AER 210 Lecture Notes

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Vector Calculus & Fluid Mechanics, Fall 2025 AER210

The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

Chapter 1

Vector Calculus

Note the section numbering is based on Stewart's book.

1.15 Double and Triple Integrals

Definiton 1.15.0.1 (Double Integral)**.** Let $f(x, y)$ be a function defined on a closed and bounded region R in the xy-plane. The double integral of f over R is denoted by

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dA \tag{1.1}$$

where dA represents an infinitesimal area element in the region R. The double integral can be interpreted as the volume under the surface defined by $z = f(x, y)$ over the region R.

1.15.1 Double Integrals in a Rectangular Region

By the point of seeing this note, you should be familiar with the simple case of rectangular, simple cases are provided as examples:

Example 1.15.1.1. Find the area under the quadric surface $z = 16 - x^2 - y^2$ over the square region $R = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 2\}$.

Note We would have to ensure that the surface is above the xy-plane in the region of interest, which is true in this case.

We can set up the double integral as follows:

$$\iint_R (16 - x^2 - y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - y^2)\, dy\, dx$$

First, we integrate with respect to y:

$$\int_0^2 (16 - x^2 - y^2)\, dy = \left[16y - x^2 y - \frac{y^3}{3}\right]_0^2 = 32 - 2x^2 - \frac{8}{3} = \frac{88}{3} - 2x^2$$

Next, we integrate with respect to x:

$$\int_0^2 \left(\frac{88}{3} - 2x^2\right)\, dx = \left[\frac{88}{3}x - \frac{2x^3}{3}\right]_0^2 = \frac{176}{3} - \frac{16}{3} = \frac{160}{3}$$

Therefore, the area under the surface is $\frac{160}{3}$.

Example 1.15.1.2. Evaluate the double integral of $f(x, y) = x - 3y^2$ over the rectangular region $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

We can set up the double integral as follows:

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx$$

First, we integrate with respect to y:

$$\int_1^2 (x - 3y^2)\, dy = \left[xy - y^3\right]_1^2 = 2x - 8 - (x - 1) = x - 7$$

Next, we integrate with respect to x:

$$\int_0^2 (x - 7)\, dx = \left[\frac{x^2}{2} - 7x\right]_0^2 = (2 - 14) - 0 = -12$$

Therefore, the value of the double integral is -12.

Theorem 1.15.1.3. If the integrand function $f(x, y)$ is seperable, i.e., $f(x, y) = g(x)h(y)$, then the double integral can be computed as follows:

$$\iint_R f(x, y)\, dA = \left(\int_a^b g(x)\, dx\right) \left(\int_c^d h(y)\, dy\right) \tag{1.2}$$

where $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$.

Proof. **Sketch:** $h(y)$ is a constant when integrating with respect to x, and vice versa. □

Example 1.15.1.4. Let $f(x, y) = \sin x \cos y$ and $R = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}\}$. Evaluate the double integral $\iint_R f(x, y)\, dA$.

Since $f(x, y)$ is seperable, we can write:

$$\iint_R f(x, y)\, dA = \left(\int_0^{\frac{\pi}{2}} \sin x\, dx\right) \left(\int_0^{\frac{\pi}{2}} \cos y\, dy\right)$$

Evaluating each integral separately gives 1 for both, so the final result is: $1 \times 1 = 1$.

1.15.2 Double Integrals in General Regions

Types of Regions When the region R is not rectangular, we can still compute the double integral by expressing the region in terms of inequalities. There are three common types of regions:

Definiton 1.15.2.1 (Type I Region)**.** A region R is called a Type I region if it can be described by the inequalities:

$$a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x)$$

where $g_1(x)$ and $g_2(x)$ are continuous functions on the interval $[a, b]$. Then, to evaluate the double integral over a Type I region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \tag{1.3}$$

Integral Order: Integrate with respect to y first, then x.

Intuition: As we traverse the outer part (x), we are summing up vertical slices (in y), and the bounds of those slices depend on x and changes.

Definiton 1.15.2.2 (Type II Region)**.** Type II reigion is similar to Type I, but the roles of x and y are swapped. A region R is called a Type II region if it can be described by the inequalities:

$$c \leq y \leq d, \quad h_1(y) \leq x \leq h_2(y)$$

where $h_1(y)$ and $h_2(y)$ are continuous functions on the interval $[c, d]$. Then, to evaluate the double integral over a Type II region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \tag{1.4}$$

The integral order and intuition is mirrored from Type I, but we are summing up horizontal slices (in x), and the bounds of those slices depend on y and changes.

Definiton 1.15.2.3 (Type III Region)**.** A region R is called a Type III region if it can be described as the union of a finite number of Type I and Type II regions. To evaluate the double integral over a Type III region for a continuous function $f(x, y)$, we can break down the integral into separate integrals over each Type I or Type II subregion and sum them up:

$$\iint_R f(x,y)\,dA = \sum_{i=1}^{n} \iint_{R_i} f(x,y)\,dA \tag{1.5}$$

where each R_i is either a Type I or Type II region. And that:

$$\bigcup_{i=1}^{n} R_i = R \quad \text{and} \quad R_i \cap R_j = \emptyset \text{ for } i \neq j$$

This approach allows us to handle more complex regions by breaking them down into simpler parts.

