ESC 195 Lecture Notes

Hei Shing Cheung

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The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

1 More on Integrals

1.1 Riemann Sum - Non-Uniform Petition

Example 1.1.1. Given the following definite integral:

$$\int_0^2 \sqrt{x}dx$$

, we cannot evaluate its Riemann sum with uniforms partition, since the series of root cannot be easily evaluated.

The definite integral of $\sqrt{x}$ from 0 to 2 using a Riemann sum with a non-uniform partition is given by:

$$\int_0^2 \sqrt{x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{x_i}\Delta x_i$$

, where:

- $x_0 = 0$, $x_n = 2$,
- $x_i = i^2 \cdot \frac{2}{n^2}$ for $i = 0, 1, 2, \ldots, n$,
- $\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2} \cdot (2i - 1)$.

The Riemann sum becomes:

$$S_n = \sum_{i=1}^{n}\sqrt{i^2 \cdot \frac{2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i-1).$$

Simplifying further:

$$S_n = \sum_{i=1}^{n}\sqrt{\frac{2i^2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i-1).$$

Taking the limit as $n \to \infty$, the sum converges to the exact value of the integral using the series of squares.

$$\int_0^2 \sqrt{x}\,dx = \frac{4\sqrt{2}}{3}.$$

Condition $n \to \infty$ Ensures $\Delta x_i \to 0$

As $n \to \infty$, the partition points x_i become increasingly dense. This ensures that the partition becomes infinitely fine.

1.2 Integration By Parts

Using the product rule:

$$\frac{d}{dx}\big[u(x)v(x)\big] = u'(x)v(x) + u(x)v'(x),$$

integrating both sides with respect to x gives:

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Rearranging this:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Integration by parts formula

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

Example 1.2.1. We want to solve the integral

$$\int xe^{2x}\,dx$$

using integration by parts.

Let:

$$u = x, \quad dv = e^{2x}\,dx.$$

Then, we compute the derivatives and integrals:

$$du = dx, \quad v = \frac{e^{2x}}{2}.$$

Now, apply the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

Substituting in the values:

$$\int xe^{2x}\,dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx.$$

Next, compute the remaining integral:

$$\int \frac{e^{2x}}{2}\,dx = \frac{e^{2x}}{4}.$$

Thus, the result is:

$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Example 1.2.2. We want to solve

$$\int x^2 \sin(2x)\,dx$$

using double integration by parts.

First, let:

$$u = x^2, \quad dv = \sin(2x)\,dx.$$

Then:

$$du = 2x\,dx, \quad v = -\frac{1}{2}\cos(2x).$$

Using the IBP formula:

$$\int u\,dv = uv - \int v\,du,$$

we get:

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\,dx.$$

Now, apply IBP again to $\int x\cos(2x)\,dx$, let:

$$u = x, \quad dv = \cos(2x)\,dx.$$

Then:

$$du = dx, \quad v = \frac{1}{2}\sin(2x).$$

Using the IBP formula again:

$$\int x\cos(2x)\,dx = \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx,$$

and solving the remaining integral:

$$\int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x).$$

Thus, the final result is:

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.$$

1.3 Trigonometric Integrals

Case I This is the case I trigonometric integrals, the strategy is using the pythagorean identity.

We want to solve the class of integrals:

$$\int \sin^n(x)\cos^m(x)\,dx \tag{2}$$

, where m or n is odd.

Example 1.3.1. We want to solve the integral:

$$I = \int \sin^3(x)\cos^2(x)dx$$

We use the identity: $\sin^2(x) = 1 - \cos^2(x)$. Thus, the integral becomes:

$$\begin{aligned} I &= \int \sin(x)(1-\cos^2(x))\cos^2(x)\,dx \\ &= \int (\cos^2(x)\sin(x) - \cos^4(x)\sin(x))\,dx \end{aligned}$$

which is now easily solvable with substution $u = \cos(x)$.

Case II The follwoing is generally solvable via case I and case III below. In general, we solve the integral by reducing the power of the trigonometric functions to arrive at a solvable integral Case I or III.

We want to solve the class of integrals:

$$\int \sin^n(x)\cos^m(x)\,dx \tag{3}$$

, where m and n is even.

Example 1.3.2. We want to solve the integral:

$$I = \int \sin^2(x)\cos^4(x)\,dx$$

We can apply the double angle formulas:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{4}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \tag{5}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \tag{6}$$

Thus, the integral becomes:

$$I = \frac{1}{8}\int \sin^2(2x)\,dx + \frac{1}{8}\int \cos(2x)\sin^2(2x)\,dx$$

Case III This is the case III trigonometric integrals, the strategy is using the reduction formula via IBP.

Reduction Formula We can solve integrals by reducing the power of the trigonometric functions. These are done using IBP and trigonometric identities.

We want to solve the classes of integrals:

$$\int \sin^n(x)\,dx, \quad \int \cos^n(x)\,dx \tag{7}$$

, where n is a positive integer. For demonstartion, We can obtain the reduction formula of $\sin^n(x)$ via IBP.

$$\begin{aligned} I_n &= \int \sin^n(x)\,dx \\ &= \int \sin^{n-1}(x)\sin(x)\,dx \\ &= \frac{-\cos(x)\sin^{n-1}(x)}{n} + \frac{(n-1)}{n}\int \cos^2(x)\sin^{n-2}(x)\,dx \\ &= \frac{-\cos(x)\sin^{n-1}(x)}{n} + \frac{(n-1)}{n}I_{n-2} \end{aligned} \tag{8}$$

Reduction for $\cos^n(x)$ We can obtain the reduction formula of $\cos^n(x)$ via IBP.

$$\begin{aligned} I_n &= \int \cos^n(x)\,dx \\ &= \int \cos^{n-1}(x)\cos(x)\,dx \\ &= \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{(n-1)}{n}\int \sin^2(x)\cos^{n-2}(x)\,dx \\ &= \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{(n-1)}{n}I_{n-2} \end{aligned} \tag{9}$$

Case IV The following is generally solvable via simple trigonometric integrals. In general, we solve the integral by applying the angle sum formulas.

We want to solve the classes of integrals:

$$\int \sin(mx)\cos(nx)\,dx, \quad \int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \tag{10}$$

We could apply the angle sum formulas:

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin((m+n)x) + \sin((m-n)x)\right] \tag{11}$$
$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right] \tag{12}$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right] \tag{13}$$

Case V The following is generally solvable via the following trigonometric identities listed below, which convert it into a reduction formula.

$$\tan^2(x) = \sec^2(x) - 1 \tag{14}$$
$$\cot^2(x) = \csc^2(x) - 1 \tag{15}$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \tag{16}$$

We want to solve the classes of integral:

$$\int \tan^n(x)\,dx, \quad \int \cot^n(x)\,dx \tag{17}$$

We know that $tan^2(x) = \sec^2(x) - 1$. Thus, we can solve the integral by reducing the power of the tangent function.

