

ESC 195 Lecture Notes

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ESC 195

The up-to-date version of this document can be found at <https://github.com/HaysonC/skulenotes>

1 More on Integrals

1.1 Riemann Sum - Non-Uniform Partition

Example 1.1.1. Given the following definite integral:

$$\int_0^2 \sqrt{x} dx$$

, we cannot evaluate its Riemann sum with uniform partition, since the series of root cannot be easily evaluated.

The definite integral of \sqrt{x} from 0 to 2 using a Riemann sum with a non-uniform partition is given by:

$$\int_0^2 \sqrt{x} dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{x_i} \Delta x_i$$

, where:

- $x_0 = 0, x_n = 2$,
- $x_i = i^2 \cdot \frac{2}{n^2}$ for $i = 0, 1, 2, \dots, n$,
- $\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2} \cdot (2i - 1)$.

The Riemann sum becomes:

$$S_n = \sum_{i=1}^n \sqrt{i^2 \cdot \frac{2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Simplifying further:

$$S_n = \sum_{i=1}^n \sqrt{\frac{2i^2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Taking the limit as $n \rightarrow \infty$, the sum converges to the exact value of the integral using the series of squares.

$$\int_0^2 \sqrt{x} dx = \frac{4\sqrt{2}}{3}.$$

Condition $n \rightarrow \infty$ Ensures $\Delta x_i \rightarrow 0$

As $n \rightarrow \infty$, the partition points x_i become increasingly dense. This ensures that the partition becomes infinitely fine.

1.2 Integration By Parts

Using the product rule:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x),$$

integrating both sides with respect to x gives:

$$u(x)v(x) = \int u'(x)v(x) dx + \int u(x)v'(x) dx.$$

Rearranging this:

$$\int u'(x)v(x) dx = u(x)v(x) - \int u(x)v'(x) dx.$$

Integration by parts formula

$$\int u dv = uv - \int v du. \tag{1}$$

Example 1.2.1. We want to solve the integral

$$\int xe^{2x} dx$$

using integration by parts.

Let:

$$u = x, \quad dv = e^{2x} dx.$$

Then, we compute the derivatives and integrals:

$$du = dx, \quad v = \frac{e^{2x}}{2}.$$

Now, apply the integration by parts formula:

$$\int u dv = uv - \int v du.$$

Substituting in the values:

$$\int xe^{2x} dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} dx.$$

1.2 Integration By Parts

Next, compute the remaining integral:

$$\int \frac{e^{2x}}{2} dx = \frac{e^{2x}}{4}.$$

Thus, the result is:

$$\int x e^{2x} dx = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Example 1.2.2. We want to solve

$$\int x^2 \sin(2x) dx$$

using double integration by parts.

First, let:

$$u = x^2, \quad dv = \sin(2x) dx.$$

Then:

$$du = 2x dx, \quad v = -\frac{1}{2} \cos(2x).$$

Using the IBP formula:

$$\int u dv = uv - \int v du,$$

we get:

$$\int x^2 \sin(2x) dx = -\frac{x^2}{2} \cos(2x) + \int x \cos(2x) dx.$$

Now, apply IBP again to $\int x \cos(2x) dx$, let:

$$u = x, \quad dv = \cos(2x) dx.$$

Then:

$$du = dx, \quad v = \frac{1}{2} \sin(2x).$$

Using the IBP formula again:

$$\int x \cos(2x) dx = \frac{x}{2} \sin(2x) - \int \frac{1}{2} \sin(2x) dx,$$

and solving the remaining integral:

$$\int \frac{1}{2} \sin(2x) dx = -\frac{1}{4} \cos(2x).$$

Thus, the final result is:

$$\int x^2 \sin(2x) dx = -\frac{x^2}{2} \cos(2x) + \frac{x}{2} \sin(2x) + \frac{1}{4} \cos(2x) + C.$$

1.3 Trigonometric Integrals

Case I This is the case I trigonometric integrals, the strategy is using the pythagorean identity.

We want to solve the class of integrals:

$$\int \sin^n(x) \cos^m(x) dx \quad (2)$$

, where m or n is odd.

Example 1.3.1. We want to solve the integral:

$$I = \int \sin^3(x) \cos^2(x) dx$$

We use the identity: $\sin^2(x) = 1 - \cos^2(x)$. Thus, the integral becomes:

$$\begin{aligned} I &= \int \sin(x)(1 - \cos^2(x)) \cos^2(x) dx \\ &= \int (\cos^2(x) \sin(x) - \cos^4(x) \sin(x)) dx \end{aligned}$$

which is now easily solvable with substitution $u = \cos(x)$.

Case II The follwoing is generally solvable via case I and case III below. In general, we solve the integral by reducing the power of the trigonometric functions to arrive at a solvable integral Case I or III.

We want to solve the class of integrals:

$$\int \sin^n(x) \cos^m(x) dx \quad (3)$$

, where m and n is even.

Example 1.3.2. We want to solve the integral:

$$I = \int \sin^2(x) \cos^4(x) dx$$

We can apply the double angle formulas:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x) \quad (4)$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad (5)$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \quad (6)$$

Thus, the integral becomes:

$$I = \frac{1}{8} \int \sin^2(2x) dx + \frac{1}{8} \int \cos(2x) \sin^2(2x) dx$$

1.3 Trigonometric Integrals

Case III This is the case III trigonometric integrals, the strategy is using the reduction formula via IBP.

Reduction Formula We can solve integrals by reducing the power of the trigonometric functions. These are done using IBP and trigonometric identities.

We want to solve the classes of integrals:

$$\int \sin^n(x) dx, \quad \int \cos^n(x) dx \quad (7)$$

, where n is a positive integer. For demonstration, We can obtain the reduction formula of $\sin^n(x)$ via IBP.

$$\begin{aligned} I_n &= \int \sin^n(x) dx \\ &= \int \sin^{n-1}(x) \sin(x) dx \\ &= \frac{-\cos(x) \sin^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \cos^2(x) \sin^{n-2}(x) dx \\ &= \frac{-\cos(x) \sin^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2} \end{aligned} \quad (8)$$

Reduction for $\cos^n(x)$ We can obtain the reduction formula of $\cos^n(x)$ via IBP.

$$\begin{aligned} I_n &= \int \cos^n(x) dx \\ &= \int \cos^{n-1}(x) \cos(x) dx \\ &= \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \sin^2(x) \cos^{n-2}(x) dx \\ &= \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2} \end{aligned} \quad (9)$$

Case IV The following is generally solvable via simple trigonometric integrals. In general, we solve the integral by applying the angle sum formulas.

We want to solve the classes of integrals:

$$\int \sin(mx) \cos(nx) dx, \quad \int \sin(mx) \sin(nx) dx, \quad \int \cos(mx) \cos(nx) dx \quad (10)$$

1.3 Trigonometric Integrals

We could apply the angle sum formulas:

$$\sin(mx) \cos(nx) = \frac{1}{2} [\sin((m+n)x) + \sin((m-n)x)] \quad (11)$$

$$\sin(mx) \sin(nx) = \frac{1}{2} [\cos((m-n)x) - \cos((m+n)x)] \quad (12)$$

$$\cos(mx) \cos(nx) = \frac{1}{2} [\cos((m-n)x) + \cos((m+n)x)] \quad (13)$$

Case V The following is generally solvable via the following trigonometric identities listed below, which convert it into a reduction formula.

$$\tan^2(x) = \sec^2(x) - 1 \quad (14)$$

$$\cot^2(x) = \csc^2(x) - 1 \quad (15)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x) \quad (16)$$

We want to solve the classes of integral:

$$\int \tan^n(x) dx, \quad \int \cot^n(x) dx \quad (17)$$

We know that $\tan^2(x) = \sec^2(x) - 1$. Thus, we can solve the integral by reducing the power of the tangent function.

Reduction for $\tan^n(x)$ We can obtain the reduction formula of $\tan^n(x)$ via the trigonometric identities.

$$\begin{aligned} I_n &= \int \tan^n(x) dx \\ &= \int \tan^{n-1}(x) \tan(x) dx \\ &= \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x) dx \\ &= \frac{\tan^{n-1}(x)}{n-1} - I_{n-2} \end{aligned} \quad (18)$$

Reduction for $\cot^n(x)$ We can obtain the reduction formula of $\cot^n(x)$ via the trigonometric identities.

$$\begin{aligned} I_n &= \int \cot^n(x) dx \\ &= \int \cot^{n-1}(x) \cot(x) dx \\ &= \frac{\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x) dx \\ &= \frac{\cot^{n-1}(x)}{n-1} - I_{n-2} \end{aligned} \quad (19)$$

1.3 Trigonometric Integrals

Case VI The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x) \sec^n(x) dx, \quad \int \cot^m(x) \csc^n(x) dx \quad (20)$$

We can solve the integral by reducing the power of the converted trigonometric functions using (14) and (15).

Case VII The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x) \sec^n(x) dx, \quad \int \cot^m(x) \csc^n(x) dx \quad (21)$$

We can solve the integral by converting between the tangent and secant functions using the trigonometric identities.

Case VIII This is the case VIII integrals, the strategy is using the trigonometric substitution.

Trigonometric Substitution We can solve integrals by substituting the trigonometric functions with other trigonometric functions.

Example 1.3.3. We want to solve the integral:

$$\int \frac{dx}{\sqrt{1-x^2}}$$

We can substitute $x = \sin(\theta)$, then $dx = \cos(\theta) d\theta$. The integral becomes:

$$\int \frac{\cos(\theta) d\theta}{\sqrt{1-\sin^2(\theta)}} = \int \frac{\cos(\theta) d\theta}{\cos(\theta)} = \int d\theta = \theta + C.$$

In general, we can use the following substitutions:

- $x = a \sin(\theta)$ for $\sqrt{a^2 - x^2}$,
- $x = a \tan(\theta)$ for $\sqrt{a^2 + x^2}$,
- $x = a \sec(\theta)$ for $\sqrt{x^2 - a^2}$.

1.4 Partial Fractions

Weierstrass Substitution We can solve integrals by using the Weierstrass substitution:

$$\tan\left(\frac{x}{2}\right) = t \Leftrightarrow \cos(x) = \frac{1-t^2}{1+t^2} \quad \sin(x) = \frac{2t}{1+t^2} \quad dx = \frac{2dt}{1+t^2} \quad (22)$$

This substitution is useful for solving integrals with trigonometric functions, by converting them into rational functions.

Summary We can solve trigonometric integrals by using the following strategies:

Case	Strategy and General Form
I	Use $\sin^2(x) + \cos^2(x) = 1$; simplify using substitution. General Form: $\int \sin^n(x) \cos^m(x) dx$, where m or n is odd.
II	Convert to Case I and Case III using double angle formulas. General Form: $\int \sin^n(x) \cos^m(x) dx$, where both m and n are even.
III	Apply reduction formulas derived via integration by parts. General Form: $\int \sin^n(x) dx$ or $\int \cos^n(x) dx$.
IV	Use angle sum formulas to simplify. General Form: $\int \sin(mx) \cos(nx) dx$.
V	Reduce tangent/cotangent powers using $\tan^2(x) = \sec^2(x) - 1$ and substitution. General Form: $\int \tan^n(x) dx$ or $\int \cot^n(x) dx$.
VI	Convert to Case V via Pythagorean identities. General Form: $\int \tan^m(x) \sec^n(x) dx$ or $\int \cot^m(x) \csc^n(x) dx$.
VII	Convert between tangent and secant functions for simplification. General Form: $\int \tan^m(x) \sec^n(x) dx$.
VIII	Use trigonometric substitution: $x = a \sin(\theta), a \tan(\theta), a \sec(\theta)$. General Form: $\int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{\sqrt{a^2+x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$.

