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Normalized frequency calculation and associated dispersion fields profiles of a multilayered FGM structure

Ismail NACIRI
Lab PETI-ERMAM, Ibn Zohr University
Polydisciplinary Faculty of Ouarzazate,
BP.638, 45000 Ouarzazate, Morocco
nacirismail@gmail.com

Rabab RAGHIB
Lab PETI-ERMAM, Ibn Zohr University
Polydisciplinary Faculty of Ouarzazate,
BP.638, 45000 Ouarzazate, Morocco
rababraghib97@gmail.com

Hassna KHALFI
Lab PETI-ERMAM, Ibn Zohr University
Polydisciplinary Faculty of Ouarzazate,
BP.638, 45000 Ouarzazate, Morocco line
khalfi97hassna@gmail.com

Lahoucine ELMAIMOUNI
Lab PETI-ERMAM, Ibn Zohr University
Polydisciplinary Faculty of Ouarzazate,
BP.638, 45000 Ouarzazate, Morocco
la_elmaimouni@yahoo.fr

Abdellah BENAMI
LM3ER-OTEA, Department of Physics,
Faculty of Sciences and Techniques,
Moulay Ismail University, BP 509
Boutalamine 52000, Errachidia, Morocco
abdou_benami@yahoo.fr

Hicham TRIBAK
Lab PETI-ERMAM, Ibn Zohr University
Polydisciplinary Faculty of Ouarzazate,
BP.638, 45000 Ouarzazate, Morocco
hicham.tribak@gmail.com

Abstract—**Functional Graded Materials (FGM) has been gaining a growing interests among research communities. They are key components in a large field of technological applications in aeronautics, automotive, biomechanics, biomedicine, *etc.* To better extend our understanding of such material's properties, this paper discusses the guided acoustic wave propagation modeling of a functionally graded inhomogeneous cylinder using Legendre polynomial approach. The studied structure is composed of three adjacent layers made of Silicon Nitride (SN) and Stainless Steel (SS) with gradient variation along the thicknesses direction presenting very dissimilar material properties. The mathematical formalism is based on linear three-dimensional theory of elasticity and on the Legendre orthogonal polynomial approach. The formulation is simple to implement thanks to an expansion of the independent mechanical variables in an appropriate series of a double Legendre orthonormal functions. The equation of motion is next converted into a matrix eigenvalue problem. Numerical calculations based on the foregoing method were the determination of normalized frequencies for longitudinal, torsion and flexural modes, normal stress components, mechanical displacement fields and phase velocity dispersion profiles.**

Keywords—Polynimial approach; FGM; Acoustic behaviour; Guided waves; Normalized frequency; dispersion profiles.

I. Introduction

Quite recently, considerable attention has been paid to multilayered structures made of functionally graded materials (FGM) thanks to their interesting mechanical properties such as wear and fatigue resistance, high strength, high modulus, *etc*. Such material's applications cover a large field of mechanical and electronic engineering technologies.

In an effort of design and optimization, appropriate theoretical models and efficient solution methods are needed to solve the guided wave propagation problem in functional graded materials. The boundary element method has been proposed to investigate the impact of the material gradient on the dispersion profiles using a specific boundary integral equation [1]. Various methods are also used to study various aspects of wave propagation such as the Peano-Series technique [2], the modified couple stress theory [3], the transfer matrix method [4], the 3D-Chebyshev spectral element [5] and the orthogonal polynomial method which will be the main modeling tool along this paper.

The idea behind the polynomial method started with the uses of Laguerre orthogonal polynomial series and was firstly proposed by Ludwig and Lengeler [6]. The method was then extended to investigate wave propagation and vibrational problems in wedges and ridges [7-9], surface acoustic waves in layered [10], inhomogeneous and semi-infinite structures. Later on, the method provides excellent precision for waveguides of various geometries such as planar and cylindrical multilayered and functionally graded structures [11–18].

Moreover, the formulation of this method is simple to implement thanks to an expansion of the independent mechanical variables in an appropriate series of a double Legendre orthonormal functions. The boundary and continuity conditions between layers are directly incorporated into the governing equations by the use of position-dependent material physical constants with a wise choice of the polynomial expansions for the mechanical displacement components. As a result, the motion equation is then converted into a matrix eigenvalue problem, leading to semi-variational determination of the different modes of frequencies and associated profiles including phase velocities, mechanical displacements and normal stress distribution.

