PHY 293 Lecture Notes

Hei Shing Cheung

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The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

Chapter 1

Waves

1.1 Harmonic Oscillators

1.1.1 Governing Equations of Harmonic Oscillators

Key Concepts & Equations

Key Equations:

- Simple Harmonic Oscillator: $m\ddot{x} + kx = 0$ or $\ddot{x} + \omega_0^2 x = 0$ where $\omega_0 = \sqrt{k/m}$
- Damped: $m\ddot{x} + b\dot{x} + kx = 0$ or $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ where $\gamma = b/m$
- Driven: $m\ddot{x} + b\dot{x} + kx = F(t)$

Tip: Remember to check the damping parameter convention (γ vs 2β) when comparing formulas!

This subsection collects the baseline ODEs for simple, damped, and driven oscillators to set notation used later.

Types of Harmonic Oscillators There are three types of harmonic oscillators: simple, damped, and driven harmonic oscillators. Consider a simple one dimensional harmonic oscillator, they are defined by the following differential equations:

Definiton 1.1.1.1 (Simple Harmonic Oscillator)**.** A simple harmonic oscillator is described by Hooke's law:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{1.1}$$

where k is the spring constant, m is the mass, and x is the displacement from equilibrium.

Definiton 1.1.1.2 (Damped Harmonic Oscillator)**.** A damped harmonic oscillator is described by the following differential equation, by adding a damping term proportional to $\dot{x}$ to the simple harmonic oscillator equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.2}$$

where b is the damping coefficient.

Note on damping parameter conventions. Different texts use different symbols and normalizations:

- Our notes use $\gamma = \frac{b}{m}$, so the ODE reads $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ and the decay envelope is $e^{-\gamma t/2}$.
- Many texts define $\beta = \frac{b}{2m}$ and may call it γ instead. In that convention the ODE is $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ with envelope $e^{-\beta t}$.

Equivalences: $\beta = \frac{\gamma}{2}$ and our underdamped condition $\gamma < 2\omega_0$ corresponds to $\beta < \omega_0$ in the 2β convention. When comparing formulas, check which definition is being used and replace $\gamma \leftrightarrow 2\beta$ accordingly.

Definiton 1.1.1.3 (Driven Harmonic Oscillator)**.** A driven harmonic oscillator is described by the following differential equation, which includes an external driving force $F(t)$:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

1.1.2 Simple Harmonic Motion

Key Concepts & Equations

Key Equations:

- General solution: $x(t) = A\cos(\omega t + \phi)$ where $\omega = \sqrt{k/m}$
- Period: $T = 2\pi/\omega = 2\pi\sqrt{m/k}$
- Frequency: $f = 1/T = \omega/(2\pi)$
- Total Energy: $E = \frac{1}{2}kA^2$ (constant for undamped oscillator)

Common Pitfall: Don't forget to match initial conditions properly when solving for amplitude and phase!

We analyze the undamped solution forms, relate constants to initial conditions, and derive period/frequency relations.

Definiton 1.1.2.1 (Simple Harmonic Motion)**.** You should have learned Hooke's law and Newton's second law, which give us the equation of motion for a simple harmonic oscillator. The same as equation (1.1), which can be rewritten as:

$$F = m\ddot{x} = -kx$$

By setting $\omega^2 = \frac{k}{m}$, a general solution can be written as:

$$x(t) = x_0 + A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

where A_1 and A_2 are constants determined by the IVP, ω is the angular frequency, and ϕ is the phase constant. x_0 is the equilibrium position (often set to 0). The unknown constants can be determined by knowing $x, \dot{x}$ at specific times.

Definiton 1.1.2.2 (Period, Frequency, and Angular Frequency)**.** The period T is the time it takes for one complete cycle of the motion, given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The frequency f is the number of cycles per unit time, given by:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The angular frequency ω is related to the frequency by:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

Example 1.1.2.3. A simple harmonic oscillator consisting of mass $m = 11.0\,\text{kg}$ attached to a spring with spring constant $k = 201\ \text{N}\,\text{m}^{-1}$. At time $t = 0\ \text{s}$ the oscillator is at position $x(0) = -0.207\ \text{m}$ and has velocity $v(0) = -1.33\ \text{m}\,\text{s}^{-1}$. Determine all coefficients of the equation describing the position $x(t)$ of the oscillator as a function of time, assuming the offset is zero.

To solve for A_1 and A_2, while we assume $x_0 = 0$, we can use the initial conditions:

$$x(0) = A_1 \cos(0) + A_2 \sin(0) = A_1 = -0.207\ \text{m}$$
$$v(0) = -A_1\omega \sin(0) + A_2\omega \cos(0) = A_2\omega = -1.33\ \text{m}\,\text{s}^{-1}$$

We can find ω from the given m and k:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{201\ \text{N}\,\text{m}^{-1}}{11.0\ \text{kg}}} \approx 4.28\ \text{rad}\,\text{s}^{-1}$$

Therefore, we can solve for A_2:

$$A_2 = \frac{v(0)}{\omega} = \frac{-1.33\ \text{m}\,\text{s}^{-1}}{4.28\ \text{rad}\,\text{s}^{-1}} \approx -0.311\ \text{m}$$

Thus, the equation describing the position $x(t)$ of the oscillator as a function of time is:

$$x(t) = -0.207\cos(4.28t) - 0.311\sin(4.28t)$$

Theorem 1.1.2.4 (A Trigonometric Identity)**.** We can also express the solution in a more compact form using a single cosine function with a phase shift:

$$x(t) = A\cos(\omega t + \phi)$$

where

$$A = \sqrt{A_1^2 + A_2^2}, \qquad \phi = \arctan\left(\frac{-A_2}{A_1}\right) = \arctan\left(\frac{-v(0)/\omega}{x(0)}\right)$$

Proof. Let $A = \sqrt{A_1^2 + A_2^2}$ and choose ϕ such that

$$\cos(\phi) = \frac{A_1}{A}, \qquad \sin(\phi) = -\frac{A_2}{A}.$$

Then, we can rewrite our original solution as

$$\begin{aligned} x(t) &= A_1\cos(\omega t) + A_2\sin(\omega t) \\ &= A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t) \\ &= A\big[\cos(\phi)\cos(\omega t) - \sin(\phi)\sin(\omega t)\big] \\ &= A\cos(\omega t + \phi), \end{aligned}$$

by the cosine addition formula. □

Example 1.1.2.5. To determine the amplitude A and phase constant ϕ for the oscillator in the previous example, we can use the values of A_1 and A_2 we found:

$$\begin{aligned} A &= \sqrt{(-0.207)^2 + (-0.311)^2} \approx 0.374 \text{ m} \\ \phi &= \arctan\left(\frac{-(-0.311)}{-0.207}\right) \approx 4.12 \text{ rad} \quad (\text{since } A_1 < 0 \text{ and } A_2 < 0) \end{aligned}$$

Therefore, the equation describing the position $x(t)$ of the oscillator as a function of time can also be written as:

$$x(t) = 0.374\cos(4.28t + 4.12)$$

Definiton 1.1.2.6 (The Energy of a Simple Harmonic Oscillator)**.** The total mechanical energy E of a simple harmonic oscillator is the sum of its kinetic energy K and potential energy U.

$$E = K + U$$

First we consider the change of potential energy from a position x_i to x_f, assuming the path is along the spring or the curve C of the oscillator. The force exerted by the spring is given by Hooke's law, $F = -kx$. The change in potential energy can be simply parametized and calculated as follows:

$$\Delta U = \int_C F \cdot ds = -\int_{x_i}^{x_f} F\,dx = \int_{x_i}^{x_f} kx\,dx = \left[\frac{1}{2}kx^2\right]_{x_i}^{x_f} = \frac{1}{2}k(x_f^2 - x_i^2)$$

Therefore, the potential energy U at a position x (taking the reference point at $x = 0$) is given by:

$$U(x) = \frac{1}{2}kx^2$$

The kinetic energy K of the oscillator is given by:

$$K = \frac{1}{2}m\dot{x}^2$$

Therefore, the total mechanical energy E of the simple harmonic oscillator is:

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{1.3}$$

The total mechanical energy E remains constant over time, as energy is conserved in the absence of non-conservative forces (like friction or air resistance).

1.1.3 Damped Harmonic Motion

Key Concepts & Equations

Key Equations:

- Damped ODE: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ where $\gamma = b/m$
- Underdamped ($\gamma < 2\omega_0$): $x(t) = A_0 e^{-\gamma t/2}\cos(\omega_r t + \phi)$ where $\omega_r = \sqrt{\omega_0^2 - \gamma^2/4}$
- Critically damped ($\gamma = 2\omega_0$): $x(t) = (A_1 t + A_2)e^{-\gamma t/2}$
- Overdamped ($\gamma > 2\omega_0$): Exponential decay without oscillation

Concept: Critical damping returns to equilibrium fastest without oscillating!

We solve the damped ODE, classify regimes (underdamped, critical, overdamped), and connect decay rates with parameters.

Definiton 1.1.3.1 (Damped Harmonic Motion)**.** For small velocities, the drag force is approximately proportional to the velocity and acts in the opposite direction. This drag force can be modeled as $F_d = -\gamma\dot{x}$, where γ is the damping coefficient. Including this drag force in the equation of motion for a harmonic oscillator leads to the damped harmonic oscillator equation (1.2). Which could be rewritten as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the undamped oscillator, and $\gamma = \frac{b}{m}$ is the damping coefficient per unit mass.

To skip the math, lets assume a solution of the form $x(t) = e^{i\omega t}$, substituting into the differential equation gives us a formulation for ω:

$$\omega = -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{1.4}$$

Also, we can characterize the real and imaginary parts of ω as:

$$\omega_r = \mathrm{Re}(\omega) = \pm\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}, \qquad \omega_i = \mathrm{Im}(\omega) = -\frac{\gamma}{2}$$

The general solution for the damped harmonic oscillator can be written as:

$$x(t) = \exp(\omega_i t)\exp(-i\omega_r t) = \exp\left(-\frac{\gamma}{2}t\right)\exp(\mp i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t)$$

- **No Damping** ($\gamma = 0$): The system behaves like a simple harmonic oscillator with angular frequency ω_0. Given by:
$$z = \exp(-i\omega_0 t)$$
- **Underdamping** ($0 < \gamma < 2\omega_0$): The system oscillates with a gradually decreasing amplitude. The angular frequency of oscillation is given by $\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. Given by:
$$z = \exp\left(-\frac{\gamma}{2}t\right)\exp(-i\omega_r t)$$
The trigonometric form of the solution is:
$$x(t) = A_0 \exp\left(-\frac{\gamma}{2}t\right)\cos(\omega_r t + \phi)$$
where A_0 and ϕ are constants determined by the initial conditions From this, we can derive the following cases:
- **Critical Damping** ($\gamma = 2\omega_0$): The system returns to equilibrium as quickly as possible without oscillating. Consider:
$$x(t) = e^{-\frac{\gamma}{2}t} f(t)$$
Inserting into the differential equation, we get:
$$\ddot{f} + \left(\omega_0^2 - \frac{\gamma^2}{4}\right) f = 0$$
Since $\gamma = 2\omega_0$, we have $\omega_0^2 - \frac{\gamma^2}{4} = 0$, leading to:
$$\ddot{f} = 0 \implies f(t) = A_1 t + A_2$$
Therefore, the general solution for the critically damped case is:
$$x(t) = (A_1 t + A_2)\exp\left(-\frac{\gamma}{2}t\right)$$
where A_1 and A_2 are constants determined by
- **Overdamping** ($\gamma > 2\omega_0$): The system returns to equilibrium without oscillating, but more slowly than in the critically damped case. The solution is given by:
$$z = \exp\left(-\frac{\gamma}{2}t\right)\exp\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2}t\right)$$
So the general solution is (the solution is via a substitution of $x(t) = e^{-\gamma t/2} f(t)$ into the differential equation, which resolves the ODE to a simple form):
$$x(t) = A_1 \exp\left[\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] + A_2 \exp\left[\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right]$$
where A_1 and A_2 are constants determined by the initial conditions.