Reduction for $\tan^n(x)$ We can obtain the reduction formula of $\tan^n(x)$ via the trigonometric identities.

$$\begin{aligned} I_n &= \int \tan^n(x)\,dx \\ &= \int \tan^{n-1}(x)\tan(x)\,dx \\ &= \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx \\ &= \frac{\tan^{n-1}(x)}{n-1} - I_{n-2} \end{aligned} \tag{18}$$

Reduction for $\cot^n(x)$ We can obtain the reduction formula of $\cot^n(x)$ via the trigonometric identities.

$$\begin{aligned} I_n &= \int \cot^n(x)\,dx \\ &= \int \cot^{n-1}(x)\cot(x)\,dx \\ &= \frac{\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x)\,dx \\ &= \frac{\cot^{n-1}(x)}{n-1} - I_{n-2} \end{aligned} \tag{19}$$

Case VI The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x)\sec^n(x)\,dx, \quad \int \cot^m(x)\csc^n(x)\,dx \tag{20}$$

We can solve the integral by reducing the power of the converted trigonometric functions using (14) and (15).

Case VII The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x)\sec^n(x)\,dx, \quad \int \cot^m(x)\csc^n(x)\,dx \tag{21}$$

We can solve the integral by converting between the tangent and secant functions using the trigonometric identities.

Case VIII This is the case VIII integrals, the strategy is using the trigonometric substitution.

Trigonometric Substitution We can solve integrals by substituting the trigonometric functions with other trigonometric functions.

Example 1.3.3. We want to solve the integral:

$$\int \frac{dx}{\sqrt{1-x^2}}$$

We can substitute $x = \sin(\theta)$, then $dx = \cos(\theta)\,d\theta$. The integral becomes:

$$\int \frac{\cos(\theta)\,d\theta}{\sqrt{1-\sin^2(\theta)}} = \int \frac{\cos(\theta)\,d\theta}{\cos(\theta)} = \int d\theta = \theta + C.$$

In general, we can use the following substitutions:

- $x = a\sin(\theta)$ for $\sqrt{a^2 - x^2}$,
- $x = a\tan(\theta)$ for $\sqrt{a^2 + x^2}$,
- $x = a\sec(\theta)$ for $\sqrt{x^2 - a^2}$.

Weierstrass Substitution We can solve integrals by using the Weierstrass substitution:

$$\tan\left(\frac{x}{2}\right) = t \Leftrightarrow \cos(x) = \frac{1-t^2}{1+t^2} \quad \sin(x) = \frac{2t}{1+t^2} \quad dx = \frac{2dt}{1+t^2} \tag{22}$$

This substitution is useful for solving integrals with trigonometric functions, by converting them into rational functions.

Summary We can solve trigonometric integrals by using the following strategies:

Case	Strategy and General Form
I	Use $\sin^2(x) + \cos^2(x) = 1$; simplify using substitution. General Form: $\int \sin^n(x)\cos^m(x)dx$, where m or n is odd.
II	Convert to Case I and Case III using double angle formulas. General Form: $\int \sin^n(x)\cos^m(x)dx$, where both m and n are even.
III	Apply reduction formulas derived via integration by parts. General Form: $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$.
IV	Use angle sum formulas to simplify. General Form: $\int \sin(mx)\cos(nx)dx$.
V	Reduce tangent/cotangent powers using $\tan^2(x) = \sec^2(x) - 1$ and substitution. General Form: $\int \tan^n(x)dx$ or $\int \cot^n(x)dx$.
VI	Convert to Case V via Pythagorean identities. General Form: $\int \tan^m(x)\sec^n(x)dx$ or $\int \cot^m(x)\csc^n(x)dx$.
VII	Convert between tangent and secant functions for simplification. General Form: $\int \tan^m(x)\sec^n(x)dx$.
VIII	Use trigonometric substitution: $x = a\sin(\theta), a\tan(\theta), a\sec(\theta)$. General Form: $\int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{\sqrt{a^2+x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$.

Table 1: Strategies and General Forms for Case I-VIII Integrals

1.4 Partial Fractions

Partial Fraction Decomposition We can decompose a rational function into partial fractions to simplify integration.

We want to solve the integral:

$$\int \frac{P(x)}{Q(x)}\,dx \tag{23}$$

, where $P(x)$ and $Q(x)$ are polynomials.

Example 1.4.1. We want to solve the integral:

$$\int \frac{2x^2+3x+1}{x^3+2x^2+x}\,dx$$

We can decompose the rational function into partial fractions:

$$\frac{2x^2+3x+1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

We can solve for the constants A, B, and C by equating the coefficients of the partial fractions to the original function. We have:

$$2x^2+3x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

Thus, we can solve it by setting $x = 0$, $x = -1$

Summary We can solve integrals using partial fraction decomposition by following these steps:

Step	Description
1	Factorize the denominator of the rational function.
2	Write the partial fraction decomposition.
3	Solve for the constants by equating the coefficients.
4	Integrate the partial fractions.

Table 2: Steps for Partial Fraction Decomposition

1.5 Improper Integrals

Improper Integral An improper integral is an integral with an infinite limit or a discontinuity in the interval of integration.

We want to solve the integral:

$$\int_a^b f(x)\,dx \tag{24}$$

, where a or b is infinite or $f(x)$ is discontinuous.

Example 1.5.1. We want to solve the integral:

$$\int_0^\infty e^{-x}\,dx$$

Evaluating the integral:

$$\begin{aligned}\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-x}\,dx \\ &= \lim_{b\to\infty} -e^{-x}\Big|_0^b \\ &= \lim_{b\to\infty} -e^{-b}+1 \\ &= 1\end{aligned}$$

1.6 Convergence Test

Convergence An improper integral converges if the limit of the integral exists.

Comparison Test We can compare the integral to another integral to determine convergence or divergence.

Spliting the Integral We can split the integral into two parts to determine convergence. We split it into two limits on the left and right, for example:

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$

2 Hyperbolic Trigonometric Functions

Definiton 2.0.1 (Hyperbolic Sine)**.** The hyperbolic sine function is defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{25}$$

Definiton 2.0.2 (Hyperbolic Cosine)**.** The hyperbolic cosine function is defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{26}$$

Properties These combinations of exponential functions have properties similar to the trigonometric functions.