Table 1: Strategies and General Forms for Case I-VIII Integrals

1.4 Partial Fractions

Partial Fraction Decomposition We can decompose a rational function into partial fractions to simplify integration.

We want to solve the integral:

$$\int \frac{P(x)}{Q(x)} dx \quad (23)$$

, where $P(x)$ and $Q(x)$ are polynomials.

Example 1.4.1. We want to solve the integral:

$$\int \frac{2x^2 + 3x + 1}{x^3 + 2x^2 + x} dx$$

1.5 Improper Integrals

We can decompose the rational function into partial fractions:

$$\frac{2x^2 + 3x + 1}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

We can solve for the constants A , B , and C by equating the coefficients of the partial fractions to the original function. We have:

$$2x^2 + 3x + 1 = A(x+1)^2 + Bx(x+1) + Cx$$

Thus, we can solve it by setting $x = 0$, $x = -1$

Summary We can solve integrals using partial fraction decomposition by following these steps:

Step	Description
1	Factorize the denominator of the rational function.
2	Write the partial fraction decomposition.
3	Solve for the constants by equating the coefficients.
4	Integrate the partial fractions.

Table 2: Steps for Partial Fraction Decomposition

1.5 Improper Integrals

Improper Integral An improper integral is an integral with an infinite limit or a discontinuity in the interval of integration.

We want to solve the integral:

$$\int_a^b f(x) dx \tag{24}$$

, where a or b is infinite or $f(x)$ is discontinuous.

Example 1.5.1. We want to solve the integral:

$$\int_0^\infty e^{-x} dx$$

Evaluating the integral:

$$\begin{aligned} \int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} -e^{-x} \Big|_0^b \\ &= \lim_{b \rightarrow \infty} -e^{-b} + 1 \\ &= 1 \end{aligned}$$

1.6 Convergence Test

Convergence An improper integral converges if the limit of the integral exists.

Comparison Test We can compare the integral to another integral to determine convergence or divergence.

Splitting the Integral We can split the integral into two parts to determine convergence. We split it into two limits on the left and right, for example:

$$\int_a^b f(x) dx = \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

2 Hyperbolic Trigonometric Functions

Definiton 2.0.1 (Hyperbolic Sine). The hyperbolic sine function is defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \quad (25)$$

Definiton 2.0.2 (Hyperbolic Cosine). The hyperbolic cosine function is defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \quad (26)$$

Properties These combinations of exponential functions have properties similar to the trigonometric functions.

Derivatives The derivatives of the hyperbolic functions are:

$$\frac{d}{dx} \sinh(x) = \cosh(x), \quad (27)$$

$$\frac{d}{dx} \cosh(x) = \sinh(x). \quad (28)$$

Identities The hyperbolic functions satisfy the following identities:

$$\cosh^2(x) - \sinh^2(x) = 1, \quad (29)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \quad (30)$$

$$\sinh(2x) = 2 \sinh(x) \cosh(x). \quad (31)$$

Hyperbola The hyperbolic functions are related to the hyperbola $x^2 - y^2 = 1$. Simiar to the circle, the hyperbola can be parametrized by the hyperbolic functions (e.g. $x = \cosh(t)$, $y = \sinh(t)$).

Area The area of a sector of the hyperbola is given by:

$$A = t/2, \quad (32)$$

where t is the angle of the sector along the parametrization $\{(x, y) \mid x = \cosh(t), y = \sinh(t)\}$.

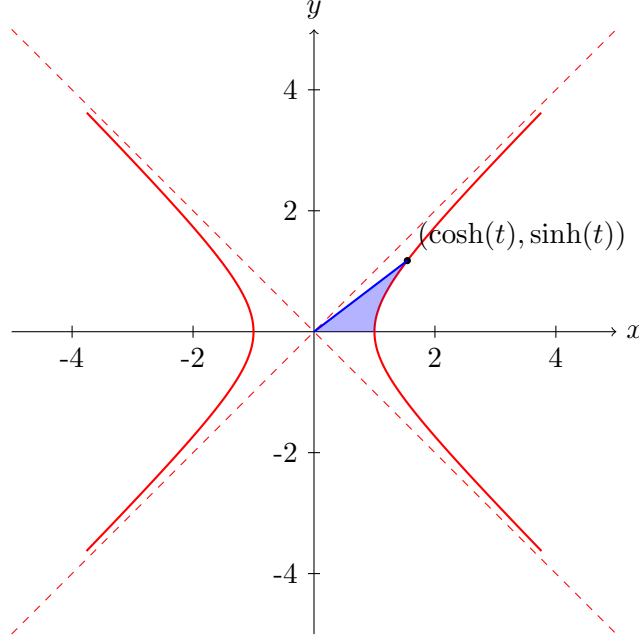


Figure 1: Sector of the hyperbola

Catenary The hyperbolic cosine function describes the shape of a hanging chain or cable. The catenary is the curve formed by a chain hanging from two points. It is given by the equation:

$$a \cosh\left(\frac{x}{a}\right), \quad (33)$$

Definiton 2.0.3 (Hyperbolic Tangent). The hyperbolic tangent function is defined as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \quad (34)$$

Derivative The derivative of the hyperbolic tangent function resembles the derivative of the regular tangent function:

$$\frac{d}{dx} \tanh(x) = \text{sech}^2(x). \quad (35)$$

Identities The hyperbolic tangent function satisfies the following identities:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad (36)$$

$$\text{sech}^2(x) = 1 - \tanh^2(x). \quad (37)$$

Secant, Cosecant, Cotangent The hyperbolic secant, cosecant, and cotangent functions are defined similarly to the regular secant, cosecant, and cotangent functions. They are the **reciprocal** of the hyperbolic cosine, sine, and tangent functions, respectively.

Inverse Hyperbolic Functions The inverse hyperbolic functions are defined as the inverse of the hyperbolic functions. They are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$, etc.

$$\operatorname{arsinh} x = \ln \left(x + \sqrt{x^2 + 1} \right) \quad (38)$$

$$\operatorname{arcosh} x = \ln \left(x + \sqrt{x^2 - 1} \right) \quad (39)$$

$$\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \quad (40)$$

$$\operatorname{arcsch} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) \quad (41)$$

$$\operatorname{arsech} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) \quad (42)$$

$$\operatorname{arcoth} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) \quad (43)$$

3 Further Applications of Integrals

3.1 Arc Length

Arc Length Given the curve $C : y = f(x)$, the arc length of the curve between two points a and b , where $y'(x)$ is continuous, we would like to compute the arc length of the curve:

$$L = \int_C ds$$

Formula We can derive the formula for the arc length of a curve $y = f(x)$ between two points a and b as:

At x_i and x_{i+1} , the length of the segment is:

$$\begin{aligned} \Delta s &\approx \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \\ &= \sqrt{\Delta_i x^2 + \Delta_i y^2} \end{aligned}$$

3.2 Surface Area of Revolution

We also have, by MVT:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= y'(c) \\ \Delta y &= y'(c) \Delta x \\ \Delta s &\approx \sqrt{\Delta x^2 + (y'(c) \Delta x)^2} \\ &= \sqrt{1 + y'(c)^2} \Delta x\end{aligned}$$

Taking a Riemann sum:

$$\begin{aligned}L &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + y'(c_i)^2} \Delta x \\ &= \int_a^b \sqrt{1 + y'(x)^2} dx\end{aligned}\tag{44}$$

Example 3.1.1. Given $f(x) = x^{\frac{3}{2}}$ between $x = 0$ and $x = 44$, we would like to compute the arc length of the curve:

$$\begin{aligned}L &= \int_0^{44} \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} dx \\ &= \int_0^{44} \sqrt{1 + \frac{9}{4}x} dx \\ &= \frac{8}{27} \left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \Big|_0^{44} = 296\end{aligned}$$

Inverse Also, we can take $x = f^{-1}(y)$ to compute the arc length of a curve.

$$L = \int_{f(a)}^{f(b)} \sqrt{1 + ((f^{-1})'(y))^2} dy\tag{45}$$

3.2 Surface Area of Revolution

Surface Area of Revolution Given the curve $C : y = f(x)$, we would like to compute the surface area of the curve between two points a and b when rotated about the x -axis:

$$A_x = \int_C 2\pi y ds$$

Formula Deriving the formula for the surface area of a curve $y = f(x)$ between two points a and b rotated about the x -axis as follows:

At x_i and x_{i+1} , the length of the segment is:

$$\Delta s \approx \sqrt{\Delta_i x^2 + \Delta_i y^2}$$

3.3 Applications to Physics and Engineering

The surface area of the segment is, by continuity:

$$A_i = \pi(y_i + y_{i+1})\Delta s = 2\pi y^* \Delta s \quad (46)$$

As the difference between y^* and y_i diminishes as $\Delta x \rightarrow 0$, taking a Riemann sum:

$$\begin{aligned} A_x &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi y_i \sqrt{\Delta_i x^2 + \Delta_i y^2} \\ &= \int_a^b 2\pi y \sqrt{1 + y'(x)^2} dx \end{aligned} \quad (47)$$

The surface area of a curve $y = f(x)$ (i.e. $x = f^{-1}$) between two points $f(a)$ and $f(b)$ rotated about the y -axis is:

$$A_y = \int_{f(a)}^{f(b)} 2\pi x \sqrt{1 + x'(y)^2} dy \quad (48)$$

Example 3.2.1. Given $y = \sqrt{x}$ between $x = 0$ and $x = 1$, we would like to compute the surface area of the curve:

$$\begin{aligned} A_x &= \int_0^1 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} dx \\ &= \pi \int_0^1 \sqrt{4x + 1} dx \end{aligned}$$

Let $u = 4x + 1$, then $du = 4dx$:

$$\begin{aligned} &= \frac{\pi}{4} \int_1^5 \sqrt{u} du \\ &= \frac{\pi}{6} \left(5^{\frac{3}{2}} - 1 \right) \end{aligned}$$

3.3 Applications to Physics and Engineering

3.3.1 Hydrostatic Pressure and Force

Hydrostatic Pressure The hydrostatic pressure and Force is given by the Archimedes' principle:

$$P = \rho g \cdot d, \quad (49a)$$

$$F = \rho g \cdot d \cdot A \quad (49b)$$

Riemann Sum Given a tank rectangular with width defined by $w = f(x)$, we can take a horizontal slice, x^* , of the tank to compute the force exerted on the tank by the water:

$$F = \sum_{i=1}^n \rho g x_i f(x_i) \Delta x \quad (50)$$

3.3 Applications to Physics and Engineering

Formula Given a tank rectangular with width defined by $w = f(x)$, where x is depth, we would like to compute the force exerted on the tank by the water:

$$F = \int_0^b \rho g x f(x) \cdot dx \quad (51)$$

, which is derived by considering dx as the depth a horizontal slice of the tank.

