PHY 293 Lecture Notes

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The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

Chapter 1

Waves

1.1 Harmonic Oscillators

1.1.1 Governing Equations of Harmonic Oscillators

Types of Harmonic Oscillators There are three types of harmonic oscillators: simple, damped, and driven harmonic oscillators. Consider a simple one dimensional harmonic oscillator, they are defined by the following differential equations:

Definiton 1.1.1.1 (Simple Harmonic Oscillator)**.** A simple harmonic oscillator is described by Hooke's law:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{1.1}$$

where k is the spring constant, m is the mass, and x is the displacement from equilibrium.

Definiton 1.1.1.2 (Damped Harmonic Oscillator)**.** A damped harmonic oscillator is described by the following differential equation, by adding a damping term proportional to $\dot{x}$ to the simple harmonic oscillator equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.2}$$

where b is the damping coefficient.

Definiton 1.1.1.3 (Driven Harmonic Oscillator)**.** A driven harmonic oscillator is described by the following differential equation, which includes an external driving force $F(t)$:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t) \tag{1.3}$$

1.1.2 The Wave Equation

Definiton 1.1.2.1 (The Wave Equation)**.** The wave equation is a second-order linear partial differential equation that describes the propagation of waves, such as sound waves, light waves, and water waves, through a medium. In one dimension, it is given by the following PDE:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

where $u(x, t)$ is the wave function, c is the speed of the wave in the medium, x is the spatial coordinate, and t is time.

1.1.3 Simple Harmonic Motion

Definiton 1.1.3.1 (Simple Harmonic Motion)**.** You should have leaned the Hooke's law and Newton's second law, which gives us the equation of motion for a simple harmonic oscillator. The same with the equation (1.1), which can be rewritten as:

$$F = m\ddot{x} = -kx \tag{1.5}$$

By setting $\omega^2 = \frac{k}{m}$, ageneral solution can be written as:

$$x(t) = x_0 + A_1 \cos(\omega t) + A_2 \sin(\omega t) \tag{1.6}$$

where A are the constants determined by the IVP, ω is the angular frequency, and ϕ is the phase constant. x_0 is the equilibrium position where we generally set it to be 0. The unknown constant can be determined by knowing $x, \dot{x}$ at specific times.

Definiton 1.1.3.2 (Period, Frequency, and Angular Frequency)**.** The period T is the time it takes for one complete cycle of the motion, given by:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1.7}$$

The frequency f is the number of cycles per unit time, given by:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.8}$$

The angular frequency ω is related to the frequency by:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \tag{1.9}$$

Example 1.1.3.3. A simple harmonic oscillator consisting of mass $m = 11.0$ kg attached to a spring with spring constant $k = 201\ \mathrm{N\,m^{-1}}$. At time $t = 0$ s the oscillator is at position $x(0) = -0.207$ m and has velocity $v(0) = -1.33\ \mathrm{m\,s^{-1}}$. Determine all coefficients of the equation describing the position $x(t)$ of the oscillator as a function of time, assuming the offset is zero.

To solve for A_1 and A_2, while we assume $x_0 = 0$, we can use the initial conditions:

$$
\begin{aligned}
x(0) &= A_1 \cos(0) + A_2 \sin(0) = A_1 = -0.207 \text{ m} \\
v(0) &= -A_1\omega \sin(0) + A_2\omega \cos(0) = A_2\omega = -1.33 \text{ m s}^{-1}
\end{aligned}
$$

We can find ω from the given m and k:

$$
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{201 \text{ N m}^{-1}}{11.0 \text{ kg}}} \approx 4.28 \text{ rad s}^{-1}
$$

Therefore, we can solve for A_2:

$$
A_2 = \frac{v(0)}{\omega} = \frac{-1.33 \text{ m s}^{-1}}{4.28 \text{ rad s}^{-1}} \approx -0.311 \text{ m}
$$