1.1.4 Energy and Quality Factor

Key Concepts & Equations

Key Equations:

- Energy decay: $E(t) = E_0 e^{-\gamma t}$ for light damping
- Time constant: $\tau = 1/\gamma$ (time for energy to drop to $1/e$)
- Quality factor: $Q = \omega_0/\gamma = \omega_0\tau$ (higher Q = less damping)
- Energy loss per cycle: $\frac{\Delta E}{E} \approx -\frac{2\pi}{Q}$

Concept: Q measures how many oscillations occur before energy decays significantly!

We study how energy decays under light damping, define the time constant and quality factor Q, and relate them to response.

Definiton 1.1.4.1 (Energy of a Very Light Damping)**.** Consider a very lightly damped harmonic oscillator, where $\gamma \ll \omega_0$. In this case, the angular frequency of oscillation ω_r can be approximated as:

$$\omega_r \approx \omega_0 \left(1 - \frac{\gamma^2}{8\omega_0^2}\right) \approx \omega_0$$

So the motion of the lightly damped oscillator can be approximated as:

$$x(t) \approx A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi)$$

Then, we can calculate the velocity of the oscillator:

$$\begin{aligned}\dot{x}(t) &= -\frac{\gamma}{2} A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi) - A_0\omega_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t + \phi) \\ &= A_0\omega_0 e^{-\frac{\gamma}{2}t} \left(-\frac{\gamma}{2\omega_0} \cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi)\right)\end{aligned}$$

The total mechanical energy $E(t)$ of the lightly damped oscillator is given by:

$$\begin{aligned}E(t) &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\left[A_0\omega_0 e^{-\frac{\gamma}{2}t}\left(-\frac{\gamma}{2\omega_0}\cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi)\right)\right]^2 + \frac{1}{2}k\left[A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_0 t + \phi)\right]^2 \\ &= \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t}\left[\left(-\frac{\gamma}{2\omega_0}\cos(\omega_0 t + \phi) - \sin(\omega_0 t + \phi)\right)^2 + \cos^2(\omega_0 t + \phi)\right] \\ &= \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t}\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi) + \frac{\gamma^2}{4\omega_0^2}\cos^2(\omega_0 t + \phi) + \frac{\gamma}{\omega_0}\sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi)\right] \\ &\approx \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t}\left[1 + \frac{\gamma^2}{4\omega_0^2}\cos^2(\omega_0 t + \phi)\right] \quad \text{(neglecting the small term } \frac{\gamma}{\omega_0}\sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi)\text{)} \\ &\approx \frac{1}{2}mA_0^2\omega_0^2 e^{-\gamma t} \quad \text{(since } \frac{\gamma^2}{4\omega_0^2} \text{ is very small)} \\ &= E_0 e^{-\gamma t} \quad \text{where } E_0 = \frac{1}{2}mA_0^2\omega_0^2 \text{ is the initial energy at } t = 0\end{aligned}$$

We can also define the time constant τ as the time it takes for the energy to decrease to $\frac{1}{e}$ of its initial value:

$$\tau = \frac{1}{\gamma}$$

So we have, for very light damping:

$$E(t) = E_0 e^{-\gamma t} = E_0 e^{-\frac{t}{\tau}} \tag{1.6}$$

Definiton 1.1.4.2 (Rate of Energy Loss)**.** Taking the time derivative of the total mechanical energy $E(t)$:

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) \\ &= (ma + kx)\dot{x}\end{aligned}$$

For a undamped harmonic oscillator, $ma + kx = 0$, so $\frac{dE}{dt} = 0$, indicating that the total mechanical energy is conserved and obeys Hooke's law completely. However, for a damped harmonic oscillator, $ma + kx = -b\dot{x}$, leading to:

$$\frac{dE}{dt} = -b\dot{x}^2$$

Definiton 1.1.4.3 (Quality Factor (Q-Factor))**.** The quality factor Q is a dimensionless parameter that characterizes the damping of a harmonic oscillator. It is defined as:

$$Q = \frac{\omega}{\gamma} = \omega\tau \tag{1.7}$$

And for very light damping, we can approximate $\omega \approx \omega_0$, leading to:

$$Q \approx \frac{\omega_0}{\gamma} = \omega_0\tau$$

This allows us to rewrite the equation of a damped harmonic oscillator as:

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0$$

and:

$$\omega = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

We can also consider the ratio between the energy at one time and the energy one period later:

$$\begin{aligned}\frac{E(t+T)}{E(t)} &= \frac{E_0 e^{-\gamma(t+T)}}{E_0 e^{-\gamma t}} \\ &= e^{-\gamma T} \approx 1 - \gamma T \\ \frac{E(t+T) - E(t)}{E(t)} &\approx -\gamma T = -\frac{2\pi}{\omega_0}\gamma = -\frac{2\pi}{Q}\end{aligned}$$

Example 1.1.4.4. *What is the number of radians through which the damped system oscillates as its energy decreases to* $1/e$ *of its initial value?*

We have

$$\frac{E}{E_0} = e^{-\gamma t} = \frac{1}{e} \implies \gamma t = 1 \implies t = \frac{1}{\gamma} = \tau$$

So the number of radians is

$$\theta = \omega\tau = \frac{\omega}{\gamma} = Q$$

1.1.5 Undamped Forced Oscillations

Key Concepts & Equations

Key Equations:

- Driven ODE: $m\ddot{x} + kx = F_0 \cos(\omega t)$
- Steady-state solution: $x(t) = A(\omega)\cos(\omega t - \delta)$
- Amplitude: $A(\omega) = \frac{\omega_0^2 a}{|\omega_0^2 - \omega^2|}$ where $a = F_0/k$

Common Pitfall: At resonance ($\omega \approx \omega_0$), amplitude becomes infinite without damping!

We examine steady-state response to sinusoidal driving without losses and identify resonance behavior.

Definiton 1.1.5.1 (Undamped Forced Oscillations)**.** Consider a driver force acting on the mass-spring system; the ODE becomes:

$$m\ddot{x} + kx = \eta$$

where $\eta = F_0 \cos(\omega t)$ is the driving force with amplitude F_0 and angular frequency ω. The steady-state particular solution is:

$$x(t) = A(\omega)\cos(\omega t - \delta)$$

To derive $A(\omega)$ and δ, use the equation of motion:

$$A(\omega)(-\omega^2 + \omega_0^2)\cos(\delta) = \omega_0^2 a$$

where $a = \frac{F_0}{k}$ is the static displacement of the mass when the driving force is constant. We also have:

$$A(\omega)(-\omega^2 + \omega_0^2)\sin(\delta) = 0$$

Now, consider the case when $\delta = 0$, we have:

$$A(\omega) = \frac{\omega_0^2 a}{\omega_0^2 - \omega^2}$$

and the case where $\delta = \pi$, we have:

$$A(\omega) = -\frac{\omega_0^2 a}{\omega_0^2 - \omega^2}$$

Observe, when $\omega \approx \omega_0$, the amplitude $A(\omega)$ becomes very large, indicating resonance. At resonance, the system oscillates with maximum amplitude, which can lead to significant energy transfer from the driving force to the oscillator.

1.1.6 Damped Forced Oscillations

Key Concepts & Equations

Key Equations:

- Amplitude: $A(\omega) = \frac{\omega_0^2 a}{\sqrt{\omega^2\gamma^2+(\omega^2-\omega_0^2)^2}}$
- Resonance amplitude: $A_{\max} = Qa$ where $Q = \omega_0/\gamma$
- Phase shift: $\tan(\delta) = \frac{\omega\gamma}{\omega_0^2-\omega^2}$
- FWHM: $\Delta\omega = \gamma = \omega_0/Q$

Concept: Damping prevents infinite amplitude at resonance and causes phase lag!

We add damping to the driven case, derive the frequency response amplitude and phase, and study bandwidth and Q.

Definiton 1.1.6.1 (Damped Forced Oscillations)**.** Consider a damped harmonic oscillator subjected to an external driving force. It is described by the following differential equation:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

where F_0 is the amplitude of the driving force, ω is the angular frequency of the driving force, b is the damping coefficient, m is the mass, and k is the spring constant. The general solution to this non-homogeneous differential equation is given by:

$$x(t) = A(\omega)\cos(\omega t - \delta)$$

and the amplitude $A(\omega)$ is given by:

$$A(\omega) = \frac{\omega_0^2 a}{\sqrt{\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2}} \tag{1.8}$$

We can derive the following 4 cases:

1. **No Damping** ($\gamma = 0$): In this case, the amplitude $A(\omega)$ simplifies to the undamped case we discussed earlier.
2. **Low-Frequency Limit** ($\omega \ll \omega_0$): In this limit, the amplitude $A(\omega)$ approaches the static displacement $a = \frac{F_0}{k}$. This means that at very low frequencies, the system behaves like a static spring, and the mass is displaced by an amount proportional to the applied force.

3. **High-Frequency Limit** ($\omega \gg \omega_0$): In this limit, the amplitude $A(\omega)$ decreases with increasing frequency, following the relation $A(\omega) \approx \frac{\omega_0^2 a}{\omega^2}$. This indicates that at very high frequencies, the mass cannot respond quickly enough to the rapidly oscillating driving force, resulting in a smaller amplitude of oscillation.

4. **Resonance** ($\omega \approx \omega_0$): At resonance, the amplitude $A(\omega)$ reaches its maximum value but it does not become infinite due to the presence of damping:

$$A_{\max} = \frac{\omega_0^2 a}{\omega_0 \gamma} = \frac{\omega_0 a}{\gamma} = Qa$$

We also have the phase shift δ given by:

$$\tan(\delta) = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$$

The phase shift δ indicates how much the oscillation of the mass lags behind the driving force. The behavior of δ can be summarized as follows:

- At low frequencies ($\omega \ll \omega_0$), δ approaches 0, meaning the mass oscillates in phase with the driving force.
- At resonance ($\omega = \omega_0$), $\delta = \frac{\pi}{2}$, indicating that the mass oscillates a quarter cycle behind the driving force.
- At high frequencies ($\omega \gg \omega_0$), δ approaches π, meaning the mass oscillates out of phase with the driving force.

