

PHY 293 Lecture Notes

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PHY293

The up-to-date version of this document can be found at <https://github.com/HaysonC/skulenotes>

Chapter 1

Waves

1.1 Harmonic Oscillators

1.1.1 Governing Equations of Harmonic Oscillators

Key Concepts & Equations

Key Equations:

- Simple Harmonic Oscillator: $m\ddot{x} + kx = 0$ or $\ddot{x} + \omega_0^2 x = 0$ where $\omega_0 = \sqrt{k/m}$
- Damped: $m\ddot{x} + b\dot{x} + kx = 0$ or $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ where $\gamma = b/m$
- Driven: $m\ddot{x} + b\dot{x} + kx = F(t)$

Tip: Remember to check the damping parameter convention (γ vs 2β) when comparing formulas!

This subsection collects the baseline ODEs for simple, damped, and driven oscillators to set notation used later.

Types of Harmonic Oscillators There are three types of harmonic oscillators: simple, damped, and driven harmonic oscillators. Consider a simple one dimensional harmonic oscillator, they are defined by the following differential equations:

Definiton 1.1.1.1 (Simple Harmonic Oscillator). A simple harmonic oscillator is described by Hooke's law:

$$m \frac{d^2 x}{dt^2} + kx = 0 \quad (1.1.1)$$

where k is the spring constant, m is the mass, and x is the displacement from equilibrium.

Definiton 1.1.1.2 (Damped Harmonic Oscillator). A damped harmonic oscillator is described by the following differential equation, by adding a damping term proportional to \dot{x} to the simple harmonic oscillator equation:

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad (1.1.2)$$

where b is the damping coefficient.

Note on damping parameter conventions. Different texts use different symbols and normalizations:

- Our notes use $\gamma = \frac{b}{m}$, so the ODE reads $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ and the decay envelope is $e^{-\gamma t/2}$.
- Many texts define $\beta = \frac{b}{2m}$ and may call it γ instead. In that convention the ODE is $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ with envelope $e^{-\beta t}$.

Equivalences: $\beta = \frac{\gamma}{2}$ and our underdamped condition $\gamma < 2\omega_0$ corresponds to $\beta < \omega_0$ in the 2β convention. When comparing formulas, check which definition is being used and replace $\gamma \leftrightarrow 2\beta$ accordingly.

Definiton 1.1.1.3 (Driven Harmonic Oscillator). A driven harmonic oscillator is described by the following differential equation, which includes an external driving force $F(t)$:

$$m\ddot{x} + b\dot{x} + kx = F(t) \quad (1.1.3)$$

1.1.2 Simple Harmonic Motion

Key Concepts & Equations

Key Equations:

- General solution: $x(t) = A \cos(\omega t + \phi)$ where $\omega = \sqrt{k/m}$
- Period: $T = 2\pi/\omega = 2\pi\sqrt{m/k}$
- Frequency: $f = 1/T = \omega/(2\pi)$
- Total Energy: $E = \frac{1}{2}kA^2$ (constant for undamped oscillator)

Common Pitfall: Don't forget to match initial conditions properly when solving for amplitude and phase!

We analyze the undamped solution forms, relate constants to initial conditions, and derive period/frequency relations.

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Definiton 1.1.2.1 (Simple Harmonic Motion). You should have learned Hooke's law and Newton's second law, which give us the equation of motion for a simple harmonic oscillator. The same as equation (1.1.1), which can be rewritten as:

$$F = m\ddot{x} = -kx \quad (1.1.4)$$

xs By setting $\omega^2 = \frac{k}{m}$, a general solution can be written as:

$$x(t) = x_0 + A_1 \cos(\omega t) + A_2 \sin(\omega t) \quad (1.1.5)$$

where A_1 and A_2 are constants determined by the IVP, ω is the angular frequency, and ϕ is the phase constant. x_0 is the equilibrium position (often set to 0). The unknown constants can be determined by knowing x, \dot{x} at specific times.

Definiton 1.1.2.2 (Period, Frequency, and Angular Frequency). The period T is the time it takes for one complete cycle of the motion, given by:

$$T = 2\pi \sqrt{\frac{m}{k}} \quad (1.1.6)$$

The frequency f is the number of cycles per unit time, given by:

$$f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (1.1.7)$$

The angular frequency ω is related to the frequency by:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \quad (1.1.8)$$

Example 1.1.2.3. A simple harmonic oscillator consisting of mass $m = 11.0$ kg attached to a spring with spring constant $k = 201$ N m⁻¹. At time $t = 0$ s the oscillator is at position $x(0) = -0.207$ m and has velocity $v(0) = -1.33$ m s⁻¹. Determine all coefficients of the equation describing the position $x(t)$ of the oscillator as a function of time, assuming the offset is zero.

To solve for A_1 and A_2 , while we assume $x_0 = 0$, we can use the initial conditions:

$$\begin{aligned} x(0) &= A_1 \cos(0) + A_2 \sin(0) = A_1 = -0.207 \text{ m} \\ v(0) &= -A_1 \omega \sin(0) + A_2 \omega \cos(0) = A_2 \omega = -1.33 \text{ m s}^{-1} \end{aligned}$$

We can find ω from the given m and k :

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{201 \text{ N m}^{-1}}{11.0 \text{ kg}}} \approx 4.28 \text{ rad s}^{-1}$$

Therefore, we can solve for A_2 :

$$A_2 = \frac{v(0)}{\omega} = \frac{-1.33 \text{ m s}^{-1}}{4.28 \text{ rad s}^{-1}} \approx -0.311 \text{ m}$$

Thus, the equation describing the position $x(t)$ of the oscillator as a function of time is:

$$x(t) = -0.207 \cos(4.28t) - 0.311 \sin(4.28t)$$

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Theorem 1.1.2.4 (A Trigonometric Identity). We can also express the solution in a more compact form using a single cosine function with a phase shift:

$$x(t) = A \cos(\omega t + \phi)$$

where

$$A = \sqrt{A_1^2 + A_2^2}, \quad \phi = \arctan\left(\frac{-A_2}{A_1}\right) = \arctan\left(\frac{-v(0)/\omega}{x(0)}\right)$$

Proof. Let $A = \sqrt{A_1^2 + A_2^2}$ and choose ϕ such that

$$\cos(\phi) = \frac{A_1}{A}, \quad \sin(\phi) = -\frac{A_2}{A}.$$

Then, we can rewrite our original solution as

$$\begin{aligned} x(t) &= A_1 \cos(\omega t) + A_2 \sin(\omega t) \\ &= A \cos(\phi) \cos(\omega t) - A \sin(\phi) \sin(\omega t) \\ &= A [\cos(\phi) \cos(\omega t) - \sin(\phi) \sin(\omega t)] \\ &= A \cos(\omega t + \phi), \end{aligned}$$

by the cosine addition formula. □

Example 1.1.2.5. To determine the amplitude A and phase constant ϕ for the oscillator in the previous example, we can use the values of A_1 and A_2 we found:

$$\begin{aligned} A &= \sqrt{(-0.207)^2 + (-0.311)^2} \approx 0.374 \text{ m} \\ \phi &= \arctan\left(\frac{-(-0.311)}{-0.207}\right) \approx 4.12 \text{ rad} \quad (\text{since } A_1 < 0 \text{ and } A_2 < 0) \end{aligned}$$

Therefore, the equation describing the position $x(t)$ of the oscillator as a function of time can also be written as:

$$x(t) = 0.374 \cos(4.28t + 4.12)$$

Definiton 1.1.2.6 (The Energy of a Simple Harmonic Oscillator). The total mechanical energy E of a simple harmonic oscillator is the sum of its kinetic energy K and potential energy U .

$$E = K + U$$

First we consider the change of potential energy from a position x_i to x_f , assuming the path is along the spring or the curve C of the oscillator. The force exerted by the spring is given by Hooke's law, $F = -kx$. The change in potential energy can be simply parametrized and calculated as follows:

$$\Delta U = \int_C F \cdot ds = - \int_{x_i}^{x_f} F dx = \int_{x_i}^{x_f} kx dx = \left[\frac{1}{2} kx^2 \right]_{x_i}^{x_f} = \frac{1}{2} k(x_f^2 - x_i^2)$$

Therefore, the potential energy U at a position x (taking the reference point at $x = 0$) is given by:

$$U(x) = \frac{1}{2} kx^2$$

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The kinetic energy K of the oscillator is given by:

$$K = \frac{1}{2}m\dot{x}^2$$

Therefore, the total mechanical energy E of the simple harmonic oscillator is:

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \quad (1.1.9)$$

The total mechanical energy E remains constant over time, as energy is conserved in the absence of non-conservative forces (like friction or air resistance).