Thus, the equation describing the position $x(t)$ of the oscillator as a function of time is:

$$
x(t) = -0.207 \cos(4.28t) - 0.311 \sin(4.28t)
$$

Theorem 1.1.3.4 (A Trigonometric Identity)**.** We can also express the solution in a more compact form using a single cosine function with a phase shift:

$$
x(t) = A\cos(\omega t + \phi) \tag{1.10}
$$

where

$$
A = \sqrt{A_1^2 + A_2^2}, \tag{1.11a}
$$

$$
\phi = \arctan\left(\frac{-A_2}{A_1}\right) = \arctan\left(\frac{-v(0)/\omega}{x(0)}\right). \tag{1.11b}
$$

Proof. Let $A = \sqrt{A_1^2 + A_2^2}$ and choose ϕ such that

$$
\cos(\phi) = \frac{A_1}{A}, \qquad \sin(\phi) = -\frac{A_2}{A}.
$$

Then, we can rewrite our original solution as

$$
\begin{aligned}
x(t) &= A_1 \cos(\omega t) + A_2 \sin(\omega t) \\
&= A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t) \\
&= A\big[\cos(\phi)\cos(\omega t) - \sin(\phi)\sin(\omega t)\big] \\
&= A\cos(\omega t + \phi),
\end{aligned}
$$

by the cosine addition formula. □

Example 1.1.3.5. To determine the amplitude A and phase constant ϕ for the oscillator in the previous example, we can use the values of A_1 and A_2 we found:

$$
\begin{aligned}
A &= \sqrt{(-0.207)^2 + (-0.311)^2} \approx 0.374 \text{ m} \\
\phi &= \arctan\left(\frac{-(-0.311)}{-0.207}\right) \approx 4.12 \text{ rad} \quad (\text{since } A_1 < 0 \text{ and } A_2 < 0)
\end{aligned}
$$

Therefore, the equation describing the position $x(t)$ of the oscillator as a function of time can also be written as:

$$
x(t) = 0.374 \cos(4.28t + 4.12)
$$

Definiton 1.1.3.6 (The Energy of a Simple Harmonic Oscillator)**.** The total mechanical energy E of a simple harmonic oscillator is the sum of its kinetic energy K and potential energy U.

$$E = K + U \tag{1.12}$$

First we consider the change of potential energy from a position x_i to x_f, assuming the path is along the spring or the curve C of the oscillator. The force exerted by the spring is given by Hooke's law, $F = -kx$. The change in potential energy can be simply parametized and calculated as follows:

$$\Delta U = \int_C F \cdot ds = -\int_{x_i}^{x_f} F\,dx = \int_{x_i}^{x_f} kx\,dx = \left[\frac{1}{2}kx^2\right]_{x_i}^{x_f} = \frac{1}{2}k(x_f^2 - x_i^2) \tag{1.13}$$

Therefore, the potential energy U at a position x (taking the reference point at $x = 0$) is given by:

$$U(x) = \frac{1}{2}kx^2 \tag{1.14}$$

The kinetic energy K of the oscillator is given by:

$$K = \frac{1}{2}m\dot{x}^2 \tag{1.15}$$

Therefore, the total mechanical energy E of the simple harmonic oscillator is:

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{1.16}$$

The total mechanical energy E remains constant over time, as energy is conserved in the absence of non-conservative forces (like friction or air resistance).

1.1.4 Damped Harmonic Motion

Definiton 1.1.4.1 (Damped Harmonic Motion)**.** For small velocities, the drag force is approximately proportional to the velocity and acts in the opposite direction. This drag force can be modeled as $F_d = -\gamma\dot{x}$, where γ is the damping coefficient. Including this drag force in the equation of motion for a harmonic oscillator leads to the damped harmonic oscillator equation (1.2). Which could be rewritten as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{1.17}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the undamped oscillator, and $\gamma = \frac{b}{m}$ is the damping coefficient per unit mass.