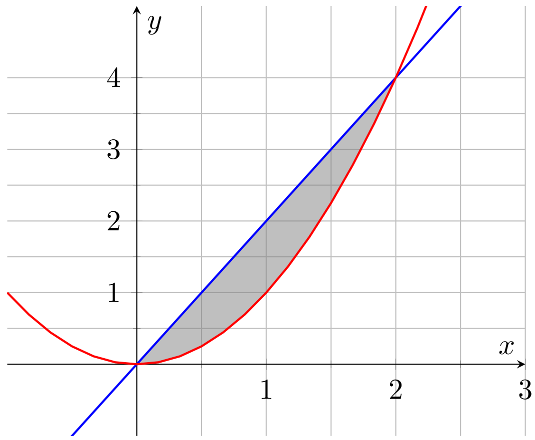

Figure 1.1: Region bounded by $y = 2x$ and $y = x^2$

Example 1.15.2.4. Find the volume of the solid that lies under the paraboloid $z = f(x, y) = x^2 + y^2$ and above the region R bounded by $y = 2x$ and $y = x^2$.

First, you would sketch the region to understand its shape and boundaries at Figure 1.1.

We can tell that this is a Type I regionwhere $0 \leq x \leq 2$, and $x^2 \leq y \leq 2x$. Thus, we can set up the double integral as follows:

$$\iint_R (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

First, we integrate with respect to y:

$$\int_{x^2}^{2x} (x^2 + y^2)\, dy = \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} = 2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3} = \frac{14x^3}{3} - x^4 - \frac{x^6}{3}$$

Next, we integrate with respect to x:

$$\int_0^2 \left(\frac{14x^3}{3} - x^4 - \frac{x^6}{3} \right) dx = \left[\frac{14x^4}{12} - \frac{x^5}{5} - \frac{x^7}{21} \right]_0^2 = \frac{216}{35}$$

Therefore, the volume of the solid is $\frac{216}{35}$.

Example 1.15.2.5. Consider the above example, but we want to set it up as a Type II region. The region R can be described by $0 \leq y \leq 4$, and $\frac{y}{2} \leq x \leq \sqrt{y}$. Thus, we can set up the double integral as follows:

$$\iint_R (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

First, we integrate with respect to x:

$$\int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx = \left[\frac{x^3}{3} + y^2 x \right]_{x=\frac{y}{2}}^{x=\sqrt{y}} = \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2} = \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{13y^3}{24}$$

Next, we integrate with respect to y:

$$\int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{13y^3}{24}\right) dy = \left[\frac{2y^{\frac{5}{2}}}{15} + \frac{2y^{\frac{7}{2}}}{7} - \frac{13y^4}{96}\right]_0^4 = \frac{216}{35}$$

Therefore, the volume of the solid is $\frac{216}{35}$, which is consistent with the previous result. This is also consistent with Fubini's Theorem.

Example 1.15.2.6. Integrate the surface given by $z = e^{x^2}$ over the triangular region with vertices at $(0,0)$, $(1,0)$, and $(1,1)$. We can describe the region as either a Type I or Type II region:

(✗) Here, we will describe it as a Type II regionwhere $0 \leq y \leq 1$, and $y \leq x \leq 1$. Thus, we can set up the double integral as follows:

$$\iint_R e^{x^2}\,dA = \int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$

We can tell that e^{x^2} does not have an elementary antiderivative, so we cannot integrate with respect to x directly.

(✓) However, we can change the order of integration to make it a Type I regionwhere $0 \leq x \leq 1$, and $0 \leq y \leq x$. Thus, we can set up the double integral as follows:

$$\iint_R e^{x^2}\,dA = \int_0^1 \int_0^x e^{x^2}\,dy\,dx$$

First, we integrate with respect to y:

$$\int_0^x e^{x^2}\,dy = \left[ye^{x^2}\right]_0^x = xe^{x^2}$$

Next, we integrate with respect to x:

$$\int_0^1 xe^{x^2}\,dx$$

This is now obvious, a simple u-substitution with $u = x^2$, $du = 2x\,dx$:

$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\left[e^u\right]_0^1 = \frac{e-1}{2}$$

Therefore, the value of the double integral is $\frac{e-1}{2}$.

Intuition When the integrand is difficult to integrate with respect to one variable, consider changing the order of integration. You should be able to tell that e^{x^2} has no elementary antiderivative, so you would have ruled out integrating with respect to x first.

Formal Definition of Double Integrals

There is two definitions of double integrals in this course, due to the discrepancy between Stewart's book and the lectures.