3.3.2 Center of Mass and Moments of Inertia

Properties of the Center of Mass The center of mass of a region R has the following properties:

1. **Symetry:** For all axis of symmetry, the center of mass lies on the axis.
2. **Additivity:** The center of mass of a region is the weighted average of the centers of mass of its parts. The weights are the areas of the parts.

$$\bar{x} = \sum_i \frac{A_i \bar{x}_i}{A} \quad (52)$$

Formula The centroid of a uniformly-dense region R bounded by $y = f(x)$, $x \in [a, b]$, $y \in [f(a), f(b)]$ is derived as follows:

We first have x_i^* and A_i as follows:

$$x_i^* = \frac{1}{2}(x_i + x_{i+1}) \quad A_i = f(x_i)\Delta x$$

For \bar{x} , we have:

$$A\bar{x} \approx \sum_i x_i^* A_i$$

Taking a Riemann sum:

$$\begin{aligned} A\bar{x} &= \lim_{n \rightarrow \infty} \sum_i x_i^* A_i \\ \bar{x} &= \frac{1}{A} \int_a^b x f(x) dx \end{aligned} \quad (53)$$

Similarly, for \bar{y} :

$$\begin{aligned} A\bar{y} &= \lim_{n \rightarrow \infty} \sum_i f(x_i^*) A_i \\ \bar{y} &= \frac{1}{A} \int_a^b \frac{1}{2} (f(x))^2 dx \end{aligned} \quad (54)$$

3.4 Applications to Economics and Biology

Theorem 3.3.1 (Pappus's Theorem). The volume of a solid of revolution is given by the product of the area of the region and the distance, R , traveled by the centroid of the region from the axis of rotation.

$$V = 2\pi RA \quad (55)$$

Example 3.3.2 (Elliptical Torus). Given an elliptical torus (tall donut) with major axis a and minor axis b , we would like to compute the volume of the torus:

$$\begin{aligned} V &= 2\pi RA \\ &= 2\pi R \cdot \pi ab = 2\pi^2 abR \end{aligned}$$

This could be, alternatively, shown via the washer method and the shell method.

3.4 Applications to Economics and Biology

“Biologists are so bad at math” - Prof. Davis as he proceeds to erase the board and skip to the next section.

4 Parametric Equations and Polar Coordinates

4.1 Curves Defined by Parametric Equations

Parametric Equations For $t \in \mathbb{R}$, we have:

$$x = x(t), \quad y = y(t) \quad (56)$$

Example 4.1.1 (Newton's Laws of Motion). Given $\ddot{x} = 0$, $\ddot{y} = -g$, we have:

$$\begin{aligned} x(t) &= A_1 t + A_2, \\ y(t) &= -\frac{1}{2}gt^2 + B_1 t + B_2 \end{aligned}$$

At initial conditions $t = 0$, we have $x(0) = 0$, $y(0) = 0$, $\dot{x}(0) = v_0 \cos(\theta)$, $\dot{y}(0) = v_0 \sin(\theta) - \frac{1}{2}gt^2$.

Example 4.1.2 (Ellipse). Given $x = a \cos(t)$, $y = b \sin(t)$, we have:

$$\begin{aligned} x^2 &= a^2 \cos^2(t), \\ y^2 &= b^2 \sin^2(t) \end{aligned}$$

, which satisfies $x^2/a^2 + y^2/b^2 = 1$.

Intersections of Parametric Curves Given two parametric curves C_1 , C_2 , defined by:

$$\begin{aligned} C_1 : x_1 &= x_1(t), \quad y_1 = y_1(t) \\ C_2 : x_2 &= x_2(t), \quad y_2 = y_2(t) \end{aligned}$$

, we can find the intersection points by solving the system of equations:

$$\begin{aligned}x_1(t) &= x_2(t) \\ y_1(t) &= y_2(t)\end{aligned}$$

Example 4.1.3. Given $x_1 = 2t + t$, $y_1 = 5 - 4t$, $x_2 = 3 - 5 \cos(\pi t)$, $y_2 = 1 + 5 \sin(\pi t)$, we have:

$$\begin{aligned}2t + 1 &= 3 - 5 \cos(\pi t) \\ 5 - 4t &= 1 + 5 \sin(\pi t)\end{aligned}$$

4.2 Calculus with Parametric Curves

Tangents The tangent to a parametric curve C at a point t is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \quad (57)$$

Equation of the Tangent The equation of the tangent to a parametric curve C at a point t_0 is given by:

$$y'(t_0)(x - x(t_0)) - x'(t_0)(y - y(t_0)) = 0 \quad (58)$$

Hence, if $x'(t_0) = 0$, we have a vertical tangent; if $y'(t_0) = 0$, we have a horizontal tangent. However, if $x'(t_0) = 0$ and $y'(t_0) = 0$, we have no information.

Example 4.2.1. Let $x = \sin 2t$, $y = \sin t$, we have:

$$\begin{aligned}x'(t) &= 2 \cos 2t, \\ y'(t) &= \cos t\end{aligned}$$

Vertical Tangent: $2 \cos 2t = 0 \Rightarrow t = \frac{\pi}{4} + \frac{n\pi}{2}$. **Horizontal Tangent:** $\cos t = 0 \Rightarrow t = \frac{\pi}{2} + n\pi$. **At** $t = 0$ we have $x' = y' = 0$. Thus, we have no information. On the graph, we see that the curve intersects itself at $t = 0$.

Area under Parametric Curve The area under a parametric curve C between t_1 and t_2 is given by:

$$A = \int_{x(t_1)}^{x(t_2)} y(x) dx = \int_{t_1}^{t_2} y(t)x'(t) dt \quad (59)$$

Definiton 4.2.2 (Orientation). The orientation of a parametric curve C is given by the direction of the curve as t increases. If the enclosed area is to the left of the trace of t (counter-clockwise) the orientation is positive; otherwise, it is negative.

The sign of the area of a parametric curve C is given by the orientation of the curve.

Area of Closed Curves The area of a closed parametric curve C is given by:

$$A = \oint_C dA = \int_{t_1}^{t_2} y(t)x'(t) dt = \int_{t_1}^{t_2} x(t)y'(t) dt \quad (60)$$

, where

$x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$ and t_2 is the smallest $t > t_1$ to satisfy the condition.

Arc Length The arc length of a parametric curve C is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (61)$$

Example 4.2.3. Given $x = t \cos t$, $y = t \sin t$, we have:

$$\begin{aligned} \frac{dx}{dt} &= \cos t - t \sin t, \\ \frac{dy}{dt} &= \sin t + t \cos t \end{aligned}$$

Thus, the arc length is given by:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} dt \\ &= \int_0^{2\pi} \sqrt{1 + t^2} dt \end{aligned}$$

Using the substitution $t = \tan \theta$, we have:

$$= \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln(2\pi + \sqrt{1 + 4\pi^2})$$

Surface Area of Revolution The surface area of a parametric curve C rotated about the x -axis is given by:

$$A = \int_a^b 2\pi y(t) ds$$

From the arc length formula, we have $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$, thus:

$$= \int_a^b 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (62)$$

Similarly, the surface area of a parametric curve C rotated about the y -axis is given by:

$$A = \int_a^b 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (63)$$

Example 4.2.4 (Surface Area of of a Ellipse). Given $x = a \sin t$, $y = b \cos t$, we have:

$$\begin{aligned}\frac{dx}{dt} &= a \cos t, \\ \frac{dy}{dt} &= -b \sin t\end{aligned}$$

Thus, the surface area of the ellipse is given by:

$$\begin{aligned}A &= \int_0^{2\pi} 2\pi b \cos t \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} dt \\ &= 2\pi b \int_0^{2\pi} \sqrt{a^2(1 - \sin^2 t) + b^2 \sin^2 t} dt \\ &= 2\pi b \int_0^{2\pi} \sqrt{a^2 + b^2 - (a^2 - b^2) \sin^2 t} dt\end{aligned}$$

This has no analytical solution, but can be solved numerically. Typically, we take $\epsilon = \sqrt{\frac{a^2 - b^2}{a^2}}$.

5 Polar Coordinates

Polar Coordinates Given a point P in the plane, we can define the polar coordinates of P as (r, θ) , where r is the distance from the origin to P and θ is the angle between the positive x -axis and the line segment from the origin to P .

Transformation We can convert between polar and Cartesian coordinates as follows:

$$x = r \cos \theta, \quad y = r \sin \theta \tag{64}$$

Reverse Transformation We can convert between Cartesian and polar coordinates as follows:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) \tag{65}$$

Note The angle θ is not unique, as $\theta + 2\pi n$ for $n \in \mathbb{Z}$ represents the same point, beware of this when converting between polar and Cartesian coordinates.

Example 5.0.1 (Lines).

$$\begin{aligned}y = mx + b &\implies \theta = \alpha = \arctan m \\ x = a &\implies r = a \sec \theta \\ y = a &\implies r = a \csc \theta\end{aligned}$$

Example 5.0.2 (Circles).

$$x^2 + y^2 = a^2 \implies r = a$$

5.1 Graphing in Polar Coordinates

Let $r = 6 \sin \theta$, we have:

$$\begin{aligned}r^2 &= 6r \sin \theta \\x^2 + y^2 &= 6y \\x^2 + (y - 3)^2 &= 9\end{aligned}$$

We deduce that the curve is a circle with radius 3 and center $(0, 3)$.

Cylindrical Coordinates Given a point P in space, we can define the cylindrical coordinates of P as (r, θ, z) , where r is the distance from the z -axis to P , θ is the angle between the positive x -axis and the projection of the line segment from the origin to P onto the xy -plane, and z is the distance from the xy -plane to P .

Spherical Coordinates Given a point P in space, we can define the spherical coordinates of P as (ρ, θ, ϕ) , where ρ is the distance from the origin to P , θ is the angle between the positive x -axis and the projection of the line segment from the origin to P onto the xy -plane, and ϕ is the angle between the positive z -axis and the line segment from the origin to P .

5.1 Graphing in Polar Coordinates

Example 5.1.1. Let $r = \frac{1}{2} + \cos \theta$, we first figure out the origin of the curve by setting $r = 0$:

$$\begin{aligned}0 &= \frac{1}{2} + \cos \theta \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}\end{aligned}$$

We then deduce the maximum and minimum of the curve to the origin by setting $\frac{dr}{d\theta} = 0$:

$$\begin{aligned}\frac{dr}{d\theta} &= -\sin \theta = 0 \\ \theta &= 0, \pi, 2\pi\end{aligned}$$

We also look for symmetry in the curve by checking $r(\theta) = r(-\theta)$:

$$\frac{1}{2} + \cos \theta = \frac{1}{2} + \cos(-\theta)$$

Thus, the curve is symmetric about the x -axis. We also look for symmetry about the y -axis by checking $r(\theta) = r(\pi + \theta)$:

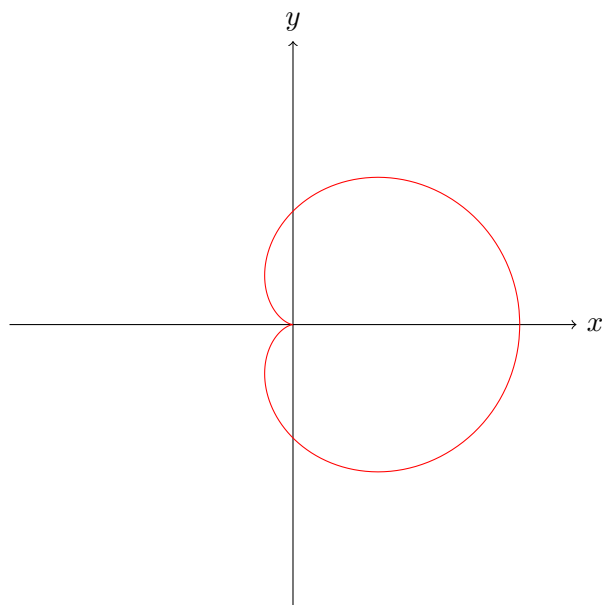
$$\frac{1}{2} + \cos \theta \neq \frac{1}{2} + \cos(\pi + \theta)$$

Thus, the curve is not symmetric about the y -axis. We also look for intervals in which the curve is increasing or decreasing by checking $r'(\theta) > 0$ or $r'(\theta) < 0$.

