ESC 195 Lecture Notes

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1 More on Integrals

1.1 Riemann Sum - Non-Uniform Petition

Example 1.1.1. Given the following definite integral:

$$\int_0^2 \sqrt{x}\,dx$$

, we cannot evaluate its Riemann sum with uniforms partition, since the series of root cannot be easily evaluated.

The definite integral of $\sqrt{x}$ from 0 to 2 using a Riemann sum with a non-uniform partition is given by:

$$\int_0^2 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^n \sqrt{x_i}\Delta x_i$$

where:

- $x_0 = 0$, $x_n = 2$,
- $x_i = i^2 \cdot \frac{2}{n^2}$ for $i = 0, 1, 2, \ldots, n$,
- $\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2} \cdot (2i - 1)$.

The Riemann sum becomes:

$$S_n = \sum_{i=1}^n \sqrt{i^2 \cdot \frac{2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Simplifying further:

$$S_n = \sum_{i=1}^n \sqrt{\frac{2i^2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Taking the limit as $n \to \infty$, the sum converges to the exact value of the integral using the series of squares.

$$\int_0^2 \sqrt{x}\,dx = \frac{4\sqrt{2}}{3}.$$

Condition $n \to \infty$ Ensures $\Delta x_i \to 0$

As $n \to \infty$, the partition points x_i become increasingly dense. This ensures that the partition becomes infinitely fine.

1.2 Integration By Parts

Using the product rule:
$$\frac{d}{dx}\big[u(x)v(x)\big] = u'(x)v(x) + u(x)v'(x),$$

integrating both sides with respect to x gives:
$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Rearranging this:
$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Integration by parts formula
$$\int u\,dv = uv - \int v\,du. \tag{1}$$

Example 1.2.1. We want to solve the integral
$$\int xe^{2x}\,dx$$

using integration by parts.

Let:
$$u = x, \quad dv = e^{2x}\,dx.$$

Then, we compute the derivatives and integrals:
$$du = dx, \quad v = \frac{e^{2x}}{2}.$$

Now, apply the integration by parts formula:
$$\int u\,dv = uv - \int v\,du.$$

Substituting in the values:
$$\int xe^{2x}\,dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx.$$

Next, compute the remaining integral:

$$\int \frac{e^{2x}}{2}\, dx = \frac{e^{2x}}{4}.$$

Thus, the result is:

$$\int xe^{2x}\, dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Example 1.2.2. We want to solve

$$\int x^2 \sin(2x)\, dx$$

using double integration by parts.

First, let:

$$u = x^2, \quad dv = \sin(2x)\, dx.$$

Then:

$$du = 2x\, dx, \quad v = -\frac{1}{2}\cos(2x).$$

Using the IBP formula:

$$\int u\, dv = uv - \int v\, du,$$

we get:

$$\int x^2 \sin(2x)\, dx = -\frac{x^2}{2}\cos(2x) + \int x \cos(2x)\, dx.$$

Now, apply IBP again to $\int x \cos(2x)\, dx$, let:

$$u = x, \quad dv = \cos(2x)\, dx.$$

Then:

$$du = dx, \quad v = \frac{1}{2}\sin(2x).$$

Using the IBP formula again:

$$\int x \cos(2x)\, dx = \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\, dx,$$

and solving the remaining integral:

$$\int \frac{1}{2}\sin(2x)\, dx = -\frac{1}{4}\cos(2x).$$

Thus, the final result is:

$$\int x^2 \sin(2x)\, dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.$$

1.3   Trigonometric Integrals

Case I   This is the case I trigonometric integrals, the strategy is using the pythagorean identity.

We want to solve the class of integrals:

$$\int \sin^n(x) \cos^m(x)\, dx \tag{2}$$

, where m or n is odd.

Example 1.3.1. We want to solve the integral:

$$I = \int \sin^3(x) \cos^2(x) dx$$

We use the identity: $\sin^2(x) = 1 - \cos^2(x)$. Thus, the integral becomes:

$$I = \int \sin(x)(1 - \cos^2(x)) \cos^2(x)\, dx$$
$$= \int (\cos^2(x) \sin(x) - \cos^4(x) \sin(x))\, dx$$

which is now easily solvable with substution $u = \cos(x)$.

Case II   The follwoing is generally solvable via case I and case III below. In general, we solve the integral by reducing the power of the trigonometric functions to arrive at a solvable integral Case I or III.