Definiton 1.1.6.2 (Power absorbed during forced oscillations)**.** We can calculate the velocity of the oscillator:

$$\dot{x}(t) = -\omega A(\omega)\sin(\omega t - \delta) = -v_0 \sin(\omega t - \delta)$$

where $v_0 = \omega A(\omega)$ is the maximum speed of the oscillator. Energy is lost at the following rate:

$$P(t) = bv(t)^2$$

Substituteing the expression for $v(t)$ into the power equation gives:

$$P(t) = bv_0^2 \sin^2(\omega t - \delta)$$

The average power $\langle P \rangle$ over one complete cycle of the driving force is given by:

$$\bar{P}(\omega) = \frac{bv_0^2}{2} = \frac{\omega^2 F_0^2 \gamma}{2m\left[(\omega^2 - \omega_0^2)^2 + \omega^2\gamma^2\right]} \tag{1.9}$$

If the driving frequency ω is close to ω_0, use $\omega^2 - \omega_0^2 \approx 2\omega_0\Delta\omega$ with $\Delta\omega = \omega - \omega_0$ to get the Lorentzian form:

$$\bar{P}(\omega) \approx \frac{\omega_0 F_0^2}{2m\gamma\left[1 + \left(\frac{2\Delta\omega}{\gamma}\right)^2\right]}$$

The maximum average power occurs at resonance:

$$\bar{P}_{\max} = \frac{\omega_0 F_0^2}{2m\gamma}$$

Full Width Half Height The graph of $\bar{P}(\omega)$ versus ω has a peak at $\omega = \omega_0$ with a maximum value of $\bar{P}_{\max}$. The full width at half maximum (FWHM) is the width of the peak at half of its maximum height. The FWHM is given by:

$$\omega_{\text{FWHM}} = 2\Delta\omega = 2\gamma = \frac{2\omega_0}{Q}$$

To illustrate the graph, we can plot $\bar{P}(\omega)$ versus ω for different values of the quality factor Q:

Average Power vs Driving Frequency for Different Quality Factors, Power normalized to $\bar{P}_{\max} = 1$

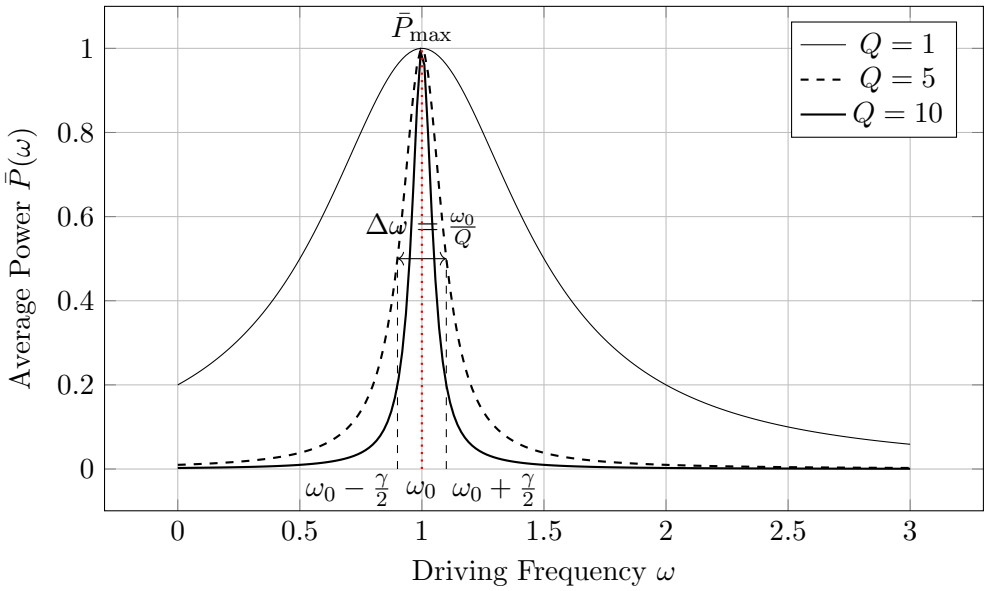

Figure 1.1: Average Power vs Driving Frequency for Different Quality Factors

1.1.7 Simple Pendulum

Key Concepts & Equations

Key Equations:

- Small angle approximation: $\ddot{\theta} + \frac{g}{l}\theta = 0$ (valid for $\theta \ll 1$ rad)
- Period: $T = 2\pi\sqrt{l/g}$ (independent of mass!)
- Physical pendulum: $\omega = \sqrt{\frac{mgd}{I}}$ where d is distance to CM

Tip: LC circuits are analogous to oscillators: $L \leftrightarrow m$, $1/C \leftrightarrow k$, $R \leftrightarrow b$

Consider the arc length s of a pendulum bob, we have:

$$s = l\theta \quad m\ddot{s} = -mg\sin(\theta)$$

For small angles, we can approximate $\sin(\theta) \approx \theta$, leading to:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

And the ODE has solution:

$$\theta(t) = \theta_0 \cos(\omega t + \phi)$$

Energy For small angle θ, the total mechanical energy E of the pendulum is given by:

$$E = K + U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos(\theta)) \approx \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2$$

This shows that the pendulum behaves like a simple harmonic oscillator with an effective spring constant $k = \frac{mg}{l}$.

The physical pendulum For a rigid body swinging about a pivot point, the equation of motion is given by:

$$I\ddot{\theta} = \tau$$

For simple rod of length L and mass m pivoted at one end, the moment of inertia about the pivot point is $I = \frac{1}{3}mL^2$. The torque due to gravity when the rod is displaced by an angle θ from the vertical is $\tau = -mg\frac{L}{2}\sin(\theta)$. For small angles, we can approximate $\sin(\theta) \approx \theta$, leading to:

$$\ddot{\theta} + \frac{3g}{2L}\theta = 0$$

And the small angle approximates leads to:

$$\ddot{\theta} + \omega^2\theta = 0$$

where $\omega = \sqrt{\frac{3g}{2L}}$. The solution to this ODE is:

$$\theta(t) = \theta_0 \cos(\omega t + \phi)$$

The angular frequency of oscillation for the physical pendulum is:

$$\omega = \sqrt{\frac{3g}{2L}}$$

The LC Circuit An LC circuit consists of an inductor L and a capacitor C connected in series. The charge $q(t)$ on the capacitor satisfies the differential equation:

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

This is analogous to the equation of motion for a simple harmonic oscillator, with the angular frequency given by:

$$\omega = \frac{1}{\sqrt{LC}}$$

The RLC Circuit An RLC circuit consists of a resistor R, inductor L, and capacitor C connected in series. The charge $q(t)$ on the capacitor satisfies the differential equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

This is analogous to the resistor behaving as a damped harmonic oscillator, with the angular frequency given by:

$$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

The following are equations that summarize the analogies between mechanical and electrical oscillators:

Mechanical Oscillator	**Electrical Oscillator**	**Analogy**
Mass m	Inductance L	Inertia
Spring constant k	Inverse capacitance $\frac{1}{C}$	Restoring force
Damping coefficient b	Resistance R	Energy dissipation
Displacement x	Charge q	Position
Velocity $\dot{x}$	Current $I = \frac{dq}{dt}$	Rate of change of position
Force F	Voltage V	Driving force

Table 1.1: Analogies between Mechanical and Electrical Oscillators

1.1.8 Coupled Oscillators

Key Concepts & Equations

Key Concepts:

- Normal coordinates $q_1 = x_1 + x_2$, $q_2 = x_1 - x_2$ decouple the equations
- Each normal mode has its own frequency: ω_1 (in-phase), ω_2 (out-of-phase)
- General motion = superposition of normal modes
- Energy doesn't flow between modes (they're independent)

Tip: Beat frequency $= (\omega_2 - \omega_1)/2$ describes energy transfer between oscillators!

Definiton 1.1.8.1 (Coupled Oscillators)**.** Consider two masses m_1 and m_2 on two pendulum connected by springs with spring constant k. The equations of motion for the two masses are given by:

$$\begin{cases} \ddot{x}_1 &= -\frac{g}{L}x_1 + \frac{k}{m_1}(x_1 - x_2) \quad (a) \\ \ddot{x}_2 &= -\frac{g}{L}x_2 - \frac{k}{m_2}(x_1 - x_2) \quad (b) \end{cases}$$

where x_1 and x_2 are the displacements of masses m_1 and m_2 from their equilibrium positions, respectively. Now if we add and subtract the two equations, we get:

$$\begin{cases} \ddot{x}_1 + \ddot{x}_2 &= -\frac{g}{L}(x_1 + x_2) \quad (c) \\ \ddot{x}_1 - \ddot{x}_2 &= -\left(\frac{g}{L} + \frac{k}{m_1} + \frac{k}{m_2}\right)(x_1 - x_2) \quad (d) \end{cases}$$

Let:

$$q_1 = x_1 + x_2 \quad \text{and} \quad q_2 = x_1 - x_2$$

also, we let:

$$\omega_1 = \sqrt{\frac{g}{L}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{L} + \frac{k}{m_1} + \frac{k}{m_2}}$$

Then, the equations of motion can be rewritten as:

$$\begin{cases} \ddot{q}_1 &= -\omega_1^2 q_1 \\ \ddot{q}_2 &= -\omega_2^2 q_2 \end{cases}$$

Observe that q_1 and q_2 are decoupled, and each behaves like a simple harmonic oscillator with angular frequencies ω_1 and ω_2, respectively. The general solutions for $q_1(t)$ and $q_2(t)$ are:

$$\begin{cases} q_1(t) &= A_1 \cos(\omega_1 t + \phi_1) \\ q_2(t) &= A_2 \cos(\omega_2 t + \phi_2) \end{cases}$$

where A_1, A_2, ϕ_1, and ϕ_2 are constants determined by the initial conditions. Finally, we can expressions for $x_1(t)$ and $x_2(t)$ in terms of $q_1(t)$ and $q_2(t)$:

$$\begin{cases} x_1(t) &= \frac{q_1(t)+q_2(t)}{2} = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) + \frac{A_2}{2}\cos(\omega_2 t + \phi_2) \\ x_2(t) &= \frac{q_1(t)-q_2(t)}{2} = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) - \frac{A_2}{2}\cos(\omega_2 t + \phi_2) \end{cases}$$

All parts of the system oscillate with the same normal frequency (eigenvalue) in a normal mode (eigenvector).

Definiton 1.1.8.2 (Normal Modes)**.** A **normal mode** is a natural pattern of oscillation in which every part of the system moves in sync with a single frequency.

For the two-mass pendulum system, there are two normal modes:

- **In-phase mode** (ω_1): both masses swing together, reaching maximum displacement on the same side at the same time.
$$x_1 = x_2$$
- **Out-of-phase mode** (ω_2): the two masses swing in opposite directions, so when one moves left, the other moves right.
$$x_1 = -x_2$$

Any general motion of the system can be written as a combination (superposition) of these two normal modes.

Energy in Coupled Oscillators The total mechanical energy E of the coupled oscillator system is given by:

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}m_1\frac{g}{L}x_1^2 + \frac{1}{2}m_2\frac{g}{L}x_2^2$$

Substituting the expressions for $x_1 = \frac{q_1+q_2}{2}$ and $x_2 = \frac{q_1-q_2}{2}$ into the energy equation, two independent energies terms for q_1 and q_2 emerge:

$$E = \left[\frac{1}{4}m\dot{q}_1^2 + \frac{1}{4}\frac{m_1 g}{L}q_1^2\right] + \left[\frac{1}{4}m\dot{q}_2^2 + \frac{1}{4}\frac{m_2 g}{L}q_2^2\right]$$

This shows that the total energy is the sum of the energies associated with each normal mode, and that no cross term between the modes exists. Thus energy does not flow from one mode to another.

Example 1.1.8.3. Given two coupled pendulums with $x_0(0) = A$, $x_1(0) = 0$, and both masses at rest initially, we have:

$$q_0(t) = C_0\cos(\omega_0 t)$$
$$q_1(t) = C_1\cos(\omega_1 t)$$

where C_0 and C_1 are constants determined by the initial conditions. Using the initial conditions. Using the cosine addition formula, which is:

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

we can express the motion of each pendulum as:

$$x_0(t) = \frac{q_0 + q_1}{2} = A\cos\left(\frac{\omega_0+\omega_1}{2}t\right)\cos\left(\frac{\omega_0-\omega_1}{2}t\right)$$
$$x_1(t) = \frac{q_0 - q_1}{2} = A\sin\left(\frac{\omega_0+\omega_1}{2}t\right)\sin\left(\frac{\omega_0-\omega_1}{2}t\right)$$

Due to the formula for the product of cosines and sines, we see that energy oscillates between the two pendulums with a beat frequency of $\frac{\omega_1-\omega_0}{2}$. The time for all the energy to transfer from one pendulum to the other is given by:

$$T = \frac{2\pi}{\frac{\omega_1-\omega_0}{2}} = \frac{4\pi}{\omega_1 - \omega_0}$$

1.1.9 Normal Modes

Key Concepts & Equations

Key Concepts:

- Normal mode = pattern where all parts oscillate at same frequency
- Find frequencies by solving eigenvalue problem: $\det(K - \omega^2 M) = 0$
- Number of normal modes = number of degrees of freedom
- Eigenvectors give amplitude ratios for each mode

Tip: For symmetric systems, look for in-phase and out-of-phase modes!

