

## **MAT185 – Linear Algebra**

### **Assignment 4**

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#### **Academic Integrity Statement:**

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## Preamble

This assignment explores two applications for eigenvectors and eigenvalues: (i) the PageRank algorithm and (ii) solving linear constant-coefficient second-order ordinary differential equations.

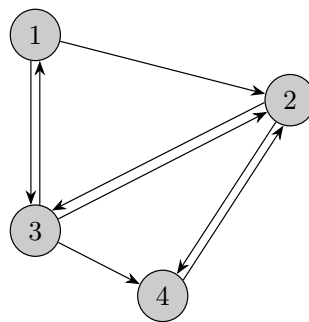
In Questions 1 and 2, you will explore using eigenvectors to calculate the ranking of nodes in a directed graph, which is useful for ranking websites by how important they are, e.g., to show a sorted list of websites based on a search criterion. The main problem with ranking websites is that not every website is connected to every other website.<sup>1</sup> We are introducing a technique which is the basis for the PageRank-algorithm, which is the main algorithm used by Google to sort its search results in the 90s and is still in use today. Bryan and Leise [1] describe this algorithm in more detail and it is a great insight if you are interested in more technical details. Reading this publication is not necessary to solve the problems in this assignment. However, we have uploaded [the publication to Quercus](#) for the curious.

In Question 3 we are using eigenvalues and eigenvectors to solve an ordinary differential equation (ODE). In that case, we are using a mass-spring system, representing other mechanics ODEs you will solve in the following terms.

## The PageRank Algorithm

The PageRank algorithm is an important part of the internal functionality of web search engines. It was established by Google and its founders Sergey Brin and Larry Page in the 1990s. Its main functionality is based on their publication in Reference [2]. We present here a key idea behind the PageRank algorithm, which involves eigenvectors, eigenvalues, and directed graphs. You will work through a small example. In reality, one needs to apply efficient numerical algorithms to estimate the PageRank eigenvector.

As an example, the directed graph in Figure 1 represents a small network of four websites. This graph has 4 *nodes*, each representing a website. If a website links to another website, this is shown by a directed edge. For example, website #1 has two links: one to website #2 and one to website #3. Website #2 also has two links: one to website #3 and one to website #4. We will only consider graphs for which every website links to some other website and no website links to itself. (Although we encourage the curious to explore such situations once they've worked through Question 1 and 2.)



**Figure 1:** Directed graph, representing a small network of linked websites.

One ranking strategy would be to use the number of incoming hyperlinks. If lots of other websites are linking to one website, surely that one website must be very useful and reliable, right? This strategy is flawed because it can be easily manipulated by someone who creates lots of spurious websites all with hyperlinks pointing to the one website they want to boost up in the ranking. When creating a ranking, not all hyperlinks should carry the same weight!

A hyperlink from a low-ranked website should have less of a contribution to a website's ranking than a hyperlink from a high-ranked website. Separately, there should be some sort of dilution effect: if a website has two hyperlinks pointing to other websites, those two links should count for more (on the receiving ends) than if the website had ten hyperlinks pointing to other websites.

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<sup>1</sup>A similar problem would be if you consider a tournament (e.g. a chess tournament), in which not every player is playing against every other player. How can we now calculate a ranking of the best players if we only have insufficient data?

The PageRank approach encapsulates these three ideas:

- The rank of a website is the sum of the weights of all incoming hyperlinks.
- The higher the ranking of a website, the higher the weights are of its outgoing hyperlinks.
- The more outgoing hyperlinks a website has, the lower the weight each of those hyperlinks has.

With this weighing strategy, we calculate the PageRank for each site as follows: The PageRank  $p_i$  of website  $i$  is a positive scalar and is the weighted sum of the PageRanks of all the websites that have hyperlinks pointing to it:

$$p_i = \sum_{j \rightarrow i} \frac{p_j}{N_j} \quad (1)$$

where  $j \rightarrow i$  denotes there is a hyperlink from site  $j$  to site  $i$ ,  $p_j$  is the PageRank of website  $j$ , and  $N_j$  is the total number of outgoing hyperlinks from site  $j$ . For example, to calculate the PageRank of website 2 in Figure 1, there is one incoming hyperlink from site 1 (which has 2 outgoing links), one from site 3 (which has 3 outgoing links), and one from site 4 (which has only one outgoing link). And so, the PageRank of  $p_2$  is defined as

$$p_2 = \frac{1}{2}p_1 + \frac{1}{3}p_3 + p_4. \quad (2)$$

In Question 1, you will find the PageRank of every website shown in Figure 1. In Question 2, you will continue studying the PageRank model for this network and develop tools that will work for much, much larger networks.

