MAT 292 Lecture Notes

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Ordinary Differential Equations, Fall 2025 MAT292

The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

"*ODEs are the bread and butter of engineering.* "

Key Concepts:

- **Conflicting Definitions.** This course has multiple definitions for key terms, which may vary between different texts and contexts.
- **Practice Derivatives.** Regular practice with derivatives is essential for mastering ODEs.

  Example 0.0.1. The following is a good example of good intuition:

  What is the antiderivative of $f(x) = \frac{\ln x}{x}$?

  We know that this is of the form $g \cdot g'$where $g(x) = \ln x$ and $g'(x) = \frac{1}{x}$. Thus, by the rule that:

  $$\frac{1}{2}\frac{d}{dx}[g(x)]^2 = g(x)g'(x) \tag{1}$$

  we can deduce that:

  $$\frac{1}{2}\frac{d}{dx}[(\ln x)^2] = \ln x \cdot \frac{1}{x}$$

- **Practice Linear Algebra.** Familiarity with linear algebra concepts is crucial for understanding ODEs.

1 Examples and Review

1.1 What is a Differential Equation?

Definiton 1.1.1 (Differential Equation)**.** Any relationship between a variable and its derivatives is called a differential equation.

Example 1.1.2 (Newton Second Law)**.** Newton's second law states that the force acting on an object is equal to the mass of the object multiplied by its acceleration. Mathematically, this can be expressed as:

$$F = m\frac{d^2x}{dt^2}$$

where F is the force, m is the mass, and $\frac{d^2x}{dt^2}$ is the acceleration (the second derivative of position with respect to time). This is a second-order ordinary differential equation.

If force is constant, this is a simple form of a differential equation, which we simply rearrange and integrate it (twice), which gives, simply:

$$x(t) = \frac{F}{2m}t^2 + C_1 t + C_0$$

where C_1 and C_0 are constants determined by initial conditions.

Example 1.1.3. Consider the following ODE:

$$x' = f(t)$$

So we have:

$$\frac{dx}{dt} = f(t)$$

Integrating both sides with respect to t gives:

$$\int_0^t dx = \int_0^t f(t)\,dt$$

So we have:

$$x(t) = \int_0^t f(t)\,dt + C$$

where C is a constant of integration. In which initial conditions can be used to determine the value of C.

Example 1.1.4 (Standard Trick: Turning Higher Order into a System)**.** Consider Hooke's Law:

$$F = m\ddot{x} = -kx$$

Then, we let:

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 \end{cases}$$

This system can be solved using the techniques for first-order ODEs, as a system by its eigenvalues and eigenvectors:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The above idea scales:

Example 1.1.5. Take $F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0$. Then, we can let:

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \\ \vdots \\ x_n = x^{(n)} \end{cases}$$

This allows us to rewrite the original equation as a system of first-order equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2}$$

Where we take A to be the appropriate matrix.

Example 1.1.6 (Exponential Growth)**.** You should have already know, that:

$$\frac{dx}{dt} = kx$$

is a first-order linear ODE. The solution to this equation is given by:

$$x(t) = Ce^{kt}$$

where C is a constant determined by initial conditions.

We could understand the eigenproblems associated with systems of ODEs as an extension of the abovewhere we look for solutions of the form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = e^{At} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} \tag{3}$$

where A is the matrix associated with the system of ODEs. $\exp(At)$ is the matrix exponential of At, which can be computed using various methods, including power series or diagonalization.

Example 1.1.7 (Superposition)**.** You should also have already known that the solution to:

$$\ddot{x} = -x$$

is given by:

$$x(t) = A\cos(t) + B\sin(t)$$

where A and B are constants determined by initial conditions. The solutions $x_1 = A\cos(t)$ and $x_2 = B\sin(t)$ can be combined to form the general solution. Thus, additional conditions must be satisfied to determine the values of A and B.

You might have observed that:

Theorem 1.1.8 (Number of Initial Conditions)**.** You need as many initial conditions as the order of the ODE to uniquely determine a solution.

Example 1.1.9 (Newton's Law of Cooling)**.** Let $u(t)$ be the temperature of the object at time t. Then, according to Newton's Law of Cooling, we have:

$$\frac{du}{dt} = -k(u - T_a)$$

where T_a is the ambient temperature, and k is a positive constant (the transmission coefficient). This is a first-order linear ODE that can be solved using the techniques discussed earlier.

Fixed-Point Solution We denote the trivial case where $u(0) = T_a$ as the fixed-point solutionwhere the temperature of the object is equal to the ambient temperature at time $t = 0$. It is easy to see that $u' = 0$ in this case so $u(t) = T_a$ for all t.

One way of solving the general solution is:

$$\begin{aligned}
\frac{du}{dt} &= -k(u - T_a) \\
\int \frac{du}{u - T_a} &= -k \int dt \\
\ln|u - T_a| &= -kt + C \\
u - T_a &= e^C e^{-kt} \\
u &= Be^{-kt} + T_a
\end{aligned}$$

Definiton 1.1.10 (Phase Portraits)**.** A phase portrait is a graphical representation of the trajectories of a dynamical system in the phase plane. Each point in the phase plane corresponds to a unique state of the system, and the trajectories represent the evolution of the system over time. Phase portraits are useful for visualizing the behavior of systems of ODEs, particularly in understanding stability and equilibrium points.

2 Qualitative Analysis of ODEs

2.1 Fixed Points

We first consider the autonomous system:

$$\frac{dx}{dt} = f(x) \tag{4}$$

Definiton 2.1.1 (Autonomous System)**.** An autonomous system is a system of ordinary differential equations (ODEs) in which the independent variable (usually time) does not explicitly appear in the equations. In other words, the rate of change of the dependent variable(s) depends only on the current state of the system and not on time itself. Autonomous systems can be expressed in the form:

$$\frac{dx}{dt} = f(x)$$

where x is the state vector and $f(x)$ is a function that describes how the state changes over time.

Definiton 2.1.2 (Fixed Point)**.** A fixed point (or equilibrium point) of a dynamical system is a point in the phase space where the system remains unchanged over time. Mathematically, for a system described by the differential equation $\frac{dx}{dt} = f(x)$, a fixed point x^* satisfies:

$$f(x^*) = 0$$

This means that if the system starts at x^*, it will stay at x^* for all future times.

Example 2.1.3. Consider $x' = \sin(x)$. The fixed points are given by:

$$\sin(x) = 0 \implies x = n\pi, \quad n \in \mathbb{Z}$$

Thus, the fixed points are $x = 0, \pm\pi, \pm 2\pi, \ldots$

2.1.1 Stability of Fixed Points

Intuition Consider a simple pendulum parametized by the angle θ from the vertical. The fixed points of this system occur when the pendulum is at rest, which happens at $\theta = 0$ (hanging straight down) and $\theta = \pi$ (inverted position). The case where $\theta = 0$ **is stable**, as small perturbations will cause the pendulum to oscillate around this point. In contrast, $\theta = \pi$ **is unstable,** as any small perturbation will cause the pendulum to fall away from this position.

What determines stability? A fixed point x^* of a dynamical system $\frac{dx}{dt} = f(x)$ is classified as:

stable if:

$$\left.\frac{df}{dx}\right|_{x=x^*} < 0 \tag{5a}$$

unstable if:

$$\left.\frac{df}{dx}\right|_{x=x^*} > 0 \tag{5b}$$

semi-stable if:

$$\left.\frac{df}{dx}\right|_{x=x^*} = 0 \tag{5c}$$

Proof. A demonstration of this will be given in the next section on Linear Stability Analysis 2.1.7. □

Example 2.1.4. Consider the system $x' = \sin(x)$. The fixed points are at $x = n\pi$where $n \in \mathbb{Z}$. To determine the stability of these fixed points, we compute the derivative of $f(x) = \sin(x)$:

$$\frac{df}{dx} = \cos(x)$$

Evaluating this derivative at the fixed points $x = n\pi$ gives:

$$\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even (unstable)} \\ -1 & \text{if } n \text{ is odd (stable)} \end{cases}$$

Thus, the fixed points at $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ are unstable, while the fixed points at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ are stable.

Drawing Fixed Point Diagrams and Phase Portraits By visualizing the direction of x due to $x' = f(x)$ on a x, $f(x)$ plane (fixed points diagram), we can draw a phase portrait. For the phase portrait, we can demonstrate the fixed points as horizontal lines, and draw points of x that converge to or diverge from these fixed points according to the direction of x'.

We draw the fixed point diagram with filled circles for stable fixed points and open circles for unstable fixed points. Arrows indicate the direction of flow towards or away from the fixed points.

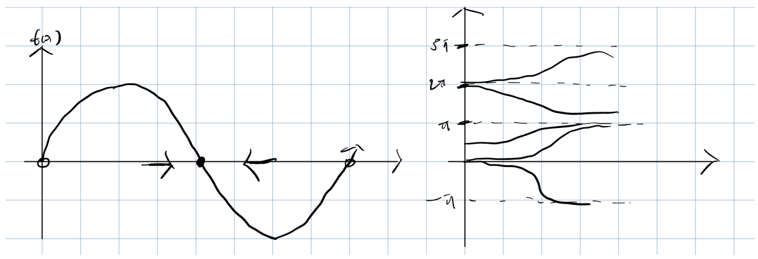


Figure 1: Fixed Point Diagram and Phase Portrait for $x' = \sin(x)$

Theorem 2.1.5 (Non-Intersection of Solutions)**.** In a dynamical system described by an ordinary differential equation (ODE) with a unique solution for given initial conditions, the trajectories of different solutions cannot intersect in the phase space. This means that if two solutions start from different initial conditions, they will never cross each other at any point in time.

Proof. **Sketch:** If the solutions intersect, then at the point of intersection, the solution is not unique, which contradicts the assumption of uniqueness. □

What Happens When We Consider $t \to -\infty$?

Theorem 2.1.6 (Stability Reversal)**.** A fixed point that is stable as $t \to \infty$ becomes unstable as $t \to -\infty$, and vice versa.

Proof. This can be understood by considering the time-reversed system. If we denote the time-reversed variable as $t' = -t$, then the original system $\frac{dx}{dt} = f(x)$ transforms, by the chain rule, into:

$$\frac{dx}{dt'} = -f(x)$$

In this new system, the direction of flow is reversed. Therefore, if a fixed point x^* is stable in the original system (i.e., trajectories approach x^* as $t \to \infty$), it will be unstable in the time-reversed system (i.e., trajectories move away from x^* as $t' \to \infty$). Conversely, an unstable fixed point in the original system becomes stable in the time-reversed system. This demonstrates the reversal of stability when considering $t \to -\infty$. □

2.1.2 Linear Stability Analysis

Definiton 2.1.7 (Linear Stability Analysis)**.** Consider the system $x' = f(x)$. Let ϵ be any perturbation around a fixed point x^*, such that:

$$x = x^* + \epsilon$$

To analyze the stability of the fixed point, we analyze the behavior of the perturbation $\epsilon(t)$ over time, since x^* is fixed, we have:

$$\epsilon' = x' = f(x) = f(x^* + \epsilon)$$

Ignoring higher order terms in Taylor expansion, we have:

$$\begin{aligned} &\approx f(x^*) + \epsilon f'(x^*) + \cancel{\frac{1}{2}\epsilon^2 f''(x^*) + \dots} \\ &= \epsilon f'(x^*) \quad (\text{since } f(x^*) = 0) \\ \Rightarrow \epsilon' &= f'(x^*)\epsilon \end{aligned}$$

This is a linear ODE in ϵ, which can be solved as:

$$\epsilon(t) = \epsilon e^{f'(x^*)t}$$

The behavior of $\epsilon(t)z$ depends on the sign of $f'(x^*)$:

- If $f'(x^*) < 0$, then $\epsilon(t) \to 0$ as $t \to \infty$, indicating that the fixed point is stable.
- If $f'(x^*) > 0$, then $\epsilon(t) \to \infty$ as $t \to \infty$, indicating that the fixed point is unstable.
- If $f'(x^*) = 0$, the linear analysis is inconclusive, and higher-order terms must be considered.

Thus, the stability of the fixed point can be determined by the sign of $f'(x^*)$.

Example 2.1.8. Consider the system $x' = x^2$. The only fix point is at $x = 0$. We have $f'(x) = 2x$, and at the fixed point:

$$f'(0) = 0$$

Thus, the linear stability analysis is inconclusive. We draw the fixed point half filled on the left, and half empty on the right. This is becuase for $x < 0$, $f(x) = x^2 > 0$, so points to the left of the fixed point move away from it (unstable). For $x > 0$, $f(x) = x^2 > 0$, so points to the right of the fixed point also move away from it (unstable). Therefore, the fixed point at $x = 0$ is unstable.

2.2 Classification of ODEs

ODE vs PDE An ordinary differential equation (ODE) contains functions of a single variable and their derivatives. A partial differential equation (PDE) contains functions of multiple variables and their partial derivatives.

Example 2.2.1 (Heat Equation)**.** The heat equation is a PDE that describes how heat diffuses through a given region over time. It is given by:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

where $u(x, t)$ is the temperature distribution function, α is the thermal diffusivity constant, and ∇^2 is the Laplacian operator.

Example 2.2.2 (Laplace's Equation)**.** Laplace's equation is a PDE that describes the behavior of scalar fields such as electric potential and fluid flow. It is given by:

$$\nabla^2 \phi = 0$$

where $\phi(x, y, z)$ is the scalar potential function, and ∇^2 is the Laplacian operator.

Definiton 2.2.3 (Order of an ODE)**.** The order of an ordinary differential equation (ODE) is determined by the highest derivative present in the equation.

2.2.1 Linearity and Homogeneity of ODEs

Definiton 2.2.4 (Linearity of an ODE)**.** An ordinary differential equation (ODE) is considered linear if it can be expressed in the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where y is the dependent variable, x is the independent variable, $a_i(x)$ are functions of x, and $g(x)$ is a known function. In a linear ODE, the dependent variable y and its derivatives appear to the first power and are not multiplied together. Otherwise, the ODE is considered nonlinear.

Definiton 2.2.5 (Homogenity of Linear ODEs)**.** A linear ordinary differential equation (ODE) is said to be homogeneous if the function $g(x)$ on the right-hand side of the equation is equal to zero for all values of x. In other words, a homogeneous linear ODE has the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

If $g(x)$ is not identically zero, then the ODE is considered non-homogeneous. Homogeneous linear ODEs have special properties and solution methods that differ from those of non-homogeneous linear ODEs.

Theorem 2.2.6 (Principle of Superposition)**.** Let:

$$a_0(t)y + \cdots + a_n(t)\frac{d^n y}{dt^n} = 0$$

Assuming that $y_1(t)$ and $y_2(t)$ are two solutions to the above equation, then any linear combination of these solutions $Ay_1(t) + By_2(t)$, where A and B are constants, is also a solution.