Oscilating Masses Connected by Springs Consider two masses $m_1 = m_2 = m$ connected by three spring with spring constant k and fixed to walls on either side. The equations of motion for the two masses are given by:

$$\begin{cases} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \quad (a) = m\ddot{x}_1 = -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -kx_2 - k(x_2 - x_1) \quad (b) = m\ddot{x}_2 = kx_1 - 2kx_2 \end{cases}$$

To solve these equations, we assume solutions of the form:

$$\begin{cases} x_1(t) &= A\cos(\omega t + \phi) \\ x_2(t) &= B\cos(\omega t + \phi) \end{cases}$$

Substituting these assumed solutions into the equations of motion, we get:

$$\begin{cases} \frac{A}{B} &= \frac{k}{m\omega^2 - 2k} \\ \frac{A}{B} &= \frac{m\omega^2 - 2k}{k} \end{cases} \tag{1.10}$$

We can deduce that $A = \pm B$ We can set the equation:

$$(m\omega^2 - 2k)^2 = k^2 \implies m\omega^2 - 2k = \pm k$$

So we can derive two normal mode frequencies:

$$\begin{cases} \omega_1 &= \sqrt{\frac{k}{m}} \quad \text{(in-phase mode)} \\ \omega_2 &= \sqrt{\frac{3k}{m}} \quad \text{(out-of-phase mode)} \end{cases}$$

Solving as an Eigenproblem Alternatively, we can create a system of equations from (1.10):

$$\begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

To find the eigenvalues ω^2, we solve the characteristic equation:

$$\det\left(\begin{bmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{bmatrix}\right) = 0$$

Calculating the determinant, we have:

$$\left(\frac{2k}{m} - \omega^2\right)^2 - \left(-\frac{k}{m}\right)^2 = 0$$

Expanding and simplifying, we get:

$$\omega^4 - \frac{4k}{m}\omega^2 + \frac{5k^2}{m^2} = 0$$

This gives us the two normal mode frequencies:

$$\begin{cases} \omega_1^2 &= \frac{k}{m} \implies \omega_1 = \sqrt{\frac{k}{m}} \\ \omega_2^2 &= \frac{3k}{m} \implies \omega_2 = \sqrt{\frac{3k}{m}} \end{cases}$$

Example 1.1.9.1. CConsider two equal masses m suspended from identical springs of spring constant k. The masses are hanged on the celing. We can solve this system's frequency via the eigenvalue method. The equations of motion are:

$$\begin{cases} m\ddot{x}_2 &= -k(x_2 - x_1) \\ m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \end{cases}$$

Assume solutions of the form:

$$\begin{cases} x_1(t) &= A\cos(\omega t + \phi) \\ x_2(t) &= B\cos(\omega t + \phi) \end{cases}$$

Substituting these assumed solutions into the equations of motion, we get:

$$\begin{cases} -A\omega^2 &= -\frac{k}{m}(-2A + B) \\ -B\omega^2 &= -\frac{k}{m}(A - B) \end{cases}$$

We can rewrite this as an eigenvalue problem:

$$\begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

To find the eigenvalues ω^2, we solve the characteristic equation:

$$\det\left(\begin{bmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - \omega^2 \end{bmatrix}\right) = 0$$

Calculating the determinant, we have:

$$\omega^2 = \frac{k}{2m}(3 \pm \sqrt{5})$$

Example 1.1.9.2. Consider three equal masses m connected by four identical springs with spring constant k and fixed to walls on either side. The equations of motion for the three masses are given by:

$$\begin{cases} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -2kx_2 + kx_1 + kx_3 \\ m\ddot{x}_3 &= -2kx_3 + kx_2 \end{cases}$$

Assume solutions of the form:

$$\begin{cases} x_1(t) &= A\cos(\omega t + \phi) \\ x_2(t) &= B\cos(\omega t + \phi) \\ x_3(t) &= C\cos(\omega t + \phi) \end{cases}$$

Substituting these assumed solutions into the equations of motion, we get:

$$\begin{cases} A\omega^2 &= -\frac{k}{m}(-2A + B) \\ B\omega^2 &= -\frac{k}{m}(A - 2B + C) \\ C\omega^2 &= -\frac{k}{m}(B - 2C) \end{cases}$$

We can rewrite this as an eigenvalue problem:

$$\begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

To find the eigenvalues ω^2, we solve the characteristic equation:

$$\det\left(\begin{bmatrix} -\frac{2k}{m} - \omega^2 & \frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} - \omega^2 & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{2k}{m} - \omega^2 \end{bmatrix}\right) = 0$$

Calculating the determinant, we have:

$$\left(\frac{2k}{m} - \omega^2\right)\left[\left(-\frac{2k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2\right] - \left(\frac{k}{m}\right)^2\left(\frac{2k}{m} - \omega^2\right) = 0$$

This gives us the three normal mode frequencies:

$$\begin{cases} \omega_1 &= \sqrt{\frac{2k}{m}} \\ \omega_2 &= \sqrt{\frac{2k}{m}(2 - \sqrt{2})} \\ \omega_3 &= \sqrt{\frac{2k}{m}(2 + \sqrt{2})} \end{cases}$$

Intuition For any perturbation, we can expressed it as a linear combination of the excitations of several normal modes. Each normal mode oscillates at its own frequency, and the overall motion is a superposition of these modes.

1.2 Waves

1.2.1 Travelling Waves

Key Concepts & Equations

Key Equations:

- Wave function: $y(x,t) = A\cos(kx - \omega t + \phi_0)$ (travelling right)
- Wave speed: $v = \omega/k = f\lambda$
- Wave number: $k = 2\pi/\lambda$, Angular frequency: $\omega = 2\pi f$
- String wave speed: $v = \sqrt{\tau/\mu}$ where τ is tension, μ is linear density

Common Pitfall: Don't confuse particle velocity ($\partial y/\partial t$) with wave velocity v!

Definiton 1.2.1.1 (Travelling Pulse)**.** A travelling pulse is a disturbance that moves through a medium. For any wave function $f(x,t)$, if it satisfies the property:

$$y(x,t) = f(x \pm vt) \tag{1.11}$$

then it represents a wave travelling in the positive (for $x - vt$) or negative (for $x + vt$) x-direction with speed v. y could represent displacement, pressure, electric field, etc.

Definiton 1.2.1.2 (Sinusoidal Wave)**.** A sinusoidal wave is a wave that can be described by a sine or cosine function. The general form of a sinusoidal wave travelling in the positive x-direction is:

$$y(x,t) = A\cos(kx \pm \omega t + \phi_0)$$

where:

- A is the amplitude (maximum displacement)
- $k = \frac{2\pi}{\lambda}$ is the wave number, with λ being the wavelength
- $\omega = 2\pi f$ is the angular frequency, with f being the frequency
- ϕ_0 is the initial phase.

So we can write the wave function as a form of (1.11):

$$y(x,t) = A\cos\left[k(x \pm vt) + \phi_0\right]$$

The wave speed v is related to the angular frequency and wave number by:

$$v = \frac{\omega}{k} = f\lambda \tag{1.12}$$

Transverse and Longitudinal Waves There is two (or three) main categories of waves:

- **Transverse Waves**: In transverse waves, the oscillations are perpendicular to the direction of wave propagation.
  (Examples) Light waves, water waves, and waves on a string.
- **Longitudinal Waves**: In longitudinal waves, the oscillations are parallel to the direction of wave propagation.
  (Examples) Sound waves and pressure waves in fluids.
- **Both**: Some waves can exhibit both transverse and longitudinal characteristics.
  (Examples) Rayleigh surface waves in seismology.

Definiton 1.2.1.3 (Velocity of a Fixed Particle)**.** For a fixed particle, we have:

$$v_y = \frac{\partial y}{\partial t} = A\omega \cos(kx \pm \omega t + \phi_0)$$

Example 1.2.1.4 (Vibrating String)**.** Consider a string in the xy-plane, stretched along the x-axis with tension τ and linear mass density μ. Now, consider at the end of the string at x makes a small angle θ with the x-axis. Then the components of the tension are:

$$\begin{cases} \tau_x &= \tau\cos(\theta) \approx \tau \\ \tau_y &= \tau\sin(\theta) \approx \tau\frac{\partial y}{\partial x} \end{cases}$$

Now, consider a small segment of the string between x and $x + \delta x$. The vetical force on the other end is given, via linear approximation, by:

$$\tau_y(x+\delta x) \approx \tau_y(x)\left[\frac{\partial y}{\partial x}_x + \frac{\partial^2 y}{\partial x^2}\delta x\right]$$

So the net force is given by:

$$F_y = \tau_y(x+\delta x) - \tau_y(x) = \tau\frac{\partial^2 y}{\partial x^2}\delta x$$

Because $F = ma$, we have:

$$\tau\frac{\partial^2 y}{\partial x^2}\delta x = \mathrm{d}m\frac{\partial^2 y}{\partial t^2}$$

where $\mathrm{d}m = \mu\delta x$ is the mass of the small segment. Thus, we have:

$$\frac{\partial^2 y}{\partial t^2} - v^2\frac{\partial^2 y}{\partial x^2} = 0$$

where $v = \sqrt{\frac{\tau}{\mu}}$ is the wave speed on the string. This is the one-dimensional wave equation.

Definiton 1.2.1.5 (The Wave Equation)**.** The wave equation is a second-order linear partial differential equation that describes the propagation of waves (sound, EM, water) through a medium. In one dimension, it is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.13}$$

where $u(x,t)$ is the disturbance, c is the wave speed, x is position, and t is time.

Or in general:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

where ∇^2 is the Laplacian operator.