## References

- [1] K. Bryan and T. Leise, *The \$25,000,000,000 Eigenvector: The Linear Algebra behind Google*, SIAM Review, 48:3, 569-581, 2006, [\[online\]](#).
- [2] S. Brin and L. Page, *The anatomy of a large-scale hypertextual Web search engine*, Computer Networks and ISDN Systems, 30:1-7, 107-117, 1998.

**Question 1:**

- (a) Equation (2) determines the PageRank of node 2. Give the system of PageRank equations for the four nodes in Figure 1, as defined by Equation (1). *No justification is required.*

$$\begin{cases} p_1 &= \frac{1}{3}p_3 \\ p_2 &= \frac{1}{2}p_1 + \frac{1}{3}p_3 + p_4 \\ p_3 &= \frac{1}{2}p_1 + \frac{1}{2}p_2 \\ p_4 &= \frac{1}{2}p_2 + \frac{1}{3}p_3 \end{cases}$$

In matrix form, this is:

$$\begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

- (b) Formulate the linear system you found in part (a) as an eigenproblem. Identify the matrix  $\mathbf{A}$ , the eigenvalue and the eigenvector. *Note: you'll solve for the eigenvectors in the next part of this question.*

For some eigenvector of  $A$ :  $\mathbf{p} = [p_1 \ p_2 \ p_3 \ p_4]^T$ , we can write the system of equations as

$$A\mathbf{p} = \mathbf{p} = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \lambda \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

where  $\lambda = 1$ . The matrix  $\mathbf{A}$  is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix}$$

- (c) Solve the eigenproblem you formulated in part (b). Calculate the PageRank vector  $\mathbf{p}$  for the graph in Figure 1. You will need to use that the components of a PageRank vector must add up to 1.

Identifying  $A\mathbf{p} = \mathbf{p}$  as an eigenvalue problem, we can solve for the eigenvector  $\mathbf{p}$  by solving the system of equations  $A\mathbf{p} - \mathbf{p} = 0$ . This is equivalent to solving the system of equations  $(A - I)\mathbf{p} = 0$ . So, we have:

$$\begin{bmatrix} -1 & 0 & 1/3 & 0 \\ 1/2 & -1 & 1/3 & 1 \\ 1/2 & 1/2 & -1 & 0 \\ 0 & 1/2 & 1/3 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can solve this system of equations using computers method. I used the `scipy.nullspace` function in Python to find the null space of the matrix. The result is:

$$\mathbf{p} = \text{span} \left( [0.14547859 \ 0.72739297 \ 0.43643578 \ 0.50917508]^T \right)$$

We can normalize this vector by dividing each component by the sum of all components:

$$\sum_{i=1}^4 p_i = 0.14547859 + 0.72739297 + 0.43643578 + 0.50917508 = 1.81848242$$

So, the PageRank vector is:

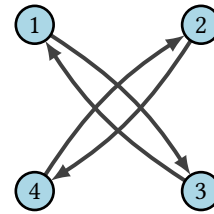
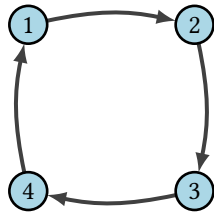
$$\mathbf{p} = [0.14547859 \ 0.72739297 \ 0.43643578 \ 0.50917508]^T / 1.81848242 = [0.08 \ 0.4 \ 0.24 \ 0.28]^T$$

(d) Which website of the graph in Figure 1 has the highest and which has the lowest PageRank?

- Website 2 has the highest PageRank of 0.4
- Website 1 has the lowest PageRank of 0.08

(e) Think about other possible networks. What would be a network that would lead to all four webpages having the same PageRank? Add the hyperlinks to the figure below on the left. What would be a *different* network would lead to all four webpages having the same PageRank? Add the hyperlinks to the figure on the right. *No justification is required.*

*There are more than two possible answers to this question. Please make sure that your hyperlinks are clearly legible in your scan, including the direction they're pointing.*



**Question 2:**

Now that you are more familiar with the eigenproblem from Question 1, you're going to study the eigenvalues and eigenvectors of the matrix  $\mathbf{A}$  from Question 1.

