MAT185 – Linear Algebra
Assignment 3

Instructions:

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2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I have not used generative AI in writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

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Question 1:

Consider the real vector space $V = \text{span}\{x^2e^x, xe^x, e^x\}$. *No justification needed for parts (a)–(d); just fill in the boxes. :)*

(a) Let $T : V \to V$ be the linear transformation corresponding to differentiation: $T(v) = v'$. Using the basis $\alpha = \{x^2e^x, xe^x, e^x\}$ for the domain and the codomain, what matrix $[T]_{\alpha\alpha}$ represents this linear transformation?

$$[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{2} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{1} \end{bmatrix}$$

(b) Demonstrate that your matrix is correct by computing $[T]_{\alpha\alpha}[v]_\alpha$ for $v = 3x^2e^x - 2xe^x + 6e^x$ to find $[T(v)]_\alpha = [3x^2e^x + 4xe^x + 4e^x]_\alpha$.

$$[T]_{\alpha\alpha}[v]_\alpha = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{2} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{1} \end{bmatrix} \begin{bmatrix} \boxed{3} \\ \boxed{-2} \\ \boxed{6} \end{bmatrix} = \begin{bmatrix} \boxed{3} \\ \boxed{4} \\ \boxed{4} \end{bmatrix} = [T(v)]_\alpha$$

(c) What is $[T]^{-1}_{\alpha\alpha}$ for your matrix?

$$[T]^{-1}_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{-2} & \boxed{1} & \boxed{0} \\ \boxed{2} & \boxed{-1} & \boxed{1} \end{bmatrix}$$

(d) Using your $[T]^{-1}_{\alpha\alpha}$, find $v \in V$ such that $v'(x) = x^2e^x$.

$$[T]^{-1}_{\alpha\alpha}[v]_\alpha = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{-2} & \boxed{1} & \boxed{0} \\ \boxed{2} & \boxed{-1} & \boxed{1} \end{bmatrix} \begin{bmatrix} \boxed{1} \\ \boxed{0} \\ \boxed{0} \end{bmatrix} = \begin{bmatrix} \boxed{1} \\ \boxed{-2} \\ \boxed{2} \end{bmatrix} = [T^{-1}(v)]_\alpha$$

$v(x) = \boxed{x^2e^x - 2xe^x + 2e^x}$

(e) You've learnt that a function has infinitely many antiderivatives. Why is it that this method has found only one of them?

In general, we have:

$$\int f(x)dx = F(x) + c$$

,where $F(x)$ is the antiderivative of $f(x)$.
The linear transformation T maps to the special case $c = 0$. The constant function is not spanned by the bases of vector space V, so the antiderivative found is unique with $c = 0$. In general, linear transformation that represent the antiderivative cannot have non-zero constant such that the additivity property of the linear transformation is preserved. (i. e. $T(f+g) = F+G+c = T(f)+T(g) = F+G+2c$)

Question 2:

(a) What is a parametrized curve that represents the line segment that connects (x_0, y_0) to (x_1, y_1)? That is, what is an $\ell(t)$ such that $\ell(0) = (x_0, y_0)$, $\ell(1) = (x_1, y_1)$, and $\{\ell(t) \mid 0 \leq t \leq 1\}$ is a line segment?

$$\ell(t) = (x(t), y(t)) = (1-t)(x_0, y_0) + t(x_1, y_1),\ 0 \leq t \leq 1$$

Putting into standard form, we have:

$$\ell(t) = (x_0, y_0) + t(x_1 - x_0, y_1 - y_0)$$

(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T((x, y)) = (2x + 3y, -6x + y)$. What is the image of the above line segment $\{\ell(t) \mid 0 \leq t \leq 1\}$ under this linear transformation?