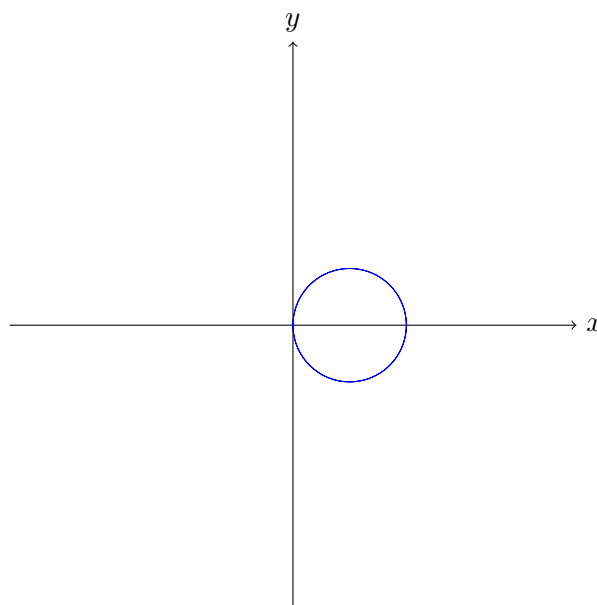
Common Polar Curves Some common polar curves are:

1. **Cardioid:** $r = a(1 + \cos \theta)$ (heart-shaped)
2. **Circle:** $r = a \cos \theta$
3. **Limasçons:** $r = a + b \sin \theta$ (heart with a hole)
4. **Lemniscate:** $r^2 = a^2 \sin 2\theta$ (infinity symbol, diagonal) $r^2 = a^2 \cos 2\theta$ (infinity symbol, horizontal)
5. **Petal Curves:** $r = a \cos n\theta$ (mn -petal flower) $r = a \sin n\theta$ (mn -petal flower); $m = 2$ if n is even, $m = 1$ if n is odd.

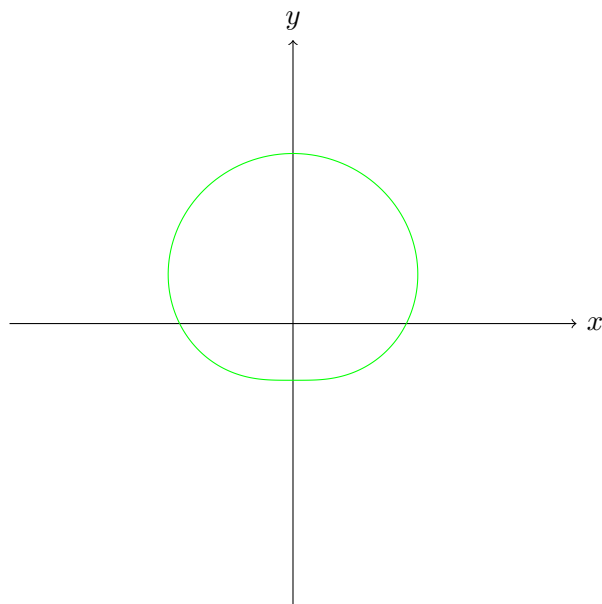
Below are some examples of polar curves:



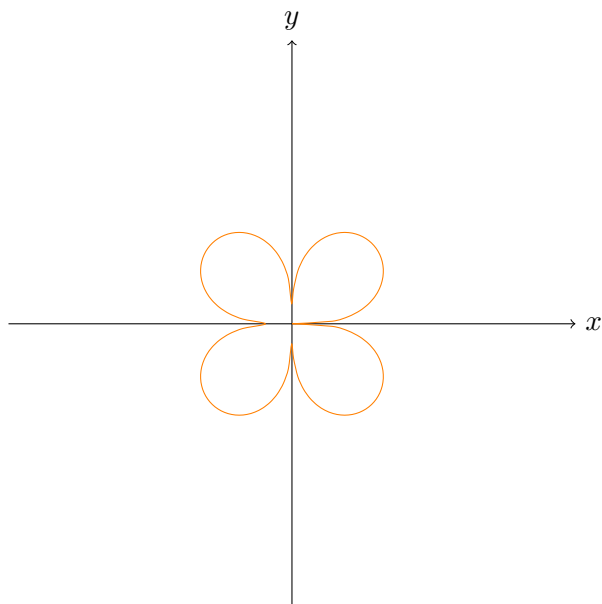
Cardioid: $r = 1 + \cos \theta$



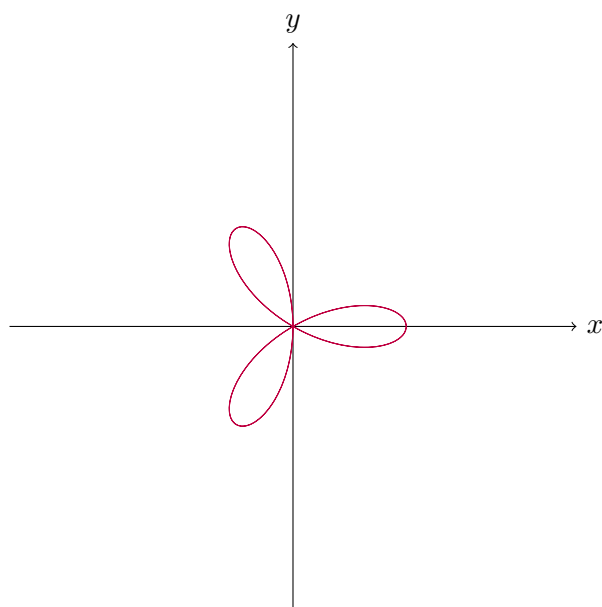
Circle: $r = \cos \theta$



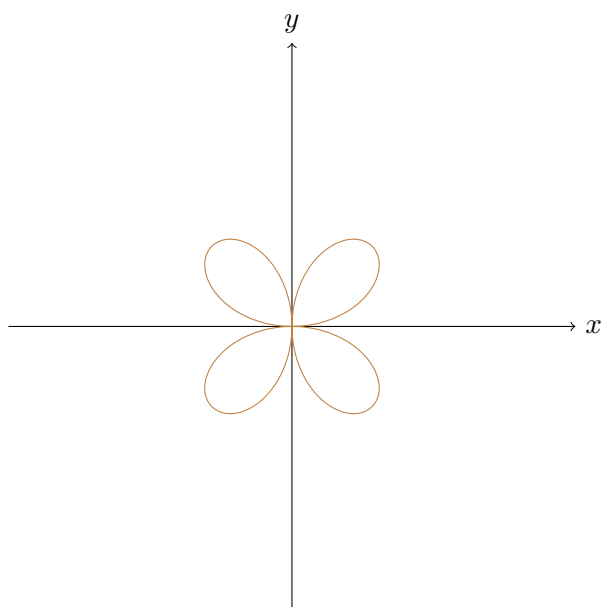
Limaçon: $r = 1 + 0.5 \sin \theta$



Lemniscate: $r^2 = \sin 2\theta$



Petal Curve: $r = \cos 3\theta$



Petal Curve: $r = \sin 2\theta$

A detailed list of polar curves and their properties could be found in Appendix B.

5.2 The intersection of Polar Curves

Example 5.2.1. Let $r_1 = \sin \theta$, $r_2 = -\cos \theta$, we have:

$$\begin{aligned}\sin \theta &= -\cos \theta \\ \tan \theta &= -1 \\ \theta &= \frac{3\pi}{4}, \frac{7\pi}{4}\end{aligned}$$

At these points, we can find the x and y coordinates of the intersection points. But also, we must check for the origin point $(x, y) = (0, 0)$. Also $r_1 = r_2$ is not always reliable so we have to check it intuitively.

5.3 Area and Length in Polar Coordinates

Area in Polar Coordinates The area of a region $R : r = g(\theta)$ in polar coordinates is given by:

$$A \approx \sum_i \frac{1}{2} [g(\theta_i)]^2 \Delta \theta_i$$

Taking a Riemann sum:

$$= \int_{\alpha}^{\beta} \frac{1}{2} [g(\theta)]^2 d\theta \quad (66)$$

Example 5.3.1. Given $r = 1 + \cos \theta$, we have:

$$\begin{aligned}A &= \int_0^{2\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 2 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{1}{2} \left(2\pi + 2 \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \Big|_0^{2\pi} = \frac{3\pi}{2}\end{aligned}$$

Area between Two Polar Curves The area between two polar curves $r^2 = f(\theta)$ and $r = g(\theta)$ is given by:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)^2 - g(\theta)^2] d\theta \quad (67)$$

Example 5.3.2. Given lemniscate $r^2 = 4 \cos 2\theta$, circle $r = 1$, to find the area between the two curves, outside the circle, and inside the lemniscate. We have:

We first consider the intersection points

$$\begin{aligned}4 \cos 2\theta &= 1 \\ \cos 2\theta &= \frac{1}{4}\end{aligned}$$

5.3 Area and Length in Polar Coordinates

We have points at $\theta = \pm 0.659$. Also, since the lemniscate is symmetric about the y -axis, we have:

$$\begin{aligned}\frac{1}{2}A &= \int_{-0.659}^{0.659} 4 \cos^2 2\theta - 1 \, d\theta \\ A &= 2.554\end{aligned}$$

Example 5.3.3. Given $r = \sin \theta$, $r = \cos \theta$, we have:

We first consider the intersection points

$$\begin{aligned}\sin \theta &= \cos \theta \\ \tan \theta &= 1 \\ \theta &= \frac{\pi}{4}\end{aligned}$$

We then consider the area between the two curves, notice that $(0, 0)$ is also an intersection point.

$$\begin{aligned}A &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (\sin \theta)^2 - (\cos \theta)^2 \, d\theta \\ &= \frac{\pi - 2}{8}\end{aligned}$$

Tangent Lines in Polar Coordinates The tangent line to a polar curve $r = r(\theta)$ is derived as follows:

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta$$

By chain rule, we have and product rule, we have:

$$\frac{dy}{dx} = \frac{r'(\theta) \sin \theta + r(\theta) \cos \theta}{r'(\theta) \cos \theta - r(\theta) \sin \theta} \quad (68)$$

$$(69)$$

Length of a Polar Curve Derived from above, the length of a polar curve $r = r(\theta)$ between α and β is given by:

$$L = \int_{\alpha}^{\beta} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2} \, d\theta \quad (70)$$

Example 5.3.4. Given $r = a - a \cos \theta$, we have:

$$\begin{aligned}r' &= a \sin \theta \\ L &= \int_0^{2\pi} \sqrt{(a - a \cos \theta)^2 + a^2 \sin^2 \theta} \, d\theta \\ &= a \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} \, d\theta\end{aligned}$$

We can use the double angle formula to simplify the integral, with $\cos 2\theta = 1 - 2 \sin^2 \theta$:

$$= 8a$$

6 Infinite Sequences and Series

6.1 Sequences

Sequences A sequence is a list of numbers written in a specific order.

Example 6.1.1. Let $f : A \rightarrow \mathbb{R}$ be a function, and $f(x) = \frac{1}{x}$. We can restrict the domain of f to $A = \{1, 2, 3, \dots\} = \mathbb{N}$, and we can define the sequence $a_n = f(n) = \frac{1}{n}$.

