equivalent to $\mathbb{Z}_2$, so for any $\mathbf{b}$ in $\mathbb{Z}_2^9$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.

37. Grace is 15, and Hans is 5.

39. 1200 and 600 square yards

41. **(a)** $a = 4 - d, b = 5 - d, c = -2 + d, d$ is arbitrary
(b) No solution

43. **(a)** No solution
(b) $[a, b, c, d, e, f] = [4, 5, 6, -3, -1, 0] + f[-1, -1, -1, 1, 1, 1]$

45. **(a)** $y = x^2 - 2x + 1$ **(b)** $y = x^2 + 6x + 10$

47. $A = 1, B = 2$

49. $A = -\frac{1}{5}, B = \frac{1}{3}, C = 0, D = -\frac{2}{15}, E = -\frac{1}{5}$

51. $a = \frac{1}{2}, b = \frac{1}{2}, c = 0$

Exercises 2.5

1.

n	0	1	2	3	4	5
x_1	0	0.8571	0.9714	0.9959	0.9991	0.9998
x_2	0	0.8000	0.9714	0.9943	0.9992	0.9998

Exact solution: $x_1 = 1, x_2 = 1$

3.

n	0	1	2	3	4	5	6
x_1	0	0.2222	0.2539	0.2610	0.2620	0.2622	0.2623
x_2	0	0.2857	0.3492	0.3582	0.3603	0.3606	0.3606

Exact solution (to four decimal places): $x_1 = 0.2623$, $x_2 = 0.3606$

5.

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997
x_2	0	0.2500	0.0834	0.1250	0.0972	0.1042	0.0996	0.1008	0.1000
x_3	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997

Exact solution: $x_1 = 0.3, x_2 = 0.1, x_3 = 0.3$

7.

n	0	1	2	3	4
x_1	0	0.8571	0.9959	0.9998	1.0000
x_2	0	0.9714	0.9992	1.0000	1.0000

After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

9.

n	0	1	2	3	4
x_1	0	0.2222	0.2610	0.2622	0.2623
x_2	0	0.3492	0.3603	0.3606	0.3606

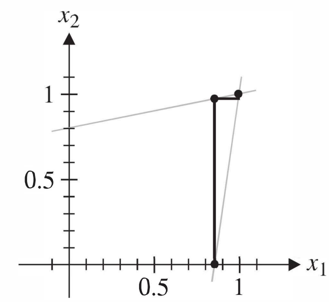
After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

11.

n	0	1	2	3	4	5	6
x_1	0	0.3333	0.2777	0.2962	0.2993	0.2998	0.3000
x_2	0	0.1667	0.1112	0.1020	0.1004	0.1000	0.1000
x_3	0	0.2777	0.2962	0.2993	0.2998	0.3000	0.3000

After four iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took seven iterations to reach the same accuracy.

13.



15.

n	0	1	2	3	4
x_1	0	3	-5	19	-53
x_2	0	-4	8	-28	80

If the equations are interchanged and the Gauss-Seidel method is applied to the equivalent system

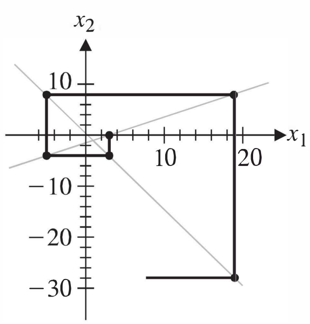
$$\begin{aligned} 3x_1 + 2x_2 &= 1 \\ x_1 - 2x_2 &= 3 \end{aligned}$$

we obtain

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	1.2222	0.9260	1.0247	0.9918	1.0027	0.9991	1.0003
x_2	0	-1.3333	-0.8889	-1.0370	-0.9876	-1.0041	-0.9986	-1.0004	-0.9998

After seven iterations, the process has converged to within 0.001 of the exact solution $x_1 = 1, x_2 = -1$.

17.



19.

n	0	1	2	3	4	5	6
x_1	0	-1.6	14.97	8.550	10.740	9.839	10.120
x_2	0	25.9	11.408	14.051	11.615	11.718	11.249
x_3	0	-10.35	-9.311	-11.200	-11.322	-11.721	-11.816

n	7	8	9	10	11	12
x_1	9.989	10.022	10.002	10.005	10.001	10.001
x_2	11.187	11.082	11.052	11.026	11.015	11.008
x_3	-11.912	-11.948	-11.973	-11.985	-11.992	-11.996

After 12 iterations, the Gauss-Seidel method has converged to within 0.01 of the exact solution $x_1 = 10, x_2 = 11, x_3 = -12$.

21.

n	13	14	15	16
x_1	10.0004	10.0003	10.0001	10.0001
x_2	11.0043	11.0023	11.0014	11.0007
x_3	−11.9976	−11.9986	−11.9993	−11.9996

23. The Gauss-Seidel method produces

n	0	1	2	3	4	5	6	7	8	9
x_1	0	0	12.5	21.875	24.219	24.805	24.951	24.988	24.997	24.999
x_2	0	0	18.75	21.438	24.609	24.902	24.976	24.994	24.998	24.999
x_3	0	50	68.75	73.438	74.609	74.902	74.976	74.994	74.998	74.999
x_4	0	62.5	71.875	74.219	74.805	74.951	74.988	74.997	74.999	75.000

The exact solution is $x_1 = 25$, $x_2 = 25$, $x_3 = 75$, $x_4 = 75$.

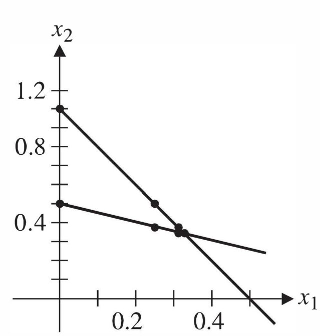
25. The Gauss-Seidel method produces the following iterates:

n	0	1	2	3	4	5	6
t_1	0	20	21.25	22.8125	23.3301	23.6596	23.7732
t_2	0	5	11.25	13.3203	14.6386	15.0926	15.2732
t_3	0	21.25	24.6094	26.9873	27.7303	27.9626	28.0352
t_4	0	2.5	5.8594	8.2373	8.9804	9.2126	9.2852
t_5	0	7.1875	14.6289	16.2829	16.7578	16.9036	16.9491
t_6	0	23.0469	24.9072	25.3207	25.4394	25.4759	25.4873

n	7	8	9	10	11	12
t_1	23.8093	23.8206	23.8242	23.8252	23.8256	23.8257
t_2	15.2824	15.2966	15.3010	15.3024	15.3029	15.3029
t_3	28.0579	28.0650	28.0671	28.0678	28.0681	28.0681
t_4	9.3079	9.3150	9.3172	9.3178	9.3181	9.3181
t_5	16.9633	16.9677	16.9690	16.9695	16.9696	16.9696
t_6	25.4908	25.4919	25.4922	25.4924	25.4924	25.4924

27. (a)

n	0	1	2	3	4	5	6
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$



(b) $2x_1 + x_2 = 1$
$x_1 + 2x_2 = 1$

(c)

n	0	1	2	3	4	5	6	7
x_1	0	0	0.25	0.3125	0.3281	0.3320	0.3330	0.3332
x_2	1	0.5	0.375	0.3438	0.3360	0.3340	0.3335	0.3334

[Columns 1, 2, and 3 of this table are the *odd-numbered* columns 1, 3, and 5 from the table in part (a).] The iterates are converging to $x_1 = x_2 = 0.3333$.

(d) $x_1 = x_2 = \frac{1}{3}$

Review Questions

1. (a) F **(c)** F **(e)** T **(g)** T **(i)** F

3. $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ **5.** $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **7.** $k = -1$ **9.** $(0, 3, 1)$

11. $x - 2y + z = 0$ **13. (a)** Yes **15.** 1 or 2

17. If $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, then $(c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = \mathbf{0}$. Linear independence of $\mathbf{u}$ and $\mathbf{v}$ implies $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Solving this system, we get $c_1 = c_2 = 0$. Hence $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.

19. Their ranks must be equal.

Chapter 3

Exercises 3.1

1. $\begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}$ **3.** Not possible

5. $\begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}$ **7.** $\begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}$

9. $[10]$ **11.** $\begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **15.** $\begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

19. $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$, $BA = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}$

Column i corresponds to warehouse i, row 1 contains the costs of shipping by truck, and row 2 contains the costs of shipping by train.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

23. $AB = [2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \quad 3\mathbf{a}_1 - \mathbf{a}_2 + 6\mathbf{a}_3 \quad \mathbf{a}_2 + 4\mathbf{a}_3]$ (where $\mathbf{a}_i$ is the ith column of A)

25. $\begin{bmatrix} 2 & 3 & 0 \\ -6 & -9 & 0 \\ 4 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -12 & -8 \\ -1 & 6 & 4 \\ 1 & -6 & -4 \end{bmatrix}$

27. $BA = \begin{bmatrix} 2\mathbf{A}_1 + 3\mathbf{A}_2 \\ \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 \\ -\mathbf{A}_1 + 6\mathbf{A}_2 + 4\mathbf{A}_3 \end{bmatrix}$ (where $\mathbf{A}_i$ is the ith row of A)

29. If $\mathbf{b}_i$ is the ith column of B, then $A\mathbf{b}_i$ is the ith column of AB. If the columns of B are linearly dependent, then there are scalars $c_1, \ldots, c_n$ (not all zero) such that $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$. But then $c_1(A\mathbf{b}_1) + \cdots + c_n(A\mathbf{b}_n) = A(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) = A\mathbf{0} = \mathbf{0}$, so the columns of AB are linearly dependent.

31. $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$ **33.** $\left[\begin{array}{cc|cc} 1 & 2 & 2 & 0 \\ 3 & 4 & 5 & 3 \\ \hline 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{array}\right]$

35. (a) $A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $A^{2001} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

37. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

39. (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \end{bmatrix}$

Exercises 3.2

1. $X = \begin{bmatrix} 5 & 4 \\ 3 & 5 \end{bmatrix}$ **3.** $X = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} \\ \frac{10}{3} & 4 \end{bmatrix}$

5. $B = 2A_1 + A_2$ **7.** Not possible

9. $\text{span}(A_1, A_2) = \left\{\begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix}\right\} = \left\{\begin{bmatrix} w & x \\ 2x - 5w & x - w \end{bmatrix}\right\}$

11. $\text{span}(A_1, A_2, A_3) = \left\{\begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} -3b + 4c + 5e & b & c \\ 0 & e & 0 \end{bmatrix}\right\}$

13. Linearly independent **15.** Linearly independent

23. $a = d, c = 0$ **25.** $3b = 2c, a = d - c$

27. $a = d, b = c = 0$

29. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $i > j$. Then, by the definition of an upper triangular matrix,

$a_{i1} = a_{i2} = \cdots = a_{i,\,i-1} = 0$ and
$b_{ij} = b_{i+1,\,j} = \cdots = b_{nj} = 0$

Now let $C = AB$. Then

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{i,\,i-1}b_{i-1,\,j} + a_{ii}b_{ij} \\ &\quad + a_{i,\,i+1}b_{i+1,\,j} + \cdots + a_{in}b_{nj} \\ &= 0 \cdot b_{1j} + 0 \cdot b_{2j} + \cdots 0 \cdot b_{i-1,\,j} + a_{ii} \cdot 0 \\ &\quad + a_{i,\,i+1} \cdot 0 + \cdots + a_{in} \cdot 0 = 0 \end{aligned}$$

from which it follows that C is upper triangular.