Derivatives The derivatives of the hyperbolic functions are:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \tag{27}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x). \tag{28}$$

Identities The hyperbolic functions satisfy the following identities:

$$\cosh^2(x) - \sinh^2(x) = 1, \tag{29}$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \tag{30}$$

$$\sinh(2x) = 2\sinh(x)\cosh(x). \tag{31}$$

Hyperbola The hyperbolic functions are related to the hyperbola $x^2-y^2 = 1$. Simiar to the circle, the hyperbola can be parametrized by the hyperbolic functions (e.g. $x = \cosh(t)$, $y = \sinh(t)$).

Area The area of a sector of the hyperbola is given by:

$$A = t/2, \tag{32}$$

where t is the angle of the sector along the parametrization $\{(x, y) \mid x = \cosh(t), y = \sinh(t)\}$.

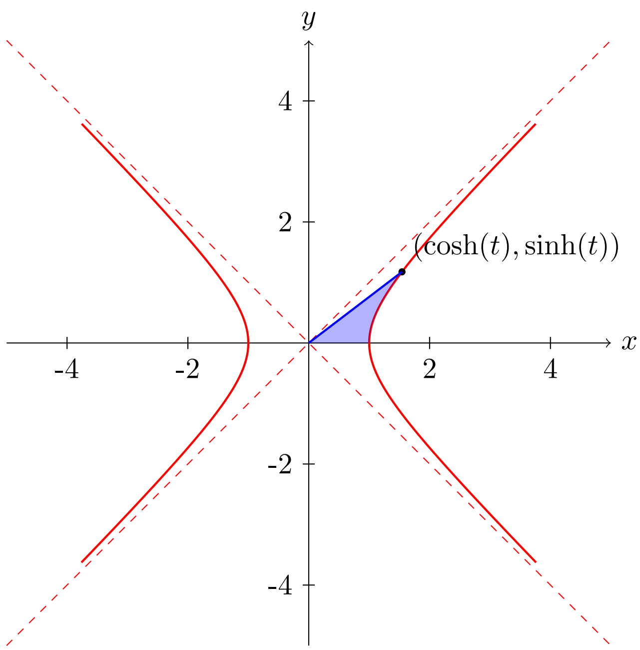


Figure 1: Sector of the hyperbola

Catenary The hyperbolic cosine function describes the shape of a hanging chain or cable. The catenary is the curve formed by a chain hanging from two points. It is given by the equation:

$$a \cosh\left(\frac{x}{a}\right), \tag{33}$$

Definiton 2.0.3 (Hyperbolic Tangent)**.** The hyperbolic tangent function is defined as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{34}$$

Derivative The derivative of the hyperbolic tangent function resembles the derivative of the regular tangent function:

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x). \tag{35}$$

Identities The hyperbolic tangent function satisfies the following identities:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \tag{36}$$

$$\operatorname{sech}^2(x) = 1 - \tanh^2(x). \tag{37}$$

Secant, Cosecant, Cotangent The hyperbolic secant, cosecant, and cotangent functions are defined similarly to the regular secant, cosecant, and cotangent functions. They are the **reciprocal** of the hyperbolic cosine, sine, and tangent functions, respectively.

Inverse Hyperbolic Functions The inverse hyperbolic functions are defined as the inverse of the hyperbolic functions. They are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$, etc.

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{38}$$

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right) \tag{39}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{40}$$

$$\operatorname{arcsch} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) \tag{41}$$

$$\operatorname{arsech} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right) \tag{42}$$

$$\operatorname{arcoth} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \tag{43}$$

3 Further Applications of Integrals

3.1 Arc Length

Arc Length Given the curve $C : y = f(x)$, the arc length of the curve between two points a and b, where $y'(x)$ is continuous, we would like the compute the arc length of the curve:

$$L = \int_C ds$$

Formula We can derive the formula for the arc length of a curve $y = f(x)$ between two points a and b as:

At x_i and x_{i+1}, the length of the segment is:

$$\begin{aligned} \Delta s &\approx \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} \\ &= \sqrt{\Delta_i x^2 + \Delta_i y^2} \end{aligned}$$

We also have, by MVT:

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= y'(c) \\
\Delta y &= y'(c)\Delta x \\
\Delta s &\approx \sqrt{\Delta x^2 + (y'(c)\Delta x)^2} \\
&= \sqrt{1 + y'(c)^2}\Delta x
\end{aligned}$$

Taking a Riemann sum:

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + y'(c_i)^2}\Delta x \\
&= \int_a^b \sqrt{1 + y'(x)^2}\, dx \qquad (44)
\end{aligned}$$

Example 3.1.1. Given $f(x) = x^{\frac{3}{2}}$ between $x = 0$ and $x = 44$, we would like to compute the arc length of the curve:

$$\begin{aligned}
L &= \int_0^{44} \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx \\
&= \int_0^{44} \sqrt{1 + \frac{9}{4}x}\, dx \\
&= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \Big|_0^{44} = 296
\end{aligned}$$

Inverse Also, we can take $x = f^{-1}(y)$ to compute the arc length of a curve.

$$L = \int_{f(a)}^{f(b)} \sqrt{1 + ((f^{-1})'(y))^2}\, dy \qquad (45)$$

3.2 Surface Area of Revolution

Surface Area of Revolution Given the curve $C : y = f(x)$, we would like to compute the surface area of the curve between two points a and b when rotated about the x-axis:

$$A_x = \int_C 2\pi y\, ds$$

Formula Deriving the formula for the surface area of a curve $y = f(x)$ between two points a and b rotated about the x-axis as follows:

At x_i and x_{i+1}, the length of the segment is:

$$\Delta s \approx \sqrt{\Delta_i x^2 + \Delta_i y^2}$$

The surface area of the segment is, by continuity:

$$A_i = \pi(y_i + y_{i+1})\Delta s = 2\pi y^\star \Delta s \tag{46}$$

As the difference between $y^\star$ and y_i diminishes as $\Delta x \to 0$, taking a Riemann sum:

$$\begin{aligned} A_x &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi y_i \sqrt{\Delta_i x^2 + \Delta_i y^2} \\ &= \int_a^b 2\pi y \sqrt{1 + y'(x)^2}\, dx \end{aligned} \tag{47}$$

