MAT 185 Lecture Notes

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1 Vector Space

Definiton 1.0.1 (Vector Space in $\mathbb{R}$)**.** A vectors space,V, over $\mathbb{R}$ is a collection of **object** $\mathbf{v} \in V$ s.t. the follow axioms are followed

1. **Addition Axioms**
   (a) **Closure Under Addition:**
   $\mathbf{x}, \mathbf{y} \in V \implies \mathbf{x} + \mathbf{y} \in V$
   (b) **Associativity of Addition:**
   $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \implies (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
   (c) **Existence of Additive Identity:**
   $\exists\, \mathbf{0} \in V$ such that $\mathbf{x} \in V \implies \mathbf{x} + \mathbf{0} = \mathbf{x}$
   (d) **Existence of Additive Inverse:**
   $\forall\, \mathbf{x} \in V$, $\exists$**-x** $\in V$ such that $\mathbf{x} + \textbf{-x} = \mathbf{0}$
2. **Scalar Multiplication Axioms**
   (a) **Closure Under Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha \in \mathbb{R}$, $\alpha\mathbf{x} \in V$
   (b) **Associativity of Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
   (c) **Distributive Property of Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
   (d) **Existence of Multiplicative Identity:**
   $\forall\, \mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$

Note It could be shown that the axiom imply the commutativity of in addition, namely $\forall \mathbf{x} \implies \mathbf{y} \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

Example $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

Theorem 1.0.2 (Cancelation, Part 1)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then:

$$\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$$
$$\mathbf{x} = \mathbf{y}$$

Proof.

$$\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$$

From additive inverse we know that **-z** exists

$$(\mathbf{x} + \mathbf{z}) + \textbf{-z} = (\mathbf{y} + \mathbf{z}) + \textbf{-z}$$

By order of addition we have:

$$\begin{aligned} \mathbf{x} + (\mathbf{z} + \textbf{-z}) = \mathbf{y} + (\mathbf{z} + \textbf{-z}) \\ \mathbf{x} + \mathbf{0} = \mathbf{y} + \mathbf{0} \end{aligned}$$

By additive identity we have:

$$\mathbf{x} = \mathbf{y}$$

$\square$

Theorem 1.0.3 (Cancelation, Part 2)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then:

$$\begin{aligned} \mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y} \\ \mathbf{x} = \mathbf{y} \end{aligned}$$

To proof that, it would require the following propostion:

Lemma 1.0.4. Let V be a vector space and $\mathbf{z} \in V$, then $\textbf{-z} + \mathbf{z} = 0$

Proof.

We know:

$$\begin{aligned} \textbf{-z} + \mathbf{z} &= (\textbf{-z} + \mathbf{z}) + 0 \\ &= (\textbf{-z} + \mathbf{z}) + (\textbf{-z} + \textbf{-(-z)}) \end{aligned}$$

By the order of additon,

$$= \textbf{-z} + (\mathbf{z} + (\textbf{-z} + \textbf{-(-z)}))$$

Again, by the order of additon,

$$\begin{aligned} &= \textbf{-z} + ((\mathbf{z} + \textbf{-z}) + \textbf{-(-z)}) \\ &= \textbf{-z} + \textbf{-(-z)} \\ &= 0 \end{aligned}$$

$\square$

Now, to prove the part 2 of the Cancelation Theorem:

Proof.

$$\mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y}$$

From additive inverse we know that **-z** exists

$$\textbf{-z} + (\mathbf{z} + \mathbf{x}) = \textbf{-z} + (\mathbf{z} + \mathbf{y})$$
$$(\textbf{-z} + \mathbf{z}) + \mathbf{x} = (\textbf{-z} + \mathbf{z}) + \mathbf{y}$$

From above, we have $\textbf{-z} + \mathbf{z} = 0$

$$0 + \mathbf{x} = 0 + \mathbf{y}$$
$$\mathbf{x} = \mathbf{y}$$

□

Lemma 1.0.5 (Inverse of an inverse)**.** Let V be a vector space and $\mathbf{x} \in V$, then:

$$\textbf{-(-x)} = \mathbf{x}$$

Proof. Assume 0, $0^\star$ are the additive identity of V and **-x**, $\textbf{-x}^\star$ are the additive inverse of **x**. We have:

$$u + 0 = u + 0^\star$$

By Cancelation Theorem, we have $0 = 0^\star$. Thus, the additive identity is unique. Now, we have:

$$\mathbf{x} + \textbf{-x} = 0$$
$$\mathbf{x} + \textbf{-x}^\star = 0$$

Thus, by the uniqueness of the additive inverse, we have:

$$\mathbf{x} + \textbf{-x} = \mathbf{x} + \textbf{-x}^\star$$

By the Cancelation Theorem, we have:

$$\textbf{-x} = \textbf{-x}^\star$$

Now, we have that the additive inverse is unique. Thus, **-(-x)** must be unique and nessarily **x**. □

Results from above

1. The additive identity is unique
2. The additive inverse is unique

Definiton 1.0.6 (Subtraction)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y} \in V$, then:

$$\mathbf{x} - \mathbf{y} \overset{\text{def}}{=} \mathbf{x} + \textbf{-y}$$

Theorem 1.0.7 (Commutativity of Addition)**.** Let V be a vector space and $\mathbf{x}, \mathbf{y} \in V$, then:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

Proof.

$$\mathbf{x} + \mathbf{y} =$$

□