# MAT 185 Lecture Notes

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The up-to-date version of this document can be found at `https://github.com/HaysonC/skulenotes`

## 1 Vector Space

### 1.1 Foundamental Properties

**Definiton 1.1.1** (Vector Space in $\mathbb{R}$)**.** This course concerns with real vector spaces. A vectors space, $V$, over $\mathbb{R}$ is a collection of **object** $\mathbf{v} \in V$ s.t. the follow axioms are followed

1. **Addition Axioms**
   (a) **Closure Under Addition:**
   $\mathbf{x}, \mathbf{y} \in V \implies \mathbf{x} + \mathbf{y} \in V$
   (b) **Associativity of Addition:**
   $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \implies (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
   (c) **Existence of Additive Identity:**
   $\exists\, \mathbf{0} \in V$ such that $\mathbf{x} \in V \implies \mathbf{x} + \mathbf{0} = \mathbf{x}$
   (d) **Existence of Additive Inverse:**
   $\forall\, \mathbf{x} \in V$, $\exists\,$**-x** $\in V$ such that $\mathbf{x} +$ **-x** $= \mathbf{0}$
2. **Scalar Multiplication Axioms**
   (a) **Closure Under Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha \in \mathbb{R}$, $\alpha\mathbf{x} \in V$
   (b) **Associativity of Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
   (c) **Distributive Property of Scalar Multiplication:**
   $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
   (d) **Existence of Multiplicative Identity:**
   $\forall\, \mathbf{x} \in V$, $\mathbf{1x} = \mathbf{x}$

**Note** It could be shown that the axiom imply the commutativity of in addition, namely $\forall \mathbf{x} \implies \mathbf{y} \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

**Example** $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ and $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

**Theorem 1.1.2** (Cancelation, Part 1)**.** Let $V$ be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then:

$$\begin{aligned} \mathbf{x} + \mathbf{z} &= \mathbf{y} + \mathbf{z} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

*Proof.*

$$\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$$

From additive inverse we know that **-z** exists

$$(\mathbf{x} + \mathbf{z}) + \mathbf{\text{-}z} = (\mathbf{y} + \mathbf{z}) + \mathbf{\text{-}z}$$

By order of addition we have:

$$\begin{aligned} \mathbf{x} + (\mathbf{z} + \mathbf{\text{-}z}) &= \mathbf{y} + (\mathbf{z} + \mathbf{\text{-}z}) \\ \mathbf{x} + \mathbf{0} &= \mathbf{y} + \mathbf{0} \end{aligned}$$

By additive identity we have:

$$\mathbf{x} = \mathbf{y}$$

□

**Theorem 1.1.3** (Cancelation, Part 2)**.** Let $V$ be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then:

$$\begin{aligned} \mathbf{z} + \mathbf{x} &= \mathbf{z} + \mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

To proof that, it would require the following propostion:

**Lemma 1.1.4.** Let $V$ be a vector space and $\mathbf{z} \in V$, then $\mathbf{\text{-}z} + \mathbf{z} = 0$

*Proof.*

We know:

$$\begin{aligned} \mathbf{\text{-}z} + \mathbf{z} &= (\mathbf{\text{-}z} + \mathbf{z}) + 0 \\ &= (\mathbf{\text{-}z} + \mathbf{z}) + (\mathbf{\text{-}z} + \mathbf{\text{-}(\text{-}z)}) \end{aligned}$$

By the order of additon,

$$= \mathbf{\text{-}z} + (\mathbf{z} + (\mathbf{\text{-}z} + \mathbf{\text{-}(\text{-}z)}))$$

Again, by the order of additon,

$$\begin{aligned} &= \mathbf{\text{-}z} + ((\mathbf{z} + \mathbf{\text{-}z}) + \mathbf{\text{-}(\text{-}z)}) \\ &= \mathbf{\text{-}z} + \mathbf{\text{-}(\text{-}z)} \\ &= 0 \end{aligned}$$

□

Now, to prove the part 2 of the Cancelation Theorem:

