AER 210 Lecture Notes

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Vector Calculus & Fluid Mechanics, Fall 2025 AER210

The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

Chapter 1

Vector Calculus

Note the section numbering is based on Stewart's book.

1.14 Partial Derivatives

Continuing from Calc II We consider Taylor series of multivariable functions, and we would need to consider partial derivatives.

1.14.1 Taylor Series for Multivariable Functions

Review. Taylor Series for Single Variable Functions

Let $f(x)$ be a function that is infinitely differentiable at $x = a$. The Taylor series of $f(x)$ about the point $x = a$ is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (1.1)$$

Also, consider the taylor approximation of $f(x + \Delta x)$ about x:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \frac{f'''(x)}{3!}(\Delta x)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n \quad (1.2)$$

Definiton 1.14.1.1 (Taylor Series for Multivariable Functions)**.** Let $f(x,y)$ be a function that is infinitely differentiable at the point (a,b). Consider the increment Δx in the x-direction and Δy in the y-direction centred at (x_0,y_0). We have the following parametric equations:

$$\begin{cases} x = x_0 + \Delta x t \\ y = y_0 + \Delta y t \end{cases} \qquad t \in [0,1]$$

Define a new function $g(t) = f(x_0 + \Delta x t, y_0 + \Delta y t)$, which is a single-variable function in terms of t. We can then apply the Taylor series for single-variable functions to $g(t)$ about $t = 0$:

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2!}t^2 + \frac{g'''(0)}{3!}t^3 + \cdots = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}t^n$$

Using the chain rule, we can compute the taylor series of $f(x,y)$ about the point (x_0,y_0):

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) &= f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right) \\ &\quad + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right) + \cdots \qquad (1.3) \\ &= f(x_0,y_0) + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\frac{\partial^n f}{\partial x^k \partial y^{n-k}}(\Delta x)^k(\Delta y)^{n-k} \qquad (1.4) \end{aligned}$$

Or equivalently, we can write:

$$f(x,y) = f(x_0,y_0) + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\frac{\partial^n f}{\partial x^k \partial y^{n-k}}(x-x_0)^k(y-y_0)^{n-k} \qquad (1.5)$$

We could also use the fact:

$$\frac{1}{k!(n-k)!} = \frac{1}{n!}\binom{n}{k}$$

we rewrite the above equations, such that we can use the pascal's triangle to help us remember the coefficients:

$$f(x,y) = f(x_0,y_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n f}{\partial x^k \partial y^{n-k}}(x-x_0)^k(y-y_0)^{n-k} \qquad (1.6)$$

WLOG, this can be extended to functions of more than two variables.

Example 1.14.1.2 (Thrid (3^{rd}) order Taylor Polynomial)**.** For some function $f(x,y)$, the third order Taylor polynomial about the point (x_0,y_0) is given by:

$$\begin{aligned} f(x,y) = f(x_0,y_0) &+ \left(\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0)\right) \\ &+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(y-y_0)^2\right) \\ &+ \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}(x-x_0)^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(x-x_0)^2(y-y_0) + 3\frac{\partial^3 f}{\partial x \partial y^2}(x-x_0)(y-y_0)^2 + \frac{\partial^3 f}{\partial y^3}(y-y_0)^3\right) \\ &+ O\left(||(x-x_0, y-y_0)||^4\right) \end{aligned}$$

where $O\left(||(x-x_0, y-y_0)||^4\right)$ represents the higher order terms that are of order 4 or higher in the Taylor series expansion.

Example 1.14.1.3 (Three Variable Case)**.** For some function $f(x,y,z)$, the second order Taylor polynomial about the point (x_0, y_0, z_0) is given by:

$$\begin{aligned}
f(x,y,z) = f(x_0,y_0,z_0) + &\left(\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0)\right) + \\
&\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-x_0)(y-y_0) + 2\frac{\partial^2 f}{\partial x \partial z}(x-x_0)(z-z_0) + \right. \\
&\left. \frac{\partial^2 f}{\partial y^2}(y-y_0)^2 + 2\frac{\partial^2 f}{\partial y \partial z}(y-y_0)(z-z_0) + \frac{\partial^2 f}{\partial z^2}(z-z_0)^2\right) + \\
&O\left(||(x-x_0, y-y_0, z-z_0)||^3\right)
\end{aligned}$$

where $O\left(||(x-x_0, y-y_0, z-z_0)||^3\right)$ represents the higher order terms that are of order 3 or higher in the Taylor series expansion.

Example 1.14.1.4. *Find the second order Taylor polynomial of* $f(x,y) = \sqrt{x^2+y^3}$ *about the point* $(1,2)$.

First, we compute the necessary partial derivatives at the point $(1,2)$:

$$\begin{aligned}
f(x,y) &= \sqrt{x^2+y^3} \implies f(1,2) = \sqrt{1^2+2^3} = \sqrt{9} = 3 \\
f_x(x,y) &= \frac{x}{\sqrt{x^2+y^3}} \implies f_x(1,2) = \frac{1}{3} \\
f_y(x,y) &= \frac{3y^2}{2\sqrt{x^2+y^3}} \implies f_y(1,2) = \frac{6}{3} = 2 \\
f_{xx}(x,y) &= \frac{y^3}{(x^2+y^3)^{3/2}} \implies f_{xx}(1,2) = \frac{8}{27} \\
f_{yy}(x,y) &= \frac{3x^2}{4(x^2+y^3)^{3/2}} \implies f_{yy}(1,2) = \frac{3}{27} = \frac{1}{9} \\
f_{xy}(x,y) &= -\frac{3xy^2}{2(x^2+y^3)^{3/2}} \implies f_{xy}(1,2) = -\frac{6}{27} = -\frac{2}{9}
\end{aligned}$$

Now, we can plug these values into the second order Taylor polynomial formula:

$$\begin{aligned}
f(x,y) &\approx f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\
&\quad + \frac{1}{2}\left(f_{xx}(1,2)(x-1)^2 + 2f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)(y-2)^2\right) \\
&\approx 3 + \frac{1}{3}(x-1) + 2(y-2) + \frac{1}{2}\left(\frac{8}{27}(x-1)^2 - \frac{4}{9}(x-1)(y-2) + \frac{1}{9}(y-2)^2\right)
\end{aligned}$$

Example 1.14.1.5. *Find the third-order taylor expansion for* $f(x,y) = e^{x-2y}$ *about the point* $(0,0)$.

Method 1 (using multivariable taylor series) We compute the necessary partial derivatives at the point $(0,0)$:

$$
\begin{aligned}
f(x,y) &= e^{x-2y} \implies f(0,0) = e^0 = 1 \\
f_x(x,y) &= e^{x-2y} \implies f_x(0,0) = 1 \\
f_y(x,y) &= -2e^{x-2y} \implies f_y(0,0) = -2 \\
f_{xx}(x,y) &= e^{x-2y} \implies f_{xx}(0,0) = 1 \\
f_{yy}(x,y) &= 4e^{x-2y} \implies f_{yy}(0,0) = 4 \\
f_{xy}(x,y) &= -2e^{x-2y} \implies f_{xy}(0,0) = -2 \\
f_{xxx}(x,y) &= e^{x-2y} \implies f_{xxx}(0,0) = 1 \\
f_{yyy}(x,y) &= -8e^{x-2y} \implies f_{yyy}(0,0) = -8 \\
f_{xxy}(x,y) &= -2e^{x-2y} \implies f_{xxy}(0,0) = -2 \\
f_{xyy}(x,y) &= 4e^{x-2y} \implies f_{xyy}(0,0) = 4
\end{aligned}
$$

Now, we can plug these values into the third order Taylor polynomial formula:

$$
\begin{aligned}
f(x,y) \approx\ & f(0,0) + f_x(0,0)x + f_y(0,0)y \\
& + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) \\
& + \frac{1}{6}\left(f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3\right) \\
\approx\ & 1 + x - 2y + \frac{1}{2}\left(x^2 - 4xy + 4y^2\right) + \frac{1}{6}\left(x^3 - 6x^2y + 12xy^2 - 8y^3\right)
\end{aligned}
$$

Method 2 (using single variable taylor series) We know that $e^{x-2y} = e^x e^{-2y}$. We can find the taylor series of e^x and e^{-2y} about 0 separately, and then multiply them together. Due to margin, this is left as an exercise to the reader.

1.15 Multiple Integrals

Definiton 1.15.0.1 (Double Integral)**.** Let $f(x,y)$ be a function defined on a closed and bounded region R in the xy-plane. The double integral of f over R is denoted by

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dA \tag{1.7}$$

where dA represents an infinitesimal area element in the region R. The double integral can be interpreted as the volume under the surface defined by $z = f(x,y)$ over the region R.

1.15.1 Double Integrals in a Rectangular Region

By the point of seeing this note, you should be familiar with the simple case of rectangular, simple cases are provided as examples:

Example 1.15.1.1. Find the area under the quadric surface $z = 16 - x^2 - y^2$ over the square region $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$.

Note We would have to ensure that the surface is above the xy-plane in the region of interest, which is true in this case.

We can set up the double integral as follows:

$$\iint_R (16 - x^2 - y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - y^2)\, dy\, dx$$

First, we integrate with respect to y:

$$\int_0^2 (16 - x^2 - y^2)\, dy = \left[16y - x^2 y - \frac{y^3}{3}\right]_0^2 = 32 - 2x^2 - \frac{8}{3} = \frac{88}{3} - 2x^2$$

Next, we integrate with respect to x:

$$\int_0^2 \left(\frac{88}{3} - 2x^2\right)\, dx = \left[\frac{88}{3}x - \frac{2x^3}{3}\right]_0^2 = \frac{176}{3} - \frac{16}{3} = \frac{160}{3}$$

Therefore, the area under the surface is $\frac{160}{3}$.