- (a) Compute  $\mathbf{Ax}$  where  $\mathbf{x} \in {}^4\mathbb{R}$ . How is  $\sum_{i=1}^4 (\mathbf{Ax})_i$  related to  $\sum_{i=1}^4 x_i$ ?

We consider the sum of the components of  $\mathbf{Ax}$ . First, we compute  $\mathbf{Ax}$ :

$$\mathbf{Ax} = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (1/3)x_3 \\ (1/2)x_1 + (1/3)x_3 + x_4 \\ (1/2)x_1 + (1/2)x_2 \\ (1/2)x_2 + (1/3)x_3 \end{bmatrix}$$

$$\sum_{i=1}^4 (\mathbf{Ax})_i = (1/3)x_3 + (1/2)x_1 + (1/3)x_3 + x_4 + (1/2)x_1 + (1/2)x_2 + (1/2)x_2 + (1/3)x_3 = x_1 + x_2 + x_3 + x_4$$

Thus, we have  $\sum_{i=1}^4 (\mathbf{Ax})_i = \sum_{i=1}^4 x_i$

Assume  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$  with eigenvalue  $\lambda$ . What can you say about  $\lambda$  and/or  $\mathbf{x}$ ?

Since  $\mathbf{Ax} = \lambda\mathbf{x}$ , we can write:  $\sum_{i=1}^4 (\mathbf{Ax})_i = \sum_{i=1}^4 \lambda x_i$ . We know from the previous part that  $\sum_{i=1}^4 (\mathbf{Ax})_i = \sum_{i=1}^4 x_i$ . So, we have  $\sum_{i=1}^4 \lambda x_i = \sum_{i=1}^4 x_i$ . So, one of the following must be true:

- $\lambda = 1$  and  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$ .
- $\sum_{i=1}^4 x_i = 0$  and  $\lambda$  can be any value.

So, we can conclude that if  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$ , then either  $\lambda = 1$  or  $\sum_{i=1}^4 x_i = 0$ .

- (b) Using your computation of  $\mathbf{Ax}$  where  $\mathbf{x} \in {}^4\mathbb{R}$ , how is  $\sum_{i=1}^4 |(\mathbf{Ax})_i|$  related to  $\sum_{i=1}^4 |x_i|$ ? *You will want to use the triangle inequality: if  $a, b \in \mathbb{R}$  then  $|a + b| \leq |a| + |b|$ .*

For all not necessarily eigenvectors  $\mathbf{x} \in {}^4\mathbb{R}$ , we have:  $(Ax)_i = \sum_{j=1}^4 A_{ij}x_j$ . Using the triangle inequality, we have:

$$|(Ax)_i| = \left| \sum_{j=1}^4 A_{ij}x_j \right| \leq \sum_{j=1}^4 |A_{ij}||x_j|$$

Taking the sum over  $i$  gives:  $\sum_{i=1}^4 |(Ax)_i| \leq \sum_{i=1}^4 \sum_{j=1}^4 |A_{ij}||x_j| = \sum_{j=1}^4 |x_j| \sum_{i=1}^4 |A_{ij}|$

Since  $\sum_{i=1}^4 |A_{ij}| = 1$  for all  $j$ , we have:

$$\sum_{i=1}^4 |(Ax)_i| \leq \sum_{j=1}^4 |x_j|$$

If  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$ , what can you say about its eigenvalue  $\lambda$ ?

From the previous part, we know that  $\sum_{i=1}^4 |(Ax)_i| \leq \sum_{j=1}^4 |x_j|$ . Since  $\mathbf{Ax} = \lambda\mathbf{x}$ , we have:

$$\sum_{i=1}^4 |(Ax)_i| = |\lambda| \sum_{i=1}^4 |x_i| \leq \sum_{i=1}^4 |x_i|$$

If  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$ , by definition we have  $\sum_{i=1}^4 |x_i| \neq 0$ . So, we can divide both sides by  $\sum_{i=1}^4 |x_i|$ :

$$|\lambda| \leq 1$$

- (c) Combining your answer to (a) and (b) what can you say about  $\mathbf{A}$ 's eigenvalues and eigenvectors?

