

ESC 195 Lecture Notes

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Calculus II, Winter 2024

ESC 195

1 More on Integrals

1.1 Riemann Sum - Non-Uniform Partition

Example 1.1.1. Given the following definite integral:

$$\int_0^2 \sqrt{x} dx$$

, we cannot evaluate its Riemann sum with uniform partition, since the series of root cannot be easily evaluated.

The definite integral of \sqrt{x} from 0 to 2 using a Riemann sum with a non-uniform partition is given by:

$$\int_0^2 \sqrt{x} dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{x_i} \Delta x_i$$

where:

- $x_0 = 0, x_n = 2$,
- $x_i = i^2 \cdot \frac{2}{n^2}$ for $i = 0, 1, 2, \dots, n$,
- $\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2} \cdot (2i - 1)$.

The Riemann sum becomes:

$$S_n = \sum_{i=1}^n \sqrt{i^2 \cdot \frac{2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Simplifying further:

$$S_n = \sum_{i=1}^n \sqrt{\frac{2i^2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Taking the limit as $n \rightarrow \infty$, the sum converges to the exact value of the integral using the series of squares.

$$\int_0^2 \sqrt{x} dx = \frac{4\sqrt{2}}{3}.$$

Condition $n \rightarrow \infty$ Ensures $\Delta x_i \rightarrow 0$

As $n \rightarrow \infty$, the partition points x_i become increasingly dense. This ensures that the partition becomes infinitely fine.

1.2 Integration By Parts

Using the product rule:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x),$$

integrating both sides with respect to x gives:

$$u(x)v(x) = \int u'(x)v(x) dx + \int u(x)v'(x) dx.$$

Rearranging this:

$$\int u'(x)v(x) dx = u(x)v(x) - \int u(x)v'(x) dx.$$

Integration by parts formula

$$\int u dv = uv - \int v du. \tag{1}$$

Example 1.2.1. We want to solve the integral

$$\int xe^{2x} dx$$

using integration by parts.

Let:

$$u = x, \quad dv = e^{2x} dx.$$

Then, we compute the derivatives and integrals:

$$du = dx, \quad v = \frac{e^{2x}}{2}.$$

Now, apply the integration by parts formula:

$$\int u dv = uv - \int v du.$$

Substituting in the values:

$$\int xe^{2x} dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} dx.$$

Next, compute the remaining integral:

$$\int \frac{e^{2x}}{2} dx = \frac{e^{2x}}{4}.$$

Thus, the result is:

$$\int x e^{2x} dx = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Example 1.2.2. We want to solve

$$\int x^2 \sin(2x) dx$$

using double integration by parts.

First, let:

$$u = x^2, \quad dv = \sin(2x) dx.$$

Then:

$$du = 2x dx, \quad v = -\frac{1}{2} \cos(2x).$$

Using the IBP formula:

$$\int u dv = uv - \int v du,$$

we get:

$$\int x^2 \sin(2x) dx = -\frac{x^2}{2} \cos(2x) + \int x \cos(2x) dx.$$

Now, apply IBP again to $\int x \cos(2x) dx$, let:

$$u = x, \quad dv = \cos(2x) dx.$$

Then:

$$du = dx, \quad v = \frac{1}{2} \sin(2x).$$

Using the IBP formula again:

$$\int x \cos(2x) dx = \frac{x}{2} \sin(2x) - \int \frac{1}{2} \sin(2x) dx,$$

and solving the remaining integral:

$$\int \frac{1}{2} \sin(2x) dx = -\frac{1}{4} \cos(2x).$$

Thus, the final result is:

$$\int x^2 \sin(2x) dx = -\frac{x^2}{2} \cos(2x) + \frac{x}{2} \sin(2x) + \frac{1}{4} \cos(2x) + C.$$

1.3 Trigonometric Integrals

Case I This is the case I trigonometric integrals, the strategy is using the pythagorean identity.

We want to solve the class of integrals:

$$\int \sin^n(x) \cos^m(x) dx \quad (2)$$

, where m or n is odd.