Example 1.15.1.2. Evaluate the double integral of $f(x, y) = x - 3y^2$ over the rectangular region $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

We can set up the double integral as follows:

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx$$

First, we integrate with respect to y:

$$\int_1^2 (x - 3y^2)\, dy = \left[xy - y^3\right]_1^2 = 2x - 8 - (x - 1) = x - 7$$

Next, we integrate with respect to x:

$$\int_0^2 (x - 7)\, dx = \left[\frac{x^2}{2} - 7x\right]_0^2 = (2 - 14) - 0 = -12$$

Therefore, the value of the double integral is -12.

Theorem 1.15.1.3. If the integrand function $f(x, y)$ is seperable, i.e., $f(x, y) = g(x)h(y)$, then the double integral can be computed as follows:

$$\iint_R f(x, y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right) \tag{1.8}$$

where $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$.

Proof. **Sketch:** $h(y)$ is a constant when integrating with respect to x, and vice versa. □

Example 1.15.1.4. Let $f(x, y) = \sin x \cos y$ and $R = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}\}$. Evaluate the double integral $\iint_R f(x, y)\, dA$.

Since $f(x, y)$ is seperable, we can write:

$$\iint_R f(x, y)\, dA = \left(\int_0^{\frac{\pi}{2}} \sin x\, dx \right) \left(\int_0^{\frac{\pi}{2}} \cos y\, dy \right)$$

Evaluating each integral separately gives 1 for both, so the final result is: $1 \times 1 = 1$.

1.15.2 Double Integrals in General Regions

Types of Regions When the region R is not rectangular, we can still compute the double integral by expressing the region in terms of inequalities. There are three common types of regions:

Definiton 1.15.2.1 (Type I Region)**.** A region R is called a Type I region if it can be described by the inequalities:

$$a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x)$$

where $g_1(x)$ and $g_2(x)$ are continuous functions on the interval $[a, b]$. Then, to evaluate the double integral over a Type I region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{1.9}$$

Integral Order: Integrate with respect to y first, then x.

Intuition: As we traverse the outer part (x), we are summing up vertical slices (in y), and the bounds of those slices depend on x and changes.

Definiton 1.15.2.2 (Type II Region)**.** Type II reigion is similar to Type I, but the roles of x and y are swapped. A region R is called a Type II region if it can be described by the inequalities:

$$c \leq y \leq d, \quad h_1(y) \leq x \leq h_2(y)$$

where $h_1(y)$ and $h_2(y)$ are continuous functions on the interval $[c, d]$. Then, to evaluate the double integral over a Type II region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{1.10}$$

The integral order and intuition is mirrored from Type I, but we are summing up horizontal slices (in x), and the bounds of those slices depend on y and changes.

Definiton 1.15.2.3 (Type III Region)**.** A region R is called a Type III region if it can be described as the union of a finite number of Type I and Type II regions. To evaluate the double integral over a Type III region for a continuous function $f(x, y)$, we can break down the integral into separate integrals over each Type I or Type II subregion and sum them up:

$$\iint_R f(x,y)\,dA = \sum_{i=1}^{n} \iint_{R_i} f(x,y)\,dA \tag{1.11}$$

where each R_i is either a Type I or Type II region. And that:

$$\bigcup_{i=1}^{n} R_i = R \quad \text{and} \quad R_i \cap R_j = \emptyset \text{ for } i \neq j$$

This approach allows us to handle more complex regions by breaking them down into simpler parts.

Example 1.15.2.4. Find the volume of the solid that lies under the paraboloid $z = f(x, y) = x^2 + y^2$ and above the region R bounded by $y = 2x$ and $y = x^2$.

First, you would sketch the region to understand its shape and boundaries at Figure 1.1.

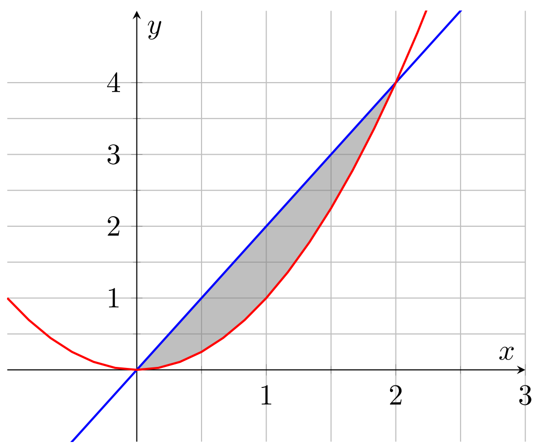

Figure 1.1: Region bounded by $y = 2x$ and $y = x^2$

We can tell that this is a Type I regionwhere $0 \leq x \leq 2$, and $x^2 \leq y \leq 2x$. Thus, we can set up the double integral as follows:

$$\iint_R (x^2 + y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\,dy\,dx$$

First, we integrate with respect to y:

$$\int_{x^2}^{2x} (x^2 + y^2)\,dy = \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} = 2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3} = \frac{14x^3}{3} - x^4 - \frac{x^6}{3}$$

Next, we integrate with respect to x:

$$\int_0^2 \left(\frac{14x^3}{3} - x^4 - \frac{x^6}{3}\right)\,dx = \left[\frac{14x^4}{12} - \frac{x^5}{5} - \frac{x^7}{21}\right]_0^2 = \frac{216}{35}$$

Therefore, the volume of the solid is $\frac{216}{35}$.

Example 1.15.2.5. Consider the above example, but we want to set it up as a Type II region. The region R can be described by $0 \leq y \leq 4$, and $\frac{y}{2} \leq x \leq \sqrt{y}$. Thus, we can set up the double integral as follows:

$$\iint_R (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

First, we integrate with respect to x:

$$\int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx = \left[\frac{x^3}{3} + y^2 x\right]_{x=\frac{y}{2}}^{x=\sqrt{y}} = \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2} = \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{13y^3}{24}$$

Next, we integrate with respect to y:

$$\int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{13y^3}{24}\right) dy = \left[\frac{2y^{\frac{5}{2}}}{15} + \frac{2y^{\frac{7}{2}}}{7} - \frac{13y^4}{96}\right]_0^4 = \frac{216}{35}$$

Therefore, the volume of the solid is $\frac{216}{35}$, which is consistent with the previous result. This is also consistent with Fubini's Theorem.

Example 1.15.2.6. Integrate the surface given by $z = e^{x^2}$ over the triangular region with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$. We can describe the region as either a Type I or Type II region:

(✗) Here, we will describe it as a Type II regionwhere $0 \leq y \leq 1$, and $y \leq x \leq 1$. Thus, we can set up the double integral as follows:

$$\iint_R e^{x^2}\, dA = \int_0^1 \int_y^1 e^{x^2}\, dx\, dy$$

We can tell that e^{x^2} does not have an elementary antiderivative, so we cannot integrate with respect to x directly.

(✓) However, we can change the order of integration to make it a Type I regionwhere $0 \leq x \leq 1$, and $0 \leq y \leq x$. Thus, we can set up the double integral as follows:

$$\iint_R e^{x^2}\, dA = \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

First, we integrate with respect to y:

$$\int_0^x e^{x^2}\, dy = \left[ye^{x^2}\right]_0^x = xe^{x^2}$$

Next, we integrate with respect to x:

$$\int_0^1 xe^{x^2}\, dx$$

This is now obvious, a simple u-substitution with $u = x^2$, $du = 2x\, dx$:

$$\int_0^1 xe^{x^2}\, dx = \frac{1}{2}\int_0^1 e^u\, du = \frac{1}{2}\left[e^u\right]_0^1 = \frac{e-1}{2}$$

Therefore, the value of the double integral is $\frac{e-1}{2}$.

Intuition When the integrand is difficult to integrate with respect to one variable, consider changing the order of integration. You should be able to tell that e^{x^2} has no elementary antiderivative, so you would have ruled out integrating with respect to x first.

Formal Definition of Double Integrals

There is two definitions of double integrals in this course, due to the discrepancy between Stewart's book and the lectures.

Review. Formal Definition of Definite Integral (Single Variable)

Consider $y = f(x) \geq 0$ on the interval $x \in [a, b]$. We divide the interval into n subintervals of equal width $\Delta x = \frac{b-a}{n}$, and let x_i^* be a sample point in the i-th subinterval. The Riemann sum is given by:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Now, for any x_i^*, we consider the minimum and maximum values of $f(x_i^*)$ in the i-th subinterval, denoted as m_i and M_i respectively. We can then define the lower sum L_n and upper sum U_n as follows:

$$L_n = \sum_{i=1}^{n} m_i \Delta x \quad \text{and} \quad U_n = \sum_{i=1}^{n} M_i \Delta x$$

To satisfy the squeeze theorem, for all i, we would need:

$$\lim_{n\to\infty} M_i - m_i = \lim_{\delta x \to 0} M_i - m_i = 0$$

If $f(x)$ is continuous on $[a, b]$. Then, we have:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \int_a^b f(x)\, dx$$

For the case of discontinuous functions, if the set of discontinuities has measure zero, then the function is still integrable.