Proof. This could be demonstrated by the following derivation:

$$\begin{aligned}
&a_0(t)(Ay_1 + By_2) + \cdots + a_n(t)\frac{d^n}{dt^n}(Ay_1 + By_2) \\
&= A\left(a_0(t)y_1 + \cdots + a_n(t)\frac{d^n y_1}{dt^n}\right) + B\left(a_0(t)y_2 + \cdots + a_n(t)\frac{d^n y_2}{dt^n}\right) \\
&= A \cdot 0 + B \cdot 0 \\
&= 0
\end{aligned}$$

Thus, $Ay_1(t) + By_2(t)$ is also a solution to the original equation. WLOG, this can be extended to any finite number of solutions. □

Conisder the non-homogeneous case:

Theorem 2.2.7 (General Solution of Non-Homogeneous Linear ODEs)**.** Let:

$$a_0(t)y + \cdots + a_n(t)\frac{d^n y}{dt^n} = g(t)$$

If $y_p(t)$ is a particular solution to the non-homogeneous equation, and $y_h(t)$ is any solution to the corresponding homogeneous equation, then:

$$y(t) = Ay_h(t) + y_p(t)$$

is the general solution to the non-homogeneous equation, where A is an arbitrary constant.

Proof. **Sketch:** The solution for the homogeneous part would simply be 0, so the particular solution would be the only solution to the non-homogeneous equation. The general solution is then the sum of the homogeneous and particular solutions. □

2.2.2 Separable ODEs

Definiton 2.2.8 (Separable ODE)**.** An ordinary differential equation (ODE) is said to be separable if it can be expressed in the form:

$$\frac{dy}{dx} = g(x)h(y) = f(x, y)$$

where $g(x)$ is a function of the independent variable x only, and $h(y)$ is a function of the dependent variable y only. This allows the variables to be separated on opposite sides of the equation, enabling integration with respect to each variable independently:

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\int \frac{1}{h(y)}dy &= \int g(x)dx
\end{aligned}$$

Let $H(y)$ be the antiderivative of $\frac{1}{h(y)}$ and $G(x)$ be the antiderivative of $g(x)$, we have:

$$\begin{aligned} H(y) &= G(x) + C \\ \Rightarrow y &= H^{-1}(G(x) + C) \end{aligned}$$

where C is the constant of integration.

Remarks If g and h are continuous, then there need not exist solution in a neighbourhood (open ϵ-ball) of any point (x_0, y_0) so long as $h(y_0) \neq 0$.

Example 2.2.9 (Proof of Uniqueness by Integrating Factor Method)**.** Consider the IVP

$$\frac{dx}{dt} = ax, \qquad x(0) = x_0.$$

Although we know the solution is $x(t) = x_0 e^{at}$, we will prove that this solution is unique.

Suppose $w(t)$ is any solution of this IVP. Define a change of variables:

$$y(t) = e^{-at} w(t).$$

Then, by the product rule,

$$\frac{dy}{dt} = \frac{d}{dt}\big(e^{-at} w(t)\big) = -ae^{-at} w(t) + e^{-at} \frac{dw}{dt}.$$

Since $\frac{dw}{dt} = aw(t)$, this becomes

$$\frac{dy}{dt} = -ae^{-at} w(t) + e^{-at}(aw(t)) = 0.$$

Thus, $y(t)$ is constant for all t. Evaluating at $t = 0$ gives

$$y(0) = e^{-a\cdot 0} w(0) = x_0,$$

so $y(t) \equiv x_0$. Therefore,

$$e^{-at} w(t) = x_0 \quad\implies\quad w(t) = x_0 e^{at}.$$

Hence, the only possible solution of the IVP is

$$x(t) = x_0 e^{at}.$$

This shows the solution is unique.

Remark The change of variable

$$y(t) = e^{-at} x(t)$$

allows us to prove the uniqueness of the solution. This is essentially the same idea as the integrating factor method.

Example 2.2.10 (Logistic Equation)**.** The logistic equation is a first-order nonlinear separable autonomous ODE that models population growth with a carrying capacity. It is given by:

$$\frac{dx}{dt} = \alpha x\,(1-x)$$

where α is the growth rate.

To solve this equation, we can separate the variables:

$$\begin{aligned}\frac{dx}{\alpha x(1-x)} &= 1dt\\ \int \frac{1}{\alpha x(1-x)}dx &= \int 1dt\end{aligned}$$

Using partial fraction decomposition, we have:

$$\begin{aligned}\frac{1}{\alpha x(1-x)} &= \frac{A}{x} + \frac{B}{1-x}\\ 1 &= A(1-x) + Bx\\ 1 &= A + (B-A)x\\ \Rightarrow A &= 1, \quad B - A = 0\\ \Rightarrow B &= 1\end{aligned}$$

Thus, we have:

$$\begin{aligned}\int \left(\frac{1}{\alpha x} + \frac{1}{\alpha(1-x)}\right) dx &= \int 1dt\\ \frac{1}{\alpha}\left(\ln|x| - \ln|1-x|\right) &= t + C\\ \ln\left|\frac{x}{1-x}\right| &= \alpha t + C'\\ \frac{x}{1-x} &= e^{\alpha t + C'} = Ce^{\alpha t}\\ \Rightarrow x(t) &= \frac{Ce^{\alpha t}}{1 + Ce^{\alpha t}}\end{aligned}$$

To analyze the stability of the fixed points, we first find the fixed points by setting $\frac{dx}{dt} = 0$:

$$\alpha x(1-x) = 0 \implies x = 0 \text{ or } x = 1$$

Next, we compute the derivative of $f(x) = \alpha x(1-x)$:

$$\frac{df}{dx} = \alpha(1-2x)$$

Evaluating this derivative at the fixed points, assumeing $\alpha > 0$:

- At $x = 0$:

$$\left.\frac{df}{dx}\right|_{x=0} = \alpha > 0 \quad \text{(unstable)}$$

- At $x = 1$:

$$\left.\frac{df}{dx}\right|_{x=1} = -\alpha < 0 \quad \text{(stable)}$$

Thus, the fixed point at $x = 0$ is unstable, while the fixed point at $x = 1$ is stable.

2.2.3 First-Order Linear ODEs - Integrating Factor Method

Definiton 2.2.11 (First-Order Linear ODE)**.** A first-order linear ordinary differential equation (ODE) is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{6}$$

where $P(x)$ and $Q(x)$ are functions of the independent variable x.

Definiton 2.2.12 (Integrating Factor Method)**.** To solve a first-order linear ODE of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we can use the integrating factor method. The steps are as follows:

1. Compute the integrating factor $\mu(x)$:

$$\mu(x) = \exp\left(\int P(x)\,dx\right) \tag{7a}$$

This is from the fact that:

$$\begin{aligned}
\frac{d\mu}{dx} &= P(x)\mu(x) \quad \text{(by requirement of step 3)} \\
\Rightarrow \frac{1}{\mu}\frac{d\mu}{dx} &= P(x) \\
\Rightarrow \int \frac{1}{\mu}\frac{d\mu}{dx}\,dx &= \int P(x)\,dx \\
\Rightarrow \ln|\mu| &= \int P(x)\,dx \\
\Rightarrow \mu(x) &= \exp\left(\int P(x)\,dx\right)
\end{aligned}$$

2. Multiply both sides of the original ODE by the integrating factor:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \tag{7b}$$

3. Recognize that the left-hand side is the derivative of $\mu(x)y$:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \tag{7c}$$

4. Integrate both sides with respect to x:

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C \tag{7d}$$

where C is the constant of integration.

5. Finally, solve for $y(x)$:

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right) \tag{7e}$$

Example 2.2.13. Consider the IVP involving a first-order linear ODE:

$$y' + \frac{2}{t}y = \frac{\sin t}{t^2}, \quad y(\pi) = 1$$

To solve this IVP using the integrating factor method, we first identify $P(t) = \frac{2}{t}$ and $Q(t) = \frac{\sin t}{t^2}$.

Next, we compute the integrating factor:

$$\begin{aligned}\mu(t) &= \exp\left(\int \frac{2}{t}\,dt\right)\\ &= \exp\left(2\ln|t|\right)\\ &= |t|^2 = t^2\end{aligned}$$

Multiplying both sides of the original ODE by the integrating factor:

$$t^2y' + 2ty = \sin t$$

Recognizing the left-hand side as the derivative of $|t|^2y$:

$$\begin{aligned}\frac{d}{dt}[t^2y] &= \sin t\\ t^2y &= -\cos t + C\\ y &= \frac{-\cos t + C}{t^2}\end{aligned}$$

To determine the constant C, we use the initial condition $y(\pi) = 1$:

$$\begin{aligned}1 &= \frac{-\cos(\pi) + C}{\pi^2}\\ \Rightarrow C &= \pi^2 - 1\end{aligned}$$

Thus, the solution to the IVP is:

$$y(t) = \frac{-\cos t + \pi^2 - 1}{t^2}$$

2.2.4 Exact ODEs

Definiton 2.2.14 (Exact ODE)**.** An ordinary differential equation (ODE) of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{or equivalently} \quad M(x,y)dx + N(x,y)dy = 0 \tag{8a}$$

is said to be exact iff there exists a function $\psi(x, y)$ such that:

$$M_y = \frac{\partial M}{\partial y} = N_x = \frac{\partial N}{\partial x} \tag{8b}$$

This implies a potential function $\psi(x, y)$ exists such that:

$$M_y = \psi_{xy} = \psi_{yx} = N_x \tag{8c}$$

The general solution to the exact ODE is given by:

$$d\psi(x,y) = 0 \implies \psi(x,y) = C \quad \text{(By chain rule of } \psi) \tag{8d}$$

In this case, we can solve the ODE by finding the potential function $\psi(x, y)$:

$$\psi(x,y) = \int M(x,y)\,dx + g(y) \tag{9}$$

to find $g(y)$, we can use the fact that:

$$\frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) = \int M_y\,dx = \int N_x\,dx = N(x,y) \tag{10}$$

We can also solve this from the other direction. Or verify your answer this way.

Example 2.2.15. Consider the exact ODE:

$$(3x^2y^2 + x + \cos y)\frac{dy}{dx} = -(2xy^3 + y)$$

We identify $N(x, y) = 3x^2y^2 + x + \cos y$ and $M(x, y) = 2xy^3 + y$. So $M_y = N_x = 6x^2y + 1$, thus the ODE is exact. Then we compute the potential function:

$$\begin{aligned}\psi(x,y) &= \int M(x,y)\,dx \\ &= \int (2xy^3 + y)\,dx \\ &= x^2y^3 + xy + g(y)\end{aligned}$$

To find $g(y)$, we compute:

$$\begin{aligned}\frac{\partial\psi}{\partial y} &= 3x^2y^2 + x + g'(y) \\ &= N(x,y) = 3x^2y^2 + x + \cos y \\ \Rightarrow g'(y) &= \cos y \\ \Rightarrow g(y) &= \sin y\end{aligned}$$

Thus, the potential function is:

$$\psi(x, y) = x^2y^3 + xy + \sin y$$

The general solution to the ODE is then:

$$\begin{aligned}\psi(x, y) &= C \\ \Rightarrow x^2y^3 + xy + \sin y &= C\end{aligned}$$

We can verify our answer by computing from the other direction:

$$\begin{aligned}\psi(x, y) &= \int N(x, y)\, dy \\ &= \int (3x^2y^2 + x + \cos y)\, dy \\ &= x^2y^3 + xy + \sin y + h(x)\end{aligned}$$

To find $h(x)$, we compute:

$$\begin{aligned}\frac{\partial \psi}{\partial x} &= 2xy^3 + y + h'(x) \\ &= M(x, y) = 2xy^3 + y \\ \Rightarrow h'(x) &= 0 \\ \Rightarrow h(x) &= C'\end{aligned}$$

Thus, we obtain the same potential function:

$$\psi(x, y) = x^2y^3 + xy + \sin y + C'$$

2.3 Bifurcations

Definiton 2.3.1 (Bifurcation)**.** A bifurcation in a dynamical system occurs when a small change in the system's parameters causes a sudden qualitative change in its behavior.

This can lead to the emergence or disappearance of fixed points, changes in stability, or the onset of periodic or chaotic behavior.

Definiton 2.3.2 (Bifurcation Point)**.** A bifurcation point is a specific value of a parameter in a dynamical system at which the qualitative behavior of the system changes. At this point, the system may undergo a bifurcation, leading to changes in the number or stability of fixed points, periodic orbits, or other dynamical features.

Types of Bifurcations There are several common types of bifurcations in dynamical systems, including:

- Saddle-Node Bifurcation: Two fixed points (one stable and one unstable) collide and annihilate each other as a parameter is varied.

- Transcritical Bifurcation: Two fixed points exchange their stability as a parameter is varied.
- Pitchfork Bifurcation: A single fixed point splits into three fixed points (one stable and two unstable, or vice versa) as a parameter is varied.
- Hopf Bifurcation: A fixed point loses stability and a small amplitude limit cycle (periodic orbit) emerges as a parameter is varied.

Example 2.3.3. Consider the ODE:

$$x' = ax(1 - x)$$

This is a classic example of a bifurcation scenario, where the behavior of the system changes qualitatively as the parameter a is varied. Consider the fix points of the system:

$$x^* = 0, \quad x^* = 1$$

So, we have:

- For $a < 0$:
  - $x^* = 0$ is stable (since $f'(0) = a < 0$)
  - $x^* = 1$ is unstable (since $f'(1) = -a > 0$)
- For $a > 0$:
  - $x^* = 0$ is unstable (since $f'(0) = a > 0$)
  - $x^* = 1$ is stable (since $f'(1) = -a < 0$)
- For $a = 0$:
  - Both $x^* = 0$ and $x^* = 1$ are semi-stable (since $f'(0) = 0$ and $f'(1) = 0$)

  This indicates a bifurcation at $a = 0$, where the stability of the fixed points changes.

Example 2.3.4 (Saddle-Node Bifurcation)**.** Consider the ODE:

$$x' = x^2 + a$$

This equation has a family of fixed points given by:

$$x^* = \pm\sqrt{-a}$$

For $a < 0$, we have two real fixed points, while for $a > 0$, there are no real fixed points. At $a = 0$, the fixed points collide and disappear, indicating a bifurcation point at $a = 0$.

Definiton 2.3.5 (Bifurcation Diagram)**.** A bifurcation diagram is a visual representation that illustrates how the fixed points of a dynamical system change as a parameter is varied.