For $0 \leq t \leq 1$, we have:

$T(\ell(t)) = (2x(t)+3y(t), -6x(t)+y(t)) = (2(1-t)x_0+2tx_1+3(1-t)y_0+3ty_1, -6(1-t)x_0-6tx_1+(1-t)y_0+ty_1)$

Simplifying, we have the line segment connecting $(2x_0 + 3y_0, -6x_0 + y_0)$ to $(2x_1 + 3y_1, -6x_1 + y_1)$:

$$T(\ell(t)) = ((1-t)(2x_0 + 3y_0) + t(2x_1 + 3y_1), (1-t)(-6x_0 + y_0) + t(-6x_1 + y_1))$$

Putting into standard form, we have:

$$T(\ell(t)) = (2x_0 + 3y_0, -6x_0 + y_0) + t(2(x_1 - x_0) + 3(y_1 - y_0), -6(x_1 - x_0) + (y_1 - y_0))$$

(c) Consider a general linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$; that is, $T((x, y)) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$ where $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$. Let $\ell \subseteq \mathbb{R}^2$ be a line in the plane. In ten words or less, what is the image of ℓ? That is, what is $T(\ell)$?

$T(\ell)$ is a line or a point in $\mathbb{R}^2$.

(d) *No justification is needed. You can select more than one answer.* Let $P \subseteq \mathbb{R}^2$ be a polygon[1] and consider a general linear transformation as in part (c). The image of P under T could be

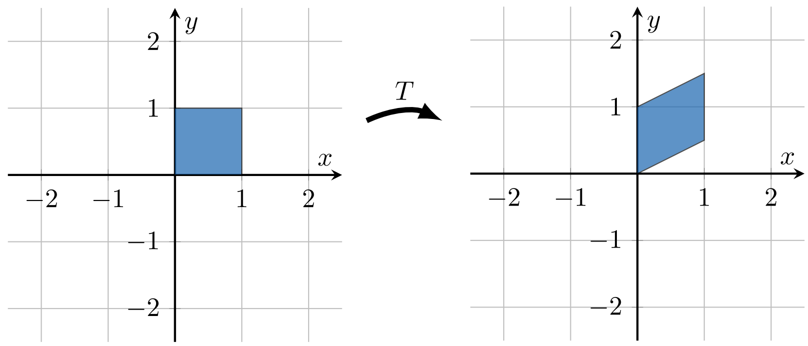
- ○ a circle
- ● a point
- ● a line segment
- ● a polygon
- ○ something else

[1] Here are the types of polygons we mean here. Think about both regular and irregular ones, please.

Question 3:

In the following, you will consider a sequence of linear transformations. For each linear transformation you will asked to draw the image of a square under the linear transformation. *No justification needed for parts (a)–(f); just fill in the boxes. :)*

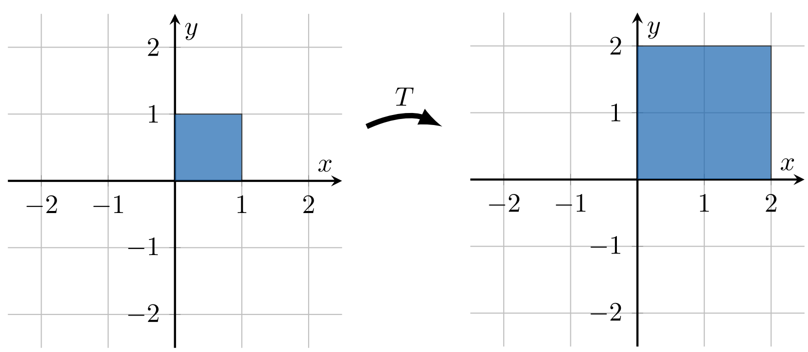
(a) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (x, x/2 + y)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{1}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{0} \\ \boxed{\frac{1}{2}} & \boxed{1} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{1}$

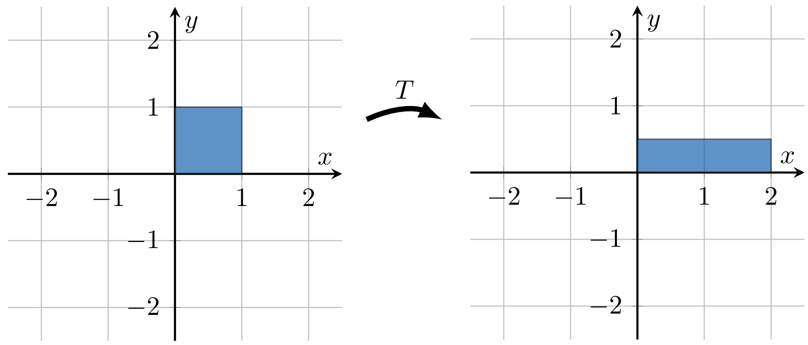
(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (2x, 2y)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{4}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{2} & \boxed{0} \\ \boxed{0} & \boxed{2} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{4}$