We have the following cases:

- If  $\sum_{i=1}^4 x_i \neq 0$ , then  $|\lambda| = 1$ .
- If  $\sum_{i=1}^4 x_i = 0$ , then  $|\lambda| < 1$ .

So, we can conclude that the eigenvalues of  $\mathbf{A}$  are all less than or equal to 1.

- (d) Using matlab or python or wolframalpha or whatever, what are the eigenvalues of  $\mathbf{A}$ ? Please give the eigenvalues in order of decreasing magnitude, largest to smallest. Remember,  $|a + ib| = \sqrt{a^2 + b^2}$ . Round to two decimal places. *No justification is required.*

$$\lambda_1 \approx \boxed{1.00} \quad \lambda_2 \approx \boxed{-0.6 + 0.22i} \quad \lambda_3 \approx \boxed{-0.6 - 0.22i} \quad \lambda_4 \approx \boxed{0.20}$$

$[|\lambda_2| \approx 0.64] \quad [|\lambda_3| \approx 0.64]$

Is  $\mathbf{A} \in {}^4\mathbb{R}^4$  diagonalizable? ☐ Yes ☒ No

Is  $\mathbf{A} \in {}^4\mathbb{C}^4$  diagonalizable? ☒ Yes ☐ No

- (e) Let  $\mathbf{x} \in {}^4\mathbb{C}$  be a linear combination of the eigenvectors of  $\mathbf{A}$ ; that is,  $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4$  where  $c_i \in \mathbb{C}$ . What is  $\mathbf{A}\mathbf{x}$ ? What is  $\mathbf{A}^2\mathbf{x}$ ? What is  $\mathbf{A}^{100}\mathbf{x}$ ? *Your answers should include the symbols  $\lambda_i$  when referring to eigenvalues, rather than the approximate values you found for eigenvalues.*

Since  $\mathbf{x}_i$  are eigenvectors of  $\mathbf{A}$ , we have:

$$\mathbf{A}\mathbf{x} = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + c_3\lambda_3\mathbf{x}_3 + c_4\lambda_4\mathbf{x}_4$$

Since  $\mathbf{A}$  is a linear operator,  $\mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4) = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + c_3\mathbf{A}\mathbf{x}_3 + c_4\mathbf{A}\mathbf{x}_4$  and we can write:

$$\begin{aligned} \mathbf{A}^2\mathbf{x} &= c_1\lambda_1^2\mathbf{x}_1 + c_2\lambda_2^2\mathbf{x}_2 + c_3\lambda_3^2\mathbf{x}_3 + c_4\lambda_4^2\mathbf{x}_4 \\ \mathbf{A}^{100}\mathbf{x} &= c_1\lambda_1^{100}\mathbf{x}_1 + c_2\lambda_2^{100}\mathbf{x}_2 + c_3\lambda_3^{100}\mathbf{x}_3 + c_4\lambda_4^{100}\mathbf{x}_4 \end{aligned}$$

As  $n \rightarrow \infty$ , what does  $\mathbf{A}^n\mathbf{x}$  do, and why?

Since  $|\lambda_1| = 1$  and  $|\lambda_2|, |\lambda_3|, |\lambda_4| < 1$ , we have:

$$\mathbf{A}^n\mathbf{x} = c_1\lambda_1^n\mathbf{x}_1 + c_2\lambda_2^n\mathbf{x}_2 + c_3\lambda_3^n\mathbf{x}_3 + c_4\lambda_4^n\mathbf{x}_4$$

As  $n \rightarrow \infty$ , the terms with  $\lambda_2, \lambda_3$  and  $\lambda_4$  will go to 0, and we will be left with  $\mathbf{A}^n\mathbf{x} \rightarrow c_1\mathbf{x}_1$ .

- (f) You have the matrix  $\mathbf{A}$  and you would like to find an approximation of an eigenvector for the eigenvalue 1. You have access to matlab and python but not to subroutines that compute eigenvalues and eigenvectors. Playing around, you discover that if you choose a random vector  $\mathbf{x} \in {}^4\mathbb{R}$  and you compute  $\mathbf{A}^n\mathbf{x}$  for larger and larger values of  $n$  then  $\mathbf{A}^n\mathbf{x}$  seems to stop changing (so far as the computer can see). If you call that computational limit  $\mathbf{x}_\infty$ , is  $\mathbf{x}_\infty$  (approximately) an eigenvector for  $\mathbf{A}$  with eigenvalue 1?

