MIE 286 Lecture Notes

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The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

1 Random Variables

Definiton 1.0.1 (Random Variable)**.** A random variable is a variable whose possible values are numerical outcomes of a random phenomenon. There are two main types of random variables:

- Discrete Random Variable: Takes on a countable number of distinct values. Examples include the outcome of rolling a die or the number of heads in a series of coin flips.
- Continuous Random Variable: Takes on an infinite number of possible values within a given range. Examples include the height of individuals or the time it takes to complete a task.
- Mixed Random Variable: Exhibits properties of both discrete and continuous random variables. It can take on specific discrete values as well as a continuous range of values.

Definiton 1.0.2 (Probability Mass Function (PMF))**.** A Probability Mass Function (PMF) is a function that gives the probability that a discrete random variable is exactly equal to some value. The PMF must satisfy the following properties:

- Non-negativity: $P(X = x) \geq 0$ for all x.
- Normalization: The sum of the probabilities over all possible values of the random variable must equal 1:

$$\sum_x P(X = x) = 1$$

Definiton 1.0.3 (Probability Density Function (PDF))**.** A Probability Density Function (PDF) is a function that describes the likelihood of a continuous random variable to take on a particular value. The PDF must satisfy the following properties:

- Non-negativity: $f(x) \geq 0$ for all x.
- Normalization: The total area under the PDF curve must equal 1:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

2 Common Distributions and Visualization

2.1 Uniform Distribution

Definiton 2.1.1 (Uniform Distribution)**.** Discrete uniform on a finite set $\{x_1, \ldots, x_n\}$: $P(X = x_i) = 1/n$ for each i. Continuous uniform on an interval $[a, b]$: the density is

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b,$$

with mean $\mu = (a+b)/2$ and variance $\sigma^2 = (b-a)^2/12$.

2.2 Normal Distribution

Definiton 2.2.1 (Normal Distribution)**.** A random variable X is normal with mean μ and variance σ^2, written $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The distribution is symmetric about μ, and the standard normal is $\mathcal{N}(0, 1)$.

2.3 Addign Distributions

We can add random variables or constants. For example, we can do $Y = 2 + \text{Uni}(0, 1)$ or $Z = \text{Norm}(0, 1) + \text{Norm}(0, 1)$. The resulting distribution can be found by convolution of the original distributions.

2.4 Symmetric Distributions

A distribution is symmetric about a point c if for all x the density or mass satisfies $f(c+x) = f(c-x)$. Symmetry implies the mean (when it exists) equals the center c.

2.5 Stem-and-Leaf Plot

A stem-and-leaf plot is a simple textual visualization which preserves raw data while showing shape. Example for the data

$$\{12, 14, 15, 17, 21, 22, 22, 24, 31\}$$

is shown with stems (tens) and leaves (ones):

```
1 | 2 4 5 7
2 | 1 2 2 4
3 | 1
```

This quickly reveals distributional shape and outliers.

2.6 Experiments and Sample Space

Definiton 2.6.1 (Experiment)**.** An experiment is a process that leads to the occurrence of one and only one of several possible outcomes. Simply, it generates data.

Definiton 2.6.2 (Sample Space)**.** The sample space, denoted by Ω, is the set of all possible outcomes of an experiment. Each outcome in the sample space is called a sample point.

3 Sets and Sample Space

The sample space Ω is a set and events are subsets of Ω. For sets $A, B \subseteq \Omega$ we define:

- **Union:** $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- **Intersection:** $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- **Complement:** $A^c = \Omega \setminus A = \{x : x \notin A\}$.
- **Set difference:** $A \setminus B = \{x : x \in A, x \notin B\}$.
- **Disjoint:** A and B are disjoint if $A \cap B = \varnothing$.
- **Subset:** $A \subseteq B$ means every element of A is also in B.

Basic probability properties follow:

- $P(\varnothing) = 0$, $P(\Omega) = 1$.
- For any event A, $P(A^c) = 1 - P(A)$.
- For any events A, B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

De Morgan's laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.