Definiton 1.15.2.7 (Definition of Double Integral)**.** Let R be a rectangular region in the xy-plane given by $R = [a, b] \times [c, d]$. The double integral of a function $f(x, y)$ over the region R is defined as:

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i \quad \text{(Riemann Definition)} \tag{1.12a}$$

where ΔA_i is the area of the i-th subrectangle, and (x_i^*, y_i^*) is a sample point in it. The limit is taken as the maximum diameter of the subrectangles approaches zero.

$$\iint_R f(x, y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta A_{ij} \quad \text{(Grid Formulation)} \tag{1.12b}$$

where ΔA_{ij} is the area of the ij-th subrectangle, and (x_i^*, y_j^*) is a sample point in it. Note that the ΔA_{ij} may be non-uniform. The limit is taken as the maximum diameter of the subrectangles approaches zero.

Similarly, the lower and upper sums for double integrals are:

$$L_n = \sum_{i=1}^{n} m_i \Delta A_i \quad \text{and} \quad U_n = \sum_{i=1}^{n} M_i \Delta A_i \quad \text{(Riemann Definition)} \tag{1.13a}$$

$$L_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} \Delta A_{ij} \quad \text{and} \quad U_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} \Delta A_{ij} \quad \text{(Grid Formulation)} \tag{1.13b}$$

Here, m_{ij} and M_{ij} are the minimum and maximum values of $f(x, y)$ in the ij-th subrectangle. Define $||P|| = \max ||(\Delta x_i, \Delta y_j)||$ as the maximum diameter of the subrectangles. For the squeeze theorem, we require:

$$\lim_{n,m\to\infty} (M_{ij} - m_{ij}) = \lim_{||P||\to 0} (M_{ij} - m_{ij}) = 0$$

If $f(x, y)$ is continuous on R, then:

$$\lim_{n,m\to\infty} L_{n,m} = \lim_{n,m\to\infty} U_{n,m} = \iint_R f(x, y)\, dA$$

The Riemann definition and grid formulation are similar.

The following is the analogue of the squeeze theorem for double integrals:

Definiton 1.15.2.8 (Squeeze Theorem for Double Integrals)**.** For the first defintion Consider region R subdivided into N subregions $R_1, R_2, \ldots, R_N$, such that all subregions $\bigcup_{i=1}^{N} R_i \subset R$ (They are all inside). For both cases, we require that $R_i \cap R_j = \emptyset$ for $i \neq j$, and then some of the area would be omited and the following would be garanteed:

$$\sum_{i=1}^{N} \Delta A \leq \text{Area}(R), \quad \sum_{i=1}^{N} m_i \Delta A_i \leq \iint_R f(x, y)\, dA$$

where m_i and M_i are the minimum and maximum values of $f(x, y)$ in the i-th subregion. Similarly, if $\bigcup_{i=1}^{N} R_i \supset R$ (They all cover R), and that we garantee that $R_i \cap R \neq \emptyset$ for all i. Then, some of the area would be double counted and the following would be garanteed:

$$\sum_{i=1}^{N} \Delta A \geq \text{Area}(R), \quad \sum_{i=1}^{N} M_i \Delta A_i \geq \iint_R f(x, y)\, dA$$

For the second definition, the same logic applies, but we consider subrectangles that creates grid that is either inside or covering R.

Example 1.15.2.9. Estimate the volume that lies above the square $R = [0, 2] \times [0, 2]$ and below the surface $z = f(x, y) = 16 - x^2 - 2y^2$ by dividing the R into four subrectangles of equal area and using the value of the function at the upper right corner of each subrectangle to form a Riemann sum. Choose the upper right corner of each subrectangle as the sample point.

We divide the square R into four subrectangles, each with an area of 1. We obtain the sum:

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*) \Delta A = \sum_{i=1}^{2} \sum_{j=1}^{2} f(i, j) \cdot 1$$

where the sample points are $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$. Evaluating the function at these points gives:

$$V \approx f(1, 1) + f(1, 2) + f(2, 1) + f(2, 2) = 34$$

Therefore, the estimated volume is approximately 34.

1.15.3 Double Integrals in Non-Rectangular Regions

Theorem 1.15.3.1 (Change of Variable to Polar Coordinates)**.** Consider the double integral of a function $f(x, y)$ over a region R in the xy-plane. If we change the variables from Cartesian coordinates (x, y) to polar coordinates (r, θ) using the transformations:

$$x = r\cos\theta, \quad y = r\sin\theta \implies r = \sqrt{x^2 + y^2}$$

then the double integral can be expressed in polar coordinates as follows:

$$\iint_R f(x, y)\, dA = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta \tag{1.14}$$

where R' is the corresponding region in the $r\theta$-plane, and the term r arises from the Jacobian determinant of the transformation from Cartesian to polar coordinates.

Proof. **Note.** *This change of variable can be derived using the Jacobian determinant of the transformation from Cartesian to polar coordinates, which will be covered in Section 1.15.9, which can fully prove this theorem in the case where $g < 0$ for some input.*

Geometric Sketch: Assume $f(r\cos\theta, r\sin\theta) = g(r, \theta) \geq 0$. Consider a small rectangle ΔA_i in the xy-plane with dimensions Δx_i and Δy_i. When we transform this rectangle into polar coordinates, it becomes a small sector of a circle with radius r_i and angle $\Delta\theta_i$. The area of this sector is given by:

$$\Delta A_i = r_i \Delta r_i \Delta\theta_i \left(1 + \frac{\Delta r_i}{2r_i}\right)$$

this is derived from the geormetric formula of the area of a sector of a circle. Then, as $\Delta r_i \to 0$, the term $\frac{\Delta r_i}{2r_i} \to 0$, and we have:

$$\Delta A_i \approx r_i \Delta r_i \Delta\theta_i$$

Therefore, the double integral in polar coordinates can be approximated as:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{n} f(r_i\cos\theta_i, r_i\sin\theta_i) r_i \Delta r_i \Delta\theta_i$$

Taking the limit as the maximum diameter of the subrectangles approaches zero, we obtain the exact double integral in polar coordinates. □

Definiton 1.15.3.2 (Reigion Defined by Varying r with θ)**.** Consider a region R in the xy-plane that can be described in polar coordinates by the inequalities:

$$\alpha \leq \theta \leq \beta, \quad g_1(\theta) \leq r \leq g_2(\theta)$$

where $g_1(\theta)$ and $g_2(\theta)$ are continuous functions on the interval $[\alpha, \beta]$. Then, to evaluate the double integral over this region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{1.15}$$

Definiton 1.15.3.3 (Reigion Defined by Varying θ with r)**.** Similarly, consider a region R in the xy-plane that can be described in polar coordinates by the inequalities:

$$a \le r \le b, \quad h_1(r) \le \theta \le h_2(r)$$

where $h_1(r)$ and $h_2(r)$ are continuous functions on the interval $[a, b]$. Then, to evaluate the double integral over this region for a continuous function $f(x, y)$, we set up the integral as follows:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr \tag{1.16}$$

When to Use Polar Coordinates

Polar coordinates are particularly useful for regions with circular or radial symmetry, as they simplify integration by transforming variables into a more natural form. They are also advantageous for integrands that are difficult in Cartesian coordinates, especially those involving terms like x^2+y^2.

Example 1.15.3.4. Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$ (Donut region). We can describe the region R in polar coordinates as $1 \le r \le 2$ and $0 \le \theta \le \pi$. Thus, we can set up the double integral as follows:

$$\begin{aligned}
\iint_R (3x + 4y^2)\,dA &= \int_0^\pi \int_1^2 (3(r\cos\theta) + 4(r\sin\theta)^2)\,r\,dr\,d\theta \\
&= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta \\
&= \int_0^\pi (r^3\cos\theta + r^4\sin^2\theta)\Big|_{r=1}^{r=2}\,d\theta \\
&= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\,d\theta = \frac{15}{2}\pi \qquad \left(\text{Using } \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)
\end{aligned}$$

Example 1.15.3.5. Find the volume of the solid bounded by the $z = 0$ plane and the paraboloid $z = 1 - x^2 - y^2$. We first consider the projection of the paraboloid onto the xy-plane, which is the circle $1 - x^2 - y^2 = 0$. We can describe the region R in polar coordinates as $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Thus, we can set up the double integral as follows:

$$\begin{aligned}
V &= \iint_R (1 - x^2 - y^2)\,dA \\
&= \int_0^{2\pi} \int_0^1 (1 - r^2)\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \left(\int_0^1 (r - r^3)\,dr\right) d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta = \frac{\pi}{2}
\end{aligned}$$

Example 1.15.3.6. *Find the area of the region R enclosed by one petal of the rose given by $r = \cos 3\theta$.*

We know that the petal has upper bound at $\theta = \pm\frac{\pi}{6}$. We can describe the region R in polar coordinates as $0 \le r \le \cos 3\theta$ and $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. Thus, we can set up the double integral as follows:

$$\begin{aligned}
\text{Area}(R) &= \iint_R 1\, dA \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\, dr\, d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{r^2}{2}\Big|_{r=0}^{r=\cos 3\theta} \right) d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^2 3\theta}{2}\, d\theta = \frac{\pi}{12} \qquad \left(\text{Using } \cos^2 x = \frac{1+\cos 2x}{2} \right)
\end{aligned}$$

Example 1.15.3.7. *Find the volume trapped between the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$.*

First, we find the intersection of the cone and the sphere:

$$z = \sqrt{x^2+y^2} \implies z^2 = x^2+y^2$$

Substituting into the sphere equation:

$$\begin{aligned}
x^2+y^2+z^2 = 1 &\implies z^2+z^2 = 1 \\
&\implies 2z^2 = 1 \implies z = \frac{1}{\sqrt{2}} \quad x^2+y^2 = \frac{1}{2}
\end{aligned}$$