☐ Yes ☒ No

If yes, why? If no, why not?

No, since  $\mathbf{x}_\infty$  is the image of the random vector  $\mathbf{x}$  under the transformation  $\mathbf{A}^n$ , it is not guaranteed that  $\mathbf{x}$  not be spanned only by  $\mathbf{x}_2, \mathbf{x}_3$  and  $\mathbf{x}_4$ . If  $c_1 = 0$ , then  $\mathbf{A}^n\mathbf{x}_\infty$  will not converge to an eigenvector of  $\mathbf{A}$  and converge to the zero vector. For instance, we pick  $\mathbf{x} = \mathbf{x}_3$ , then  $\mathbf{A}^n\mathbf{x}_\infty = c_3\lambda_3^n\mathbf{x}_3$  and as  $n \rightarrow \infty$ ,  $\mathbf{A}^n\mathbf{x}_\infty$  will converge to 0. So, we can conclude that  $\mathbf{x}_\infty$  is not guaranteed to converge to the eigenvector of  $\mathbf{A}$  associated with the eigenvalue 1.

**For the curious:** The linear system in question 1(a) is a small linear system: four equations in four unknowns. Your immediate reaction upon seeing the system should have been to set up an augmented matrix, do some elementary row operations, and find all solutions. Finding solutions by viewing it as an eigenproblem seems pretty artificial! And even if you do decide to take the eigenproblem view of things; you can find the null space of  $\mathbf{I} - \mathbf{A}$  via row reduction. So row reduction must be the right approach, right?

For a internet search engine, how big is the matrix  $\mathbf{A}$ ? Its size depends on the number of considered webpages, which is roughly on the order of tens of billions for a search engine. We do know that the  $\mathbf{A}$  is very sparse because, on average, there are fewer than 100 links per webpage pointing to different page. So,  $\mathbf{A}$  is a sparse matrix of size  $10 \text{ billion} \times 10 \text{ billion}$ . The number of operations needed to transform an  $n \times n$  matrix to reduced row echelon form is  $\mathcal{O}(n^3)$ . But the number of operations needed to compute  $\mathbf{A}\mathbf{x}$  once is  $\mathcal{O}(n^2)$ . When you have  $\mathbf{A}\mathbf{x}$ , computing  $\mathbf{A}^2\mathbf{x}$  is a matter of computing  $\mathbf{A}(\mathbf{A}\mathbf{x})$  — this will be another  $\mathcal{O}(n^2)$ . Computing  $\mathbf{A}^k \mathbf{x}$  is  $\mathcal{O}(kn^2)$  and so, if  $k$  isn't large, this will be faster than computing the reduced row echelon form. Using this approach to approximate an eigenvector for the largest eigenvalue is called *power method*. It works well as long as the geometric multiplicity of the largest eigenvalue is 1. The speed with which it converges has to do with how big the gap is between  $|\lambda_1|$  and  $|\lambda_2|$  where  $\lambda_2$  is an eigenvalue with “next largest” size. The bigger  $|\lambda_1| - |\lambda_2|$  is, the faster the power method converges. You can find Python implementations of the power method to calculate the dominant eigenvalue and eigenvector [online](#).

Numerical methods for computing eigenvalues have been an area of active research and continue to be so. The power method is a snowflake on top of an iceberg when it comes to the topic!

A separate question is: what properties of a network ensure that 1 will be an eigenvalue of the matrix  $\mathbf{A}$ ? What properties ensure that its eigenspace will be one dimensional?

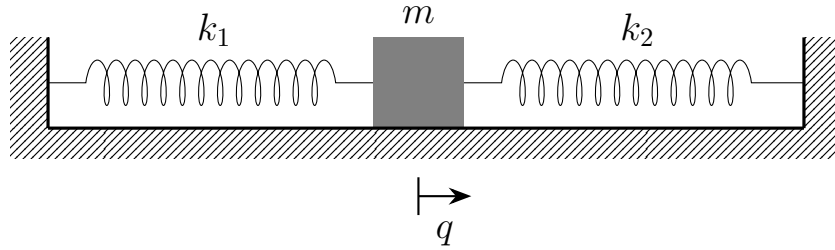


**Question 3:**

The typical ODEs in mechanics are second-order ODEs, because the acceleration is the second derivative of the position in respect to time. If the ODEs are linear, they can be reformulated as two linear first-order ODEs. If the ODEs have constant coefficients and if the forcing is constant, we can solve the system using the methods from this course.

