MAT 185 Lecture Notes

Hei Shing Cheung
Linear Algebra, Winter 2024 MAT 185

The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

1 Vector Space

1.1 Foundamental Properties

Definiton 1.1.1 (Vector Space in $\mathbb{R}$)**.** This course concerns with real vector spaces. A vectors space, V, over $\mathbb{R}$ is a collection of **object** $\mathbf{v} \in V$ s.t. the follow axioms are followed

1. **Addition Axioms**
   (a) **Closure Under Addition:**
   $\mathbf{x}, \mathbf{y} \in V \implies \mathbf{x} + \mathbf{y} \in V$
   (b) **Associativity of Addition:**
   $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \implies (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
   (c) **Existence of Additive Identity:**
   $\exists\, \mathbf{0} \in V$ such that $\mathbf{x} \in V \implies \mathbf{x} + \mathbf{0} = \mathbf{x}$
   (d) **Existence of Additive Inverse:**
   $\forall\, \mathbf{x} \in V$, $\exists$**-x** $\in V$ such that $\mathbf{x} + \textbf{-x} = \mathbf{0}$
2. **Scalar Multiplication Axioms**
   (a) **Closure Under Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha \in \mathbb{R}$, $\alpha\mathbf{x} \in V$
   (b) **Associativity of Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
   (c) **Distributive Property of Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
   (d) **Existence of Multiplicative Identity:**
   $\forall\, \mathbf{x} \in V$, $\mathbf{1x} = \mathbf{x}$

Note It could be shown that the axiom imply the commutativity of in addition, namely $\forall\mathbf{x} \implies \mathbf{y} \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

Example $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

Theorem 1.1.2 (Cancelation, Part 1)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then:

$$\begin{aligned}\mathbf{x} + \mathbf{z} &= \mathbf{y} + \mathbf{z} \\ \mathbf{x} &= \mathbf{y}\end{aligned}$$

Proof.

$$\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$$

From additive inverse we know that **-z** exists

$$(\mathbf{x} + \mathbf{z}) + \textbf{-z} = (\mathbf{y} + \mathbf{z}) + \textbf{-z}$$

By order of addition we have:

$$\begin{aligned}\mathbf{x} + (\mathbf{z} + \textbf{-z}) &= \mathbf{y} + (\mathbf{z} + \textbf{-z}) \\ \mathbf{x} + \mathbf{0} &= \mathbf{y} + \mathbf{0}\end{aligned}$$

By additive identity we have:

$$\mathbf{x} = \mathbf{y}$$

□

Theorem 1.1.3 (Cancelation, Part 2)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then:

$$\begin{aligned}\mathbf{z} + \mathbf{x} &= \mathbf{z} + \mathbf{y} \\ \mathbf{x} &= \mathbf{y}\end{aligned}$$

To proof that, it would require the following propostion:

Lemma 1.1.4. Let V be a vector space and $\mathbf{z} \in V$, then $\textbf{-z} + \mathbf{z} = 0$

Proof.

We know:

$$\begin{aligned}\textbf{-z} + \mathbf{z} &= (\textbf{-z} + \mathbf{z}) + 0 \\ &= (\textbf{-z} + \mathbf{z}) + (\textbf{-z} + \textbf{-(-z)})\end{aligned}$$

By the order of additon,

$$= \textbf{-z} + (\mathbf{z} + (\textbf{-z} + \textbf{-(-z)}))$$

Again, by the order of additon,

$$\begin{aligned}&= \textbf{-z} + ((\mathbf{z} + \textbf{-z}) + \textbf{-(-z)}) \\ &= \textbf{-z} + \textbf{-(-z)} \\ &= 0\end{aligned}$$

□

Now, to prove the part 2 of the Cancelation Theorem:

Proof.

$$\mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y}$$

From additive inverse we know that **-z** exists

$$\begin{aligned} \textbf{-z} + (\mathbf{z} + \mathbf{x}) &= \textbf{-z} + (\mathbf{z} + \mathbf{y}) \\ (\textbf{-z} + \mathbf{z}) + \mathbf{x} &= (\textbf{-z} + \mathbf{z}) + \mathbf{y} \end{aligned}$$

From above, we have $\textbf{-z} + \mathbf{z} = 0$

$$\begin{aligned} 0 + \mathbf{x} &= 0 + \mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

□

Lemma 1.1.5 (Inverse of an inverse)**.** Let V be a vector space and $\mathbf{x} \in V$, then:

$$\textbf{-(-x)} = \mathbf{x}$$

Proof. Assume 0, $0^\star$ are the additive identity of V and **-x**, $\textbf{-x}^\star$ are the additive inverse of **x**. We have:

$$u + 0 = u + 0^\star$$

By Cancelation Theorem, we have $0 = 0^\star$. Thus, the additive identity is unique. Now, we have:

$$\begin{aligned} \mathbf{x} + \textbf{-x} &= 0 \\ \mathbf{x} + \textbf{-x}^\star &= 0 \end{aligned}$$