Thus, we have the region $R = \{(r,\theta) \mid 0 \le r \le \frac{1}{\sqrt{2}}, 0 \le \theta \le 2\pi\}$. We can set up the double integral as follows:

$$\begin{aligned}
V &= \iint_R (\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2})\, dA \\
&= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1-r^2} - r)\, r\, dr\, d\theta \\
&= 2\pi \left[\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}\, dr - \int_0^{\frac{1}{\sqrt{2}}} r^2 dr \right] \\
&= 2\pi \left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^{\frac{1}{\sqrt{2}}} - \frac{r^3}{3}\Big|_0^{\frac{1}{\sqrt{2}}} \right] = \frac{2\pi}{3}(1 - \frac{1}{\sqrt{2}})
\end{aligned}$$

1.15.4 Applications of Double Integrals

Review: Moment of Inertia using Single Integral

Consider a thin rod of length L with a linear mass density $\rho(x)$, where x is the distance from one end of the rod. We can use the following formula to find the moment of inertia I of the rod about an axis perpendicular to the rod and passing through one end:

$$I = \int_0^L x^2 \rho(x)\, dx \tag{1.17}$$

where x^2 is the square of the distance from the axis of rotation, and $\rho(x)\, dx$ represents the mass element of the rod at position x. This comes from the definition of moment of inertia, which is the following sum with point masses:

$$I = \sum m_i r_i^2 \quad \text{s.t.} \quad \text{KE} = \frac{1}{2} I\omega^2$$

Definiton 1.15.4.1 (Mass of a Lamina)**.** Consider a lamina occupying the region R in the xy-plane with a surface mass density $\sigma(x, y)$, where $\sigma(x, y)$ is the mass per unit area at the point (x, y). The mass M of the lamina can be found using the following double integral:

$$M = \iint_R \sigma(x, y)\, dA \tag{1.18}$$

where dA represents an infinitesimal area element in the region R.

Definiton 1.15.4.2 (Moment of a Lamina)**.** The moment of the lamina about the x-axis (M_x) and y-axis (M_y) can be found using the following formulas:

$$M_x = \iint_R y\sigma(x, y)\, dA, \quad M_y = \iint_R x\sigma(x, y)\, dA \tag{1.19}$$

where y and x are the distances from the respective axes of rotation, and $\sigma(x, y)\, dA$ represents the mass element of the lamina at the point (x, y).

Definiton 1.15.4.3 (Center of Mass of a Lamina)**.** The center of mass $(\bar{x}, \bar{y})$ of the lamina can be found using the following formulas:

$$\bar{x} = \frac{1}{M} M_y = \frac{1}{M} \iint_R x\sigma(x, y)\, dA, \quad \bar{y} = \frac{1}{M} M_x = \frac{1}{M} \iint_R y\sigma(x, y)\, dA \tag{1.20}$$

where M is the total mass of the lamina as calculated in the previous example.

Example 1.15.4.4. *Find the cnetre of mass of the following plate with density funciton* $\sigma(x, y) = x + y$ *over the reigion bounded by the axis and* $y = \sqrt{x}$ *and* $x = 1$.

First, we find the mass of the lamina:

$$
\begin{aligned}
M &= \iint_R (x+y)\, dA \\
&= \int_0^1 \int_0^{\sqrt{x}} (x+y)\, dy\, dx \\
&= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(x\sqrt{x} + \frac{x}{2} \right) dx = \left[\frac{2}{5} x^{\frac{5}{2}} + \frac{1}{4} x^2 \right]_0^1 = \frac{13}{20}
\end{aligned}
$$

Now, we find the moments about the x-axis and y-axis:

$$
\begin{aligned}
M_x &= \iint_R y(x+y)\, dA \\
&= \int_0^1 \int_0^{\sqrt{x}} y(x+y)\, dy\, dx \\
&= \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(\frac{x^2}{2} + \frac{x^{\frac{3}{2}}}{3} \right) dx = \left[\frac{x^3}{6} + \frac{2}{15} x^{\frac{5}{2}} \right]_0^1 = \frac{3}{10}
\end{aligned}
$$

and,

$$
\begin{aligned}
M_y &= \iint_R x(x+y)\, dA \\
&= \int_0^1 \int_0^{\sqrt{x}} x(x+y)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(x^2\sqrt{x} + \frac{x^{\frac{3}{2}}}{2} \right) dx = \left[\frac{2}{7} x^{\frac{7}{2}} + \frac{1}{6} x^3 \right]_0^1 = \frac{19}{42}
\end{aligned}
$$

Finally, we can find the center of mass:

$$
\bar{x} = \frac{M_y}{M} = \frac{190}{273}, \quad \bar{y} = \frac{M_x}{M} = \frac{6}{13}
$$

Definiton 1.15.4.5 (Geometric Center of a Lamina)**.** If the surface mass density $\sigma(x, y)$ is constant, then the center of mass is also known as the geometric center (or centroid) of the lamina. The formulas for the geometric center are:

$$
x_c = \frac{1}{\text{Area}(R)} \iint_R x\, dA, \quad y_c = \frac{1}{\text{Area}(R)} \iint_R y\, dA \tag{1.21}
$$

where Area(R) is the area of the region R.

Definiton 1.15.4.6 (Moment of Inertia of a Lamina)**.** The moment of inertia of the lamina about the x-axis (I_x) and y-axis (I_y) can be found using the following formulas:

$$I_x = \iint_R y^2\sigma(x,y)\,dA, \quad I_y = \iint_R x^2\sigma(x,y)\,dA \tag{1.22}$$

where y^2 and x^2 are the squares of the distances from the respective axes of rotation, and $\sigma(x,y)\,dA$ represents the mass element of the lamina at the point (x,y). Also for the moment of inertia about the origin (I_o):

$$I_o = \iint_R (x^2+y^2)\sigma(x,y)\,dA \tag{1.23}$$

In general, for an axis defined by a line $ax+by+c=0$, the moment of inertia about that axis (I_l) can be found using the following formula:

$$I_l = \iint_R \left(\frac{ax+by+c}{\sqrt{a^2+b^2}}\right)^2 \sigma(x,y)\,dA \tag{1.24}$$

Example 1.15.4.7. A rectangular plate of mass m, length L and width W is rotated about a vertical line on its left side with width W. Find the moment of inertia of the plate about this line in two cases:

1. The plate has uniform density $\sigma(x,y) = \frac{m}{LW}$.
2. The density varies at a point proportional to the square of the distance from the right most side.
3. It has uniform density, but rotated its center.

Solution. 1. Uniform Density We can describe the region R in Cartesian coordinates as $0 \le x \le L$ and $0 \le y \le W$. The surface mass density is $\sigma(x,y) = \frac{m}{LW}$. Thus, we can set up the double integral as it rotates around the y-axis:

$$\begin{aligned}
I_y &= \iint_R x^2\sigma(x,y)\,dA \\
&= \int_0^W \int_0^L x^2 \cdot \frac{m}{LW}\,dx\,dy \\
&= \int_0^W \left[\frac{m}{LW}\cdot\frac{x^3}{3}\right]_{x=0}^{x=L} dy = \int_0^W \frac{mL^2}{3W}dy = \left[\frac{mL^2}{3W}y\right]_{y=0}^{y=W} = \frac{mL^2}{3}
\end{aligned}$$

Solution. 2. Varying Density We can describe the region R in Cartesian coordinates as $0 \le x \le L$ and $0 \le y \le W$. The surface mass density is $\sigma(x,y) = k(L-x)^2$. To find the constant k, we have:

$$\begin{aligned}
I_y &= \iint_R x^2\sigma(x,y)\,dA \\
&= \int_0^W \int_0^L x^2 \cdot k(L-x)^2\,dx\,dy \\
&= \int_0^W \left[k\int_0^L (x^2L^2 - 2Lx^3 + x^4)\,dx\right] dy \\
&= \int_0^W \left[k\left(\frac{L^5}{3} - \frac{L^5}{2} + \frac{L^5}{5}\right)\right] dy = \int_0^W \left[k\cdot\frac{L^5}{30}\right] dy = k\cdot\frac{L^5}{30}W
\end{aligned}$$

Solution. 3. Rotated Center We can describe the region R in Cartesian coordinates as $-\frac{L}{2} \leq x \leq \frac{L}{2}$ and $-W \leq y \leq W$. The surface mass density is $\sigma(x,y) = \frac{m}{LW}$. Thus, we can set up the double integral as it rotates around the origin:

$$
\begin{aligned}
I_o &= \iint_R (x^2+y^2)\sigma(x,y)\,dA \\
&= \int_{-W}^{W}\int_{-L/2}^{L/2} (x^2+y^2)\cdot\frac{m}{LW}\,dx\,dy \\
&= \int_{-W}^{W}\left[\frac{m}{LW}\left(\frac{x^3}{3}+y^2x\right)\Bigg|_{x=-L/2}^{x=L/2}\right]dy \\
&= \int_{-W}^{W}\left[\frac{mL^2}{12W}+\frac{my^2}{W}\right]dy \\
&= \left[\frac{mL^2}{12W}y+\frac{my^3}{3W}\right]_{y=-W}^{y=W} = \frac{mL^2}{6}+\frac{2mW^2}{3}
\end{aligned}
$$

1.15.5 Surface Area

Theorem 1.15.5.1 (Surface Area)**.** Given $z = f(x,y)$, where f is a differentiable function over the region R in the xy-plane, the surface area S of the surface above the region R is given by:

$$
S = \iint_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA \tag{1.25}
$$