35. (a) A, B symmetric $\Rightarrow (A + B)^T = A^T + B^T = A + B \Rightarrow A + B$ is symmetric

37. Matrices (b) and (c) are skew-symmetric.

41. Either A or B (or both) must be the zero matrix.

43. (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

47. *Hint:* Use the trace.

Exercises 3.3

1. $\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$ **3.** Not invertible

5. Not invertible **7.** $\begin{bmatrix} -1.6 & -2.8 \\ 0.\overline{3} & 1 \end{bmatrix}$

9. $\begin{bmatrix} a/(a^2 + b^2) & b/(a^2 + b^2) \\ -b/(a^2 + b^2) & a/(a^2 + b^2) \end{bmatrix}$ **11.** $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$

13. (a) $\mathbf{x}_1 = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(c) The method in part (b) uses fewer multiplications.

17. (b) $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$

21. $X = A^{-1}(BA)^2B^{-1}$

23. $X = (AB)^{-1}BA + A$

25. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **27.** $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

29. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **31.** $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **35.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

43. (a) If A is invertible, then $BA = CA \Rightarrow (BA)A^{-1} = (CA)A^{-1} \Rightarrow B(AA^{-1}) = C(AA^{-1}) \Rightarrow BI = CI \Rightarrow B = C$.

45. *Hint:* Rewrite $A^2 - 2A + I = O$ as $A(2I - A) = I$.

47. If AB is invertible, then there exists a matrix X such that $(AB)X = I$. But then $A(BX) = I$ too, so A is invertible (with inverse BX).

49. $\begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$ **51.** $\begin{bmatrix} 1/(a^2 + 1) & -a/(a^2 + 1) \\ a/(a^2 + 1) & 1/(a^2 + 1) \end{bmatrix}$

53. Not invertible

55. $\begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}, a \neq 0$

57. $\begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a/d & -b/d & -c/d & 1/d \end{bmatrix}, d \neq 0$

61. Not invertible **63.** $\begin{bmatrix} 4 & 6 & 4 \\ 5 & 3 & 2 \\ 0 & 6 & 5 \end{bmatrix}$

69. $\left[\begin{array}{rr|rr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right]$ **71.** $\left[\begin{array}{rr|rr} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$

Exercises 3.4

1. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **3.** $\begin{bmatrix} -3/2 \\ -2 \\ -1 \end{bmatrix}$ **5.** $\begin{bmatrix} -7 \\ -15 \\ -2 \\ 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

15. $L^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, U^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix}$,

$A^{-1} = \begin{bmatrix} -5/12 & 1/12 \\ 1/6 & 1/6 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -16 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

27. $\begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$ **31.** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Exercises 3.5

1. Subspace **3.** Subspace

5. Subspace **7.** Not a subspace

11. $\mathbf{b}$ is in col(A), $\mathbf{w}$ is not in row(A).

15. No

17. $\{[1 \quad 0 \quad -1], [0 \quad 1 \quad 2]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A); $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for null(A).

19. $\{[1 \quad 0 \quad 1 \quad 0], [0 \quad 1 \quad -1 \quad 0], [0 \quad 0 \quad 0 \quad 1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for col(A); $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for null(A).

21. $\{[1 \quad 0 \quad -1], [1 \quad 1 \quad 1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A)

23. $\{[1 \quad 1 \quad 0 \quad 1], [0 \quad 1 \quad -1 \quad 1], [0 \quad 1 \quad -1 \quad -1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A)

25. Both $\{[1 \quad 0 \quad -1], [0 \quad 1 \quad 2]\}$ and $\{[1 \quad 0 \quad -1], [1 \quad 1 \quad 1]\}$ are linearly independent spanning sets for row(A) $= \{[a \quad b \quad -a + 2b]\}$. Both $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ are linearly independent spanning sets for col(A) $= \mathbb{R}^2$.

27. $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

29. $\{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [0 \quad 0 \quad 1]\}$

31. $\{[2 \quad -3 \quad 1], [1 \quad -1 \quad 0], [4 \quad -4 \quad 1]\}$

35. rank(A) $= 2$, nullity(A) $= 1$

37. rank(A) $= 3$, nullity(A) $= 1$

39. If A is 3×5, then rank(A) ≤ 3, so there cannot be more than three linearly independent columns.

41. nullity(A) $= 2, 3, 4$, or 5

43. If $a = -1$, then $\operatorname{rank}(A) = 1$; if $a = 2$, then $\operatorname{rank}(A) = 2$; for $a \neq -1, 2$, $\operatorname{rank}(A) = 3$.

45. Yes **47.** Yes **49.** No

51. $\mathbf{w}$ is in $\operatorname{span}(\mathcal{B})$ if and only if the linear system with augmented matrix $[\mathcal{B} \mid \mathbf{w}]$ is consistent, which is true in this case, since

$$[\mathcal{B} \mid \mathbf{w}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

From this reduced row echelon form, it is also clear that $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

53. $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 1$

55. $\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 1$

57. Let $\mathbf{A}_1, \ldots, \mathbf{A}_m$ be the row vectors of A so that $\operatorname{row}(A) = \operatorname{span}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$. If $\mathbf{x}$ is in $\operatorname{null}(A)$, then, since $A\mathbf{x} = \mathbf{0}$, we also have $\mathbf{A}_i \cdot \mathbf{x} = 0$ for $i = 1, \ldots, m$, by the row-column definition of matrix multiplication. If $\mathbf{r}$ is in $\operatorname{row}(A)$, then $\mathbf{r}$ is of the form $\mathbf{r} = c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m$. Therefore,

$$\begin{aligned} \mathbf{r} \cdot \mathbf{x} &= (c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m) \cdot \mathbf{x} \\ &= c_1(\mathbf{A}_1 \cdot \mathbf{x}) + \cdots + c_m(\mathbf{A}_m \cdot \mathbf{x}) = 0 \end{aligned}$$

59. (a) If a set of columns of AB is linearly independent, then the corresponding columns of B are linearly independent (by an argument similar to that needed to prove Exercise 29 in Section 3.1). It follows that the *maximum* number k of linearly independent columns of AB [i.e., $k = \operatorname{rank}(AB)$] is not more than the *maximum* number r of linearly independent columns of B [i.e., $r = \operatorname{rank}(B)$]. In other words, $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

61. (a) From Exercise 59(a), $\operatorname{rank}(UA) \leq \operatorname{rank}(A)$ and $\operatorname{rank}(A) = \operatorname{rank}((U^{-1}U)A) = \operatorname{rank}(U^{-1}(UA)) \leq \operatorname{rank}(UA)$. Hence, $\operatorname{rank}(UA) = \operatorname{rank}(A)$.

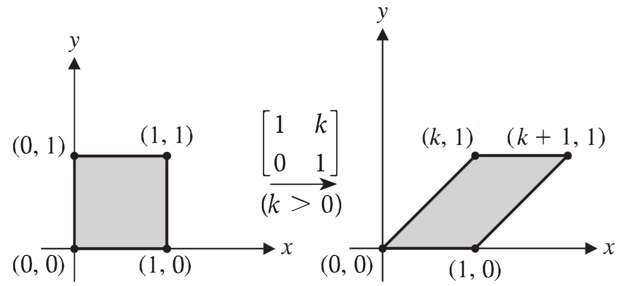
Exercises 3.6

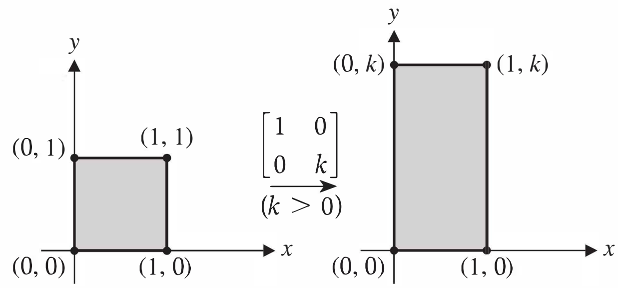
1. $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **13.** $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$

15. $[F] - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **17.** $[D] - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

19. $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ stretches or contracts in the x-direction (combined with a reflection in the y-axis if $k < 0$); $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ stretches or contracts in the y-direction (combined with a reflection in the x-axis if $k < 0$); $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection in the line $y = x$; $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ is a *shear* in the x-direction; $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ is a *shear* in the y-direction. For example,





21. $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ **23.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

25. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **27.** $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

31. $[S \circ T] = \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}$

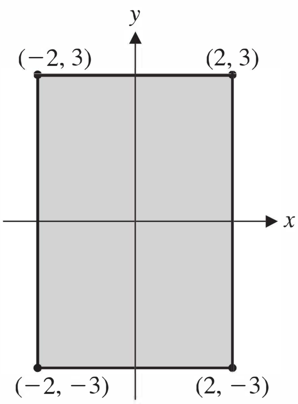
33. $[S \circ T] = \begin{bmatrix} 0 & 6 & -6 \\ 1 & -2 & 2 \end{bmatrix}$

35. $[S \circ T] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

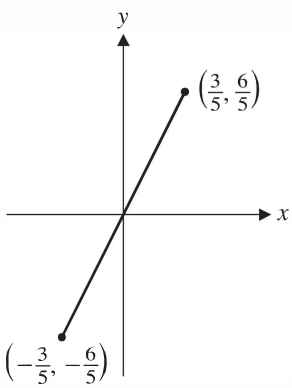
37. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ **39.** $\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

45. In vector form, let the parallel lines be given by $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and $\mathbf{x}' = \mathbf{p}' + t\mathbf{d}$. Their images are $T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{d}) = T(\mathbf{p}) + tT(\mathbf{d})$ and $T(\mathbf{x}') = T(\mathbf{p}' + t\mathbf{d}) = T(\mathbf{p}') + tT(\mathbf{d})$. Suppose $T(\mathbf{d}) \neq \mathbf{0}$. If $T(\mathbf{p}') - T(\mathbf{p})$ is parallel to $T(\mathbf{d})$, then the images represent the same line; otherwise the images represent distinct parallel lines. On the other hand, if $T(\mathbf{d}) = \mathbf{0}$, then the images represent two distinct points if $T(\mathbf{p}') \neq T(\mathbf{p})$ and single point otherwise.

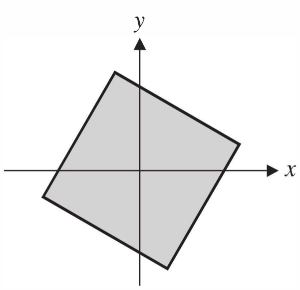
47.



49.



51.



Exercises 3.7

1. $\mathbf{x}_1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0.38 \\ 0.62 \end{bmatrix}$ **3.** 64%

5. $\mathbf{x}_1 = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}$ **7.** $\frac{5}{18}$

9. **(a)** $P = \begin{bmatrix} 0.662 & 0.250 \\ 0.338 & 0.750 \end{bmatrix}$ **(b)** 0.353

(c) 42.5% wet, 57.5% dry

11. **(a)** $P = \begin{bmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{bmatrix}$

(b) 0.08, 0.1062, 0.1057, 0.1057, 0.1057

(c) 10.6% good, 5.5% fair, 83.9% poor

13. The entries of the vector $\mathbf{j}P$ are just the column sums of the matrix P. So P is stochastic if and only if $\mathbf{j}P = \mathbf{j}$.

