ESC 195 Lecture Notes

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1 More on Integrals

1.1 Riemann Sum - Non-Uniform Petition

Example 1.1.1. Given the following definite integral:

$$\int_0^2 \sqrt{x}dx$$

, we cannot evaluate its Riemann sum with uniforms partition, since the series of root cannot be easily evaluated.

The definite integral of $\sqrt{x}$ from 0 to 2 using a Riemann sum with a non-uniform partition is given by:

$$\int_0^2 \sqrt{x}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i} \Delta x_i$$

where:

- $x_0 = 0$, $x_n = 2$,
- $x_i = i^2 \cdot \frac{2}{n^2}$ for $i = 0, 1, 2, \dots, n$,
- $\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2} \cdot (2i - 1)$.

The Riemann sum becomes:

$$S_n = \sum_{i=1}^{n} \sqrt{i^2 \cdot \frac{2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Simplifying further:

$$S_n = \sum_{i=1}^{n} \sqrt{\frac{2i^2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Taking the limit as $n \to \infty$, the sum converges to the exact value of the integral using the series of squares.

$$\int_0^2 \sqrt{x}\, dx = \frac{4\sqrt{2}}{3}.$$

Condition $n \to \infty$ Ensures $\Delta x_i \to 0$

As $n \to \infty$, the partition points x_i become increasingly dense. This ensures that the partition becomes infinitely fine.

1.2 Integration By Parts

Using the product rule:

$$\frac{d}{dx}\big[u(x)v(x)\big] = u'(x)v(x) + u(x)v'(x),$$

integrating both sides with respect to x gives:

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Rearranging this:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Integration by parts formula

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

Example 1.2.1. We want to solve the integral

$$\int xe^{2x}\,dx$$

using integration by parts.

Let:

$$u = x, \quad dv = e^{2x}\,dx.$$

Then, we compute the derivatives and integrals:

$$du = dx, \quad v = \frac{e^{2x}}{2}.$$

Now, apply the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

Substituting in the values:

$$\int xe^{2x}\,dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx.$$

Next, compute the remaining integral:

$$\int \frac{e^{2x}}{2}\,dx = \frac{e^{2x}}{4}.$$

Thus, the result is:

$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Example 1.2.2. We want to solve

$$\int x^2 \sin(2x)\,dx$$

using double integration by parts.

First, let:

$$u = x^2, \quad dv = \sin(2x)\,dx.$$

Then:

$$du = 2x\,dx, \quad v = -\frac{1}{2}\cos(2x).$$

Using the IBP formula:

$$\int u\,dv = uv - \int v\,du,$$

we get:

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\,dx.$$

Now, apply IBP again to $\int x\cos(2x)\,dx$, let:

$$u = x, \quad dv = \cos(2x)\,dx.$$

Then:

$$du = dx, \quad v = \frac{1}{2}\sin(2x).$$

Using the IBP formula again:

$$\int x\cos(2x)\,dx = \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx,$$

and solving the remaining integral:

$$\int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x).$$

Thus, the final result is:

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.$$

1.3 Trigonometric Integrals

Case I This is the case I trigonometric integrals, the strategy is using the pythagorean identity.

We want to solve the class of integrals:

$$\int \sin^n(x)\cos^m(x)\,dx \tag{2}$$

, where m or n is odd.

Example 1.3.1. We want to solve the integral:

$$I = \int \sin^3(x)\cos^2(x)dx$$

We use the identity: $\sin^2(x) = 1 - \cos^2(x)$. Thus, the integral becomes:

$$\begin{aligned} I &= \int \sin(x)(1-\cos^2(x))\cos^2(x)\,dx \\ &= \int (\cos^2(x)\sin(x) - \cos^4(x)\sin(x))\,dx \end{aligned}$$

which is now easily solvable with substution $u = \cos(x)$.

Case II The follwoing is generally solvable via case I and case III below. In general, we solve the integral by reducing the power of the trigonometric functions to arrive at a solvable integral Case I or III.

We want to solve the class of integrals:

$$\int \sin^n(x)\cos^m(x)\,dx \tag{3}$$

, where m and n is even.

Example 1.3.2. We want to solve the integral:

$$I = \int \sin^2(x)\cos^4(x)\,dx$$

We can apply the double angle formulas:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{4}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \tag{5}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \tag{6}$$

Thus, the integral becomes:

$$I = \frac{1}{8}\int \sin^2(2x)\,dx + \frac{1}{8}\int \cos(2x)\sin^2(2x)\,dx$$

Case III This is the case III trigonometric integrals, the strategy is using the reduction formula via IBP.