Example 1.3.1. We want to solve the integral:

$$I = \int \sin^3(x) \cos^2(x) dx$$

We use the identity: $\sin^2(x) = 1 - \cos^2(x)$. Thus, the integral becomes:

$$\begin{aligned} I &= \int \sin(x)(1 - \cos^2(x)) \cos^2(x) dx \\ &= \int (\cos^2(x) \sin(x) - \cos^4(x) \sin(x)) dx \end{aligned}$$

which is now easily solvable with substitution $u = \cos(x)$.

Case II The follwoing is generally solvable via case I and case III below. In general, we solve the integral by reducing the power of the trigonometric functions to arrive at a solvable integral Case I or III.

We want to solve the class of integrals:

$$\int \sin^n(x) \cos^m(x) dx \quad (3)$$

, where m and n is even.

Example 1.3.2. We want to solve the integral:

$$I = \int \sin^2(x) \cos^4(x) dx$$

We can apply the double angle formulas:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x) \quad (4)$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad (5)$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \quad (6)$$

Thus, the integral becomes:

$$I = \frac{1}{8} \int \sin^2(2x) dx + \frac{1}{8} \int \cos(2x) \sin^2(2x) dx$$

1.3 Trigonometric Integrals

Case III This is the case III trigonometric integrals, the strategy is using the reduction formula via IBP.

Reduction Formula We can solve integrals by reducing the power of the trigonometric functions. These are done using IBP and trigonometric identities.

We want to solve the classes of integrals:

$$\int \sin^n(x) dx, \quad \int \cos^n(x) dx \quad (7)$$

, where n is a positive integer. For demonstration, We can obtain the reduction formula of $\sin^n(x)$ via IBP.

$$\begin{aligned} I_n &= \int \sin^n(x) dx \\ &= \int \sin^{n-1}(x) \sin(x) dx \\ &= \frac{-\cos(x) \sin^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \cos^2(x) \sin^{n-2}(x) dx \\ &= \frac{-\cos(x) \sin^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2} \end{aligned} \quad (8)$$

Reduction for $\cos^n(x)$ We can obtain the reduction formula of $\cos^n(x)$ via IBP.

$$\begin{aligned} I_n &= \int \cos^n(x) dx \\ &= \int \cos^{n-1}(x) \cos(x) dx \\ &= \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \sin^2(x) \cos^{n-2}(x) dx \\ &= \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n-1)}{n} I_{n-2} \end{aligned} \quad (9)$$

Case IV The following is generally solvable via simple trigonometric integrals. In general, we solve the integral by applying the angle sum formulas.

We want to solve the classes of integrals:

$$\int \sin(mx) \cos(nx) dx, \quad \int \sin(mx) \sin(nx) dx, \quad \int \cos(mx) \cos(nx) dx \quad (10)$$

1.3 Trigonometric Integrals

We could apply the angle sum formulas:

$$\sin(mx) \cos(nx) = \frac{1}{2} [\sin((m+n)x) + \sin((m-n)x)] \quad (11)$$

$$\sin(mx) \sin(nx) = \frac{1}{2} [\cos((m-n)x) - \cos((m+n)x)] \quad (12)$$

$$\cos(mx) \cos(nx) = \frac{1}{2} [\cos((m-n)x) + \cos((m+n)x)] \quad (13)$$

Case V The following is generally solvable via the following trigonometric identities listed below, which convert it into a reduction formula.

$$\tan^2(x) = \sec^2(x) - 1 \quad (14)$$

$$\cot^2(x) = \csc^2(x) - 1 \quad (15)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x) \quad (16)$$

We want to solve the classes of integral:

$$\int \tan^n(x) dx, \quad \int \cot^n(x) dx \quad (17)$$

We know that $\tan^2(x) = \sec^2(x) - 1$. Thus, we can solve the integral by reducing the power of the tangent function.