We want to solve the class of integrals:

$$\int \sin^n(x) \cos^m(x)\, dx \tag{3}$$

, where m and n is even.

Example 1.3.2. We want to solve the integral:

$$I = \int \sin^2(x) \cos^4(x)\, dx$$

We can apply the double angle formulas:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x) \tag{4}$$
$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \tag{5}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \tag{6}$$

Thus, the integral becomes:

$$I = \frac{1}{8} \int \sin^2(2x)\, dx + \frac{1}{8} \int \cos(2x) \sin^2(2x)\, dx$$

Case III This is the case III trigonometric integrals, the strategy is using the reduction formula via IBP.

Reduction Formula We can solve integrals by reducing the power of the trigonometric functions. These are done using IBP and trigonometric identities.

We want to solve the classes of integrals:

$$\int \sin^n(x)\, dx, \quad \int \cos^n(x)\, dx \tag{7}$$

, where n is a positive integer. For demonstartion, We can obtain the reduction formula of $\sin^n(x)$ via IBP.

$$
\begin{aligned}
I_n &= \int \sin^n(x)\, dx \\
&= \int \sin^{n-1}(x) \sin(x)\, dx \\
&= \frac{-\cos(x) \sin^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \cos^2(x) \sin^{n-2}(x)\, dx \\
&= \frac{-\cos(x) \sin^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2}
\end{aligned}
\tag{8}
$$

Reduction for $\cos^n(x)$ We can obtain the reduction formula of $\cos^n(x)$ via IBP.

$$
\begin{aligned}
I_n &= \int \cos^n(x)\, dx \\
&= \int \cos^{n-1}(x) \cos(x)\, dx \\
&= \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \sin^2(x) \cos^{n-2}(x)\, dx \\
&= \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2}
\end{aligned}
\tag{9}
$$

Case IV The following is generally solvable via simple trigonometric integrals. In general, we solve the integral by applying the angle sum formulas.

We want to solve the classes of integrals:

$$\int \sin(mx) \cos(nx)\, dx, \quad \int \sin(mx) \sin(nx)\, dx, \quad \int \cos(mx) \cos(nx)\, dx \tag{10}$$

We could apply the angle sum formulas:

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin((m+n)x) + \sin((m-n)x)\right] \tag{11}$$

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right] \tag{12}$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right] \tag{13}$$

Case V   The following is generally solvable via the following trigonometric identities listed below, which convert it into a reduction formula.

$$\tan^2(x) = \sec^2(x) - 1 \tag{14}$$

$$\cot^2(x) = \csc^2(x) - 1 \tag{15}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \tag{16}$$

We want to solve the classes of integral:

$$\int \tan^n(x)\,dx, \quad \int \cot^n(x)\,dx \tag{17}$$

We know that $tan^2(x) = \sec^2(x) - 1$. Thus, we can solve the integral by reducing the power of the tangent function.

Reduction for $\tan^n(x)$ We can obtain the reduction formula of $\tan^n(x)$ via the trigonometric identities.

$$\begin{aligned}
I_n &= \int \tan^n(x)\,dx \\
&= \int \tan^{n-1}(x)\tan(x)\,dx \\
&= \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx \\
&= \frac{\tan^{n-1}(x)}{n-1} - I_{n-2}
\end{aligned} \tag{18}$$

Reduction for $\cot^n(x)$ We can obtain the reduction formula of $\cot^n(x)$ via the trigonometric identities.

$$\begin{aligned}
I_n &= \int \cot^n(x)\,dx \\
&= \int \cot^{n-1}(x)\cot(x)\,dx \\
&= \frac{\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x)\,dx \\
&= \frac{\cot^{n-1}(x)}{n-1} - I_{n-2}
\end{aligned} \tag{19}$$

Case VI   The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x) \sec^n(x)\, dx, \quad \int \cot^m(x) \csc^n(x)\, dx \tag{20}$$

We can solve the integral by reducing the power of the converted trigonometric functions using (14) and (15).

Case VII   The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x) \sec^n(x)\, dx, \quad \int \cot^m(x) \csc^n(x)\, dx \tag{21}$$

We can solve the integral by converting between the tangent and secant functions using the trigonometric identities.

Case VIII   This is the case VIII integrals, the strategy is using the trigonometric substitution.

Trigonometric Substitution   We can solve integrals by substituting the trigonometric functions with other trigonometric functions.