Proof. Consider a small rectangle ΔA_i in the xy-plane with dimensions Δx_i and Δy_i. When we project this rectangle onto the surface $z = f(x,y)$, it becomes a parallelogram T_i tangent to the surface. Using the partial derivatives of the surface, we can deduce the two vectors that define the parallelogram:

$$
\vec{u} = (\Delta x_i, 0, f_x(x_i^*, y_i^*)\Delta x_i), \quad \vec{v} = (0, \Delta y_i, f_y(x_i^*, y_i^*)\Delta y_i)
$$

The area of this parallelogram is given by the magnitude of the cross product of these two vectors:

$$
\begin{aligned}
\Delta T_i = ||\vec{u}\times\vec{v}|| &= \left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x_i & 0 & f_x(x_i^*, y_i^*)\Delta x_i \\ 0 & \Delta y_i & f_y(x_i^*, y_i^*)\Delta y_i \end{vmatrix}\right\| \\
&= \sqrt{(-f_x(x_i^*, y_i^*)\Delta x_i\Delta y_i)^2+(-f_y(x_i^*, y_i^*)\Delta x_i\Delta y_i)^2+(\Delta x_i\Delta y_i)^2} \\
&= \sqrt{1+(f_x(x_i^*, y_i^*))^2+(f_y(x_i^*, y_i^*))^2}\,\Delta x_i\Delta y_i
\end{aligned}
$$

Therefore, the surface area can be approximated as:

$$
S \approx S_n = \sum_{i=1}^{n}\Delta T_i = \sum_{i=1}^{n}\sqrt{1+(f_x(x_i^*, y_i^*))^2+(f_y(x_i^*, y_i^*))^2}\,\Delta A_i
$$

Taking the limit as the maximum diameter of the subrectangles approaches zero, we obtain the exact surface area:

$$
S = \lim_{|P|\to 0} S_n = \iint_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA
$$

which completes our derivation. $\square$

Example 1.15.5.2. *Find the surface of the sphere $x^2 + y^2 + z^2 = a^2$.*

We first consider the first octant, where $x, y, z \geq 0$. Then the total volumn would be eight times the volumn of the first octant. We first express it in polar coordinates:

$$x^2 + y^2 = r^2 \implies z = \sqrt{a^2 - r^2}$$

taking the partial derivatives:

$$\frac{\partial z}{\partial r} = \frac{-r}{\sqrt{a^2 - r^2}}, \quad \frac{\partial z}{\partial \theta} = 0$$

Thus, we can set up the double integral as follows:

$$\begin{aligned} S_{\text{1st octant}} &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}\, dA \\ &= \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{1 + \left(\frac{-r}{\sqrt{a^2 - r^2}}\right)^2 + 0^2}\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{1 + \frac{r^2}{a^2 - r^2}}\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-a\sqrt{a^2 - r^2}\right]_{r=0}^{r=a} d\theta = \int_0^{\frac{\pi}{2}} a^2\, d\theta = \frac{\pi a^2}{2} \end{aligned}$$

Therefore, the total surface area of the sphere is:

$$S = 8 \cdot S_{\text{1st octant}} = 4\pi a^2$$

Example 1.15.5.3. *Let R be the triangular region with vertices at $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 0)$. Find the surface area of the portion of $z = 3x + y^2$ that lies above the region R.*

We first express the function and take the partial derivatives:

$$z = 3x + y^2, \quad \frac{\partial z}{\partial x} = 3, \quad \frac{\partial z}{\partial y} = 2y$$

Thus, we can set up the double integral as follows:

$$\begin{aligned} S &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \int_0^1 \int_0^y \sqrt{1 + 3^2 + (2y)^2}\, dx\, dy \\ &= \int_0^1 \int_0^y \sqrt{10 + 4y^2}\, dx\, dy \\ &= \int_0^1 \sqrt{10 + 4y^2} \cdot y\, dy \end{aligned}$$

The extra y is useful for substitution. Let $u = 10 + 4y^2$, then $du = 8y\,dy$ (that's why we integrate w.r.t x first). Thus, we have:

$$\begin{aligned} S &= \int_{u=10}^{u=14} \sqrt{u} \cdot \frac{du}{8} = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=10}^{u=14} \\ &= \frac{1}{12}(14^{\frac{3}{2}} - 10^{\frac{3}{2}}) \approx 1.7 \end{aligned}$$

1.15.6 Triple Integrals in Rectangular Coordinates

The idea of a triple integral (in fact n-tuple integral) could be extend from the idea of double integral, similar to Definition 1.15.2.7. We can define a triple integral as follows:

Definiton 1.15.6.1. Consider a function $f(x, y, z)$ that is continuous a 3-D reigion with volumne V. We can partition the reigion into n subreigons with volume ΔV_i. Then, we can define the triple integral of f over the reigion V as follows:

$$\iiint_V f(x, y, z)\,dV = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta V_i \tag{1.26}$$

where (x_i^*, y_i^*, z_i^*) is a point in the i-th subregion, and $|P|$ is the maximum diameter of the subregions. This limit exists and equals to the triple integral if f is continuous on V, since the following squeeze theorem holds, similar to the double integral case:

$$\sum_{i=1}^{n} m_i \Delta V_i \leq \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta V_i \leq \sum_{i=1}^{n} M_i \Delta V_i$$

where m and M are the minimum and maximum values of f on V, respectively.

In the retangluar case, we can express ΔV_i as:

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$$

Thus we have:

$$dV = dx\,dy\,dz$$

Definiton 1.15.6.2 (Triple Integral in Rectangular Region)**.** Consider a function $f(x, y, z)$ that is continuous over a rectangular box B defined by the inequalities:

$$a \leq x \leq b, \quad c \leq y \leq d, \quad r \leq z \leq s$$

Then, the triple integral of f over the box B can be computed as an iterated integral in any order of integration:

$$\iiint_B f(x, y, z)\,dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\,dz\,dy\,dx \tag{1.27}$$

or any other permutation of the order of integration.

Example 1.15.6.3. Consider $f(x, y, z)$ over a box $Q = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$. We would form the triple integral as follows:

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

Of course, the order of integration can be change (there are ${}_3P_3 = 6$ ways).

Definiton 1.15.6.4 (Triple Integrals in General Reigion)**.** Consider a function $f(x, y, z)$ that is continuous over a general reigion V in the 3-D space. We can describe the reigion V using the following inequalities:

$$a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x), \quad h_1(x, y) \leq z \leq h_2(x, y)$$

where $g_1(x)$, $g_2(x)$, $h_1(x, y)$, and $h_2(x, y)$ are continuous functions. Then, the triple integral of f over the reigion V can be computed as an iterated integral:

$$\iiint_V f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dy\, dx = \iint_R \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dA \quad (1.28)$$

where $R = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ is the projection of the reigion V onto the xy-plane.

The order of integration and their associated bounds can be changed based on the description of the region.

Example 1.15.6.5. *Evaluate $\iiint_Q 6xy dV$, where Q is the tetrahedron bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $2x + y + z = 4$.*

Let R be the projection of the tetrahedron onto the xy-plane. Then, we consider how the tetrahedron change in the z direction over R:

$$z = 4 - 2x - y \implies 0 \leq z \leq 4 - 2x - y$$

So we can set up:

$$\begin{aligned}\iiint_Q 6xy\, dV &= \iint_R \int_0^{4-2x-y} 6xy\, dz\, dA \\ &= \iint_R 6xy(4 - 2x - y)\, dA\end{aligned}$$

Now, we consider how the projection R change in the y direction over x:

$$2x + y = 4 \implies 0 \leq y \leq 4 - 2x, \quad 0 \leq x \leq 2$$

So we can set up:

$$\begin{aligned}\iiint_Q 6xy\, dV &= \int_0^2 \int_0^{4-2x} 6xy(4 - 2x - y)\, dy\, dx \\ &= \int_0^2 \left[12xy^2 - 6x(4 - 2x)y^2 - 2y^3\right]_{y=0}^{y=4-2x} dx \\ &= \int_0^2 (192x - 144x^2 + 24x^3 - 32x^4 + 8x^5)\, dx = \frac{64}{5}\end{aligned}$$

Example 1.15.6.6. *Evaluate the integral given in the previous Example 1.15.6.5 by intergrating w.r.t. x first.*

We consider how the tetrahedron change in the x direction over R, which is now the projection of the tetrahedron onto the yz-plane:

$$2x + y + z = 4 \implies 0 \le x \le \frac{4 - y - z}{2}$$

So we can set up:

$$\begin{aligned}\iiint_Q 6xy\, dV &= \iint_R \int_0^{\frac{4-y-z}{2}} 6xy\, dx\, dA \\ &= \frac{3}{4} \iint_R y(4 - y - z)^2\, dA\end{aligned}$$

Now, we consider how the projection R change in the y direction over z:

$$y + z = 4 \implies 0 \le y \le 4 - z, \quad 0 \le z \le 4$$

So we can set up:

$$\begin{aligned}\iiint_Q 6xy\, dV &= \frac{3}{4} \int_0^4 \int_0^{4-z} y(4 - y - z)^2\, dy\, dz \\ &= \frac{64}{5}\end{aligned}$$

Example 1.15.6.7. *Find the volume of the solid bounded by the surface $z = 4 - y^2$ and planes given by $x + z = 4$, $x = 0$, and $z = 0$.*

First, we select the $z = 4 - y^2$ surface as our base reigion R. Then, we consider how the solid change in the x direction over R:

$$x + z = 4 \implies 0 \le x \le 4 - z$$

So we can set up:

$$\begin{aligned}V &= \iint_R \int_0^{4-z} 1\, dx\, dA \\ &= \iint_R (4 - z)\, dA\end{aligned}$$