It is a plot of the fixed points against the parameter, showing regions of stability and instability, and we denote stable fixed points with solid lines and unstable fixed points with dashed lines. A bifurcation diagram of Example 2.3.4 is shown below:

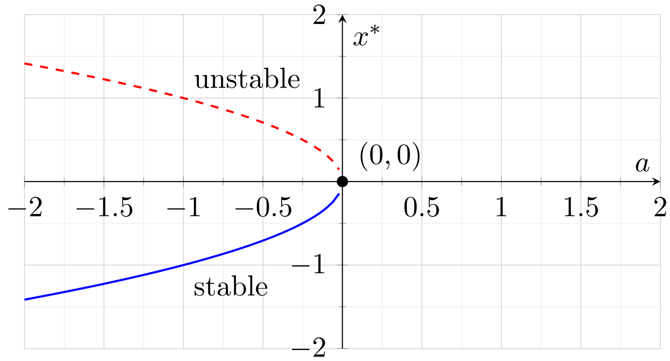


Figure 2: Bifurcation Diagram for $x' = x^2 + a$

Example 2.3.6 (Pitchfork Bifurcation)**.** Consider the ODE:

$$x' = ax - x^3$$

This equation has a family of fixed points given by:

$$x^* = 0, \quad x^* = \pm\sqrt{a}$$

For $a \leq 0$, there is one real fixed point at $x^* = 0$. For $a > 0$, there are three real fixed points: $x^* = 0$ (unstable) and $x^* = \pm\sqrt{a}$ (stable). At $a = 0$, the fixed point at $x^* = 0$ changes stability, indicating a bifurcation point at $a = 0$.

We can plot the bifurcation diagram for this system as follows:

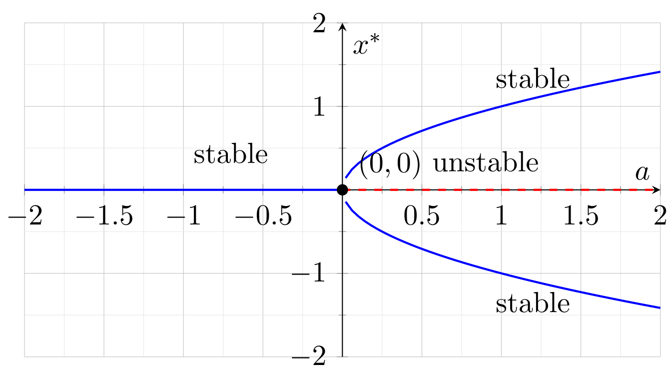


Figure 3: Bifurcation Diagram for $x' = ax - x^3$

Example 2.3.7 (Transcritical Bifurcation)**.** Consider the ODE:

$$x' = x(r - x)$$

This equation has a family of fixed points given by:

$$x^* = 0, \quad x^* = r$$

For $r < 0$, $x^* = 0$ is stable. For $r > 0$, $x^* = 0$ is unstable and $x^* = r$ is stable. At $r = 0$, the stability of the fixed point at $x^* = 0$ changes, indicating a bifurcation point at $r = 0$. The bifurcation diagram is as follows:

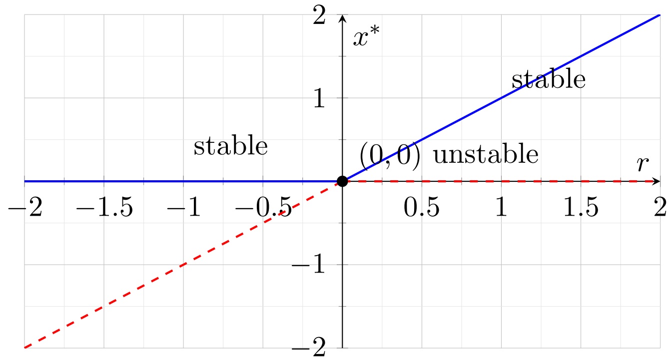


Figure 4: Bifurcation Diagram for $x' = x(r - x)$

Linear Stability Analysis and Normal Forms Consider the ODE in the above example 2.3.7: We can verify the stability of the fixed points by performing linear stability analysis. Consider $x \sim x^*$ and $r \sim r^*$, such that $|x - x^*| = \epsilon$ and $|r - r^*| = \delta$ for small $\epsilon, \delta > 0$. Then,

$$\begin{aligned} f(x,r) &= f(x^*,r^*) + (x - x^*)f_x(x^*,r^*) + (r - r^*)f_r(x^*,r^*) + \\ &\quad \frac{(x - x^*)^2}{2} f_{xx}(x^*,r^*) + O(\epsilon^3, \delta^2, \epsilon\delta) \\ &\approx a(x - x^*)^2 + b(r - r^*) \end{aligned}$$

which we refer to as the normal form of the system. Here, $a = \frac{1}{2} f_{xx}(x^*, r^*)$ and $b = f_r(x^*, r^*)$. And that $f_x(x^*, r^*) = 0$ is called a saddle node.

Example 2.3.8. Consider the ODE:

$$x' = rx + x^3 - x^5$$

We can find the fixed points by setting $x' = 0$, and we have

$$x^\star = 0 \quad \text{or} \quad x^\star = \pm\sqrt{\frac{1 \pm \sqrt{1 + 4r}}{2}}$$

Now, we see the bifurcation points are at $r = -\frac{1}{4}$ and $r = 0$. We have:

- For $r < -\frac{1}{4}$:
  - $x^* = 0$ is stable (since $f'(0) = r < 0$)
- For $r = -\frac{1}{4}$:
  - $x^* = 0$ is stable (since $f'(0) = r < 0$)
  - $x^* = \pm\frac{1}{2}$ are semi-stable (since $f'(\pm\frac{1}{2}) = 0$)
- For $-\frac{1}{4} < r < 0$:
  - $x^* = 0$ is stable (since $f'(0) = r < 0$)
  - $x^* = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$ is stable (since $f'(-\sqrt{\frac{1+\sqrt{1+4r}}{2}}) < 0$)

  - $x^* = \sqrt{\frac{1+\sqrt{1+4r}}{2}}$ is unstable (since $f'(\sqrt{\frac{1+\sqrt{1+4r}}{2}}) > 0$)

- For $r = 0$:
  - $x^* = 0$ is semi-stable (since $f'(0) = 0$)
  - $x^* = -1$ is stable (since $f'(-1) < 0$)
  - $x^* = 1$ is stable (since $f'(1) < 0$)

- For $r > 0$:
  - $x^* = 0$ is unstable (since $f'(0) = r > 0$)
  - $x^* = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$ is stable (since $f'(-\sqrt{\frac{1+\sqrt{1+4r}}{2}}) < 0$)
  - $x^* = \sqrt{\frac{1+\sqrt{1+4r}}{2}}$ is stable (since $f'(\sqrt{\frac{1+\sqrt{1+4r}}{2}}) < 0$)

The bifurcation diagram is as follows:

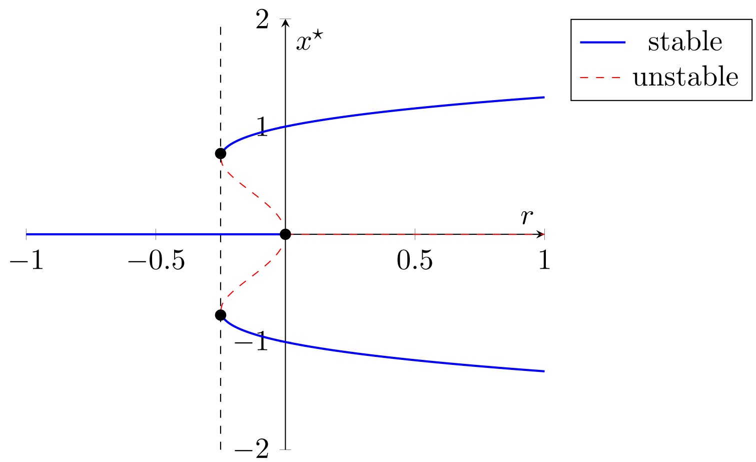


Figure 5: Bifurcation diagram for $x' = rx + x^3 - x^5$.

Conditions for Bifurcations Consider $x' = f(x, r)$. Suppose (x^c, r^c) is a candidate for a bifurcation point. Then, the following conditions must hold:

- **Equilibrium Condition** $f(x^c, r^c) = 0$ (i.e., x^c is a fixed point for parameter value r^c)
- **Non Hyperbolicity Condition** $f_x(x^c, r^c) = 0$ (i.e., the Jacobian at the fixed point has a zero eigenvalue)
- **Transversality Condition** $f_r(x^c, r^c) \neq 0$ (i.e., the fixed point changes with respect to the parameter)
- **Types of Bifurcations** For different types of bifurcations, we have the following additional conditions:
  - **Saddle-Node Bifurcation** $f_{xx}(x^c, r^c) \neq 0$
  - **Pitchfork Bifurcation** $f_{xx}(x^c, r^c) = 0$ and $f_{xxx}(x^c, r^c) \neq 0$

2.4 Existence and Uniqueness of Solutions

Example 2.4.1. Consider the following IVP:

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

Apart from the trivial solution $y(t) = 0$, we can also solve this ODE by separating variables:

$$\begin{aligned} \frac{dy}{y^{1/3}} &= dt \\ \int y^{-1/3}\, dy &= \int 1\, dt \\ \frac{3}{2} y^{2/3} &= t + C \end{aligned}$$

Using the initial condition $y(0) = 0$, we find $C = 0$. Thus, the solution is:

$$y(t) = \left(\frac{2}{3}(t)\right)^{3/2}$$

Thus, we have two solutions to the IVP. Existence is guaranteed, but uniqueness is not.

Example 2.4.2. Consider the following IVP:

$$y' = y^2, \quad y(0) = 1$$

We can solve this ODE by separating variables:

$$\begin{aligned} \frac{dy}{y^2} &= dt \\ \int y^{-2}\, dy &= \int 1\, dt \\ -y^{-1} &= t + C \end{aligned}$$

Using the initial condition $y(0) = 1$, we find $C = -1$. Thus, the solution is:

$$y(t) = \frac{1}{1-t}$$

The solution exists but not globally, as it blows up at $t = 1$. Due to the nature of the initial condition, the solution is defined only for $t < 1$ to guarantee continuousness.

Theorem 2.4.3 (Existence and Uniqueness Theorem for Linear ODEs)**.** Consider the IVP:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

where $p(t)$ and $g(t)$ are continuous on an open interval $I = (\alpha, \beta)$ containing t_0. Then, there exists a unique solution $y(t)$ defined on the entire interval I.

Proof. (**Existence**) We can use the integrating factor method to find a solution. The integrating factor is given by:

$$\mu(t) = \exp\left(\int p(t)\, dt\right)$$

Multiplying both sides of the ODE by the integrating factor, we have:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

Integrating both sides with respect to t:

$$\mu(t)y = \int \mu(s)g(s)\,ds + C$$

We can determine C using the initial condition $y(t_0) = y_0$:

$$y_0 = \frac{1}{\mu(t_0)}\left(\int_{t_0}^{t_0} \mu(s)g(s)\,ds + y_0\right)$$

Thus, we have a well defined solution $y(t)$ defined on the interval I, as long as p and g are continuous on I.

(**Uniqueness**) Suppose there are two solutions $y_1(t)$ and $y_2(t)$ to the IVP, then they must satisfy $y_1(t_0) = y_2(t_0) = y_0$. Define $z(t) = y_1(t) - y_2(t)$, then $z(t)$ satisfies:

$$z' + p(t)z = g - g = 0, \quad z(t_0) = y_1(t_0) - y_2(t_0) = 0$$

Apply the same integrating factor method, we have:

$$\mu(t)z' + \mu(t)p(t)z = 0$$

Integrating both sides with respect to t:

$$\mu(t)z = \int_{t_0}^{t} 0\,ds + C = C$$

Using the initial condition $z(t_0) = 0$, we find $C = 0$. Then, since μ is a exponential, $\mu > 0$, $z(t) = 0$ for all $t \in I$, which implies $y_1(t) = y_2(t)$ for all $t \in I$. Therefore, the solution to the IVP is unique. □

Theorem 2.4.4 (Cauchy-Lipschitz Theorem)**.** Consider the IVP:

$$y' = f(t, y), \quad y(t_0) = y_0$$

where $f(t, y)$ and $f_y(t, y)$ are continuous on a rectangle $R = [\alpha, \beta] \times [\gamma, \delta]$. Then, for some point $(t_0, y_0) \in \text{int}(R)$, there exists an interval $I = (t_0 - h, t_0 + h) \subseteq [\alpha, \beta]$ for some $0 < h \leq \min(t_0 - \alpha, \beta - t_0)$, such that there exists a unique solution $y(t)$ defined on the interval I.

Proof. The proof involves the method of successive approximations (Picard iterations). Which we will not cover here. □

Example 2.4.5. To demonstrate that the Cauchy-Lipschitz theorem applies in the linear case, consider the IVP:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

where $p(t)$ and $g(t)$ are continuous. We also have $f_y = -p(t)y + g(t) = -p(t)$. So both f and f_y are continuous, thus the Cauchy-Lipschitz theorem guarantees the existence and uniqueness of the solution on some interval around t_0.

Example 2.4.6. Consider the IVP:

$$y' = y^{1/3}, \quad y(0) = 0$$

Here, $f(t, y) = y^{1/3}$ and $f_y(t, y) = \frac{1}{3}y^{-2/3}$. The function $f(t, y)$ is continuous everywhere, but $f_y(t, y)$ is not continuous at $y = 0$. Therefore, the Cauchy-Lipschitz theorem does not apply at the initial condition $(0, 0)$, which explains why we have multiple solutions to this IVP.

2.5 Population Dynamics

Example 2.5.1 (Simple Model)**.** Consider the ODE:

$$P' = rP$$

where $P(t)$ is the population at time t and r is the growth rate. This model assumes that the population grows exponentially without any constraints. The solution to this ODE is:

$$P(t) = P_0 e^{rt}$$

Is this a good model? Not really, because it assumes unlimited resources and no environmental constraints, which is unrealistic in real-world scenarios.

Example 2.5.2 (A More Realistic Model)**.** Consider the logistic growth model:

$$P' = rP\left(1 - \frac{P}{K}\right)$$

where $K > 0$ is the carrying capacity of the environment. This model accounts for limited resources and environmental constraints. When $P = K$, the population growth rate becomes zero, indicating that the population has reached its maximum sustainable size. The solution to this ODE is:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right) e^{-rt}}$$

This model is more realistic as it predicts that the population will grow rapidly when small, but will slow down and stabilize as it approaches the carrying capacity K.

Note This is the same ODE considered in Example 2.2.10.