Review. Formal Definition of Definite Integral (Single Variable)

Consider $y = f(x) \geq 0$ on the interval $x \in [a, b]$. We divide the interval into n subintervals of equal width $\Delta x = \frac{b-a}{n}$, and let x_i^* be a sample point in the i-th subinterval. The Riemann sum is given by:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Now, for any x_i^*, we consider the minimum and maximum values of $f(x_i^*)$ in the i-th subinterval, denoted as m_i and M_i respectively. We can then define the lower sum L_n and upper sum U_n as follows:

$$L_n = \sum_{i=1}^{n} m_i \Delta x \quad \text{and} \quad U_n = \sum_{i=1}^{n} M_i \Delta x$$

To satisfy the squeeze theorem, for all i, we would need:

$$\lim_{n\to\infty} M_i - m_i = \lim_{\delta x \to 0} M_i - m_i = 0$$

If $f(x)$ is continuous on $[a, b]$. Then, we have:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \int_a^b f(x)\, dx$$

For the case of discontinuous functions, if the set of discontinuities has measure zero, then the function is still integrable.

Definiton 1.15.2.7 (Definition of Double Integral)**.** Let R be a rectangular region in the xy-plane given by $R = [a, b] \times [c, d]$. The double integral of a function $f(x, y)$ over the region R is defined as:

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i \quad \text{(Riemann Definition)} \tag{1.6a}$$

where ΔA_i is the area of the i-th subrectangle, and (x_i^*, y_i^*) is a sample point in it. The limit is taken as the maximum diameter of the subrectangles approaches zero.

$$\iint_R f(x, y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta A_{ij} \quad \text{(Grid Formulation)} \tag{1.6b}$$

where ΔA_{ij} is the area of the ij-th subrectangle, and (x_i^*, y_j^*) is a sample point in it. Note that the ΔA_{ij} may be non-uniform. The limit is taken as the maximum diameter of the subrectangles approaches zero.

Similarly, the lower and upper sums for double integrals are:

$$L_n = \sum_{i=1}^{n} m_i \Delta A_i \quad \text{and} \quad U_n = \sum_{i=1}^{n} M_i \Delta A_i \quad \text{(Riemann Definition)} \tag{1.7a}$$

$$L_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\Delta A_{ij} \quad \text{and} \quad U_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\Delta A_{ij} \quad \text{(Grid Formulation)} \tag{1.7b}$$

Here, m_{ij} and M_{ij} are the minimum and maximum values of $f(x, y)$ in the ij-th subrectangle. Define $||P|| = \max ||(\Delta x_i, \Delta y_j)||$ as the maximum diameter of the subrectangles. For the squeeze theorem, we require:

$$\lim_{n,m\to\infty}(M_{ij} - m_{ij}) = \lim_{||P||\to 0}(M_{ij} - m_{ij}) = 0$$

If $f(x, y)$ is continuous on R, then:

$$\lim_{n,m\to\infty} L_{n,m} = \lim_{n,m\to\infty} U_{n,m} = \iint_R f(x, y)\, dA$$

The Riemann definition and grid formulation are similar.

The following is the analogue of the squeeze theorem for double integrals:

Definiton 1.15.2.8 (Squeeze Theorem for Double Integrals)**.** For the first defintion Consider region R subdivided into N subregions $R_1, R_2, \dots, R_N$, such that all subregions $\bigcup_{i=1}^{N} R_i \subset R$ (They are all inside). For both cases, we require that $R_i \cap R_j = \emptyset$ for $i \neq j$, and then some of the area would be omited and the following would be garanteed:

$$\sum_{i=1}^{N} \Delta A \leq \text{Area}(R), \quad \sum_{i=1}^{N} m_i \Delta A_i \leq \iint_R f(x, y)\, dA$$

where m_i and M_i are the minimum and maximum values of $f(x, y)$ in the i-th subregion. Similarly, if $\bigcup_{i=1}^{N} R_i \supset R$ (They all cover R), and that we garantee that $R_i \cap R \neq \emptyset$ for all i. Then, some of the area would be double counted and the following would be garanteed:

$$\sum_{i=1}^{N} \Delta A \geq \text{Area}(R), \quad \sum_{i=1}^{N} M_i \Delta A_i \geq \iint_R f(x, y)\, dA$$

For the second definition, the same logic applies, but we consider subrectangles that creates grid that is either inside or covering R.

Example 1.15.2.9. Estimate the volume that lies above the square $R = [0, 2] \times [0, 2]$ and below the surface $z = f(x, y) = 16 - x^2 - 2y^2$ by dividing the R into four subrectangles of equal area and using the value of the function at the upper right corner of each subrectangle to form a Riemann sum. Choose the upper right corner of each subrectangle as the sample point.

We divide the square R into four subrectangles, each with an area of 1. We obtain the sum:

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*) \Delta A = \sum_{i=1}^{2} \sum_{j=1}^{2} f(i, j) \cdot 1$$

where the sample points are $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$. Evaluating the function at these points gives:

$$V \approx f(1, 1) + f(1, 2) + f(2, 1) + f(2, 2) = 34$$

Therefore, the estimated volume is approximately 34.

1.15.3 Double Integrals in Non-Rectangular Regions

Theorem 1.15.3.1 (Change of Variable to Polar Coordinates)**.** Consider the double integral of a function $f(x,y)$ over a region R in the xy-plane. If we change the variables from Cartesian coordinates (x,y) to polar coordinates (r,θ) using the transformations:

$$x = r\cos\theta, \quad y = r\sin\theta \implies r = \sqrt{x^2+y^2}$$

then the double integral can be expressed in polar coordinates as follows:

$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{1.8}$$

where R' is the corresponding region in the $r\theta$-plane, and the term r arises from the Jacobian determinant of the transformation from Cartesian to polar coordinates.