15. 4 **17.** 9.375

19. Yes, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **21.** No

23. No **25.** Yes, $\mathbf{x} = \begin{bmatrix} 10 \\ 27 \\ 35 \end{bmatrix}$

27. Productive **29.** Not productive

31. $\mathbf{x} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$ **33.** Yes, $\mathbf{x} = \begin{bmatrix} 10 \\ 6 \\ 8 \end{bmatrix}$

37. $\mathbf{x}_1 = \begin{bmatrix} 45 \\ 6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 120 \\ 27 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 375 \\ 72 \end{bmatrix}$

39. $\mathbf{x}_1 = \begin{bmatrix} 500 \\ 70 \\ 50 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 720 \\ 350 \\ 35 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1175 \\ 504 \\ 175 \end{bmatrix}$

41. **(a)** For L_1, we have $\mathbf{x}_1 = \begin{bmatrix} 50 \\ 8 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 40 \\ 40 \end{bmatrix}$,

$\mathbf{x}_3 = \begin{bmatrix} 200 \\ 32 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 160 \\ 160 \end{bmatrix}, \mathbf{x}_5 = \begin{bmatrix} 800 \\ 128 \end{bmatrix}, \mathbf{x}_6 = \begin{bmatrix} 640 \\ 640 \end{bmatrix}$,

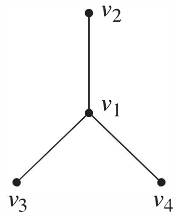
$\mathbf{x}_7 = \begin{bmatrix} 3200 \\ 512 \end{bmatrix}, \mathbf{x}_8 = \begin{bmatrix} 2560 \\ 2560 \end{bmatrix}, \mathbf{x}_9 = \begin{bmatrix} 12800 \\ 2048 \end{bmatrix}$,

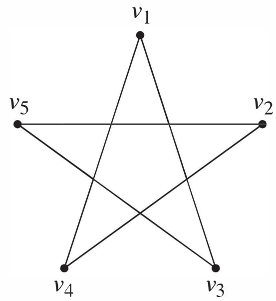
$\mathbf{x}_{10} = \begin{bmatrix} 10240 \\ 10240 \end{bmatrix}$.

(b) The first population oscillates between two states,while the second approaches a steady state.

43. The population oscillates through a cycle of three states (for the relative population): If $0.1 < s \leq 1$, the actual population is growing; if $s = 0.1$, the actual population goes through a cycle of length 3; and if $0 \leq s < 0.1$, the actual population is declining (and will eventually die out).

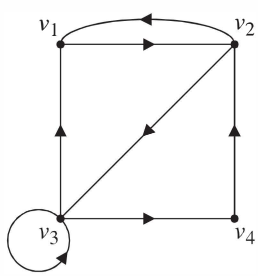
45. $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **47.** $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

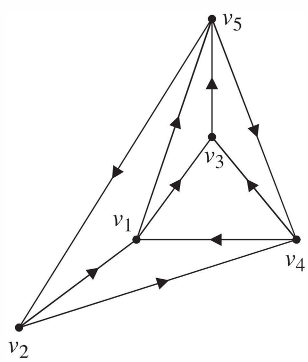
49.


51.


53. $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

55. $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

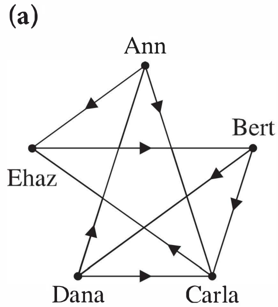
57.


59.


61. 2 **63.** 3 **65.** 0 **67.** 3

69. **(a)** Vertex i is not adjacent to any other vertices.

71. If we use direct wins only, P_2 is in first place; P_3, P_4, and P_6 tie for second place; and P_1 and P_5 tie for third place. If we combine direct and indirect wins, the players rank as follows: P_2 in first place, followed by P_6, P_4, P_3, P_5, and P_1.

73. **(a)**


$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

(b) two steps; all of the off-diagonal entries of the second row of $A + A^2$ are nonzero.

(d) If the graph has n vertices, check the (i, j) entry of the powers A^k for $k = 1, \ldots, n - 1$. Vertex i is connected to vertex j by a path of length k if and only if $(A^k)_{ij} \neq 0$.

75. $(AA^T)_{ij}$ counts the number of vertices adjacent to *both* vertex i and vertex j.

77. Bipartite **79.** Bipartite

Review Questions

1. **(a)** T **(c)** F **(e)** T **(g)** T **(i)** T

3. Impossible

5. $\begin{bmatrix} \frac{17}{83} & -\frac{1}{83} \\ -\frac{1}{83} & \frac{5}{166} \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix}$

9. $\begin{bmatrix} 0 & -9 \\ 2 & 4 \\ 1 & -6 \end{bmatrix}$

11. Because $(I - A)(I + A + A^2) = I - A^3 = I - O = I$, $(I - A)^{-1} = I + A + A^2$.

13. A basis for row(A) is $\{[1, -2, 0, -1, 0], [0, 0, 1, 2, 0], [0, 0, 0, 0, 1]\}$; a basis for col($A$) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}$ (or the standard basis for $\mathbf{R}^3$); and a basis for null(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

15. An invertible matrix has a trivial (zero) null space. If A is invertible, then so is A^T, and so both A and A^T have trivial null spaces. If A is not invertible, then A and A^T need not have the same null space. For example, take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

17. Because A has n linearly independent columns, rank(A) $= n$. Hence rank(A^TA) $= n$ by Theorem 3.28. Because A^TA is $n \times n$, this implies that A^TA is invertible, by the Fundamental Theorem of Invertible Matrices. AA^T need not be invertible. For example, take $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

19. $\begin{bmatrix} -1/5\sqrt{2} & -3/5\sqrt{2} \\ 2/5\sqrt{2} & 6/5\sqrt{2} \end{bmatrix}$

Chapter 4

Exercises 4.1

1. $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

3. $A\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\mathbf{v}, \lambda = -3$

5. $A\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

7. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **9.** $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

13. $\lambda = 1, E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right); \lambda = -1, E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

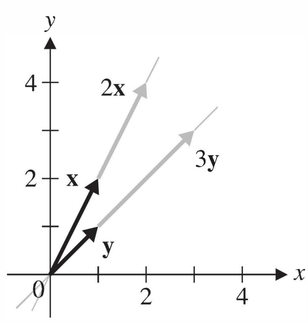
15. $\lambda = 0, E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right); \lambda = 1, E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

17. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right); \lambda = 3, E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

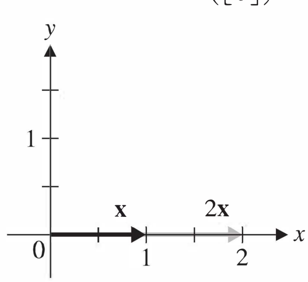
19. $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 1; \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda = 2$

21. $\mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda = 2; \mathbf{v} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda = 0$

23. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right); \lambda = 3, E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



25. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$



27. $\lambda = 1 + i, E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right); \lambda = 1 - i, E_{1-i} = \text{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$

29. $\lambda = 1 + i, E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right); \lambda = 1 - i, E_{1-i} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

31. $\lambda = 1, 2$ **33.** $\lambda = 4$

Exercises 4.2

1. 16 **3.** 0 **5.** -18 **7.** 6 **9.** -12 **11.** $a^2b + ab^2$ **13.** 4 **15.** $abdg$ **17.** 0 **25.** 2 **27.** -24 **29.** 0 **31.** 0 **33.** -24 **35.** 8 **37.** -4 **39.** -8 **45.** $k \neq 0, 2$ **47.** -6 **49.** $-\frac{3}{2}$

51. $(-2)3^n$

53. $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$

55. 0, 1 **57.** $x = \frac{3}{2}, y = -\frac{1}{2}$

59. $x = -1, y = 0, z = 1$ **61.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

63. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Exercises 4.3

1. (a) $\lambda^2 - 7\lambda + 12$ **(b)** $\lambda = 3, 4$

(c) $E_3 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right); E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(d) The algebraic and geometric multiplicities are all 1.

3. (a) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$

(b) $\lambda = -2, 1, 3$

(c) $E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right); E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right);$

$E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right)$

(d) The algebraic and geometric multiplicities are all 1.

5. (a) $-\lambda^3 + \lambda^2$ **(b)** $\lambda = 0, 1$

(c) $E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right); E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(d) $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 1.

7. (a) $-\lambda^3 + 9\lambda^2 - 27\lambda + 27$

(b) $\lambda = 3$

(c) $E_3 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

(d) $\lambda = 3$ has algebraic multiplicity 3 and geometric multiplicity 2.

9. (a) $\lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12$

(b) $\lambda = -1, 2, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$; $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right)$;

$E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

11. (a) $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 4\lambda - 3$

(b) $\lambda = -1, 1, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$;

$E_1 = \text{span}\left(\begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \end{bmatrix}\right)$;

$E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 2.

15. $\begin{bmatrix} 2^{-9} + 3 \cdot 2^{10} \\ -2^{-9} + 3 \cdot 2^{10} \end{bmatrix}$ **17.** $\begin{bmatrix} 2 \\ (2 \cdot 3^{20} - 1)/3^{20} \\ 2 \end{bmatrix}$

23. (a) $\lambda = -2, E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$; $\lambda = 5, E_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(b) (i) $\lambda = -\frac{1}{2}, E_{-1/2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$; $\lambda = \frac{1}{5}, E_{1/5} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(iii) $\lambda = 0, E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$; $\lambda = 7$, $E_7 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

27. $\begin{bmatrix} -3 & 4 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, -\lambda^3 - 3\lambda^2 + 4\lambda - 12$

35. $A^2 = 4A - 5I, A^3 = 11A - 20I$

$A^4 = 24A - 55I$

37. $A^{-1} = -\dfrac{1}{5}A + \dfrac{4}{5}I, A^{-2} = -\dfrac{4}{25}A + \dfrac{11}{25}I$

Exercises 4.4

1. The characteristic polynomial of A is $\lambda^2 - 5\lambda + 1$, but that of B is $\lambda^2 - 2\lambda + 1$.

3. The eigenvalues of A are $\lambda = 2$ and $\lambda = 4$, but those of B are $\lambda = 1$ and $\lambda = 4$.

5. $\lambda_1 = 4, E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$; $\lambda_2 = 3, E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

7. $\lambda_1 = 6, E_6 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}\right)$; $\lambda_2 = -2, E_{-2} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

9. Not diagonalizable

11. $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. Not diagonalizable

15. $P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 35839 & -69630 \\ -11605 & 24234 \end{bmatrix}$

19. $\begin{bmatrix} (3^k + 3(-1)^k)/4 & (3^{k+1} - 3(-1)^k)/4 \\ (3^k - (-1)^k)/4 & (3^{k+1} + (-1)^k)/4 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

23.

$$\begin{bmatrix} (5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (-5 + 2^{k+2} + (-3)^k)/10 \\ (2^{k+1} - 2(-3)^k)/5 & (2^k + 4(-3)^k)/5 & (2^{k+1} - 2(-3)^k)/5 \\ (-5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (5 + 2^{k+2} + (-3)^k)/10 \end{bmatrix}$$

25. $k = 0$ **27.** $k = 0$

29. All real values of k

37. If $A \sim B$, then there is an invertible matrix P such that $B = P^{-1}AP$. Therefore, we have

$$\operatorname{tr}(B) = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}(P^{-1}(AP)) = \operatorname{tr}((AP)P^{-1}) = \operatorname{tr}(APP^{-1}) = \operatorname{tr}(AI) = \operatorname{tr}(A)$$

using Exercise 45 in Section 3.2.