The surface area of a curve $y = f(x)$ (i.e. $x = f^{-1}$)between two points $f(a)$ and $f(b)$ rotated about the y-axis is:

$$A_y = \int_{f(a)}^{f(b)} 2\pi x \sqrt{1 + x'(y)^2}\, dy \tag{48}$$

Example 3.2.1. Given $y = \sqrt{x}$ between $x = 0$ and $x = 1$, we would like to compute the surface area of the curve:

$$\begin{aligned} A_x &= \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx \\ &= \pi \int_0^1 \sqrt{4x+1}\, dx \end{aligned}$$

Let $u = 4x + 1$, then $du = 4dx$:

$$\begin{aligned} &= \frac{\pi}{4} \int_1^5 \sqrt{u}\, du \\ &= \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right) \end{aligned}$$

3.3 Applications to Physics and Engineering

3.3.1 Hydrostatic Pressure and Force

Hydrostatic Pressure The hydrostatic pressure and Force is given by the Archimedes' principle:

$$P = \rho g \cdot d, \tag{49a}$$

$$F = \rho g \cdot d \cdot A \tag{49b}$$

Riemann Sum Given a tank rectangular with width defined by $w = f(x)$, we can take a horizontal slice, $x^\star$, of the tank to compute the force exerted on the tank by the water:

$$F = \sum_{i=1}^{n} \rho g x_i f(x_i) \Delta x \tag{50}$$

Formula Given a tank rectangular with width defined by $w = f(x)$, where x is depth, we would like to compute the force exerted on the tank by the water:

$$F = \int_0^b \rho g x f(x) \cdot dx \tag{51}$$

, which is derived by considering dx as the depth a horizontal slice of the tank.

3.3.2 Center of Mass and Moments of Inertia

Properties of the Center of Mass The center of mass of a region R has the following properties:

1. **Symetry**: For all axis of symmetry, the center of mass lies on the axis.
2. **Additivity**: The center of mass of a region is the weighted average of the centers of mass of its parts. The weights are the areas of the parts.

$$\bar{x} = \sum_i \frac{A_i \bar{x}_i}{A} \tag{52}$$

Formula The centroid of a uniformly-dense region R bounded by $y = f(x)$, $x \in [a, b]$, $y \in [f(a), f(b)]$ is derived as follows:

We first have $x_i^\star$ and A_i as follows:

$$x_i^\star = \frac{1}{2}(x_i + x_{i+1}) \quad A_i = f(x_i)\Delta x$$

For $\bar{x}$, we have:

$$A\bar{x} \approx \sum_i x_i^\star A_i$$

Taking a Riemann sum:

$$A\bar{x} = \lim_{n\to\infty} \sum_i x_i^\star A_i$$

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx \tag{53}$$

Similarly, for $\bar{y}$:

$$A\bar{y} = \lim_{n\to\infty} \sum_i f(x_i^\star) A_i$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\, dx \tag{54}$$

Theorem 3.3.1 (Pappus's Theorem)**.** The volume of a solid of revolution is given by the product of the area of the region and the distance, R, traveled by the centroid of the region from the axis of rotation.

$$V = 2\pi RA \tag{55}$$

Example 3.3.2 (Eliptical Torus)**.** Given an eliptical torus (tall donut) with major axis a and minor axis b, we would like to compute the volume of the torus:

$$\begin{aligned} V &= 2\pi RA \\ &= 2\pi R \cdot \pi ab = 2\pi^2 abR \end{aligned}$$

This could be, alternatively, shown via the washer method and the shell method.

3.4 Applications to Economics and Biology

"Biologists are so bad at math" - Prof. Davis as he proceeds to erase the board and skip to the next section.

4 Parametric Equations and Polar Coordinates

4.1 Curves Defined by Parametric Equations

Parametric Equations For $t \in \mathbb{R}$, we have:

$$x = x(t), \quad y = y(t) \tag{56}$$

Example 4.1.1 (Newton's Laws of Motion)**.** Given $\ddot{x} = 0$, $\ddot{y} = -g$, we have:

$$\begin{aligned} x(t) &= A_1 t + A_2, \\ y(t) &= -\frac{1}{2} g t^2 + B_1 t + B_2 \end{aligned}$$

At initial conditions $t = 0$, we have $x(0) = 0$, $y(0) = 0$, $\dot{x}(0) = v_0 \cos(\theta)$, $\dot{y}(0) = v_0 \sin(\theta) - \frac{1}{2} g t^2$.

Example 4.1.2 (Ellipse)**.** Given $x = a\cos(t)$, $y = b\sin(t)$, we have:

$$\begin{aligned} x^2 &= a^2 \cos^2(t), \\ y^2 &= b^2 \sin^2(t) \end{aligned}$$

, which satisfies $x^2/a^2 + y^2/b^2 = 1$.

Intersections of Parametric Curves Given two parametric curves C_1, C_2, defined by:

$$\begin{aligned} C_1 &: x_1 = x_1(t), \quad y_1 = y_1(t) \\ C_2 &: x_2 = x_2(t), \quad y_2 = y_2(t) \end{aligned}$$

, we can find the intersection points by solving the system of equations:

$$x_1(t) = x_2(t)$$
$$y_1(t) = y_2(t)$$

Example 4.1.3. Given $x_1 = 2t + t$, $y_1 = 5 - 4t$, $x_2 = 3 - 5\cos(\pi t)$, $y_2 = 1 + 5\sin(\pi t)$, we have:

$$2t + 1 = 3 - 5\cos(\pi t)$$
$$5 - 4t = 1 + 5\sin(\pi t)$$

4.2 Calculus with Parametric Curves

Tangents The tangent to a parametric curve C at a point t is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \tag{57}$$

Equation of the Tangent The equation of the tangent to a parametric curve C at a point t_0 is given by:

$$y'(t_0)(x - x(t_0)) - x'(t_0)(y - y(t_0)) = 0 \tag{58}$$

Hence, if $x'(t_0) = 0$, we have a vertical tangent; if $y'(t_0) = 0$, we have a horizontal tangent. However, if $x'(t_0) = 0$ and $y'(t_0) = 0$, we have no information.

Example 4.2.1. Let $x = \sin 2t$, $y = \sin t$, we have:

$$x'(t) = 2\cos 2t,$$
$$y'(t) = \cos t$$

Vertical Tangent: $2\cos 2t = 0 \Rightarrow t = \frac{\pi}{4} + \frac{n\pi}{2}$. **Horizontal Tangent**: $\cos t = 0 \Rightarrow t = \frac{\pi}{2} + n\pi$. **At** $t = 0$ we have $x' = y' = 0$. Thus, we have no information. On the graph, we see that the curve intersects itself at $t = 0$.