Proof. **Note.** *This change of variable can be derived using the Jacobian determinant of the transformation from Cartesian to polar coordinates, which will be covered in Section 1.15.9, which can fully prove this theorem in the case where $g < 0$ for some input.*

Geometric Sketch: Assume $f(r\cos\theta, r\sin\theta) = g(r,\theta) \geq 0$. Consider a small rectangle ΔA_i in the xy-plane with dimensions Δx_i and Δy_i. When we transform this rectangle into polar coordinates, it becomes a small sector of a circle with radius r_i and angle $\Delta\theta_i$. The area of this sector is given by:

$$\Delta A_i = r_i \Delta r_i \Delta\theta_i \left(1 + \frac{\Delta r_i}{2r_i}\right)$$

this is derived from the geormetric formula of the area of a sector of a circle. Then, as $\Delta r_i \to 0$, the term $\frac{\Delta r_i}{2r_i} \to 0$, and we have:

$$\Delta A_i \approx r_i \Delta r_i \Delta\theta_i$$

Therefore, the double integral in polar coordinates can be approximated as:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{n} f(r_i\cos\theta_i, r_i\sin\theta_i) r_i \Delta r_i \Delta\theta_i$$

Taking the limit as the maximum diameter of the subrectangles approaches zero, we obtain the exact double integral in polar coordinates. □

Definiton 1.15.3.2 (Reigion Defined by Varying r with θ)**.** Consider a region R in the xy-plane that can be described in polar coordinates by the inequalities:

$$\alpha \leq \theta \leq \beta, \quad g_1(\theta) \leq r \leq g_2(\theta)$$

where $g_1(\theta)$ and $g_2(\theta)$ are continuous functions on the interval $[\alpha, \beta]$. Then, to evaluate the double integral over this region for a continuous function $f(x,y)$, we set up the integral as follows:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{1.9}$$

Definiton 1.15.3.3 (Reigion Defined by Varying θ with r)**.** Similarly, consider a region R in the xy-plane that can be described in polar coordinates by the inequalities:

$$a \leq r \leq b, \quad h_1(r) \leq \theta \leq h_2(r)$$

where $h_1(r)$ and $h_2(r)$ are continuous functions on the interval $[a, b]$. Then, to evaluate the double integral over this region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x,y)\, dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr \tag{1.10}$$

When to Use Polar Coordinates

Polar coordinates are particularly useful for regions with circular or radial symmetry, as they simplify integration by transforming variables into a more natural form. They are also advantageous for integrands that are difficult in Cartesian coordinates, especially those involving terms like x^2+y^2.

Example 1.15.3.4. Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$ (Donut region). We can describe the region R in polar coordinates as $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$. Thus, we can set up the double integral as follows:

$$\begin{aligned}
\iint_R (3x + 4y^2)\, dA &= \int_0^\pi \int_1^2 (3(r\cos\theta) + 4(r\sin\theta)^2)\, r\, dr\, d\theta \\
&= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta \\
&= \int_0^\pi (r^3\cos\theta + r^4\sin^2\theta)\Big|_{r=1}^{r=2} d\theta \\
&= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta = \frac{15}{2}\pi \qquad \left(\text{Using } \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)
\end{aligned}$$

Example 1.15.3.5. Find the volume of the solid bounded by the $z = 0$ plane and the paraboloid $z = 1 - x^2 - y^2$. We first consider the projection of the paraboloid onto the xy-plane, which is the circle $1 - x^2 - y^2 = 0$. We can describe the region R in polar coordinates as $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Thus, we can set up the double integral as follows:

$$\begin{aligned}
V &= \iint_R (1 - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\int_0^1 (r - r^3)\, dr \right) d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4} \right) d\theta = \frac{\pi}{2}
\end{aligned}$$

Example 1.15.3.6. *Find the area of the region R enclosed by one petal of the rose given by $r = \cos 3\theta$.*

We know that the petal has upper bound at $\theta = \pm\frac{\pi}{6}$. We can describe the region R in polar coordinates as $0 \le r \le \cos 3\theta$ and $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. Thus, we can set up the double integral as follows:

$$\begin{aligned}
\text{Area}(R) &= \iint_R 1\, dA \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\, dr\, d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{r^2}{2} \Big|_{r=0}^{r=\cos 3\theta} \right) d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^2 3\theta}{2}\, d\theta = \frac{\pi}{12} \qquad \left(\text{Using } \cos^2 x = \frac{1+\cos 2x}{2} \right)
\end{aligned}$$

Example 1.15.3.7. *Find the volume trapped between the cone* $z = \sqrt{x^2+y^2}$ *and the sphere* $x^2+y^2+z^2 = 1$.