39. $P = \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix}$

41. $P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$

51. (b) $\dim E_{-1} = 1, \dim E_1 = 2, \dim E_2 = 3$

Exercises 4.5

1. (a) $\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$, 6.000

(b) $\lambda_1 = 6$

3. (a) $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$, 2.618

(b) $\lambda_1 = (3 + \sqrt{5})/2 \approx 2.618$

5. (a) $m_5 = 11.001, \mathbf{y}_5 = \begin{bmatrix} -0.333 \\ 1.000 \end{bmatrix}$

7. (a) $m_8 = 10.000, \mathbf{y}_8 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

9.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 26 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 17.692 \\ 5.923 \end{bmatrix}$	$\begin{bmatrix} 18.018 \\ 6.004 \end{bmatrix}$	$\begin{bmatrix} 17.999 \\ 6.000 \end{bmatrix}$	$\begin{bmatrix} 18.000 \\ 6.000 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.308 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.335 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$
m_k	1	26	17.692	18.018	17.999	18.000

Therefore, $\lambda_1 \approx 18, \mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$.

11.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7.571 \\ 2.857 \end{bmatrix}$	$\begin{bmatrix} 7.755 \\ 3.132 \end{bmatrix}$	$\begin{bmatrix} 7.808 \\ 3.212 \end{bmatrix}$	$\begin{bmatrix} 7.823 \\ 3.234 \end{bmatrix}$	$\begin{bmatrix} 7.827 \\ 3.240 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.286 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.377 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.404 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.411 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.413 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$
m_k	1	7	7.571	7.755	7.808	7.823	7.827

Therefore, $\lambda_1 \approx 7.827, \mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$.

13.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}21\\15\\13\end{bmatrix}$	$\begin{bmatrix}16.809\\12.238\\10.714\end{bmatrix}$	$\begin{bmatrix}17.011\\12.371\\10.824\end{bmatrix}$	$\begin{bmatrix}16.999\\12.363\\10.818\end{bmatrix}$	$\begin{bmatrix}17.000\\12.363\\10.818\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.714\\0.619\end{bmatrix}$	$\begin{bmatrix}1\\0.728\\0.637\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$
m_k	1	21	16.809	17.011	16.999	17.000

Therefore, $\lambda_1 \approx 17, \mathbf{v}_1 \approx \begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$.

15. $\lambda_1 \approx 5, \mathbf{v}_1 \approx \begin{bmatrix}1\\0\\0.333\end{bmatrix}$

17.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}7\\2\end{bmatrix}$	$\begin{bmatrix}7.571\\2.857\end{bmatrix}$	$\begin{bmatrix}7.755\\3.132\end{bmatrix}$	$\begin{bmatrix}7.808\\3.212\end{bmatrix}$	$\begin{bmatrix}7.823\\3.234\end{bmatrix}$	$\begin{bmatrix}7.827\\3.240\end{bmatrix}$
$R(\mathbf{x}_k)$	7	7.755	7.823	7.828	7.828	7.828	7.828
$\mathbf{y}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0.286\end{bmatrix}$	$\begin{bmatrix}1\\0.377\end{bmatrix}$	$\begin{bmatrix}1\\0.404\end{bmatrix}$	$\begin{bmatrix}1\\0.411\end{bmatrix}$	$\begin{bmatrix}1\\0.413\end{bmatrix}$	$\begin{bmatrix}1\\0.414\end{bmatrix}$

19.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}21\\15\\13\end{bmatrix}$	$\begin{bmatrix}16.809\\12.238\\10.714\end{bmatrix}$	$\begin{bmatrix}17.011\\12.371\\10.824\end{bmatrix}$	$\begin{bmatrix}16.999\\12.363\\10.818\end{bmatrix}$	$\begin{bmatrix}17.000\\12.363\\10.818\end{bmatrix}$
$R(\mathbf{x}_k)$	16.333	16.998	17.000	17.000	17.000	17.000
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.714\\0.619\end{bmatrix}$	$\begin{bmatrix}1\\0.728\\0.637\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$

21.

k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}5\\4\end{bmatrix}$	$\begin{bmatrix}4.8\\3.2\end{bmatrix}$	$\begin{bmatrix}4.667\\2.667\end{bmatrix}$	$\begin{bmatrix}4.571\\2.286\end{bmatrix}$	$\begin{bmatrix}4.500\\2.000\end{bmatrix}$	$\begin{bmatrix}4.444\\1.778\end{bmatrix}$	$\begin{bmatrix}4.400\\1.600\end{bmatrix}$	$\begin{bmatrix}4.364\\1.455\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.8\end{bmatrix}$	$\begin{bmatrix}1\\0.667\end{bmatrix}$	$\begin{bmatrix}1\\0.571\end{bmatrix}$	$\begin{bmatrix}1\\0.500\end{bmatrix}$	$\begin{bmatrix}1\\0.444\end{bmatrix}$	$\begin{bmatrix}1\\0.400\end{bmatrix}$	$\begin{bmatrix}1\\0.364\end{bmatrix}$	$\begin{bmatrix}1\\0.333\end{bmatrix}$
m_k	1	5	4.8	4.667	4.571	4.500	4.444	4.400	4.364

Since $\lambda_1 = \lambda_2 = 4, \mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, m_k is converging slowly to the exact answer.

23.

k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}5\\4\\1\end{bmatrix}$	$\begin{bmatrix}4.2\\3.2\\0.2\end{bmatrix}$	$\begin{bmatrix}4.048\\3.048\\0.048\end{bmatrix}$	$\begin{bmatrix}4.012\\3.012\\0.012\end{bmatrix}$	$\begin{bmatrix}4.003\\3.003\\0.003\end{bmatrix}$	$\begin{bmatrix}4.001\\3.001\\0.001\end{bmatrix}$	$\begin{bmatrix}4.000\\3.000\\0.000\end{bmatrix}$	$\begin{bmatrix}4.000\\3.000\\0.000\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.8\\0.2\end{bmatrix}$	$\begin{bmatrix}1\\0.762\\0.048\end{bmatrix}$	$\begin{bmatrix}1\\0.753\\0.012\end{bmatrix}$	$\begin{bmatrix}1\\0.751\\0.003\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0.001\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0\end{bmatrix}$
m_k	1	5	4.2	4.048	4.012	4.003	4.001	4.000	4.000

In this case, $\lambda_1 = \lambda_2 = 4$ and $E_4 = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

Clearly, m_k is converging to 4 and $\mathbf{y}_k$ is converging to a

vector in the eigenspace E_4—namely, $\begin{bmatrix}1\\0\\0\end{bmatrix} + 0.75\begin{bmatrix}0\\1\\0\end{bmatrix}$.

25.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
m_k	1	1	-1	1	-1	1

The exact eigenvalues are complex (i and $-i$), so the power method cannot possibly converge to either the dominant eigenvalue or the dominant eigenvector if we start with a *real* initial iterate. Instead, the power method oscillates between two sets of real vectors.

27.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}3\\4\\3\end{bmatrix}$	$\begin{bmatrix}2.500\\4.000\\2.500\end{bmatrix}$	$\begin{bmatrix}2.250\\4.000\\2.250\end{bmatrix}$	$\begin{bmatrix}2.125\\4.000\\2.125\end{bmatrix}$	$\begin{bmatrix}2.063\\4.000\\2.063\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0.750\\1\\0.750\end{bmatrix}$	$\begin{bmatrix}0.625\\1\\0.625\end{bmatrix}$	$\begin{bmatrix}0.562\\1\\0.562\end{bmatrix}$	$\begin{bmatrix}0.531\\1\\0.531\end{bmatrix}$	$\begin{bmatrix}0.516\\1\\0.516\end{bmatrix}$
m_k	1	4	4	4	4	4

The eigenvalues are $\lambda_1 = -12$, $\lambda_2 = 4$, $\lambda_3 = 2$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\mathbf{x}_0 = \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3$, the initial vector $\mathbf{x}_0$ has a zero component in the direction of the dominant eigenvector, so the power method cannot converge to the dominant eigenvalue/eigenvector. Instead, it converges to a *second* eigenvalue/eigenvector pair, as the calculations show.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix}$	$\begin{bmatrix} -18 \\ 9 \\ 18 \end{bmatrix}$	$\begin{bmatrix} -18 \\ 9 \\ 18 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$
m_k	1	4	-18	-18
$R(\mathbf{x}_k)$	-0.667	-18	-18	-18

In this case, there is no dominant eigenvalue. (We could choose either 18 or -18 for m_k, $k \geq 2$.) However, the Rayleigh quotient method (Exercises 17–20) converges to -18. Thus, -18 is the dominant eigenvalue of $A - 17I$, and $\lambda_2 = -18 + 17 = -1$ is the second eigenvalue of A.

29. Apply the power method to $A - 18I = \begin{bmatrix} -4 & 12 \\ 5 & -15 \end{bmatrix}$.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ -10 \end{bmatrix}$	$\begin{bmatrix} 15.2 \\ -19 \end{bmatrix}$	$\begin{bmatrix} 15.2 \\ -19 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$
m_k	1	-10	-19	-19

Thus, -19 is the dominant eigenvalue of $A - 18I$, and $\lambda_2 = -19 + 18 = -1$ is the second eigenvalue of A.

31. Apply the power method to $A - 17I = \begin{bmatrix} -8 & 4 & 8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{bmatrix}$.

33.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} -0.833 \\ 1.056 \end{bmatrix}$	$\begin{bmatrix} 0.798 \\ -0.997 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ -1.000 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ -1.000 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.789 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.801 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.800 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.800 \\ 1 \end{bmatrix}$
m_k	1	0.5	1.056	-0.997	-1.000	-1.000

Thus, the eigenvalue of A that is smallest in magnitude is $1/(-1) = -1$.

35.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\-1\end{bmatrix}$	$\begin{bmatrix}-0.500\\0.000\\0.500\end{bmatrix}$	$\begin{bmatrix}0.500\\0.333\\-0.500\end{bmatrix}$	$\begin{bmatrix}0.500\\0.111\\-0.500\end{bmatrix}$	$\begin{bmatrix}0.500\\0.259\\-0.500\end{bmatrix}$	$\begin{bmatrix}0.500\\0.160\\-0.500\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\-1\end{bmatrix}$	$\begin{bmatrix}-1.000\\0.000\\1.000\end{bmatrix}$	$\begin{bmatrix}-1.000\\-0.667\\1.000\end{bmatrix}$	$\begin{bmatrix}-1.000\\-0.222\\1.000\end{bmatrix}$	$\begin{bmatrix}-1.000\\-0.518\\1.000\end{bmatrix}$	$\begin{bmatrix}-1.000\\-0.321\\1.000\end{bmatrix}$
m_k	1	-0.500	-0.500	-0.500	-0.500	-0.500

Clearly, m_k converges to -0.5, so the smallest eigenvalue of A is $1/(-0.5) = -2$.