Thus, by the uniqueness of the additive inverse, we have:

$$\mathbf{x} + \textbf{-x} = \mathbf{x} + \textbf{-x}^\star$$

By the Cancelation Theorem, we have:

$$\textbf{-x} = \textbf{-x}^\star$$

Now, we have that the additive inverse is unique. Thus, **-(-x)** must be unique and nessarily **x**. □

Additional Results from Above

1. The additive identity is unique
2. The additive inverse is unique

Definiton 1.1.6 (Subtraction)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y} \in V$, then:

$$\mathbf{x} - \mathbf{y} \stackrel{\text{def}}{=} \mathbf{x} + \textbf{-y}$$

Theorem 1.1.7 (Addition is Commutative)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y} \in V$, then:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

Proof.

$$\mathbf{x} + \mathbf{y} =$$

□

1.2 Vector Subspace

Definiton 1.2.1 (Vector Subspace)**.** Let V be a vector space and $W \subseteq V$, then W is a vector subspace of V if W is a vector space.

Theorem 1.2.2 (Subspace Test, I)**.** Let V be a vector space over $\mathbb{R}$ and $W \subseteq V$ with the usual addition and scalar multiplication, then W is a subspace of V iff for all $\mathbf{x}, \mathbf{y} \in W$ and $\alpha \in \mathbb{R}$:

1. **Closure Under Addition:**
   $\mathbf{x} + \mathbf{y} \in W$
2. **Closure Under Scalar Multiplication:**
   $\alpha\mathbf{x} \in W$
3. Additive Identity:
   $\mathbf{0} \in W$

Proof. ($\Rightarrow$) Assume W is a subspace of V, then W is a vector space. Thus, the axioms of vector space are satisfied.
($\Leftarrow$) Assume the three conditions are satisfied, then W is a vector space. Thus, W is a subspace of V. □

Definiton 1.2.3 (Null Space)**.** Let V be a vector space and $A \in {}^m\mathbb{R}^n$[1], then:

$$\text{null}(A) = \{\mathbf{x} \in {}^n\mathbb{R} \mid A\mathbf{x} = \mathbf{0}\} \tag{1}$$

is the null space of A, otherwise known as the **kernel** of A or the solution space of $A\mathbf{x} = \mathbf{0}$

We can use the Subspace Test I to show that the null space of a matrix is a subspace of ${}^n\mathbb{R}$.

1. **Existence of Additive Identity:**
   The zero vector is in the null space of A as the trivial solution to the equation $A\mathbf{x} = \mathbf{0}$
2. **Closure Under Addition:**
   Let $\mathbf{x}, \mathbf{y} \in \text{null}(A)$, then $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus, $\mathbf{x} + \mathbf{y} \in \text{null}(A)$. This holds since ${}^n\mathbb{R}$ is a vector space.

[1] Same as $\mathbb{R}^{m\times n}$, The set of all $m \times n$ matrices with real entries

3. **Closure Under Scalar Multiplication:**
   Let $\mathbf{x} \in \text{null}(A)$ and $\alpha \in \mathbb{R}$, then $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$. Thus, $\alpha\mathbf{x} \in \text{null}(A)$. This holds since ${}^n\mathbb{R}^m$ is a vector space under usual addition and scalar multiplication.

Theorem 1.2.4 (Subspace Test, II)**.** Let V be a vector space over $\mathbb{R}$ and $W \subseteq V$ with the usual addition and scalar multiplication, then W is a subspace of V iff for all $\mathbf{x}, \mathbf{y} \in W$ and $\alpha, \beta \in \mathbb{R}$:

1. $\alpha\mathbf{x} + \beta\mathbf{y} \in W$

Proof. ($\Rightarrow$) Assume W is a subspace of V, then W is a vector space. Thus, the axioms of vector space are satisfied.
($\Leftarrow$) Assume the condition is satisfied, then W is a vector space. Thus, W is a subspace of V. $\square$

Definiton 1.2.5 (Intersection of Sets)**.** Let A and B be sets, then:

$$A \cap B = \{x \,|\, x \in A \,\text{and}\, x \in B\} \tag{2}$$

Definiton 1.2.6 (Union of Sets)**.** Let A and B be sets, then:

$$A \cup B = \{x \,|\, x \in A \,\text{or}\, x \in B\} \tag{3}$$

1.3 Linear combinations

Definiton 1.3.1 (Linear Combination)**.** Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, then:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$

is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Definiton 1.3.2 (Linear independence)**.** Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. If and only if the equation:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

has **only** the trivial solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Then, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are said to be linearly independent; otherwise, they are linearly dependent.

Definiton 1.3.3 (Span)**.** Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, then:

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \,|\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\} \tag{4}$$

is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Definiton 1.3.4 (Basis)**.** Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = V$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of V.