In this problem, you will explore solving a second-order ODE which describes a mass-spring system using the techniques discussed in this course. *For the curious: the methods used in this problem work equally well if there is damping in the system and these methods apply for other oscillatory systems such as RL circuits and RLC circuits.*

Consider the following mass-spring system with one mass and two springs:



**Figure 2:** Mass-spring system with the mass  $m$  and two springs of stiffness  $k_1$  and  $k_2$ .

We will ignore the effect of gravity and of friction and we will assume the displacements are small enough that Hooke's law applies. Let  $m$  be the mass, and  $k_1$  and  $k_2$  be the spring constants (i.e. the stiffnesses) of the springs. The equation of motion for the system in Figure 2 is

$$m\ddot{q}(t) = -k_1q(t) - k_2q(t) = -(k_1 + k_2)q(t). \quad (3)$$

Here,  $q$  is the displacement from rest; if the system is at rest then  $q(t) = 0$  for all time  $t$ . To understand the force: a displacement of the mass  $m$  in positive  $q$  direction result in the left spring to lengthen, which leads to pulling in  $-q$  direction, and the right spring to compress, which leads to pushing in  $-q$  direction.

- (a) Using the substitution of  $x_1(t) = q(t)$  and  $x_2(t) = \dot{q}(t)$ , show that Equation (3) can be expressed as the following system of two first-order ODEs:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1+k_2}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{A}\mathbf{x}(t). \quad (4)$$

From Equation (3), we have:

$$m\dot{x}(t) = m\ddot{q}(t) = -k_1q(t) - k_2q(t) = -(k_1 + k_2)q(t) = -kq(t)$$

So, we have:

$$\dot{x}(t) = -\frac{k_1 + k_2}{m}x(t) \cdot (1)$$

Also, we have the definition of  $x_1$  and  $x_2$ :  $x_1(t) = q(t)$  and  $x_2(t) = \dot{q}(t)$ , so we can write:

$$\dot{x}_1(t) = \dot{q}(t) = x_2(t)$$

Constructing the system of equations, we have:

$$\begin{bmatrix} 0 & 1 \\ -\frac{k_1+k_2}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$$

You will now solve Equation (3), using Equation (4) and the methods introduced in this course.

- (b) View  $\mathbf{A}$  as a complex matrix, rather than a real one,  $\mathbf{A} \in {}^2\mathbb{C}^2$ , and compute the eigenvalues of  $\mathbf{A}$  from Equation (4). To keep your computations simple, replace  $k_1 + k_2$  with  $k$  (the effective spring constant). Your eigenvalues should be of the form  $\pm i\omega$  where  $\omega$  is a function of  $k$  and  $m$ . The expression you will find for  $\omega$  is called the “natural angular frequency” or the “natural frequency”. For the rest of Question 3, refer to  $\omega$  rather than to expressions involving  $k$  and  $m$ .

The eigenvalues of  $A$  are the solutions of the characteristic polynomial:

$$\det(A - \lambda I) = 0$$

We have:

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{bmatrix}$$

So, we have:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{bmatrix} = (-\lambda)(-\lambda) - (1)\left(-\frac{k}{m}\right) = \lambda^2 + \frac{k_1 + k_2}{m}$$

Setting the determinant to 0, we have:

$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda^2 = -\frac{k}{m}$$

$$\lambda = \pm i\sqrt{\frac{k}{m}}$$

Setting  $\omega = \sqrt{\frac{k}{m}}$ , we have:

$$\lambda = \pm i\omega$$

- (c) Compute the eigenvectors for the two complex eigenvalues calculated in Question 3 (b). **Hint:** The eigenvectors will be complex as well. *You will need to rewrite  $\mathbf{A}$  so that it no longer involves  $k$  and  $m$ , only  $\omega$ .*

We consider the matrix:

$$A \pm i\omega I = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \pm i \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \pm i\omega & 1 \\ -\omega^2 & \pm i\omega \end{bmatrix}$$

Considering the first eigenvalue  $\lambda_1 = i\omega$ , and consider any  $c_1, c_2 \in \mathbb{C}$ , we have:

$$\begin{bmatrix} i\omega & 1 \\ -\omega^2 & i\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