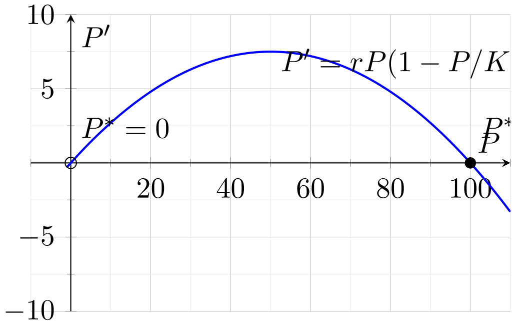
Example 2.5.3 (Logistics with Threshold)**.** Now we introduce a threshold T below which the population cannot sustain itself:

$$P' = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{T} - 1\right)$$

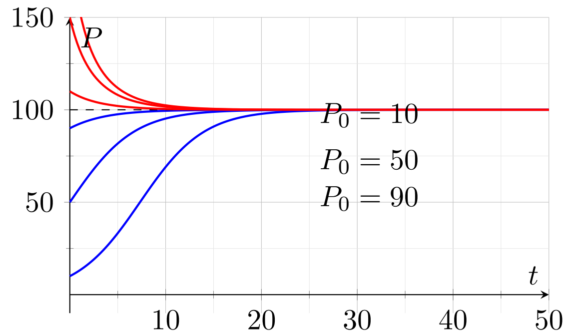
This model introduces an Allee effect, where the population growth rate becomes negative when the population is below the threshold T. The fixed points of this system are:

$$P^* = 0, \quad P^* = T, \quad P^* = K$$

The stability of these fixed points can be analyzed as follows:



(a) Fixed Point Diagram for the Logistic Growth Model

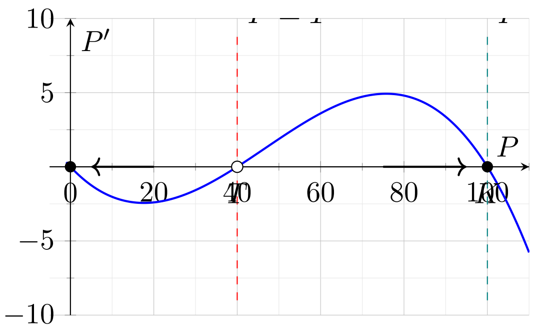


(b) Sample solution trajectories (different P_0)

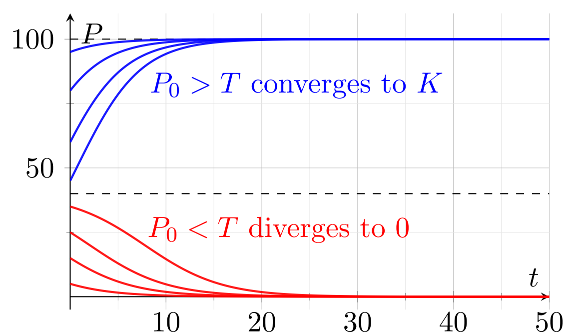
Figure 6: Logistic growth: (a) fixed-point diagram; (b) sample solution trajectories for different initial conditions.

- $P^* = 0$ is stable (since $f'(0) < 0$)
- $P^* = T$ is unstable (since $f'(T) > 0$)
- $P^* = K$ is stable (since $f'(K) < 0$)

A phase plot and sample solution trajectories are shown below (the sketch shows guides at $P = T$ and $P = K$, and sample trajectories that converge to K or diverge to 0):



(a) Phase diagram (fixed points and flow)



(b) Sample time trajectories: convergence to K or collapse to 0

Figure 7: Logistic growth with threshold: (a) phase portrait with horizontal guides at $P = T$ and $P = K$; (b) qualitative solution trajectories showing convergence and divergence relative to the threshold.

Example 2.5.4 (Harvesting Model)**.** Consider a population model with harvesting:

$$P' = rP\left(1 - \frac{P}{K}\right) - H$$

where H is the constant harvesting rate. The fixed points of this system are given by solving:

$$rP\left(1 - \frac{P}{K}\right) - H = 0$$

This leads to a quadratic equation in P:

$$P^2 - KP + \frac{KH}{r} = 0$$

The solutions to this equation are:

$$P^* = \frac{K \pm \sqrt{K^2 - 4\frac{KH}{r}}}{2}$$

The stability of these fixed points can be analyzed as follows:

- If $H < \frac{rK}{4}$, there are two real fixed points: one stable and one unstable.
- If $H = \frac{rK}{4}$, there is one real fixed point (a saddle-node bifurcation point).
- If $H > \frac{rK}{4}$, there are no real fixed points, indicating that the population will decline to extinction.

Fixed Point Diagram A fixed point diagram is shown below:

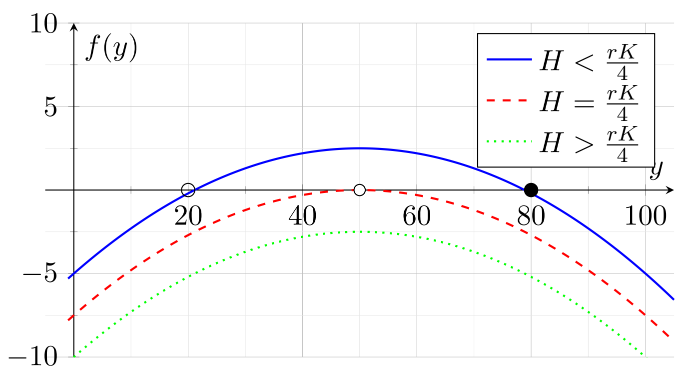


Figure 8: Fixed point diagram for the harvesting model with different harvesting rates H.

Bifurcation Diagram This is a classic example of a saddle-node bifurcation (0 to 2 fixed points). The bifurcation point is at $(k/2, 4h/K)$. The bifurcation diagram is shown below:

Theorem 2.5.5 (ID Autonomous ODEs are Monotonic)**.** Consider the ODE:

$$y' = f(y)$$

where $f(y)$ is continuous. Then, the solution $y(t)$ is monotonic on its interval of existence.

Proof. WLOG, suppose $f(y_0) > 0$. For the sake of contradiction, suppose $y(t)$ is not monotonic. Then, there exists $t_1 > t_0$ such that $y(t_1) < y(t_0)$. By the Intermediate Value Theorem, there exists $t_2 \in (t_0, t_1)$ such that $y(t_2) = y(t_0)$. By the Mean Value Theorem, there exists $c \in (t_0, t_2)$ such that:

$$\frac{y(t_2) - y(t_0)}{t_2 - t_0} = y'(c)$$

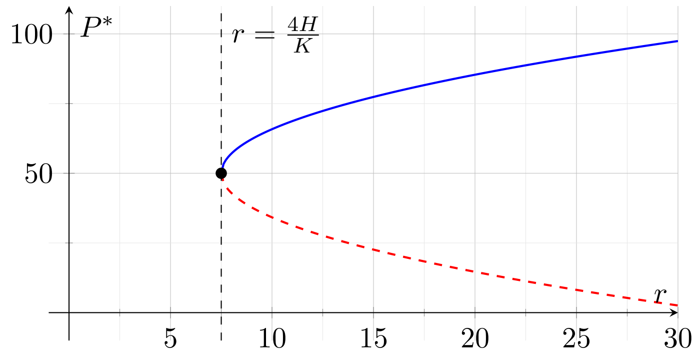


Figure 9: Bifurcation diagram for the harvesting model showing the saddle-node bifurcation at $H = \frac{rK}{4}$.

But $y(t_2) = y(t_0)$, so the left-hand side is zero. Thus, $y'(c) = 0$. However, since $f(y)$ is continuous and $f(y_0) > 0$, there exists a neighborhood around y_0 where $f(y) > 0$. This contradicts the assumption that $y'(c) = 0$. Therefore, $y(t)$ must be monotonic. $\square$

Definiton 2.5.6 (System of ODEs)**.** A system of ODEs is a set of coupled first-order ODEs of the form:

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \dots, x_n) \\ x_2' = f_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ x_n' = f_n(t, x_1, x_2, \dots, x_n) \end{cases}$$

where $x_i(t)$ are the state variables and $f_i(t, x_1, x_2, \dots, x_n)$ are given functions.

2.6 Systems of ODEs

Definiton 2.6.1 (Two-Species Interaction Model)**.** Consider a model of two species interacting, such as predator-prey dynamics:

$$\begin{cases} x' = x(r_1 - ax - by) \\ y' = y(r_2 + cx - dy) \end{cases}$$

where $x(t)$ and $y(t)$ are the populations of the two species, r_1, r_2, a, b, c, d are constants representing:

- r_1, r_2: intrinsic growth rates of species x and y
- a, d: intrinsic carrying capacity coefficients
- b, c: interspecific coefficients representing the effect of one species on the other

Example 2.6.2 (Lotka-Volterra Model)**.** Now, consider the classic Lotka-Volterra predator-prey model:

$$\begin{cases} x' & = ax - bxy \\ y' & = -cy + dxy \\ (x(0), y(0)) & = (x_0, y_0) \end{cases}$$

where $x(t)$ is the prey population, $y(t)$ is the predator population, and a, b, c, d are positive constants representing:

- a: growth rate of prey in the absence of predators
- b: predator efficiency (how many prey are eaten per predator per unit time)
- c: natural death rate of predators in the absence of prey
- d: conversion efficiency (how many predators are born per prey eaten)

Consider the fixed points of this system by setting $x' = 0$ and $y' = 0$. Then we have:

$$(x^*, y^*) = (0, 0) \quad \text{and} \quad (x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right)$$

The case where $(x^*, y^*) = (0, 0)$ is trivial (extinction of both species). The non-trivial fixed point $\left(\frac{c}{d}, \frac{a}{b}\right)$ represents a coexistence equilibrium where the rate of the prey being eaten is the rate of the prey being born.

We can further analyze this by considering Nullclines.

Definiton 2.6.3 (Nullclines)**.** Nullclines are curves in the phase plane where the derivative of **one of the variables** is zero. For the Lotka-Volterra model, the nullclines are given by:

- Prey nullcline ($x' = 0$): $y = \frac{a}{b}$ (horizontal line)
- Predator nullcline ($y' = 0$): $x = \frac{c}{d}$ (vertical line)

The intersection of these nullclines gives the fixed points of the system.

Example 2.6.4 (Nullclines of the Lotka-Volterra Model)**.** The x-nullclines ($x' = 0$) are given by:

$$x' = ax - bxy = 0 \implies x(a - by) = 0 \implies x = 0 \text{ or } y = \frac{a}{b}$$

The y-nullclines ($y' = 0$) are given by:

$$y' = -cy + dxy = 0 \implies y(-c + dx) = 0 \implies y = 0 \text{ or } x = \frac{c}{d}$$

Also, consider the growth of each species in the nullcline regions:

- For $x' = 0$ (prey nullcline):
  - If $x = 0$, then $y' < 0$ (predator population decreases).
  - If $y < \frac{a}{b}$, then $x' > 0$ (prey population increases).
  - If $y > \frac{a}{b}$, then $x' < 0$ (prey population decreases).
- For $y' = 0$ (predator nullcline):

- If $y = 0$, then $x' > 0$ (prey population increases).
- If $x < \frac{c}{d}$, then $y' < 0$ (predator population decreases).
- If $x > \frac{c}{d}$, then $y' > 0$ (predator population increases).

These directions would ultimately leads to a circular flow around the fixed point $\left(\frac{c}{d}, \frac{a}{b}\right)$.

The intersection of these nullclines gives the fixed points of the system. A phase portrait showing the nullclines and the fixed point is illustrated below: And a figure for the vector field is shown

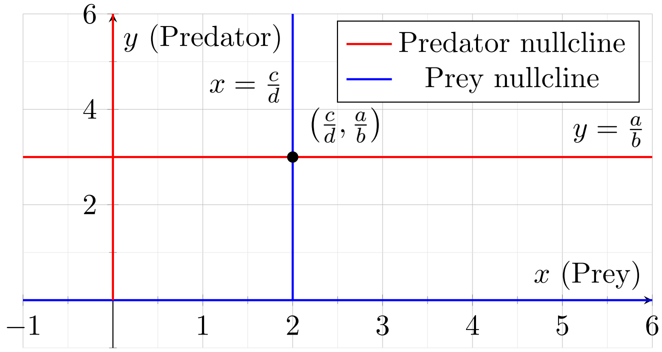


Figure 10: Nullclines of the Lotka-Volterra model showing the prey nullcline (horizontal line) and predator nullcline (vertical line). The intersection point represents the coexistence equilibrium.

below:

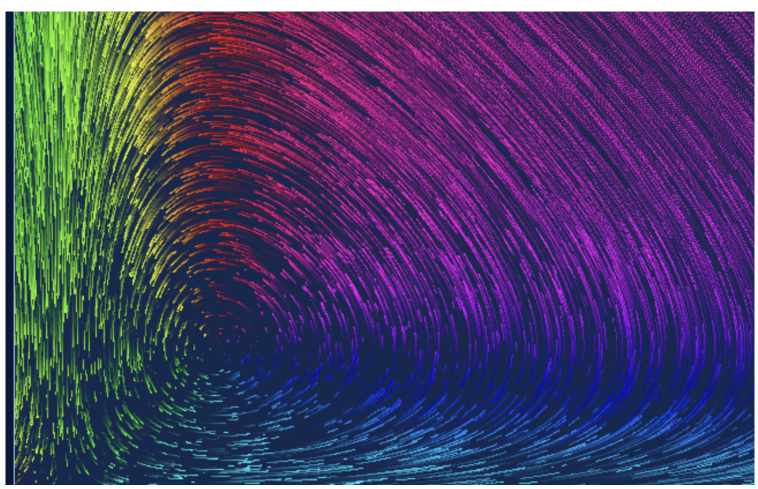


Figure 11: Vector field of the Lotka-Volterra model showing the circular flow around the fixed point $\left(\frac{c}{d}, \frac{a}{b}\right)$.

Example 2.6.5 (Simple Pendulum)**.** Consider a simple pendulum of length L and mass m under the influence of gravity g. The angle $\theta(t)$ that the pendulum makes with the vertical satisfies the second-order ODE:

$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$

To convert this into a system of first-order ODEs, we let:

$$\begin{cases} x := \theta & x' = \theta' = y \\ y := \theta' & y' = \theta'' = -\frac{g}{L}\sin(x) \end{cases}$$

Consider the fixed points of this system by setting $x' = 0$ and $y' = 0$. Then we have:

$$(x^*, y^*) = (n\pi, 0) \quad n \in \mathbb{Z}$$

The case where n is even (e.g., $0, 2, 4, \ldots$) corresponds to the pendulum hanging straight down (stable equilibrium), while the case where n is odd (e.g., $1, 3, 5, \ldots$) corresponds to the pendulum being inverted (unstable equilibrium).

Consider the nullclines:

- $x' = 0$ (horizontal nullcline): $y = 0$
- $y' = 0$ (vertical nullcline): $\sin(x) = 0 \implies x = n\pi, n \in \mathbb{Z}$ And we have, for y:

A vector field of the system is shown below:

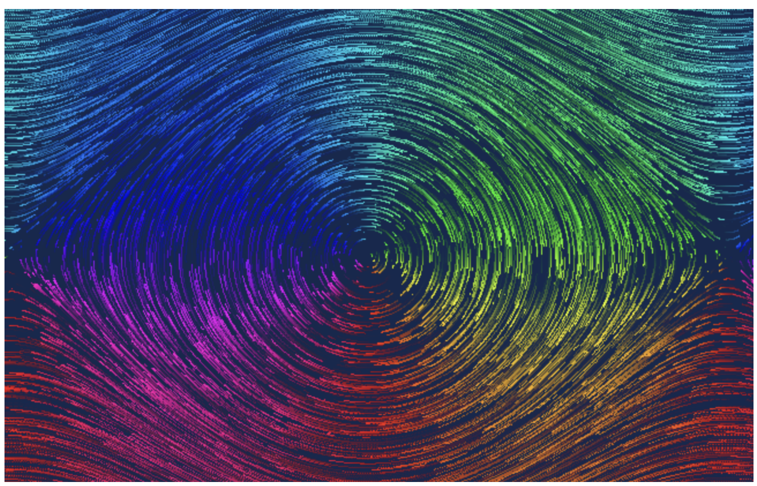


Figure 12: Vector field of the simple pendulum system showing the fixed points at $(n\pi, 0)$.