Area under Parametric Curve The area under a parametric curve C between t_1 and t_2 is given by:

$$A = \int_{x(t_1)}^{x(t_2)} y(x)\,dx = \int_{t_1}^{t_2} y(t)x'(t)\,dt \tag{59}$$

Definiton 4.2.2 (Orientation)**.** The orientation of a parametric curve C is given by the direction of the curve as t increases. If the enclosed area is to the left of the trace of t (counter-clockwise) the orientation is positive; otherwise, it is negative.

The sign of the area of a parametric curve C is given by the orientation of the curve.

Area of Closed Curves The area of a closed parametric curve C is given by:

$$A = \oint_C dA = \int_{t_1}^{t_2} y(t)x'(t)\,dt = \int_{t_1}^{t_2} x(t)y'(t)\,dt \tag{60}$$

, where

$$x(t_1) = x(t_2), \quad y(t_1) = y(t_2) \quad \text{and} \quad t_2 \text{ is the smallest } t > t_1 \text{ to satisfy the condition.}$$

Arc Length The arc length of a parametric curve C is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{61}$$

Example 4.2.3. Given $x = t\cos t$, $y = t\sin t$, we have:

$$\begin{aligned} \frac{dx}{dt} &= \cos t - t\sin t, \\ \frac{dy}{dt} &= \sin t + t\cos t \end{aligned}$$

Thus, the arc length is given by:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{1+t^2}\,dt \end{aligned}$$

Using the substitution $t = \tan\theta$, we have:

$$= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln(2\pi + \sqrt{1+4\pi^2})$$

Surface Area of Revolution The surface area of a parametric curve C rotated about the x-axis is given by:

$$A = \int_a^b 2\pi y(t) ds$$

From the arc length formula, we have $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$, thus:

$$= \int_a^b 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{62}$$

Similarly, the surface area of a parametric curve C rotated about the y-axis is given by:

$$A = \int_a^b 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{63}$$

Example 4.2.4 (Surface Area of of a Ellipse)**.** Given $x = a \sin t$, $y = b \cos t$, we have:

$$\frac{dx}{dt} = a \cos t,$$
$$\frac{dy}{dt} = -b \sin t$$

Thus, the surface area of the ellipse is given by:

$$\begin{aligned} A &= \int_0^{2\pi} 2\pi b \cos t \sqrt{a^2 \cos^2 t + b^2 \sin^2 t}\, dt \\ &= 2\pi b \int_0^{2\pi} \sqrt{a^2(1 - \sin^2 t) + b^2 \sin^2 t}\, dt \\ &= 2\pi b \int_0^{2\pi} \sqrt{a^2 + b^2 - (a^2 - b^2) \sin^2 t}\, dt \end{aligned}$$

This has no analytical solution, but can be solved numerically. Typically, we take $\epsilon = \sqrt{\frac{a^2 - b^2}{a^2}}$.

5 Polar Coordinates

Polar Coordinates Given a point P in the plane, we can define the polar coordinates of P as (r, θ), where r is the distance from the origin to P and θ is the angle between the positive x-axis and the line segment from the origin to P.

Transformation We can convert between polar and Cartesian coordinates as follows:

$$x = r \cos \theta, \quad y = r \sin \theta \tag{64}$$

Reverse Transformation We can convert between Cartesian and polar coordinates as follows:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) \tag{65}$$

Note The angle θ is not unique, as $\theta + 2\pi n$ for $n \in \mathbb{Z}$ represents the same point, beware of this when converting between polar and Cartesian coordinates.

Example 5.0.1 (Lines)**.**

$$\begin{aligned} y = mx + b &\implies \theta = \alpha = \arctan m \\ x = a &\implies r = a \sec \theta \\ y = a &\implies r = a \csc \theta \end{aligned}$$

Example 5.0.2 (Cirlces)**.**

$$x^2 + y^2 = a^2 \implies r = a$$

Let $r = 6\sin\theta$, we have:

$$r^2 = 6r\sin\theta$$
$$x^2 + y^2 = 6y$$
$$x^2 + (y-3)^2 = 9$$

We deduce that the curve is a circle with radius 3 and center $(0, 3)$.

Cylinrical Coordinates Given a point P in space, we can define the cylindrical coordinates of P as (r, θ, z), where r is the distance from the z-axis to P, θ is the angle between the positive x-axis and the projection of the line segment from the origin to P onto the xy-plane, and z is the distance from the xy-plane to P.

Spherical Coordinates Given a point P in space, we can define the spherical coordinates of P as (ρ, θ, ϕ), where ρ is the distance from the origin to P, θ is the angle between the positive x-axis and the projection of the line segment from the origin to P onto the xy-plane, and ϕ is the angle between the positive z-axis and the line segment from the origin to P.

5.1 Graphing in Polar Coordinates

Example 5.1.1. Let $r = \frac{1}{2} + \cos\theta$, we first figure out the origin of the curve by setting $r = 0$:

$$0 = \frac{1}{2} + \cos\theta$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

We then deduce the maximum and minimum of the curve to the origin by setting $\frac{dr}{d\theta} = 0$:

$$\frac{dr}{d\theta} = -\sin\theta = 0$$
$$\theta = 0, \pi, 2\pi$$

We also look for symmetry in the curve by checking $r(\theta) = r(-\theta)$:

$$\frac{1}{2} + \cos\theta = \frac{1}{2} + \cos(-\theta)$$

Thus, the curve is symmetric about the x-axis. We also look for symmetry about the y-axis by checking $r(\theta) = r(\pi + \theta)$:

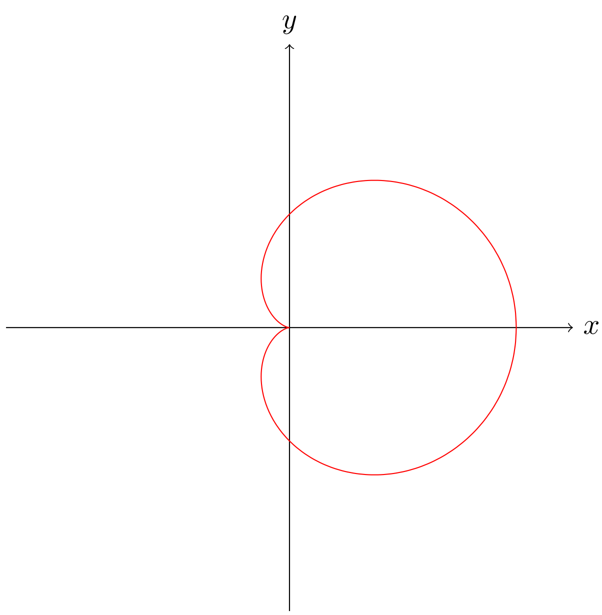
$$\frac{1}{2} + \cos\theta \neq \frac{1}{2} + \cos(\pi + \theta)$$

Thus, the curve is not symmetric about the y-axis. We also look for intervals in which the curve is increasing or decreasing by checking $r'(\theta) > 0$ or $r'(\theta) < 0$.