Example 1.3.3. We want to solve the integral:

$$\int \frac{dx}{\sqrt{1 - x^2}}$$

We can substitute $x = \sin(\theta)$, then $dx = \cos(\theta)\, d\theta$. The integral becomes:

$$\int \frac{\cos(\theta)\, d\theta}{\sqrt{1 - \sin^2(\theta)}} = \int \frac{\cos(\theta)\, d\theta}{\cos(\theta)} = \int d\theta = \theta + C.$$

In general, we can use the following substitutions:

- $x = a\sin(\theta)$ for $\sqrt{a^2 - x^2}$,
- $x = a\tan(\theta)$ for $\sqrt{a^2 + x^2}$,
- $x = a\sec(\theta)$ for $\sqrt{x^2 - a^2}$.

Weierstrass Substitution   We can solve integrals by using the Weierstrass substitution:

$$\tan\left(\frac{x}{2}\right) = t \Leftrightarrow \cos(x) = \frac{1 - t^2}{1 + t^2} \quad \sin(x) = \frac{2t}{1 + t^2} \quad dx = \frac{2dt}{1 + t^2} \tag{22}$$

This substitution is useful for solving integrals with trigonometric functions, by converting them into rational functions.

Summary   We can solve trigonometric integrals by using the following strategies:

Case	Strategy and General Form
I	Use $\sin^2(x) + \cos^2(x) = 1$; simplify using substitution. General Form: $\int \sin^n(x)\cos^m(x)dx$, where m or n is odd.
II	Convert to Case I and Case III using double angle formulas. General Form: $\int \sin^n(x)\cos^m(x)dx$, where both m and n are even.
III	Apply reduction formulas derived via integration by parts. General Form: $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$.
IV	Use angle sum formulas to simplify. General Form: $\int \sin(mx)\cos(nx)dx$.
V	Reduce tangent/cotangent powers using $\tan^2(x) = \sec^2(x) - 1$ and substitution. General Form: $\int \tan^n(x)dx$ or $\int \cot^n(x)dx$.
VI	Convert to Case V via Pythagorean identities. General Form: $\int \tan^m(x)\sec^n(x)dx$ or $\int \cot^m(x)\csc^n(x)dx$.
VII	Convert between tangent and secant functions for simplification. General Form: $\int \tan^m(x)\sec^n(x)dx$.
VIII	Use trigonometric substitution: $x = a\sin(\theta), a\tan(\theta), a\sec(\theta)$. General Form: $\int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{\sqrt{a^2+x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$.

Table 1: Strategies and General Forms for Case I-VIII Integrals

1.4   Partial Fractions

Partial Fraction Decomposition   We can decompose a rational function into partial fractions to simplify integration.

We want to solve the integral:

$$\int \frac{P(x)}{Q(x)} \, dx \tag{23}$$

, where $P(x)$ and $Q(x)$ are polynomials.

Example 1.4.1. We want to solve the integral:

$$\int \frac{2x^2 + 3x + 1}{x^3 + 2x^2 + x} \, dx$$

We can decompose the rational function into partial fractions:

$$\frac{2x^2 + 3x + 1}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

We can solve for the constants A, B, and C by equating the coefficients of the partial fractions to the original function. We have:

$$2x^2 + 3x + 1 = A(x+1)^2 + Bx(x+1) + Cx$$

Thus, we can solve it by setting $x = 0$, $x = -1$

Summary   We can solve integrals using partial fraction decomposition by following these steps:

Step	Description
1	Factorize the denominator of the rational function.
2	Write the partial fraction decomposition.
3	Solve for the constants by equating the coefficients.
4	Integrate the partial fractions.

Table 2: Steps for Partial Fraction Decomposition

1.5   Improper Integrals

Improper Integral   An improper integral is an integral with an infinite limit or a discontinuity in the interval of integration.

We want to solve the integral:

$$\int_a^b f(x)\,dx \tag{24}$$

, where a or b is infinite or $f(x)$ is discontinuous.

Example 1.5.1. We want to solve the integral:

$$\int_0^\infty e^{-x}\,dx$$

Evaluating the integral:

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty} -e^{-x}\Big|_0^b$$
$$= \lim_{b\to\infty} -e^{-b} + 1$$
$$= 1$$

2 Hyperbolic Trigonometric Functions

Definiton 2.0.1 (Hyperbolic Sine). The hyperbolic sine function is defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{25}$$

Definiton 2.0.2 (Hyperbolic Cosine). The hyperbolic cosine function is defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{26}$$

Properties These combinations of exponential functions have properties similar to the trigonometric functions.