Reduction Formula We can solve integrals by reducing the power of the trigonometric functions. These are done using IBP and trigonometric identities.

We want to solve the classes of integrals:

$$\int \sin^n(x)\,dx, \quad \int \cos^n(x)\,dx \tag{7}$$

, where n is a positive integer. For demonstartion, We can obtain the reduction formula of $\sin^n(x)$ via IBP.

$$\begin{aligned}
I_n &= \int \sin^n(x)\,dx \\
&= \int \sin^{n-1}(x)\sin(x)\,dx \\
&= \frac{-\cos(x)\sin^{n-1}(x)}{n} + \frac{(n-1)}{n}\int \cos^2(x)\sin^{n-2}(x)\,dx \\
&= \frac{-\cos(x)\sin^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2}
\end{aligned}$$

Case IV The following is generally solvable via simple trigonometric integrals. In general, we solve the integral by applying the angle sum formulas.

We want to solve the classes of integrals:

$$\int \sin(mx)\cos(nx)\,dx, \quad \int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \tag{8}$$

We could apply the angle sum formulas:

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin((m+n)x) + \sin((m-n)x)\right] \tag{9}$$

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right] \tag{10}$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right] \tag{11}$$

Case V The following is generally solvable via the following trigonometric identities listed below, which convert it into a reduction formula.

$$\tan^2(x) = \sec^2(x) - 1 \tag{12}$$

$$\cot^2(x) = \csc^2(x) - 1 \tag{13}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \tag{14}$$

We want to solve the classes of integral:

$$\int \tan^n(x)\,dx, \quad \int \cot^n(x)\,dx \tag{15}$$

We know that $tan^2(x) = \sec^2(x) - 1$. Thus, we can solve the integral by reducing the power of the tangent function.

Case VI The following is generally solvable via the trigonometric identities (12) and (13).

We want to solve the classes of integral:

$$\int \tan^m(x)\sec^n(x)\,dx, \quad \int \cot^m(x)\csc^n(x)\,dx \tag{16}$$

We can solve the integral by reducing the power of the converted trigonometric functions using (12) and (13).

Case VII The following is generally solvable via the trigonometric identities (12) and (13).

We want to solve the classes of integral:

$$\int \tan^m(x)\sec^n(x)\,dx, \quad \int \cot^m(x)\csc^n(x)\,dx \tag{17}$$

We can solve the integral by converting between the tangent and secant functions using the trigonometric identities.

Case VIII This is the case VIII integrals, the strategy is using the trigonometric substitution.

Trigonometric Substitution We can solve integrals by substituting the trigonometric functions with other trigonometric functions.

Example 1.3.3. We want to solve the integral:

$$\int \frac{dx}{\sqrt{1-x^2}}$$

We can substitute $x = \sin(\theta)$, then $dx = \cos(\theta)\,d\theta$. The integral becomes:

$$\int \frac{\cos(\theta)\,d\theta}{\sqrt{1-\sin^2(\theta)}} = \int \frac{\cos(\theta)\,d\theta}{\cos(\theta)} = \int d\theta = \theta + C.$$

In general, we can use the following substitutions:

- $x = a\sin(\theta)$ for $\sqrt{a^2 - x^2}$,
- $x = a\tan(\theta)$ for $\sqrt{a^2 + x^2}$,
- $x = a\sec(\theta)$ for $\sqrt{x^2 - a^2}$.

Summary We can solve trigonometric integrals by using the following strategies:

Case	Strategy and General Form
I	Use $\sin^2(x) + \cos^2(x) = 1$; simplify using substitution. General Form: $\int \sin^n(x)\cos^m(x)dx$, where m or n is odd.
II	Convert to Case I and Case III using double angle formulas. General Form: $\int \sin^n(x)\cos^m(x)dx$, where both m and n are even.
III	Apply reduction formulas derived via integration by parts. General Form: $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$.
IV	Use angle sum formulas to simplify. General Form: $\int \sin(mx)\cos(nx)dx$.
V	Reduce tangent/cotangent powers using $\tan^2(x) = \sec^2(x) - 1$ and substitution. General Form: $\int \tan^n(x)dx$ or $\int \cot^n(x)dx$.
VI	Convert to Case V via Pythagorean identities. General Form: $\int \tan^m(x)\sec^n(x)dx$ or $\int \cot^m(x)\csc^n(x)dx$.
VII	Convert between tangent and secant functions for simplification. General Form: $\int \tan^m(x)\sec^n(x)dx$.
VIII	Use trigonometric substitution: $x = a\sin(\theta), a\tan(\theta), a\sec(\theta)$. General Form: $\int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{\sqrt{a^2+x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$.