37. The calculations are the same as for Exercise 33.

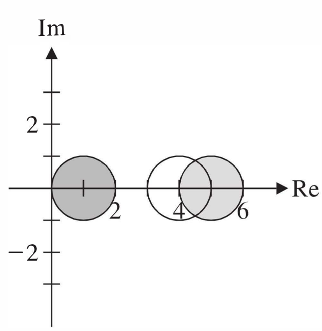
39. We apply the inverse power method to $A - 5I = \begin{bmatrix}-1 & 0 & 6\\-1 & -2 & 1\\6 & 0 & -1\end{bmatrix}$. Taking $\mathbf{x}_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, we have

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0.200\\-0.500\\0.200\end{bmatrix}$	$\begin{bmatrix}-0.080\\-0.500\\-0.080\end{bmatrix}$	$\begin{bmatrix}0.032\\-0.500\\0.032\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}-0.400\\1\\-0.400\end{bmatrix}$	$\begin{bmatrix}0.160\\1\\0.160\end{bmatrix}$	$\begin{bmatrix}-0.064\\1\\-0.064\end{bmatrix}$
m_k	1	-0.500	-0.500	-0.500

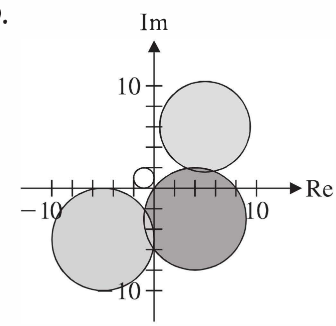
Clearly, m_k converges to -0.5, so the eigenvalue of A closest to 5 is $5 + 1/(-0.5) = 5 - 2 = 3$.

41. 0.732 **43.** -0.619

47.



49.



51. *Hint:* Show that 0 is not contained in any Gerschgorin disk and then apply Theorem 4.16.

53. Exercise 52 implies that $|\lambda|$ is less than or equal to all of the column sums of A for every eigenvalue λ. But for a stochastic matrix, all column sums are 1. Hence $|\lambda| \le 1$.

Exercises 4.6

1. Not regular **3.** Regular

5. Not regular **7.** $L = \begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ \frac{4}{5} & \frac{4}{5}\end{bmatrix}$

9. $L = \begin{bmatrix}0.304 & 0.304 & 0.304\\0.354 & 0.354 & 0.354\\0.342 & 0.342 & 0.342\end{bmatrix}$

11. 1, $\begin{bmatrix}2\\1\end{bmatrix}$ **13.** 2, $\begin{bmatrix}16\\4\\1\end{bmatrix}$

15. The population is increasing, decreasing, and constant, respectively.

17.
$$P^{-1}LP = \begin{bmatrix} b_1 & b_2s_1 & b_3s_1s_2 & \cdots & b_{n-1}s_1s_2\cdots s_{n-2} & b_ns_1s_2\cdots s_{n-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$
The characteristic polynomial of L is $(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - b_3s_1s_2\lambda^{n-3} - \cdots - b_ns_1s_2\cdots s_{n-1})(-1)^n$.

19. $\lambda \approx 1.746$, $\mathbf{p} \approx \begin{bmatrix} 0.660 \\ 0.264 \\ 0.076 \end{bmatrix}$

21. $\lambda \approx 1.092$, $\mathbf{p} \approx \begin{bmatrix} 0.535 \\ 0.147 \\ 0.094 \\ 0.078 \\ 0.064 \\ 0.053 \\ 0.029 \end{bmatrix}$

25. (a) $h \approx 0.082$

29. 3, $\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$

31. 3, $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$

33. Reducible

35. Irreducible

43. 1, 2, 4, 8, 16

45. 0, 1, 1, 0, -1

47. $x_n = 4^n - (-1)^n$

49. $y_n = (n - \frac{1}{2})2^n$

51. $b_n = \dfrac{1}{2\sqrt{3}}[(1 + \sqrt{3})^n - (1 - \sqrt{3})^n]$

57. (a) $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 8$
(b) $d_n = d_{n-1} + d_{n-2}$
(c) $d_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

59. The general solution is $x(t) = -3C_1e^{-t} + C_2e^{4t}$, $y(t) = 2C_1e^{-t} + C_2e^{4t}$. The specific solution is $x(t) = -3e^{-t} + 3e^{4t}$, $y(t) = 2e^{-t} + 3e^{4t}$.

61. The general solution is $x_1(t) = (1 + \sqrt{2})C_1e^{\sqrt{2}t} + (1 - \sqrt{2})C_2e^{-\sqrt{2}t}$, $x_2(t) = C_1e^{\sqrt{2}t} + C_2e^{-\sqrt{2}t}$. The specific solution is $x_1(t) = (2 + \sqrt{2})e^{\sqrt{2}t}/4 + (2 - \sqrt{2})e^{-\sqrt{2}t}/4$, $x_2(t) = \sqrt{2}e^{\sqrt{2}t}/4 - \sqrt{2}e^{-\sqrt{2}t}/4$.

63. The general solution is $x(t) = -C_1 + C_3e^{-t}$, $y(t) = C_1 + C_2e^t - C_3e^{-t}$, $z(t) = C_1 + C_2e^t$. The specific solution is $x(t) = 2 - e^{-t}$, $y(t) = -2 + e^t + e^{-t}$, $z(t) = -2 + e^t$.

65. (a) $x(t) = -120e^{8t/5} + 520e^{11t/10}$, $y(t) = 240e^{8t/5} + 260^{11t/10}$. Strain X dies out after approximately 2.93 days; strain Y continues to grow.

67. $a = 10$, $b = 20$; $x(t) = 10e^t(\cos t + \sin t) + 10$, $y(t) = 10e^t(\cos t - \sin t) + 20$. Species Y dies out when $t \approx 1.22$.

71. $x(t) = C_1e^{2t} + C_2e^{3t}$

77. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 27 \\ 27 \end{bmatrix}$ **(c)** Repeller

79. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **(c)** Neither

81. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}, \begin{bmatrix} 1.75 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 3.125 \\ -1.75 \end{bmatrix}$ **(c)** Saddle point

83. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0.36 \\ 0.36 \end{bmatrix}, \begin{bmatrix} 0.216 \\ 0.216 \end{bmatrix}$ **(c)** Attractor

85. $r = \sqrt{2}$, $\theta = 45°$, spiral repeller

87. $r = 2$, $\theta = -60°$, spiral repeller

89. $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}$, spiral attractor

91. $P = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, orbital center

Review Questions

1. (a) F **(c)** F **(e)** F **(g)** T **(i)** F

3. -18

5. Since $A^T = -A$, we have $\det A = \det(A^T) = \det(-A) = (-1)^n \det A = -\det A$ by Theorem 4.7 and the fact that n is odd. It follows that $\det A = 0$.

7. $A\mathbf{x} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\mathbf{x}$, $\lambda = 5$

9. (a) $4 - 3\lambda^2 - \lambda^3$

(c) $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$, $E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

11. $\begin{bmatrix} 162 \\ 158 \end{bmatrix}$

13. Not similar

15. Not similar

17. 0, 1, or -1

19. If $A\mathbf{x} = \lambda\mathbf{x}$, then $(A^2 - 5A + 2I)\mathbf{x} = A^2\mathbf{x} - 5A\mathbf{x} + 2\mathbf{x} = 3^2\mathbf{x} - 5(3\mathbf{x}) + 2\mathbf{x} = -4\mathbf{x}$.

Chapter 5

Exercises 5.1

1. Orthogonal **3.** Not orthogonal **5.** Orthogonal

7. $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$ **9.** $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ **11.** Orthonormal

13. $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3\sqrt{5} \\ 4/3\sqrt{5} \\ -5/3\sqrt{5} \end{bmatrix}$

15. Orthonormal

17. Orthogonal, $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

19. Orthogonal, $\begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$

21. Not orthogonal

27. $$\cos(\angle(Q\mathbf{x}, Q\mathbf{y})) = \frac{Q\mathbf{x}\cdot Q\mathbf{y}}{\|Q\mathbf{x}\|\|Q\mathbf{y}\|} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$$
$= \cos(\angle(\mathbf{x}, \mathbf{y}))$ by Theorem 5.6

29. Rotation, $\theta = 45°$ **31.** Reflection, $y = \sqrt{3}x$

33. (a) $A(A^T + B^T)B = AA^TB + AB^TB = IB + AI = B + A = A + B$

(b) From part (a),
$$\begin{aligned}\det(A + B) &= \det(A(A^T + B^T)B) \\ &= \det A \det(A^T + B^T)\det B \\ &= \det A \det((A + B)^T)\det B \\ &= \det A \det(A + B)\det B\end{aligned}$$

Assume that $\det A + \det B = 0$ (so that $\det B = -\det A$) but that $A + B$ is invertible. Then $\det(A + B) \neq 0$, so $1 = \det A \det B = \det A(-\det A) = -(\det A)^2$. This is impossible, so we conclude that $A + B$ cannot be invertible.

Exercises 5.2

1. $W^\perp = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

3. $W^\perp = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = t, z = -t\right\},$

$\mathcal{B}^\perp = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$

5. $W^\perp = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right\}$

7. row(A): $\{[1 \;\; 0 \;\; 1], [0 \;\; 1 \;\; -2]\}$, null($A$): $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$

9. col(A): $\left\{\begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix}\right\}$, null($A^T$):

$\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -7 \\ 0 \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix}\right\}$ **13.** $\left\{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

15. $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ **17.** $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$

19. $\mathbf{v} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ -\frac{4}{5} \end{bmatrix}$ **21.** $\mathbf{v} = \begin{bmatrix} \frac{7}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

25. No

Exercises 5.3

1. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix},$

$\mathbf{q}_2 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

5. $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}\right\}$ **7.** $\mathbf{v} = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{9} \end{bmatrix} + \begin{bmatrix} \frac{38}{9} \\ -\frac{38}{9} \\ \frac{19}{9} \end{bmatrix}$

9. $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}-\frac{3}{35}\\ \frac{34}{35}\\ -\frac{1}{7}\end{bmatrix}, \begin{bmatrix}-\frac{15}{34}\\ 0\\ \frac{9}{34}\end{bmatrix}\right\}$

13. $Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & 1/\sqrt{3} & -2/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix}$

15. $\begin{bmatrix}0 & 2/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 3/\sqrt{6} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$

17. $R = \begin{bmatrix}3 & 9 & \frac{1}{3}\\ 0 & 6 & \frac{2}{3}\\ 0 & 0 & \frac{7}{3}\end{bmatrix}$

19. $A = AI$

21. $A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T =$

$$\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3}\\ 0 & 2/\sqrt{6} & -1/2\sqrt{3}\\ 0 & 0 & 3/2\sqrt{3}\end{bmatrix}\cdot \begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\end{bmatrix}$$

23. Let $R\mathbf{x} = \mathbf{0}$. Then $A\mathbf{x} = QR\mathbf{x} = Q\mathbf{0} = \mathbf{0}$. Since $A\mathbf{x}$ represents a linear combination of the columns of A (which are linearly independent), we must have $\mathbf{x} = \mathbf{0}$. Hence, R is invertible, by the Fundamental Theorem.

Exercises 5.4

1. $Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}, D = \begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}$

3. $Q = \begin{bmatrix}2/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{3} & -2/\sqrt{6}\end{bmatrix}, D = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$

5. $Q = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}, D = \begin{bmatrix}5 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & -2\end{bmatrix}$

7. $Q = \begin{bmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}, D = \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$

9. $Q = \begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0\\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2}\end{bmatrix},$

$D = \begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

11. $Q^TAQ = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}a & b\\ b & a\end{bmatrix}\cdot$

$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}a+b & 0\\ 0 & a-b\end{bmatrix} = D$

13. **(a)** If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Therefore, $A + B$ is symmetric, by Exercise 35 in Section 3.2, and so is orthogonally diagonalizable, by the Spectral Theorem.

15. If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Since $AB = BA$, AB is also symmetric, by Exercise 36 in Section 3.2. Hence, AB is orthogonally diagonalizable, by the Spectral Theorem.