Reduction for $\tan^n(x)$ We can obtain the reduction formula of $\tan^n(x)$ via the trigonometric identities.

$$\begin{aligned} I_n &= \int \tan^n(x) dx \\ &= \int \tan^{n-1}(x) \tan(x) dx \\ &= \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x) dx \\ &= \frac{\tan^{n-1}(x)}{n-1} - I_{n-2} \end{aligned} \quad (18)$$

Reduction for $\cot^n(x)$ We can obtain the reduction formula of $\cot^n(x)$ via the trigonometric identities.

$$\begin{aligned} I_n &= \int \cot^n(x) dx \\ &= \int \cot^{n-1}(x) \cot(x) dx \\ &= \frac{\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x) dx \\ &= \frac{\cot^{n-1}(x)}{n-1} - I_{n-2} \end{aligned} \quad (19)$$

1.3 Trigonometric Integrals

Case VI The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x) \sec^n(x) dx, \quad \int \cot^m(x) \csc^n(x) dx \quad (20)$$

We can solve the integral by reducing the power of the converted trigonometric functions using (14) and (15).

Case VII The following is generally solvable via the trigonometric identities (14) and (15).

We want to solve the classes of integral:

$$\int \tan^m(x) \sec^n(x) dx, \quad \int \cot^m(x) \csc^n(x) dx \quad (21)$$

We can solve the integral by converting between the tangent and secant functions using the trigonometric identities.

Case VIII This is the case VIII integrals, the strategy is using the trigonometric substitution.

Trigonometric Substitution We can solve integrals by substituting the trigonometric functions with other trigonometric functions.

Example 1.3.3. We want to solve the integral:

$$\int \frac{dx}{\sqrt{1-x^2}}$$

We can substitute $x = \sin(\theta)$, then $dx = \cos(\theta) d\theta$. The integral becomes:

$$\int \frac{\cos(\theta) d\theta}{\sqrt{1-\sin^2(\theta)}} = \int \frac{\cos(\theta) d\theta}{\cos(\theta)} = \int d\theta = \theta + C.$$

In general, we can use the following substitutions:

- $x = a \sin(\theta)$ for $\sqrt{a^2 - x^2}$,
- $x = a \tan(\theta)$ for $\sqrt{a^2 + x^2}$,
- $x = a \sec(\theta)$ for $\sqrt{x^2 - a^2}$.

1.4 Partial Fractions

Weierstrass Substitution We can solve integrals by using the Weierstrass substitution:

$$\tan\left(\frac{x}{2}\right) = t \Leftrightarrow \cos(x) = \frac{1-t^2}{1+t^2} \quad \sin(x) = \frac{2t}{1+t^2} \quad dx = \frac{2dt}{1+t^2} \quad (22)$$

This substitution is useful for solving integrals with trigonometric functions, by converting them into rational functions.

Summary We can solve trigonometric integrals by using the following strategies:

Case	Strategy and General Form
I	Use $\sin^2(x) + \cos^2(x) = 1$; simplify using substitution. General Form: $\int \sin^n(x) \cos^m(x) dx$, where m or n is odd.
II	Convert to Case I and Case III using double angle formulas. General Form: $\int \sin^n(x) \cos^m(x) dx$, where both m and n are even.
III	Apply reduction formulas derived via integration by parts. General Form: $\int \sin^n(x) dx$ or $\int \cos^n(x) dx$.
IV	Use angle sum formulas to simplify. General Form: $\int \sin(mx) \cos(nx) dx$.
V	Reduce tangent/cotangent powers using $\tan^2(x) = \sec^2(x) - 1$ and substitution. General Form: $\int \tan^n(x) dx$ or $\int \cot^n(x) dx$.
VI	Convert to Case V via Pythagorean identities. General Form: $\int \tan^m(x) \sec^n(x) dx$ or $\int \cot^m(x) \csc^n(x) dx$.
VII	Convert between tangent and secant functions for simplification. General Form: $\int \tan^m(x) \sec^n(x) dx$.
VIII	Use trigonometric substitution: $x = a \sin(\theta), a \tan(\theta), a \sec(\theta)$. General Form: $\int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{\sqrt{a^2+x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$.