Table 1: Strategies and General Forms for Case I-VIII Integrals

2 Hyperbolic Trigonometric Functions

Definiton 2.0.1 (Hyperbolic Sine)**.** The hyperbolic sine function is defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{18}$$

Definiton 2.0.2 (Hyperbolic Cosine)**.** The hyperbolic cosine function is defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{19}$$

Properties These combinations of exponential functions have properties similar to the trigonometric functions.

Derivatives The derivatives of the hyperbolic functions are:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \tag{20}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x). \tag{21}$$

Identities The hyperbolic functions satisfy the following identities:

$$\cosh^2(x) - \sinh^2(x) = 1, \tag{22}$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \tag{23}$$

$$\sinh(2x) = 2\sinh(x)\cosh(x). \tag{24}$$

Hyperbola The hyperbolic functions are related to the hyperbola $x^2 - y^2 = 1$. Simiar to the circle, the hyperbola can be parametrized by the hyperbolic functions (e.g. $x = \cosh(t)$, $y = \sinh(t)$).

Area The area of a sector of the hyperbola is given by:

$$A = t/2, \tag{25}$$

where t is the angle of the sector along the parametrization $\{(x, y) \mid x = \cosh(t), y = \sinh(t)\}$.

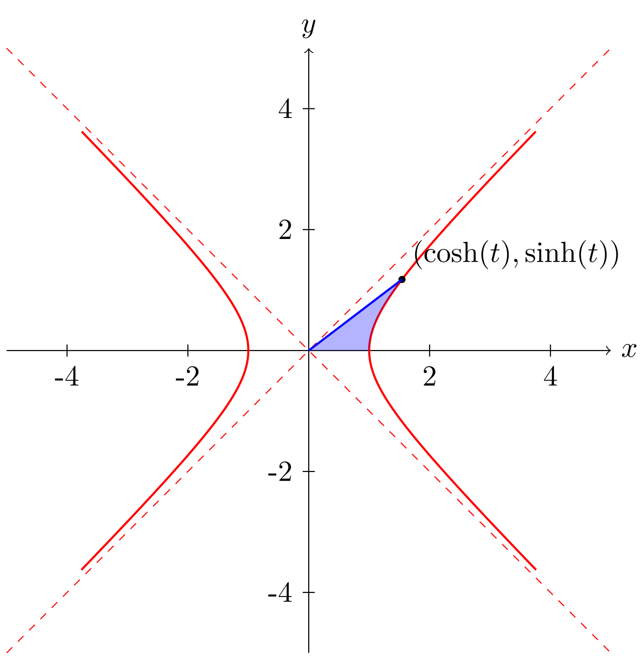

Figure 1: Sector of the hyperbola

Catenary The hyperbolic cosine function describes the shape of a hanging chain or cable. The catenary is the curve formed by a chain hanging from two points. It is given by the equation:

$$a \cosh\left(\frac{x}{a}\right), \tag{26}$$

Definiton 2.0.3 (Hyperbolic Tangent)**.** The hyperbolic tangent function is defined as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{27}$$

Derivative The derivative of the hyperbolic tangent function resembles the derivative of the regular tangent function:

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x). \tag{28}$$

Identities The hyperbolic tangent function satisfies the following identities:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \tag{29}$$

$$\operatorname{sech}^2(x) = 1 - \tanh^2(x). \tag{30}$$

Secant, Cosecant, Cotangent The hyperbolic secant, cosecant, and cotangent functions are defined similarly to the regular secant, cosecant, and cotangent functions. They are the **reciprocal** of the hyperbolic cosine, sine, and tangent functions, respectively.

Inverse Hyperbolic Functions The inverse hyperbolic functions are defined as the inverse of the hyperbolic functions. They are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$, etc.

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \tag{31}$$

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \tag{32}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{33}$$

$$\operatorname{arcsch} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) \tag{34}$$

$$\operatorname{arsech} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) \tag{35}$$

$$\operatorname{arcoth} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \tag{36}$$