

# MAT 185 Lecture Notes

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MAT 185

The up-to-date version of this document can be found at <https://github.com/HaysonC/skulenotes>

## 1 Vector Space

**Definiton 1.0.1** (Vector Space in  $\mathbb{R}$ ). This course concerns with real vector spaces. A vectors space,  $V$ , over  $\mathbb{R}$  is a collection of **object**  $\mathbf{v} \in V$  s.t. the follow axioms are followed

### 1. Addition Axioms

- (a) **Closure Under Addition:**  
 $\mathbf{x}, \mathbf{y} \in V \implies \mathbf{x} + \mathbf{y} \in V$
- (b) **Associativity of Addition:**  
 $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \implies (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
- (c) **Existence of Additive Identity:**  
 $\exists \mathbf{0} \in V$  such that  $\mathbf{x} \in V \implies \mathbf{x} + \mathbf{0} = \mathbf{x}$
- (d) **Existence of Additive Inverse:**  
 $\forall \mathbf{x} \in V, \exists -\mathbf{x} \in V$  such that  $\mathbf{x} + -\mathbf{x} = \mathbf{0}$

### 2. Scalar Multiplication Axioms

- (a) **Closure Under Scalar Multiplication:**  
 $\forall \mathbf{x} \in V$  and  $\forall \alpha \in \mathbb{R}, \alpha \mathbf{x} \in V$
- (b) **Associativity of Scalar Multiplication:**  
 $\forall \mathbf{x} \in V$  and  $\forall \alpha, \beta \in \mathbb{R}, (\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
- (c) **Distributive Property of Scalar Multiplication:**  
 $\forall \mathbf{x} \in V$  and  $\forall \alpha, \beta \in \mathbb{R}, (\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- (d) **Existence of Multiplicative Identity:**  
 $\forall \mathbf{x} \in V, 1\mathbf{x} = \mathbf{x}$

**Note** It could be shown that the axiom imply the commutativity of in addition, namely  $\forall \mathbf{x} \implies \mathbf{y} \in \mathbb{R}, \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

**Example**  $\mathbb{R}^n$  is a vector space over  $\mathbb{R}$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$  and  $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

**Theorem 1.0.2** (Cancelation, Part 1). Let  $V$  be a vector space and  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ , then:

$$\begin{aligned}\mathbf{x} + \mathbf{z} &= \mathbf{y} + \mathbf{z} \\ \mathbf{x} &= \mathbf{y}\end{aligned}$$

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*Proof.*

$$\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$$

From additive inverse we know that  $-\mathbf{z}$  exists

$$(\mathbf{x} + \mathbf{z}) + (-\mathbf{z}) = (\mathbf{y} + \mathbf{z}) + (-\mathbf{z})$$

By order of addition we have:

$$\begin{aligned}\mathbf{x} + (\mathbf{z} + (-\mathbf{z})) &= \mathbf{y} + (\mathbf{z} + (-\mathbf{z})) \\ \mathbf{x} + \mathbf{0} &= \mathbf{y} + \mathbf{0}\end{aligned}$$

By additive identity we have:

$$\mathbf{x} = \mathbf{y}$$

□

**Theorem 1.0.3** (Cancellation, Part 2). Let  $V$  be a vector space and  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ , then:

$$\begin{aligned}\mathbf{z} + \mathbf{x} &= \mathbf{z} + \mathbf{y} \\ \mathbf{x} &= \mathbf{y}\end{aligned}$$

To prove that, it would require the following proposition:

**Lemma 1.0.4.** Let  $V$  be a vector space and  $\mathbf{z} \in V$ , then  $-\mathbf{z} + \mathbf{z} = \mathbf{0}$

*Proof.*

We know:

$$\begin{aligned}-\mathbf{z} + \mathbf{z} &= (-\mathbf{z} + \mathbf{z}) + \mathbf{0} \\ &= (-\mathbf{z} + \mathbf{z}) + (-\mathbf{z} + -(-\mathbf{z}))\end{aligned}$$

By the order of addition,

$$= -\mathbf{z} + (\mathbf{z} + (-\mathbf{z} + -(-\mathbf{z})))$$

Again, by the order of addition,

$$\begin{aligned}&= -\mathbf{z} + ((\mathbf{z} + (-\mathbf{z})) + -(-\mathbf{z})) \\ &= -\mathbf{z} + -(-\mathbf{z}) \\ &= \mathbf{0}\end{aligned}$$

□

Now, to prove the part 2 of the Cancellation Theorem:

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*Proof.*

$$\mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y}$$

From additive inverse we know that  $-\mathbf{z}$  exists

$$-\mathbf{z} + (\mathbf{z} + \mathbf{x}) = -\mathbf{z} + (\mathbf{z} + \mathbf{y})$$

$$(-\mathbf{z} + \mathbf{z}) + \mathbf{x} = (-\mathbf{z} + \mathbf{z}) + \mathbf{y}$$

From above, we have  $-\mathbf{z} + \mathbf{z} = 0$

$$0 + \mathbf{x} = 0 + \mathbf{y}$$

$$\mathbf{x} = \mathbf{y}$$

□

**Lemma 1.0.5** (Inverse of an inverse). Let  $V$  be a vector space and  $\mathbf{x} \in V$ , then:

$$-(-\mathbf{x}) = \mathbf{x}$$

*Proof.* Assume  $0, 0^*$  are the additive identity of  $V$  and  $-\mathbf{x}, -\mathbf{x}^*$  are the additive inverse of  $\mathbf{x}$ . We have:

$$u + 0 = u + 0^*$$

By Cancellation Theorem, we have  $0 = 0^*$ . Thus, the additive identity is unique. Now, we have:

$$\mathbf{x} + -\mathbf{x} = 0$$

$$\mathbf{x} + -\mathbf{x}^* = 0$$

Thus, by the uniqueness of the additive inverse, we have:

$$\mathbf{x} + -\mathbf{x} = \mathbf{x} + -\mathbf{x}^*$$

By the Cancellation Theorem, we have:

$$-\mathbf{x} = -\mathbf{x}^*$$

Now, we have that the additive inverse is unique. Thus,  $-(-\mathbf{x})$  must be unique and necessarily  $\mathbf{x}$ . □