*Proof.*

$$\mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y}$$

From additive inverse we know that **-z** exists

$$\begin{aligned} \textbf{-z} + (\mathbf{z} + \mathbf{x}) &= \textbf{-z} + (\mathbf{z} + \mathbf{y}) \\ (\textbf{-z} + \mathbf{z}) + \mathbf{x} &= (\textbf{-z} + \mathbf{z}) + \mathbf{y} \end{aligned}$$

From above, we have $\textbf{-z} + \mathbf{z} = 0$

$$\begin{aligned} 0 + \mathbf{x} &= 0 + \mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

□

**Lemma 1.1.5** (Inverse of an inverse)**.** Let $V$ be a vector space and $\mathbf{x} \in V$, then:

$$\textbf{-(-x)} = \mathbf{x}$$

*Proof.* Assume 0, $0^\star$ are the additive identity of $V$ and **-x**, $\textbf{-x}^\star$ are the additive inverse of **x**. We have:

$$u + 0 = u + 0^\star$$

By Cancelation Theorem, we have $0 = 0^\star$. Thus, the additive identity is unique. Now, we have:

$$\begin{aligned} \mathbf{x} + \textbf{-x} &= 0 \\ \mathbf{x} + \textbf{-x}^\star &= 0 \end{aligned}$$

Thus, by the uniqueness of the additive inverse, we have:

$$\mathbf{x} + \textbf{-x} = \mathbf{x} + \textbf{-x}^\star$$

By the Cancelation Theorem, we have:

$$\textbf{-x} = \textbf{-x}^\star$$

Now, we have that the additive inverse is unique. Thus, **-(-x)** must be unique and nessarily **x**. □

#### Additional Results from Above

1. The additive identity is unique
2. The additive inverse is unique

**Definiton 1.1.6** (Subtraction)**.** Let $V$ be a vector space and $\mathbf{x}, \mathbf{y} \in V$, then:

$$\mathbf{x} - \mathbf{y} \overset{\text{def}}{=} \mathbf{x} + \textbf{-y}$$

**Theorem 1.1.7** (Addition is Commutative)**.** Let $V$ be a vector space and $\mathbf{x}, \mathbf{y} \in V$, then:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

*Proof.*

$$\mathbf{x} + \mathbf{y} =$$

□

## 1.2 Vector Subspace

**Definiton 1.2.1** (Vector Subspace)**.** Let $V$ be a vector space and $W \subseteq V$, then $W$ is a vector subspace of $V$ if $W$ is a vector space.

**Theorem 1.2.2** (Subspace Test, I)**.** Let $V$ be a vector space over $\mathbb{R}$ and $W \subseteq V$ with the usual addition and scalar multiplication, then $W$ is a subspace of $V$ iff for all $\mathbf{x}, \mathbf{y} \in W$ and $\alpha \in \mathbb{R}$:

1. **Closure Under Addition:**
   $\mathbf{x} + \mathbf{y} \in W$
2. **Closure Under Scalar Multiplication:**
   $\alpha\mathbf{x} \in W$
3. Additive Identity:
   $\mathbf{0} \in W$

*Proof.* ($\Rightarrow$) Assume $W$ is a subspace of $V$, then $W$ is a vector space. Thus, the axioms of vector space are satisfied.
($\Leftarrow$) Assume the three conditions are satisfied, then $W$ is a vector space. Thus, $W$ is a subspace of $V$. □

**Definiton 1.2.3** (Null Space)**.** Let $V$ be a vector space and $A \in {}^m\mathbb{R}^n$[1], then:

$$\text{null}(A) = \{\mathbf{x} \in {}^n\mathbb{R} \mid A\mathbf{x} = \mathbf{0}\} \tag{1}$$

is the null space of $A$, otherwise known as the **kernel** of $A$ or the solution space of $A\mathbf{x} = \mathbf{0}$

We can use the Subspace Test I to show that the null space of a matrix is a subspace of ${}^n\mathbb{R}$.