Definiton 1.2.1.6 (Energy of a String)**.** The kinetic energy of a string segment of mass $\mathrm{d}m = \mu \mathrm{d}x$ is given by:

$$\mathrm{d}K = \frac{1}{2}\mathrm{d}m v_y^2 = \frac{1}{2}\mu \mathrm{d}x \left(\frac{\partial y}{\partial t}\right)^2$$

The potential energy stored in the string due to its tension is given by:

$$\mathrm{d}U = \frac{1}{2}\tau(\delta s - \delta x)$$

where δs is the actual length of the string segment and δx is the horizontal length. For small displacements, we can approximate:

$$\delta s \approx \delta x \left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right)$$

Thus, the potential energy becomes:

$$\mathrm{d}U \approx \frac{1}{2}\tau \delta x \left(\frac{\partial y}{\partial x}\right)^2$$

The total energy E of the string segment is the sum of its kinetic and potential energies:

$$\mathrm{d}E = \frac{1}{2}\mu \mathrm{d}x \left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\tau \mathrm{d}x \left(\frac{\partial y}{\partial x}\right)^2$$

For a sinusoidal wave $y(x,t) = A\cos(kx - \omega t)$, the change in energy is:

$$\mathrm{d}E = \frac{1}{2}\mu A^2 \omega^2 \sin^2(kx - \omega t)\mathrm{d}x + \frac{1}{2}\tau A^2 k^2 \sin^2(kx - \omega t)\mathrm{d}x$$

Integrating over one wavelength λ, we find the total energy in one wavelength:

$$E = \int_0^\lambda \mathrm{d}E = \frac{1}{4}\mu A^2 \omega^2 \lambda + \frac{1}{4}\tau A^2 k^2 \lambda$$

Definiton 1.2.1.7 (Power Transmitted by a String)**.** The power carried by a transverse wave could be calculated from $P = Fv$. The transverse force on the string is given by:

$$P(x,t) = A^2 \mu v \omega^2 \sin^2(kx - \omega t + \phi_0)$$

The average power is then given by:

$$\bar{P} = \frac{1}{2}A^2 \mu v \omega^2 = \frac{1}{2}\mu A^2 \omega^2 \frac{\lambda}{T}$$

And the maximum power is given by:

$$P_{max} = A^2 \mu v \omega^2$$

Speed of Sound The speed of sound waves through various meida is as follows:

- In ideal gas, it is given by:
$$v = \sqrt{\frac{\gamma RT}{m}} = \sqrt{\frac{\gamma k_B T}{m}}$$
where γ is the adiabatic index, R is the universal gas constant, k_B is the Boltzmann constant, T is the absolute temperature, and m is the molar mass of the gas. γ varies based on the type of gas:
  - Monatomic gases (e.g., helium, neon): $\gamma = \frac{5}{3}$
  - Diatomic gases (e.g., nitrogen, oxygen): $\gamma = \frac{7}{5}$
  - Polyatomic gases (e.g., carbon dioxide, methane): γ is typically between 1.3 and 1.4
- In a liquid, it is given by:
$$v = \sqrt{\frac{B_{\text{adiabatic}}}{m}} = \sqrt{\frac{B}{\rho}}$$
where $B_{\text{adiabatic}}$ is the adiabatic bulk modulus, B is the bulk modulus, ρ is the density of the liquid and m is the molar mass of the liquid.
- In a solid, it is given by:
$$v = \sqrt{\frac{E}{\rho}}$$
where E is the Young's modulus and ρ is the density of the solid. It could also be given by:
$$v = \sqrt{\frac{B}{\rho}}$$
where B is the bulk modulus.

Definiton 1.2.1.8 (Impedance)**.** Similar to electrical impedance, we also have acoustic impedance and mechanical, they measure how much a structure recisits motion when subjected to a harmonic force. They are defined as follows:

- Electical impedance Z_e:
$$Z_e = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$
- Acoustic impedance Z_a, it is the ratio of acoustic pressure to particle velocity:
$$Z_a = \frac{p}{v} = \rho v = \sqrt{B\rho}$$
- Mechanical impedance Z_m:
$$Z_m = v\mu = \sqrt{\tau\mu}$$

Wave boundry Conditions When a wave encounters a boundary between two different media, part of the wave is reflected back into the original medium, and part is transmitted into the new medium. This is constrined by the following boundary conditions:

- The displacement of the wave must be continuous across the boundary:

$$y_1(x_b, t) = y_2(x_b, t) = y_3(x_b, t)$$

- The derivative of the displacement with respect to position must also be continuous across the boundary:

$$\left.\frac{\partial y_1}{\partial x}\right|_{x_b} = \left.\frac{\partial y_2}{\partial x}\right|_{x_b} = \left.\frac{\partial y_3}{\partial x}\right|_{x_b}$$

and then from these conditions, we can derive the amplitudes of the reflected and transmitted waves:

$$A_i = A_r + A_t \tag{1.14}$$

and

$$A_i Z_1 = A_r Z_1 + A_t Z_2$$

From these two equations, we can solve for the reflection and transmission coefficients:

$$R = \frac{A_r}{A_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{1.15}$$

and

$$T = \frac{A_t}{A_i} = \frac{2Z_2}{Z_2 + Z_1} \tag{1.16}$$

Special Cases There are two special cases:

- **Fixed End**: If the wave encounters a boundary where the medium is fixed (e.g., a string tied to a wall), the reflected wave undergoes a phase change of π (inversion). The reflection coefficient is $R = -1$ and the transmission coefficient is $T = 0$.
- **Free End**: If the wave encounters a boundary where the medium is free to move (e.g., a string attached to a ring that can slide along a rod), the reflected wave does not undergo a phase change. The reflection coefficient is $R = 1$ and the transmission coefficient is $T = 2$.

Example 1.2.1.9 (Transmission of 64% Energy)**.** Suppose a wave transmits 64% of its incident energy across a boundary. Then:

$$T_e = 0.64, \quad R_e = 1 - T_e = 0.36$$

so that the amplitude reflection coefficient is

$$R = +\sqrt{R_e} = 0.6$$

Important the positive sign indicates the reflected wave is *in phase* with the incident wave.

From the boundary condition in Eq. (1.14),

$$T = 1 + R = 1 + 0.6 = 1.6$$

Thus, the amplitude transmission coefficient is $T = 1.6$, even though only 64% of the energy is transmitted. This apparent paradox arises because the transmitted energy depends on both the amplitude squared and the ratio of impedances $\frac{Z_2}{Z_1}$.

1.2.2 Superposition and Standing Waves

Key Concepts & Equations

Key Equations:

- Standing wave: $y(x,t) = 2A\sin(kx)\cos(\omega t)$ (for fixed-fixed ends)
- Allowed wavelengths: $\lambda_n = 2L/n$ where $n = 1, 2, 3, ...$
- Frequencies: $f_n = nv/(2L) = nf_1$ (harmonics)
- Constructive interference: path difference $= m\lambda$
- Destructive interference: path difference $= (m + 1/2)\lambda$

Tip: Nodes are spaced $\lambda/2$ apart; antinodes too!

Definiton 1.2.2.1 (Modulation of two waves)**.** Consider two monochromatic waves:

$$\begin{cases} y_1(x,t) &= A\cos(k_1 x - \omega_1 t) \\ y_2(x,t) &= A\cos(k_2 x - \omega_2 t) \end{cases}$$

Then, the resultant wave $y_1 + y_2$ can be expressed as:

$$y(x,t) = 2A\cos\left(\frac{(k_1 - k_2)x - (\omega_1 - \omega_2)t}{2}\right)\cos\left(\frac{(k_1 + k_2)x - (\omega_1 + \omega_2)t}{2}\right)$$

Definiton 1.2.2.2 (Coherence)**.** Two waves are said to be coherent if they are monochromatic and maintain a constant phase relationship over time. This means that the difference in phase between the two waves does not change as they propagate:

$$\Delta\phi = \phi_1 - \phi_2 = \text{constant}$$

Coherent waves can produce stable interference patterns, such as constructive and destructive interference.

Definiton 1.2.2.3 (Phase and Phase Difference)**.** Phase of the wave is an argument $kx \pm \omega t + \phi_0$ of the wave function. The phase difference between two waves at a given point in space and time is given by:

$$\Delta\phi = (k_1 x - \omega_1 t + \phi_1) - (k_2 x - \omega_2 t + \phi_2)$$

where k_1, k_2 are the wave numbers, ω_1, ω_2 are the angular frequencies, and ϕ_1, ϕ_2 are the initial phases of the two waves.

If we consider two waves of the same frequency and wave number, the phase difference simplifies to:

$$\Delta\phi = \Delta\phi + k\Delta x + \omega\Delta t$$

Definiton 1.2.2.4 (Waves in Two and Three Dimensions)**.** In two or three dimensions, waves can propagate in various directions. The general form of a wave in two dimensions is:

$$z(x, y, t) = A\cos(k_x x + k_y y - \omega t + \phi_0)$$

where k_x and k_y are the wave numbers in the x and y directions, respectively, and ϕ_0 is the initial phase. We also define v and k as:

$$v = \frac{\omega}{k}, \quad k = \sqrt{k_x^2 + k_y^2}$$

Definiton 1.2.2.5 (Spherical Waves)**.** A spherical wave is a wave that propagates outward in all directions from a point source. The general form of a spherical wave is:

$$y(r, t) = \frac{A}{r}\cos(kr - \omega t + \phi_0)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance from the source, A is the amplitude at a reference distance, k is the wave number, ω is the angular frequency, and ϕ_0 is the initial phase. The amplitude decreases with distance as $1/r$ due to the spreading of the wavefront.

Definiton 1.2.2.6 (Interference)**.** Consider two coherent in phase sources S_1 and S_2 separated by a distance d. The path difference Δr between the two waves at a point P is given by:

$$\Delta r = r_2 - r_1$$

where r_1 and r_2 are the distances from the sources S_1 and S_2 to the point P, respectively.

We have the following two cases for constructive and destructive interference:

- **Constructive Interference**: Occurs when the path difference is an integer multiple of the wavelength:
$$\Delta r = m\lambda, \quad m = 0, 1, 2, \ldots$$
- **Destructive Interference**: Occurs when the path difference is an odd multiple of half the wavelength:
$$\Delta r = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, 1, 2, \ldots$$

Nodes and Antinodes In a standing wave, nodes are points where the medium does not move (zero displacement), while antinodes are points where the medium experiences maximum displacement. The distance between two consecutive nodes or antinodes is half the wavelength ($\frac{\lambda}{2}$). Nodes occur at positions where $kx = n\pi$ (for integer n), and antinodes occur at positions where $kx = (n + \frac{1}{2})\pi$.

Definiton 1.2.2.7 (Nodes)**.** Nodes are points along a standing wave where the wave has zero amplitude. At these points, destructive interference occurs, and the medium remains at rest. Nodes occur at:

$$x_n = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots$$

Definiton 1.2.2.8 (Antinodes)**.** Antinodes are points along a standing wave where the wave has maximum amplitude. At these points, constructive interference occurs, and the medium oscillates with the greatest displacement. Antinodes occur at:

$$x_a = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots$$

Antinodal Curves In two-dimensional standing waves, antinodal curves are the loci of points where the amplitude of the wave is maximum. These curves are formed by the constructive interference of waves from multiple sources. The shape and spacing of antinodal curves depend on the geometry of the wave sources and the wavelength of the waves. An antindal curve could be visuallized as following:

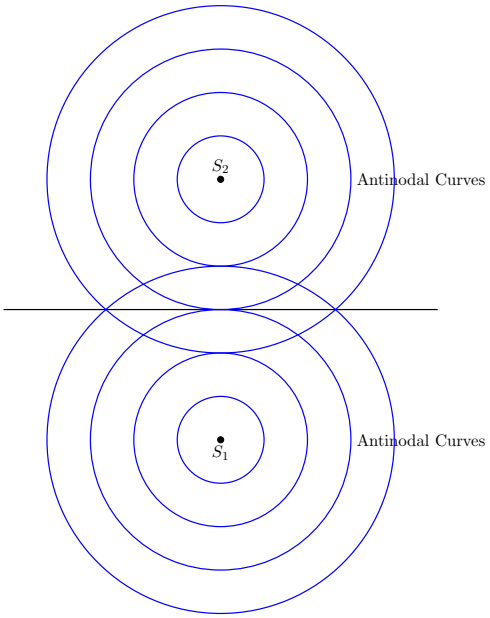

Figure 1.2: Antinodal Curves

Definiton 1.2.2.9 (Standing Waves)**.** Standing waves are formed by the superposition of two waves of the same frequency and amplitude traveling in opposite directions. The solution of the standing waves must satisfy a quantization condition, which leads to discrete frequencies and wavelengths. This discretized solution is given by:

$$y(x, t) = A_n \sin(kx)\cos(\omega t) \tag{1.17}$$

where $k = \frac{n\pi}{L}$ for $n = 1, 2, 3, \ldots$, and L is the length of the medium (e.g., a string fixed at both ends). The corresponding frequencies are given by:

$$f_n = \frac{nv}{2L}$$

In turn, the wavelengths are given by:

$$\lambda_n = \frac{2L}{n}$$

where v is the wave speed. The distance between two consecutive nodes or antinodes is given by:

$$d = \frac{\lambda}{2}$$

Systems with no Fixed Ends For systems with no fixed ends, the general solution is given by:

$$y(x,t) = A_n \cos(kx)\cos(\omega_n t)$$

where $k = \frac{n\pi}{L}$ for $n = 0, 1, 2, \ldots$. the postion of the nodes are:

$$x_n = n\frac{\lambda}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

and the position of the antinodes are:

$$x_a = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Systems with no free Ends For systems with no free ends, the general solution is given by:

$$y(x,t) = A_n \sin(kx)\cos(\omega_n t)$$

The postion of the nodes are:

$$x_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots$$

and the position of the antinodes are:

$$x_a = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots$$

System with Mixed Ends For systems with mixed ends (one fixed and one free), the general solution has two cases:

$$y(x,t) = \begin{cases} A_n \sin(kx)\cos(\omega_n t) & \text{if the fixed end is at } x = 0 \\ A_n \cos(kx)\cos(\omega_n t) & \text{if the fixed end is at } x = L \end{cases}$$

The postion of the nodes are:

$$x_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots$$

and the position of the antinodes are:

$$x_a = n\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots$$

Definiton 1.2.2.10 (Energy of Standing Waves)**.** The total energy stored in a standing wave on a string of length L is given by:

$$E = \frac{1}{4}\mu A^2 \omega^2 L$$

where μ is the linear mass density of the string, A is the amplitude of the wave, and ω is the angular frequency. The energy is equally divided between kinetic and potential energy, with each contributing half of the total energy.