To skip the math, lets assume a solution of the form $x(t) = e^{i\omega t}$, substituting into the differential equation gives us a formulation for ω:

$$\begin{aligned} -\omega^2 - i\gamma\omega + \omega_0^2 &= 0 \\ \omega &= -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \end{aligned} \tag{1.18}$$

Also, we can characterize the real and imaginary parts of ω as:

$$\omega_r = \mathrm{Re}(\omega) = \pm\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{1.19a}$$

$$\omega_i = \mathrm{Im}(\omega) = -\frac{\gamma}{2} \tag{1.19b}$$

The general solution for the damped harmonic oscillator can be written as:

$$x(t) = \exp(\omega_i t)\exp(-i\omega_r t) = \exp\left(-\frac{\gamma}{2}t\right)\exp(\mp i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t) \tag{1.20}$$

- **No Damping** ($\gamma = 0$): The system behaves like a simple harmonic oscillator with angular frequency ω_0. Given by:
$$z = \exp(-i\omega_0 t) \tag{1.21a}$$

- **Underdamping** ($0 < \gamma < 2\omega_0$): The system oscillates with a gradually decreasing amplitude. The angular frequency of oscillation is given by $\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. Given by:
$$z = \exp\left(-\frac{\gamma}{2}t\right)\exp(-i\omega_r t) \tag{1.21b}$$
The trigonometric form of the solution is:
$$x(t) = A_0\exp\left(-\frac{\gamma}{2}t\right)\cos(\omega_r t + \phi) \tag{1.21c}$$
where A_0 and ϕ are constants determined by the initial conditions From this, we can derive the following cases:

- **Critical Damping** ($\gamma = 2\omega_0$): The system returns to equilibrium as quickly as possible without oscillating. Consider:
$$x(t) = e^{-\frac{\gamma}{2}t}f(t)$$
Inserting into the differential equation, we get:
$$\ddot{f} + \left(\omega_0^2 - \frac{\gamma^2}{4}\right)f = 0$$
Since $\gamma = 2\omega_0$, we have $\omega_0^2 - \frac{\gamma^2}{4} = 0$, leading to:
$$\ddot{f} = 0 \implies f(t) = A_1 t + A_2$$
Therefore, the general solution for the critically damped case is:
$$x(t) = (A_1 t + A_2)\exp\left(-\frac{\gamma}{2}t\right) \tag{1.21d}$$
where A_1 and A_2 are constants determined by

- **Overdamping** ($\gamma > 2\omega_0$): The system returns to equilibrium without oscillating, but more slowly than in the critically damped case. The solution is given by:

$$z = \exp\left(-\frac{\gamma}{2}t\right)\exp\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2 t}\right) \tag{1.21e}$$

So the general solution is (the solution is via a substitution of $x(t) = e^{-\gamma t/2}f(t)$ into the differential equation, which resolves the ODE to a simple form):

$$x(t) = A_1 \exp\left[\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] + A_2 \exp\left[\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] \tag{1.21f}$$

where A_1 and A_2 are constants determined by the initial conditions.

1.1.5 Energy and Quality Factor

Definiton 1.1.5.1 (Energy of a Very Light Damping)**.** Consider a very lightly damped harmonic oscillator, where $\gamma \ll \omega_0$. In this case, the angular frequency of oscillation ω_r can be approximated as:

$$\omega_r \approx \omega_0\left(1 - \frac{\gamma^2}{8\omega_0^2}\right) \approx \omega_0$$

So the motion of the lightly damped oscillator can be approximated as:

$$x(t) \approx A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_0 t + \phi)$$

Then, we can calculate the velocity of the oscillator:

$$\begin{aligned}\dot{x}(t) &= -\frac{\gamma}{2}A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_0 t + \phi) - A_0\omega_0 e^{-\frac{\gamma}{2}t}\sin(\omega_0 t + \phi)\\ &= A_0\omega_0 e^{-\frac{\gamma}{2}t}\left(-\frac{\gamma}{2\omega_0}\cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi)\right)\end{aligned}$$

