PHY 293 Lecture Notes

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The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

Chapter 1

Waves

1.1 Harmonic Oscillators

1.1.1 Governing Equations of Harmonic Oscillators

Types of Harmonic Oscillators There are three types of harmonic oscillators: simple, damped, and driven harmonic oscillators. Consider a simple one dimensional harmonic oscillator, they are defined by the following differential equations:

Definiton 1.1.1.1 (Simple Harmonic Oscillator)**.** A simple harmonic oscillator is described by Hooke's law:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{1.1}$$

where k is the spring constant, m is the mass, and x is the displacement from equilibrium.

Definiton 1.1.1.2 (Damped Harmonic Oscillator)**.** A damped harmonic oscillator is described by the following differential equation, by adding a damping term proportional to $\dot{x}$ to the simple harmonic oscillator equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.2}$$

where b is the damping coefficient.

Definiton 1.1.1.3 (Driven Harmonic Oscillator)**.** A driven harmonic oscillator is described by the following differential equation, which includes an external driving force $F(t)$:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t) \tag{1.3}$$

1.1.2 The Wave Equation

Definiton 1.1.2.1 (The Wave Equation)**.** The wave equation is a second-order linear partial differential equation that describes the propagation of waves, such as sound waves, light waves, and water waves, through a medium. In one dimension, it is given by the following PDE:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

where $u(x, t)$ is the wave function, c is the speed of the wave in the medium, x is the spatial coordinate, and t is time.

1.1.3 Simple Harmonic Motion

Definiton 1.1.3.1 (Simple Harmonic Motion)**.** You should have leaned the Hooke's law and Newton's second law, which gives us the equation of motion for a simple harmonic oscillator. The same with the equation (1.1), which can be rewritten as:

$$F = m\ddot{x} = -kx \tag{1.5}$$

By setting $\omega^2 = \frac{k}{m}$, ageneral solution can be written as:

$$x(t) = x_0 + A_1 \cos(\omega t) + A_2 \sin(\omega t) \tag{1.6}$$

where A are the constants determined by the IVP, ω is the angular frequency, and ϕ is the phase constant. x_0 is the equilibrium position where we generally set it to be 0. The unknown constant can be determined by knowing $x, \dot{x}$ at specific times.

Definiton 1.1.3.2 (Period, Frequency, and Angular Frequency)**.** The period T is the time it takes for one complete cycle of the motion, given by:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1.7}$$

The frequency f is the number of cycles per unit time, given by:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.8}$$

The angular frequency ω is related to the frequency by:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \tag{1.9}$$

Example 1.1.3.3. A simple harmonic oscillator consisting of mass $m = 11.0$ kg attached to a spring with spring constant $k = 201\ \mathrm{N\,m^{-1}}$. At time $t = 0$ s the oscillator is at position $x(0) = -0.207$ m and has velocity $v(0) = -1.33\ \mathrm{m\,s^{-1}}$. Determine all coefficients of the equation describing the position $x(t)$ of the oscillator as a function of time, assuming the offset is zero.

To solve for A_1 and A_2, while we assume $x_0 = 0$, we can use the initial conditions:

$$\begin{aligned} x(0) &= A_1 \cos(0) + A_2 \sin(0) = A_1 = -0.207 \text{ m} \\ v(0) &= -A_1\omega \sin(0) + A_2\omega \cos(0) = A_2\omega = -1.33 \text{ m}\,\text{s}^{-1} \end{aligned}$$

We can find ω from the given m and k:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{201 \text{ N}\,\text{m}^{-1}}{11.0 \text{ kg}}} \approx 4.28 \text{ rad}\,\text{s}^{-1}$$

Therefore, we can solve for A_2:

$$A_2 = \frac{v(0)}{\omega} = \frac{-1.33 \text{ m}\,\text{s}^{-1}}{4.28 \text{ rad}\,\text{s}^{-1}} \approx -0.311 \text{ m}$$

Thus, the equation describing the position $x(t)$ of the oscillator as a function of time is:

$$x(t) = -0.207\cos(4.28t) - 0.311\sin(4.28t)$$

Theorem 1.1.3.4 (A Trigonometric Identity)**.** We can also express the solution in a more compact form using a single cosine function with a phase shift:

$$x(t) = A\cos(\omega t + \phi) \tag{1.10}$$

where

$$A = \sqrt{A_1^2 + A_2^2}, \tag{1.11a}$$

$$\phi = \arctan\left(\frac{-A_2}{A_1}\right) = \arctan\left(\frac{-v(0)/\omega}{x(0)}\right). \tag{1.11b}$$