Systems of ODEs as Matrices A system of ODEs can be expressed in matrix form as:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$$

where $\mathbf{x}$ is the state vector, A is a matrix of coefficients, and $\mathbf{b}(t)$ is a vector of functions.

Definiton 2.6.6 (General System of ODEs)**.** WLOG, for a system of 2 ODEs with x and y, we write:

$$\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases}$$

where f and g are given functions:

$$f(t,x,y) = p_{11}(t)x + p_{12}(t)y + g_1(t)g(t,x,y) = p_{21}(t)x + p_{22}(t)y + g_2(t)$$

We can express this in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{11}$$

Furthermore, we generalize this to n dimensions, and the inhomogeneous, non-autonomous system of linears ODEs is given by:

$$\mathbf{x}_1' = \sum_{j=1}^{n} P_{1j}(t)x_j + g_1(t) \quad i = 1, 2, \dots, n$$
$$\vdots$$
$$\mathbf{x}_n' = \sum_{j=1}^{n} P_{nj}(t)x_j + g_n(t)$$

or in matrix form:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{G}(t) \tag{12}$$

where $A(t)$ is an $n \times n$ matrix of functions, $\mathbf{x}$ is the state vector, and $\mathbf{G}(t)$ is the inhomogeneous term.

Theorem 2.6.7 (Existence and Uniqueness for Systems of ODEs)**.** If $p_{ij}(t)$ and $g_i(t)$ are continuous on an open interval $I = (a, b)$ containing t_0 for all $i, j = 1, 2, \dots, n$, then the initial value problem

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{G}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution on the interval I.

Example 2.6.8 (Linear, homogeneous and autonomous)**.** Consider the simple ODE system:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

We cosidner the fix points by setting $\mathbf{x}' = 0$:

$$A\mathbf{x} + \mathbf{b} = 0 \implies A\mathbf{x} = -\mathbf{b}$$

If A is invertible, then the fixed point is given by:

$$\mathbf{x}^* = -A^{-1}\mathbf{b}$$

If A is not invertible, then there are either no fixed points or infinitely many fixed points. Now, we let $\mathbf{y} = \mathbf{x} + A^{-1}\mathbf{b}$, then we have:

$$\mathbf{y}' = \mathbf{x}'$$

we have:

$$Ay = A\mathbf{x} + \mathbf{b} = \mathbf{x}' = \mathbf{y}'$$

Thus, the system reduces to:

$$\mathbf{y}' = A\mathbf{y}$$

which is a linear, homogeneous, and autonomous system. The fixed point of this system is given by:

$$\mathbf{y}^* = 0$$

Example 2.6.9 (Simple ODEs)**.** Consider the system of ODEs:

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}$$

and the initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Then we can consider the solutions:

$$x_1(t) = x_0 e^t$$

$$x_2(t) = y_0 e^{-2t}$$

or we can write the solution in vector form:

$$\mathbf{x}(t) = x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + y_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

The fixed point and nullclines are given by:

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The nullclines are given by:

- $x' = 0$: $x = 0$ (vertical line)
- $y' = 0$: $y = 0$ (horizontal line)

Stability of Nullclines The x-nullcline since it goes to the origin as $t \to -\infty$ and away from the origin as $t \to \infty$, while the y-nullcline goes to the origin as $t \to \infty$ and away from the origin as $t \to -\infty$. **Note - Stability** The stability here really means that the trajectories approach or

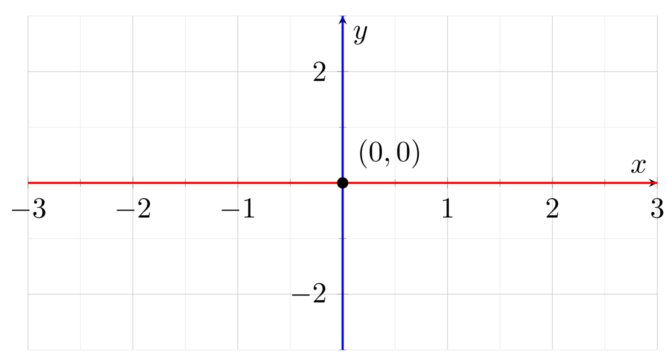


Figure 13: Phase portrait of the simple ODE system showing the nullclines and the fixed point at the origin.

move away from the fixed point along the nullclines, this does not mean for perturbation along the nullclines. **Saddle Node** The fixed point is a saddle node (since one eigenvalue is positive and the other is negative). The trajectories in the phase plane are shown below:

Example 2.6.10 (Linear Algebra)**.** Consider the system of ODEs:

$$\mathbf{x}' = A\mathbf{x}$$

where A is a constant matrix. Consider:

$$\begin{aligned}\mathbf{x} &= \vec{v}e^{\lambda t} \\ \mathbf{x}' &= \lambda\vec{v}e^{\lambda t} \\ &= A\vec{v}e^{\lambda t} = A\mathbf{x}\end{aligned}$$

Thus, we have:

$$A\vec{v} = \lambda\vec{v} \tag{13}$$

which is the eigenvalue problem.

Example 2.6.11 (Superposition)**.** Assume $x_1(t)$ and $x_2(t)$ are solutions to the system of ODEs:

$$\mathbf{x}' = A\mathbf{x}$$

Then, any linear combination of these solutions is also a solution:

$$\mathbf{x}(t) = c_1 x_1(t) + c_2 x_2(t)$$

where c_1 and c_2 are constants. This is due to the linearity of the system.

Definiton 2.6.12 (General Eigenproblem for n-dim ODEs System)**.** Combining the above ideas, consider the system:

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$. We define the *specturm* as:

$$\sigma(A) = \{(\lambda_i, \vec{v}_i) \in \mathbb{C} \times \mathbb{C}^n : A\vec{v}_i = \lambda_i\vec{v}_i, i = 1, 2, \ldots, n\} \tag{14}$$

where λ_i are the eigenvalues and $\vec{v}_i$ are the corresponding eigenvectors. The general solution to the system can be expressed as:

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i\vec{v}_i e^{\lambda_i t} \tag{15}$$

where c_i are constants determined by the initial conditions.

Example 2.6.13. Consider the following system of ODEs:

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$$

We find the eigenvalues and eigenvectors by solving the characteristic polynomial:

$$\begin{aligned}\det(A - \lambda I) &= \det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \\ &= \lambda^2 - 1 = 0 \\ \implies \lambda_1 &= 1, \quad \lambda_2 = -1\end{aligned}$$

The corresponding eigenvectors are:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus, the general solution to the system is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

where c_1 and c_2 are constants determined by the initial conditions.

Consider the phase portait. If we start on the span of $[-1, 1]^T$, then we go to the origin as $t \to \infty$. If we start on the span of $[1, 1]^T$, then we go away from the origin as $t \to \infty$. Then, the trivial fixed point is a saddle node.

Definiton 2.6.14 (Computing Eigenvalues via Trace and Determinant)**.** Consider matrix $A \in \mathbb{R}^{2\times 2}$. We compute the eigenvalues by solving the characteristic polynomial:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \\ \implies \lambda_{1,2} &= \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2} \end{aligned}$$

If the roots are real, there are four cases that coud emerge from this (WLOG):

- $\lambda_1 > \lambda_2 > 0$: We see much faster growth in the span assocaited with λ_1.
- $\lambda_1 < \lambda_2 < 0$: We see much faster decay in the span assocaited with λ_1.
- $\lambda_1 < 0 < \lambda_2$: We would see the span assocaited with λ_2 dominate as $t \to \infty$ and the span assocaited with λ_1 dominate as $t \to -\infty$. This is a saddle node.
- $\lambda_2 = 0 < \lambda_1$: Along the span of v_2, we would see horizontal line, as we growth only along the span of v_1.
- $\lambda_1 = \lambda_2 > 0$: Along the span of v_2, we would see horizontal line, as we decay only along the span of v_1.
- $\lambda_1 = \lambda_2 = 0$: The system is trivial, and all points are fixed points.
- $\lambda_1 = \lambda_2 < 0$: It is a inward star that everything goes to the origin.
- $\lambda_1 = \lambda_2 > 0$: It is an outward star that everything goes away from the origin.

Definiton 2.6.15 (Complex Eigenvalues)**.** Consider the case where the eigenvalues are complex, i.e., $\lambda = \alpha \pm i\beta$ with $\beta \neq 0$. The general solution to the system can be expressed as:

$$\mathbf{x}(t) = e^{\alpha t}\left(c_1 \operatorname{Re}(\vec{v}e^{i\beta t}) + c_2 \operatorname{Im}(\vec{v}e^{i\beta t})\right)$$

where $\vec{v}$ is the complex eigenvector corresponding to λ, and c_1, c_2 are constants determined by the initial conditions.

This would lead to the following definition of a radial and angular component:

$$v(t) = ve^{\alpha t}\text{cis}(\beta t) \tag{16}$$

where $r(t)$ represents the radial distance from the origin and $\theta(t)$ represents the angular position. So, for a general solution, we have:

$$\mathbf{x}(t) = \sum_i c_i v_i e^{\alpha_i t}\text{cis}(\beta_i t) \tag{17}$$

The behavior of the system depends on the sign of α and β:

- If $\alpha > 0$ the system spirals outward (complex source).
  - If $\beta > 0$, the systems sprial counter-clockwise.
  - If $\beta < 0$, the systems sprial clockwise.
- If $\alpha < 0$ the system spirals inward (complex sink).
  - If $\beta > 0$, the systems sprial counter-clockwise.
  - If $\beta < 0$, the systems sprial clockwise.
- If $\alpha = 0$ the system is a center (closed orbits).
  - If $\beta > 0$, the systems sprial counter-clockwise.
  - If $\beta < 0$, the systems sprial clockwise.

Complex Eigenvalues Resolves to Real Numbers Consider eigenvalues $\lambda = \alpha \pm i\beta \in \mathbb{C}$ with $v = \mathbf{p} + i\mathbf{q} \in \mathbb{C}^n$. Then, the general solution $x : \mathbb{R} \to \mathbb{R}^n$ is given by:

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\bar{\lambda} t}\bar{\mathbf{v}} \tag{18}$$

where $c_1, c_2 \in \mathbb{C}$.

We also have:

$$\bar{\mathbf{x}} = \bar{c_1} e^{\bar{\lambda} t}\bar{\mathbf{v}} + \bar{c_2} e^{\lambda t}\mathbf{v}$$

So, we can say that:

$$c_1 = \bar{c_2}, \quad c_2 = \bar{c_1}$$

Let $c_1 = \delta + i\gamma$, then $c_2 = \delta - i\gamma$. Thus, we have:

$$\mathbf{x}(t) = (\delta + i\gamma)e^{(\alpha+i\beta)t}(\mathbf{p} + i\mathbf{q}) + (\delta - i\gamma)e^{(\alpha-i\beta)t}(\mathbf{p} - i\mathbf{q}) \tag{19}$$

Seperating the real and imaginary parts, we have:

$$\mathbf{x}(t) = \delta(ve^{\lambda t} + \bar{v}e^{\bar{\lambda} t}) + i\gamma(ve^{\lambda t} - \bar{v}e^{\bar{\lambda} t}) \tag{20}$$

By the Euler's formula, we have:

$$\mathbf{x}(t) = 2\delta\text{Re}(ve^{\lambda t}) - 2\gamma\text{Im}(ve^{\lambda t}) \in \mathbb{R}^n \tag{21}$$

Generalized eigenvectors and solution for a defective matrix Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

which has a double eigenvalue $\lambda = 2$ but only one linearly independent eigenvector

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To complete a Jordan chain of length two we seek a generalized eigenvector v_g satisfying

$$(A - \lambda I)\, v_g = v. \tag{22}$$

For this A we have $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Choosing the convenient solution

$$v_g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

indeed yields $(A - 2I)v_g = v$, so (22) holds.

The two linearly independent solutions corresponding to this Jordan chain are

$$x_1(t) = v\, e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}, \tag{23}$$

and

$$x_2(t) = (v_g + t\, v)\, e^{\lambda t} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{2t}. \tag{24}$$

One checks directly that these satisfy $x_i'(t) = Ax_i(t)$ using $Av = \lambda v$ and $Av_g = \lambda v_g + v$.

Therefore the general solution of $\mathbf{x}' = A\mathbf{x}$ is the linear combination

$$\mathbf{x}(t) = C_1\, x_1(t) + C_2\, x_2(t) = e^{\lambda t} \begin{bmatrix} C_1 + C_2 t \\ C_2 \end{bmatrix} \tag{25}$$

Applying the matrix to the generalized eigenvector We can apply the matrix A to the generalized eigenvector v_g, then , Av_g would stay in the span of v and v_g: In other words, we call this subspace invariant under the transformation A.

Example 2.6.16 (Generalized Eigenvectors)**.** Consider the system of ODEs:

$$\mathbf{x}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x}$$

We find the eigenvalues and eigenvectors by solving the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \\ &= (\lambda - 3)^2 = 0 \\ \implies \lambda_1 &= 3, \quad \lambda_2 = 3 \end{aligned}$$

The corresponding eigenvector is:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since there is only one linearly independent eigenvector, we need to find a generalized eigenvector v_g satisfying:

$$(A - 3I)v_g = v_1$$

Solving this, we find:

$$v_g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, the two linearly independent solutions to the system are:

$$x_1(t) = v_1 e^{3t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$$

$$x_2(t) = (v_g + tv_1)e^{3t} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) e^{3t}$$

Therefore, the general solution to the system is:

$$\mathbf{x}(t) = c_1 x_1(t) + c_2 x_2(t)$$

where c_1 and c_2 are constants determined by the initial conditions.

Definiton 2.6.17 (Trace-Determinant Plane)**.** We found that we can find the eigenvalues of a 2×2 matrix A using:

$$\lambda_{1,2} = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2}$$

where

- $\operatorname{tr}(A) = \lambda_1 + \lambda_2 = a_{11} + a_{22}$ is the trace of A
- $\det(A) = \lambda_1 \lambda_2 = a_{11}a_{22} - a_{12}a_{21}$ is the determinant of A

And we have the following analysis:

- If $\det(A) < 0$ and $\operatorname{tr}(A) > 0$, then $\lambda_1 > 0 > \lambda_2$ (saddle node).
- Consider the discriminant boundry $\operatorname{tr}(A)^2 - 4\det(A) = 0$, it is a parabola that upward opening with vertex at the origin. Consider region above that, where there is complex eigenvalues:

  - If $\operatorname{tr}(A) > 0$ and $\det(A) > 0$, then $\alpha = \operatorname{tr}(A)/2 > 0$ (spiral source).
  - If $\operatorname{tr}(A) < 0$ and $\det(A) > 0$, then $\alpha = \operatorname{tr}(A)/2 < 0$ (spiral sink).
  - If $\operatorname{tr}(A) = 0$ and $\det(A) > 0$, then $\lambda_1 = -\lambda_2 = i\beta$ (center).