Derivatives The derivatives of the hyperbolic functions are:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \tag{27}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x). \tag{28}$$

Identities The hyperbolic functions satisfy the following identities:

$$\cosh^2(x) + \sinh^2(x) = 1, \tag{29}$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \tag{30}$$

$$\sinh(2x) = 2\sinh(x)\cosh(x). \tag{31}$$

Hyperbola The hyperbolic functions are related to the hyperbola $x^2 - y^2 = 1$. Simiar to the circle, the hyperbola can be parametrized by the hyperbolic functions (e.g. $x = \cosh(t)$, $y = \sinh(t)$).

Area The area of a sector of the hyperbola is given by:

$$A = t/2, \tag{32}$$

where t is the angle of the sector along the parametrization $\{(x, y) \mid x = \cosh(t), y = \sinh(t)\}$.

Catenary The hyperbolic cosine function describes the shape of a hanging chain or cable. The catenary is the curve formed by a chain hanging from two points. It is given by the equation:

$$a\cosh\left(\frac{x}{a}\right), \tag{33}$$

Definiton 2.0.3 (Hyperbolic Tangent). The hyperbolic tangent function is defined as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{34}$$
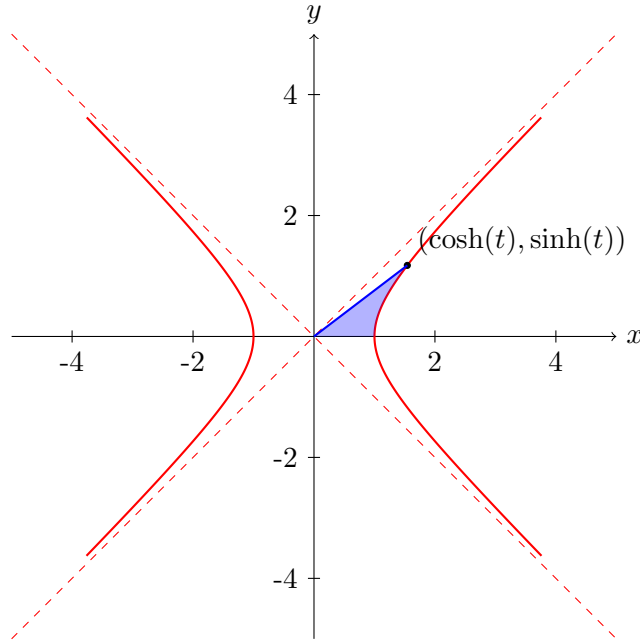
Figure 1: Sector of the hyperbola

Derivative The derivative of the hyperbolic tangent function resembles the derivative of the regular tangent function:

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x). \tag{35}$$

Identities The hyperbolic tangent function satisfies the following identities:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \tag{36}$$

$$\operatorname{sech}^2(x) = 1 - \tanh^2(x). \tag{37}$$

Secant, Cosecant, Cotangent The hyperbolic secant, cosecant, and cotangent functions are defined similarly to the regular secant, cosecant, and cotangent functions. They are the **reciprocal** of the hyperbolic cosine, sine, and tangent functions, respectively.

Inverse Hyperbolic Functions   The inverse hyperbolic functions are defined as the inverse of the hyperbolic functions. They are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$, etc.

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \tag{38}$$

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \tag{39}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{40}$$

$$\operatorname{arcsch} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) \tag{41}$$

$$\operatorname{arsech} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) \tag{42}$$

$$\operatorname{arcoth} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \tag{43}$$

3   Further Applications of Integrals

3.1   Arc Length

Arc Length   Given the curve $C : y = f(x)$, the arc length of the curve between two points a and b, where $y'(x)$ is continuous, we would like the compute the arc length of the curve:

$$L = \int_C ds$$

Formula   We can derive the formula for the arc length of a curve $y = f(x)$ between two points a and b as:

At x_i and x_{i+1}, the length of the segment is:

$$\Delta s \approx \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$
$$= \sqrt{\Delta_i x^2 + \Delta_i y^2}$$

We also have, by MVT:

$$\frac{\Delta y}{\Delta x} = y'(c)$$
$$\Delta y = y'(c)\Delta x$$
$$\Delta s \approx \sqrt{\Delta x^2 + (y'(c)\Delta x)^2}$$
$$= \sqrt{1 + y'(c)^2}\,\Delta x$$