1.1.3 Damped Harmonic Motion

Key Concepts & Equations

Key Equations:

- Damped ODE: $\ddot{x} + \gamma\dot{x} + \omega_0^2x = 0$ where $\gamma = b/m$
- Underdamped ($\gamma < 2\omega_0$): $x(t) = A_0e^{-\gamma t/2} \cos(\omega_r t + \phi)$ where $\omega_r = \sqrt{\omega_0^2 - \gamma^2/4}$
- Critically damped ($\gamma = 2\omega_0$): $x(t) = (A_1t + A_2)e^{-\gamma t/2}$
- Overdamped ($\gamma > 2\omega_0$): Exponential decay without oscillation

Concept: Critical damping returns to equilibrium fastest without oscillating!

We solve the damped ODE, classify regimes (underdamped, critical, overdamped), and connect decay rates with parameters.

Definiton 1.1.3.1 (Damped Harmonic Motion). For small velocities, the drag force is approximately proportional to the velocity and acts in the opposite direction. This drag force can be modeled as $F_d = -\gamma\dot{x}$, where γ is the damping coefficient. Including this drag force in the equation of motion for a harmonic oscillator leads to the damped harmonic oscillator equation (1.1.2). Which could be rewritten as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2x = 0 \quad (1.1.10)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the undamped oscillator, and $\gamma = \frac{b}{m}$ is the damping coefficient per unit mass.

To skip the math, lets assume a solution of the form $x(t) = e^{i\omega t}$, substituting into the differential equation gives us a formulation for ω :

$$\omega = -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \quad (1.1.11)$$

Also, we can characterize the real and imaginary parts of ω as:

$$\omega_r = \text{Re}(\omega) = \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}, \quad \omega_i = \text{Im}(\omega) = -\frac{\gamma}{2}$$

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The general solution for the damped harmonic oscillator can be written as:

$$x(t) = \exp(\omega_i t) \exp(-i\omega_r t) = \exp\left(-\frac{\gamma}{2}t\right) \exp(\mp i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t)$$

- **No Damping** ($\gamma = 0$): The system behaves like a simple harmonic oscillator with angular frequency ω_0 . Given by:

$$z = \exp(-i\omega_0 t)$$

- **Underdamping** ($0 < \gamma < 2\omega_0$): The system oscillates with a gradually decreasing amplitude. The angular frequency of oscillation is given by $\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. Given by:

$$z = \exp\left(-\frac{\gamma}{2}t\right) \exp(-i\omega_r t)$$

The trigonometric form of the solution is:

$$x(t) = A_0 \exp\left(-\frac{\gamma}{2}t\right) \cos(\omega_r t + \phi)$$

where A_0 and ϕ are constants determined by the initial conditions. From this, we can derive the following cases:

- **Critical Damping** ($\gamma = 2\omega_0$): The system returns to equilibrium as quickly as possible without oscillating. Consider:

$$x(t) = e^{-\frac{\gamma}{2}t} f(t)$$

Inserting into the differential equation, we get:

$$\ddot{f} + \left(\omega_0^2 - \frac{\gamma^2}{4}\right) f = 0$$

Since $\gamma = 2\omega_0$, we have $\omega_0^2 - \frac{\gamma^2}{4} = 0$, leading to:

$$\ddot{f} = 0 \implies f(t) = A_1 t + A_2$$

Therefore, the general solution for the critically damped case is:

$$x(t) = (A_1 t + A_2) \exp\left(-\frac{\gamma}{2}t\right)$$

where A_1 and A_2 are constants determined by

- **Overdamping** ($\gamma > 2\omega_0$): The system returns to equilibrium without oscillating, but more slowly than in the critically damped case. The solution is given by:

$$z = \exp\left(-\frac{\gamma}{2}t\right) \exp\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2}t\right)$$

So the general solution is (the solution is via a substitution of $x(t) = e^{-\gamma t/2} f(t)$ into the differential equation, which resolves the ODE to a simple form):

$$x(t) = A_1 \exp\left[\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] + A_2 \exp\left[\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right]$$

where A_1 and A_2 are constants determined by the initial conditions.

1.1.4 Energy and Quality Factor

Key Concepts & Equations

Key Equations:

- Energy decay: $E(t) = E_0 e^{-\gamma t}$ for light damping
- Time constant: $\tau = 1/\gamma$ (time for energy to drop to $1/e$)
- Quality factor: $Q = \omega_0/\gamma = \omega_0\tau$ (higher Q = less damping)
- Energy loss per cycle: $\frac{\Delta E}{E} \approx -\frac{2\pi}{Q}$

Concept: Q measures how many oscillations occur before energy decays significantly!

We study how energy decays under light damping, define the time constant and quality factor Q , and relate them to response.

Definiton 1.1.4.1 (Energy of a Very Light Damping). Consider a very lightly damped harmonic oscillator, where $\gamma \ll \omega_0$. In this case, the angular frequency of oscillation ω_r can be approximated as:

$$\omega_r \approx \omega_0 \left(1 - \frac{\gamma^2}{8\omega_0^2} \right) \approx \omega_0$$

So the motion of the lightly damped oscillator can be approximated as:

$$x(t) \approx A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi)$$

Then, we can calculate the velocity of the oscillator:

$$\begin{aligned} \dot{x}(t) &= -\frac{\gamma}{2} A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi) - A_0 \omega_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t + \phi) \\ &= A_0 \omega_0 e^{-\frac{\gamma}{2}t} \left(-\frac{\gamma}{2\omega_0} \cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi) \right) \end{aligned}$$

The total mechanical energy $E(t)$ of the lightly damped oscillator is given by:

$$\begin{aligned} E(t) &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \\ &= \frac{1}{2} m \left[A_0 \omega_0 e^{-\frac{\gamma}{2}t} \left(-\frac{\gamma}{2\omega_0} \cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi) \right) \right]^2 + \frac{1}{2} k \left[A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi) \right]^2 \\ &= \frac{1}{2} m A_0^2 \omega_0^2 e^{-\gamma t} \left[\left(-\frac{\gamma}{2\omega_0} \cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi) \right)^2 + \cos^2(\omega_0 t + \phi) \right] \\ &= \frac{1}{2} m A_0^2 \omega_0^2 e^{-\gamma t} \left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi) + \frac{\gamma^2}{4\omega_0^2} \cos^2(\omega_0 t + \phi) + \frac{\gamma}{\omega_0} \sin(\omega_0 t + \phi) \cos(\omega_0 t + \phi) \right] \\ &\approx \frac{1}{2} m A_0^2 \omega_0^2 e^{-\gamma t} \left[1 + \frac{\gamma^2}{4\omega_0^2} \cos^2(\omega_0 t + \phi) \right] \quad (\text{neglecting the small term } \frac{\gamma}{\omega_0} \sin(\omega_0 t + \phi) \cos(\omega_0 t + \phi)) \\ &\approx \frac{1}{2} m A_0^2 \omega_0^2 e^{-\gamma t} \quad (\text{since } \frac{\gamma^2}{4\omega_0^2} \text{ is very small}) \\ &= E_0 e^{-\gamma t} \quad \text{where } E_0 = \frac{1}{2} m A_0^2 \omega_0^2 \text{ is the initial energy at } t = 0 \end{aligned}$$

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We can also define the time constant τ as the time it takes for the energy to decrease to $\frac{1}{e}$ of its initial value:

$$\tau = \frac{1}{\gamma}$$

So we have, for very light damping:

$$E(t) = E_0 e^{-\gamma t} = E_0 e^{-\frac{t}{\tau}} \quad (1.1.13)$$

Definiton 1.1.4.2 (Rate of Energy Loss). Taking the time derivative of the total mechanical energy $E(t)$:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \right) \\ &= (m\dot{x} + kx)\dot{x} \end{aligned}$$

For a undamped harmonic oscillator, $m\dot{x} + kx = 0$, so $\frac{dE}{dt} = 0$, indicating that the total mechanical energy is conserved and obeys Hooke's law completely. However, for a damped harmonic oscillator, $m\dot{x} + kx = -b\dot{x}$, leading to:

$$\frac{dE}{dt} = -b\dot{x}^2$$

Definiton 1.1.4.3 (Quality Factor (Q-Factor)). The quality factor Q is a dimensionless parameter that characterizes the damping of a harmonic oscillator. It is defined as:

$$Q = \frac{\omega}{\gamma} = \omega\tau \quad (1.1.14)$$

And for very light damping, we can approximate $\omega \approx \omega_0$, leading to:

$$Q \approx \frac{\omega_0}{\gamma} = \omega_0\tau$$

This allows us to rewrite the equation of a damped harmonic oscillator as:

$$\ddot{x} + \frac{\omega_0}{Q} \dot{x} + \omega_0^2 x = 0$$

and:

$$\omega = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$$

We can also consider the ratio between the energy at one time and the energy one period later:

$$\begin{aligned} \frac{E(t+T)}{E(t)} &= \frac{E_0 e^{-\gamma(t+T)}}{E_0 e^{-\gamma t}} \\ &= e^{-\gamma T} \approx 1 - \gamma T \\ \frac{E(t+T) - E(t)}{E(t)} &\approx -\gamma T = -\frac{2\pi}{\omega_0} \gamma = -\frac{2\pi}{Q} \end{aligned}$$

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Example 1.1.4.4. *What is the number of radians through which the damped system oscillates as its energy decreases to $1/e$ of its initial value?*

We have

$$\frac{E}{E_0} = e^{-\gamma t} = \frac{1}{e} \implies \gamma t = 1 \implies t = \frac{1}{\gamma} = \tau$$

So the number of radians is

$$\theta = \omega \tau = \frac{\omega}{\gamma} = Q$$

1.1.5 Undamped Forced Oscillations

Key Concepts & Equations

Key Equations:

- Driven ODE: $m\ddot{x} + kx = F_0 \cos(\omega t)$
- Steady-state solution: $x(t) = A(\omega) \cos(\omega t - \delta)$
- Amplitude: $A(\omega) = \frac{\omega_0^2 a}{|\omega_0^2 - \omega^2|}$ where $a = F_0/k$

Common Pitfall: At resonance ($\omega \approx \omega_0$), amplitude becomes infinite without damping!