Notation Typically, we denote a sequence as $\{a_n\}$, where a_n is the n th term of the sequence. We can also denote with curly bracket as $\{a_n\} = \{a_1, a_2, a_3, \dots\} = \{f(n)\}$.

Ratio The ratio of two sequences $\{a_n\}$ and $\{b_n\}$ for some $n \leq N$ is a sequence $\{c_n\}$ such that $c_n = \frac{a_n}{b_n}$.

Example 6.1.2. Let $a_n = n^2$, $b_n = e^n$, we have:

$$\frac{a_n}{b_n} = \left\{ \frac{1}{e}, \frac{4}{e^2}, \frac{9}{e^3}, \dots \right\}$$

Bounded Above A sequence $\{a_n\}$ is said to be bounded above if there exists a number M such that $|a_n| \leq M$ for all n . (Similarly for bounded below).

Definiton 6.1.3 (Monotonic Sequences). A sequence $\{a_n\}$ is said to be non decreasing if $a_{n+1} \geq a_n$ for all n . Similarly, a sequence is said to be non increasing if $a_{n+1} \leq a_n$ for all n . A sequence is said to be increasing if $a_{n+1} > a_n$ for all n . Similarly, a sequence is said to be decreasing if $a_{n+1} < a_n$ for all n .

A sequence $\{a_n\}$ is said to be monotonic if it is either increasing, decreasing, non increasing, or non decreasing.

Definiton 6.1.4 (Unbounded Sequences). A sequence $\{a_n\}$ is said to be unbounded if it is not bounded above or below. We can prove it using a epsilon-delta proof or considering the derivatives of the function that defines the sequence.

Example 6.1.5. Proof that the sequence $\{a_n\} = 2^n$ is unbounded. Let M be a number, we can find an n such that $2^n > M$. We can find this n by taking the logarithm of M to the base 2.

Example 6.1.6. Proof that the sequence $\{a_n\} = \frac{x^2}{e^x}$ is bounded above. We show that the derivative is negative after a certain point ($x = 2$), thus the sequence is bounded above.

6.2 The Limit of a Sequence

Definiton 6.2.1 (Limit of a Sequence). The limit of a sequence $\{a_n\}$ is given by:

$$\lim_{n \rightarrow \infty} a_n = L$$

if for every $\epsilon > 0$, there exists an N such that $|a_n - L| < \epsilon$ for all $n > N$. We denote this as $a_n \rightarrow L$ as $n \rightarrow \infty$.

6.2 The Limit of a Sequence

Convergence A sequence $\{a_n\}$ is said to converge to L if the limit of the sequence is L . And the following are true:

1. If a sequence converges, then it is bounded.
2. If a sequence is unbounded, then it does not converge, i.e. it diverges.
3. However, if a sequence is bounded, it may or may not converge.

Example 6.2.2. Prove that the sequence $\{a_n\} = \frac{n}{n+1}$ converges to 1. We can show that $|a_n - 1| < \epsilon$ for all $n > N$ by taking $N = \frac{1}{\epsilon}$ by some algebraic manipulation. We have:
Given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$, then for all $n > N$:

$$\begin{aligned} |a_n - 1| &= \left| \frac{n}{n+1} - 1 \right| \\ &= \left| \frac{n - n - 1}{n+1} \right| \\ &= \frac{1}{n+1} < \frac{1}{n} < \epsilon \end{aligned}$$

Theorem 6.2.3 (Uniqueness of Limits). The limit of a sequence is unique. That is, if $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} a_n = M$, then necessarily $L = M$.

Theorem 6.2.4 (Monotone Convergence Theorem). A monotonic sequence is convergent if and only if it is bounded. And necessarily the following are true if the sequence converges:

1. If the sequence is increasing or non decreasing, then the limit is the supremum of the sequence.
2. If the sequence is decreasing or non increasing, then the limit is the infimum of the sequence.

Theorem 6.2.5 (Limit Operations). Given two sequences $\{a_n\}$ and $\{b_n\}$, and $c \in \mathbb{R}$, we have:

1. **Sum of Limits:** $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
2. **Product of Limits:** $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$
3. **Scalar Multiple of Limits:** $\lim_{n \rightarrow \infty} c \cdot a_n = c \cdot \lim_{n \rightarrow \infty} a_n$
4. **reciprocal of Limits:** $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{\lim_{n \rightarrow \infty} a_n}$
5. **Quotient of Limits:** $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$, provided $\lim_{n \rightarrow \infty} b_n \neq 0$

Theorem 6.2.6 (Squeeze Theorem¹). Given three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, if $a_n \leq b_n \leq c_n$ for all n , and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then necessarily $\lim_{n \rightarrow \infty} b_n = L$. This is useful for proving limits of sequences.

Theorem 6.2.7 (Composite Function). Given sequence $\{a_n\}$ converges to L and continuous function f at L , we have:

$$\lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right) \quad (71)$$

¹Also known as the Sandwich Theorem or the Pinching Theorem

Some Common Limits Some common limits of sequences are:

1. for $a > 0$, $\lim_{n \rightarrow \infty} a^{1/n} = 1$
2. for $|a| < 1$, $\lim_{n \rightarrow \infty} a^n = 0$
3. for $a > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n^a} = 0$
4. for $a > 0$, $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$

6.3 Series

Definiton 6.3.1 (Infinite Series). An infinite series is the sum of the terms of an infinite sequence. We denote the sum of the series as:

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \quad (72)$$

We also denote it as just $\sum a_n$.

Definiton 6.3.2 (Partial Sum). The partial sum of an infinite series is the sum of the first n terms of the series. We denote the partial sum as:

$$S_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n \quad (73)$$

If the sequence $\{S_n\}$ converges to S , then the series is said to converge to S ; otherwise, it is said to diverge.

Geometric Series A geometric series is a series of the form:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots \quad (74)$$

Theorem 6.3.3 (Geometric Series). A geometric series converges if $|r| < 1$, and the sum of the series is given by:

$$S = \frac{a}{1-r} \quad (75)$$

If $|r| \geq 1$, then the series diverges.

Theorem 6.3.4 (Addition of Series). Given two series $\sum a_n$ and $\sum b_n$, we have:

$$\sum a_n + \sum b_n = \sum (a_n + b_n) \quad (76)$$

Theorem 6.3.5 (Convergence). $\sum a_n$ converges if and only if $\sum_{k=j}^{\infty} a_k$ converges for all j .

6.4 The Integral Test and Estimates of Sums

Example 6.3.6. Given $\sum \frac{3^{k-1}}{4^{3k+1}}$ converges. Then we have:

$$\sum \frac{3^{k-1}}{4^{3k+1}} = \frac{1}{12} + \frac{1}{64} + \frac{1}{4^7} + \cdots + \sum_{k=n}^{\infty} \frac{3^{k-1}}{4^{3k+1}}$$

converges for all n .

Theorem 6.3.7 (Zero Test, Divergence Test). If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum a_n$ diverges. Note that the contrapositive is true, i.e. if $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

6.4 The Integral Test and Estimates of Sums

Theorem 6.4.1 (Integral Test). Let f be a continuous, positive, and decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then the series $\sum a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x) dx$ converges.

Example 6.4.2 ($p=2$ -series). Given $\sum \frac{1}{n^2}$, we can use the integral test to show that the series converges. We have:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} dx &= -\frac{1}{x} \Big|_1^{\infty} \\ &= 1 \end{aligned}$$

Thus, the series converges.

Note that the “iff” statement allow us to determine the convergence of an integral.

Example 6.4.3 (Harmonic Series). Given $\sum \frac{1}{n}$, we can use the integral test to show that the series diverges. We have:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &= \ln x \Big|_1^{\infty} \\ &= \infty \end{aligned}$$

Thus, the series diverges. The harmonic series is a classic example of a divergent series.

Definiton 6.4.4 (Remainder). The remainder of a series $\sum a_n$ is given by:

$$R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k \tag{77}$$

We can use the remainder to estimate the sum of a series.

Estimating Sums We can estimate the sum of a series by using the remainder of the series. We can use the remainder to estimate the sum of a series. Note that we always have:

$$R_n \leq \int_n^{\infty} f(x) dx \text{ /quad and } R_n \geq \int_{n+1}^{\infty} f(x) dx \tag{78}$$

We can use it to estimate the sum of a series from a partial sum.

$$S_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq S_n + \int_n^{\infty} f(x) dx \tag{79}$$

6.5 Other Comparison Tests

Example 6.4.5. Given $\sum \frac{1}{n^2}$, we can use the remainder to estimate the sum of the series. We have:

$$\begin{aligned} S_n + \int_{n+1}^{\infty} \frac{1}{x^2} dx &\leq S \leq S_n + \int_n^{\infty} \frac{1}{x^2} dx \\ \frac{1}{n} + \int_{n+1}^{\infty} \frac{1}{x^2} dx &\leq S \leq \frac{1}{n} + \int_n^{\infty} \frac{1}{x^2} dx \end{aligned}$$

We can then evaluate the integrals to obtain the value bounds.

6.5 Other Comparison Tests

Theorem 6.5.1 (Basic Comparison Test). Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $0 \leq a_n \leq b_n$ for all n . Then:

1. If $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof. Assume, WLOG, that $a_k \leq b_k$ for all k . Define $S_n = \sum_{k=1}^n a_k$ and $T_n = \sum_{k=1}^n b_k$. Also, $T = \sum b_k$ exists. We have:

$$S_{n+1} = S_n + a_{n+1} > S_n$$

Thus, $\{S_n\}$ is increasing. Also, we have:

$$S_n \leq T_n < T$$

Thus, $\{S_n\}$ is bounded above. Thus, $\{S_n\}$ converges by the Monotone Convergence Theorem.

□

Example 6.5.2. Let $S = \sum \frac{7}{17n^2 + 3n^{1/3} + 5}$. We know that $17n^2 > 17n^{1/3} + 5$ for all n . Thus, we have:

$$\frac{7}{17n^2} > \frac{7}{17n^2 + 3n^{1/3} + 5}$$

We can then use the comparison test to show that the series converges.

Theorem 6.5.3 (Limit Comparison Test). Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n, b_n > 0$ for all n . Then:

1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then either both series $\sum a_n$ and $\sum b_n$ converge or both diverge.
2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

6.6 Alternating Series and Absolute Convergence

3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof. We can prove the first part of the theorem by epsilon-delta proof.

Proof of part 1: Let $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$. Then, for all $\epsilon > 0$, for all $n > N$, we have:

$$\left| \frac{a_n}{b_n} - c \right| < \epsilon$$

Choose $\epsilon = \frac{c}{2}$, then we have:

$$\begin{aligned} \left| \frac{a_n}{b_n} - c \right| &< \frac{c}{2} \\ \frac{c}{2} &< \frac{a_n}{b_n} < \frac{3c}{2} \end{aligned}$$

We can then use the basic comparison test to show that the series converge or diverge. \square

Example 6.5.4. Let $a_n = \frac{1}{n^3 - n}$ and choose $b_n = \frac{1}{n^3}$. We can use the limit comparison test to show that the series $\sum a_n$ converges. We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n^3}{n^3 - n} \\ &= 1 \end{aligned}$$

Since $\sum b_n$ converges, then $\sum a_n$ converges.

Example 6.5.5. Let $a_n = \frac{\ln(n/1000)}{n}$ and choose $b_n = \frac{1}{n}$. We can use the limit comparison test to show that the series $\sum a_n$ diverges. We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \ln(n/1000) \\ &\rightarrow \infty \end{aligned}$$

Since $\sum b_n$ diverges, then $\sum a_n$ diverges.

