Due: 10:00pm, Tuesday Jan. 21, 2025

MAT185 – Linear Algebra
Assignment 1

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I have not used generative AI in writing this assignment.
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Question 1:

In this problem, you will prove that the elementary operations you learnt in ESC103 do not change the set of solutions of a system of linear equations. Consider the four linear systems

$$\mathcal{A}: \quad \begin{cases} 3x + 2y + 2z = 9 \\ 11x + 7y + 3z = 15 \\ 3x + 2y + z = 5 \end{cases} \qquad \mathcal{B}: \quad \begin{cases} 3x + 2y + 2z = 9 \\ 11x + 7y + 3z = 15 \\ (3+11\beta)x + (2+7\beta)y + (1+3\beta)z = 5+15\beta \end{cases}$$

$$\mathcal{C}: \quad \begin{cases} 3x + 2y + 2z = 9 \\ 11\alpha x + 7\alpha y + 3\alpha z = 15\alpha \\ 3x + 2y + z = 5 \end{cases} \qquad \mathcal{D}: \quad \begin{cases} 11x + 7y + 3z = 15 \\ 3x + 2y + 2z = 9 \\ 3x + 2y + z = 5 \end{cases}$$

Above, x, y, and z are all **variables**. The parameters α and β are real numbers.

Let $A \subseteq \mathbb{R}^3$ be the **set of solutions** of the linear system $\mathcal{A}$. A point $(a, b, c) \in A$ means that

$$\mathcal{A}: \quad \begin{cases} 3a + 2b + 2c = 9 & \textbf{TRUE} \\ 11a + 7b + 3c = 15 & \textbf{TRUE} \\ 3a + 2b + c = 5 & \textbf{TRUE} \end{cases}$$

Because a, b, and c are all **real numbers**, equations involving them are either true (as in $1 + 1 = 2$ is true) or false (as in $1 + 1 = 3$ is false). And so, (a, b, c) is in A (is a solution of $\mathcal{A}$) if all three equations are true when the variables x, y, and z take on the values a, b, and c respectively. If there are no points (a, b, c) for which the three equations are all true then $\mathcal{A}$ has no solutions and $A = \varnothing$. *Note: this does not contradict the statement that $A \subseteq \mathbb{R}^3$ because $\varnothing \subseteq \mathbb{R}^3$.*

Pro tip: If you have an equation where the left-hand side and right-hand side are real numbers (or are numbers in any field) then the equation $LHS = RHS$ is **TRUE** if and only if $LHS - RHS = 0$ and is **FALSE** if and only if $LHS - RHS \neq 0$.

Finally, there is the question of how you could show that two sets, A and B, are equal.

- If you want to prove that $A = B$ and $A = \varnothing$ then you need to prove that $B = \varnothing$.
- If you want to prove that $A = B$ and $B = \varnothing$ then you need to prove that $A = \varnothing$.
- If you want to prove that $A = B$ and neither A nor B are nonempty then you can do this by proving
  - if $a \in A$ then $a \in B$ (this proves $A \subseteq B$) **and**
  - if $b \in B$ then $b \in A$ (this proves $B \subseteq A$).

If you have sets A and B and you don't know whether or not they are empty or nonempty, you have do all three steps above.

For the curious: The third case is related to another common proof technique: if you want to prove that two real numbers are equal, $a = b$, you first use one idea/approach to prove $a \leq b$ and a different idea/approach to prove $a \geq b$. This technique is jarring the first few times one sees it. This approach works for any totally-ordered field; it doesn't work for the complex numbers, for example.

(a) Let A be the set of solutions of $\mathcal{A}$ and B be the set of solutions of $\mathcal{B}$. Prove that $A = B$. *Note: you don't know whether or not $\mathcal{A}$ or $\mathcal{B}$ have any solutions. This means that you must address all three possible cases: $A = \varnothing$, $B = \varnothing$, and neither A nor B are the empty set.*

Please start your answer on the next page, not on this one. The grader will not look at this page. You can continue your answer on the top of page 4, if needed.

case 1: $A = \varnothing$

Note I denote the k^{th} equation of $\mathcal{P}$ as $\mathcal{P}_k$ for linear system $\mathcal{P} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$.