First, we find the intersection of the cone and the sphere:

$$z = \sqrt{x^2+y^2} \implies z^2 = x^2+y^2$$

Substituting into the sphere equation:

$$\begin{aligned}
x^2+y^2+z^2 = 1 &\implies z^2+z^2 = 1 \\
&\implies 2z^2 = 1 \implies z = \frac{1}{\sqrt{2}} \quad x^2+y^2 = \frac{1}{2}
\end{aligned}$$

Thus, we have the region $R = \{(r,\theta) \mid 0 \le r \le \frac{1}{\sqrt{2}}, 0 \le \theta \le 2\pi\}$. We can set up the double integral as follows:

$$\begin{aligned}
V &= \iint_R (\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2})\, dA \\
&= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1-r^2} - r)\, r\, dr\, d\theta \\
&= 2\pi \left[\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}\, dr - \int_0^{\frac{1}{\sqrt{2}}} r^2 dr \right] \\
&= 2\pi \left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}} \Big|_0^{\frac{1}{\sqrt{2}}} - \frac{r^3}{3} \Big|_0^{\frac{1}{\sqrt{2}}} \right] = \frac{2\pi}{3}(1 - \frac{1}{\sqrt{2}})
\end{aligned}$$

1.15.4 Applications of Double Integrals

Review: Moment of Inertia using Single Integral

Consider a thin rod of length L with a linear mass density $\rho(x)$, where x is the distance from one end of the rod. We can use the following formula to find the moment of inertia I of the rod about an axis perpendicular to the rod and passing through one end:

$$I = \int_0^L x^2 \rho(x)\, dx \tag{1.11}$$

where x^2 is the square of the distance from the axis of rotation, and $\rho(x)\, dx$ represents the mass element of the rod at position x. This comes from the definition of moment of inertia, which is the following sum with point masses:

$$I = \sum m_i r_i^2 \quad \text{s.t.} \quad \text{KE} = \frac{1}{2} I \omega^2$$

Definiton 1.15.4.1 (Mass of a Lamina)**.** Consider a lamina occupying the region R in the xy-plane with a surface mass density $\sigma(x, y)$, where $\sigma(x, y)$ is the mass per unit area at the point (x, y). The mass M of the lamina can be found using the following double integral:

$$M = \iint_R \sigma(x, y)\, dA \tag{1.12}$$

where dA represents an infinitesimal area element in the region R.

Definiton 1.15.4.2 (Moment of a Lamina)**.** The moment of the lamina about the x-axis (M_x) and y-axis (M_y) can be found using the following formulas:

$$M_x = \iint_R y\sigma(x, y)\, dA, \quad M_y = \iint_R x\sigma(x, y)\, dA \tag{1.13}$$

where y and x are the distances from the respective axes of rotation, and $\sigma(x, y)\, dA$ represents the mass element of the lamina at the point (x, y).

Definiton 1.15.4.3 (Center of Mass of a Lamina)**.** The center of mass $(\bar{x}, \bar{y})$ of the lamina can be found using the following formulas:

$$\bar{x} = \frac{1}{M} M_y = \frac{1}{M} \iint_R x\sigma(x, y)\, dA, \quad \bar{y} = \frac{1}{M} M_x = \frac{1}{M} \iint_R y\sigma(x, y)\, dA \tag{1.14}$$

where M is the total mass of the lamina as calculated in the previous example.

Example 1.15.4.4. *Find the cnetre of mass of the following plate with density funciton $\sigma(x, y) = x + y$ over the reigion bounded by the axis and $y = \sqrt{x}$ and $x = 1$.*

First, we find the mass of the lamina:

$$\begin{aligned} M &= \iint_R (x + y)\, dA \\ &= \int_0^1 \int_0^{\sqrt{x}} (x + y)\, dy\, dx \\ &= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^1 \left(x\sqrt{x} + \frac{x}{2} \right) dx = \left[\frac{2}{5} x^{\frac{5}{2}} + \frac{1}{4} x^2 \right]_0^1 = \frac{13}{20} \end{aligned}$$

Now, we find the moments about the x-axis and y-axis:

$$\begin{aligned}
M_x &= \iint_R y(x+y)\,dA \\
&= \int_0^1 \int_0^{\sqrt{x}} y(x+y)\,dy\,dx \\
&= \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(\frac{x^2}{2} + \frac{x^{\frac{3}{2}}}{3}\right) dx = \left[\frac{x^3}{6} + \frac{2}{15}x^{\frac{5}{2}}\right]_0^1 = \frac{3}{10}
\end{aligned}$$

and,

$$\begin{aligned}
M_y &= \iint_R x(x+y)\,dA \\
&= \int_0^1 \int_0^{\sqrt{x}} x(x+y)\,dy\,dx \\
&= \int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(x^2\sqrt{x} + \frac{x^{\frac{3}{2}}}{2}\right) dx = \left[\frac{2}{7}x^{\frac{7}{2}} + \frac{1}{6}x^3\right]_0^1 = \frac{19}{42}
\end{aligned}$$

Finally, we can find the center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{190}{273}, \quad \bar{y} = \frac{M_x}{M} = \frac{6}{13}$$

Definiton 1.15.4.5 (Geometric Center of a Lamina)**.** If the surface mass density $\sigma(x, y)$ is constant, then the center of mass is also known as the geometric center (or centroid) of the lamina. The formulas for the geometric center are:

$$x_c = \frac{1}{\text{Area}(R)} \iint_R x\,dA, \quad y_c = \frac{1}{\text{Area}(R)} \iint_R y\,dA \tag{1.15}$$

where $\text{Area}(R)$ is the area of the region R.