- Consider the region below the discriminant boundry, where there is real eigenvalues:
  - If $\operatorname{tr}(A) > 0$ and $\det(A) > 0$, then $\lambda_1 \geq \lambda_2 > 0$ (nodal source).
  - If $\operatorname{tr}(A) < 0$ and $\det(A) > 0$, then $0 > \lambda_1 \geq \lambda_2$ (nodal sink).

- **Degenerate Star Node** Consider the line $\operatorname{tr}(A)^2 - 4\det(A) = 0$.
- **Lines of Nodes** Consider the line $\det(A) = 0$.

2.7 Wronskian and Fundamental Matrix

Definiton 2.7.1 (Functionals)**.** A functional is a mapping from a vector space of functions to the real numbers. For example, consider the space of continuous functions on an interval $[a, b]$, denoted by $C[a, b]$. A functional $J : C[a, b] \to \mathbb{R}$ could be defined as:

$$J[f] = \int_a^b f(x)\,dx$$

for any function $f \in C[a, b]$.

Motivation For an nth-order differential equation we need n linearly independent solutions to form the general solution. These solutions constitute a basis of the solution space, so we must check whether a given set of solutions is linearly independent.

Definiton 2.7.2 (Wronskian)**.** Consider n functions $x_1(t), x_2(t), \ldots, x_n(t)$ for which $x : \mathbb{R} \to \mathbb{R}^n$ that are solutions of a nth-order linear differential equation on an interval I. The Wronskian $W(t)$ is defined as the determinant of the matrix whose columns are formed by the functions and their derivatives up to order $n - 1$:

$$W(t) = \det \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{bmatrix} \tag{26}$$

Iff there exist t in interval I such that $W(t) \neq 0$, then the functions $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent on I.

Example 2.7.3. Let $f(x) = \cos(mx)$ and $g(x) = \sin(mx)$ for some constant $m \neq 0$. We compute the Wronskian:

$$\begin{aligned} W(f, g)(x) &= \det \begin{bmatrix} \cos(mx) & \sin(mx) \\ -m\sin(mx) & m\cos(mx) \end{bmatrix} \\ &= m\cos^2(mx) + m\sin^2(mx) \\ &= m(\cos^2(mx) + \sin^2(mx)) \\ &= m \neq 0 \end{aligned}$$

Since $W(f, g)(x) \neq 0$ for all x, the functions $\cos(mx)$ and $\sin(mx)$ are linearly independent on $\mathbb{R}$.

Trading order for dimension Consider the nth-order ODE:

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + \ldots + a_1(t)x' + a_0(t)x = g(t)$$

where $a_n(t) \neq 0$ on an interval I. We can convert this into a system of first-order ODEs by defining:

$$\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = x'' \\ \vdots \\ x_n = x^{(n-1)} \end{cases}$$

Then, we have solution $x : \mathbb{R} \to \mathbb{R}^n$ given by:

$$x = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t} + \ldots + v_n e^{\lambda_n t}$$

where v_i are the eigenvectors and λ_i are the eigenvalues of the corresponding matrix A. Then, we can define the Wronskian for this system as:

$$W(v_1 e^{\lambda_1 t}, v_2 e^{\lambda_2 t}, \ldots, v_n e^{\lambda_n t}) = \begin{vmatrix} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & \ldots & v_n e^{\lambda_n t} \end{vmatrix} = e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)t} \begin{vmatrix} v_1 & v_2 & \ldots & v_n \end{vmatrix} \tag{27}$$

Which the first term is never zero, so the Wronskian is non-zero iff the eigenvectors are linearly independent. The basis $\{v_1 e^{\lambda_1 t}, v_2 e^{\lambda_2 t}, \ldots, v_n e^{\lambda_n t}\}$ is a fundamental called the fundamental set of solutions.

General Planer Systems Consider the system of ODEs:

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{2\times 2}$. We define the fundamental matrix $\Phi(t)$ as:

$$\Phi(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix} \tag{28}$$

where $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two linearly independent solutions of the system. The Wronskian of the system is given by:

$$W(t) = \det(\Phi(t)) = x_1(t)y_2(t) - x_2(t)y_1(t) \tag{29}$$

The Wronskian satisfies the differential equation:

$$W'(t) = \mathrm{tr}(A)W(t) \tag{30}$$

with the solution:

$$W(t) = W(0)e^{\mathrm{tr}(A)t} \tag{31}$$

where $W(0)$ is the Wronskian at $t = 0$.

2.8 Stability of Nonlinear Systems in Local Linear Approximation

Definiton 2.8.1 (Change of Variables)**.** The change of variable in a higher dimensions, the change of variables related by the Jacobian matrix. WLOG, consider change of variables $F : \mathbb{R}^2 \to \mathbb{R}^2$ and system $x' = Ax$. Let:

$$F(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

Then, the Jacobian matrix J_F is given by:

$$J_F = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \tag{32}$$

The change of variables transforms the system $x' = Ax$ into a new system $u' = Bu$ where:

$$B = J_F A J_F^{-1} \tag{33}$$

provided that J_F is invertible.

A reference for simple linear stability analysis.

Linear Stablity Analysis Consider the simple linear one-dimensional ODE:

$$y' = f(y)$$

where $f(y)$ is a differentiable function. Let y^* be a fixed point, i.e., $f(y^*) = 0$. We can perform a linear stability analysis by considering a small perturbation $\eta(t) = y(t) - y^*$. Then, we have:

$$\begin{aligned} \eta' &= y' - (y^*)' \\ &= f(y) - 0 \\ &= f(y^* + \eta) \\ &\approx f(y^*) + f'(y^*)\eta \\ &= f'(y^*)\eta = A\eta \end{aligned}$$

So, we have the linearized system:

$$\eta' = A\eta$$

where $A = f'(y^*)$. The stability of the fixed point y^* is determined by the sign of A.

Definiton 2.8.2 (Linear Stability Analysis in Higher Dimensions)**.** Consider the two-dimensional system:

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

where $f(x,y)$ and $g(x,y)$ are differentiable functions. Let (x^*,y^*) be a fixed point. We let perturbations $u(t)=x(t)-x^*$ and $v(t)=y(t)-y^*$. Then, we have:

$$\begin{cases} u' &= x' = \cancel{f(x,y)} + \frac{\partial f}{\partial x}(x^*,y^*)u + \frac{\partial f}{\partial y}(x^*,y^*)v + \cancel{O(u^2,v^2,uv)} \\ v' &= y' = \cancel{g(x,y)} + \frac{\partial g}{\partial x}(x^*,y^*)u + \frac{\partial g}{\partial y}(x^*,y^*)v + \cancel{O(u^2,v^2,uv)} \end{cases}$$

So we can write the linearized system as:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = J \begin{bmatrix} u \\ v \end{bmatrix}$$

where J is the Jacobian matrix evaluated at the fixed point (x^*,y^*). This describes the local behavior of the system near the fixed point. Then the stability and behavior is determined by the determinant and trace of the Jacobian matrix J, and hence the eigenvalues and eigenvectors of J.

Example 2.8.3. Consider the nonlinear pendulum equation:

$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$

where θ is the angle of the pendulum, g is the acceleration due to gravity, and L is the length of the pendulum. We can rewrite this as a system of first-order ODEs by defining:

$$\begin{cases} x_1 = \theta \\ x_2 = \theta' \end{cases}$$

Then, we have:

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{g}{L}\sin(x_1) \end{cases}$$

Now, consider the Jacaobian matrix:

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(x_1) & 0 \end{bmatrix}$$

Consider the fixed point $(n\pi, 0)$ for $n \in \mathbb{Z}$. We have:

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(n\pi) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}(-1)^n & 0 \end{bmatrix}$$

So the derminant is:

$$\det(J) = -\frac{g}{L}(-1)^n$$

and the trace is:

$$\operatorname{tr}(J) = 0$$

When n is even, we have a positive determinant, so the fixed point is a center. When n is odd, we have a negative determinant, so the fixed point is a saddle node.

Definiton 2.8.4 (Polar Coordinates)**.** Consider the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) given by:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \implies x^2 + y^2 = r^2, \quad \tan(\theta) = \frac{y}{x} \tag{34}$$

The inverse transformation is given by:

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \tag{35}$$

A trick we can do is to consider the time derivatives of both sides. We have:

$$\begin{aligned} 2rr' &= 2xx' + 2yy' \\ r' &= \frac{xx' + yy'}{r} \\ \theta' &= \frac{1}{1 + (y/x)^2}\left(\frac{xy' - yx'}{x^2}\right) \\ &= \frac{xy' - yx'}{r^2} \end{aligned}$$

Then we can transform a system of ODEs in Cartesian coordinates to polar coordinates:

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} \implies \begin{cases} r' = F(r, \theta) = \frac{xf(x,y) + yg(x,y)}{r} \\ \theta' = G(r, \theta) = \frac{xg(x,y) - yf(x,y)}{r^2} \end{cases} \tag{36}$$

Relationship to Eigenvectors Consider the nullclines $\theta' = 0$. Along this nullcline, the direction on the xy-plane is purely radial (either inward or outward). This occurs when:

$$xy' - yx' = 0 \implies \frac{y'}{x'} = \frac{y}{x}$$

This means that the slope of the trajectory is equal to the slope of the position vector, which corresponds to the direction of an eigenvector of the system. Thus, the nullclines in polar coordinates correspond to the directions of the eigenvectors in the Cartesian coordinate system.

Clockwise Parity For clockwise rotation, we have $\theta' < 0$. For counter-clockwise rotation, we have $\theta' > 0$.

Example 2.8.5 (Limit Cycles and Closed Orbits)**.** Consider the following system of ODEs:

$$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \end{cases}$$

The system is nonlinear. There is a trivial (in fact, it is the only) fixed point at the origin. We can determine the stability of this fixed point by analyzing the Jacobian matrix:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 1 - 3x^2 - y^2 & -1 - 2xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{bmatrix} \\ J(0,0) &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ \det(J(0,0)) &= 2 > 0 \\ \operatorname{tr}(J(0,0)) &= 2 > 0 \end{aligned}$$

Then, we conclude that this is a outward spiral (spiral source). We can also convert this system into polar coordinates:

$$\begin{aligned} r' &= \left[r\cos\theta[-r\sin\theta + r\cos\theta(1-r^2)] + r\sin\theta[r\cos\theta + r\sin\theta(1-r^2)] \right] / r \\ &= r(1-r^2) \\ \theta' &= \left[r\cos\theta[r\cos\theta + r\sin\theta(1-r^2)] - r\sin\theta[-r\sin\theta + r\cos\theta(1-r^2)] \right] / r^2 \\ &= 1 \end{aligned}$$

So, we have a decoupled system with constant angular velocity of 1 (counter-clockwise rotation) and a radial component that is positive when $r < 1$ and negative when $r > 1$. On $r = 1$, we have a have a closed orbit. Now, we can solve this system:

$$\begin{aligned} \frac{dr}{dt} &= r(1-r^2) \\ \int \frac{1}{r(1-r^2)} dr &= \int dt \\ \ln \left| \frac{r}{\sqrt{1-r^2}} \right| &= t + C \\ \frac{r}{\sqrt{1-r^2}} &= Ke^t \\ r(t) &= \frac{Ke^t}{\sqrt{1+K^2e^{2t}}} \end{aligned}$$

also, obviously, $\theta(t) = t + \theta(0)$.

where $K = e^C$ is a constant determined by the initial condition. The phase portrait is shown below:

Definiton 2.8.6 (Limit Cycle)**.** A limit cycle is a closed trajectory in the phase space of a dynamical system that is isolated. This means that there are no other closed trajectories in its immediate vicinity. Limit cycles can be stable, unstable, or semi-stable:

- A stable limit cycle attracts nearby trajectories, meaning that trajectories starting close to the limit cycle will converge to it as time goes to infinity.

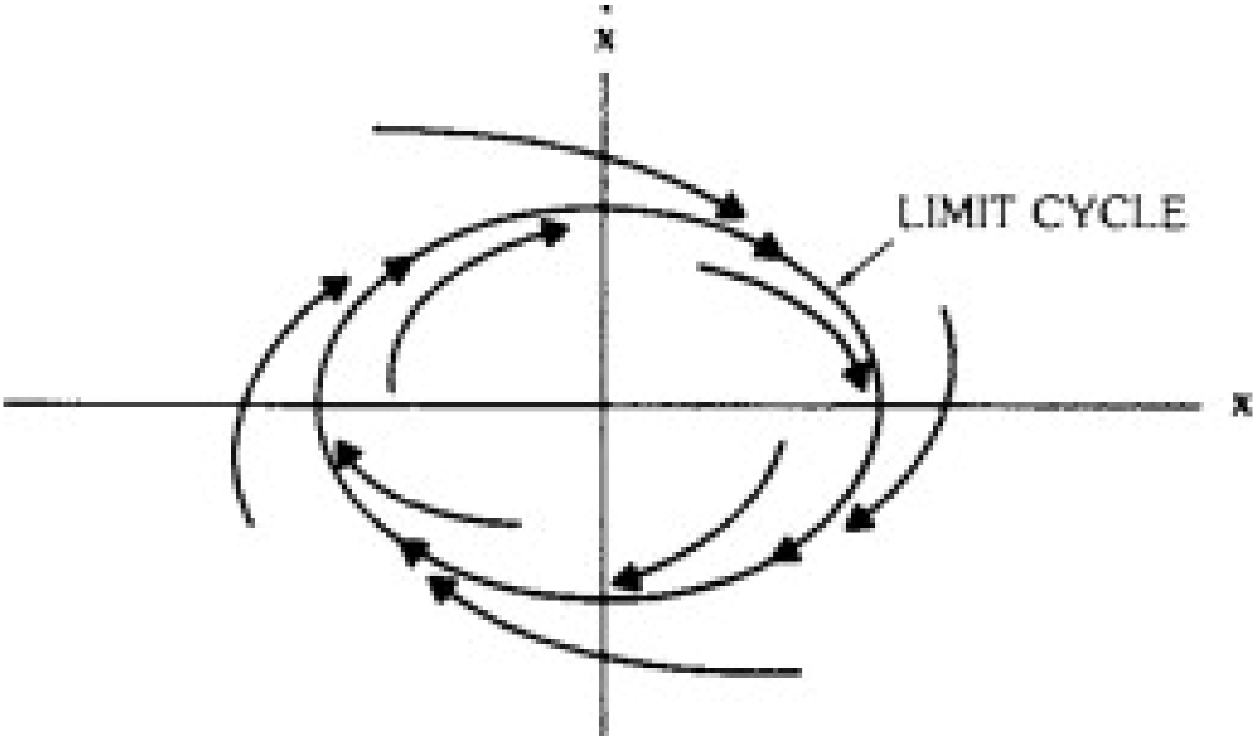


Figure 14: Phase Portrait of the Nonlinear Pendulum System

- An unstable limit cycle repels nearby trajectories, meaning that trajectories starting close to the limit cycle will diverge away from it as time goes to infinity.
- A semi-stable limit cycle has both attracting and repelling properties depending on the direction from which nearby trajectories approach it.