6.6 Alternating Series and Absolute Convergence

Definiton 6.6.1 (Alternating Series). An alternating series is a series of the form:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots \quad (80)$$

where $a_n > 0$ for all n .

Proof. The proof for this is straight forward. We can show that the series converges by showing that the sequence $\{S_n\}$ is bounded above and below. Left as an exercise to the reader (Hint: Consider both the odd and even partial sums). \square

Example 6.6.2. Let $a_n = \frac{1}{n}$, then the series $\sum (-1)^{n-1} \frac{1}{n}$ is an alternating series.

6.7 Ratio and Root Tests

Note $\sum \cos n$ is not an alternating series, though it alternates between positive and negative values, the negative and positive values are not necessarily next to each other.

Theorem 6.6.3 (Alternating Series Test). Let $\{a_n\}$ be a sequence such that $a_n > 0$ for all n and $\lim_{n \rightarrow \infty} a_n = 0$. Then the alternating series $\sum (-1)^{n-1} a_n$ converges if:

1. $a_{n+1} \leq a_n$ for all n .
2. $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem 6.6.4 (Value of Alternating Series). The value of an alternating series is given by:

$$S = a_1 - a_2 + a_3 - a_4 + \dots = \lim_{n \rightarrow \infty} S_n \quad (81)$$

where S_n is the n th partial sum of the series. Additionally, the limit would always be between two consecutive partial sums.

Example 6.6.5. Let $S = e^{-1} = \sum (-1)^{n-1} \frac{1}{n!}$, we can use the above to estimate the value of e^{-1} . For $n = 5$ we have:

$$e^{-1} = 0.366 \pm 0.0015$$

Definition 6.6.6 (Absolute Convergence). A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ converges. Conversely, a series is said to be conditionally convergent if the series converges but the absolute value of the series diverges.

Example 6.6.7. Let $a_n = \frac{(-1)^{n-1}}{n}$, then the series $\sum \frac{(-1)^{n-1}}{n}$ is conditionally convergent (Hint: $1/2 L$).

Theorem 6.6.8 (Absolute Convergence Test). If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

Proof. We can prove this by using the comparison test. We can show that $-|a_n| \leq a_n \leq |a_n|$ and $0 \leq a_n + |a_n| \leq 2|a_n|$. Thus, we can use the comparison test to show that the series converges. Since $\sum a_n + |a_n|$ converges, then $\sum a_n$ converges. \square

6.7 Ratio and Root Tests

Theorem 6.7.1 (Root Test). Let $\{a_n\}$ be a sequence, and let $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then:

1. If $L < 1$, then the series $\sum a_n$ converges.
2. If $L > 1$, then the series $\sum a_n$ diverges.
3. If $L = 1$, then the test is inconclusive.

Proof. Proof of part 1: Let $L < 1$. Then, choose μ such that $L < \mu < 1$. Then for some N , $\sqrt[n]{|a_n|} < \mu$ for all $n > N$, then $a_n < \mu^n$ for all $n > N$. We can then use the comparison test to show that the series converges since $\sum \mu^n$ converges for $|\mu| < 1$. \square

6.7 Ratio and Root Tests

Example 6.7.2. Let $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$. We can use the root test to show that the series $\sum a_n$ converges. We have:

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} &= \lim_{n \rightarrow \infty} \frac{n^2+1}{2n^2+1} \\ &= \frac{1}{2}\end{aligned}$$

Thus, the series converges.

Example 6.7.3. Let $a_n = \frac{2^n}{n}$. So $\sqrt[n]{\frac{2^n}{n}} = \frac{2}{n^{1/n}}$. So the limit 2 (denom. converges to 2), is greater than 1, thus the series diverges.

Theorem 6.7.4 (Ratio Test). Let $\{a_n\}$ be a sequence, and let $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then:

1. If $L < 1$, then the series $\sum a_n$ converges.
2. If $L > 1$, then the series $\sum a_n$ diverges.
3. If $L = 1$, then the test is inconclusive.

Proof. Proof of part 1: Let $L < 1$. Then, choose μ such that $L < \mu < 1$. Then for some N , $\frac{a_{n+1}}{a_n} < \mu$ for all $n > N$, then $a_n < \mu a_{n+1} < \mu^2 a_{n+2} < \dots < \mu^j a_{n+j}$ for all $n > N$. So we have:

$$\begin{aligned}\sum a_n &< \sum \frac{a_k}{\mu^k} \mu^n \\ &= \frac{a_k}{\mu_k} \sum \mu^n\end{aligned}$$

Since $\sum \mu^n$ converges, then $\sum a_n$ converges. □

Example 6.7.5. Let $a_n = \frac{k^2}{e^k}$ for some k . We can use the ratio test to show that the series $\sum a_n$ converges. We have:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(k+1)^2}{k^2} \cdot \frac{e^k}{e^{k+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(k+1)^2}{k^2} \cdot \frac{1}{e} \right| \\ &= \frac{1}{e} < 1\end{aligned}$$

Thus, the series converges.

6.8 Power Series

Definiton 6.8.1 (Power Series). A power series is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \quad (82)$$

where c_n are constants and x is a variable.

Example 6.8.2 (Geometric Series). Let $c_n = 1$ for all n , then the series $\sum x^n$ is a geometric series:

$$\sum x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

Definiton 6.8.3 (Power Series at a Point). A power series at a point a is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots \quad (83)$$

where c_n are constants and x is a variable.

A special convergent limit for the power series when $x = a$ is called the radius of convergence. This always converges to c_0 .

Example 6.8.4. Let $a_n = \frac{x^n}{n^2}$ for some x . We can use the ratio test to show that the series $\sum a_n$ converges. We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \cdot n^2}{x^n \cdot (n+1)^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| x \cdot \frac{n^2}{(n+1)^2} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 \\ &= |x| \end{aligned}$$

According to the ratio test, the series $\sum a_n$ converges if $|x| < 1$. And if $x = 1$ it is a $p = 2$ series which converges. And for $x = -1$ we can use the alternating series test to show that the series converges since $|a_{n+1}| \leq |a_n|$ and $\lim_{n \rightarrow \infty} a_n = 0$.

Example 6.8.5. Let $a_n = \frac{x^n(1+5^n)}{n!}$ for some x . We can use the ratio test to show that the series $\sum a_n$ converges. We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}(1+5^{n+1})}{(n+1)!} \cdot \frac{n!}{x^n(1+5^n)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{1+n} \cdot \frac{1+5^{n+1}}{1+5^n} \right| \end{aligned}$$

The first part goes to 5, and the second part goes to 0. Thus, the limit is 0.

$$= 0$$

According to the ratio test, the series $\sum a_n$ converges.

Example 6.8.6. Let $a_n = x^n n!$ for some x . We can use the ratio test to show that the series $\sum a_n$ converges only when $x = 0$. We have:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}(n+1)!}{x^n n!} \right| \\ &= \lim_{n \rightarrow \infty} |x(n+1)|\end{aligned}$$

Thus, the series converges only when $x = 0$.

Theorem 6.8.7. For a power series $\sum c_n(x-a)^n$, there are three possibilities in respect to convergence:

1. The series converges only when $x = a$.
2. The series converges for all x .
3. There exists a radius of convergence R such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Interval of Convergence The interval of convergence could also have different types of convergence at the endpoints. It could be either open or closed on either side.

Example 6.8.8. Let $a_n = \frac{(-2)^n(x-1)^n}{n+2}$, we can use the ratio test to show that the series $\sum a_n$ converges for $|x-1| < 2$. We have:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}(x-1)^{n+1}}{n+3} \cdot \frac{n+2}{(-2)^n(x-1)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2(x-1)}{n+3} \right|\end{aligned}$$

It would require $|x-1| < 2$ for the series to converge, so $R = 1/2$ or $x \in (1/2, 3/2)$. However, we also need to check the endpoints. For $x = 1/2$, we have:

$$\begin{aligned}\sum \frac{(-2)^n(1/2-1)^n}{n+2} &= \sum \frac{(-2)^n(-1/2)^n}{n+2} \\ &= \sum \frac{1}{n+2}\end{aligned}$$

This is a $p = 1$ series, thus it diverges. For $x = 3/2$, we have:

$$\begin{aligned}\sum \frac{(-2)^n(3/2-1)^n}{n+2} &= \sum \frac{(-2)^n(1/2)^n}{n+2} \\ &= \sum \frac{(-1)^n}{n+2}\end{aligned}$$

This is an alternating series, thus it converges. Thus, the interval of convergence is $(1/2, 3/2]$.

6.9 Representing Functions as Power Series

Representing Functions We can represent functions as power series. We have, for some analytic function $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \quad (84)$$

Example 6.9.1. Let $f(x) = \frac{1}{1-x}$, we can represent the function as a power series. We have:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \\ &= \sum x^n \end{aligned}$$

Example 6.9.2. Let $f(x) = \frac{x}{x-3}$, we can represent the function as a power series. We have:

$$\begin{aligned} \frac{x}{x-3} &= x \cdot \frac{1}{x-3} = -x \cdot \frac{1}{3-x} \\ &= -x \cdot \frac{1}{-3(1-x/3)} \end{aligned}$$

Using the geometric series, we have:

$$\begin{aligned} &= -x \cdot \sum \frac{1}{3} \left(\frac{x}{3}\right)^n \\ &= -\sum \frac{x^{n+1}}{3^{n+1}}, \quad |x/3| < 1 \end{aligned}$$

Theorem 6.9.3 (Term-by-term Differentiation and Integration). Consider the power series $\sum c_n x^n$ with radius of convergence R . Then:

1. The power series $\sum n c_n x^{n-1}$ has the same radius of convergence R .
2. The power series $\sum \frac{c_n}{n+1} x^{n+1}$ has the same radius of convergence R .

However, the **interval endpoints may change**.

Order of Operation of Differentiation, Integration and Summation The following are true:

1. $\frac{d}{dx} \sum c_n x^n = \sum \frac{d}{dx} c_n x^n$
2. $\int \sum c_n x^n dx = \sum \int c_n x^n dx$

However, note that the constnat of integration is important in the integration case.

6.9 Representing Functions as Power Series

Example 6.9.4. Let $f(x) = \frac{1}{(1+x)^2}$, we can represent the function as a power series. Since:

$$\frac{d}{dx} \frac{-1}{1+x} = \frac{1}{(1+x)^2}$$

We can then use the power series of $\frac{-1}{1+x}$ to find the power series of $\frac{1}{(1+x)^2}$.

$$\begin{aligned} \frac{d}{dx} \sum (-1)^n x^n &= \sum \frac{d}{dx} (-1)^n x^n \\ &= \sum (-1)^n n x^{n-1} \end{aligned}$$

Thus, the power series of $\frac{1}{(1+x)^2}$ is $\sum (-1)^n n x^{n-1}$. The radius of convergence is the same as the original power series (1).

Example 6.9.5. Let $f(x) = \ln(1-x)$, we can represent the function as a power series. Since:

$$\begin{aligned} \int \frac{1}{1-x} dx &= -\ln(1-x) \\ \ln(1-x) &= -\int \sum x^n dx \\ &= C - \sum \frac{x^{n+1}}{n+1} \end{aligned}$$

Let $x = 0$, then $C = 0$. Thus, the power series of $\ln(1-x)$ is $\sum \frac{x^{n+1}}{n+1}$. The radius of convergence is the same as the original power series (1). Also, we can show that $x = 1$ diverges and $x = -1$ conditionally converges.