We prove that $B = \varnothing$. Assume, for the sake of contradiction, that $B \neq \varnothing$. Then there exists a point $(a, b, c) \in B$ such that $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are **TRUE**. Thus:

- For all $\beta \in \mathbb{R}$, $\beta\mathcal{B}_2 : 11\beta a + 7\beta b + 3\beta c = 15\beta$ is **TRUE**; also
- $\mathcal{B}_3$ is **TRUE**.

Hence, $\mathcal{B}_3 - \beta\mathcal{B}_2$ is **TRUE** for all $\beta \in \mathbb{R}$. This implies there exist (a, b, c) s.t.: $3a + 2b + c = 5$ is **TRUE**. It follows from $\mathcal{A}_3 : 3a + 2b + c = 5$ and that $\mathcal{B}_2 = \mathcal{A}_2$ and that $\mathcal{B}_1 = \mathcal{A}_1$ are all **TRUE**. Hence $(a, b, c) \in A$. Thus, $A \neq \varnothing$, a contradiction. Therefore, $B = \varnothing$.

case 2: $B = \varnothing$

We prove that $A = \varnothing$. Assume, for the sake of contradiction, that $A \neq \varnothing$. Then there exists a point $(a, b, c) \in A$ such that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are **TRUE**. Thus:

- For all $\beta \in \mathbb{R}$, $\beta\mathcal{A}_2 : 11\beta a + 7\beta b + 3\beta c = 15\beta$ is **TRUE**; also
- $\mathcal{A}_3$ is **TRUE**.

Hence, $\mathcal{B}_3 = \mathcal{A}_3 + \beta\mathcal{A}_2$ is **TRUE** for all $\beta \in \mathbb{R}$. It follows from $\mathcal{A}_2 = \mathcal{B}_2$ and that $\mathcal{A}_1 = \mathcal{B}_1$ are all **TRUE**. Hence $(a, b, c) \in B$. Thus, $B \neq \varnothing$, a contradiction. Therefore, $A = \varnothing$.

case 3: both A and B are nonempty

$(A \subseteq B)$ Take $(a_1, a_2, a_3) \in A$. Then $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are **TRUE**. Thus:

- For all $\beta \in \mathbb{R}$, $\beta\mathcal{A}_2 : 11\beta a_1 + 7\beta a_2 + 3\beta a_3 = 15\beta$ is **TRUE**; also
- $\mathcal{A}_3$ is **TRUE**.

Hence, $\mathcal{B}_3 = \mathcal{A}_3 + \beta\mathcal{A}_2$ is **TRUE** for all $\beta \in \mathbb{R}$. It follows from $\mathcal{A}_2 = \mathcal{B}_2$ and that $\mathcal{A}_1 = \mathcal{B}_1$ are all **TRUE**. Thus, $(a_1, a_2, a_3) \in B$. Therefore, $A \subseteq B$.

$(B \subseteq A)$ Take $(b_1, b_2, b_3) \in B$. Then $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are **TRUE**. Thus:

- For all $\beta \in \mathbb{R}$, $\beta\mathcal{B}_2 : 11\beta b_1 + 7\beta b_2 + 3\beta b_3 = 15\beta$ is **TRUE**; also
- $\mathcal{B}_3$ is **TRUE**.

Hence, $\mathcal{A}_3 = \mathcal{B}_3 - \beta\mathcal{B}_2$ is **TRUE** for all $\beta \in \mathbb{R}$. It follows from $\mathcal{B}_2 = \mathcal{A}_2$ and that $\mathcal{B}_1 = \mathcal{A}_1$ are all **TRUE**. Thus, $(b_1, b_2, b_3) \in A$. Therefore, $B \subseteq A$.

You can continue your answer here. If you do so, please make it clear which of the three cases you're continuing!