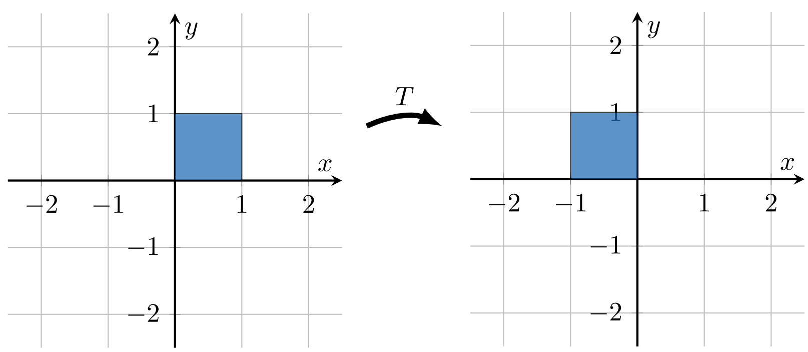
(c) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (2x, y/2)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{1}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{2} & \boxed{0} \\ \boxed{0} & \boxed{\frac{1}{2}} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{1}$

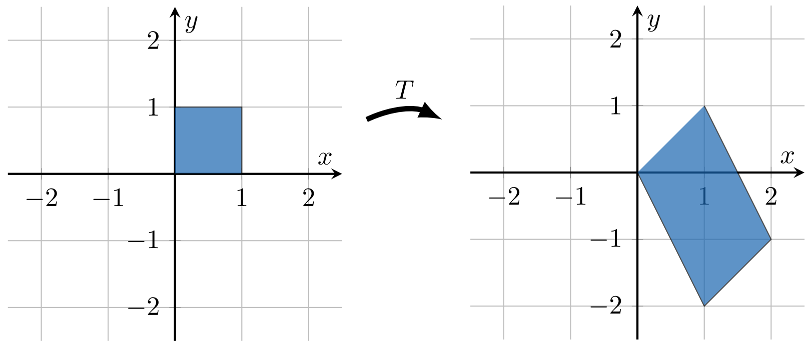
(d) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (-y, x)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{1}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{1}$

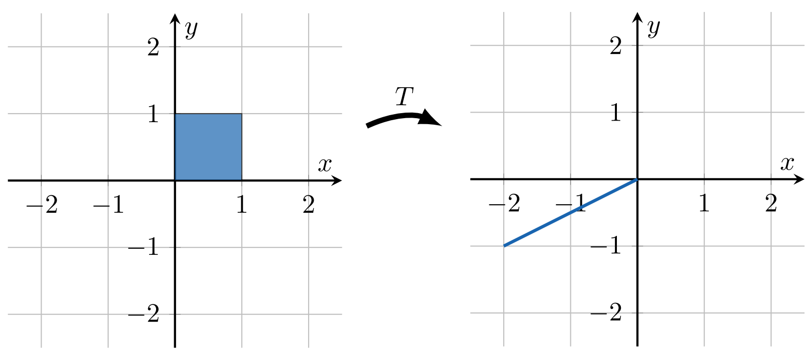
(e) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (x + y, x - 2y)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{3}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{-2} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{-3}$

(f) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (-x - y, -x/2 - y/2)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{0}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{-1} & \boxed{-1} \\ \boxed{-\frac{1}{2}} & \boxed{-\frac{1}{2}} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{0}$