Definiton 1.2.2.11 (Average Power of Standing Waves)**.** The average power transmitted by a standing wave on a string is given by:

$$\bar{P} = \frac{1}{2}\mu A^2 \omega^2 v$$

where μ is the linear mass density of the string, A is the amplitude of the wave, ω is the angular frequency, and v is the wave speed. This power represents the rate at which energy is transferred along the string.

1.2.3 Fourier analysis

Definiton 1.2.3.1 (Standing Waves as Normal Modes of a Vibrating String)**.** The superposition principle states that if $y_1(x,t)$ and $y_2(x,t)$ are two solutions to the wave equation, then their sum $y(x,t) = y_1(x,t) + y_2(x,t)$ is also a solution. For a string fixed at both ends, the n^{th} normal mode of vibration can be expressed as:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) \tag{1.18}$$

The motion of the string will be a superposition of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) \tag{1.19}$$

where A_n are the amplitudes of the respective modes, and $\omega_n = \frac{n\pi v}{L}$ are the angular frequencies of the modes.

Definiton 1.2.3.2 (The amplitude of normal modes – Fourier analysis)**.** If we look at the initial shape of the string:

$$y(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$$

Any initial shape of the string can be expressed as a Fourier sine series. Also, for fixed ends ($f(0) = f(L) = 0$), we can write the shape as the following:

$$f(x) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$$

To find the coefficients A_n, we use:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{1.20}$$

Chapter 2

Modern Physics

2.1 Quantum Mechanics

Definiton 2.1.0.1 (Quantization)**.** Quantization is the constraint that energy and other physical properties can only take on discrete values (*quanta*) rather than a continuous range.

Definiton 2.1.0.2 (Probabilistic Framework)**.** Predictions in quantum mechanics are inherently probabilistic. The exact outcome of a measurement cannot be determined, only the probability of different outcomes.

Definiton 2.1.0.3 (Black Body)**.** A black body is an idealized physical object that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence. It also emits radiation in a characteristic spectrum that depends solely on its temperature.

Definiton 2.1.0.4 (Classical View of Black Body Radiation)**.** According to classical physics, Reyleigh-Jeans law predicts that the energy density of black body radiation increases indefinitely with frequency, leading to the "ultraviolet catastrophe." The classical formula is given by:

$$u(\nu, T) = \frac{8\pi\nu^2 k_B T}{c^3}$$

where $u(\nu, T)$ is the energy density per unit frequency, ν is the frequency, k_B is the Boltzmann constant, T is the absolute temperature, and c is the speed of light.

Definiton 2.1.0.5 (Quantum View of Black Body Radiation)**.** Max Planck resolved the ultraviolet catastrophe by proposing that electromagnetic energy is quantized. He introduced the concept that energy is emitted or absorbed in discrete packets called **quanta**, with energy given by:

$$E_n = nhf \quad n = 0, 1, 2, \ldots \tag{2.1}$$

where h is Planck's constant and f is the frequency of the radiation. This leads to Planck's law for black body radiation. The resulting law for the black body spectrum is:

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{2.2}$$

2.1.1 Quantization of Light

Definiton 2.1.1.1 (De Broglie Wavelength)**.** The de Broglie wavelength is the wavelength associated with a particle and is given by:

$$\lambda = \frac{h}{p} \tag{2.3}$$

where h is Planck's constant and p is the momentum of the particle. This concept suggests that all matter exhibits wave-like properties, with the wavelength inversely proportional to the momentum.

Definiton 2.1.1.2 (Energy of a Wave)**.** The energy of a photon (quantum of light) is given by:

$$E = hf = \frac{hc}{\lambda} \tag{2.4}$$

where E is the energy of the photon, h is Planck's constant, and f is the frequency of the electromagnetic wave.

And the momentum of a photon is given by:

$$p = \frac{E}{c} = \frac{hf}{c} \tag{2.5}$$

Definiton 2.1.1.3 (Photoelectric Effect)**.** The photoelectric effect is the phenomenon where electrons are emitted from a material (usually a metal) when it is exposed to light of sufficient frequency. The key observations of the photoelectric effect are:

- Electrons are emitted only if the incident light has a frequency above a certain threshold frequency f_0.
- The kinetic energy of the emitted electrons depends on the frequency of the incident light, not its intensity.
- The number of emitted electrons is proportional to the intensity of the incident light, provided the frequency is above the threshold.

The energy of the incident photons is given by:

$$E = hf$$

where h is Planck's constant and f is the frequency of the light. The maximum kinetic energy K_{max} of the emitted electrons is given by:

$$K_{max} = hf - \phi$$

where ϕ is the work function of the material, representing the minimum energy required to liberate an electron from the surface.

Definiton 2.1.1.4 (Campton Effect)**.** The Compton effect is the increase in wavelength (or decrease in energy) of X-rays or gamma rays when they are scattered by electrons. This phenomenon provides evidence for the particle-like behavior of light. The change in wavelength $\Delta\lambda$ of the scattered photon is given by the Compton formula:

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{2.6}$$

where:

- λ is the initial wavelength of the photon,
- λ' is the wavelength after scattering,
- h is Planck's constant,
- m_e is the rest mass of the electron,
- c is the speed of light,
- θ is the angle at which the photon is scattered.

Example 2.1.1.5 (Photoelectric Experiment)**.** Consider the folling $V - f$ graph of a photoelectric experiment: From the graph, we can determine the work function ϕ and Planck's constant h:

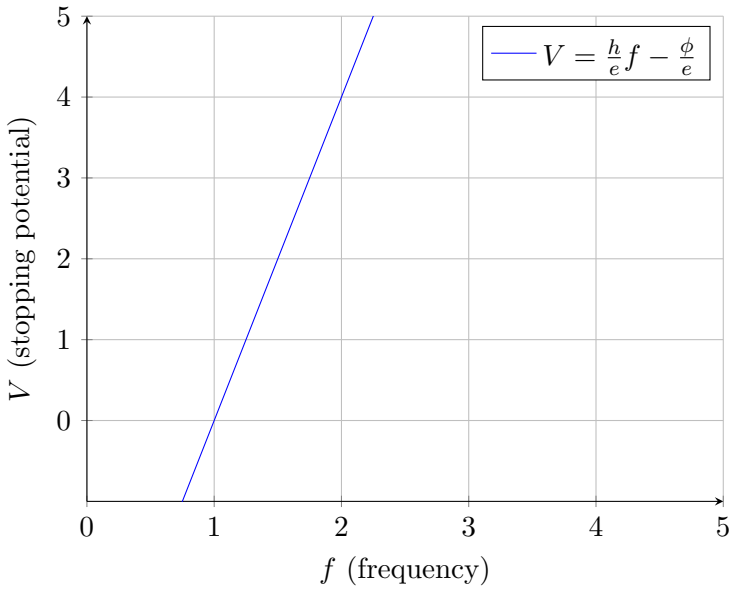

Figure 2.1: Photoelectric Experiment $V - f$ Graph

First, we find the planks constant from the slope of the line:

$$\text{slope} = \frac{h}{e} = 4\,\text{V·s}$$

Thus, we have:

$$h = 4e\,\text{V·s} = 4 \times 1.602 \times 10^{-19}\,\text{C·V·s} = 6.408 \times 10^{-19}\,\text{J·s}$$

which is approximately equal to the known value of Planck's constant $h \approx 6.626 \times 10^{-34}\,\text{J·s}$. We can find the work function from the x-intercept, and covnert to energy using the experimental value of h:

$$f_0 = 1\,\text{Hz} \implies \phi = hf_0 = 6.408 \times 10^{-19}\,\text{J}$$

converting to electron volts:

$$\phi = \frac{6.408 \times 10^{-19}\,\text{J}}{1.602 \times 10^{-19}\,\text{J/eV}} \approx 4\,\text{eV}$$

2.1.2 Quantization of Atomic Energy Level

Definiton 2.1.2.1 (Bohr Model of the Hydrogen Atom)**.** The Bohr model describes the hydrogen atom as a system where the electron orbits the nucleus in discrete energy levels. The key postulates of the Bohr model are:

- The electron moves in circular orbits around the nucleus under the influence of the Coulomb force.
- Only certain discrete orbits are allowed, corresponding to quantized angular momentum:

$$L = n\hbar = n\frac{h}{2\pi}, \quad n = 1, 2, 3, \ldots \tag{2.7}$$

- The electron does not radiate energy while in a stable orbit.
- Energy is emitted or absorbed when the electron transitions between these quantized orbits, with the energy difference given by:

$$\Delta E = E_f - E_i = hf \tag{2.8}$$

Definiton 2.1.2.2 (Energy Levels of the Hydrogen Atom)**.** The energy levels of the hydrogen atom are quantized and given by the formula:

$$E_n = -\frac{13.6\,\text{eV}}{n^2}, \quad n = 1, 2, 3, \ldots \tag{2.9}$$

where E_n is the energy of the electron in the n^{th} energy level. The negative sign indicates that the electron is bound to the nucleus. The energy difference between two levels n_i and n_f corresponds to the emission or absorption of a photon with frequency f:

$$hf = E_{n_f} - E_{n_i} = -13.6\,\text{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{2.10}$$

Definiton 2.1.2.3 (Energy Spectrum of Hydrogen-Like Atoms)**.** The energy spectrum of hydrogen-like atoms (atoms with a single electron, such as He^+, Li^{2+}, etc.) is given by:

$$E_n = -\frac{Z^2 \cdot 13.6\,\text{eV}}{n^2}, \quad n = 1, 2, 3, \ldots \tag{2.11}$$

where Z is the atomic number of the nucleus (number of protons). The energy difference between two levels n_i and n_f corresponds to the emission or absorption of a photon with frequency f:

$$hf = E_{n_f} - E_{n_i} = -Z^2 \cdot 13.6\,\text{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{2.12}$$

Hydrogen-Like Atoms Atoms with one valence electron, such as He^+, Li^{2+}, Be^{3+}, etc. Or, alkaline metals like Na, K, Rb, Cs, Fr.