Taking a Riemann sum:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + y'(c_i)^2} \Delta x$$

$$= \int_a^b \sqrt{1 + y'(x)^2} \, dx \tag{44}$$

Example 3.1.1. Given $f(x) = x^{\frac{3}{2}}$ between $x = 0$ and $x = 44$, we would like to compute the arc length of the curve:

$$L = \int_0^{44} \sqrt{1 + \left(\frac{3}{2} x^{\frac{1}{2}} \right)^2} \, dx$$

$$= \int_0^{44} \sqrt{1 + \frac{9}{4} x} \, dx$$

$$= \frac{8}{27} \left(1 + \frac{9}{4} x \right)^{\frac{3}{2}} \Big|_0^{44} = 296$$

3.2   Surface Area of Revolution

Surface Area of Revolution   Given the curve $C : y = f(x)$, the surface area of the curve between two points a and b, where $y'(x)$ is continuous, we would like the compute the surface area of the curve:

$$A = \int_C 2\pi y \, ds$$

Formula   Deriving the formula for the surface area of a curve $y = f(x)$ between two points a and b is similar to the arc length formula:

At x_i and x_{i+1}, the length of the segment is:

$$\Delta s \approx \sqrt{\Delta_i x^2 + \Delta_i y^2}$$

The surface area of the segment is, by continuity:

$$A_i = \pi (y_i + y_{i+1}) \Delta s = 2\pi y^{\star} \Delta s \tag{45}$$

As the difference between $y^{\star}$ and y_i diminishes as $\Delta x \to 0$, taking a Riemann sum:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi y_i \sqrt{\Delta_i x^2 + \Delta_i y^2}$$

$$= \int_a^b 2\pi y \sqrt{1 + y'(x)^2} \, dx \tag{46}$$

Example 3.2.1. Given $y = \sqrt{x}$ between $x = 0$ and $x = 1$, we would like to compute the surface area of the curve:

$$A = \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx$$
$$= \pi \int_0^1 \sqrt{4x + 1}\,dx$$

Let $u = 4x + 1$, then $du = 4dx$:

$$= \frac{\pi}{4} \int_1^5 \sqrt{u}\,du$$
$$= \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right)$$

3.3   Applications to Physics and Engineering

3.3.1   Hydrostatic Pressure and Force

Hydrostatic Pressure   The hydrostatic pressure and Force is given by the Archimedes' principle:

$$P = \rho g \cdot d, \tag{47a}$$
$$F = \rho g \cdot d \cdot A \tag{47b}$$

Formula   Given a tank rectangular with width defined by $w = f(x)$, where x is depth, we would like to compute the force exerted on the tank by the water:

$$F = \int_0^b \rho g x f(x) \cdot dx \tag{48}$$

, which is derived by considering dx as the depth a horizontal slice of the tank.

3.3.2   Center of Mass and Moments of Inertia

Properties of the Center of Mass   The center of mass of a region R has the following properties:

1. **Symetry**: For all axis of symmetry, the center of mass lies on the axis.

2. **Additivity**: The center of mass of a region is the weighted average of the centers of mass of its parts. The weights are the areas of the parts.

$$\bar{x} = \sum_i \frac{A_i \bar{x}_i}{A} \tag{49}$$

Formula The centroid of a uniformly-dense region R bounded by $y = f(x)$, $x \in [a, b]$, $y \in [f(a), f(b)]$ is derived as follows:

We first have $x_i^\star$ and A_i as follows:

$$x_i^\star = \frac{1}{2}(x_i + x_{i+1}) \quad A_i = f(x_i)\Delta x$$

For $\bar{x}$, we have:

$$A\bar{x} \approx \sum_i x_i^\star A_i$$

Taking a Riemann sum:

$$A\bar{x} = \lim_{n \to \infty} \sum_i x_i^\star A_i$$

$$\bar{x} = \frac{1}{A} \int_a^b x f(x)\, dx \tag{50}$$

Similarly, for $\bar{y}$:

$$A\bar{y} = \lim_{n \to \infty} \sum_i f(x_i^\star) A_i$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2}(f(x))^2\, dx \tag{51}$$

Theorem 3.3.1 (Pappus's Theorem)**.** The volume of a solid of revolution is given by the product of the area of the region and the distance, R, traveled by the centroid of the region from the axis of rotation.

$$V = 2\pi R A \tag{52}$$

Example 3.3.2 (Eliptical Torus)**.** Given an eliptical torus (tall donut) with major axis a and minor axis b, we would like to compute the volume of the torus:

$$V = 2\pi R A$$
$$= 2\pi R \cdot \pi a b = 2\pi^2 a b R$$

This could be, alternatively, shown via the washer method and the shell method.