Table 1: Strategies and General Forms for Case I-VIII Integrals

1.4 Partial Fractions

Partial Fraction Decomposition We can decompose a rational function into partial fractions to simplify integration.

We want to solve the integral:

$$\int \frac{P(x)}{Q(x)} dx \quad (23)$$

, where $P(x)$ and $Q(x)$ are polynomials.

Example 1.4.1. We want to solve the integral:

$$\int \frac{2x^2 + 3x + 1}{x^3 + 2x^2 + x} dx$$

1.5 Improper Integrals

We can decompose the rational function into partial fractions:

$$\frac{2x^2 + 3x + 1}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

We can solve for the constants A , B , and C by equating the coefficients of the partial fractions to the original function. We have:

$$2x^2 + 3x + 1 = A(x+1)^2 + Bx(x+1) + Cx$$

Thus, we can solve it by setting $x = 0$, $x = -1$

Summary We can solve integrals using partial fraction decomposition by following these steps:

Step	Description
1	Factorize the denominator of the rational function.
2	Write the partial fraction decomposition.
3	Solve for the constants by equating the coefficients.
4	Integrate the partial fractions.

Table 2: Steps for Partial Fraction Decomposition

1.5 Improper Integrals

Improper Integral An improper integral is an integral with an infinite limit or a discontinuity in the interval of integration.

We want to solve the integral:

$$\int_a^b f(x) dx \tag{24}$$

, where a or b is infinite or $f(x)$ is discontinuous.

Example 1.5.1. We want to solve the integral:

$$\int_0^\infty e^{-x} dx$$

Evaluating the integral:

$$\begin{aligned} \int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} -e^{-x} \Big|_0^b \\ &= \lim_{b \rightarrow \infty} -e^{-b} + 1 \\ &= 1 \end{aligned}$$

2 Hyperbolic Trigonometric Functions

Definiton 2.0.1 (Hyperbolic Sine). The hyperbolic sine function is defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \quad (25)$$

Definiton 2.0.2 (Hyperbolic Cosine). The hyperbolic cosine function is defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \quad (26)$$

Properties These combinations of exponential functions have properties similar to the trigonometric functions.

Derivatives The derivatives of the hyperbolic functions are:

$$\frac{d}{dx} \sinh(x) = \cosh(x), \quad (27)$$

$$\frac{d}{dx} \cosh(x) = \sinh(x). \quad (28)$$

Identities The hyperbolic functions satisfy the following identities:

$$\cosh^2(x) - \sinh^2(x) = 1, \quad (29)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \quad (30)$$

$$\sinh(2x) = 2 \sinh(x) \cosh(x). \quad (31)$$

Hyperbola The hyperbolic functions are related to the hyperbola $x^2 - y^2 = 1$. Simiar to the circle, the hyperbola can be parametrized by the hyperbolic functions (e.g. $x = \cosh(t)$, $y = \sinh(t)$).

Area The area of a sector of the hyperbola is given by:

$$A = t/2, \quad (32)$$

where t is the angle of the sector along the parametrization $\{(x, y) \mid x = \cosh(t), y = \sinh(t)\}$.

Catenary The hyperbolic cosine function describes the shape of a hanging chain or cable. The catenary is the curve formed by a chain hanging from two points. It is given by the equation:

$$a \cosh\left(\frac{x}{a}\right), \quad (33)$$

Definiton 2.0.3 (Hyperbolic Tangent). The hyperbolic tangent function is defined as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \quad (34)$$