(g) Continuing with the unit square $\mathcal{S}$, as shown in parts (a)–(f), the area of $T(\mathcal{S})$ and $\det[T]_{\alpha\alpha}$ are related. What is the relationship you observed? Prove this relationship holds for a general linear transformation T. *See question 2(c) for what we mean by a general linear transformation. Your answer should not need to refer to the entries of T; there should be no a_{12} or the like in your answer.*

Area of $T(\mathcal{S})$ = Area of $\mathcal{S} \times \det[T]_{\alpha\alpha} = \det[T]_{\alpha\alpha}$ Let V be a 2-dimensional vector space and $T : V \to V$ be a general linear transformation for standard basis α. We have $\alpha = \{\alpha_1, \alpha_2\} = \{(1,0),(0,1)\}$ hence α_1 and α_2 encloses the unit square $\mathcal{S}$. Let v_1, v_2 be the images of α_1, α_2 under T respectively. Then v_1, v_2 encloses the image of the unit square $T(\mathcal{S})$ which is a parallelogram. Let θ be the angle between v_1 and v_2, we have:

$$P = \text{Area of parallelogram } T(\mathcal{S}) = |v_1||v_2|\sin\theta$$

Hence, we have:

$$P = |v_1||v_2|\sqrt{1 - \left(\frac{v_1 \cdot v_2}{|v_1||v_2|}\right)^2} = \sqrt{|v_1|^2|v_2|^2 - (v_1 \cdot v_2)^2}$$

Expanding, we have:

$$P = \sqrt{(v_{11}v_{21})^2 + (v_{12}v_{22})^2 + (v_{11}v_{22})^2 + (v_{12}v_{21})^2 - (v_{11}v_{21} + v_{12}v_{22})^2}$$
$$= \sqrt{v_{11}^2v_{22}^2 + v_{12}^2v_{21}^2 - 2v_{11}v_{21}v_{12}v_{22}}$$

Simplifying, we have:

$$P = \sqrt{(v_{11}v_{21})^2 + (v_{12}v_{22})^2 - 2v_{11}v_{21}v_{12}v_{22}} = \sqrt{(v_{11}v_{22} - v_{12}v_{21})^2} = |v_{11}v_{22} - v_{12}v_{21}|$$

Let $V = \begin{bmatrix} v_1^{\mathbf{T}} & v_2^{\mathbf{T}} \end{bmatrix}$ and $\mathcal{A} = \begin{bmatrix} \alpha_1^{\mathbf{T}} & \alpha_2^{\mathbf{T}} \end{bmatrix}$, then $V = \mathcal{A}T$. We have:

$$P = |v_{11}v_{22} - v_{12}v_{21}| = |\det(V)| = |\det(\mathcal{A}T) = |\det(\mathcal{A})\det(T)|.$$

Since $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, we have $\det(\mathcal{A}) = 1$. Therefore, we have:

$$P = \det(T) = \det[T]_{\alpha\alpha}$$

Hence, the area of $T(\mathcal{S})$ is equal to $|\det[T]_{\alpha\alpha}|$.

(h) If you started with a square of area 4, $\mathcal{S}_1 = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$ how would you use the determinant to compute the area of $T(\mathcal{S}_1)$ for a general linear transformation?

I would use the determinant to compute the area of $T(\mathcal{S}_1)$ for a general linear transformation by:

$$\text{Area of } T(\mathcal{S}_1) = \text{Area of } \mathcal{S}_1 \times \det[T]_{\alpha\alpha} = 4\det[T]_{\alpha\alpha}$$

For all parallelogram $\mathcal{S}_r$ enclosed by two vectors in V with area A_r, we can find a linear transformation $Q_r : V \to V$ such that $Q_r(\mathcal{S}_0) = \mathcal{S}_r$ where $\mathcal{S}_0$ is the unit square. Therefore, the area of $\mathcal{S}_r$ is $A_r = \det[Q]_{\alpha\alpha}$. Hence $T(\mathcal{S}_r) = T(Q(\mathcal{S}_0))$. From part g, we have $T(\mathcal{S}_r) = T(Q(\mathcal{S}_0)) = \det[TQ]_{\alpha\alpha} = \det[T]_{\alpha\alpha}\det[Q]_{\alpha\alpha} = A_r\det[T]_{\alpha\alpha}$. Also, since $Q_r(\mathcal{S}_0) = \mathcal{S}_r$ from part g, we have $\det[Q]_{\alpha\alpha} = A_r$. Therefore, the area of $T(\mathcal{S}_r)$ is $A_r\det[T]_{\alpha\alpha}$.
In this case where $\mathcal{S}_1$ is a square of area 4, the area of $T(\mathcal{S}_1)$ is $4\det[T]_{\alpha\alpha}$.