Example 6.9.6 (Estimation). We would like to estimate the following definite integral:

$$I = \int_0^0 .1 \frac{dx}{1+x^4}$$

We can use the power series of $\frac{1}{1-x}$ to estimate the integral. Take $-x = -(-x)^4$, then we have:

$$\begin{aligned} I &= \int_0^0 .1 \frac{dx}{1+x^4} = \int_0^0 .1 \sum (-x)^{4n} dx \\ &= \sum \int_0^0 .1 (-x)^{4n} dx \\ &= \sum \frac{(-0.1)^{4n+1}}{4n+1} \end{aligned}$$

By alternate series estimation, we can show that that $I \approx 0.099998 \pm 1.1 \times 10^{-10}$.

6.10 Taylor and Maclaurin Series

Example 6.9.7. Let $f(x) = \arctan(x)$, we can represent the function as a power series. Since:

$$\begin{aligned}\frac{d}{dx} \arctan(x) &= \frac{1}{1+x^2} \\ \int \frac{1}{1+x^2} dx &= \arctan(x) \\ \arctan(x) &= \int \sum (-1)^n x^{2n} dx \\ &= C + \sum \frac{(-1)^n x^{2n+1}}{2n+1}\end{aligned}$$

Let $x = 0$, then $C = 0$. Thus, the power series of $\arctan(x)$ is $\sum \frac{(-1)^n x^{2n+1}}{2n+1}$. The radius of convergence is the same as the original power series (1).

Example 6.9.8 (Leibniz Formula). The result from Example 6.9.7 can be used estimate the value of π . Since $\arctan(1) = \frac{\pi}{4}$, we have:

$$\begin{aligned}\arctan(1) &= \sum \frac{(-1)^n}{2n+1} \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \\ &= \frac{\pi}{4}\end{aligned}$$

6.10 Taylor and Maclaurin Series

“The amazing thing about Taylor series is that they swiftly converges” - Prof. J. Davis

We begin by assuming all function discussed in this section has a power series representation (i.e. it is infinitely differentiable), and that there is a non zero radius of convergence.

Example 6.10.1 (n-th derivatives of power series). Let $f(x) = \sum c_n x^n$, we can find the n-th derivative of the function. We have:

$$\begin{aligned}f(x) &= \sum c_n x^n \\ f'(x) &= \sum c_n n x^{n-1} \\ f''(x) &= \sum c_n n(n-1) x^{n-2} \\ &\vdots \\ f^{(n)}(x) &= \sum c_n n(n-1) \dots (n-n+1) x^{n-n} \\ &= \sum c_n n!\end{aligned}$$

Thus, by taking $f^{(n)}(0)$, we can find the n-th coefficient of the power series, i.e. $c_n = \frac{f^{(n)}(0)}{n!}$.

Theorem 6.10.2. If f has a power series representation about a , then the coefficients of the power series are given by:

$$c_n = \frac{f^{(n)}(a)}{n!} \quad (85)$$

Definiton 6.10.3 (Taylor Series). The Taylor series of a function f about a is given by:

$$f(x) = \sum \frac{f^{(n)}(a)}{n!} (x - a)^n \quad (86)$$

Definiton 6.10.4 (Maclaurin Series). The Maclaurin series of a function f is the Taylor series of f about $a = 0$.

$$f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n \quad (87)$$

Definiton 6.10.5 (Analytic Function). A function f is said to be analytic if can be represented as a power series about 0. In turn, it is infinitely differentiable and has a power series representation about any point.

Example 6.10.6. Let $f(x) = e^x$, we can find the Maclaurin series of the function. We have:

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ &\vdots \\ f^{(n)}(x) &= e^x \\ f^{(n)}(0) &= 1 \end{aligned}$$

Thus, the Maclaurin series of e^x is $\sum \frac{x^n}{n!}$. Also, by the ratio test, we can show that the series converges for all x (i.e. the radius of convergence is ∞).

“When is it that $f(x) = \sum f^{(n)}(a)(x - a)^n/n!$?”

Definiton 6.10.7 (n-th Order Taylor Polynomial). The n-th order Taylor polynomial of a function f about a is given by:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k \quad (88)$$

Definiton 6.10.8 (Remainder). The remainder of the n-th order Taylor polynomial is given by:

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1} \quad (89)$$

for some c between a and x .

Theorem 6.10.9 (Remainder Convergence). If $f(a) = P_n(x) + R_n(x)$ and $\lim_{n \rightarrow \infty} R_n(x) = 0$, then $f(x) = \sum \frac{f^{(n)}(a)}{n!} (x - a)^n$.

Theorem 6.10.10 (Taylor Theorem). Let f be a function that is $n + 1$ times differentiable on an interval I containing a . If the n -th derivative of f is continuous on I , then for all x in I :

$$f(x) = f(a) + f'(a)(x - a) + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n + R_n(x) \quad (90)$$

where $R_n(x) = \int \frac{f^{(n+1)}(c)}{n!} (x - a)^{n+1} dx = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$.

And for bounded derivatives, i.e. $|f^{(n+1)}(x)| \leq M$ for all x in I , we have:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad (91)$$

We can quickly obtain an error bound for the Taylor polynomial by using the above inequality.

Taylor Series of Common Functions

$$e^x = \sum \frac{x^n}{n!} \quad (92)$$

$$\sin(x) = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (93)$$

$$\cos(x) = \sum \frac{(-1)^n x^{2n}}{(2n)!} \quad (94)$$

$$\ln(1+x) = \sum \frac{(-1)^{n+1} x^n}{n} \quad (95)$$

$$(1+x)^p = \sum \binom{p}{n} x^n \quad (96)$$

Example 6.10.11. Let $f(x) = \cos(x)$, we want to find the Taylor series of the function about $a = 17\pi/4$. The derivatives and their respective value at a are:

$$f(x=a) = \cos(a) = \frac{1}{\sqrt{2}}$$

$$f'(x=a) = -\sin(a) = -\frac{1}{\sqrt{2}}$$

$$f''(x=a) = -\cos(a) = -\frac{1}{\sqrt{2}}$$

$$f'''(x=a) = \sin(a) = \frac{1}{\sqrt{2}}$$

We can see that the derivatives repeat every 4th derivative, and that we could separate it into two alternating series, we have:

$$\cos(x) = \sum \frac{1}{\sqrt{2}} (-1)^n \frac{(x-17\pi/4)^{2n}}{(2n)!} + \sum \frac{1}{\sqrt{2}} (-1)^n \frac{(x-17\pi/4)^{2n+1}}{(2n+1)!}$$

Multiplication and Division of Power Series We can multiply and divide power series as we would with polynomials. We can also represent functions as products of known power series. We have some examples below.

Example 6.10.12. Let $f(x) = \frac{e^x}{1+x}$, we can represent the function as product of two known power

series. We have:

$$\begin{aligned}
 f(x) &= e^x \cdot \frac{1}{1+x} \\
 &= \sum \frac{x^n}{n!} \cdot \sum (-1)^n x^n \\
 &= (1 + x + \frac{x^2}{2} + \dots) \cdot (1 - x + x^2 - \dots) \\
 &= \sum \sum \frac{(-1)^n x^{2n}}{n!}
 \end{aligned}$$

We shall just deduce it with powers of terms.

Example 6.10.13. Let $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, we can represent the function as division of two known power series. We have:

$$\begin{aligned}
 f(x) &= \frac{\sin(x)}{\cos(x)} \\
 &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} \\
 &= x - \frac{x^3}{3} + \frac{2x^5}{15} - \dots
 \end{aligned}$$

We shall just deduce it with long division, note that the radius of convergence is not the same as the original power series.

6.11 Applications of Taylor Series

Approximation We can use the Taylor series to approximate functions. We can use the first few terms of the Taylor series to approximate the function. We can also use the remainder to find an error bound for the approximation.

Error Estimation We can summarize two ways:

1. If we know that it is an alternating series, then $|R_n(x)| \leq a_{n+1}$.
2. Otherwise, we can use the remainder formula to find an error bound. $|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$.

Example 6.11.1. Let $f(x) = \sqrt{x}$, we can use the Taylor series to approximate $\sqrt{1.25}$.

$$\begin{aligned} f(x) &= \sqrt{x} = 1 \\ f'(x) &= \frac{1}{2\sqrt{x}} = \frac{1}{2} \\ f''(x) &= -\frac{1}{4x^{3/2}} = -\frac{1}{4} \\ f'''(x) &= \frac{3}{8x^{5/2}} = \frac{3}{8} \end{aligned}$$

We can then use the Taylor series to approximate $\sqrt{1.25}$.

$$\begin{aligned} f(1.25) &\approx 1 + \frac{1}{2}(1.25 - 1) - \frac{1}{8}(1.25 - 1)^2 + \frac{1}{16}(1.25 - 1)^3 \\ &= 1 + \frac{1}{2}(0.25) - \frac{1}{8}(0.25)^2 + \frac{1}{16}(0.25)^3 \\ &= 1 + \frac{1}{8} - \frac{1}{32} + \frac{1}{128} \\ &= 1.1182 \end{aligned}$$

Since it is an alternating series, we also have that the error is less than $\frac{3}{8} \cdot \frac{1}{16}(0.25)^4 = 0.00015$. Thus, the error is less than 0.00015.

Example 6.11.2. Let $f(x) = \ln(1 - x)$, we would like to find a term such that the estimate of $\ln(1.4)$ is less than 10^{-3} . Since the Maclaurin series of $\ln(1 - x)$ is $\sum \frac{x^n}{n}$, we can use the remainder for alternating series to find the error bound. We have:

$$\begin{aligned} |R_1(1.4)| &\leq \frac{1}{2}(0.4)^2 = 0.08 \\ &\vdots \\ |R_4(1.4)| &\leq \frac{1}{5}(0.4)^5 = 0.002 \\ |R_5(1.4)| &\leq \frac{1}{6}(0.4)^6 = 0.0007 \end{aligned}$$

Thus, we can use the first 5 terms of the Maclaurin series to approximate $\ln(1.4)$ to less than 10^{-3} .

6.12 Binomial Series

Definiton 6.12.1 (Binomial Series). The binomial series is a power series of the form:

$$(1 + x)^p = \sum \binom{p}{n} x^n \quad (97)$$

, where $\binom{p}{n} = \frac{p(p-1)\dots(p-n+1)}{n!}$ and $\binom{k}{0} = 1$.

Example 6.12.2 $((1 + x)^k)$. Let $f(x) = (1 + x)^k$, we can find the binomial series of the function. We have:

$$(1 + x)^k = \sum \binom{k}{n} x^n$$

This could be derived, also, via Maclaurin series, we have:

$$\begin{aligned}
 f(x) &= (1+x)^k & f(0) &= 1 \\
 f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
 f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
 &\vdots \\
 f^{(n)}(x) &= k(k-1)\dots(k-n+1)(1+x)^{k-n} \\
 f^{(n)}(0) &= k(k-1)\dots(k-n+1)
 \end{aligned}$$

Thus, the binomial series of $(1+x)^k$ is $\sum \binom{k}{n} x^n$.

We can also show that the binomial series converges for $|x| < 1$. With the ratio test, we have:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{\binom{k}{n+1} x^{n+1}}{\binom{k}{n} x^n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{k-n}{n+1} x \right| \\
 &= |x|
 \end{aligned}$$

Thus, the series converges for $|x| < 1$. Also for the endpoints, Prof. J. Davis didn't show the process, but we have:

$$I = \begin{cases} (-1, 1) & k \leq -1 \\ (-1, 1] & k \in (-1, 0) \\ [-1, 1] & k \geq 0 \end{cases}$$

6.13 Fourier Series

Definiton 6.13.1 (Periodic Function). A function f is said to be periodic if there exists a positive number P such that $f(x+P) = f(x)$ for all x .