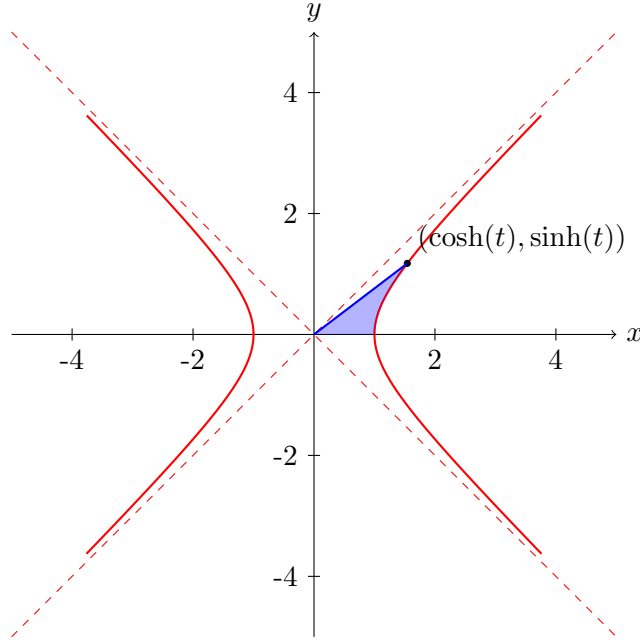


Figure 1: Sector of the hyperbola

Derivative The derivative of the hyperbolic tangent function resembles the derivative of the regular tangent function:

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x). \quad (35)$$

Identities The hyperbolic tangent function satisfies the following identities:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad (36)$$

$$\operatorname{sech}^2(x) = 1 - \tanh^2(x). \quad (37)$$

Secant, Cosecant, Cotangent The hyperbolic secant, cosecant, and cotangent functions are defined similarly to the regular secant, cosecant, and cotangent functions. They are the **reciprocal** of the hyperbolic cosine, sine, and tangent functions, respectively.

Inverse Hyperbolic Functions The inverse hyperbolic functions are defined as the inverse of the hyperbolic functions. They are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$, etc.

$$\operatorname{arsinh} x = \ln \left(x + \sqrt{x^2 + 1} \right) \quad (38)$$

$$\operatorname{arcosh} x = \ln \left(x + \sqrt{x^2 - 1} \right) \quad (39)$$

$$\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \quad (40)$$

$$\operatorname{arsch} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) \quad (41)$$

$$\operatorname{arsech} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) \quad (42)$$

$$\operatorname{arcoth} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) \quad (43)$$

A Trigonometric Derivatives

Trigonometric Function	Inverse Trigonometric Function
$\frac{d}{dx} \sin(x) = \cos(x)$	$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx} \cos(x) = -\sin(x)$	$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx} \tan(x) = \sec^2(x)$	$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$
$\frac{d}{dx} \cot(x) = -\csc^2(x)$	$\frac{d}{dx} \operatorname{arccot}(x) = \frac{-1}{1+x^2}$
$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$	$\frac{d}{dx} \operatorname{arcsec}(x) = \frac{1}{ x \sqrt{x^2-1}}$
$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$	$\frac{d}{dx} \operatorname{arccsc}(x) = \frac{-1}{ x \sqrt{x^2-1}}$
Hyperbolic Function	Inverse Hyperbolic Function
$\frac{d}{dx} \sinh(x) = \cosh(x)$	$\frac{d}{dx} \operatorname{arsinh}(x) = \frac{1}{\sqrt{x^2+1}}$
$\frac{d}{dx} \cosh(x) = \sinh(x)$	$\frac{d}{dx} \operatorname{arcosh}(x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$	$\frac{d}{dx} \operatorname{artanh}(x) = \frac{1}{1-x^2} \quad (x < 1)$
$\frac{d}{dx} \coth(x) = -\operatorname{csch}^2(x)$	$\frac{d}{dx} \operatorname{arcoth}(x) = \frac{1}{1-x^2} \quad (x > 1)$
$\frac{d}{dx} \operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$	$\frac{d}{dx} \operatorname{arsech}(x) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x \leq 1)$
$\frac{d}{dx} \operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$	$\frac{d}{dx} \operatorname{arcsch}(x) = \frac{-1}{ x \sqrt{x^2+1}}$