We examine steady-state response to sinusoidal driving without losses and identify resonance behavior.

Definiton 1.1.5.1 (Undamped Forced Oscillations). Consider a driver force acting on the mass-spring system; the ODE becomes:

$$m\ddot{x} + kx = \eta \tag{1.1.15}$$

where $\eta = F_0 \cos(\omega t)$ is the driving force with amplitude F_0 and angular frequency ω . The steady-state particular solution is:

$$x(t) = A(\omega) \cos(\omega t - \delta) \tag{1.1.16}$$

To derive $A(\omega)$ and δ , use the equation of motion:

$$A(\omega)(-\omega^2 + \omega_0^2) \cos(\delta) = \omega_0^2 a$$

where $a = \frac{F_0}{k}$ is the static displacement of the mass when the driving force is constant. We also have:

$$A(\omega)(-\omega^2 + \omega_0^2) \sin(\delta) = 0$$

Now, consider the case when $\delta = 0$, we have:

$$A(\omega) = \frac{\omega_0^2 a}{\omega_0^2 - \omega^2}$$

and the case where $\delta = \pi$, we have:

$$A(\omega) = -\frac{\omega_0^2 a}{\omega_0^2 - \omega^2}$$

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Observe, when $\omega \approx \omega_0$, the amplitude $A(\omega)$ becomes very large, indicating resonance. At resonance, the system oscillates with maximum amplitude, which can lead to significant energy transfer from the driving force to the oscillator.

1.1.6 Damped Forced Oscillations

Key Concepts & Equations

Key Equations:

- Amplitude: $A(\omega) = \frac{\omega_0^2 a}{\sqrt{\omega^2 \gamma^2 + (\omega^2 - \omega_0^2)^2}}$
- Resonance amplitude: $A_{\max} = Qa$ where $Q = \omega_0/\gamma$
- Phase shift: $\tan(\delta) = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$
- FWHM: $\Delta\omega = \gamma = \omega_0/Q$

Concept: Damping prevents infinite amplitude at resonance and causes phase lag!

We add damping to the driven case, derive the frequency response amplitude and phase, and study bandwidth and Q .

Definiton 1.1.6.1 (Damped Forced Oscillations). Consider a damped harmonic oscillator subjected to an external driving force. It is described by the following differential equation:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

where F_0 is the amplitude of the driving force, ω is the angular frequency of the driving force, b is the damping coefficient, m is the mass, and k is the spring constant. The general solution to this non-homogeneous differential equation is given by:

$$x(t) = A(\omega) \cos(\omega t - \delta)$$

and the amplitude $A(\omega)$ is given by:

$$A(\omega) = \frac{\omega_0^2 a}{\sqrt{\omega^2 \gamma^2 + (\omega^2 - \omega_0^2)^2}} \quad (1.1.17)$$

We can derive the following 4 cases:

1. **No Damping** ($\gamma = 0$): In this case, the amplitude $A(\omega)$ simplifies to the undamped case we discussed earlier.
2. **Low-Frequency Limit** ($\omega \ll \omega_0$): In this limit, the amplitude $A(\omega)$ approaches the static displacement $a = \frac{F_0}{k}$. This means that at very low frequencies, the system behaves like a static spring, and the mass is displaced by an amount proportional to the applied force.

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3. **High-Frequency Limit** ($\omega \gg \omega_0$): In this limit, the amplitude $A(\omega)$ decreases with increasing frequency, following the relation $A(\omega) \approx \frac{\omega_0^2 a}{\omega^2}$. This indicates that at very high frequencies, the mass cannot respond quickly enough to the rapidly oscillating driving force, resulting in a smaller amplitude of oscillation.
4. **Resonance** ($\omega \approx \omega_0$): At resonance, the amplitude $A(\omega)$ reaches its maximum value but it does not become infinite due to the presence of damping:

$$A_{\max} = \frac{\omega_0^2 a}{\omega_0 \gamma} = \frac{\omega_0 a}{\gamma} = Qa$$

We also have the phase shift δ given by:

$$\tan(\delta) = \frac{\omega \gamma}{\omega_0^2 - \omega^2}$$

The phase shift δ indicates how much the oscillation of the mass lags behind the driving force. The behavior of δ can be summarized as follows:

- At low frequencies ($\omega \ll \omega_0$), δ approaches 0, meaning the mass oscillates in phase with the driving force.
- At resonance ($\omega = \omega_0$), $\delta = \frac{\pi}{2}$, indicating that the mass oscillates a quarter cycle behind the driving force.
- At high frequencies ($\omega \gg \omega_0$), δ approaches π , meaning the mass oscillates out of phase with the driving force.

Definiton 1.1.6.2 (Power absorbed during forced oscillations). We can calculate the velocity of the oscillator:

$$\dot{x}(t) = -\omega A(\omega) \sin(\omega t - \delta) = -v_0 \sin(\omega t - \delta)$$

where $v_0 = \omega A(\omega)$ is the maximum speed of the oscillator. Energy is lost at the following rate:

$$P(t) = bv(t)^2$$

Substituteing the expression for $v(t)$ into the power equation gives:

$$P(t) = bv_0^2 \sin^2(\omega t - \delta)$$

The average power $\langle P \rangle$ over one complete cycle of the driving force is given by:

$$\bar{P}(\omega) = \frac{bv_0^2}{2} = \frac{\omega^2 F_0^2 \gamma}{2m [(\omega^2 - \omega_0^2)^2 + \omega^2 \gamma^2]} \quad (1.1.18)$$

If the driving frequency ω is close to ω_0 , use $\omega^2 - \omega_0^2 \approx 2\omega_0 \Delta\omega$ with $\Delta\omega = \omega - \omega_0$ to get the Lorentzian form:

$$\bar{P}(\omega) \approx \frac{\omega_0 F_0^2}{2m\gamma \left[1 + \left(\frac{2\Delta\omega}{\gamma}\right)^2\right]}$$

The maximum average power occurs at resonance:

$$\bar{P}_{\max} = \frac{\omega_0 F_0^2}{2m\gamma}$$

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Full Width Half Height The graph of $\bar{P}(\omega)$ versus ω has a peak at $\omega = \omega_0$ with a maximum value of \bar{P}_{\max} . The full width at half maximum (FWHM) is the width of the peak at half of its maximum height. The FWHM is given by:

$$\omega_{\text{FWHM}} = 2\Delta\omega = 2\gamma = \frac{2\omega_0}{Q}$$

To illustrate the graph, we can plot $\bar{P}(\omega)$ versus ω for different values of the quality factor Q :

Average Power vs Driving Frequency for Different Quality Factors, Power normalized to $\bar{P}_{\max} = 1$

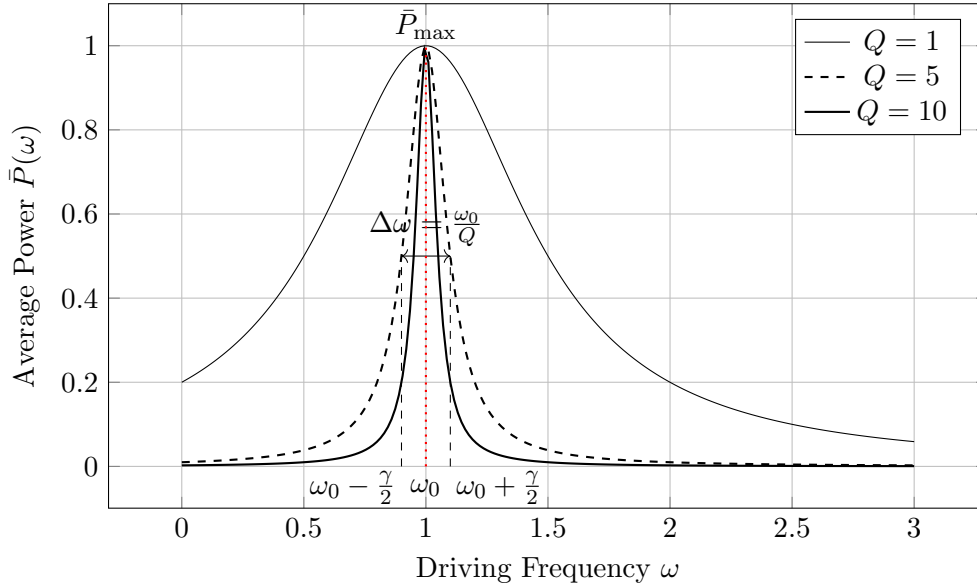


Figure 1.1: Average Power vs Driving Frequency for Different Quality Factors

1.1.7 Simple Pendulum

Key Concepts & Equations

Key Equations:

- Small angle approximation: $\ddot{\theta} + \frac{g}{l}\theta = 0$ (valid for $\theta \ll 1$ rad)
- Period: $T = 2\pi\sqrt{l/g}$ (independent of mass!)
- Physical pendulum: $\omega = \sqrt{\frac{mgd}{I}}$ where d is distance to CM

Tip: LC circuits are analogous to oscillators: $L \leftrightarrow m$, $1/C \leftrightarrow k$, $R \leftrightarrow b$

Consider the arc length s of a pendulum bob, we have:

$$\begin{aligned} s &= l\theta \\ m\ddot{s} &= -mg\sin\theta \end{aligned} \tag{1.1.19}$$

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For small angles, we can approximate $\sin(\theta) \approx \theta$, leading to:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad (1.1.20)$$

And the ODE has solution:

$$\theta(t) = \theta_0 \cos(\omega t + \phi) \quad (1.1.21)$$

Energy For small angle θ , the total mechanical energy E of the pendulum is given by:

$$E = K + U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos \theta) \approx \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2 \quad (1.1.22)$$

This shows that the pendulum behaves like a simple harmonic oscillator with an effective spring constant $k = \frac{mg}{l}$.

The physical pendulum For a rigid body swinging about a pivot point, the equation of motion is given by:

$$I\ddot{\theta} = \tau \quad (1.1.23)$$

For simple rod of length L and mass m pivoted at one end, the moment of inertia about the pivot point is $I = \frac{1}{3}mL^2$. The torque due to gravity when the rod is displaced by an angle θ from the vertical is $\tau = -mg\frac{L}{2}\sin \theta$. For small angles, we can approximate $\sin \theta \approx \theta$, leading to:

$$\ddot{\theta} + \frac{3g}{2L}\theta = 0 \quad (1.1.24)$$

And the small angle approximation leads to:

$$\ddot{\theta} + \omega^2\theta = 0 \quad (1.1.25)$$

where $\omega = \sqrt{\frac{3g}{2L}}$. The solution to this ODE is:

$$\theta(t) = \theta_0 \cos(\omega t + \phi) \quad (1.1.26)$$

The angular frequency of oscillation for the physical pendulum is:

$$\omega = \sqrt{\frac{3g}{2L}} \quad (1.1.27)$$

The LC Circuit An LC circuit consists of an inductor L and a capacitor C connected in series. The charge $q(t)$ on the capacitor satisfies the differential equation:

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (1.1.28)$$

This is analogous to the equation of motion for a simple harmonic oscillator, with the angular frequency given by:

$$\omega = \frac{1}{\sqrt{LC}} \quad (1.1.29)$$

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The RLC Circuit An RLC circuit consists of a resistor R , inductor L , and capacitor C connected in series. The charge $q(t)$ on the capacitor satisfies the differential equation:

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0 \quad (1.1.30)$$

This is analogous to the resistor behaving as a damped harmonic oscillator, with the angular frequency given by:

$$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad (1.1.31)$$

The following are equations that summarize the analogies between mechanical and electrical oscillators:

Mechanical Oscillator	Electrical Oscillator	Analogy
Mass m	Inductance L	Inertia
Spring constant k	Inverse capacitance $\frac{1}{C}$	Restoring force
Damping coefficient b	Resistance R	Energy dissipation
Displacement x	Charge q	Position
Velocity \dot{x}	Current $I = \frac{dq}{dt}$	Rate of change of position
Force F	Voltage V	Driving force

Table 1.1: Analogies between Mechanical and Electrical Oscillators

1.1.8 Coupled Oscillators

Key Concepts & Equations

Key Concepts:

- Normal coordinates $q_1 = x_1 + x_2$, $q_2 = x_1 - x_2$ decouple the equations
- Each normal mode has its own frequency: ω_1 (in-phase), ω_2 (out-of-phase)
- General motion = superposition of normal modes
- Energy doesn't flow between modes (they're independent)

Tip: Beat frequency = $(\omega_2 - \omega_1)/2$ describes energy transfer between oscillators!

Definiton 1.1.8.1 (Coupled Oscillators). Consider two masses m_1 and m_2 on two pendulum connected by springs with spring constant k . The equations of motion for the two masses are given by:

$$\begin{cases} \ddot{x}_1 = -\frac{g}{L}x_1 + \frac{k}{m_1}(x_1 - x_2) \\ \ddot{x}_2 = -\frac{g}{L}x_2 - \frac{k}{m_2}(x_1 - x_2) \end{cases} \quad (1.1.32)$$

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where x_1 and x_2 are the displacements of masses m_1 and m_2 from their equilibrium positions, respectively. Now if we add and subtract the two equations, we get:

$$\begin{cases} \ddot{x}_1 + \ddot{x}_2 = -\frac{g}{L}(x_1 + x_2) \\ \ddot{x}_1 - \ddot{x}_2 = -\left(\frac{g}{L} + \frac{k}{m_1} + \frac{k}{m_2}\right)(x_1 - x_2) \end{cases} \quad (1.1.33)$$

Let:

$$q_1 = x_1 + x_2, \quad q_2 = x_1 - x_2 \quad (1.1.34)$$

also, we let:

$$\omega_1 = \sqrt{\frac{g}{L}}, \quad \omega_2 = \sqrt{\frac{g}{L} + \frac{k}{m_1} + \frac{k}{m_2}} \quad (1.1.35)$$

Then, the equations of motion can be rewritten as:

$$\begin{cases} \ddot{q}_1 = -\omega_1^2 q_1 \\ \ddot{q}_2 = -\omega_2^2 q_2 \end{cases} \quad (1.1.36)$$

Observe that q_1 and q_2 are decoupled, and each behaves like a simple harmonic oscillator with angular frequencies ω_1 and ω_2 , respectively. The general solutions for $q_1(t)$ and $q_2(t)$ are:

$$\begin{cases} q_1(t) = A_1 \cos(\omega_1 t + \phi_1) \\ q_2(t) = A_2 \cos(\omega_2 t + \phi_2) \end{cases} \quad (1.1.37)$$

where A_1, A_2, ϕ_1 , and ϕ_2 are constants determined by the initial conditions. Finally, we can expressions for $x_1(t)$ and $x_2(t)$ in terms of $q_1(t)$ and $q_2(t)$:

$$\begin{cases} x_1(t) = \frac{q_1(t) + q_2(t)}{2} = \frac{A_1}{2} \cos(\omega_1 t + \phi_1) + \frac{A_2}{2} \cos(\omega_2 t + \phi_2) \\ x_2(t) = \frac{q_1(t) - q_2(t)}{2} = \frac{A_1}{2} \cos(\omega_1 t + \phi_1) - \frac{A_2}{2} \cos(\omega_2 t + \phi_2) \end{cases} \quad (1.1.38)$$

All parts of the system oscillate with the same normal frequency (eigenvalue) in a normal mode (eigenvector).

Definiton 1.1.8.2 (Normal Modes). A **normal mode** is a natural pattern of oscillation in which every part of the system moves in sync with a single frequency.

For the two-mass pendulum system, there are two normal modes:

- **In-phase mode** (ω_1): both masses swing together, reaching maximum displacement on the same side at the same time.

$$x_1 = x_2 \quad (1.1.39)$$

- **Out-of-phase mode** (ω_2): the two masses swing in opposite directions, so when one moves left, the other moves right.

$$x_1 = -x_2 \quad (1.1.40)$$

Any general motion of the system can be written as a combination (superposition) of these two normal modes.

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Energy in Coupled Oscillators The total mechanical energy E of the coupled oscillator system is given by:

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}m_1\frac{g}{L}x_1^2 + \frac{1}{2}m_2\frac{g}{L}x_2^2 \quad (1.1.41)$$

Substituting the expressions for $x_1 = \frac{q_1+q_2}{2}$ and $x_2 = \frac{q_1-q_2}{2}$ into the energy equation, two independent energies terms for q_1 and q_2 emerge:

$$E = \left[\frac{1}{4}m\dot{q}_1^2 + \frac{1}{4}\frac{m_1g}{L}q_1^2 \right] + \left[\frac{1}{4}m\dot{q}_2^2 + \frac{1}{4}\frac{m_2g}{L}q_2^2 \right] \quad (1.1.42)$$

This shows that the total energy is the sum of the energies associated with each normal mode, and that no cross term between the modes exists. Thus energy does not flow from one mode to another.