(b) Let A be the set of solutions of $\mathcal{A}$ and C be the set of solutions of $\mathcal{C}$.

i. The system $\mathcal{C}$ has a parameter $\alpha \in \mathbb{R}$. Under what conditions on α is $A \subseteq C$? *No justification needed, just one complete sentence.*

$\alpha \in \mathbb{R}$. It can be any real number.

ii. Under what conditions on α is $C \subseteq A$? *No justification needed, just one complete sentence.*

$\alpha \neq 0$. It can be any non zero real number.

iii. Assuming the condition of part ii. holds, prove that $C \subseteq A$. *You may assume that $C \neq \varnothing$; we'll assume that if you can address the $C = \varnothing$ case based on your methods in part (a).*

Given nonempty A, C, take $(a, b, c) \in C$ such that $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are **TRUE**. For all $\alpha \neq 0$, we have $\frac{1}{\alpha} \in \mathbb{R}$. So $\frac{1}{\alpha}\mathcal{C}_2$ is **TRUE**. Thus $\mathcal{B}_2 = \mathcal{C}_2$ is **TRUE**. It follows that $\mathcal{C}_3 = \mathcal{B}_3$ is **TRUE** and that $\mathcal{C}_1 = \mathcal{B}_1$ is **TRUE**. Therefore, $(a, b, c) \in A$. Hence, $C \subseteq A$.

(c) Let D be the set of solutions of $\mathcal{D}$. Prove that $A = D$. *You may assume that both A and D are nonempty; we'll assume you can address the case where either of them is the empty set based on your work in part (a).*

This part is worth zero points. Your proof will not be graded. If you got stuck, or wrote a proof you aren't quite sure of, please post to piazza or come to office hours.

YOU MUST UPLOAD THIS PAGE EVEN IF YOU WROTE NOTHING ON IT.

We have $\mathcal{A}_1 = \mathcal{D}_2$ and $\mathcal{A}_2 = \mathcal{D}_1$ and $\mathcal{A}_3 = \mathcal{D}_3$. Given nonempty A, B, we have:

($A \subseteq D$) Take $(a_1, a_2, a_3) \in A$. Then $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are **TRUE**. Thus, in respective order:

- $\mathcal{D}_2$ is **TRUE**; also
- $\mathcal{D}_1$ is **TRUE**; also
- $\mathcal{D}_3$ is **TRUE**,

so, it follows that $(a_1, a_2, a_3) \in D$. Therefore, $A \subseteq D$.

($D \subseteq A$) Take $(d_1, d_2, d_3) \in D$. Then $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are **TRUE**. Thus, in respective order:

- $\mathcal{A}_2$ is **TRUE**; also
- $\mathcal{A}_1$ is **TRUE**; also
- $\mathcal{A}_3$ is **TRUE**,

so, it follows that $(d_1, d_2, d_3) \in A$. Therefore, $D \subseteq A$.

(d) At this point, you should have proven that the three elementary operations did not change the set of solutions of the linear system $\mathcal{A}$. The key insights of your proofs didn't rely on $\mathcal{A}$'s having exactly three equations and three variables and having coefficients 3, 2, 11 and so forth. We asked you to think about a simple, concrete example so that you could focus on the elementary operations and not be distracted by notation. Now we are asking you to face the notation and prove something about a general linear system.

Consider a system of m linear equations with n variables

$$\mathcal{A}: \quad \begin{cases} \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad 1 \leq i \leq m \end{cases}$$

where the coefficients a_{ij} and constants b_i are all real numbers. Let $\mathcal{B}$ be the linear system where the i_0th equation of $\mathcal{A}$ has been multiplied by $\alpha \neq 0$.

Assume that both $\mathcal{A}$ and $\mathcal{B}$ have solutions. Prove that $\mathcal{A}$ and $\mathcal{B}$ have the same set of solutions.

This part is worth zero points. Your proof will not be graded. If you got stuck, or wrote a proof you aren't quite sure of, please post to piazza or come to office hours.

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We have $\mathcal{A}_i = \mathcal{B}_i$ for all $i \neq i_0$ and $\mathcal{B}_{i_0} = \alpha\mathcal{A}_{i_0}$. Given nonempty A, B, we have:

$$\mathcal{A}_{i_0} \text{ is } \textbf{TRUE} \implies \alpha\mathcal{A}_{i_0} \text{ is } \textbf{TRUE} \implies \mathcal{B}_{i_0} \text{ is } \textbf{TRUE}.$$

Therefore, $\mathcal{A}$ and $\mathcal{B}$ have the same set of solutions.