Now, we consider how the projection R change in the z direction over y:

$$z = 4 - y^2 \implies 0 \le z \le 4 - y^2, \quad -2 \le y \le 2$$

So we can set up:

$$\begin{aligned}V &= \int_{-2}^{2} \int_0^{4-y^2} (4 - z)\, dz\, dy \\ &= \int_{-2}^{2} \left[4z - \frac{z^2}{2}\right]_{z=0}^{z=4-y^2} dy \\ &= \int_{-2}^{2} \left(8 - \frac{y^2}{2}\right) dy \\ &= \frac{128}{5}\end{aligned}$$

Example 1.15.6.8. *Change the order of Integration in the following triple integral such that the integration is performed w.r.t. x, then y, then z:*

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$$

The region is limited in the z direction by $0 \leq z \leq 1-y$:

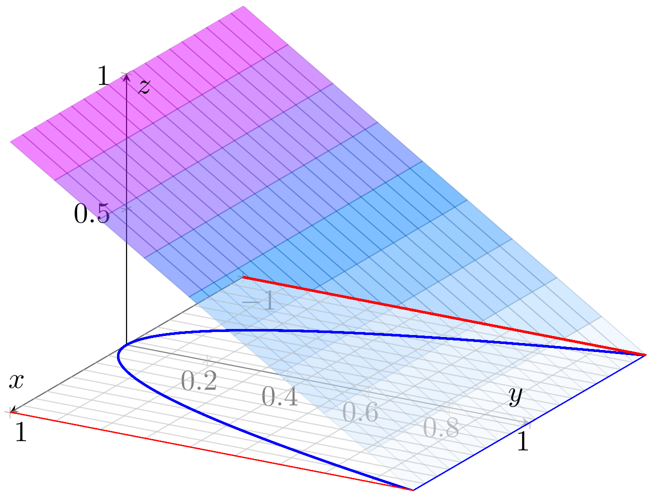

Figure 1.2: The region bounded by $z = 1-y$, $z = 0$, $y = x^2$, $x = -1$, and $x = 1$.

We can also visualize the R that is the reigion projected in the xy-plane as follows

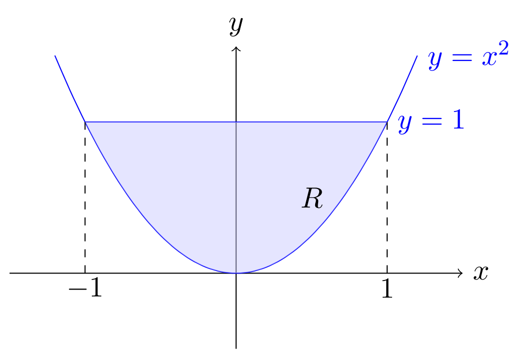

Figure 1.3: The projection of the region onto the xy-plane.

So, we can consider the projection of the triangle on the zy-plane as our base region R'. Then, we consider how the region changes in the x direction over R:

$$x^2 = y \implies -\sqrt{y} \leq x \leq \sqrt{y}$$

So we can set up:

$$\begin{aligned}\iiint_V f(x,y,z)\,dV &= \iint_{R'}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dA\\ &= \iint_{R'}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz\end{aligned}$$

Now, we consider how the projection R' change in the y direction over z:

$$z = 1 - y \implies 0 \leq y \leq 1 - z, \quad 0 \leq z \leq 1$$

So we can set up:

$$\iiint_V f(x, y, z)\, dV = \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$$

Applications of Triple Integrals

Physical Applications Similar to double integrals, triple integrals can be used to compute physical quantities such as mass, center of mass, and moment of inertia for three-dimensional objects with variable density.

Definiton 1.15.6.9 (Mass of a Solid)**.** Consider a solid V with a variable density function $\rho(x, y, z)$ that is continuous over the solid. The mass M of the solid can be computed using the following triple integral:

$$M = \iiint_V \rho(x, y, z)\, dV \tag{1.29}$$

Definiton 1.15.6.10 (Center of Mass of a Solid)**.** Consider a solid V with a variable density function $\rho(x, y, z)$ that is continuous over the solid. The coordinates of the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the solid can be computed using the following formulas:

$$\bar{x} = \frac{1}{M} \iiint_V x\rho(x, y, z)\, dV, \quad \bar{y} = \frac{1}{M} \iiint_V y\rho(x, y, z)\, dV, \quad \bar{z} = \frac{1}{M} \iiint_V z\rho(x, y, z)\, dV \tag{1.30}$$

where M is the mass of the solid.

Definiton 1.15.6.11 (Moment of Inertia of a Solid)**.** Consider a solid V with a variable density function $\rho(x, y, z)$ that is continuous over the solid. The moment of inertia of the solid about the x-axis (I_x), y-axis (I_y), and z-axis (I_z) can be computed using the following formulas:

$$I_x = \iiint_V (y^2+z^2)\rho(x, y, z)\, dV, \quad I_y = \iiint_V (x^2+z^2)\rho(x, y, z)\, dV, \quad I_z = \iiint_V (x^2+y^2)\rho(x, y, z)\, dV \tag{1.31}$$

where $y^2 + z^2$, $x^2 + z^2$, and $x^2 + y^2$ are the squares of the distances from the respective axes of rotation, and $\rho(x, y, z)\, dV$ represents the mass element of the solid at the point (x, y, z).

Also for the moment of inertia about the origin (I_o):

$$I_o = \iiint_V (x^2 + y^2 + z^2)\rho(x, y, z)\, dV \tag{1.32}$$

In general, for an axis defined by a line $ax + by + cz + d = 0$, the moment of inertia about that axis (I_l) can be found using the following formula:

$$I_l = \iiint_V \left(\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} \right)^2 \rho(x, y, z)\, dV \tag{1.33}$$

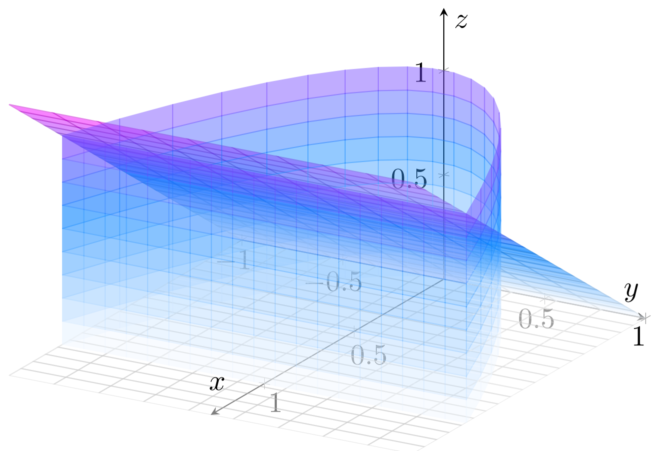

Figure 1.4: The solid bounded by the surfaces $x = y^2$, $z = 0$, $x = z$, and $x = 1$.

Example 1.15.6.12. *Find the center of mass of a solid of constant density ρ_0 that is bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$, $x = z$ and $x = 1$.* We visualize the solid as follows: We can find the mass of the solid:

$$\begin{aligned} M &= \iiint_V \rho_0 \, dV = \rho_0 \iiint_V 1 \, dV \\ &= \rho_0 \int_{-1}^{1} \int_{y^2}^{1} \int_0^x 1 \, dz \, dx \, dy \\ &= \rho_0 \cdot \frac{4}{5} \end{aligned}$$

Next, we can find the coordinates of the center of mass:

$$\begin{aligned} \bar{x} &= \frac{1}{M} \iiint_V x\rho_0 \, dV = \frac{\rho_0}{M} \iiint_V x \, dV = 5/7 \\ \bar{y} &= \frac{1}{M} \iiint_V y\rho_0 \, dV = \frac{\rho_0}{M} \iiint_V y \, dV = 0 \text{ (by symmetry)} \\ \bar{z} &= \frac{1}{M} \iiint_V z\rho_0 \, dV = \frac{\rho_0}{M} \iiint_V z \, dV = 5/14 \end{aligned}$$

Example 1.15.6.13. *Find the moment of inertia of a cyliner with radius a and height h about its central axis, assuming the cylinder has a constant density ρ_0.* We consider the base region R as the circular base of the cylinder in the xy-plane. Then, we consider how the cylinder change in the z direction over R:

$$0 \le z \le h$$

So we can set up:

$$\begin{aligned} I_z &= \iiint_V (x^2 + y^2)\rho_0 \, dV = \rho_0 \iint_R \int_0^h (x^2 + y^2) \, dz \, dA \\ &= \rho_0 h \iint_R (x^2 + y^2) \, dA \end{aligned}$$

Then, we can express the base region R in polar coordinates:

$$x^2 + y^2 = r^2 \implies 0 \le r \le a, \quad 0 \le \theta \le 2\pi$$

So we can set up:

$$\begin{aligned} I_z &= \rho_0 h \int_0^{2\pi} \int_0^a r^2 \cdot r \, dr \, d\theta \\ &= \rho_0 h \int_0^{2\pi} \left[\frac{r^4}{4} \right]_{r=0}^{r=a} d\theta = \frac{\pi \rho_0 h a^4}{2} \end{aligned}$$

This also leads to the well-known formula for the moment of inertia of a solid cylinder about its central axis:

$$I_z = \frac{1}{2} M a^2$$

where $M = \pi a^2 h \rho_0$ is the mass of the cylinder.

1.15.7 Triple Integrals in Cylindrical Coordinates

Definiton 1.15.7.1 (Cylindrical Coordinates)**.** The cylindrical coordinates of a point P in 3-D space are given by the ordered triple (r, θ, z), where:

- r is the distance from the z-axis to the projection of P onto the xy-plane,
- θ is the angle between the positive x-axis and the line segment from the origin to the projection of P onto the xy-plane,
- z is the same as in rectangular coordinates, representing the height of point P above the xy-plane.