17. $A = \begin{bmatrix}\frac{5}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{5}{2}\end{bmatrix} + \begin{bmatrix}\frac{3}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \frac{3}{2}\end{bmatrix}$

19. $A = \begin{bmatrix}5 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 2\\ 0 & 2 & 2\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & -1\end{bmatrix}$

21. $\begin{bmatrix}\frac{1}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{bmatrix}$ **23.** $\begin{bmatrix}\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3}\\ -\frac{2}{3} & \frac{5}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{8}{3}\end{bmatrix}$

Exercises 5.5

1. $2x^2 + 6xy + 4y^2$ **3.** 123 **5.** -5

7. $\begin{bmatrix}1 & 3\\ 3 & 2\end{bmatrix}$ **9.** $\begin{bmatrix}3 & -\frac{3}{2}\\ -\frac{3}{2} & -1\end{bmatrix}$ **11.** $\begin{bmatrix}5 & 1 & -2\\ 1 & -1 & 2\\ -2 & 2 & 2\end{bmatrix}$

13. $Q = \begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & -2/\sqrt{5}\end{bmatrix}, y_1^2 + 6y_2^2$

15. $Q = \begin{bmatrix}2/\sqrt{5} & 2/3\sqrt{5} & -1/3\\ 0 & 5/3\sqrt{5} & 2/3\\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3\end{bmatrix}, 9y_1^2 + 9y_2^2 - 9y_3^2$

17. $Q = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ -1/\sqrt{3} & 0 & 2/\sqrt{6}\\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\end{bmatrix}, 2(x')^2 +$ $(y')^2 - (z')^2$

19. Positive definite **21.** Negative definite

23. Positive definite **25.** Indefinite

27. For any vector $\mathbf{x}$, we have $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T B^T B\mathbf{x} = (B\mathbf{x})^T(B\mathbf{x}) = \|B\mathbf{x}\|^2 \geq 0$. If $\mathbf{x}^T A\mathbf{x} = 0$, then $\|B\mathbf{x}\|^2 = 0$, so $B\mathbf{x} = \mathbf{0}$. Since B is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and hence $A = B^T B$ is positive definite.

29. **(a)** Every eigenvalue of cA is of the form $c\lambda$ for some eigenvalue λ of A. By Theorem 5.24, $\lambda > 0$, so $c\lambda > 0$, since c is positive. Hence, cA is positive definite, by Theorem 5.24.

(c) Let $\mathbf{x} \neq \mathbf{0}$. Then $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{x}^T B\mathbf{x} > 0$, since A and B are positive definite. But then $\mathbf{x}^T(A + B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0$, so $A + B$ is positive definite.

31. The maximum value of $f(\mathbf{x})$ is 2 when $\mathbf{x} = \pm \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$;

the minimum value of $f(\mathbf{x})$ is 0 when $\mathbf{x} = \pm \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

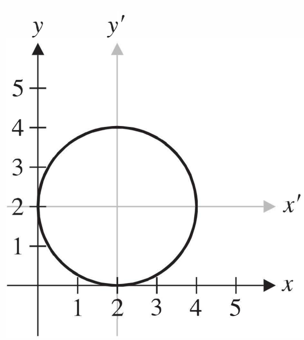
33. The maximum value of $f(\mathbf{x})$ is 4 when $\mathbf{x} = \pm \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$;

the minimum value of $f(\mathbf{x})$ is 1 when $\mathbf{x} =$

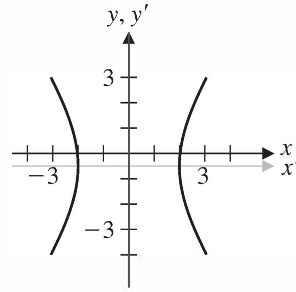
$$\pm \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \text{ or } \pm \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

35. Ellipse **37.** Parabola **39.** Hyperbola

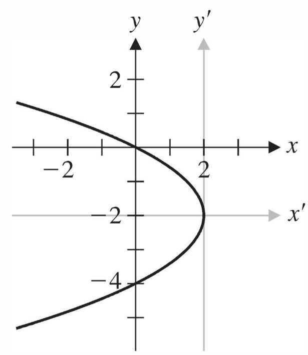
41. Circle, $x' = x - 2$, $y' = y - 2$, $(x')^2 + (y')^2 = 4$



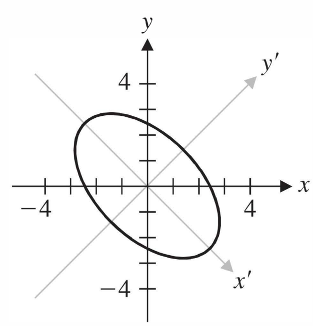
43. Hyperbola, $x' = x$, $y' = y + \frac{1}{2}$, $(x')^2/4 - (y')^2/9 = 1$



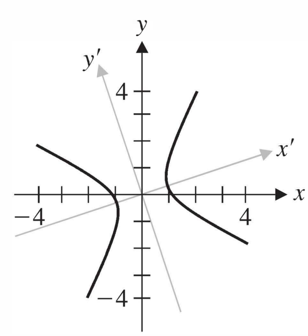
45. Parabola, $x' = x - 2$, $y' = y + 2$, $x' = -\frac{1}{2}(y')^2$



47. Ellipse, $(x')^2/4 + (y')^2/12 = 1$



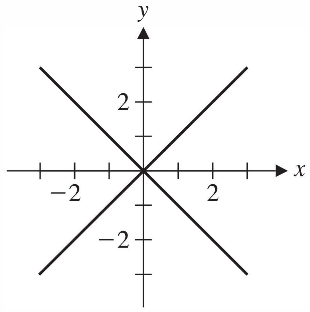
49. Hyperbola, $(x')^2 - (y')^2 = 1$



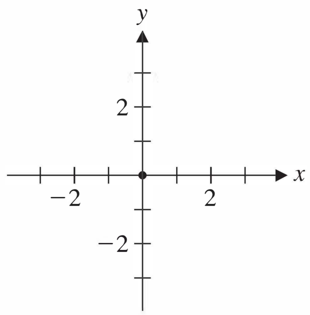
51. Ellipse, $(x'')^2/50 + (y'')^2/10 = 1$

53. Hyperbola, $(x'')^2 - (y'')^2 = 1$

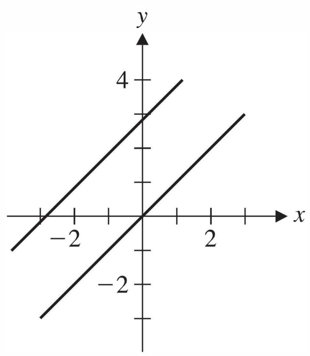
55. Degenerate (two lines)



57. Degenerate (a point)



59. Degenerate (two lines)



61. Hyperboloid of one sheet, $(x')^2 - (y')^2 + 3(z')^2 = 1$

63. Hyperbolic paraboloid, $z = -(x')^2 + (y')^2$

65. Hyperbolic paraboloid, $x' = -\sqrt{3}(y')^2 + \sqrt{3}(z')^2$

67. Ellipsoid, $3(x'')^2 + (y'')^2 + 2(z'')^2 = 4$

Review Questions

1. **(a)** T **(c)** T **(e)** F **(g)** F **(i)** F

3. $\begin{bmatrix} 9/2 \\ 2/3 \\ -11/6 \end{bmatrix}$

5. Verify that $Q^TQ = I$.

7. Theorem 5.6(c) shows that if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, then $Q\mathbf{v}_i \cdot Q\mathbf{v}_j = 0$. Theorem 5.6(b) shows that $\{Q\mathbf{v}_1, \ldots, Q\mathbf{v}_k\}$ consists of unit vectors, because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ does. Hence, $\{Q\mathbf{v}_1, \ldots, Q\mathbf{v}_k\}$ is an orthonormal set.

9. $\left\{\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right\}$

13. row(A): $\{[1 \;\; 0 \;\; 2 \;\; 3 \;\; 4], [0 \;\; 1 \;\; 0 \;\; 2 \;\; 1]\}$

col(A): $\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -5 \end{bmatrix}\right\}$

null(A): $\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

null(A^T): $\left\{\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$

15. **(a)** $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}\right\}$

17. $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}\right\}$

19. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Chapter 6

Exercises 6.1

1. Vector space

3. Not a vector space; axiom 1 fails.

5. Not a vector space; axiom 8 fails.

7. Vector space

9. Vector space

11. Vector space

15. Complex vector space

17. Not a complex vector space; axiom 6 fails.

19. Not a vector space; axioms 1, 4, and 6 fail.

21. Not a vector space; the operations of addition and multiplication are not even the same.

25. Subspace **27.** Not a subspace

29. Not a subspace **31.** Subspace

33. Subspace **35.** Subspace

37. Not a subspace **39.** Subspace

41. Subspace **43.** Not a subspace

45. Not a subspace

47. Take U to be the x-axis and W the y-axis, for example. Then $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are in $U \cup W$, but $\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$ is not.

51. No

53. Yes; $s(x) = (3 + 2t)p(x) + (1 + t)q(x) + tr(x)$ for any scalar t.

55. Yes; $h(x) = f(x) + g(x)$

57. No

59. No

61. Yes

Exercises 6.2

1. Linearly independent

3. Linearly dependent; $\begin{bmatrix}-1 & 0\\-1 & 7\end{bmatrix} = 4\begin{bmatrix}-1 & 1\\-2 & 2\end{bmatrix} + \begin{bmatrix}3 & 0\\1 & 1\end{bmatrix} - 2\begin{bmatrix}0 & 2\\-3 & 1\end{bmatrix}$

5. Linearly independent

7. Linearly dependent; $3x + 2x^2 = 7x - 2(2x - x^2)$

9. Linearly independent

11. Linearly dependent; $1 = \sin^2 x + \cos^2 x$

13. Linearly dependent; $\ln(x^2) = -2\ln 2 \cdot 1 + 2 \cdot \ln(2x)$

17. (a) Linearly independent

(b) Linearly dependent

19. Basis **21.** Not a basis

23. Not a basis **25.** Not a basis

27. $[A]_{\mathcal{B}} = \begin{bmatrix}-1\\-1\\-1\\4\end{bmatrix}$ **29.** $[p(x)]_{\mathcal{B}} = \begin{bmatrix}6\\-1\\3\end{bmatrix}$

35. $\dim V = 2$, $\mathcal{B} = \{1 - x, 1 - x^2\}$

37. $\dim V = 3$, $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$

39. $\dim V = 2$, $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

41. $(n^2 - n)/2$

43. (a) $\dim(U \times V) = \dim U + \dim V$

(b) Show that if $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis for W, then $\{(\mathbf{w}_1, \mathbf{w}_1), \ldots, (\mathbf{w}_n, \mathbf{w}_n)\}$ is a basis for Δ.