Example 1.1.8.3. Given two coupled pendulums with $x_0(0) = A$, $x_1(0) = 0$, and both masses at rest initially, we have:

$$\begin{aligned} q_0(t) &= C_0 \cos(\omega_0 t) \\ q_1(t) &= C_1 \cos(\omega_1 t) \end{aligned}$$

where C_0 and C_1 are constants determined by the initial conditions. Using the initial conditions. Using the cosine addition formula, which is:

$$\begin{aligned} \cos(A) + \cos(B) &= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\ \cos(A) - \cos(B) &= -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \end{aligned}$$

we can express the motion of each pendulum as:

$$\begin{aligned} x_0(t) &= \frac{q_0 + q_1}{2} = A \cos\left(\frac{\omega_0 + \omega_1}{2}t\right) \cos\left(\frac{\omega_0 - \omega_1}{2}t\right) \\ x_1(t) &= \frac{q_0 - q_1}{2} = A \sin\left(\frac{\omega_0 + \omega_1}{2}t\right) \sin\left(\frac{\omega_0 - \omega_1}{2}t\right) \end{aligned}$$

Due to the formula for the product of cosines and sines, we see that energy oscillates between the two pendulums with a beat frequency of $\frac{\omega_1 - \omega_0}{2}$. The time for all the energy to transfer from one pendulum to the other is given by:

$$T = \frac{2\pi}{\frac{\omega_1 - \omega_0}{2}} = \frac{4\pi}{\omega_1 - \omega_0}$$

1.1.1.9 Normal Modes

Key Concepts & Equations

Key Concepts:

- Normal mode = pattern where all parts oscillate at same frequency
- Find frequencies by solving eigenvalue problem: $\det(K - \omega^2 M) = 0$
- Number of normal modes = number of degrees of freedom
- Eigenvectors give amplitude ratios for each mode

Tip: For symmetric systems, look for in-phase and out-of-phase modes!

Oscilating Masses Connected by Springs Consider two masses $m_1 = m_2 = m$ connected by three spring with spring constant k and fixed to walls on either side. The equations of motion for the two masses are given by:

$$\begin{cases} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) & (a) &= m\ddot{x}_1 = -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -kx_2 - k(x_2 - x_1) & (b) &= m\ddot{x}_2 = kx_1 - 2kx_2 \end{cases}$$

To solve these equations, we assume solutions of the form:

$$\begin{cases} x_1(t) &= A \cos(\omega t + \phi) \\ x_2(t) &= B \cos(\omega t + \phi) \end{cases}$$

Substituting these assumed solutions into the equations of motion, we get:

$$\begin{cases} \frac{A}{B} &= \frac{k}{m\omega^2 - 2k} \\ \frac{A}{B} &= \frac{m\omega^2 - 2k}{k} \end{cases} \quad (1.1.43)$$

We can deduce that $A = \pm B$ We can set the equation:

$$(m\omega^2 - 2k)^2 = k^2 \implies m\omega^2 - 2k = \pm k$$

So we can derive two normal mode frequencies:

$$\begin{cases} \omega_1 &= \sqrt{\frac{k}{m}} & (\text{in-phase mode}) \\ \omega_2 &= \sqrt{\frac{3k}{m}} & (\text{out-of-phase mode}) \end{cases}$$

Solving as an Eigenproblem Alternatively, we can create a system of equations from (1.1.43):

$$\begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

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To find the eigenvalues ω^2 , we solve the characteristic equation:

$$\det \left(\begin{bmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{bmatrix} \right) = 0$$

Calculating the determinant, we have:

$$\left(\frac{2k}{m} - \omega^2 \right)^2 - \left(-\frac{k}{m} \right)^2 = 0$$

Expanding and simplifying, we get:

$$\omega^4 - \frac{4k}{m}\omega^2 + \frac{5k^2}{m^2} = 0$$

This gives us the two normal mode frequencies:

$$\begin{cases} \omega_1^2 = \frac{k}{m} \implies \omega_1 = \sqrt{\frac{k}{m}} \\ \omega_2^2 = \frac{3k}{m} \implies \omega_2 = \sqrt{\frac{3k}{m}} \end{cases}$$

Example 1.1.9.1. Consider two equal masses m suspended from identical springs of spring constant k . The masses are hanged on the celing. We can solve this system's frequency via the eigenvalue method. The equations of motion are:

$$\begin{cases} m\ddot{x}_2 = -k(x_2 - x_1) \\ m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \end{cases}$$

Assume solutions of the form:

$$\begin{cases} x_1(t) = A \cos(\omega t + \phi) \\ x_2(t) = B \cos(\omega t + \phi) \end{cases}$$

Substituting these assumed solutions into the equations of motion, we get:

$$\begin{cases} -A\omega^2 = -\frac{k}{m}(-2A + B) \\ -B\omega^2 = -\frac{k}{m}(A - B) \end{cases}$$

We can rewrite this as an eigenvalue problem:

$$\begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

To find the eigenvalues ω^2 , we solve the characteristic equation:

$$\det \left(\begin{bmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - \omega^2 \end{bmatrix} \right) = 0$$

Calculating the determinant, we have:

$$\omega^2 = \frac{k}{2m}(3 \pm \sqrt{5})$$

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Example 1.1.9.2. Consider three equal masses m connected by four identical springs with spring constant k and fixed to walls on either side. The equations of motion for the three masses are given by:

$$\begin{cases} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -2kx_2 + kx_1 + kx_3 \\ m\ddot{x}_3 &= -2kx_3 + kx_2 \end{cases}$$

Assume solutions of the form:

$$\begin{cases} x_1(t) &= A \cos(\omega t + \phi) \\ x_2(t) &= B \cos(\omega t + \phi) \\ x_3(t) &= C \cos(\omega t + \phi) \end{cases}$$

Substituting these assumed solutions into the equations of motion, we get:

$$\begin{cases} A\omega^2 &= -\frac{k}{m}(-2A + B) \\ B\omega^2 &= -\frac{k}{m}(A - 2B + C) \\ C\omega^2 &= -\frac{k}{m}(B - 2C) \end{cases}$$

We can rewrite this as an eigenvalue problem:

$$\begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

To find the eigenvalues ω^2 , we solve the characteristic equation:

$$\det \left(\begin{bmatrix} -\frac{2k}{m} - \omega^2 & \frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} - \omega^2 & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{2k}{m} - \omega^2 \end{bmatrix} \right) = 0$$

Calculating the determinant, we have:

$$\left(\frac{2k}{m} - \omega^2 \right) \left[\left(-\frac{2k}{m} - \omega^2 \right)^2 - \left(\frac{k}{m} \right)^2 \right] - \left(\frac{k}{m} \right)^2 \left(\frac{2k}{m} - \omega^2 \right) = 0$$

This gives us the three normal mode frequencies:

$$\begin{cases} \omega_1 &= \sqrt{\frac{2k}{m}} \\ \omega_2 &= \sqrt{\frac{2k}{m}(2 - \sqrt{2})} \\ \omega_3 &= \sqrt{\frac{2k}{m}(2 + \sqrt{2})} \end{cases}$$

Intuition For any perturbation, we can express it as a linear combination of the excitations of several normal modes. Each normal mode oscillates at its own frequency, and the overall motion is a superposition of these modes.

1.2 Waves

1.2.1 Travelling Waves

Key Concepts & Equations

Key Equations:

- Wave function: $y(x, t) = A \cos(kx - \omega t + \phi_0)$ (travelling right)
- Wave speed: $v = \omega/k = f\lambda$
- Wave number: $k = 2\pi/\lambda$, Angular frequency: $\omega = 2\pi f$
- String wave speed: $v = \sqrt{\tau/\mu}$ where τ is tension, μ is linear density

Common Pitfall: Don't confuse particle velocity ($\partial y/\partial t$) with wave velocity v !

Definiton 1.2.1.1 (Travelling Pulse). A travelling pulse is a disturbance that moves through a medium. For any wave function $f(x, t)$, if it satisfies the property:

$$y(x, t) = f(x \pm vt) \quad (1.2.1)$$

then it represents a wave travelling in the positive (for $x - vt$) or negative (for $x + vt$) x-direction with speed v . y could represent displacement, pressure, electric field, etc.

Definiton 1.2.1.2 (Sinusoidal Wave). A sinusoidal wave is a wave that can be described by a sine or cosine function. The general form of a sinusoidal wave travelling in the positive x-direction is:

$$y(x, t) = A \cos(kx \pm \omega t + \phi_0)$$

where:

- A is the amplitude (maximum displacement)
- $k = \frac{2\pi}{\lambda}$ is the wave number, with λ being the wavelength
- $\omega = 2\pi f$ is the angular frequency, with f being the frequency
- ϕ_0 is the initial phase.