Example 6.13.2. Let $f(x) = \cos(nt) + \frac{1}{2}\sin(2\pi t)$ for some n . We can show that the function is periodic. We have:

$$\begin{aligned}
 f(x+P) &= \cos(n(t+P)) + \frac{1}{2}\sin(2\pi(t+P)) \\
 &= \cos(nt+nP) + \frac{1}{2}\sin(2\pi t + 2\pi P) \\
 &= \cos(nt) + \frac{1}{2}\sin(2\pi t) \\
 &= f(x)
 \end{aligned}$$

Thus, the function is periodic.

Theorem 6.13.3. For periodic function f , with period P , continuous and piecewise differentiable, then the Fourier series of the function converges to the function.

Definiton 6.13.4 (Fourier Series). The Fourier series of a function f is a series of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + b_n \sin(n\omega x) \quad (98)$$

, where a_0, a_n, b_n are the Fourier coefficients of the function, and $\omega = \frac{2\pi}{P}$.

Integral Properties of Fourier Series First, we have these integral properites of cos and sin:

$$\int_{-P/2}^{P/2} \cos(n\omega x) dx = \begin{cases} 0 & n \neq 0 \\ P & n = 0 \end{cases} \quad (99)$$

$$\int_{-P/2}^{P/2} \sin(n\omega x) dx = 0 \quad (100)$$

Also, we have the product of cos and sin:

$$\int_{-P/2}^{P/2} \sin(m\omega x) \sin(n\omega x) dx = \begin{cases} 0 & m \neq n \\ \frac{P}{2} & m = n \end{cases} \quad (101)$$

$$\int_{-P/2}^{P/2} \cos(m\omega x) \cos(n\omega x) dx = \begin{cases} 0 & m \neq n \\ \frac{P}{2} & m = n \end{cases} \quad (102)$$

Theorem 6.13.5 (Fourier Coefficients). The Fourier coefficients of a function f are given by:

$$a_0 = \frac{1}{P} \int_{-P/2}^{P/2} f(x) dx \quad (103)$$

$$a_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \cos(n\omega x) dx \quad (104)$$

$$b_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \sin(n\omega x) dx \quad (105)$$

A Trigonometric Derivatives

Trigonometric Function	Inverse Trigonometric Function
$\frac{d}{dx} \sin(x) = \cos(x)$	$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx} \cos(x) = -\sin(x)$	$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx} \tan(x) = \sec^2(x)$	$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$
$\frac{d}{dx} \cot(x) = -\csc^2(x)$	$\frac{d}{dx} \operatorname{arccot}(x) = \frac{-1}{1+x^2}$
$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$	$\frac{d}{dx} \operatorname{arcsec}(x) = \frac{1}{ x \sqrt{x^2-1}}$
$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$	$\frac{d}{dx} \operatorname{arccsc}(x) = \frac{-1}{ x \sqrt{x^2-1}}$
Hyperbolic Function	Inverse Hyperbolic Function
$\frac{d}{dx} \sinh(x) = \cosh(x)$	$\frac{d}{dx} \operatorname{arsinh}(x) = \frac{1}{\sqrt{x^2+1}}$
$\frac{d}{dx} \cosh(x) = \sinh(x)$	$\frac{d}{dx} \operatorname{arcosh}(x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$	$\frac{d}{dx} \operatorname{artanh}(x) = \frac{1}{1-x^2} \quad (x < 1)$
$\frac{d}{dx} \coth(x) = -\operatorname{csch}^2(x)$	$\frac{d}{dx} \operatorname{arcoth}(x) = \frac{1}{1-x^2} \quad (x > 1)$
$\frac{d}{dx} \operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$	$\frac{d}{dx} \operatorname{arsech}(x) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x \leq 1)$
$\frac{d}{dx} \operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$	$\frac{d}{dx} \operatorname{arcsch}(x) = \frac{-1}{ x \sqrt{x^2+1}}$

B Polar Curves

Below are some common polar curves: Click: [5.1](#) to go back to the section on polar coordinates.

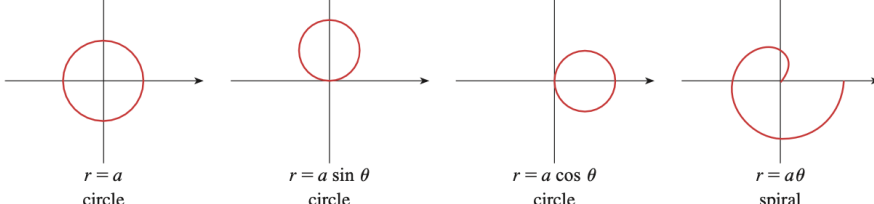
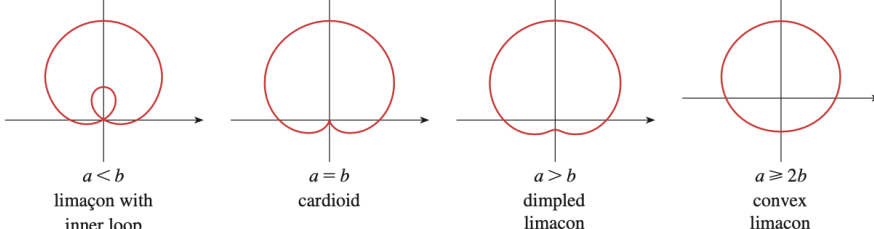
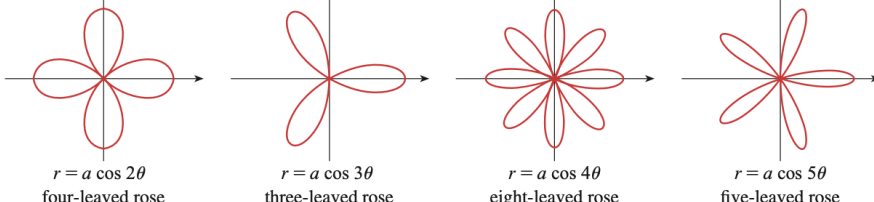
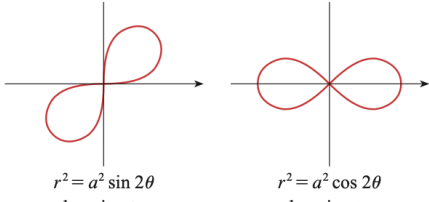
Circles and Spiral	 <div> $r = a$ circle </div> <div> $r = a \sin \theta$ circle </div> <div> $r = a \cos \theta$ circle </div> <div> $r = a \theta$ spiral </div>
Limaçons $r = a \pm b \sin \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 <div> $a < b$ limaçon with inner loop </div> <div> $a = b$ cardioid </div> <div> $a > b$ dimpled limaçon </div> <div> $a \geq 2b$ convex limaçon </div>
Roses $r = a \sin n\theta$ $r = a \cos n\theta$ n -leaved if n is odd $2n$ -leaved if n is even	 <div> $r = a \cos 2\theta$ four-leaved rose </div> <div> $r = a \cos 3\theta$ three-leaved rose </div> <div> $r = a \cos 4\theta$ eight-leaved rose </div> <div> $r = a \cos 5\theta$ five-leaved rose </div>
Lemniscates Figure-eight-shaped curves	 <div> $r^2 = a^2 \sin 2\theta$ lemniscate </div> <div> $r^2 = a^2 \cos 2\theta$ lemniscate </div>

Figure 2: Polar Curves

C Trigonometric Identities

Identity	Value
$\sin^2 \theta + \cos^2 \theta = 1$	1
$\tan^2 \theta = \sec^2 \theta - 1$	$\sec^2 \theta - 1$
$\tan \theta = \frac{\sin \theta}{\cos \theta}$	$\frac{\sin \theta}{\cos \theta}$
$\cot \theta = \frac{\cos \theta}{\sin \theta}$	$\frac{\cos \theta}{\sin \theta}$
$\sec \theta = \frac{1}{\cos \theta}$	$\frac{1}{\cos \theta}$
$\csc \theta = \frac{1}{\sin \theta}$	$\frac{1}{\sin \theta}$
$\sin(-\theta) = -\sin \theta$	$-\sin \theta$
$\cos(-\theta) = \cos \theta$	$\cos \theta$
$\tan(-\theta) = -\tan \theta$	$-\tan \theta$
$\cot(-\theta) = -\cot \theta$	$-\cot \theta$
$\sec(-\theta) = \sec \theta$	$\sec \theta$
$\csc(-\theta) = -\csc \theta$	$-\csc \theta$
$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$	$\sin \theta \cos \phi \pm \cos \theta \sin \phi$
$\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi$	$\cos \theta \cos \phi \mp \sin \theta \sin \phi$
$\tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$	$\frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$
$\sin(2\theta) = 2 \sin \theta \cos \theta$	$2 \sin \theta \cos \theta$
$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$	$\cos^2 \theta - \sin^2 \theta$
$2 \cos(2\theta) = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$	$2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$	$\frac{2 \tan \theta}{1 - \tan^2 \theta}$

Table 3: Trigonometric Identities

Category	Integral Formula
Basic Trigonometric Integrals	
$\int \sin x \, dx$	$-\cos x + C$
$\int \cos x \, dx$	$\sin x + C$
$\int \tan x \, dx$	$\ln \sec x + C$
$\int \cot x \, dx$	$\ln \sin x + C$
$\int \sec x \, dx$	$\ln \sec x + \tan x + C$
$\int \csc x \, dx$	$\ln \csc x - \cot x + C$
Inverse Trigonometric Integrals	
$\int \arcsin x \, dx$	$x \arcsin x + \sqrt{1 - x^2} + C$
$\int \arccos x \, dx$	$x \arccos x - \sqrt{1 - x^2} + C$
$\int \arctan x \, dx$	$x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$
$\int \operatorname{arccot} x \, dx$	$x \operatorname{arccot} x + \frac{1}{2} \ln(1 + x^2) + C$
$\int \operatorname{arcsec} x \, dx$	$x \operatorname{arcsec} x - \ln x + \sqrt{x^2 - 1} + C$
$\int \operatorname{arccsc} x \, dx$	$x \operatorname{arccsc} x + \ln x + \sqrt{x^2 - 1} + C$
Basic Hyperbolic Trigonometric Integrals	
$\int \sinh x \, dx$	$\cosh x + C$
$\int \cosh x \, dx$	$\sinh x + C$
$\int \tanh x \, dx$	$\ln \cosh x + C$
$\int \coth x \, dx$	$\ln \sinh x + C$
$\int \operatorname{sech}^2 x \, dx$	$\tanh x + C$
$\int \operatorname{csch}^2 x \, dx$	$-\coth x + C$
Trigonometric Substitution Choices	
$\sqrt{a^2 - x^2}$	Set $x = a \sin \theta$, then $dx = a \cos \theta \, d\theta$
$\sqrt{a^2 + x^2}$	Set $x = a \sinh \theta$, then $dx = a \cosh \theta \, d\theta$
$\sqrt{x^2 - a^2}$	Set $x = a \cosh \theta$, then $dx = a \sinh \theta \, d\theta$
Reduction Formulas	
$\int \sin^n x \, dx$	$-\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$
$\int \cos^n x \, dx$	$\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$