Definiton 1.15.4.6 (Moment of Inertia of a Lamina)**.** The moment of inertia of the lamina about the x-axis (I_x) and y-axis (I_y) can be found using the following formulas:

$$I_x = \iint_R y^2\sigma(x, y)\,dA, \quad I_y = \iint_R x^2\sigma(x, y)\,dA \tag{1.16}$$

where y^2 and x^2 are the squares of the distances from the respective axes of rotation, and $\sigma(x, y)\,dA$ represents the mass element of the lamina at the point (x, y). Also for the moment of inertia about the origin (I_o):

$$I_o = \iint_R (x^2 + y^2)\sigma(x, y)\,dA \tag{1.17}$$

In general, for an axis defined by a line $ax + by + c = 0$, the moment of inertia about that axis (I_l) can be found using the following formula:

$$I_l = \iint_R \left(\frac{ax + by + c}{\sqrt{a^2 + b^2}}\right)^2 \sigma(x, y)\,dA \tag{1.18}$$

Example 1.15.4.7. A rectangular plate of mass m, length L and width W is rotated about a vertical line on its left side with width W. Find the moment of inertia of the plate about this line in two cases:

1. The plate has uniform density $\sigma(x, y) = \frac{m}{LW}$.

2. The density varies at a point proportional to the square of the distance from the right most side.

3. It has uniform density, but rotated its center.

Solution. 1. Uniform Density We can describe the region R in Cartesian coordinates as $0 \leq x \leq L$ and $0 \leq y \leq W$. The surface mass density is $\sigma(x, y) = \frac{m}{LW}$. Thus, we can set up the double integral as it rotates around the y-axis:

$$\begin{aligned}
I_y &= \iint_R x^2 \sigma(x, y)\, dA \\
&= \int_0^W \int_0^L x^2 \cdot \frac{m}{LW}\, dx\, dy \\
&= \int_0^W \left[\frac{m}{LW} \cdot \frac{x^3}{3}\right]_{x=0}^{x=L} dy = \int_0^W \frac{mL^2}{3W} dy = \left[\frac{mL^2}{3W} y\right]_{y=0}^{y=W} = \frac{mL^2}{3}
\end{aligned}$$

Solution. 2. Varying Density We can describe the region R in Cartesian coordinates as $0 \leq x \leq L$ and $0 \leq y \leq W$. The surface mass density is $\sigma(x, y) = k(L - x)^2$. To find the constant k, we have:

$$\begin{aligned}
I_y &= \iint_R x^2 \sigma(x, y)\, dA \\
&= \int_0^W \int_0^L x^2 \cdot k(L - x)^2\, dx\, dy \\
&= \int_0^W \left[k \int_0^L (x^2 L^2 - 2Lx^3 + x^4)\, dx\right] dy \\
&= \int_0^W \left[k \left(\frac{L^5}{3} - \frac{L^5}{2} + \frac{L^5}{5}\right)\right] dy = \int_0^W \left[k \cdot \frac{L^5}{30}\right] dy = k \cdot \frac{L^5}{30} W
\end{aligned}$$

Solution. 3. Rotated Center We can describe the region R in Cartesian coordinates as $-\frac{L}{2} \leq x \leq \frac{L}{2}$ and $-W \leq y \leq W$. The surface mass density is $\sigma(x, y) = \frac{m}{LW}$. Thus, we can set up the double integral as it rotates around the origin:

$$
\begin{aligned}
I_o &= \iint_R (x^2+y^2)\sigma(x,y)\,dA \\
&= \int_{-W}^{W}\int_{-L/2}^{L/2}(x^2+y^2)\cdot\frac{m}{LW}\,dx\,dy \\
&= \int_{-W}^{W}\left[\frac{m}{LW}\left(\frac{x^3}{3}+y^2x\right)_{x=-L/2}^{x=L/2}\right]dy \\
&= \int_{-W}^{W}\left[\frac{mL^2}{12W}+\frac{my^2}{W}\right]dy \\
&= \left[\frac{mL^2}{12W}y+\frac{my^3}{3W}\right]_{y=-W}^{y=W} = \frac{mL^2}{6}+\frac{2mW^2}{3}
\end{aligned}
$$

1.15.5 Surface Area

Theorem 1.15.5.1 (Surface Area)**.** Given $z = f(x,y)$, where f is a differentiable function over the region R in the xy-plane, the surface area S of the surface above the region R is given by:

$$S = \iint_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA \tag{1.19}$$

Proof. Consider a small rectangle ΔA_i in the xy-plane with dimensions Δx_i and Δy_i. When we project this rectangle onto the surface $z = f(x,y)$, it becomes a parallelogram T_i tangent to the surface. Using the partial derivatives of the surface, we can deduce the two vectors that define the parallelogram:

$$\vec{u} = (\Delta x_i, 0, f_x(x_i^*, y_i^*)\Delta x_i)\,, \quad \vec{v} = (0, \Delta y_i, f_y(x_i^*, y_i^*)\Delta y_i)$$