So we can write the wave function as a form of (1.2.1):

$$y(x, t) = A \cos[k(x \pm vt) + \phi_0]$$

The wave speed v is related to the angular frequency and wave number by:

$$v = \frac{\omega}{k} = f\lambda \quad (1.2.2)$$

Transverse and Longitudinal Waves There is two (or three) main categories of waves:

- **Transverse Waves:** In transverse waves, the oscillations are perpendicular to the direction of wave propagation.
(Examples) Light waves, water waves, and waves on a string.
- **Longitudinal Waves:** In longitudinal waves, the oscillations are parallel to the direction of wave propagation.
(Examples) Sound waves and pressure waves in fluids.
- **Both:** Some waves can exhibit both transverse and longitudinal characteristics.
(Examples) Rayleigh surface waves in seismology.

Definiton 1.2.1.3 (Velocity of a Fixed Particle). For a fixed particle, we have:

$$v_y = \frac{\partial y}{\partial t} = A\omega \cos(kx \pm \omega t + \phi_0)$$

Example 1.2.1.4 (Vibrating String). Consider a string in the xy -plane, stretched along the x -axis with tension τ and linear mass density μ . Now, consider at the end of the string at x makes a small angle θ with the x -axis. Then the components of the tension are:

$$\begin{cases} \tau_x &= \tau \cos(\theta) \approx \tau \\ \tau_y &= \tau \sin(\theta) \approx \tau \frac{\partial y}{\partial x} \end{cases}$$

Now, consider a small segment of the string between x and $x + \delta x$. The vetical force on the other end is given, via linear approximation, by:

$$\tau_y(x + \delta x) \approx \tau_y(x) \left[\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} \delta x \right]$$

So the net force is given by:

$$F_y = \tau_y(x + \delta x) - \tau_y(x) = \tau \frac{\partial^2 y}{\partial x^2} \delta x$$

Because $F = ma$, we have:

$$\tau \frac{\partial^2 y}{\partial x^2} \delta x = dm \frac{\partial^2 y}{\partial t^2}$$

where $dm = \mu \delta x$ is the mass of the small segment. Thus, we have:

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

where $v = \sqrt{\frac{\tau}{\mu}}$ is the wave speed on the string. This is the one-dimensional wave equation.

Definiton 1.2.1.5 (The Wave Equation). The wave equation is a second-order linear partial differential equation that describes the propagation of waves (sound, EM, water) through a medium. In one dimension, it is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (1.2.3)$$

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where $u(x, t)$ is the disturbance, c is the wave speed, x is position, and t is time.

Or in general:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

where ∇^2 is the Laplacian operator.

Definiton 1.2.1.6 (Energy of a String). The kinetic energy of a string segment of mass $dm = \mu dx$ is given by:

$$dK = \frac{1}{2} dm v_y^2 = \frac{1}{2} \mu dx \left(\frac{\partial y}{\partial t} \right)^2$$

The potential energy stored in the string due to its tension is given by:

$$dU = \frac{1}{2} \tau (\delta s - \delta x)$$

where δs is the actual length of the string segment and δx is the horizontal length. For small displacements, we can approximate:

$$\delta s \approx \delta x \left(1 + \frac{1}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right)$$

Thus, the potential energy becomes:

$$dU \approx \frac{1}{2} \tau \delta x \left(\frac{\partial y}{\partial x} \right)^2$$

The total energy E of the string segment is the sum of its kinetic and potential energies:

$$dE = \frac{1}{2} \mu dx \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} \tau dx \left(\frac{\partial y}{\partial x} \right)^2$$

For a sinusoidal wave $y(x, t) = A \cos(kx - \omega t)$, the change in energy is:

$$dE = \frac{1}{2} \mu A^2 \omega^2 \sin^2(kx - \omega t) dx + \frac{1}{2} \tau A^2 k^2 \sin^2(kx - \omega t) dx$$

Integrating over one wavelength λ , we find the total energy in one wavelength:

$$E = \int_0^\lambda dE = \frac{1}{4} \mu A^2 \omega^2 \lambda + \frac{1}{4} \tau A^2 k^2 \lambda$$

Definiton 1.2.1.7 (Power Transmitted by a String). The power carried by a transverse wave could be calculated from $P = Fv$. The transverse force on the string is given by:

$$P(x, t) = A^2 \mu v \omega^2 \sin^2(kx - \omega t + \phi_0)$$

The average power is then given by:

$$\bar{P} = \frac{1}{2} A^2 \mu v \omega^2 = \frac{1}{2} \mu A^2 \omega^2 \frac{\lambda}{T}$$

And the maximum power is given by:

$$P_{max} = A^2 \mu v \omega^2$$

Speed of Sound The speed of sound waves through various media is as follows:

- In ideal gas, it is given by:

$$v = \sqrt{\frac{\gamma RT}{m}} = \sqrt{\frac{\gamma k_B T}{m}}$$

where γ is the adiabatic index, R is the universal gas constant, k_B is the Boltzmann constant, T is the absolute temperature, and m is the molar mass of the gas. γ varies based on the type of gas:

- Monatomic gases (e.g., helium, neon): $\gamma = \frac{5}{3}$
- Diatomic gases (e.g., nitrogen, oxygen): $\gamma = \frac{7}{5}$
- Polyatomic gases (e.g., carbon dioxide, methane): γ is typically between 1.3 and 1.4

- In a liquid, it is given by:

$$v = \sqrt{\frac{B_{\text{adiabatic}}}{m}} = \sqrt{\frac{B}{\rho}}$$

where $B_{\text{adiabatic}}$ is the adiabatic bulk modulus, B is the bulk modulus, ρ is the density of the liquid and m is the molar mass of the liquid.

- In a solid, it is given by:

$$v = \sqrt{\frac{E}{\rho}}$$

where E is the Young's modulus and ρ is the density of the solid. It could also be given by:

$$v = \sqrt{\frac{B}{\rho}}$$

where B is the bulk modulus.

Definiton 1.2.1.8 (Impedance). Similar to electrical impedance, we also have acoustic impedance and mechanical, they measure how much a structure resists motion when subjected to a harmonic force. They are defined as follows:

- Electrical impedance Z_e :

$$Z_e = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

- Acoustic impedance Z_a , it is the ratio of acoustic pressure to particle velocity:

$$Z_a = \frac{p}{v} = \rho v = \sqrt{B\rho}$$

- Mechanical impedance Z_m :

$$Z_m = v\mu = \sqrt{\tau\mu}$$

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Wave boundary Conditions When a wave encounters a boundary between two different media, part of the wave is reflected back into the original medium, and part is transmitted into the new medium. This is constrained by the following boundary conditions:

- The displacement of the wave must be continuous across the boundary:

$$y_1(x_b, t) = y_2(x_b, t) = y_3(x_b, t)$$

- The derivative of the displacement with respect to position must also be continuous across the boundary:

$$\left. \frac{\partial y_1}{\partial x} \right|_{x_b} = \left. \frac{\partial y_2}{\partial x} \right|_{x_b} = \left. \frac{\partial y_3}{\partial x} \right|_{x_b}$$

and then from these conditions, we can derive the amplitudes of the reflected and transmitted waves:

$$A_i = A_r + A_t \tag{1.2.4}$$

and

$$A_i Z_1 = A_r Z_1 + A_t Z_2$$

From these two equations, we can solve for the reflection and transmission coefficients:

$$R = \frac{A_r}{A_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{1.2.5}$$

and

$$T = \frac{A_t}{A_i} = \frac{2Z_2}{Z_2 + Z_1} \tag{1.2.6}$$

Special Cases There are two special cases:

- **Fixed End:** If the wave encounters a boundary where the medium is fixed (e.g., a string tied to a wall), the reflected wave undergoes a phase change of π (inversion). The reflection coefficient is $R = -1$ and the transmission coefficient is $T = 0$.
- **Free End:** If the wave encounters a boundary where the medium is free to move (e.g., a string attached to a ring that can slide along a rod), the reflected wave does not undergo a phase change. The reflection coefficient is $R = 1$ and the transmission coefficient is $T = 2$.

Energy Reflection and Transmission The energy reflection and transmission coefficients are given by:

$$R_e = R^2 = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2, \quad T_e = \frac{P_t}{P_i} = \frac{Z_1}{Z_2} T^2 = \frac{4Z_1 Z_2}{(Z_2 + Z_1)^2} \tag{1.2.7}$$

Note that $R_e + T_e = 1$, which means that the total energy is conserved across the boundary.

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Example 1.2.1.9 (Transmission of 64% Energy). Suppose a wave transmits 64% of its incident energy across a boundary. Then:

$$T_e = 0.64, \quad R_e = 1 - T_e = 0.36$$

so that the amplitude reflection coefficient is

$$R = +\sqrt{R_e} = 0.6$$

Important the positive sign indicates the reflected wave is *in phase* with the incident wave.