45. $\{1 + x, 1 + x + x^2, 1\}$

47. $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right\}$

49. $\{1, 1 + x\}$

51. $\{1 - x, x - x^2\}$

53. $\{\sin^2 x, \cos^2 x\}$

59. (a) $p_0(x) = \frac{1}{2}x^2 - \frac{5}{2}x + 3$, $p_1(x) = -x^2 + 4x - 3$, $p_2(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$

61. (c) (i) $3x^2 - 16x + 19$ (ii) $x^2 - 4x + 5$

63. $(p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})$

Exercises 6.3

1. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$, $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}\frac{5}{2}\\-\frac{1}{2}\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

3. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$

5. $[p(x)]_{\mathcal{B}} = \begin{bmatrix}2\\-1\end{bmatrix}$, $[p(x)]_{\mathcal{C}} = \begin{bmatrix}-3\\2\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$

7. $[p(x)]_{\mathcal{B}} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$, $[p(x)]_{\mathcal{C}} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}$

9. $[A]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}, [A]_{\mathcal{C}} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ -3 \\ \frac{9}{2} \end{bmatrix}, P_{\mathcal{C}\leftarrow\mathcal{B}} =$

$\begin{bmatrix} \frac{1}{2} & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ \frac{3}{2} & -1 & -2 & -\frac{1}{2} \end{bmatrix}, P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$

11. $[f(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, [f(x)]_{\mathcal{C}} = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix},$

$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}, P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

13. **(a)** $\begin{bmatrix} (3 - 2\sqrt{3})/2 \\ (-3\sqrt{3} + 2)/2 \end{bmatrix} \approx \begin{bmatrix} 3.232 \\ -1.598 \end{bmatrix}$

(b) $\begin{bmatrix} 2 + 2\sqrt{3} \\ 2\sqrt{3} - 2 \end{bmatrix} \approx \begin{bmatrix} 5.464 \\ 1.464 \end{bmatrix}$

15. $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

17. $-2 - 8(x - 1) - 5(x - 1)^2$

19. $-1 + 3(x + 1) - 3(x + 1)^2 + (x + 1)^3$

Exercises 6.4

1. Linear transformation **3.** Linear transformation

5. Linear transformation

7. Not a linear transformation

9. Linear transformation

11. Not a linear transformation

13. We have

$$\begin{aligned} S(p(x) + q(x)) &= S((p + q)(x)) = x((p + q)(x)) \\ &= x(p(x) + q(x)) = xp(x) + xq(x) \\ &= S(p(x)) + S(q(x)) \end{aligned}$$

and
$$\begin{aligned} S(cp(x)) &= S((cp)(x)) = x((cp)(x)) \\ &= x(cp(x)) = cxp(x) = cS(p(x)) \end{aligned}$$

Therefore, S is linear. Similarly,

$$\begin{aligned} T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) &= T\begin{bmatrix} a + c \\ b + d \end{bmatrix} \\ &= (a + c) + ((a + c) + (b + d))x \\ &= (a + (a + b)x) + (c + (c + d)x) \\ &= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \end{aligned}$$

and
$$\begin{aligned} T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) &= T\begin{bmatrix} ka \\ kb \end{bmatrix} = (ka) + (ka + kb)x \\ &= k(a + (a + b)x) = kT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \end{aligned}$$

Therefore, T is linear.

15. $T\begin{bmatrix} -7 \\ 9 \end{bmatrix} = 5 - 14x - 8x^2, T\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a + 3b}{4}\right) - \left(\frac{a + 7b}{4}\right)x + \left(\frac{a - b}{2}\right)x^2$

17. $T(4 - x + 3x^2) = 4 + 3x + 5x^2, T(a + bx + cx^2) = a + cx + \left(\frac{3a - b - c}{2}\right)x^2$

19. *Hint:* Let $a = T(E_{11}), b = T(E_{12}), c = T(E_{21}), d = T(E_{22})$.

23. *Hint:* Consider the effect of T and D on the standard basis for $\mathscr{P}_n$.

25. $(S \circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 6 \end{bmatrix}, (S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x & -y \\ 0 & 2x + 2y \end{bmatrix}$.

$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix}$ does not make sense.

27. $(S \circ T)(p(x)) = p'(x + 1), (T \circ S)(p(x)) = (p(x + 1))' = p'(x + 1)$

29. $(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(\begin{bmatrix} x - y \\ -3x + 4y \end{bmatrix}\right) = \begin{bmatrix} 4(x - y) + (-3x + 4y) \\ 3(x - y) + (-3x + 4y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = T\left(S\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} 4x + y \\ 3x + y \end{bmatrix}\right) = \begin{bmatrix} (4x + y) - (3x + y) \\ -3(4x + y) + 4(3x + y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Therefore, $S \circ T = I$ and $T \circ S = I$, so S and T are inverses.

Exercises 6.5

1. **(a)** Only (ii) is in $\ker(T)$.
(b) Only (iii) is in $\operatorname{range}(T)$.
(c) $\ker(T) = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \right\}, \operatorname{range}(T) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\}$

3. **(a)** Only (iii) is in $\ker(T)$.
(b) All of them are in $\operatorname{range}(T)$.
(c) $\ker(T) = \{a + bx + cx^2 : a = -c, b = -c\} = \{t + tx - tx^2\}, \operatorname{range}(T) = \mathbb{R}^2$

5. A basis for $\ker(T)$ is $\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$, and a basis for $\operatorname{range}(T)$ is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$; $\operatorname{rank}(T) = \operatorname{nullity}(T) = 2$, and $\operatorname{rank}(T) + \operatorname{nullity}(T) = 4 = \dim M_{22}$.

7. A basis for $\ker(T)$ is $\{1 + x - x^2\}$, and a basis for $\operatorname{range}(T)$ is $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$; $\operatorname{rank}(T) = 2$, $\operatorname{nullity}(T) = 1$, and $\operatorname{rank}(T) + \operatorname{nullity}(T) = 3 = \dim \mathscr{P}_2$.

9. $\operatorname{rank}(T) = \operatorname{nullity}(T) = 2$

11. $\operatorname{rank}(T) = \operatorname{nullity}(T) = 2$

13. $\operatorname{rank}(T) = 1$, $\operatorname{nullity}(T) = 2$

15. One-to-one and onto

17. Neither one-to-one nor onto

19. One-to-one but not onto

21. Isomorphic, $T\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

23. Not isomorphic

25. Isomorphic, $T(a + bi) = \begin{bmatrix}a\\b\end{bmatrix}$

31. *Hint:* Define $T : \mathscr{C}[0, 1] \to \mathscr{C}[0, 2]$ by letting $T(f)$ be the function whose value at x is $(T(f))(x) = f(x/2)$ for x in $[0, 2]$.

33. (a) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in V and let $(S \circ T)(\mathbf{v}_1) = (S \circ T)(\mathbf{v}_2)$. Then $S(T(\mathbf{v}_1)) = S(T(\mathbf{v}_2))$, so $T(\mathbf{v}_1) = T(\mathbf{v}_2)$, since S is one-to-one. But now $\mathbf{v}_1 = \mathbf{v}_2$, since T is one-to-one. Hence, $S \circ T$ is one-to-one.

35. (a) By the Rank Theorem, $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$. If T is onto, then $\operatorname{range}(T) = W$, so $\operatorname{rank}(T) = \dim(\operatorname{range}(T)) = \dim W$. Therefore,

$$\begin{aligned}\dim V + \operatorname{nullity}(T) &< \dim W + \operatorname{nullity}(T)\\ &= \operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V\end{aligned}$$

so nullity $(T) < 0$, which is impossible. Therefore, T cannot be onto.

Exercises 6.6

1. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[4 + 2x]_{\mathcal{B}} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix} = [2 - 4x]_{\mathcal{C}} = [T(4 + 2x)]_{\mathcal{C}}$

3. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} = [a + b(x + 2) + c(x + 2)^2)]_{\mathcal{C}} = [T(a + bx + cx^2)]_{\mathcal{C}}$

5. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&0\\1&1&1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix}1&0&0\\1&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\a+b+c\end{bmatrix}$
$= \begin{bmatrix}a + b\cdot 0 + c\cdot 0^2\\a + b\cdot 1 + c\cdot 1^2\end{bmatrix}_{\mathcal{C}} = [T(a + bx + cx^2)]_{\mathcal{C}}$

7. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}-7\\7\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\7\end{bmatrix} = \begin{bmatrix}7\\7\\7\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}7\\7\\7\end{bmatrix}_{\mathcal{C}} = \left[T\begin{bmatrix}-7\\7\end{bmatrix}\right]_{\mathcal{C}}$

9. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}a\\c\\b\\d\end{bmatrix} = \left[\begin{bmatrix}a&c\\b&d\end{bmatrix}\right]_{\mathcal{C}} = [T(A)]_{\mathcal{C}}$

11. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}0&-1&1&0\\-1&0&0&1\\1&0&0&-1\\0&1&-1&0\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix}0&-1&1&0\\-1&0&0&1\\1&0&0&-1\\0&1&-1&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}c-b\\d-a\\a-d\\b-c\end{bmatrix} = \left[\begin{bmatrix}c-b&d-a\\a-d&b-c\end{bmatrix}\right]_{\mathcal{C}} = [AB - BA]_{\mathcal{C}} = [T(A)]_{\mathcal{C}}$

13. (b) $[D]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $[D]_{\mathcal{B}}[3\sin x - 5\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [3\cos x + 5\sin x]_{\mathcal{B}} = [D(3\sin x - 5\cos x)]_{\mathcal{B}}$

15. (a) $[D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

17. $[S \circ T]_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}$

19. Invertible, $T^{-1}(a + bx) = -b + ax$

21. Invertible, $T^{-1}(p(x)) = p(x - 2)$

23. Invertible, $T^{-1}(a + bx + cx^2) = (a - b + 2c) + (b - 2c)x + cx^2$ or $T^{-1}(p(x)) = p(x) - p'(x) + p''(x)$

25. Not invertible **27.** $-3\sin x - \cos x + C$

29. $\frac{4}{5}e^{2x}\cos x - \frac{3}{5}e^{2x}\sin x + C$

31. $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}$ **33.** $\mathcal{C} = \{1 - x, 2 + x\}$

35. $\mathcal{C} = \{1, x\}$

37. $[T]_{\mathcal{E}} = \begin{bmatrix} (d_1^2 - d_2^2)/(d_1^2 + d_2^2) & 2d_1d_2/(d_1^2 + d_2^2) \\ 2d_1d_2/(d_1^2 + d_2^2) & (d_2^2 - d_1^2)/(d_1^2 + d_2^2) \end{bmatrix}$

Exercises 6.7

1. $y(t) = 2e^{3t}/e^3$

3. $y(t) = ((1 - e^4)e^{3t} + (e^3 - 1)e^{4t})/(e^3 - e^4)$

5. $f(t) = \left(\dfrac{e^{(\sqrt{5}-1)/2}}{e^{\sqrt{5}} - 1}\right)[e^{(1+\sqrt{5})t/2} - e^{(1-\sqrt{5})t/2}]$

7. $y(t) = e^t - (1 - e^{-1})te^t$

9. $y(t) = ((k + 1)e^{kt} + (k - 1)e^{-kt})/2k$

11. $y(t) = e^t\cos(2t)$

13. (a) $p(t) = 100e^{\ln(16)t/3} \approx 100e^{0.924t}$

(b) 45 minutes **(c)** In 9.968 hours

15. (a) $m(t) = 50e^{-ct}$, where $c = \ln 2/1590 \approx 4.36 \times 10^{-4}$; 32.33 mg remain after 1000 years.

(b) After 3691.9 years

17. $x(t) = \dfrac{5 - 10\cos(10\sqrt{K})}{\sin(10\sqrt{K})}\sin(\sqrt{K}t) + 10\cos(\sqrt{K}t)$

19. (b) No

Review Questions

1. (a) F **(c)** T **(e)** F **(g)** F **(i)** T

3. Subspace **5.** Subspace

7. Let $c_1A + c_2B = O$. Then $c_1A - c_2B = c_1A^T + c_2B^T = (c_1A + c_2B)^T = O$. Adding, we have $2c_1A = O$, so $c_1 = 0$ because A is nonzero. Hence $c_2B = O$, and so $c_2 = 0$. Thus, $\{A, B\}$ is linearly independent.