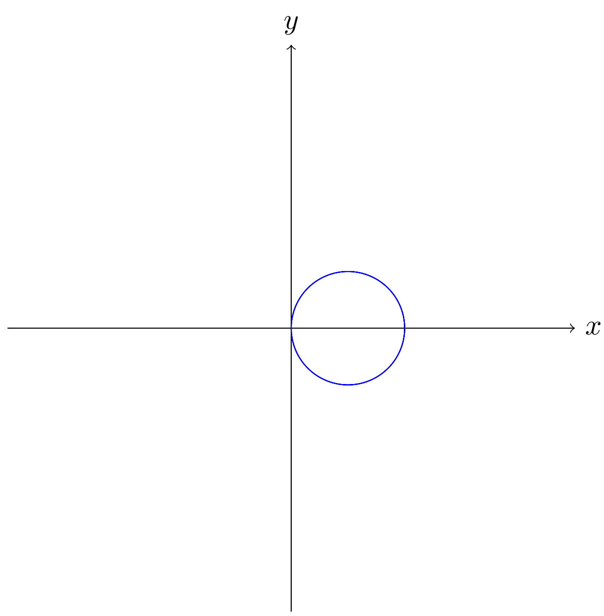
Common Polar Curves Some common polar curves are:

1. **Cardioid**: $r = a(1 + \cos\theta)$ (heart-shaped)
2. **Circle**: $r = a\cos\theta$
3. **Limaşçons**: $r = a + b\sin\theta$ (heart with a hole)
4. **Lemniscate**: $r^2 = a^2 \sin 2\theta$ (infinity symbol, diagonal) $r^2 = a^2\cos 2\theta$ (infinity symbol, horizontal)
5. **Petal Curves**: $r = a\cos n\theta$ (mn-petal flower) $r = a\sin n\theta$ (mn-petal flower); $m = 2$ if n is even, $m = 1$ if n is odd.

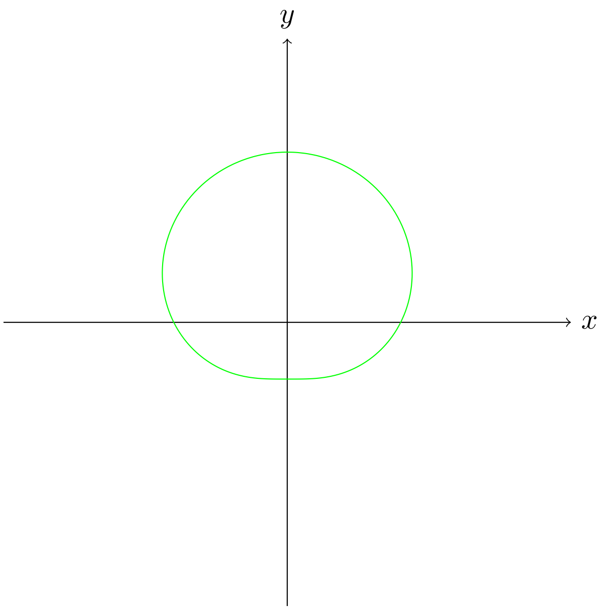
Below are some examples of polar curves:



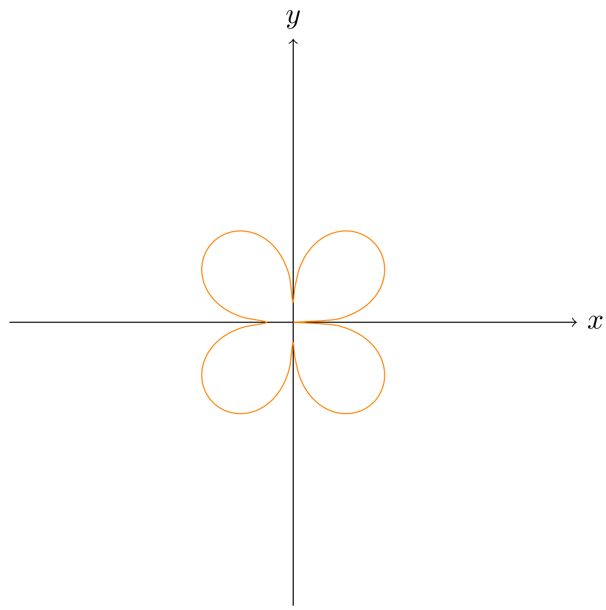
Cardioid: $r = 1 + \cos\theta$



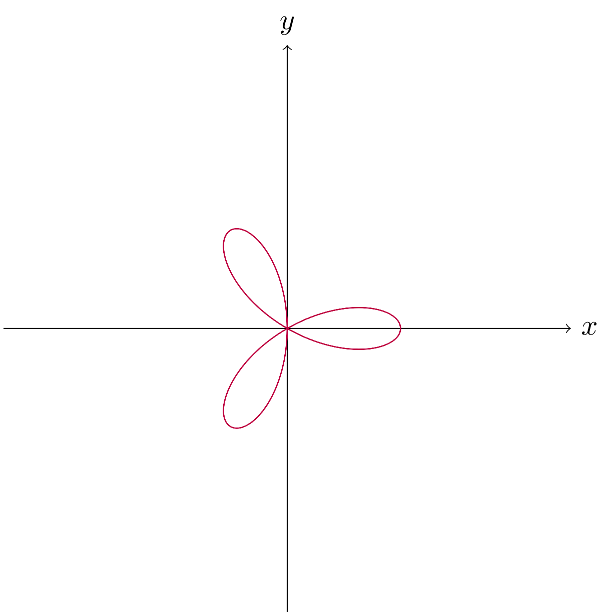
Circle: $r = \cos\theta$



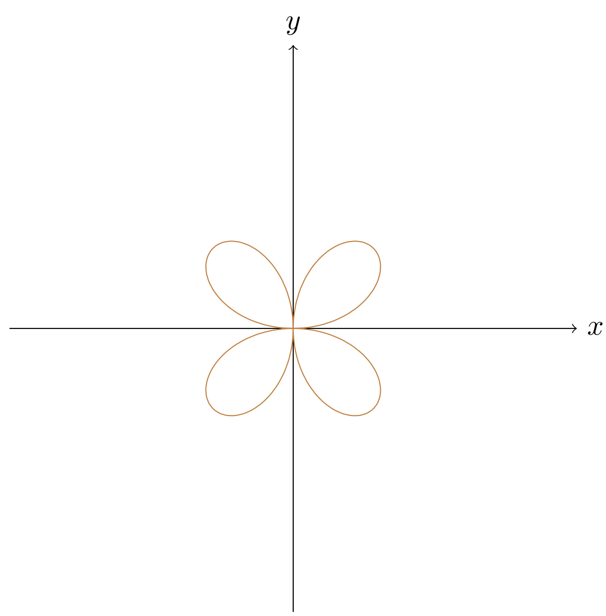
Limaçon: $r = 1 + 0.5 \sin\theta$



Lemniscate: $r^2 = \sin 2\theta$



Petal Curve: $r = \cos 3\theta$



Petal Curve: $r = \sin 2\theta$

A detailed list of polar curves and their properties could be found in Appendix B.

5.2 The intersection of Polar Curves

Example 5.2.1. Let $r_1 = \sin\theta$, $r_2 = -\cos\theta$, we have:

$$\begin{aligned}\sin\theta &= -\cos\theta\\ \tan\theta &= -1\\ \theta &= \frac{3\pi}{4}, \frac{7\pi}{4}\end{aligned}$$

At these points, we can find the x and y coordinates of the intersection points. But also, we must check for the origin point $(x, y) = (0, 0)$. Also $r_1 = r_2$ is not always reliable so we have to check it intuitively.

5.3 Area and Length in Polar Coordinates

Area in Polar Coordinates The area of a region $R : r = g(\theta)$ in polar coordinates is given by:

$$A \approx \sum_i \frac{1}{2}[g(\theta_i)]^2 \Delta\theta_i$$

Taking a Riemann sum:

$$= \int_\alpha^\beta \frac{1}{2}[g(\theta)]^2\, d\theta \tag{66}$$

Example 5.3.1. Given $r = 1 + \cos\theta$, we have:

$$\begin{aligned}A &= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta\\ &= \int_0^{2\pi} \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta)\, d\theta\\ &= \frac{1}{2}\int_0^{2\pi} 2 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta\\ &= \frac{1}{2}\left(2\pi + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = \frac{3\pi}{2}\end{aligned}$$

Area between Two Polar Curves The area between two polar curves $r^2 = f(\theta)$ and $r = g(\theta)$ is given by:

$$A = \int_\alpha^\beta \frac{1}{2}[f(\theta)^2 - g(\theta)^2]\, d\theta \tag{67}$$

Example 5.3.2. Given lemniscate $r^2 = 4\cos 2\theta$, circle $r = 1$, to find the area between the two curves, outside the circle, and inside the lemniscate. We have:

We first consider the intersection points

$$\begin{aligned}4\cos 2\theta &= 1\\ \cos 2\theta &= \frac{1}{4}\end{aligned}$$

We have points at $\theta = \pm 0.659$. Also, since the lemniscate is symmetric about the y-axis, we have:

$$\begin{aligned} \frac{1}{2}A &= \int_{-0.659}^{0.659} 4\cos^2 2\theta - 1\, d\theta \\ A &= 2.554 \end{aligned}$$

Example 5.3.3. Given $r = \sin\theta$, $r = \cos\theta$, we have:

We first consider the intersection points

$$\begin{aligned} \sin\theta &= \cos\theta \\ \tan\theta &= 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

We then consider the area between the two curves, notice that (0, 0) is also an intersection point.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin\theta)^2 - (\cos\theta)^2\, d\theta \\ &= \frac{\pi - 2}{8} \end{aligned}$$

Tangent Lines in Polar Coordinates The tangent line to a polar curve $r = r(\theta)$ is derived as follows:

$$x(\theta) = r(\theta)\cos\theta, \quad y(\theta) = r(\theta)\sin\theta$$

By chain rule, we have and product rule, we have:

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta} \tag{68}$$

$$\tag{69}$$

Length of a Polar Curve Derived from above, the length of a polar curve $r = r(\theta)$ between α and β is given by:

$$L = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta \tag{70}$$

Example 5.3.4. Given $r = a - a\cos\theta$, we have:

$$\begin{aligned} r' &= a\sin\theta \\ L &= \int_0^{2\pi} \sqrt{(a - a\cos\theta)^2 + a^2\sin^2\theta}\, d\theta \\ &= a\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \end{aligned}$$

We can use the double angle formula to simplify the integral, with $\cos 2\theta = 1 - 2\sin^2\theta$:

$$= 8a$$

6 Infinite Sequences and Series

6.1 Sequences

Sequences A sequence is a list of numbers written in a specific order.

Example 6.1.1. Let $f : A \to \mathbb{R}$ be a function, and $f(x) = \frac{1}{x}$. We can restrict the domain of f to $A = \{1, 2, 3, \ldots\} = \mathbb{N}$, and we can define the sequence $a_n = f(n) = \frac{1}{n}$.

Notation Typcially, we denote a sequence as $\{a_n\}$, where a_n is the nth term of the sequence. We can also denote with curly bracket as $\{a_n\} = \{a_1, a_2, a_3, \ldots\} = \{f(n)\}$.

Ratio The ratio of two sequences $\{a_n\}$ and $\{b_n\}$ for some $n \leq N$ is a sequence $\{c_n\}$ such that $c_n = \frac{a_n}{b_n}$.

Example 6.1.2. Let $a_n = n^2$, $b_n = e^n$, we have:

$$\frac{a_n}{b_n} = \{\frac{1}{e}, \frac{4}{e^2}, \frac{9}{e^3}, \ldots\}$$

Bounded Above A sequence $\{a_n\}$ is said to be bounded above if there exists a number M such that $|a_n| \leq M$ for all n. (Similarly for bounded below).

Definiton 6.1.3 (Monotonic Sequences)**.** A sequence $\{a_n\}$ is said to be non decreasing if $a_{n+1} \geq a_n$ for all n. Similarly, a sequence is said to be non increasing if $a_{n+1} \leq a_n$ for all n. A sequence is said to be increasing if $a_{n+1} > a_n$ for all n. Similarly, a sequence is said to be decreasing if $a_{n+1} < a_n$ for all n.
A sequence $\{a_n\}$ is said to be monotonic if it is either increasing, decreasing, non increasing, or non decreasing.