So the first eigenvector is  $c_2[1, i\omega]^T$ . Considering the second eigenvalue  $\lambda_2 = -i\omega$ , we have:

$$\begin{bmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

So the second eigenvector is  $c_2[1, -i\omega]^T$ . So the eigenvectors are:

$$c_1 \begin{bmatrix} 1 \\ i\omega \end{bmatrix} \text{ and } c_2 \begin{bmatrix} 1 \\ -i\omega \end{bmatrix}$$

- (d) Using the eigenvalues and eigenvectors you found, what is the general solution,  $\mathbf{x}(t)$ , for the system in Equation (4)? What is each component,  $x_1(t)$  and  $x_2(t)$ ? *Please keep working with  $\omega$  rather than  $k$  and  $m$ . Also, you want to check your work by checking on a piece of scratch paper that  $\dot{x}_1(t) = x_2(t)$ . There's no need to present that verification here.*

Given  $\lambda_1 = i\omega$  and  $\lambda_2 = -i\omega$ , we have:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{i\omega t} \begin{bmatrix} 1 \\ i\omega \end{bmatrix} + c_2 e^{-i\omega t} \begin{bmatrix} 1 \\ -i\omega \end{bmatrix}$$

So, we have:

$$\begin{aligned} x_1(t) &= c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ x_2(t) &= c_1 i\omega e^{i\omega t} - c_2 i\omega e^{-i\omega t} = i\omega(c_1 e^{i\omega t} - c_2 e^{-i\omega t}) \end{aligned}$$

- (e) Assume that the mass is initially at rest,  $q(0) = q_0$  and  $\dot{q}(0) = 0$ . This determines  $\mathbf{x}(0)$ . What is  $\mathbf{x}(t)$ ? At this point, we haven't discussed what it means to have a complex number in the exponent but you do still know that  $e^0 = 1$ .

First, we solve for  $c_1$  and  $c_2$  using the given initial conditions:

$$x_1(0) = c_1 + c_2 = q_0 \quad x_2(0) = i\omega(c_1 - c_2) = \dot{q}(0) = 0$$

Thus, by solving the system of equations, we have  $c_1 + c_2 = q_0/2$ . Therefore, we have:

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ i\omega(c_1 e^{i\omega t} - c_2 e^{-i\omega t}) \end{bmatrix} = \begin{bmatrix} \frac{q_0}{2} e^{i\omega t} + \frac{q_0}{2} e^{-i\omega t} \\ i\omega(\frac{q_0}{2} e^{i\omega t} - \frac{q_0}{2} e^{-i\omega t}) \end{bmatrix}$$

- (f) At this point, you have an expression for  $q(t)$  that involves  $i$ . Use Euler's formula,  $e^{i\theta} = \cos \theta + i \sin \theta$ , to express  $q(t)$  in terms of real quantities only. **For the curious:** you can use these ideas to rewrite the general solution from part (d) as something involving  $\cos$  and  $\sin$  if you assume the initial conditions  $q(0) = q_0$  and  $\dot{q}(0) = v_0$ .

Using the results of the initial conditions, using Euler's formula, we can rewrite  $q(t)$  as:

$$\begin{aligned} q(t) &= \frac{q_0}{2} e^{i\omega t} + \frac{q_0}{2} e^{-i\omega t} = \frac{q_0}{2} (\cos(\omega t) + i \sin(\omega t)) + \frac{q_0}{2} (\cos(-\omega t) + i \sin(-\omega t)) \\ &= \frac{q_0}{2} (\cos(\omega t) + i \sin(\omega t)) + \frac{q_0}{2} (\cos(\omega t) - i \sin(\omega t)) \end{aligned}$$

The imaginary parts cancel out, so we have:

$$q(t) = \frac{q_0}{2} (\cos(\omega t) + \cos(\omega t)) = q_0 \cos(\omega t)$$

So, we have:

$$q(t) = q_0 \cos(\omega t)$$

- (g) Check if your found solution for  $q(t)$  satisfies the ODE in Equation (3).

$$\begin{aligned} \ddot{q}(t) &= -\omega^2 q(t) \\ m\ddot{q}(t) &= -k_1 q(t) - k_2 q(t) = -(k_1 + k_2)q(t) \end{aligned}$$