9. $\{1, x^2, x^4\}$, $\dim W = 3$

11. Linear transformation

13. Linear transformation **15.** $n^2 - 1$

17. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

19. $S \circ T$ is the zero transformation.

Chapter 7

Exercises 7.1

1. (a) -10 **(b)** $\sqrt{14}$ **(c)** $\sqrt{93}$

3. Any nonzero scalar multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

5. (a) 1 **(b)** $\sqrt{13}$ **(c)** $\sqrt{14}$

7. x^2 is one possibility

9. (a) π **(b)** $\sqrt{\pi}$ **(c)** $\sqrt{\pi}$

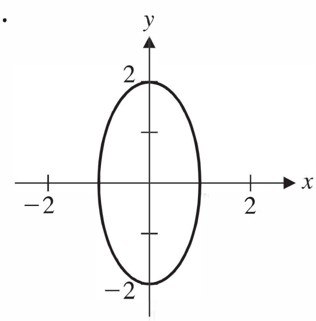
13. Axiom (4) fails: $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$, but $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

15. Axiom (4) fails: $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$, but $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

17. Axiom (4) fails: $p(x) = 1 - x$ is not the zero polynomial, but $\langle p(x), p(x) \rangle = 0$.

19. $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

21.



25. -8 **27.** $\sqrt{6}$

29. $\|\mathbf{u} + \mathbf{v} - \mathbf{w}\|^2 = \langle \mathbf{u} + \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - \mathbf{w} \rangle$
$= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle - 2\langle \mathbf{u}, \mathbf{w} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle$
$= 1 + 3 + 4 + 2 - 10 - 0 = 0$

Therefore, $\|\mathbf{u} + \mathbf{v} - \mathbf{w}\| = 0$, so, by axiom (4), $\mathbf{u} + \mathbf{v} - \mathbf{w} = \mathbf{0}$ or $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

31. $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle - \langle \mathbf{v}, \mathbf{v}\rangle = \|\mathbf{u}\|^2 - \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

33. Using Exercise 32 and a similar identity for $\|\mathbf{u} - \mathbf{v}\|^2$, we have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\quad + \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2\end{aligned}$$

Dividing by 2 yields the identity we want.

35. $$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| &\Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2 \\ &\Leftrightarrow \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\qquad = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\Leftrightarrow 2\langle \mathbf{u}, \mathbf{v}\rangle = -2\langle \mathbf{u}, \mathbf{v}\rangle \Leftrightarrow \langle \mathbf{u}, \mathbf{v}\rangle = 0\end{aligned}$$

37. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ **39.** $\{1, x, x^2\}$

41. (a) $1/\sqrt{2}, \sqrt{3}x/\sqrt{2}, \sqrt{5}(3x^2 - 1)/2\sqrt{2}$
(b) $\sqrt{7}(5x^3 - 3x)/2\sqrt{2}$

Exercises 7.2

1. $\|\mathbf{u}\|_E = \sqrt{42}$, $\|\mathbf{u}\|_s = 10$, $\|\mathbf{u}\|_m = 5$

3. $d_E(\mathbf{u}, \mathbf{v}) = \sqrt{70}$, $d_s(\mathbf{u}, \mathbf{v}) = 14$, $d_m(\mathbf{u}, \mathbf{v}) = 6$

5. $\|\mathbf{u}\|_H = 4$, $\|\mathbf{v}\|_H = 5$

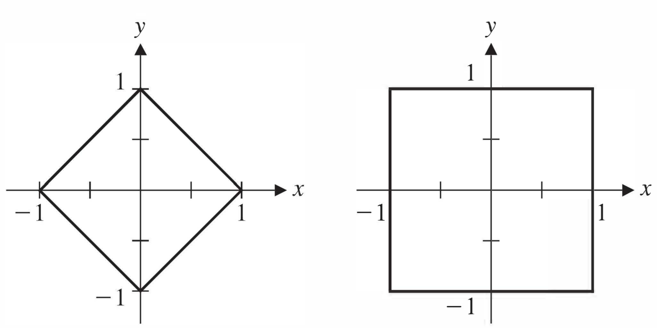
7. (a) At most one component of $\mathbf{v}$ is nonzero.

9. Suppose $\|\mathbf{v}\|_m = |v_k|$. Then $\|\mathbf{v}\|_E = \sqrt{v_1^2 + \cdots + v_k^2 + \cdots + v_n^2} \ge \sqrt{v_k^2} = |v_k| = \|\mathbf{v}\|_m$.

11. Suppose $\|\mathbf{v}\|_m = |v_k|$. Then $|v_i| \le |v_k|$ for $i = 1, \ldots, n$, so

$$\begin{aligned}\|\mathbf{v}\|_s &= |v_1| + \cdots + |v_n| \le |v_k| + \cdots + |v_k| \\ &= n|v_k| = n\|\mathbf{v}\|_m\end{aligned}$$

13.



21. $\|A\|_F = \sqrt{19}$, $\|A\|_1 = 4$, $\|A\|_\infty = 6$

23. $\|A\|_F = \sqrt{31}$, $\|A\|_1 = 6$, $\|A\|_\infty = 6$

25. $\|A\|_F = 2\sqrt{11}$, $\|A\|_1 = 7$, $\|A\|_\infty = 7$

27. $\mathbf{x} = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}-1\\1\end{bmatrix}$ **29.** $\mathbf{x} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

31. $\mathbf{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$

33. (a) By the definition of an operator norm, $\|I\| = \max\limits_{\|\mathbf{x}\|=1}\|I\mathbf{x}\| = \max\limits_{\|\mathbf{x}\|=1}\|\mathbf{x}\| = 1$.

35. $\operatorname{cond}_1(A) = \operatorname{cond}_\infty(A) = 21$; well-conditioned

37. $\operatorname{cond}_1(A) = \operatorname{cond}_\infty(A) = 400$; ill-conditioned

39. $\operatorname{cond}_1(A) = 77$, $\operatorname{cond}_\infty(A) = 128$; moderately ill-conditioned

41. (a) $\operatorname{cond}_\infty(A) = (\max\{|k| + 1, 2\}) \cdot \left(\max\left\{\left|\dfrac{k}{k-1}\right| + \left|\dfrac{1}{k-1}\right|, \left|\dfrac{2}{k-1}\right|\right\}\right)$

43. (a) $\operatorname{cond}_\infty(A) = 40$
(b) At most 400% relative change

45. Using Exercise 33(a), we have $\operatorname{cond}(A) = \|A\|\|A^{-1}\| \ge \|AA^{-1}\| = \|I\| = 1$.

49. $k \ge 6$ **51.** $k \ge 10$

Exercises 7.3

1. $\|\mathbf{e}\| = \sqrt{2} \approx 1.414$ **3.** $\|\mathbf{e}\| = \sqrt{6}/2 \approx 1.225$

5. $\|\mathbf{e}\| = \sqrt{7} \approx 2.646$

7. $y = -3 + \frac{5}{2}x$, $\|\mathbf{e}\| \approx 1.225$

9. $y = \frac{11}{3} - 2x$, $\|\mathbf{e}\| \approx 0.816$

11. $y = \frac{7}{10} + \frac{8}{25}x$, $\|\mathbf{e}\| \approx 0.447$

13. $y = -\frac{1}{5} + \frac{7}{5}x$, $\|\mathbf{e}\| \approx 0.632$

15. $y = 3 - \frac{18}{5}x + x^2$ **17.** $y = \frac{18}{5} - \frac{17}{10}x - \frac{1}{2}x^2$

19. $\overline{\mathbf{x}} = \begin{bmatrix}\frac{1}{5}\\ \frac{7}{15}\end{bmatrix}$ **21.** $\overline{\mathbf{x}} = \begin{bmatrix}\frac{4}{3}\\ -\frac{5}{6}\end{bmatrix}$

23. $\overline{\mathbf{x}} = \begin{bmatrix}4 + t\\ -5 - t\\ -5 - 2t\\ t\end{bmatrix}$ **25.** $\begin{bmatrix}\frac{42}{11}\\ \frac{19}{11}\\ \frac{42}{11}\end{bmatrix}$

27. $\bar{\mathbf{x}} = \begin{bmatrix} \frac{5}{3} \\ -2 \end{bmatrix}$

29. $y = 0.92 + 0.73x$

31. (a) If we let the year 1920 correspond to $t = 0$, then $y = 56.6 + 2.9t$; 79.9 years

33. (a) $p(t) = 150e^{0.131t}$

35. 139 days

37. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}$

39. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

41. $\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}, \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$

45. $A^+ = [\frac{1}{5} \quad \frac{2}{5}]$

47. $A^+ = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

49. $A^+ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

51. $A^+ = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

53. (a) If A is invertible, so is A^T, and we have $A^+ = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$.

Exercises 7.4

1. 2, 3 **3.** $\sqrt{2}$, 0 **5.** 5 **7.** 2, 3

9. $\sqrt{5}$, 2, 0

11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1]$

17. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$

21. $A = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1/\sqrt{2} \quad 1/\sqrt{2}] + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$[-1/\sqrt{2} \quad 1/\sqrt{2}]$ (Exercise 3)

23. (Exercise 7) $A = 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [0 \quad 1] + 2 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [1 \quad 0]$

33. The line segment $[-1, 1]$

35. The solid ellipse $\dfrac{y_1^2}{5} + \dfrac{y_2^2}{4} \le 1$

37. (a) $\|A\|_2 = \sqrt{2}$ **(b)** $\mathrm{cond}_2(A) = \infty$

39. (a) $\|A\|_2 = 1.95$ **(b)** $\mathrm{cond}_2(A) = 38.11$

41. $A^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

43. $A^+ = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{5} & 0 \end{bmatrix}$

45. $A^+ = \begin{bmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 0.52 \\ 1.04 \end{bmatrix}$

47. $A^+ = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

61. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

63. $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Exercises 7.5

1. $g(x) = \frac{1}{3}$ **3.** $g(x) = \frac{3}{5}x$

5. $g(x) = \frac{3}{16} + \frac{15}{16}x^2$ **7.** $\{1, x - \frac{1}{2}\}$

9. $g(x) = x - \frac{1}{6}$

11. $g(x) = (4e - 10) + (18 - 6e)x \approx 0.87 + 1.69x$

13. $g(x) = \frac{1}{20} - \frac{3}{5}x + \frac{3}{2}x^2$

15. $g(x) = 39e - 105 + (588 - 216e)x + (210e - 570)x^2 \approx 1.01 + 0.85x + 0.84x^2$

21. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{9}\right)$

23. $a_0 = \frac{1}{2}, a_k = 0, b_k = \dfrac{1 - (-1)^k}{k\pi}$

25. $a_0 = \pi, a_k = 0, b_k = \dfrac{2(-1)^k}{k}$

Review Questions

1. (a) T **(c)** F **(e)** T **(g)** T **(i)** T

3. Inner product **5.** $\sqrt{3}$ **7.** $\left\{\begin{bmatrix} \frac{1}{1} \\ 1 \end{bmatrix}\right\}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

9. Not a norm

11. $\text{cond}_\infty(A) \approx 2432$

13. $y = 1.7x$

15. $\begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$

17. (a) $\sqrt{2}, \sqrt{2}$

(b) $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $A^+ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$

19. The singular values of PAQ are the square roots of the eigenvalues of $(PAQ)^T(PAQ) = Q^TA^TP^TPAQ = Q^T(A^TA)Q$. But $Q^T(A^TA)Q$ is similar to A^TA because $Q^T = Q^{-1}$, and hence it has the same eigenvalues as A^TA. Thus, PAQ and A have the same singular values.

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