Convergent Sequences A sequence $\{a_n\}$ is said to be convergent if there exists a number L such that for all $\epsilon > 0$, there exists an N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

A Trigonometric Derivatives

Trigonometric Function	**Inverse Trigonometric Function**
$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$	$\frac{d}{dx}\mathrm{arccot}(x) = \frac{-1}{1+x^2}$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\frac{d}{dx}\mathrm{arcsec}(x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$	$\frac{d}{dx}\mathrm{arccsc}(x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

Hyperbolic Function	**Inverse Hyperbolic Function**
$\frac{d}{dx}\sinh(x) = \cosh(x)$	$\frac{d}{dx}\mathrm{arsinh}(x) = \frac{1}{\sqrt{x^2+1}}$
$\frac{d}{dx}\cosh(x) = \sinh(x)$	$\frac{d}{dx}\mathrm{arcosh}(x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
$\frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$	$\frac{d}{dx}\mathrm{artanh}(x) = \frac{1}{1-x^2} \quad (\lvert x\rvert < 1)$
$\frac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$	$\frac{d}{dx}\mathrm{arcoth}(x) = \frac{1}{1-x^2} \quad (\lvert x\rvert > 1)$
$\frac{d}{dx}\mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x)$	$\frac{d}{dx}\mathrm{arsech}(x) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x \leq 1)$
$\frac{d}{dx}\mathrm{csch}(x) = -\mathrm{csch}(x)\coth(x)$	$\frac{d}{dx}\mathrm{arcsch}(x) = \frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$

B Polar Curves

Below are some common polar curves: Click: 5.1 to go back to the section on polar coordinates.

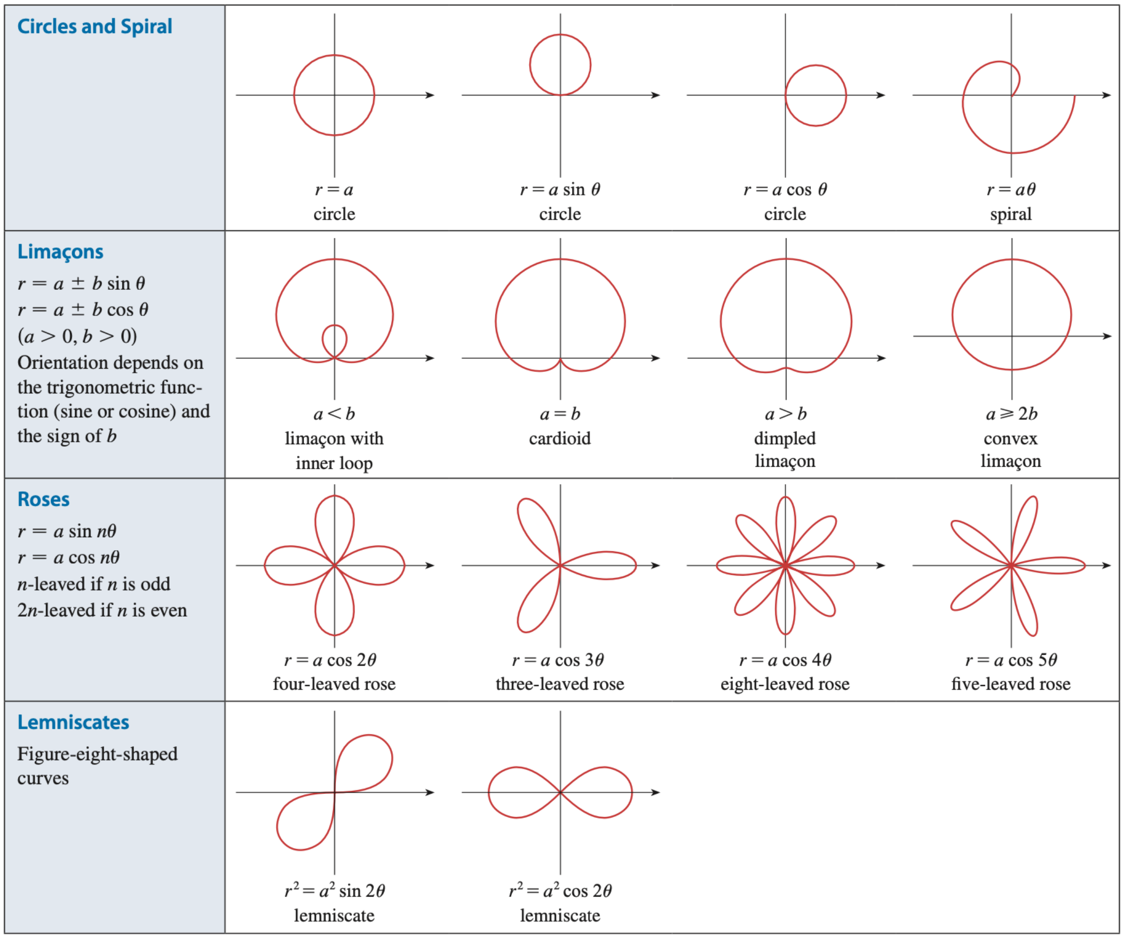


Figure 2: Polar Curves

Identity	**Value**
$\sin^2\theta + \cos^2\theta = 1$	1
$\tan^2\theta = \sec^2\theta$ - 1	$\sec^2\theta - 1$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\frac{\sin\theta}{\cos\theta}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\frac{\cos\theta}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$	$\frac{1}{\cos\theta}$
$\csc\theta = \frac{1}{\sin\theta}$	$\frac{1}{\sin\theta}$
$\sin(-\theta) = -\sin\theta$	$-\sin\theta$
$\cos(-\theta) = \cos\theta$	$\cos\theta$
$\tan(-\theta) = -\tan\theta$	$-\tan\theta$
$\cot(-\theta) = -\cot\theta$	$-\cot\theta$
$\sec(-\theta) = \sec\theta$	$\sec\theta$
$\csc(-\theta) = -\csc\theta$	$-\csc\theta$
$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$	$\sin\theta\cos\phi \pm \cos\theta\sin\phi$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$	$\cos\theta\cos\phi \mp \sin\theta\sin\phi$
$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$	$\frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$
$\sin(2\theta) = 2\sin\theta\cos\theta$	$2\sin\theta\cos\theta$
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$	$\cos^2\theta - \sin^2\theta$
$2\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$	$2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$	$\frac{2\tan\theta}{1-\tan^2\theta}$

Table 3: Trigonometric Identities

C Trigonometric Identities