The relationships between cylindrical coordinates and rectangular coordinates are given by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{1.34}$$

and conversely:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{1.35}$$

For the change of variables in triple integrals, the volume element dV in cylindrical coordinates is given by:

$$dV = dA \, dz = r \, dr \, d\theta \, dz \tag{1.36}$$

Example 1.15.7.2. Consider the triple integral in the cononical reigion V bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$. Changing to cylindrical coordinates, we consider the projection of the cone onto the xy-plane as our polar base region R. Then, we consider how the cone change in the z direction over R:

$$z = r \implies r \le z \le 2$$

So the region V_{polar} is:

$$V_{\text{polar}} = \{(r, \theta, z) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, r \leq z \leq 2\}$$

So we can set up:

$$\iiint_V f(x, y, z)\, dV = \int_0^{2\pi} \int_0^2 \int_r^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Let $f(x, y, z) = x^2 + .y^2 = r^2$. Then, we have:

$$\begin{aligned}\iiint_V (x^2 + y^2)\, dV &= \int_0^{2\pi} \int_0^2 \int_r^2 r^2 \cdot r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 r^3(2 - r)\, dr\, d\theta = \frac{16\pi}{5}\end{aligned}$$

1.15.8 Triple Integrals in Spherical Coordinates

Definiton 1.15.8.1 (Spherical Coordinates)**.** The spherical coordinates of a point P in 3-D space are given by the ordered triple (ρ, ϕ, θ), where:

- ρ is the distance from the origin to the point P,
- ϕ is the angle between the positive z-axis and the line segment from the origin to point P (also known as the polar angle or colatitude),
- θ is the angle between the positive x-axis and the projection of the line segment from the origin to point P onto the xy-plane (also known as the azimuthal angle).

The relationships between spherical coordinates and rectangular coordinates are given by:

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi \tag{1.37}$$

and conversely:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.38}$$

For the change of variables in triple integrals, the volume element dV in spherical coordinates is given by:

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \tag{1.39}$$

Example 1.15.8.2. *find the mass of a half sphere of radius a that has a density $k(2a - \rho)$, where k is a constant and ρ is the distance from the origin.*

Let $\lambda = k(2a - \rho)$ be the density function. Then, we consider the half sphere S in spherical coordinates:

$$S = \{(\rho, \phi, \theta) \mid 0 \leq \rho \leq a, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi\}$$

So we can set up:

$$
\begin{aligned}
M &= \iiint_S \lambda \, dV = \iiint_S k(2a-\rho)\, dV \\
&= k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a (2a-\rho)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[2a \cdot \frac{\rho^3}{3} - \frac{\rho^4}{4} \right]_{\rho=0}^{\rho=a} \sin\phi \, d\phi \, d\theta \\
&= k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\frac{2a^4}{3} - \frac{a^4}{4} \right) \sin\phi \, d\phi \, d\theta \\
&= k \cdot \frac{5a^4}{12} \int_0^{2\pi} [-\cos\phi]_{\phi=0}^{\phi=\frac{\pi}{2}} \, d\theta = k \cdot \frac{5a^4}{12} \cdot 2\pi \\
&= \frac{5\pi k a^4}{6}
\end{aligned}
$$

1.15.9 Change of Variables in Multiple Integrals

Change of Variables in Single Integrals

Consider the following integral:

$$\int_1^3 2x\sqrt{x^2+1}\, dx$$

We can use substitution to solve this integral. Let $u = x^2+1$, then $du = 2x\, dx$. Then, we carefully plug in to the integral, and notice how this could extend to multiple integrals:

$$
\begin{aligned}
\int_1^3 2x\sqrt{x^2+1}\, dx &= \int_{u=2}^{u=10} 2\sqrt{u}\sqrt{u-1} \cdot \frac{du}{2\sqrt{u-1}} = \int_{u=2}^{u=10} \sqrt{u}\, du \\
&= \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{u=2}^{u=10} = \frac{2}{3}(10^{\frac{3}{2}} - 2^{\frac{3}{2}})
\end{aligned}
$$

The key step is to express dx in terms of du:

$$dx = \frac{du}{2x}$$

Consider how the area under a curve shifts when we change the variable from x to u, we can visualize this as follows:

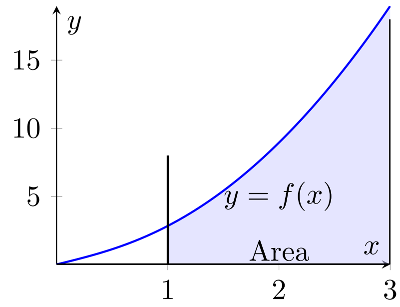

Figure 1.5: Area under the curve $y = f(x)$ from $x = 1$ to $x = 3$.

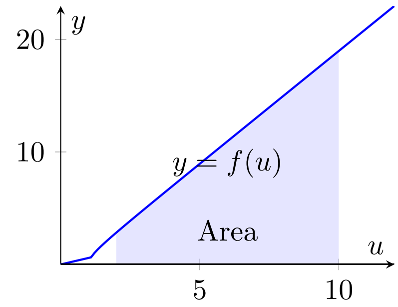

Figure 1.6: Area under the curve $y = f(u)$ from $u = 2$ to $u = 10$.

Although $2x\sqrt{x^2+1} = 2\sqrt{u}\sqrt{u-1}$, the area under the curve at any given x_0, $u_0 = x_0^2 + 1$ are not necessarily equal: $f(x_0) \neq f(u_0)$. This is because the width of each rectangle changes when we change the variable from x to u. But for small changes, we can approximate the width of each rectangle as follows:

$$\Delta u \approx \frac{du}{dx}\Delta x = 2x\Delta x$$

So, the area of each rectangle changes as follows:

$$\text{Area} \approx f(x)\Delta x = f(u)\frac{\Delta u}{2x}$$

Taking the limit as $\Delta x \to 0$ (and thus $\Delta u \to 0$), we get:

$$\text{Area} = f(x)\,dx = f(u)\frac{du}{2x}$$

This is the key idea behind the change of variables in integrals, and it can be extended to multiple integrals.

Change of Variables in Double Integrals Consider the following change of variables in double integral $\iint_R f(x,y)\,dA$:

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

Such that we call the corresponding region in the uv-plane as S. Now we assume that there exists a bijective mapping between (x,y) and (u,v), and that the functions g and h have continuous partial derivatives. Then, we can express the area element dA in terms of du and dv.

We can imagine any uniform small rectangle in the uv-plane with sides Δu and Δv. Consider how the four corners of the rectangle map to the xy-plane:

- The corner at (u,v) maps to $(x,y) = (g(u,v), h(u,v))$.
- The corner at $(u+\Delta u, v)$ maps to $(x,y) = (g(u+\Delta u, v), h(u+\Delta u, v))$.
- The corner at $(u, v+\Delta v)$ maps to $(x,y) = (g(u, v+\Delta v), h(u, v+\Delta v))$.

And we can consider the infinitesimal case as $\Delta u \to 0$ and $\Delta v \to 0$. Then, we can approximate the area of the parallelogram (we can prove this is a parallelogram as the dot product of the two sides converge to 0) formed by these four points in the xy-plane by the magnitude of the cross product of the two vectors.

First, we consider the first order taylor approximateion of g and h at the above four corners:

$$\begin{aligned} g(u+\Delta u, v) &\approx g(u,v) + g_u(u,v)\Delta u \\ h(u+\Delta u, v) &\approx h(u,v) + h_u(u,v)\Delta u \\ g(u, v+\Delta v) &\approx g(u,v) + g_v(u,v)\Delta v \\ h(u, v+\Delta v) &\approx h(u,v) + h_v(u,v)\Delta v \end{aligned}$$

And then we condsier the vecotr's form of the two sides of the parallelogram:

$$\begin{aligned} \overrightarrow{A} &= \langle g(u+\Delta u, v) - g(u,v), h(u+\Delta u, v) - h(u,v)\rangle \\ &\approx \Delta u \langle g_u(u,v), h_u(u,v)\rangle \\ \overrightarrow{B} &= \langle g(u, v+\Delta v) - g(u,v), h(u, v+\Delta v) - h(u,v)\rangle \\ &\approx \Delta v \langle g_v(u,v), h_v(u,v)\rangle \end{aligned}$$

Now, we can compute the area of the parallelogram as follows:

$$\begin{aligned} \text{Area} \approx |\overrightarrow{A} \times \overrightarrow{B}| &= \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ g_u(u,v)\Delta u & h_u(u,v)\Delta u & 0 \\ g_v(u,v)\Delta v & h_v(u,v)\Delta v & 0 \end{matrix} \right\| \\ &= |(0-0)\mathbf{i} - (0-0)\mathbf{j} + (g_u(u,v)h_v(u,v) - g_v(u,v)h_u(u,v))\Delta u \Delta v \mathbf{k}| \\ &= |g_u(u,v)h_v(u,v) - g_v(u,v)h_u(u,v)|\Delta u \Delta v \end{aligned}$$

Taking the limit as $\Delta u \to 0$ and $\Delta v \to 0$, we get:

$$dA = |g_u(u,v)h_v(u,v) - g_v(u,v)h_u(u,v)|\, du\, dv$$

which could be rewritten as the following defintion:

Definiton 1.15.9.1 (Jacobian of a Transformation)**.** Consider the transformation:

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

The Jacobian of the transformation is defined as:

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = g_u(u,v)h_v(u,v) - g_v(u,v)h_u(u,v) \tag{1.40}$$

And the area element dA in terms of du and dv is given by:

$$dA = |J|\, du\, dv$$

Additionally, we if we define the chang of variable the other way around:

$$\begin{cases} u = p(x, y) \\ v = q(x, y) \end{cases}$$

Then, the Jacobian of the transformation is defined as:

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = p_x(x, y)q_y(x, y) - p_y(x, y)q_x(x, y)$$

And the area element dA in terms of dx and dy is given by:

$$dA = \frac{1}{|J|}\, dx\, dy$$

WLOG, the Jacobian can be defined for any number of variables. The geneal Jacobian for a transformation is defined as:

$$J = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} \tag{1.41}$$

for the transformation:

$$\begin{cases} x_1 = g_1(u_1, u_2, \ldots, u_n) \\ x_2 = g_2(u_1, u_2, \ldots, u_n) \\ \vdots \\ x_n = g_n(u_1, u_2, \ldots, u_n) \end{cases}$$

Example 1.15.9.2 (Change to Polar Coordinates)**.** Consider the change of variables from rectangular coordinates (x, y) to polar coordinates (r, θ):

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

We can compute the Jacobian of this transformation as follows:

$$\begin{aligned} J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ = r(\cos^2\theta + \sin^2\theta) = r \end{aligned}$$

So, the area element dA in terms of dr and $d\theta$ is given by:

$$dA = |J|\, dr\, d\theta = r\, dr\, d\theta$$

This is consistent with the formula for the area element in polar coordinates.

Example 1.15.9.3. Cosndier the interal:

$$\iint_R (x^2 + 2xy)\, dA$$

where R the region bounded by the lines $y = 2x + 3$ and $y = 2x + 1$, $y = 5 - x$ and $y = 2 - x$.

Motivated by the bounds, we can use the change of variables:

$$\begin{cases} u = y - 2x \\ v = y + x \end{cases}$$

So, let S be the corresponding region in the uv-plane. We can find the bounds of S by plugging in the lines that bound R:

- $y = 2x + 3 \implies u = 3$
- $y = 2x + 1 \implies u = 1$
- $y = 5 - x \implies v = 5$
- $y = 2 - x \implies v = 2$

Then, we consider the Jacobian. First, we express x and y in terms of u and v:

$$\begin{cases} u = y - 2x \\ v = x + y \end{cases} \implies \begin{cases} x = \frac{1}{3}(v - u) \\ y = \frac{1}{3}(2v + u) \end{cases}$$

Then, we can compute the Jacobian of this transformation as follows:

$$\begin{aligned} J = \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} \\ &= -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3} \end{aligned}$$

So, we can express the integral in terms of u and v:

$$\begin{aligned} \iint_R (x^2 + 2xy)\, dA &= \iint_S \left(\left(\frac{1}{3}(v - u) \right)^2 + 2 \cdot \frac{1}{3}(v - u) \cdot \frac{1}{3}(2v + u) \right) |J|\, du\, dv \\ &= \iint_S \left(\frac{1}{9}(v - u)^2 + \frac{2}{9}(v - u)(2v + u) \right) \cdot \frac{1}{3}\, du\, dv \\ &= \frac{1}{27} \iint_S (5v^2 - u^2)\, du\, dv \\ &= \frac{196}{27} \end{aligned}$$

Example 1.15.9.4. *Evaluate the integral*

$$\iint_R xy\, dA$$

where R is the region bounded by curves $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$.

Motivated by the bounds, we can use the change of variables:

$$\begin{cases} u = x^2 + y^2 \\ v = x^2 - y^2 \end{cases} \implies \begin{cases} x = \sqrt{\frac{u+v}{2}} \\ y = \sqrt{\frac{u-v}{2}} \end{cases}$$

The jacobian of this transformation is:

$$\begin{aligned} J = \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{2(u+v)}} & \frac{1}{2\sqrt{2(u+v)}} \\ \frac{1}{2\sqrt{2(u-v)}} & -\frac{1}{2\sqrt{2(u-v)}} \end{vmatrix} \\ &= -\frac{1}{8}\left(\frac{1}{\sqrt{u^2-v^2}} + \frac{1}{\sqrt{u^2-v^2}}\right) = -\frac{1}{4\sqrt{u^2-v^2}} \end{aligned}$$

So, we can express the integral in terms of u and v:

$$\begin{aligned} \iint_R xy\, dA &= \iint_S \sqrt{\frac{u+v}{2}} \cdot \sqrt{\frac{u-v}{2}} \cdot |J|\, du\, dv \\ &= \iint_S \frac{\sqrt{u^2-v^2}}{2} \cdot \frac{1}{4\sqrt{u^2-v^2}}\, du\, dv = \iint_S \frac{1}{8}\, du\, dv \\ &= \int_1^4 \int_4^9 \frac{1}{8}\, dv\, du = \frac{1}{8}(4-1)(9-4) = \frac{15}{8} \end{aligned}$$

Theorem 1.15.9.5 (Back Transformation)**.** Consider the Jacobian J of a transformation:

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

If $J \neq 0$, then the Jacobian of the inverse transformation:

$$\begin{cases} u = p(x,y) \\ v = q(x,y) \end{cases}$$

is given by:

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} \tag{1.42}$$

Proof. **Sketch:** Consider a small rectangle in the uv-plane with sides Δu and Δv. The area of this rectangle is $\Delta A_{uv} = \Delta u \Delta v$. This rectangle maps to a parallelogram in the xy-plane with area $\Delta A_{xy} \approx |J| \Delta u \Delta v$. Now, consider the inverse transformation. A small rectangle in the xy-plane with sides Δx and Δy maps back to a parallelogram in the uv-plane with area $\Delta A_{uv} \approx |J^{-1}| \Delta x \Delta y$. Since these areas must be consistent under the transformations, we have:

$$|J| \Delta u \Delta v = \Delta x \Delta y \quad \text{and} \quad |J^{-1}| \Delta x \Delta y = \Delta u \Delta v$$

Dividing the first equation by the second, we get:

$$|J| = \frac{1}{|J^{-1}|}$$

□

Example 1.15.9.6. Consider the transformation and integral in 1.15.9.4. We can verify the result using the back transformation theorem. The Jacobian of the inverse transformation is:

$$\begin{aligned} J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} \\ &= -4xy - 4xy = -8xy \end{aligned}$$

So, we have:

$$J = \frac{1}{J^{-1}} = -\frac{1}{8xy} \implies I = \iint_R xy\, dA = \iint_S \frac{xy}{8xy}\, du\, dv = \iint_S \frac{1}{8}\, du\, dv = \frac{15}{8}$$

This is consistent with the result we obtained in 1.15.9.4, and results in a much simpler integral to evaluate.

Example 1.15.9.7. Consider the integral:

$$\iint_R \exp\left(\frac{x-y}{x+y}\right) dA$$

where R is the trapezoid with vertices at $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

Motivated by the integrabled, we can use the change of variables:

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

Also, we can define our bounds as follow:

- $y = x - 2 \implies u = 2$
- $y = x - 1 \implies v = 1$
- $x = 0 \implies u = -v$
- $y = 0 \implies u = v$

We can find the Jacobian of this transformation by its inverse:

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \implies J = -\frac{1}{2}$$

So, we can write the integral in terms of u and v:

$$\begin{aligned} \iint_R \exp\left(\frac{x-y}{x+y}\right) dA &= \iint_S \exp\left(\frac{v}{u}\right) \cdot |J|\, du\, dv \\ &= \frac{1}{2}\int_1^2 \int_{-v}^{v} \exp\left(\frac{v}{u}\right) du\, dv \\ &= \frac{1}{2}\int_1^2 v(e - e^{-1})\, dv = \frac{3}{4}(e - e^{-1}) \end{aligned}$$

Example 1.15.9.8. *Evaluate the integral*

$$\iint_R (x^2 - y^2) \exp(xy) \, dA$$

where R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$, and the lines $y = x$ and $y = x + 2$.

Motivated by the bounds, we can use the change of variables:

$$\begin{cases} u = xy \\ v = y - x \end{cases}$$

So, let S be the corresponding region in the uv-plane. We can find the bounds of S by plugging in the lines that bound R:

- $xy = 1 \implies u = 1$
- $xy = 4 \implies u = 4$
- $y = x \implies v = 0$
- $y = x + 2 \implies v = 2$

We find the Jacobian of this transformation by its inverse:

$$J^{-1} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -1 & 1 \end{vmatrix} = y + x \implies J = \frac{1}{y + x}$$

So, we can express the integral in terms of u and v:

$$\begin{aligned} \iint_R (x^2 - y^2) \exp(xy) \, dA &= \iint_S (x + y)(x - y) e^u \cdot |J| \, du \, dv \\ &= \int_0^2 \int_1^4 -v e^u \, du \, dv = \int_0^2 -v(e^4 - e) \, dv = 2(e - e^4) \end{aligned}$$

Example 1.15.9.9. *Find the volume of the region bounded by the hyperbolic cylinders:*

$$xy = 1, \quad xy = 3, \quad xz = 1, \quad xz = 3, \quad x = 36, \quad y = 25, \quad yz = 49$$

Motivated by the bounds, we can use the change of variables:

$$\begin{cases} u = xy \\ v = xz \\ w = yz \end{cases}$$

So, let S be the corresponding region in the uvw-space. We can find the bounds of S by plugging in the surfaces that bound the region:

$$1 \le u \le 3$$
$$1 \le v \le 3$$
$$25 \le w \le 49$$

Chapter 2

Fluid Mechanics