(e) Consider writing down the augmented matrix of a linear system $\mathcal{A}$ and applying elementary row operations to the matrix until you have a matrix in reduced row echelon form (RREF). Let $\mathcal{R}$ be a linear system represented by that RREF matrix. Explain why $\mathcal{A}$ and $\mathcal{R}$ have the same set of solutions. *There's a lot of white space here but this isn't to indicate that you need to write at length to answer this question. You should be able to write your explanation w/ 100 or fewer words (in the space above the horizontal line).*

Consider the augmented matrix of $\mathcal{A}$ and apply elementary row operations to obtain the RREF matrix of $\mathcal{A}$. For each elementary row operation, the set of solutions of $\mathcal{A}$ remains unchanged. This is proven in the previous parts. Thus, $\mathcal{A}$ and $\mathcal{R}$ have the same set of solutions.

Question 2:

One of the goals of this question is to get you into good shape for True/False questions on exams.

For questions (b), (c), and (d) below, please assume the "natural" vector addition and scalar multiplication. For example, for functions assume the addition and multiplication given in section 4.2 of Medici. The ones given in the "An Unusual Vector Space" example are completely valid but are considered "unnatural" for this problem.

(a) Prove the following lemma by showing that all the vector space axioms hold. It's very useful and once you've proven it you can use it whenever you want.

Lemma: Let $\mathbf{V}$ be a vector space over the real numbers. If $\mathbf{V}_0 \subseteq \mathbf{V}$ contains $\mathbf{0}$ and, with the same vector addition and scalar multiplication of $\mathbf{V}$, is closed under vector addition and scalar multiplication then $\mathbf{V}_0$ is a vector space over the real numbers.

Additve Axioms

1. By assumption, $\mathbf{V}_0$ is closed under vector addition. Thus, for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}_0$, $\mathbf{u} + \mathbf{v} \in \mathbf{V}_0$.
2. By assumption, $\mathbf{V}_0$ has the same vector addition as $\mathbf{V}$ and $\mathbf{V}$ is a vetor space. Thus, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_0$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. By assumption, $\mathbf{V}_0$ contains $\mathbf{0}$ and $\mathbf{V}$ is a vector space. Thus, for all $\mathbf{u} \in \mathbf{V}_0 \subseteq \mathbf{V}$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For all $\mathbf{u} \in \mathbf{V}_0 \subseteq \mathbf{V}$, a vector space. Under the usual scalar multiplication, $0\mathbf{u} = \mathbf{0}$. With $(1 + (-1)) = 0$, we have:

$$\begin{aligned} 1\mathbf{u} + (-1)\mathbf{u} &= (1 + (-1))\mathbf{u} \\ &= 0\mathbf{u} \\ &= \mathbf{0}. \end{aligned}$$

   Since $\mathbf{V}_0$ is closed under scalar multiplication, $(-1)\mathbf{u} \in \mathbf{V}_0$. Thus, $\mathbf{V}_0$ contains the additive inverse of $\mathbf{u}$.

Scalar Multiplication Axioms

1. By assumption, $\mathbf{V}_0$ is closed under scalar multiplication. Thus, for all $c \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{V}_0$, $c\mathbf{u} \in \mathbf{V}_0$.
2. By assumption, $\mathbf{V}_0$ has the same scalar multiplication as $\mathbf{V}$ and $\mathbf{V}$ is a vector space. Thus, for all $c, d \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{V}_0 \subseteq \mathbf{V}$, $(cd)\mathbf{u} = c(d\mathbf{u})$.
3. By assumption, $\mathbf{V}_0$ has the same scalar multiplication as $\mathbf{V}$ and $\mathbf{V}$ is a vector space. Thus, for all $c \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbf{V}_0 \subseteq \mathbf{V}$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
4. By assumption, $\mathbf{V}_0$ has the same scalar multiplication as $\mathbf{V}$ and $\mathbf{V}$ is a vector space. Thus, for all $\mathbf{u} \in \mathbf{V}_0 \subseteq \mathbf{V}$, $1\mathbf{u} = \mathbf{u}$.

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●).

Hint: if you think any of the following are vector spaces, ask yourself whether there's a way of proving that this is true without having to start at the definition and proving that all eight axioms hold.