1. **Existence of Additive Identity:**
   The zero vector is in the null space of $A$ as the trivial solution to the equation $A\mathbf{x} = \mathbf{0}$
2. **Closure Under Addition:**
   Let $\mathbf{x}, \mathbf{y} \in \text{null}(A)$, then $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus, $\mathbf{x} + \mathbf{y} \in \text{null}(A)$. This holds since ${}^n\mathbb{R}$ is a vector space.

[1] Same as $\mathbb{R}^{m\times n}$, The set of all $m \times n$ matrices with real entries

3. **Closure Under Scalar Multiplication:**
   Let $\mathbf{x} \in \text{null}(A)$ and $\alpha \in \mathbb{R}$, then $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$. Thus, $\alpha\mathbf{x} \in \text{null}(A)$. This holds since ${}^n\mathbb{R}^m$ is a vector space under usual addition and scalar multiplication.

**Theorem 1.2.4** (Subspace Test, II)**.** Let $V$ be a vector space over $\mathbb{R}$ and $W \subseteq V$ with the usual addition and scalar multiplication, then $W$ is a subspace of $V$ iff for all $\mathbf{x}, \mathbf{y} \in W$ and $\alpha, \beta \in \mathbb{R}$:

1. $\alpha\mathbf{x} + \beta\mathbf{y} \in W$

*Proof.* ($\Rightarrow$) Assume $W$ is a subspace of $V$, then $W$ is a vector space. Thus, the axioms of vector space are satisfied.
($\Leftarrow$) Assume the condition is satisfied, then $W$ is a vector space. Thus, $W$ is a subspace of $V$. □

**Definiton 1.2.5** (Intersection of Sets)**.** Let $A$ and $B$ be sets, then:

$$A \cap B = \{x \mid x \in A \,\text{and}\, x \in B\} \tag{2}$$

**Definiton 1.2.6** (Union of Sets)**.** Let $A$ and $B$ be sets, then:

$$A \cup B = \{x \mid x \in A \,\text{or}\, x \in B\} \tag{3}$$

# 2 Linear combinations

**Definiton 2.0.1** (Linear Combination)**.** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, then:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$

is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

## 2.1 Linear Independence, Span, and Basis

**Definiton 2.1.1** (Linear independence)**.** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. If and only if the equation:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

has **only** the trivial solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Then, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are said to be linearly independent; otherwise, they are linearly dependent.

**Definiton 2.1.2** (Span)**.** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, then:

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \mid \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\} \tag{4}$$

is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

**Theorem 2.1.3** (Unique Representation)**.** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}/0$, and that they span $V$. For any $\mathbf{v}_{n+1} \in V$, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, then the equation:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{v}_{n+1}$$

is the only way to express $\mathbf{v}_{n+1}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

We can prove by contrapositive:

*Proof.* Assume that there exists another way to express $\mathbf{v}_{n+1}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \ldots + \beta_n \mathbf{v}_n$$

Rearranging the equation, we have:

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \ldots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0}$$

Since $(\alpha_i - \beta_i)$ for $i = 1, 2, \ldots, n$ are not all zero, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. Thus, the contrapositive is true. □

**Theorem 2.1.4** (Growing and Pruning)**.** The following are the two theorems describe the relationship between linear independence and the span:

**Growing** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ be linearly independent. If $\mathbf{v}_{n+1} \in V$ and $\mathbf{v}_{n+1} \notin \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}$ are linearly independent.

**Pruning** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then there exists a $\mathbf{v}_i$ such that $\mathbf{v}_i \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$.

**Definiton 2.1.5** (Basis)**.** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = V$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of $V$.

## 2.2 Fundamental Theorem of Linear Algebra

**Theorem 2.2.1** (Fundamental Theorem of Linear Algebra)**.** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = V$, then:

1. Any basis of $V$ has the same number of elements.
2. Any linearly independent set of $V$ has at most $n$ elements.
3. Any spanning set of $V$ has at least $n$ elements.