From the boundary condition in Eq. (1.2.4),

$$T = 1 + R = 1 + 0.6 = 1.6$$

Thus, the amplitude transmission coefficient is $T = 1.6$, even though only 64% of the energy is transmitted. This apparent paradox arises because the transmitted energy depends on both the amplitude squared and the ratio of impedances $\frac{Z_2}{Z_1}$.

Example 1.2.1.10 (Tricky Case: Impedance Mismatch with Energy Conservation). Let $Z_2 = 3Z_1$. Then:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{1}{2}, \quad T = \frac{2Z_2}{Z_2 + Z_1} = 1.5$$

Amplitude transmission seems large, but power depends on both amplitude and impedance.

Energy (power) transmission and reflection coefficients are:

$$R_e = R^2 = 0.25, \quad T_e = \frac{Z_1}{Z_2} T^2 = \frac{1}{3} (1.5)^2 = 0.75$$

so

$$R_e + T_e = 1.00$$

confirming energy conservation.

Interpretation: The transmitted amplitude is higher because the second medium has greater impedance ($Z_2 > Z_1$), which means each unit of amplitude carries less energy. Therefore, amplitude ratios alone can be misleading — energy flow must always be evaluated using both amplitude and impedance.

1.2.2 Superposition and Standing Waves

Key Concepts & Equations

Key Equations:

- Standing wave: $y(x, t) = 2A \sin(kx) \cos(\omega t)$ (for fixed-fixed ends)
- Allowed wavelengths: $\lambda_n = 2L/n$ where $n = 1, 2, 3, \dots$
- Frequencies: $f_n = nv/(2L) = nf_1$ (harmonics)
- Constructive interference: path difference $= m\lambda$
- Destructive interference: path difference $= (m + 1/2)\lambda$

Tip: Nodes are spaced $\lambda/2$ apart; antinodes too!

Definiton 1.2.2.1 (Modulation of two waves). Consider two monochromatic waves:

$$\begin{cases} y_1(x, t) &= A \cos(k_1x - \omega_1t) \\ y_2(x, t) &= A \cos(k_2x - \omega_2t) \end{cases}$$

Then, the resultant wave $y_1 + y_2$ can be expressed as:

$$y(x, t) = 2A \cos\left(\frac{(k_1 - k_2)x - (\omega_1 - \omega_2)t}{2}\right) \cos\left(\frac{(k_1 + k_2)x - (\omega_1 + \omega_2)t}{2}\right)$$

Definiton 1.2.2.2 (Coherence). Two waves are said to be coherent if they are monochromatic and maintain a constant phase relationship over time. This means that the difference in phase between the two waves does not change as they propagate:

$$\Delta\phi = \phi_1 - \phi_2 = \text{constant}$$

Coherent waves can produce stable interference patterns, such as constructive and destructive interference.

Definiton 1.2.2.3 (Phase and Phase Difference). Phase of the wave is an argument $kx \pm \omega t + \phi_0$ of the wave function. The phase difference between two waves at a given point in space and time is given by:

$$\Delta\phi = (k_1x - \omega_1t + \phi_1) - (k_2x - \omega_2t + \phi_2)$$

where k_1, k_2 are the wave numbers, ω_1, ω_2 are the angular frequencies, and ϕ_1, ϕ_2 are the initial phases of the two waves.

If we consider two waves of the same frequency and wave number, the phase difference simplifies to:

$$\Delta\phi = \Delta\phi + k\Delta x + \omega\Delta t$$

Definiton 1.2.2.4 (Waves in Two and Three Dimensions). In two or three dimensions, waves can propagate in various directions. The general form of a wave in two dimensions is:

$$z(x, y, t) = A \cos(k_x x + k_y y - \omega t + \phi_0)$$

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where k_x and k_y are the wave numbers in the x and y directions, respectively, and ϕ_0 is the initial phase. We also define v and k as:

$$v = \frac{\omega}{k}, \quad k = \sqrt{k_x^2 + k_y^2}$$

Definiton 1.2.2.5 (Spherical Waves). A spherical wave is a wave that propagates outward in all directions from a point source. The general form of a spherical wave is:

$$y(r, t) = \frac{A}{r} \cos(kr - \omega t + \phi_0)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance from the source, A is the amplitude at a reference distance, k is the wave number, ω is the angular frequency, and ϕ_0 is the initial phase. The amplitude decreases with distance as $1/r$ due to the spreading of the wavefront.

Definiton 1.2.2.6 (Interference). Consider two coherent in phase sources S_1 and S_2 separated by a distance d . The path difference Δr between the two waves at a point P is given by:

$$\Delta r = r_2 - r_1$$

where r_1 and r_2 are the distances from the sources S_1 and S_2 to the point P , respectively.

We have the following two cases for constructive and destructive interference:

- **Constructive Interference:** Occurs when the path difference is an integer multiple of the wavelength:

$$\Delta r = m\lambda, \quad m = 0, 1, 2, \dots$$

- **Destructive Interference:** Occurs when the path difference is an odd multiple of half the wavelength:

$$\Delta r = \left(m + \frac{1}{2}\right) \lambda, \quad m = 0, 1, 2, \dots$$

Nodes and Antinodes In a standing wave, nodes are points where the medium does not move (zero displacement), while antinodes are points where the medium experiences maximum displacement. The distance between two consecutive nodes or antinodes is half the wavelength ($\frac{\lambda}{2}$). Nodes occur at positions where $kx = n\pi$ (for integer n), and antinodes occur at positions where $kx = (n + \frac{1}{2})\pi$.

Definiton 1.2.2.7 (Nodes). Nodes are points along a standing wave where the wave has zero amplitude. At these points, destructive interference occurs, and the medium remains at rest. Nodes occur at:

$$x_n = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \dots$$

Definiton 1.2.2.8 (Antinodes). Antinodes are points along a standing wave where the wave has maximum amplitude. At these points, constructive interference occurs, and the medium oscillates with the greatest displacement. Antinodes occur at:

$$x_a = \left(n + \frac{1}{2}\right) \frac{\lambda}{2}, \quad n = 0, 1, 2, \dots$$

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Antinodal Curves In two-dimensional standing waves, antinodal curves are the loci of points where the amplitude of the wave is maximum. These curves are formed by the constructive interference of waves from multiple sources. The shape and spacing of antinodal curves depend on the geometry of the wave sources and the wavelength of the waves. An antinodal curve could be visualized as following:

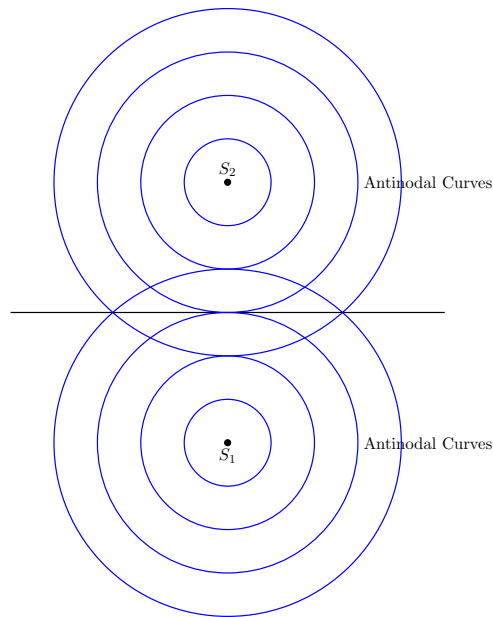


Figure 1.2: Antinodal Curves

Definiton 1.2.2.9 (Standing Waves). Standing waves are formed by the superposition of two waves of the same frequency and amplitude traveling in opposite directions. The solution of the standing waves must satisfy a quantization condition, which leads to discrete frequencies and wavelengths. This discretized solution is given by:

$$y(x, t) = A_n \sin(kx) \cos(\omega t) \quad (1.2.8)$$

where $k = \frac{n\pi}{L}$ for $n = 1, 2, 3, \dots$, and L is the length of the medium (e.g., a string fixed at both ends). The corresponding frequencies are given by:

$$f_n = \frac{nv}{2L}$$

In turn, the wavelengths are given by:

$$\lambda_n = \frac{2L}{n}$$

where v is the wave speed. The distance between two consecutive nodes or antinodes is given by:

$$d = \frac{\lambda}{2}$$

Systems with no Fixed Ends For systems with no fixed ends, the general solution is given by:

$$y(x, t) = A_n \cos(kx) \cos(\omega_n t)$$

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where $k = \frac{n\pi}{L}$ for $n = 0, 1, 2, \dots$ the position of the nodes are:

$$x_n = n\frac{\lambda}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

and the position of the antinodes are:

$$x_a = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

Systems with no free Ends For systems with no free ends, the general solution is given by:

$$y(x, t) = A_n \sin(kx) \cos(\omega_n t)$$

The position of the nodes are:

$$x_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, 1, 2, \dots$$

and the position of the antinodes are:

$$x_a = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \dots$$

System with Mixed Ends For systems with mixed ends (one fixed and one free), the general solution has two cases:

$$y(x, t) = \begin{cases} A_n \sin(kx) \cos(\omega_n t) & \text{if the fixed end is at } x = 0 \\ A_n \cos(kx) \cos(\omega_n t) & \text{if the fixed end is at } x = L \end{cases}$$

The position of the nodes are:

$$x_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, 1, 2, \dots$$

and the position of the antinodes are:

$$x_a = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \dots$$

Definiton 1.2.2.10 (Energy of Standing Waves). The total energy stored in a standing wave on a string of length L is given by:

$$E = \frac{1}{4}\mu A^2 \omega^2 L$$

where μ is the linear mass density of the string, A is the amplitude of the wave, and ω is the angular frequency. The energy is equally divided between kinetic and potential energy, with each contributing half of the total energy.