3.4 Applications to Economics and Biology

"Biologists are so bad at math" - Prof. Davis as he proceeds to erase the board and skip to the next section.

4 Parametric Equations and Polar Coordinates

4.1 Curves Defined by Parametric Equations

Parametric Equations   For $t \in \mathbb{R}$, we have:

$$x = x(t), \quad y = y(t) \tag{53}$$

Example 4.1.1 (Newton's Laws of Motion)**.** Given $\ddot{x} = 0$, $\ddot{y} = -g$, we have:

$$x(t) = A_1 t + A_2,$$
$$y(t) = -\frac{1}{2}gt^2 + B_1 t + B_2$$

At initial conditions $t = 0$, we have $x(0) = 0$, $y(0) = 0$, $\dot{x}(0) = v_0 \cos(\theta)$, $\dot{y}(0) = v_0 \sin(\theta) - \frac{1}{2}gt^2$.

Example 4.1.2 (Ellipse)**.** Given $x = a \cos(t)$, $y = b \sin(t)$, we have:

$$x^2 = a^2 \cos^2(t),$$
$$y^2 = b^2 \sin^2(t)$$

, which satisfies $x^2/a^2 + y^2/b^2 = 1$.

Intersections of Parametric Curves   Given two parametric curves C_1, C_2, defined by:

$$C_1 : x_1 = x_1(t), \quad y_1 = y_1(t)$$
$$C_2 : x_2 = x_2(t), \quad y_2 = y_2(t)$$

, we can find the intersection points by solving the system of equations:

$$x_1(t) = x_2(t)$$
$$y_1(t) = y_2(t)$$

Example 4.1.3. Given $x_1 = 2t + t$, $y_1 = 5 - 4t$, $x_2 = 3 - 5\cos(\pi t)$, $y_2 = 1 + 5\sin(\pi t)$, we have:

$$2t + 1 = 3 - 5\cos(\pi t)$$
$$5 - 4t = 1 + 5\sin(\pi t)$$

A   Trigonometric Derivatives

Trigonometric Function	Inverse Trigonometric Function
$\dfrac{d}{dx}\sin(x) = \cos(x)$	$\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$
$\dfrac{d}{dx}\cos(x) = -\sin(x)$	$\dfrac{d}{dx}\arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$
$\dfrac{d}{dx}\tan(x) = \sec^2(x)$	$\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$
$\dfrac{d}{dx}\cot(x) = -\csc^2(x)$	$\dfrac{d}{dx}\mathrm{arccot}(x) = \dfrac{-1}{1+x^2}$
$\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\dfrac{d}{dx}\mathrm{arcsec}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$
$\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x)$	$\dfrac{d}{dx}\mathrm{arccsc}(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$
Hyperbolic Function	**Inverse Hyperbolic Function**
$\dfrac{d}{dx}\sinh(x) = \cosh(x)$	$\dfrac{d}{dx}\mathrm{arsinh}(x) = \dfrac{1}{\sqrt{x^2+1}}$
$\dfrac{d}{dx}\cosh(x) = \sinh(x)$	$\dfrac{d}{dx}\mathrm{arcosh}(x) = \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$
$\dfrac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$	$\dfrac{d}{dx}\mathrm{artanh}(x) = \dfrac{1}{1-x^2} \quad (|x| < 1)$
$\dfrac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$	$\dfrac{d}{dx}\mathrm{arcoth}(x) = \dfrac{1}{1-x^2} \quad (|x| > 1)$
$\dfrac{d}{dx}\mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x)$	$\dfrac{d}{dx}\mathrm{arsech}(x) = \dfrac{-1}{x\sqrt{1-x^2}} \quad (0 < x \le 1)$
$\dfrac{d}{dx}\mathrm{csch}(x) = -\mathrm{csch}(x)\coth(x)$	$\dfrac{d}{dx}\mathrm{arcsch}(x) = \dfrac{-1}{|x|\sqrt{x^2+1}}$