## Additional Results from Above

1. The additive identity is unique
2. The additive inverse is unique

**Definiton 1.0.6** (Subtraction). Let  $V$  be a vector space and  $\mathbf{x}, \mathbf{y} \in V$ , then:

$$\mathbf{x} - \mathbf{y} \stackrel{\text{def}}{=} \mathbf{x} + -\mathbf{y}$$

**Theorem 1.0.7** (Addition is Commutative). Let  $V$  be a vector space and  $\mathbf{x}, \mathbf{y} \in V$ , then:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

*Proof.*

$$\mathbf{x} + \mathbf{y} =$$

□

## 1.1 Vector Subspace

**Definiton 1.1.1** (Vector Subspace). Let  $V$  be a vector space and  $W \subseteq V$ , then  $W$  is a vector subspace of  $V$  if  $W$  is a vector space.

**Theorem 1.1.2** (Subspace Test, I). Let  $V$  be a vector space over  $\mathbb{R}$  and  $W \subseteq V$  with the usual addition and scalar multiplication, then  $W$  is a subspace of  $V$  iff for all  $\mathbf{x}, \mathbf{y} \in W$  and  $\alpha \in \mathbb{R}$ :

1. **Closure Under Addition:**  
 $\mathbf{x} + \mathbf{y} \in W$
2. **Closure Under Scalar Multiplication:**  
 $\alpha \mathbf{x} \in W$
3. Additive Identity:  
 $\mathbf{0} \in W$

*Proof.* ( $\Rightarrow$ ) Assume  $W$  is a subspace of  $V$ , then  $W$  is a vector space. Thus, the axioms of vector space are satisfied.

( $\Leftarrow$ ) Assume the three conditions are satisfied, then  $W$  is a vector space. Thus,  $W$  is a subspace of  $V$ .  $\square$

**Definiton 1.1.3** (Null Space). Let  $V$  be a vector space and  $A \in {}^m\mathbb{R}^{n1}$ , then:

$$\text{null}(A) = \{\mathbf{x} \in {}^n\mathbb{R} \mid A\mathbf{x} = \mathbf{0}\} \quad (1)$$

is the null space of  $A$ , otherwise known as the **kernel** of  $A$  or the solution space of  $A\mathbf{x} = \mathbf{0}$

We can use the Subspace Test I to show that the null space of a matrix is a subspace of  ${}^n\mathbb{R}$ .

1. **Existence of Additive Identity:**  
The zero vector is in the null space of  $A$  as the trivial solution to the equation  $A\mathbf{x} = \mathbf{0}$
2. **Closure Under Addition:**  
Let  $\mathbf{x}, \mathbf{y} \in \text{null}(A)$ , then  $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ . Thus,  $\mathbf{x} + \mathbf{y} \in \text{null}(A)$ . This holds since  ${}^n\mathbb{R}$  is a vector space.
3. **Closure Under Scalar Multiplication:**  
Let  $\mathbf{x} \in \text{null}(A)$  and  $\alpha \in \mathbb{R}$ , then  $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$ . Thus,  $\alpha\mathbf{x} \in \text{null}(A)$ . This holds since  ${}^n\mathbb{R}^m$  is a vector space under usual addition and scalar multiplication.

**Theorem 1.1.4** (Subspace Test, II). Let  $V$  be a vector space over  $\mathbb{R}$  and  $W \subseteq V$  with the usual addition and scalar multiplication, then  $W$  is a subspace of  $V$  iff for all  $\mathbf{x}, \mathbf{y} \in W$  and  $\alpha, \beta \in \mathbb{R}$ :

1.  $\alpha\mathbf{x} + \beta\mathbf{y} \in W$

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<sup>1</sup>Same as  $\mathbb{R}^{m \times n}$ , The set of all  $m \times n$  matrices with real entries

## 1.2 Linear combinations

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*Proof.* ( $\Rightarrow$ ) Assume  $W$  is a subspace of  $V$ , then  $W$  is a vector space. Thus, the axioms of vector space are satisfied.

( $\Leftarrow$ ) Assume the condition is satisfied, then  $W$  is a vector space. Thus,  $W$  is a subspace of  $V$ .  $\square$

**Definiton 1.1.5** (Intersection of Sets). Let  $A$  and  $B$  be sets, then:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \quad (2)$$

**Definiton 1.1.6** (Union of Sets). Let  $A$  and  $B$  be sets, then:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \quad (3)$$

## 1.2 Linear combinations

**Definiton 1.2.1** (Linear Combination). Let  $V$  be a vector space and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$  and  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ , then:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n$$

is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$

**Definiton 1.2.2** (Linear independence). Let  $V$  be a vector space and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$  and  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ . If the equation:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

has only the trivial solution  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .

Then,  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are said to be linearly independent; otherwise, they are linearly dependent.

**Definiton 1.2.3** (Span). Let  $V$  be a vector space and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ , then:

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\} \quad (4)$$

is the set of all linear combinations of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$