The following question 4 is worth zero points. Your work will not be graded. The question generalizes the previous question and is within your reach if you are comfortable with dot products and cross products. It is material that you may be assumed to already know when you're in AER210. We encourage you to think about and work on this problem and we are happy to talk about it with you, but it is provided "for the curious" in that you will not be tested on it.

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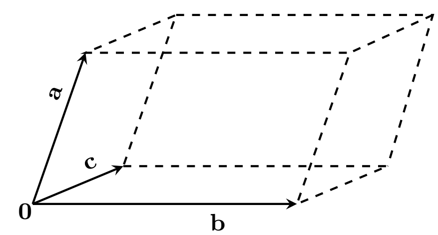
Question 4:

What about volumes of parallelepipeds?

Each face of a parallellepiped is parallel to the opposite face; the parallelepiped is determined by the three edges coming out from any one corner. In the image, we have a corner at $(0, 0, 0)$ and the three edges are represented using the vectors in $\mathbf{a}, \mathbf{b}, \mathbf{c} \in {}^3\mathbb{R}$, for example the vectors based at the origin $(0, 0, 0)$.



To find the volume of a parallelepiped[2], you need to find the area of one face and the distance of the opposite face to that one face. For example, if you found the area of the bottom face (the one determined by $\mathbf{b}$ and $\mathbf{c}$) you would need to find the height of the parallelepiped. You would then multiply the two areas.

(a) Using the above figure, explain why the volume of the parallelipiped is $| \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) |$. Explain how this is related to the determinant of the matrix whose rows are given by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$\det \begin{bmatrix} \mathbf{a}^\mathbf{T} \\ \mathbf{b}^\mathbf{T} \\ \mathbf{c}^\mathbf{T} \end{bmatrix}.$$

[2] A parallelepiped is a solid object with corners and edges and an interior. We can understand it using Cartesian coordinates the moment we say where the origin is and where the x, y, and z axes are. In the picture we're implicitly taking the view that the positive x axis is in the same direction as $\mathbf{b}$, the positive y axis is in the same direction as $\mathbf{c}$, and the positive z axis is in the same direction as $\mathbf{a}$. But we need to be careful to remember that edges are collections of points like $(0, 0, 0)$ and $(1, 0, 0)$ and so forth while the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in ${}^3\mathbb{R}$.

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(b) Explain why you would have gotten the same volume if you had based your calculation on the front of the parallelepiped (determined by **a** and **b**) or if you had based your calculation on the side of the parallelepiped (determined by **a** and **c**).

$$\text{volume of parallelepiped} = \det \begin{bmatrix} \mathbf{c}^{\mathbf{T}} \\ \mathbf{a}^{\mathbf{T}} \\ \mathbf{b}^{\mathbf{T}} \end{bmatrix} = \det \begin{bmatrix} \mathbf{b}^{\mathbf{T}} \\ \mathbf{c}^{\mathbf{T}} \\ \mathbf{a}^{\mathbf{T}} \end{bmatrix}.$$

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(c) Consider the unit cube $\mathcal{C} = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. What is the image of $\mathcal{C}$? (What is $T(\mathcal{C})$ geometrically?) If α is the standard basis for $\mathbb{R}^3$ and $[T]_{\alpha\alpha}$ is the matrix that represents the linear transformation, what is $\det [T]_{\alpha\alpha}$ and how does it relate to the volume of $T(\mathcal{C})$?

(d) If you started with a cube of volume 8, $\mathcal{C}_1 = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2\}$, how would you use the determinant to compute the volume of $T(\mathcal{C}_1)$?