ESC 195 Lecture Notes

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1 More on Integrals

1.1 Riemann Sum - Non-Uniform Petition

Example 1.1.1. Given the following definite integral:

$$\int_0^2 \sqrt{x} dx$$

, we cannot evaluate its Riemann sum with uniforms partition, since the series of root cannot be easily evaluated.

The definite integral of $\sqrt{x}$ from 0 to 2 using a Riemann sum with a non-uniform partition is given by:

$$\int_0^2 \sqrt{x}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i} \Delta x_i$$

where:

- $x_0 = 0$, $x_n = 2$,
- $x_i = i^2 \cdot \frac{2}{n^2}$ for $i = 0, 1, 2, \ldots, n$,
- $\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2} \cdot (2i - 1)$.

The Riemann sum becomes:

$$S_n = \sum_{i=1}^{n} \sqrt{i^2 \cdot \frac{2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Simplifying further:

$$S_n = \sum_{i=1}^{n} \sqrt{\frac{2i^2}{n^2}} \cdot \frac{2}{n^2} \cdot (2i - 1).$$

Taking the limit as $n \to \infty$, the sum converges to the exact value of the integral using the series of squares.

$$\int_0^2 \sqrt{x}\, dx = \frac{4\sqrt{2}}{3}.$$

Condition $n \to \infty$ Ensures $\Delta x_i \to 0$

As $n \to \infty$, the partition points x_i become increasingly dense. This ensures that the partition becomes infinitely fine.

1.2 Integration By Parts

Using the product rule:

$$\frac{d}{dx}\big[u(x)v(x)\big] = u'(x)v(x) + u(x)v'(x),$$

integrating both sides with respect to x gives:

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Rearranging this:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Integration by parts formula

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

Example 1.2.1. We want to solve the integral

$$\int xe^{2x}\,dx$$

using integration by parts.

Let:

$$u = x, \quad dv = e^{2x}\,dx.$$

Then, we compute the derivatives and integrals:

$$du = dx, \quad v = \frac{e^{2x}}{2}.$$

Now, apply the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

Substituting in the values:

$$\int xe^{2x}\,dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx.$$

Next, compute the remaining integral:

$$\int \frac{e^{2x}}{2}\,dx = \frac{e^{2x}}{4}.$$

Thus, the result is:

$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Example 1.2.2. We want to solve

$$\int x^2 \sin(2x)\,dx$$

using double integration by parts.

First, let:

$$u = x^2, \quad dv = \sin(2x)\,dx.$$

Then:

$$du = 2x\,dx, \quad v = -\frac{1}{2}\cos(2x).$$

Using the IBP formula:

$$\int u\,dv = uv - \int v\,du,$$

we get:

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\,dx.$$

Now, apply IBP again to $\int x\cos(2x)\,dx$, let:

$$u = x, \quad dv = \cos(2x)\,dx.$$

Then:

$$du = dx, \quad v = \frac{1}{2}\sin(2x).$$

Using the IBP formula again:

$$\int x\cos(2x)\,dx = \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx,$$

and solving the remaining integral:

$$\int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x).$$

Thus, the final result is:

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.$$