(b) The set $\mathbf{V}$ of all nonpositive real-valued functions on $[2,3]$ with the usual addition and scalar multiplication is a vector space over $\mathbb{R}$. *Note: a real-valued function on* $[2,3]$ *is a function* $f : [2,3] \to \mathbb{R}$. *That is, its domain is* $[2,3]$ *and its range is a subset of* $\mathbb{R}$. *Its range could be all of* $\mathbb{R}$, *of course. A function is nonpositive if* $f(x) \leq 0$ *for all* x *in its domain.*

○ True

● False

Pick $f \in \mathbf{V}$ such that there exist $x_0 \in [2,3]$, $f(x_0) \neq 0$. Choose real number $c < 0$, then $cf(x_0) > 0$, hence $c \cdot f \notin \mathbf{V}$. Thus, $\mathbf{V}$ is not closed under scalar multiplication. Therefore, $\mathbf{V}$ is not a vector space over $\mathbb{R}$.

(c) The set $\mathbf{V}$ of all real polynomials of degree exactly n with the usual addition and scalar multiplication is a vector space. *Note: A "real polynomial" is a polynomial whose coefficients are all real numbers.*

○ True

● False

Pick $p, q \in \mathbf{V}$ for which $A, -A \in \mathbb{R}$ such that $p(x) = Ax^n$ and $q(x) = -Ax^n$. Then $p(x) + q(x) = A_n x^n + (-A_n)x^n = 0x^n = 0$. However, 0 is not a polynomial of degree exactly n. Thus, $\mathbf{V}$ is not closed under vector addition. Therefore, $\mathbf{V}$ is not a vector space over $\mathbb{R}$.

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●).

Hint: if you think the following is a vector space, ask yourself whether there's a way of proving that this is true without having to start at the definition and proving that all eight axioms hold.

(d) The set of 2×2 upper triangular real matrices with the usual entry-wise addition and scalar multiplication is a vector space over the real numbers.

Notes: An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ for all $1 \leq j < i \leq n$. A matrix is real if all of its entries are real numbers. All integer matrices and all rational matrices are also real matrices. After doing this problem, ask yourself how your answer would apply to $n \times n$ upper triangular matrices.

● True

○ False

We have the following:

Additive Axioms

1. For all $A, B \in \mathbf{V}$, $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix}$ is upper triangular.
2. For all $A, B, C \in \mathbf{V}$, for $(A + B) + C$, we have:

$$\begin{aligned} (A+B)+C &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix} + C \\ &= \begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ 0 & a_{22} + b_{22} + c_{22} \end{bmatrix} = A + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ 0 & b_{22} + c_{22} \end{bmatrix} \\ &= A + (B + C). \end{aligned}$$

3. $\mathbf{0}$ is upper triangular. We take $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For all $A \in \mathbf{V}$, $A + \mathbf{0} = \begin{bmatrix} a_{11} + 0 & a_{12} + 0 \\ 0 & a_{22} + 0 \end{bmatrix} = A$.
4. For all $A \in \mathbf{V}$, We take another upper triangular matrix $-A = \begin{bmatrix} -a_{11} & -a_{12} \\ 0 & -a_{22} \end{bmatrix}$. For all $A \in \mathbf{V}$, $A + (-A) = \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{12} \\ 0 & a_{22} - a_{22} \end{bmatrix} = \mathbf{0}$.

Scalar Multiplication Axioms

1. For all $A \in \mathbf{V}$, $c \in \mathbb{R}$, $cA = \begin{bmatrix} ca_{11} & ca_{12} \\ 0 & ca_{22} \end{bmatrix}$ is upper triangular. So $cA \in \mathbf{V}$.
2. For all $A \in \mathbf{V}$, $c(dA) = \begin{bmatrix} cda_{11} & cda_{12} \\ 0 & cda_{22} \end{bmatrix} = (cd)A$.
3. For all $A \in \mathbf{V}$, $(c+d)A = \begin{bmatrix} (c+d)a_{11} & (c+d)a_{12} \\ 0 & (c+d)a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ 0 & ca_{22} \end{bmatrix} + \begin{bmatrix} da_{11} & da_{12} \\ 0 & da_{22} \end{bmatrix} = cA + dA$.
4. For all $A \in \mathbf{V}$, $1A = \begin{bmatrix} 1a_{11} & 1a_{12} \\ 0 & 1a_{22} \end{bmatrix} = A$.