Definiton 2.1.2.4 (Characteristic X-Rays and Moseley's Law)**.** When high-energy electrons strike a metal target, they can eject inner-shell electrons, leading to characteristic X-ray emission as outer-shell electrons fill the vacancies. Moseley's law relates the frequency of the emitted X-rays to the atomic number Z of the target element:

$$\sqrt{f} = K(Z - b) \tag{2.13}$$

where f is the frequency of the emitted X-ray, K is a constant, and b is a screening constant that accounts for the shielding effect of the inner electrons.

2.1.3 De Broglie's Wavelength

Definiton 2.1.3.1 (De Broglie Hypothesis)**.** The de Broglie hypothesis posits that all matter exhibits wave-like properties, with the wavelength λ of a particle being inversely proportional to its momentum p:

$$\lambda = \frac{h}{p} \tag{2.14}$$

where h is Planck's constant. This concept extends wave-particle duality to all particles, suggesting that particles such as electrons, protons, and even larger objects have associated wavelengths.

Definiton 2.1.3.2 (Wave Nature of Electrons)**.** The electron behave as wave satisfying the de Broglie relation. For a stable robit, the elevtron wave must fit exactly into the circumference of the orbit:

$$2\pi r = n\lambda, \quad n = 1, 2, 3, \ldots \tag{2.15}$$

where r is the radius of the orbit and λ is the de Broglie wavelength of the electron. This condition leads to the quantization of angular momentum in Bohr's model of the atom:

$$L = rp = n\hbar = n\frac{h}{2\pi}, \quad n = 1, 2, 3, \ldots \tag{2.16}$$

Definiton 2.1.3.3 (De Broglie Wavelength of an Electron)**.** *Calculate the de Broglie wavelength of an electron moving with a velocity of* 2×10^6 *m/s. The mass of the electron is* 9.11×10^{-31} *kg and Planck's constant is* 6.626×10^{-34} *J·s. Ignore relativistic effects.*

To calculate the de Broglie wavelength λ of the electron, we use the de Broglie relation:

$$\lambda = \frac{h}{p}$$

where p is the momentum of the electron, given by:

$$p = mv$$

where m is the mass of the electron and v is its velocity.

First, we calculate the momentum p:

$$p = (9.11 \times 10^{-31}\,\text{kg})(2 \times 10^6\,\text{m/s}) = 1.822 \times 10^{-24}\,\text{kg·m/s}$$

Now, we can calculate the de Broglie wavelength λ:

$$\lambda = \frac{6.626 \times 10^{-34}\,\text{J·s}}{1.822 \times 10^{-24}\,\text{kg·m/s}} \approx 3.64 \times 10^{-10}\,\text{m}$$

Therefore, the de Broglie wavelength of the electron is approximately 3.64×10^{-10} meters.

Example 2.1.3.4 (De Broglie Wavelength of a Baseball)**.** *Jeff Hoffman was pitching a baseball with a speed of* 30 *m/s, sending Blue Jays to the World Series. The mass of the baseball is* 0.1 *kg. Calculate the de Broglie wavelength of the baseball.*

Using the same method, we have:

$$\lambda = \frac{6.63 \times 10^{-34}\,\text{J·s}}{(0.1\,\text{kg})(30\,\text{m/s})} = 2.21 \times 10^{-34}\,\text{m}$$

Therefore, the de Broglie wavelength of the baseball is approximately 2.21×10^{-34} meters, which is extremely small and not observable in everyday life.

Example 2.1.3.5 (Momentum is Wavelength)**.** *The mass of the proton is 1836 times greater than the mass of the electron. In order for a proton to have the same momentum as an electron, how must the de Broglie wavelength of the proton compare to that of the electron?* According to the de Broglie relation:

$$\lambda = \frac{h}{p}$$

For the proton and electron to have the same momentum p, we have:

$$\lambda_p = \frac{h}{p} \quad \text{and} \quad \lambda_e = \frac{h}{p}$$

Therefore,:

$$\lambda_p = \lambda_e$$

Thus, the de Broglie wavelength of the proton must be equal to that of the electron in order for them to have the same momentum.

Example 2.1.3.6 (Davisson-Germer Experiment)**.** In the Davisson-Germer experiment, electrons were accelerated through a potential difference of 54 V before striking a nickel crystal. Calculate the de Broglie wavelength of the electrons after acceleration.

The interplaner spacing of the nickel crystal is $d = 0.091$ nm, and the first-order diffraction maximum was observed at an angle of $\theta = 50°$.

First, we calculate the kinetic energy K of the electrons after being accelerated through the potential difference V:

$$K = eV = (1.602 \times 10^{-19}\,\text{C})(54\,\text{V}) = 8.65 \times 10^{-18}\,\text{J}$$

Next, we calculate the momentum p of the electrons using the kinetic energy:

$$K = \frac{p^2}{2m} \implies p = \sqrt{2mK}$$

where m is the mass of the electron (9.11×10^{-31} kg). Thus,

$$p = \sqrt{2(9.11 \times 10^{-31}\,\text{kg})(8.65 \times 10^{-18}\,\text{J})} \approx 1.26 \times 10^{-24}\text{kg·m/s}$$

Now, we can calculate the de Broglie wavelength λ of the electrons:

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\text{J·s}}{1.26 \times 10^{-24}\,\text{kg·m/s}} \approx 1.67 \times 10^{-10}\,\text{m}$$

Finally, we can verify this wavelength using Bragg's law for diffraction:

$$n\lambda = 2d\sin\theta$$

For the first-order maximum ($n = 1$), $\theta = \frac{180-50}{2} = 65°$:

$$\lambda = 2(0.091 \times 10^{-9}\,\text{m})\sin(65°) \approx 1.67 \times 10^{-10}\,\text{m}$$

This matches our earlier calculation, confirming the de Broglie wavelength of the electrons is approximately 1.67×10^{-10} meters.

2.1.4 Matter Waves

Definiton 2.1.4.1 (Matter Wave Function)**.** In quantum mechanics, the matter wave function $\psi(x,t)$ describes the quantum state of a particle. The probability density

$$D(x,t) = |\psi(x,t)|^2 \tag{2.17}$$

gives the likelihood of finding the particle at position x and time t. The wave function must satisfy the normalization condition:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2\,dx = 1 \tag{2.18}$$

, where $|\psi(x,t)|^2$ is:

$$|\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t)$$

Here, $\psi^*(x,t)$ is the complex conjugate of

The wave function can be expressed as a superposition of plane waves:

Definiton 2.1.4.2 (Fourier Integtal)**.** The foruer transformer of a function $f(x)$ is given by:

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

The inverse fourier transform is given by:

$$f(x) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk$$

Definiton 2.1.4.3 (Postion and Momentum Space)**.** In quantum mechanics, the wave function can be represented in both position space and momentum space. In terms of fourier transformation, $f(x)$ is the wave function in position space, while $A(k)$ is the wave function in momentum space, where k is related to the momentum p by $p = \hbar k$. The probability density in momentum space is given by:

Definiton 2.1.4.4 (Localization Trade-off)**.** In fourier transform, there is a trade-off between the localization of a function in position space and its localization in momentum (wave-number) space. A function that is highly localized in position space will have a broad distribution in momentum space, and vice versa. This is a manifestation of the Heisenberg uncertainty principle. That is, if the position is well-defined, we would see a broad distribution of momenta, and if the momentum is well-defined, we would see a broad distribution of positions.

Theorem 2.1.4.5 (Fourier Uncertainty Relation)**.** The Fourier uncertainty relation states that the product of the standard deviations of a function in position space and its Fourier transform in momentum space is bounded by a constant:

$$\Delta x \Delta k \geq \frac{1}{2} \tag{2.19}$$

where Δx is the standard deviation in position space and Δk is the standard deviation in wave-number space:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \quad \Delta k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$$

Also, we have, in the time space:

$$\Delta t \Delta \omega \geq \frac{1}{2} \tag{2.20}$$

where Δt is the standard deviation in time and $\Delta \omega$ is the standard deviation in angular frequency space.

Theorem 2.1.4.6 (Heisenberg Uncertainty Principle)**.** The Heisenberg uncertainty principle states that it is impossible to simultaneously know both the exact position and exact momentum of a particle. The product of the uncertainties in position Δx and momentum Δp is bounded by a constant:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{2.21}$$

where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant. This principle reflects the fundamental limit on the precision with which certain pairs of physical properties can be known.

Proof. (Direct Proof) Starting from the Fourier uncertainty relation:

$$\Delta x \Delta k \geq \frac{1}{2}$$

We know that momentum p is related to wave-number k by:

$$p = \hbar k$$

Therefore, the uncertainty in momentum Δp can be expressed in terms of the uncertainty in wave-number Δk:

$$\Delta p = \hbar \Delta k$$

Substituting this into the Fourier uncertainty relation gives:

$$\Delta x \left(\frac{\Delta p}{\hbar} \right) \geq \frac{1}{2}$$

Multiplying both sides by $\hbar$ yields:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

This completes the proof of the Heisenberg uncertainty principle. □

Definiton 2.1.4.7 (Time-Energy Uncertainty Principle)**.** The time-energy uncertainty principle states that there is a fundamental limit to the precision with which energy and time can be simultaneously known. The product of the uncertainties in wave angular frequency $\Delta\omega$ and time Δt is bounded by a constant:

$$\Delta\omega\Delta t \geq \frac{\hbar}{2} \tag{2.22}$$

This principle implies that if a system exists in a state for a very short time, there is a corresponding uncertainty in its energy.

Proof. (Sketch) Starting from the Fourier uncertainty relation in the time domain, we use the same reasoning as in the position-momentum case. □

Definiton 2.1.4.8 (Hilbert Space)**.** A Hilbert space is a complete inner product space that provides the mathematical framework for quantum mechanics. it is the space cotnaining all physically allowed wave function that are:

- Complex, continuous, and normalizable.
- Can be added together and multiplied by complex scalars (i.e., it is a vector space).
- Equipped with an inner product that allows for the calculation of probabilities and expectation values.

Theorem 2.1.4.9 (From fourier to Hilbert Space)**.** Every wave function $\psi(x,t)$ can be represented as a vector in a Hilbert space. Where each of the basis vectors corresponds to a specific state of the system, such as a position eigenstate or momentum eigenstate. The inner product in this space allows us to calculate probabilities and expectation values of observables. Of coruse, it must satisfy the normalization condition:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2\, dx = 1$$

This idea will be explored further in the next chapters.

2.2 The Schrödinger Equation in One-Dimensional Space

Standing Waves and Stationary States

- A particle confined in space behaves like a standing wave.
- For a stationary state:
$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar},$$
giving a time-independent probability density $|\Psi|^2 = |\psi|^2$.
- Superpositions of stationary states produce time-dependent densities.

Definiton 2.2.0.1 (Time Dependent Schrödinger Equation)**.** The time-dependent Schrödinger equation describes how the quantum state of a physical system changes over time. It is given by:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) \tag{2.23}$$

where $\Psi(x,t)$ is the wave function of the particle, m is the mass of the particle, $V(x)$ is the potential energy as a function of position, and $\hbar$ is the reduced Planck's constant.

Example 2.2.0.2 (Particle in a Box)**.** Consider a particle confined in a one-dimensional box of length L with infinitely high potential walls at $x = 0$ and $x = L$. The potential $V(x)$ is given by:

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

The time-independent Schrödinger equation inside the box is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

The general solution to this equation is:

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$.