Example 2.8.7 (Von der Pol Oscillator)**.** Consider the Van der Pol oscillator, which is a classic example of a system that exhibits limit cycle behavior. The system is described by the second-order differential equation:

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

where μ is a scalar parameter indicating the nonlinearity and the strength of the damping. We can rewrite this as a system of first-order ODEs by defining:

$$\begin{cases} x_1 = x \\ x_2 = \frac{dx}{dt} \end{cases}$$

Then, we have:

$$\begin{cases} x_1' = x_2 \\ x_2' = \mu(1 - x_1^2)x_2 - x_1 \end{cases}$$

Consider the fixed point at the origin $(0, 0)$. We can also consider the nullclines:

- $x_1' = 0$ implies $x_2 = 0$ and $x_2' = -x_1$

- $x_2' = 0$ implies $x_2 = \frac{x_1}{\mu(1-x_1^2)}$ for $x_1 \neq \pm 1$

We now consider the Jacobian matrix:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 - 2\mu x_1 x_2 & \mu(1-x_1^2) \end{bmatrix} \\ J(0,0) &= \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \\ \det(J(0,0)) &= 1 > 0 \\ \operatorname{tr}(J(0,0)) &= \mu \end{aligned}$$

Consider the line $\tau^2 - 4\Delta = 0$, we have:

- If $\mu < 0$, then the fixed point is a stable spiral (spiral sink).
- If $\mu = 0$, then the fixed point is a center.
- If $\mu > 0$, then the fixed point is an unstable spiral (spiral source).

Thus, this is a limit cycle that is stable when $\mu > 0$.

Theorem 2.8.8 (Pomcoré-Bendixson Theorem)**.** Consider a two-dimensional autonomous system:

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

where f and g are continuously differentiable functions. If there exists a trajectory that remains in a closed, bounded region R of the phase plane for all future time and if R contains only unstable nodes, then all trajectories in R approach a limit cycle as $t \to \infty$.

Intuition A monotonically increasing function that gets bounded must converge. A similar idea applies to trajectories in a dynamical system: if they are confined to a bounded region and cannot escape, they must eventually settle into a steady state, such as a limit cycle.

3 Matrix Exponentials

Taylor Series Definition of Exponentials

Consider a scalar exponential, then,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Taylor series expansion of e^x around $x = 0$. This series converges for all real numbers x.

Definiton 3.0.1 (Matrix Exponential)**.** Consider a square matrix $A \in \mathbb{R}^{n \times n}$. The matrix exponential e^{At} is defined by the Taylor series expansion:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \tag{37}$$

where I is the identity matrix of the same dimension as A, and t is a scalar parameter (often representing time in applications).

Diagonalization If A is diagonalizable, i.e., there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, then the matrix exponential can be computed as:

$$e^{At} = Pe^{Dt}P^{-1} \tag{38}$$

where e^{Dt} is computed by exponentiating each diagonal element of D:

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix} \tag{39}$$

this would make it easier to compute the matrix exponential.

Definiton 3.0.2 (Properties of Matrix Exponentials)**.** The matrix exponential has several important properties:

- $e^{A \cdot 0} = I$, where 0 is the zero matrix.
- $e^{A+B} = e^A e^B$ if A and B commute (i.e., $AB = BA$).
- $e^{A(t+s)} = e^{At}e^{As}$ for any scalars t and s.
- $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.
- If A is invertible, then e^{At} is also invertible, and its inverse is given by $(e^{At})^{-1} = e^{-At}$.
- The solution to the system of linear differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is given by:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 \tag{40}$$

- The derivatives of the matrix exponential can be computed using the series expansion:

$$\frac{d^n}{dt^n}e^{At} = A^n e^{At} \tag{41}$$

Example 3.0.3 (Schrödinger Equation)**.** Consider the time-dependent Schrödinger equation in quantum mechanics:

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

where $|\Psi(t)\rangle$ is the state vector (wavefunction) of the quantum system, $\hat{H}$ is the Hamiltonian operator (which can be represented as a matrix in a finite-dimensional space for which $\hat{H} = \hat{H}^\dagger$ (Conjugate-Transpose)), i is the imaginary unit, and $\hbar$ is the reduced Planck constant. We claim that Hermitian matrices have real eigenvalues and orthogonal eigenvectors. We first rewrite the soltuion of the Schrödinger equation as:

$$|\Psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\Psi(0)\rangle$$

Let $\langle\phi(n)| = |\phi(n)\rangle^\dagger$, then we call $\langle\phi(n)|\Psi(t)\rangle$ the norm of the wavefunction in the direction of the eigenvector $|\phi(n)\rangle$. We have:

$$\begin{aligned}\langle\phi(n)|\Psi(t)\rangle &= \langle\phi(0)|\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i}{\hbar}\hat{H}t\right)^k|\Psi(0)\rangle \\ &= \langle\phi(0)|\Psi(0)\rangle\end{aligned}$$

Then, the solution lies on the same norm as the initial condition, in other words, the solution is in S^{n-1} if the initial condition is in S^{n-1}.

What if A is not diagonalizable? If A is not diagonalizable there could be two cases:

- If we know the fundamental set of solutions $\{x_1(t), x_2(t), \dots, x_n(t)\}$, then we can construct the fundamental matrix $\Phi(t)$ and compute the matrix exponential as:

$$e^{At} = \Phi(t)\Phi(0)^{-1} \tag{42}$$

  where $\Phi(0)$ is the fundamental matrix evaluated at $t = 0$. And $\phi = [x_1, x_2, \dots, x_n]$.

- Otherwise, we can consider its Jordan Canonical Form.

Jordan Canonical Form We first define the AM and GM of an eigenvalue:

Definiton 3.0.4 (Geometric and Algebraic Multiplicity)**.** Consider a square matrix $A \in \mathbb{R}^{n\times n}$ and an eigenvalue λ of A. The algebraic multiplicity (AM) of λ is the number of times λ appears as a root of the characteristic polynomial of A. The geometric multiplicity (GM) of λ is the dimension of the eigenspace corresponding to λ, which is the null space of $A - \lambda I$. Diagonalizability is characterized by the condition that for each eigenvalue, its algebraic multiplicity equals its geometric multiplicity.

Definiton 3.0.5 (Jordan Block)**.** A Jordan block $J_k(\lambda)$ of size k associated with an eigenvalue λ is a $k \times k$ matrix of the form:

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \tag{43}$$

where λ is the eigenvalue and the superdiagonal entries are all ones, while all other entries are zero.

Definiton 3.0.6 (Jordan Canonical Form)**.** A square matrix $A \in \mathbb{R}^{n\times n}$ is said to be in Jordan canonical form if it is a block diagonal matrix where each block is a Jordan block. Specifically, A can be expressed as:

$$A = PJP^{-1} \tag{44}$$

where P is an invertible matrix and J is the Jordan matrix composed of Jordan blocks along its diagonal. The Jordan canonical form provides a way to represent matrices that are not diagonalizable, allowing for the computation of functions of matrices, such as the matrix exponential.

Example 3.0.7. Consider the system of ODEs:

$$x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x$$

The characteristic polynomial is given by:

$$\begin{aligned} p(\lambda) &= \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \\ &= (1-\lambda)(3-\lambda) + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2 \end{aligned}$$

So, we have a repeated eigenvalue $\lambda = 2$ with algebraic multiplicity 2. To find the eigenvectors, we solve:

$$(A - 2I)v = 0$$
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

This gives us the eigenvector $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since the geometric multiplicity is 1, we need to find a generalized eigenvector w such that:

$$(A - 2I)w = v$$
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving this, we can take $w = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Now, we can form the Jordan matrix:

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = 2I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Obserbe that $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent since $N^2 = 0$. And that I commutes with everything. So, we can compute the matrix exponential as:

$$\begin{aligned} e^{At} &= Pe^{Jt}P^{-1} \\ &= Pe^{(2I+N)t}P^{-1} \\ &= Pe^{2tI}e^{Nt}P^{-1} \\ &= Pe^{2t}\left(I + Nt + \frac{(Nt)^2}{2!} + \dots\right)P^{-1} \\ &= Pe^{2t}\left(I + Nt\right)P^{-1} \\ &= e^{2t}P\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}P^{-1} \end{aligned}$$

where $P = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$. Thus, we have:

$$e^{At} = e^{2t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

4 Higher Order ODEs

4.1 Second Order Linear ODEs

Definiton 4.1.1 (General Form of Second Order Linear ODEs)**.** A second-order linear ordinary differential equation (ODE) has the general form:

$$y'' = f(t, y, y') \tag{45}$$

Example 4.1.2. We can convert a second-order linear ODE into a system of first-order ODEs. Consider the second-order ODE:

$$y'' + p(t)y' + q(t)y = g(t)$$

We can define:

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$$

Then, we have the system of first-order ODEs:

$$\begin{cases} x_1' = x_2 \\ x_2' = -g(t) - p(t)x_2 - q(t)x_1 \end{cases}$$

In matrix form, this can be written as:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -g(t) \end{bmatrix}$$

This transformation allows us to apply techniques for first-order systems to analyze and solve second-order ODEs.

Example 4.1.3. Consider the above Example 4.1.2 with constant coefficients:

$$y'' + py' + qy = 0$$

We can write the corresponding system of first-order ODEs as:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And you should be able to solve and analyze the system using the techniques we have learned for first-order systems, especially the trace-determinant plane.

4.2 Second Order Homogeneous ODE

Definiton 4.2.1 (Characteristic Equation)**.** The characteristic equation of a second-order linear ODE with constant coefficients:

$$y'' + py' + qy = 0$$

is given by the quadratic equation:

$$r^2 + pr + q = 0 \tag{46}$$

where r represents the roots of the characteristic polynomial. The nature of the roots (real and distinct, real and repeated, or complex conjugates) determines the form of the general solution to the ODE. This comes from the assuned solution of the form $y = e^{rt}$.

Definiton 4.2.2 (Cauchy-Euler Equation and Indicial Equation)**.** A Cauchy-Euler equation is a type of second-order linear ODE of the form:

$$t^2y'' + aty' + by = 0 \tag{47}$$

where a and b are constants. To solve this equation, we assume a solution of the form $y = t^r$, where r is a constant to be determined. Substituting this assumed solution into the Cauchy-Euler equation leads to the indicial equation:

$$r(r-1) + ar + b = 0 \tag{48}$$

The roots of the indicial equation determine the general solution of the Cauchy-Euler equation, which can take different forms depending on whether the roots are real and distinct, real and repeated, or complex conjugates.

To generalize, a Cauchy-Euler equation can be expressed as:

$$\sum_m y^{(m)}(t)a_m t^m = 0 \tag{49}$$

where a_m are constants.

And the indicial equation in this generalized form is given by:

$$\sum_m a_m r(r-1)(r-2)\dots(r-m+1) = 0 \tag{50}$$

Example 4.2.3 (Cauchy-Euler Equation)**.** Consider the following second-order ODE:

$$t^2y'' - ty' + y = 0$$

This is in the form of a Cauchy-Euler equation:

$$\sum_m y^{(m)}(t)a_m t^m = 0$$

where $a_2 = 1, a_1 = -1, a_0 = 1$. We can solve this by assuming a solution of the form $y = t^r$. Then, we have:

$$\begin{aligned} y' &= rt^{r-1} \\ y'' &= r(r-1)t^{r-2} \end{aligned}$$

Substituting these into the original equation gives:

$$\begin{aligned} t^2(r(r-1)t^{r-2}) - t(rt^{r-1}) + t^r &= 0 \\ r(r-1)t^r - rt^r + t^r &= 0 \\ (r^2 - 2r + 1)t^r &= 0 \\ (r-1)^2 t^r &= 0 \end{aligned}$$

Thus, we have a repeated root $r = 1$. The general solution for this case is given by:

$$y(t) = C_1 t + C_2 t \ln(t)$$

where C_1 and C_2 are constants determined by initial conditions.

4.3 Second Order Inhomogeneous ODE

Theorem 4.3.1 (General Solution of Inhomogeneous ODE)**.** The general solution of a second-order inhomogeneous ODE:

$$y'' + py' + qy = \sum_i g_i(t)$$

is given by:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \sum_i y_{p_i}(t) \tag{51}$$

where c_1, c_2 are constants determined by initial conditions, $y_h(t) = \{y_1(t), y_2(t)\}$ is the general solution of the corresponding homogeneous equation $y'' + py' + qy = 0$, and $y_{p_i}(t)$ are particular solutions corresponding to each $g_i(t)$. Each $y_{p_i}(t)$ satisfies:

1. $\text{Null}(L) = \{y_h\}$
2. $L(y_{p_i}) = g_i(t)$

Theorem 4.3.2 (Uniqueness of Particular Solutions)**.** If $y_{p_1}(t)$ and $y_{p_2}(t)$ are two particular solutions of the inhomogeneous ODE:

$$y'' + py' + qy = g(t)$$

then their difference $y_{p_1}(t) - y_{p_2}(t)$ is a solution of the corresponding homogeneous equation:

$$y'' + py' + qy = 0$$

And since the homongenous solution is unique up to a linear combination of basis solutions, we have:

$$y_{p_1}(t) - y_{p_2}(t) = c_1 y_1(t) + c_2 y_2(t)$$

where c_1 and c_2 are constants, and $y_1(t)$ and $y_2(t)$ are the fundamental solutions of the corresponding homogeneous equation.

Proof. We have:

$$L(y_{p_1}) = g(t)$$
$$L(y_{p_2}) = g(t)$$

Subtracting these two equations gives:

$$L(y_{p_1}) - L(y_{p_2}) = g(t) - g(t)$$
$$L(y_{p_1} - y_{p_2}) = 0$$

Thus, $y_{p_1}(t) - y_{p_2}(t)$ is indeed a solution of the corresponding homogeneous equation. □

Definiton 4.3.3 (Wronskian Derivation of y_p)**.** If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the corresponding homogeneous equation (a.k.a. the fundamental set of homogeneous solutions), then we can assume a particular solution of the form:

$$y_p(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(t)} dt$$

where $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the Wronskian determinant of y_1 and y_2.