Definiton 1.2.2.11 (Average Power of Standing Waves). The average power transmitted by a standing wave on a string is given by:

$$\bar{P} = \frac{1}{2}\mu A^2 \omega^2 v$$

where μ is the linear mass density of the string, A is the amplitude of the wave, ω is the angular frequency, and v is the wave speed. This power represents the rate at which energy is transferred along the string.

1.2.3 Fourier analysis

Definiton 1.2.3.1 (Standing Waves as Normal Modes of a Vibrating String). The superposition principle states that if $y_1(x, t)$ and $y_2(x, t)$ are two solutions to the wave equation, then their sum $y(x, t) = y_1(x, t) + y_2(x, t)$ is also a solution. For a string fixed at both ends, the n^{th} normal mode of vibration can be expressed as:

$$y_n(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t) \quad (1.2.9)$$

The motion of the string will be a superposition of normal modes:

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t) \quad (1.2.10)$$

where A_n are the amplitudes of the respective modes, and $\omega_n = \frac{n\pi v}{L}$ are the angular frequencies of the modes.

Definiton 1.2.3.2 (The amplitude of normal modes – Fourier analysis). If we look at the initial shape of the string:

$$y(x, 0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$$

Any initial shape of the string can be expressed as a Fourier sine series. Also, for fixed ends ($f(0) = f(L) = 0$), we can write the shape as the following:

$$f(x) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$$

To find the coefficients A_n , we use:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (1.2.11)$$

1.2.4 Dispersion of Waves

Definiton 1.2.4.1 (Wave Number). The wave number k is defined as:

$$k = \frac{2\pi}{\lambda} \quad (1.2.12)$$

where λ is the wavelength of the wave. The wave number represents the spatial frequency of the wave, indicating how many wavelengths fit into a unit distance.

Definiton 1.2.4.2 (Dispersion Relation). The dispersion relation describes how the angular frequency ω of a wave depends on its wave number k . It is given by:

$$\omega = \omega(k) \quad (1.2.13)$$

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The specific form of the dispersion relation depends on the type of wave and the properties of the medium through which it propagates.

Non-dispersive waves: If ω is directly proportional to k (i.e., $\omega = vk$), the wave is non-dispersive, meaning all frequency components travel at the same phase velocity v .

Definiton 1.2.4.3 (Phase and Group Velocity). The phase velocity v_p is the rate at which the phase of the wave propagates in space. It is given by:

$$v_p = \frac{\omega}{k} = \frac{c}{n} \quad (1.2.14)$$

where c is the speed of light in a vacuum and n is the refractive index of the medium.

The group velocity v_g is the rate at which the envelope of a wave packet (or group of waves) propagates. It is given by:

$$v_g = \frac{d\omega}{dk} = v - k \frac{dv}{dk} \frac{\lambda}{k} = v - \lambda \frac{dv}{d\lambda} \quad (1.2.15)$$

where v is the average of the phase velocities of the individual waves in the packet. λ is average wavelength of the wave packet.

We have the following three cases:

- If $v_p = v_g$, the wave is non-dispersive.
- If $v_p > v_g$, the wave is said to be normally dispersive.
- If $v_p < v_g$, the wave is said to be anomalously dispersive.

Definiton 1.2.4.4 (Refractive Index). The refractive index n of a medium is defined as the ratio of the speed of light in a vacuum c to the phase velocity v_p of light in the medium:

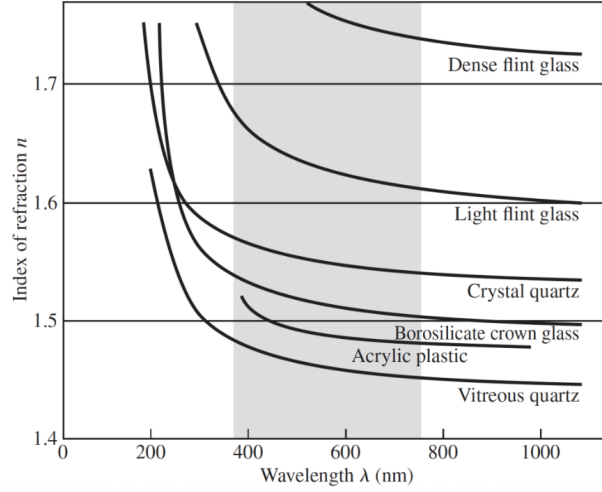
$$n = \frac{c}{v_p} \quad (1.2.16)$$

The refractive index indicates how much the light slows down when it enters the medium compared to its speed in a vacuum. A higher refractive index means that light travels more slowly in that medium.

Example 1.2.4.5. The yellow light from sodium has two components with wavelengths of 589.00 nm and 589.59 nm. The refractive index n of a particular glass at these wavelengths has values of 1.6351 and 1.6350, respectively.

1. What are the phase velocities of the light at these wavelengths?
2. What is the velocity of the narrow pulse of sodium transmitted through the glass?

Also, we are given the following data:

Figure 1.3: $n - \lambda$ data

1. The phase velocity v_p is given by:

$$v_p = \frac{c}{n}$$

For the first wavelength (589.00 nm):

$$v_{p1} = \frac{3 \times 10^8 \text{ m/s}}{1.6351} \approx 1.834 \times 10^8 \text{ m/s}$$

For the second wavelength (589.59 nm):

$$v_{p2} = \frac{3 \times 10^8 \text{ m/s}}{1.6350} \approx 1.834 \times 10^8 \text{ m/s}$$

2. To find the group velocity v_g , we first need to calculate the average phase velocity v and average wavelength λ :

$$v = \frac{v_{p1} + v_{p2}}{2} \approx 1.834 \times 10^8 \text{ m/s}$$

$$\lambda = \frac{589.00 \text{ nm} + 589.59 \text{ nm}}{2} = 589.295 \text{ nm} = 5.89295 \times 10^{-7} \text{ m}$$

We are given the $n - \lambda$ data, so we have $\frac{dn}{d\lambda} = -0.0001/0.59 \text{ nm} = -1.6949 \times 10^{-7} \text{ nm}^{-1} = -1.6949 \times 10^{-16} \text{ m}^{-1}$. And by chain rule, we have:

$$\frac{dv}{d\lambda} = \frac{dv}{dn} \frac{dn}{d\lambda} = -\frac{c}{n^2} \frac{dn}{d\lambda} = -\frac{v}{n} \frac{dn}{d\lambda}$$

Thus, we can calculate the group velocity v_g :

$$v_g = v - \lambda \frac{dv}{d\lambda} = v - \lambda \frac{v}{n} \frac{dn}{d\lambda} = 0.5742c \approx 1.722 \times 10^8 \text{ m/s}$$

1.2.5 Wave Pockets

Definiton 1.2.5.1 (Wave Packet). A wave packet is a localized disturbance that results from the superposition of multiple waves with different wavelengths and frequencies. It can be represented mathematically as:

$$y(x, t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega t)} dk \quad (1.2.17)$$

where $A(k)$ is the amplitude distribution function in wave number space, k is the wave number, and ω is the angular frequency. The wave packet travels with the group velocity $v_g = \frac{d\omega}{dk}$.

Definiton 1.2.5.2 (Width of the Wave Packet). The spatial width Δx of a wave packet is inversely related to the spread in wave numbers Δk of the constituent waves. This relationship is given by:

$$\Delta x \Delta k \approx 1 \quad (1.2.18)$$

where Δx is the uncertainty in position and Δk is the uncertainty in wave number. A narrower wave packet (smaller Δx) corresponds to a broader range of wave numbers (larger Δk), and vice versa.

Definiton 1.2.5.3 (Uncertainty Principle for Waves). The uncertainty principle for waves states that there is a fundamental limit to the precision with which certain pairs of physical properties, such as position and momentum, can be simultaneously known. For wave packets, this principle can be expressed as:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad (1.2.19)$$

where Δx is the uncertainty in position, Δp is the uncertainty in momentum, and \hbar is the reduced Planck constant. This principle highlights the trade-off between the precision of position and momentum measurements for wave-like phenomena.