Applying the boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$, we find that $B = 0$ and $kL = n\pi$ for $n = 1, 2, 3, \ldots$. Thus, the allowed wave numbers are:

$$k_n = \frac{n\pi}{L}$$

The corresponding energy levels are quantized:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1, 2, 3, \ldots$$

The normalized wave functions are:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Therefore, the stationary states of the particle in the box are given by:

$$\Psi_n(x,t) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)e^{-iE_n t/\hbar}$$

Definiton 2.2.0.3 (Energy Quantization)**.**

- Only discrete energies E_n yield normalizable solutions satisfying boundary conditions.
- Examples:

$$U(x) = 0 \qquad \text{(infinite square well)}$$
$$U(x) = \tfrac{1}{2}kx^2 \qquad \text{(harmonic oscillator)}$$
$$U(r) = -\frac{ke^2}{r} \qquad \text{(hydrogen atom)}$$

2.2.1 Eigenstates and Observables

Definiton 2.2.1.1 (Ovservables and Operators)**.** In quantum mechanics, physical observables such as position, momentum, and energy are represented by operators that act on the wave function. An operator is a mathematical entity that transforms one function into another. For example:

- Position operator: $\hat{x} = x$
- Momentum operator: $\hat{p} = -i\hbar\frac{d}{dx}$
- Hamiltonian (energy) operator: $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

From Schrödinger Equation to Observables The time-independent Schrödinger equation (TISE) governs quantum systems:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \tag{2.24}$$

Notice, that it is an **eigenvalue equation**.

Eigenvalue Equation When an operator $\hat{A}$ acts on a special function called an **eigenfunction** $\psi(x)$, it returns the function scaled by a measurable number called an **eigenvalue** λ:

$$\hat{A}\psi(x) = \lambda\psi(x) \tag{2.25}$$

The TISE is precisely this structure, where $\hat{H}$ (the Hamiltonian) is the operator and E is the eigenvalue:

$$\hat{H}\psi(x) = E\psi(x) \tag{2.26}$$

Common Operators Below is a table of common quantum operators:

Observable	Operator	Eigenvalue
Position	$\hat{x} = x$	x_0
Momentum	$\hat{p} = -i\hbar\frac{d}{dx}$	p_0
Potential Energy	$\hat{V} = V(x)$	$V(x_0)$
Kinetic Energy	$\hat{K} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$	K
Energy (Hamiltonian)	$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$	E

The choice of $V(x)$ depends on the physical system being modeled:

- Infinite square well: $V(x) = 0$ inside the well, $V(x) = \infty$ outside.
-

- Harmonic oscillator: $V(x) = \frac{1}{2}kx^2$.
- Hydrogen atom: $V(r) = -\frac{ke^2}{r}$.

Definiton 2.2.1.2 (Expected Value)**.** In quantum mechanics, the expectation value of an observable represented by operator $\hat{A}$ is:

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, \hat{A}\psi(x)\, dx \tag{2.27}$$

where $\psi(x)$ is the wave function of the system, and $\psi^*(x)$ is its complex conjugate. The expectation value represents the average outcome of measurements of the observable A on a large number of identically prepared systems.

Example 2.2.1.3. A particle of mass m is confined in a one-dimensional infinite potential well of width L. Its wave function is given by:

$$\psi_n(x) = Ae^{x/L} \quad \text{for } 0 < x < L$$

and $\psi_n(x) = 0$ elsewhere. Determine $\langle x \rangle$ for the particle and $\langle H \rangle$.

First, we need to normalize the wave function to find the constant A:

$$\int_0^L |\psi_n(x)|^2\, dx = 1$$

This gives:

$$|A|^2 \int_0^L e^{2x/L}\, dx = 1$$

Evaluating the integral:

$$|A|^2 \left[\frac{L}{2} e^{2x/L}\right]_0^L = 1$$

$$|A|^2 \left(\frac{L}{2}(e^2 - 1)\right) = 1$$

Thus, we find:

$$|A|^2 = \frac{2}{L(e^2-1)} \implies A = \sqrt{\frac{2}{L(e^2-1)}}$$

Now, we can calculate the expectation value of position $\langle x \rangle$:

$$\langle x \rangle = \int_0^L \psi_n^*(x)\, x\, \psi_n(x)\, dx$$

$$= |A|^2 \int_0^L x e^{2x/L}\, dx$$

Evaluating the integral using integration by parts:

$$\int x e^{2x/L}\, dx = \left(\frac{L}{2} x e^{2x/L} - \frac{L^2}{4} e^{2x/L}\right) + C$$

Thus,

$$\begin{aligned}\langle x\rangle &= |A|^2 \left[\frac{L}{2}xe^{2x/L} - \frac{L^2}{4}e^{2x/L}\right]_0^L \\ &= |A|^2 \left(\frac{L^2}{2}e^2 - \frac{L^2}{4}e^2 + \frac{L^2}{4}\right) \\ &= |A|^2 \frac{L^2}{4}(e^2+1)\end{aligned}$$

Substituting $|A|^2$:

$$\langle x\rangle = \frac{2}{L(e^2-1)} \cdot \frac{L^2}{4}(e^2+1) = \frac{L(e^2+1)}{2(e^2-1)}$$

Next, we calculate the expectation value of the Hamiltonian $\langle H\rangle$:

$$\langle H\rangle = \int_0^L \psi_n^*(x)\, \hat{H}\, \psi_n(x)\, dx$$

where the Hamiltonian operator $\hat{H}$ for a particle in an infinite potential well is:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

Calculating the second derivative of $\psi_n(x)$:

$$\frac{d^2\psi_n(x)}{dx^2} = \frac{A}{L^2}e^{x/L}$$

Thus,

$$\hat{H}\psi_n(x) = -\frac{\hbar^2}{2m} \cdot \frac{A}{L^2}e^{x/L} = -\frac{\hbar^2 A}{2mL^2}e^{x/L}$$

Now, we can compute $\langle H\rangle$:

$$\langle H\rangle = -\frac{\hbar^2 A^2}{2mL^2}\int_0^L e^{2x/L}\, dx$$

Evaluating the integral:

$$\int_0^L e^{2x/L}\, dx = \left[\frac{L}{2}e^{2x/L}\right]_0^L = \frac{L}{2}(e^2-1)$$

Thus,

$$\langle H\rangle = -\frac{\hbar^2 A^2}{2mL^2} \cdot \frac{L}{2}(e^2-1) = -\frac{\hbar^2 A^2(e^2-1)}{4mL}$$

Substituting A^2:

$$\langle H\rangle = -\frac{\hbar^2}{4mL} \cdot \frac{2}{L(e^2-1)}(e^2-1) = -\frac{\hbar^2}{2mL^2}$$

Therefore, the expectation values are:

$$\langle x\rangle = \frac{L(e^2+1)}{2(e^2-1)}, \quad \langle H\rangle = -\frac{\hbar^2}{2mL^2}$$

2.2.2 Wave functions

Definiton 2.2.2.1 (Properties of wave functions)**.** A valid wave function $\psi(x,t)$ must satisfy the following properties:

- **Normalization**: The total probability of finding the particle in all space must be 1:
$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 1 \tag{2.28}$$
- **Continuity**: The wave function and its first derivative must be continuous everywhere, except at points where the potential is infinite.
- **Single-valuedness**: The wave function must have a unique value at each point in space and time.
- **Square Integrability**: The integral of the square of the wave function over all space must be finite:
$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx < \infty \tag{2.29}$$

Definiton 2.2.2.2 (Energy well)**.** An energy well is a potential energy profile that confines a particle within a specific region of space. It is characterized by a lower potential energy inside the well compared to the surrounding areas. Energy wells can be classified into different types based on their shape and depth, such as infinite square wells, finite square wells, and harmonic oscillator potentials. The quantization of energy levels arises from the boundary conditions imposed by the well, leading to discrete allowed energy states for the confined particle.

Definiton 2.2.2.3 (Strödinger Equation - Time Independent)**.** The time-independent Schrödinger equation (TISE) describes the stationary states of a quantum system. It is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{2.30}$$

where $\psi(x)$ is the spatial part of the wave function, m is the mass of the particle, $V(x)$ is the potential energy as a function of position, and E is the total energy of the particle. The TISE is derived from the time-dependent Schrödinger equation by separating variables and assuming a solution of the form $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$.

Definiton 2.2.2.4 (Particle in a One-Dimensional Infinite Potential Well)**.** A particle confined in a one-dimensional infinite potential well (also known as a "particle in a box") has quantized energy levels and corresponding wave functions. The potential $V(x)$ is defined as:

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{otherwise} \end{cases} \tag{2.31}$$

The allowed energy levels are given by:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1, 2, 3, \ldots \tag{2.32}$$

The corresponding normalized wave functions are:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L \tag{2.33}$$

and $\psi_n(x) = 0$ elsewhere.

The probability density of finding the particle are the square of the constnat in the wave function:

$$|\psi_n(x)|^2 = \frac{2}{L} c_n^2 \sin^2\left(\frac{n\pi x}{L}\right) \tag{2.34}$$

Definiton 2.2.2.5 (Particle in a Finite Potential Well)**.** A particle confined in a one-dimensional finite potential well has quantized energy levels and corresponding wave functions. The potential $V(x)$ is defined as:

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ V_0 & \text{otherwise} \end{cases} \tag{2.35}$$

where V_0 is a finite positive constant representing the potential outside the well. The allowed energy levels are determined by solving the time-independent Schrödinger equation with appropriate boundary conditions. The wave functions inside and outside the well must be continuous and differentiable at the boundaries $x = 0$ and $x = L$. The energy levels are quantized, but unlike the infinite well, they depend on the depth V_0 of the well.

Definiton 2.2.2.6 (Quantum Tunneling)**.** Quantum tunneling is a phenomenon where a particle has a non-zero probability of crossing a potential barrier, even if its energy is less than the height of the barrier. This occurs due to the wave-like nature of particles in quantum mechanics, allowing the wave function to extend into and beyond the barrier region. The probability of tunneling depends on factors such as the width and height of the barrier, as well as the energy of the particle. Quantum tunneling has important applications in various fields, including nuclear fusion, semiconductor physics, and scanning tunneling microscopy.

This is a result of the wave function penetrating into the classically forbidden region, leading to a finite probability of finding the particle on the other side of the barrier.

Definiton 2.2.2.7 (Timed Case)**.** For each wave funtion, we can add the time dependent part:

$$\Psi_n(x,t) = \psi_n(x) e^{-iE_n t/\hbar} \tag{2.36}$$

where $\Psi_n(x,t)$ is the full time-dependent wave function, $\psi_n(x)$ is the spatial part of the wave function, E_n is the energy of the state, and $\hbar$ is the reduced Planck's constant.

2.3 Relativity

Definiton 2.3.0.1 (Postulates of Special Relativity)**.**

- The laws of physics are the same in all inertial frames of reference.
- The speed of light in a vacuum is constant and independent of the motion of the light source or observer.

Definiton 2.3.0.2 (Lorentz Transformation)**.** The Lorentz transformation relates the space and time coordinates of events as observed in two inertial frames of reference moving at a constant velocity relative to each other. If one frame S' is moving at a velocity v relative to another frame S, the transformations are given by:

$$x' = \gamma(x - vt) \tag{2.37}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.38}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor, and c is the speed of light in a vacuum.

These are observable when the relative velocity v approaches the speed of light c, leading to significant effects such as time dilation and length contraction.

- **Faster Object Sees Slower Clocks**: A moving clock runs slower when observed from a stationary frame. For a fast moving object, the symmetry of time dilation means each observer sees the other's clock running slow. It breaks when we goes back to the original frame. A fast travelling would be younger than the one who stayed.
- **Faster Object Sees Shorter Lengths**: An object moving at a high speed appears shorter in the direction of motion when observed from a stationary frame. This effect is known as length contraction.