The total mechanical energy $E(t)$ of the lightly damped oscillator is given by:

$$\begin{aligned}E(t) &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\\ &= \frac{1}{2}m\left[A_0\omega_0 e^{-\frac{\gamma}{2}t}\left(-\frac{\gamma}{2\omega_0}\cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi)\right)\right]^2 + \frac{1}{2}k\left[A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_0 t + \phi)\right]^2\\ &= \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t}\left[\left(-\frac{\gamma}{2\omega_0}\cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi)\right)^2 + \cos^2(\omega_0 t + \phi)\right]\\ &= \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t}\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi) + \frac{\gamma^2}{4\omega_0^2}\cos^2(\omega_0 t + \phi) + \frac{\gamma}{\omega_0}\sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi)\right]\\ &\approx \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t}\left[1 + \frac{\gamma^2}{4\omega_0^2}\cos^2(\omega_0 t + \phi)\right] \quad \text{(neglecting the small term } \frac{\gamma}{\omega_0}\sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi))\\ &\approx \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t} \quad \text{(since } \frac{\gamma^2}{4\omega_0^2} \text{ is very small)}\\ &= E_0 e^{-\gamma t} \quad \text{where } E_0 = \frac{1}{2}mA_0^2\omega_0^2 \text{ is the initial energy at } t = 0\end{aligned}$$

We can also define the time constant τ as the time it takes for the energy to decrease to $\frac{1}{e}$ of its initial value:

$$\tau = \frac{1}{\gamma} \tag{1.22}$$

So we have, for very light damping:

$$E(t) = E_0 e^{-\gamma t} = E_0 e^{-\frac{t}{\tau}} \tag{1.23}$$

Definiton 1.1.5.2 (Rate of Energy Loss)**.** Taking the time derivative of the total mechanical energy $E(t)$:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) \\ &= (ma + kx)\dot{x} \end{aligned}$$

For a undamped harmonic oscillator, $ma + kx = 0$, so $\frac{dE}{dt} = 0$, indicating that the total mechanical energy is conserved and obeys Hooke's law completely. However, for a damped harmonic oscillator, $ma + kx = -b\dot{x}$, leading to:

$$\frac{dE}{dt} = -b\dot{x}^2 \tag{1.24}$$

Definiton 1.1.5.3 (Quality Factor (Q-Factor))**.** The quality factor Q is a dimensionless parameter that characterizes the damping of a harmonic oscillator. It is defined as:

$$Q = \frac{\omega}{\gamma} = \omega\tau \tag{1.25a}$$

And for very light damping, we can approximate $\omega \approx \omega_0$, leading to:

$$Q \approx \frac{\omega_0}{\gamma} = \omega_0\tau \tag{1.25b}$$

This allows us to rewrite the equation of a damped harmonic oscillator as:

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0 \tag{1.26}$$

and:

$$\omega = \omega_0\sqrt{1 - \frac{1}{4Q^2}} \tag{1.27}$$

We can also consider the ratio between the energy at one time and the energy one period later:

$$\begin{aligned} \frac{E(t+T)}{E(t)} &= \frac{E_0 e^{-\gamma(t+T)}}{E_0 e^{-\gamma t}} \\ &= e^{-\gamma T} \approx 1 - \gamma T \\ \frac{E(t+T) - E(t)}{E(t)} &\approx -\gamma T = -\frac{2\pi}{\omega_0}\gamma = -\frac{2\pi}{Q} \end{aligned}$$

Example 1.1.5.4. *What is the number of radians through which the damped system oscillates as its energy decreases to $1/e$ of its initial value?*

We have
$$\frac{E}{E_0} = e^{-\gamma t} = \frac{1}{e} \implies \gamma t = 1 \implies t = \frac{1}{\gamma} = \tau$$
So the number of radians is
$$\theta = \omega\tau = \frac{\omega}{\gamma} = Q$$