Proof. Let $A = \sqrt{A_1^2 + A_2^2}$ and choose ϕ such that

$$\cos(\phi) = \frac{A_1}{A}, \qquad \sin(\phi) = -\frac{A_2}{A}.$$

Then, we can rewrite our original solution as

$$\begin{aligned} x(t) &= A_1\cos(\omega t) - A_2\sin(\omega t) \\ &= A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t) \\ &= A\big[\cos(\phi)\cos(\omega t) - \sin(\phi)\sin(\omega t)\big] \\ &= A\cos(\omega t + \phi), \end{aligned}$$

by the cosine addition formula. □

Example 1.1.3.5. To determine the amplitude A and phase constant ϕ for the oscillator in the previous example, we can use the values of A_1 and A_2 we found:

$$\begin{aligned} A &= \sqrt{(-0.207)^2 + (-0.311)^2} \approx 0.374 \text{ m} \\ \phi &= \arctan\left(\frac{-(-0.311)}{-0.207}\right) \approx 4.12 \text{ rad} \quad (\text{since } A_1 < 0 \text{ and } A_2 < 0) \end{aligned}$$

Therefore, the equation describing the position $x(t)$ of the oscillator as a function of time can also be written as:

$$x(t) = 0.374\cos(4.28t + 4.12)$$

Definiton 1.1.3.6 (The Energy of a Simple Harmonic Oscillator)**.** The total mechanical energy E of a simple harmonic oscillator is the sum of its kinetic energy K and potential energy U.

$$E = K + U \tag{1.12}$$

First we consider the change of potential energy from a position x_i to x_f, assuming the path is along the spring or the curve C of the oscillator. The force exerted by the spring is given by Hooke's law, $F = -kx$. The change in potential energy can be simply parametized and calculated as follows:

$$\Delta U = \int_C F \cdot ds = -\int_{x_i}^{x_f} F\,dx = \int_{x_i}^{x_f} kx\,dx = \left[\frac{1}{2}kx^2\right]_{x_i}^{x_f} = \frac{1}{2}k(x_f^2 - x_i^2) \tag{1.13}$$

Therefore, the potential energy U at a position x (taking the reference point at $x = 0$) is given by:

$$U(x) = \frac{1}{2}kx^2 \tag{1.14}$$

The kinetic energy K of the oscillator is given by:

$$K = \frac{1}{2}m\dot{x}^2 \tag{1.15}$$

Therefore, the total mechanical energy E of the simple harmonic oscillator is:

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{1.16}$$

The total mechanical energy E remains constant over time, as energy is conserved in the absence of non-conservative forces (like friction or air resistance).

1.1.4 Damped Harmonic Motion

Definiton 1.1.4.1 (Damped Harmonic Motion)**.** For small velocities, the drag force is approximately proportional to the velocity and acts in the opposite direction. This drag force can be modeled as $F_d = -\gamma\dot{x}$, where γ is the damping coefficient. Including this drag force in the equation of motion for a harmonic oscillator leads to the damped harmonic oscillator equation (1.2). Which could be rewritten as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{1.17}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the undamped oscillator, and $\gamma = \frac{b}{m}$ is the damping coefficient per unit mass.

To skip the math, lets assume a solution of the form $x(t) = e^{i\omega t}$, substituting into the differential equation gives us a formulation for ω:

$$-\omega^2 - i\gamma\omega + \omega_0^2 = 0$$

$$\omega = -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{1.18}$$

Also, we can characterize the real and imaginary parts of ω as:

$$\omega_r = \text{Re}(\omega) = \pm\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{1.19a}$$

$$\omega_i = \text{Im}(\omega) = -\frac{\gamma}{2} \tag{1.19b}$$

The general solution for the damped harmonic oscillator can be written as:

$$x(t) = \exp(\omega_i t)\exp(-i\omega_r t) = \exp\left(-\frac{\gamma}{2}t\right)\exp(\mp i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t) \tag{1.20}$$

- **No Damping** ($\gamma = 0$): The system behaves like a simple harmonic oscillator with angular frequency ω_0. Given by:

$$z = \exp(-i\omega_0 t) \tag{1.21a}$$

- **Underdamping** ($0 < \gamma < 2\omega_0$): The system oscillates with a gradually decreasing amplitude. The angular frequency of oscillation is given by $\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. Given by:

$$z = \exp\left(-\frac{\gamma}{2}t\right)\exp(-i\omega_r t) \tag{1.21b}$$

  The trigonometric form of the solution is:

$$x(t) = A_0\exp\left(-\frac{\gamma}{2}t\right)\cos(\omega_r t + \phi) \tag{1.21c}$$

  where A_0 and ϕ are constants determined by the initial conditions From this, we can derive the following cases:

- **Critical Damping** ($\gamma = 2\omega_0$): The system returns to equilibrium as quickly as possible without oscillating. The solution is given by:

$$z = \exp(-\omega_0 t) \tag{1.21d}$$

- **Overdamping** ($\gamma > 2\omega_0$): The system returns to equilibrium without oscillating, but more slowly than in the critically damped case. The solution is given by:

$$z = \exp\left(-\frac{\gamma}{2}t\right)\exp\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2 t}\right) \tag{1.21e}$$

  So the general solution is (the solution is via a substitution of $x(t) = e^{-\gamma t/2}f(t)$ into the differential equation, which resolves the ODE to a simple form):

$$x(t) = A_1\exp\left[\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] + A_2\exp\left[\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] \tag{1.21f}$$

  where A_1 and A_2 are constants determined by the initial conditions.

Chapter 2

Modern Physics