Example 4.3.4. Consider the following second-order inhomogeneous ODE:

$$y'' - 3y' - 4y = 3e^{2t} + \sin(t)$$

The corresponding homogeneous equation is:

$$y'' - 3y' - 4y = 0$$

The characteristic equation is:

$$r^2 - 3r - 4 = 0$$

Solving this gives the roots $r_1 = 4$ and $r_2 = -1$. Thus, the general solution of the homogeneous equation is:

$$y_h(t) = C_1 e^{4t} + C_2 e^{-t}$$

where C_1 and C_2 are constants determined by initial conditions.

Next, we find particular solutions for each term on the right-hand side of the inhomogeneous equation. For the term $3e^{2t}$, we can assume a particular solution of the form $y_{p_1}(t) = Ae^{2t}$. Substituting this into the left-hand side of the ODE gives:

$$\begin{aligned} y''_{p_1} - 3y'_{p_1} - 4y_{p_1} &= 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} \\ &= -6Ae^{2t} \end{aligned}$$

Setting this equal to $3e^{2t}$ gives:

$$-6Ae^{2t} = 3e^{2t} \implies A = -\frac{1}{2}$$

Thus, we have:

$$y_{p_1}(t) = -\frac{1}{2}e^{2t}$$

For the term $\sin(t)$, we can assume a particular solution of the form $y_{p_2}(t) = B\cos(t) + C\sin(t)$. Substituting this into the left-hand side of the ODE gives:

$$\begin{aligned} y''_{p_2} - 3y'_{p_2} - 4y_{p_2} &= -B\cos(t) - C\sin(t) + 3B\sin(t) - 3C\cos(t) - 4B\cos(t) - 4C\sin(t) \\ &= (-5B - 3C)\cos(t) + (3B - 5C)\sin(t) \end{aligned}$$

Setting this equal to $\sin(t)$ gives the system of equations:

$$\begin{aligned} -5B - 3C &= 0 \\ 3B - 5C &= 1 \end{aligned}$$

Solving this system gives $B = -\frac{3}{34}$ and $C = \frac{5}{34}$. Thus, we have:

$$y_{p_2}(t) = -\frac{3}{34}\cos(t) + \frac{5}{34}\sin(t)$$

Finally, the general solution of the inhomogeneous ODE is:

$$y(t) = C_1e^{4t} + C_2e^{-t} - \frac{1}{2}e^{2t} - \frac{3}{34}\cos(t) + \frac{5}{34}\sin(t)$$

Operator Theory We call the differential operator $L = D^2 + pD + qI$ where D is the differentiation operator and I is the identity operator. Then, we would call $L : C^2(\mathbb{R}) \to C(\mathbb{R})$ a transformation from the space of twice differentiable functions to the space of continuous functions. We can consider the factorization of this operator:

$$L = (D - \lambda_1 I)(D - \lambda_2 I)$$

where λ_1 and λ_2 are the roots of the characteristic equation.

Then, we can consider the eigenvalues and eigenfunctions of this operator:

Definiton 4.3.5 (Eigenvalues and Eigenfunctions of Differential Operators)**.** An eigenvalue λ of a differential operator L is a scalar such that there exists a non-trivial function $y(t)$ (called an eigenfunction) satisfying the equation:

$$L[y(t)] = \lambda y(t) \tag{52}$$

In the context of second-order linear ODEs, if we consider the operator $L = D^2 + pD + qI$, then the eigenvalue problem becomes:

$$y'' + py' + qy = \lambda y \tag{53}$$

The solutions to this equation can provide insights into the behavior of the system described by the differential equation, including stability and oscillatory properties.

Example 4.3.6 (Eigenfunctions of D)**.** Consider the differential operator D defined by $D[y(t)] = y'(t)$. We want to find the eigenvalues and eigenfunctions of this operator. The eigenvalue equation is given by:

$$D[y(t)] = \lambda y(t)$$

which translates to the first-order ODE, solving this gives:

$$f_\lambda = e^{\lambda t}$$

where λ is any complex number. Thus, the eigenfunctions of the operator D exists in a dense, uncoutable spectrum.

If we impose boundary conditions, for example, $y(0) = 1 = y(2\pi)$, then we have:

$$e^{\lambda 2\pi} = 1 \implies \lambda = ik, k \in \mathbb{Z}$$

Thus, the eigenvalues are discrete and countable, and the corresponding eigenfunctions are:

$$f_k(t) = e^{ikt}, k \in \mathbb{Z}$$

which form a basis for the space of periodic functions with period 2π.

Definiton 4.3.7 (Fourier Basis)**.** The set of functions $\{e^{ikt} \mid k \in \mathbb{Z}\}$ forms an orthogonal basis for the space of square-integrable functions on the interval $[0, 2\pi]$. This basis is known as the Fourier basis. Any function $f(t)$ in this space can be expressed as a Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{54}$$

where the coefficients c_k are given by:

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt \tag{55}$$

The Fourier basis is widely used in various fields, including signal processing, heat transfer, and quantum mechanics, due to its ability to represent periodic functions and analyze frequency components.

Definiton 4.3.8 (Generalized Eigenfunction)**.** A generalized eigenfunction of a differential operator L associated with an eigenvalue λ is a function $y(t)$ that satisfies the equation:

$$(L - \lambda I)^m [y(t)] = 0 \tag{56}$$

for some positive integer m. Here, I is the identity operator, and m is the order of the generalized eigenfunction. Generalized eigenfunctions arise in cases where the operator L has repeated eigenvalues, and they extend the concept of eigenfunctions to include functions that may not satisfy the standard eigenvalue equation but still provide valuable information about the structure of the solution space.

Definiton 4.3.9 (Spectrum)**.** The spectrum of a differential operator L is the set of all eigenvalues λ for which there exists a non-trivial solution $y(t)$ (eigenfunction):

$$\text{spec}(L) = \{\lambda \in \mathbb{C} \mid L[y(t)] = \lambda y(t) \text{ has a non-trivial solution } y(t)\} \tag{57}$$

Theorem 4.3.10 (Variation of Parameters)**.** Variation of parameters is a method used to find a particular solution to a non-homogeneous linear differential equation. For a second-order ODE of the form:

$$y'' + p(t)y' + q(t)y = g(t)$$

where $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the corresponding homogeneous equation, a particular solution $y_p(t)$ can be expressed as:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{58}$$

where $u_1(t)$ and $u_2(t)$ are functions to be determined. These functions satisfy the system of equations:

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y \end{cases} \tag{59}$$

Solving this system allows us to find $u_1(t)$ and $u_2(t)$, and thus the particular solution $y_p(t)$.

Or, we can also plug in the following formula directly:

$$\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)}dt \\ u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}dt \end{cases} \tag{60}$$

where $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the Wronskian determinant of y_1 and y_2. However, practically, we only use this when g is a sum of functions that are:

- Trigonometric functions
- Exponential functions
- Polynomials

Proof. We start with Defintion 4.3.3 and assume a particular solution of the form:

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}dt$$

where $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the Wronskian determinant of y_1 and y_2. To verify that this is indeed a particular solution, we need to show that it satisfies the original inhomogeneous ODE:

$$y'' + p(t)y' + q(t)y = g(t)$$

We can compute the first and second derivatives of $y_p(t)$ using the product rule and the chain rule. After substituting these derivatives back into the left-hand side of the ODE, we can simplify the expression. The key step involves using the properties of the Wronskian determinant and the fact that $y_1(t)$ and $y_2(t)$ are solutions of the corresponding homogeneous equation. After simplification, we find that the left-hand side reduces to $g(t)$, confirming that $y_p(t)$ is indeed a particular solution of the inhomogeneous ODE. $\square$

Example 4.3.11. Conisder the following second-order inhomogeneous ODE:

$$y'' + y = \tan(t)$$

The corresponding homogeneous equation is:

$$y'' + y = 0$$

The characteristic equation is:

$$r^2 + 1 = 0$$

Solving this gives the roots $r_1 = i$ and $r_2 = -i$. Thus, the fundamental set of solutions to the homogeneous equation is:

$$y_1(t) = \cos(t), \quad y_2(t) = \sin(t)$$

The Wronskian determinant $W(t)$ is given by:

$$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \cos(t)\cos(t) - \sin(t)(-\sin(t)) = \cos^2(t) + \sin^2(t) = 1$$

Using the variation of parameters formula, we can find a particular solution $y_p(t)$:

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)g(t)}{W(t)} dt = -\int \sin(t)\tan(t) dt = -\int \sin^2(t)/\cos(t) dt \\ &= -\int (1 - \cos^2(t))/\cos(t) dt = -\int \sec(t) dt + \int \cos(t) dt \\ &= -\ln|\sec(t) + \tan(t)| + \sin(t) + C_1 \end{aligned}$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(t)} dt = \int \cos(t)\tan(t) dt = \int \sin(t) dt = -\cos(t) + C_2$$

Thus, a particular solution is given by:

$$\begin{aligned} y_p(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= [-\ln|\sec(t) + \tan(t)| + \sin(t)]\cos(t) + [-\cos(t)]\sin(t) \\ &= -\cos(t)\ln|\sec(t) + \tan(t)| \end{aligned}$$

Finally, the general solution of the inhomogeneous ODE is:

$$y(t) = C_1\cos(t) + C_2\sin(t) - \cos(t)\ln|\sec(t) + \tan(t)|$$

Finding the fundamental set of solutions We can find the fundamental set of solutions using:

- Cauchy-Euler Equation method if applicable (see Definition 4.2.2)
- Reduction of Order if one solution is known:

  $$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2} dt \tag{61}$$

  where $y_1(t)$ is a known solution of the homogeneous equation.
- Variation of Parameters

5 Numerical Methods for ODEs

Definiton 5.0.1 (Euler's Method)**.** Euler's method is a numerical technique used to approximate the solution of ordinary differential equations (ODEs) with a given initial value. It is a first-order method that uses a simple iterative approach to estimate the value of the solution at discrete points. Given an initial value problem of the form:

$$y' = f(t, y), \quad y(t_0) = y_0$$

Euler's method approximates the solution at successive time steps $t_n = t_0 + nh$ (where h is the step size) using the formula:

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{62}$$

for $n = 0, 1, 2, \ldots$. The method provides a straightforward way to compute approximate solutions, but it may require small step sizes for accurate results, especially for stiff equations or problems with rapid changes.

The local truncation error of Euler's method is $O(h^2)$, and the global truncation error is $O(h)$. The formula, given the upperbound M on the second derivative of the true solution, is:

$$|y(t_n) - y_n| \leq \frac{hM}{2}(e^{L(t_n - t_0)} - 1) \tag{63}$$

where L is the Lipschitz constant for f with respect to y.

Definiton 5.0.2 (Improved Euler's Method (Heun's Method))**.** Improved Euler's method, also known as Heun's method, is a numerical technique used to approximate the solution of ordinary differential equations (ODEs) with a given initial value. It is a second-order method that improves upon the basic Euler's method by using an average of slopes to estimate the next value. Given an initial value problem of the form:

$$y' = f(t, y), \quad y(t_0) = y_0$$

Improved Euler's method approximates the solution at successive time steps $t_n = t_0 + nh$ (where h is the step size) using the following steps:

$$k_1 = f(t_n, y_n) \tag{64a}$$

$$k_2 = f(t_n + h, y_n + hk_1) \tag{64b}$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \tag{64c}$$

The local truncation error of Improved Euler's method is $O(h^3)$, and the global truncation error is $O(h^2)$. The formula, given the upperbound M on the second derivative of the true solution, is:

$$|y(t_n) - y_n| \leq \frac{h^2M}{2}(e^{L(t_n - t_0)} - 1) \tag{65}$$

where L is the Lipschitz constant for f with respect to y.

Definiton 5.0.3 (Runge-Kutta Method)**.** The Runge-Kutta method is a family of iterative methods used to approximate the solutions of ordinary differential equations (ODEs). The most commonly used version is the fourth-order Runge-Kutta method (RK4), which provides a good balance between accuracy and computational efficiency. Given an initial value problem of the form:

$$y' = f(t, y), \quad y(t_0) = y_0$$

the RK4 method approximates the solution at successive time steps $t_n = t_0 + nh$ (where h is the step size) using the following steps: 1. Compute the slopes:

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\ k_4 &= f(t_n + h, y_n + hk_3) \end{aligned}$$

2. Update the solution using a weighted average of these slopes:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

for $n = 0, 1, 2, \ldots$. The RK4 method is widely used due to its high accuracy and stability for a variety of ODE problems.

The local truncation error of the RK4 method is $O(h^5)$, and the global truncation error is $O(h^4)$. The formula, given the upperbound M on the fifth derivative of the true solution, is:

$$|y(t_n) - y_n| \leq \frac{h^4 M}{90}(e^{L(t_n - t_0)} - 1) \tag{66}$$

where $L = \max_{t,y} \left|\frac{\partial f}{\partial y}\right|$ is the Lipschitz constant for f with respect to y.

Definiton 5.0.4 (Order of a Method)**.** The order of a numerical method for solving ordinary differential equations (ODEs) refers to the rate at which the local truncation error decreases as the step size h approaches zero. Specifically, a method is said to be of order p if the local truncation error $\tau(h)$ satisfies:

$$\tau(h) = O(h^{p+1}) \tag{67}$$

This means that as the step size h is halved, the local truncation error decreases by a factor of approximately 2^{p+1}. Higher-order methods generally provide more accurate results for a given step size, but they may also require more computational effort per step. For example, Euler's method is a first-order method, while the fourth-order Runge-Kutta method (RK4) is a fourth-order method.

Example 5.0.5 (Estimating the Order of a Method)**.** Given the following approximations of the solution to an ODE at a specific point using different step sizes:

$$\begin{aligned} y(h) &= 1.2 \\ y(h/2) &= 1.1 \\ y(h/4) &= 1.05 \end{aligned}$$

We can estimate the order of the numerical method used. Let $E(h)$ be the error associated with step size h. Assuming the error behaves as:

$$E(h) = Ch^p$$

for some constant C and order p, we can set up the following equations based on the given approximations:

$$\begin{aligned} E(h) &= |y(h) - y_{\text{exact}}| \\ E(h/2) &= |y(h/2) - y_{\text{exact}}| \\ E(h/4) &= |y(h/4) - y_{\text{exact}}| \end{aligned}$$

By taking the ratios of these errors, we can solve for p:

$$\begin{aligned} \frac{E(h)}{E(h/2)} &= \frac{Ch^p}{C(h/2)^p} = 2^p \\ \frac{E(h/2)}{E(h/4)} &= \frac{C(h/2)^p}{C(h/4)^p} = 2^p \end{aligned}$$

Taking the logarithm of both sides gives:

$$\begin{aligned} p &= \log_2\left(\frac{E(h)}{E(h/2)}\right) \\ p &= \log_2\left(\frac{E(h/2)}{E(h/4)}\right) \end{aligned}$$

By substituting the actual error values (which can be computed if the exact solution is known), we can estimate the order p of the numerical method used. In this case, we find that $p \approx 1$, indicating that the method is first-order accurate.