1.1.6 Undamped Forced Oscillations

Definiton 1.1.6.1 (Undamped Forced Oscillations)**.** Consider a driver porves on the left side of the sping, we have the following differential equation:
$$m\ddot{x} + kx = \eta \tag{1.28}$$
where $\eta = F_0 \cos(\omega t)$ is the driving force with amplitude F_0 and angular frequency ω. The general solution to this non-homogeneous differential equation given by:
$$x(t) = A(\omega)\cos(\omega t - \delta) \tag{1.29}$$
to derive $A(\omega)$ and δ, we can use the equation of motion:
$$A(\omega)(-\omega^2 + \omega_0^2)\cos(\delta) = \omega_0^2 a$$
where $a = \frac{F_0}{k}$ is the static displacement of the mass when the driving force is constant. We also haev:
$$A(\omega)(-\omega^2 + \omega_0^2)\sin(\delta) = 0$$
Now, consider the case when $\delta = 0$, we have:
$$A(\omega) = \frac{\omega_0^2 a}{\omega_0^2 - \omega^2} \tag{1.30}$$
and the case where $\delta = \pi$, we have:
$$A(\omega) = -\frac{\omega_0^2 a}{\omega_0^2 - \omega^2} \tag{1.31}$$
Observe, when $\omega \approx \omega_0$, the amplitude $A(\omega)$ becomes very large, indicating resonance. At resonance, the system oscillates with maximum amplitude, which can lead to significant energy transfer from the driving force to the oscillator.

1.1.7 Damped Forced Oscillations

Definiton 1.1.7.1 (Damped Forced Oscillations)**.** Consider a damped harmonic oscillator subjected to an external driving force. It is described by the following differential equation:
$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega t) \tag{1.32}$$

where F_0 is the amplitude of the driving force, ω is the angular frequency of the driving force, b is the damping coefficient, m is the mass, and k is the spring constant. The general solution to this non-homogeneous differential equation is given by:

$$x(t) = A(\omega)\cos(\omega t - \delta) \tag{1.33}$$

and the amplitude $A(\omega)$ is given by:

$$A(\omega) = \frac{\omega_0^2 a}{\sqrt{\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2}} \tag{1.34}$$

We can derive the following 4 cases:

1. **No Damping** ($\gamma = 0$): In this case, the amplitude $A(\omega)$ simplifies to the undamped case we discussed earlier.

2. **Low-Frequency Limit** ($\omega \ll \omega_0$): In this limit, the amplitude $A(\omega)$ approaches the static displacement $a = \frac{F_0}{k}$. This means that at very low frequencies, the system behaves like a static spring, and the mass is displaced by an amount proportional to the applied force.

3. **High-Frequency Limit** ($\omega \gg \omega_0$): In this limit, the amplitude $A(\omega)$ decreases with increasing frequency, following the relation $A(\omega) \approx \frac{\omega_0^2 a}{\omega^2}$. This indicates that at very high frequencies, the mass cannot respond quickly enough to the rapidly oscillating driving force, resulting in a smaller amplitude of oscillation.

4. **Resonance** ($\omega \approx \omega_0$): At resonance, the amplitude $A(\omega)$ reaches its maximum value but it does not become infinite due to the presence of damping:

$$A_{\max} = \frac{\omega_0^2 a}{\omega_0\gamma} = \frac{\omega_0 a}{\gamma} = Qa \tag{1.35}$$

We also have the phase shift δ given by:

$$\tan(\delta) = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{1.36}$$

The phase shift δ indicates how much the oscillation of the mass lags behind the driving force. The behavior of δ can be summarized as follows:

- At low frequencies ($\omega \ll \omega_0$), δ approaches 0, meaning the mass oscillates in phase with the driving force.
- At resonance ($\omega = \omega_0$), $\delta = \frac{\pi}{2}$, indicating that the mass oscillates a quarter cycle behind the driving force.
- At high frequencies ($\omega \gg \omega_0$), δ approaches π, meaning the mass oscillates out of phase with the driving force.

Chapter 2

Modern Physics