The area of this parallelogram is given by the magnitude of the cross product of these two vectors:

$$
\begin{aligned}
\Delta T_i = ||\vec{u}\times\vec{v}|| &= \left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x_i & 0 & f_x(x_i^*, y_i^*)\Delta x_i \\ 0 & \Delta y_i & f_y(x_i^*, y_i^*)\Delta y_i \end{vmatrix}\right\| \\
&= \sqrt{(-f_x(x_i^*, y_i^*)\Delta x_i\Delta y_i)^2 + (-f_y(x_i^*, y_i^*)\Delta x_i\Delta y_i)^2 + (\Delta x_i\Delta y_i)^2} \\
&= \sqrt{1+(f_x(x_i^*, y_i^*))^2+(f_y(x_i^*, y_i^*))^2}\,\Delta x_i\Delta y_i
\end{aligned}
$$

Therefore, the surface area can be approximated as:

$$S \approx S_n = \sum_{i=1}^{n}\Delta T_i = \sum_{i=1}^{n}\sqrt{1+(f_x(x_i^*, y_i^*))^2+(f_y(x_i^*, y_i^*))^2}\,\Delta A_i$$

Taking the limit as the maximum diameter of the subrectangles approaches zero, we obtain the exact surface area:

$$S = \lim_{|P|\to 0} S_n = \iint_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA$$

which completes our derivation. □

Example 1.15.5.2. *Find the surface of the sphere $x^2 + y^2 + z^2 = a^2$.*

We first consider the first octant, where $x, y, z \geq 0$. Then the total volumn would be eight times the volumn of the first octant. We first express it in polar coordinates:

$$x^2 + y^2 = r^2 \implies z = \sqrt{a^2 - r^2}$$

taking the partial derivatives:

$$\frac{\partial z}{\partial r} = \frac{-r}{\sqrt{a^2 - r^2}}, \quad \frac{\partial z}{\partial \theta} = 0$$

Thus, we can set up the double integral as follows:

$$\begin{aligned} S_{\text{1st octant}} &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}\, dA \\ &= \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{1 + \left(\frac{-r}{\sqrt{a^2 - r^2}}\right)^2 + 0^2}\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{1 + \frac{r^2}{a^2 - r^2}}\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-a\sqrt{a^2 - r^2}\right]_{r=0}^{r=a} d\theta = \int_0^{\frac{\pi}{2}} a^2\, d\theta = \frac{\pi a^2}{2} \end{aligned}$$

Therefore, the total surface area of the sphere is:

$$S = 8 \cdot S_{\text{1st octant}} = 4\pi a^2$$

Example 1.15.5.3. *Let R be the triangular region with vertices at $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 0)$. Find the surface area of the portion of $z = 3x + y^2$ that lies above the region R.*

We first express the function and take the partial derivatives:

$$z = 3x + y^2, \quad \frac{\partial z}{\partial x} = 3, \quad \frac{\partial z}{\partial y} = 2y$$

Thus, we can set up the double integral as follows:

$$\begin{aligned} S &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \int_0^1 \int_0^y \sqrt{1 + 3^2 + (2y)^2}\, dx\, dy \\ &= \int_0^1 \int_0^y \sqrt{10 + 4y^2}\, dx\, dy \\ &= \int_0^1 \sqrt{10 + 4y^2} \cdot y\, dy \end{aligned}$$

The extra y is useful for substitution. Let $u = 10 + 4y^2$, then $du = 8y\,dy$ (that's why we integrate w.r.t x first). Thus, we have:

$$\begin{aligned} S &= \int_{u=10}^{u=14} \sqrt{u} \cdot \frac{du}{8} = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=10}^{u=14} \\ &= \frac{1}{12}(14^{\frac{3}{2}} - 10^{\frac{3}{2}}) \approx 1.7 \end{aligned}$$

1.15.6 Triple Integrals

The idea of a triple integral (in fact n-tuple integral) could be extend from the idea of double integral, similar to Definition 1.15.2.7. We can define a triple integral as follows:

Definiton 1.15.6.1. Consider a function $f(x, y, z)$ that is continuous a 3-D reigion with volumne V. We can partition the reigion into n subreigons with volume ΔV_i. Then, we can define the triple integral of f over the reigion V as follows:

$$\iiint_V f(x, y, z)\, dV = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta V_i \tag{1.20}$$

where (x_i^*, y_i^*, z_i^*) is a point in the i-th subregion, and $|P|$ is the maximum diameter of the subregions. This limit exists and equals to the triple integral if f is continuous on V, since the following squeeze theorem holds, similar to the double integral case:

$$\sum_{i=1}^{n} m_i \Delta V_i \leq \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta V_i \leq \sum_{i=1}^{n} M_i \Delta V_i$$

where m and M are the minimum and maximum values of f on V, respectively.

In the retangluar case, we can express ΔV_i as:

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$$

Thus we have:

$$dV = dx\, dy\, dz$$

1.15.7 Triple Integrals in Rectangular Coordinates

Definiton 1.15.7.1 (Triple Integral in Rectangular Region)**.** Consider a function $f(x, y, z)$ that is continuous over a rectangular box B defined by the inequalities:

$$a \leq x \leq b, \quad c \leq y \leq d, \quad r \leq z \leq s$$

Then, the triple integral of f over the box B can be computed as an iterated integral in any order of integration:

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx \tag{1.21}$$

or any other permutation of the order of integration.

Example 1.15.7.2. Consider $f(x,y,z)$ over a box $Q = \{(x,y,z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}$. We would form the triple integral as follows:

$$\iiint_Q f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$$

Of course, the order of integration can be change (there are ${}_3P_3 = 6$ ways).

Definiton 1.15.7.3 (Triple Integrals in General Reigion)**.** Consider a function $f(x,y,z)$ that is continuous over a general reigion V in the 3-D space. We can describe the reigion V using the following inequalities:

$$a \le x \le b, \quad g_1(x) \le y \le g_2(x), \quad h_1(x,y) \le z \le h_2(x,y)$$

where $g_1(x)$, $g_2(x)$, $h_1(x,y)$, and $h_2(x,y)$ are continuous functions. Then, the triple integral of f over the reigion V can be computed as an iterated integral:

$$\iiint_V f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx = \iint_R \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dA \tag{1.22}$$

where $R = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$ is the projection of the reigion V onto the xy-plane.

The order of integration and their associated bounds can be changed based on the description of the region.

Example 1.15.7.4. *Evaluate $\iiint_Q 6xydV$, where Q is the tetrahedron bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $2x + y + z = 4$.*

Let R be the projection of the tetrahedron onto the xy-plane. Then, we consider how the tetrahedron change in the z direction over R:

$$z = 4 - 2x - y \implies 0 \le z \le 4 - 2x - y$$

So we can set up:

$$\begin{aligned}
\iiint_Q 6xy\,dV &= \iint_R \int_0^{4-2x-y} 6xy\,dz\,dA \\
&= \iint_R 6xy(4 - 2x - y)\,dA
\end{aligned}$$

Now, we consider how the projection R change in the y direction over x:

$$2x + y = 4 \implies 0 \le y \le 4 - 2x, \quad 0 \le x \le 2$$

So we can set up:

$$\begin{aligned}
\iiint_Q 6xy\,dV &= \int_0^2 \int_0^{4-2x} 6xy(4 - 2x - y)\,dy\,dx \\
&= \int_0^2 \left[12xy^2 - 6x(4-2x)y^2 - 2y^3\right]_{y=0}^{y=4-2x} dx \\
&= \int_0^2 (192x - 144x^2 + 24x^3 - 32x^4 + 8x^5)\,dx = \frac{64}{5}
\end{aligned}$$

Example 1.15.7.5. *Evaluate the integral given in the previous Example 1.15.7.4 by intergrating w.r.t. x first.*

We consider how the tetrahedron change in the x direction over R, which is now the projection of the tetrahedron onto the yz-plane:

$$2x + y + z = 4 \implies 0 \le x \le \frac{4-y-z}{2}$$

So we can set up:

$$\begin{aligned}\iiint_Q 6xy\,dV &= \iint_R \int_0^{\frac{4-y-z}{2}} 6xy\,dx\,dA \\ &= \frac{3}{4}\iint_R y(4-y-z)^2\,dA\end{aligned}$$

Now, we consider how the projection R change in the y direction over z:

$$y + z = 4 \implies 0 \le y \le 4 - z, \quad 0 \le z \le 4$$

So we can set up:

$$\begin{aligned}\iiint_Q 6xy\,dV &= \frac{3}{4}\int_0^4 \int_0^{4-z} y(4-y-z)^2\,dy\,dz \\ &= \frac{64}{5}\end{aligned}$$

Example 1.15.7.6. *Find the volume of the solid bounded by the surface $z = 4 - y^2$ and planes given by $x + z = 4$, $x = 0$, and $z = 0$.*

First, we select the $z = 4 - y^2$ surface as our base reigion R. Then, we consider how the solid change in the x direction over R:

$$x + z = 4 \implies 0 \le x \le 4 - z$$

So we can set up:

$$\begin{aligned}V &= \iint_R \int_0^{4-z} 1\,dx\,dA \\ &= \iint_R (4-z)\,dA\end{aligned}$$

Now, we consider how the projection R change in the z direction over y:

$$z = 4 - y^2 \implies 0 \le z \le 4 - y^2, \quad -2 \le y \le 2$$

So we can set up:

$$\begin{aligned}V &= \int_{-2}^{2}\int_0^{4-y^2} (4-z)\,dz\,dy \\ &= \int_{-2}^{2} \left[4z - \frac{z^2}{2}\right]_{z=0}^{z=4-y^2} dy \\ &= \int_{-2}^{2} \left(8 - \frac{y^2}{2}\right) dy \\ &= \frac{128}{5}\end{aligned}$$

1.15.8 Triple Integrals in Spherical Coordinates

1.15.9 Change of Variables in Multiple Integrals

Chapter 2

Fluid Mechanics