Very recently, Legendre orthogonal polynomial approach has been applied to study the vibration analysis of a multilayer functionally graded cylinder with effects of graded-index and boundary conditions [19], also to study the axisymmetric free vibration modeling of a functionally graded piezoelectric resonator [20].

Based on the above bibliographic review, we can affirm that the polynomial method is highly effective in calculating guided waves propagation in FGM materials; For this raison, this paper extend the application of Legendre polynomial approach to study the dispersion behavior of guided wave propagation in inhomogeneous cylinder composed of three adjacent layers made of silicon nitride and stainless steel with gradient variation along the radial direction presenting very dissimilar material properties.

II. PROBLEM STATEMENT

The studied structure is an inhomogeneous cylinder of infinite length composed of a stainless steel layer (SS) sandwiched between two layers of silicon nitride (SN) as depicted in Figure. 1 including also the space cylindrical coordinates system $(r\varphi z)$. The material properties varying along the radial direction. R_0 and R_1 the inner and outer radius, respectively. R is the mean radius and H is the thicknesses of the cylinder.

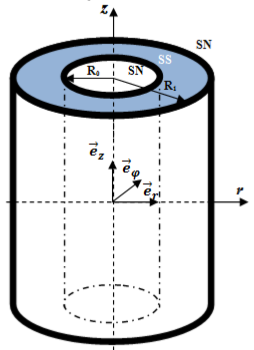

Figure 1. A multilayer FGM cylider composed of SN and SS.

Let's start our mathematic formulation, of guided acoustic wave propagation, with the basic differential governing equation of motion, under the assumption of small deformations, expressed by [21-22]:

$$\rho\frac{\partial^2 U(M,t)}{\partial t^2} = \nabla\sigma_{ij}(M) \quad (1)$$

ρ is the density of the material, $U(M,t) = (u,v,w)$ are the components of elastic displacement in radial, circumferential and axial directions respectively. σ_{ij} are the stress components.

For the elastic medium, the relationship between the general strain components and general displacement components can be expressed by Eq.2 while the relationship between stress-strain components is expressed by Eq.3:

$$\begin{cases} S_{rr} = \frac{\partial u}{\partial r} \\ S_{\varphi\varphi} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \varphi} \\ S_{zz} = \frac{\partial w}{\partial z} \end{cases} \begin{cases} 2S_{\varphi z} = \frac{1}{r}\frac{\partial w}{\partial \varphi} + \frac{\partial v}{\partial z} \\ 2S_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ 2S_{\varphi r} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \varphi} - \frac{v}{r} \end{cases} \quad (2)$$

S_{ij} are the strain components.

$$\begin{cases} \sigma_{rr} = c_{11}S_{rr} + c_{12}S_{\varphi\varphi} + c_{13}S_{zz} \\ \sigma_{\varphi\varphi} = c_{12}S_{rr} + c_{23}S_{\varphi\varphi} + c_{23}S_{zz} \\ \sigma_{zz} = c_{13}S_{rr} + c_{23}S_{\varphi\varphi} + c_{33}S_{zz} \\ \sigma_{\varphi z} = 2c_{44}S_{\varphi z} \\ \sigma_{rz} = 2c_{55}S_{rz} \\ \sigma_{\varphi z} = 2c_{66}S_{r\varphi} \end{cases} \quad (3)$$

In the system of cylindrical coordinates systems, Eq.1 can be rewritten as follows:

$$\frac{\partial\sigma_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{\partial\sigma_{\varphi z}}{\partial z} + \frac{2\sigma_{r\varphi}}{r} = \rho\frac{\partial^2 u}{\partial t^2} \quad (4a)$$

$$\frac{\partial\sigma_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{\partial\sigma_{\varphi z}}{\partial z} + \frac{2\sigma_{r\varphi}}{r} = \rho\frac{\partial^2 v}{\partial t^2} \quad (4a)$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\varphi z}}{\partial\varphi} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho\frac{\partial^2 w}{\partial t^2} \quad (4c)$$

For mathematical convenience, we introduce the following change of variables: $q_1 = kr$, $q_2 = \varphi$ et $q_3 = kz$, where k is the magnitude of the wave vector in the propagation direction.

From now, the FGM cylinder is defined as follow:

$kR_0 \le q_1 \le kR_1$; $0 \le q_2 \le 2\pi$, $-\infty \le q_3 \le +\infty$

The corresponding boundary and continuity conditions of the FGM structure must satisfy these requirements: At the upper and bottom surfaces, the normal components of stress field should equal zero while in the interfaces between layers, the mechanical displacement and the normal stress components must be continuous.

In order to introduce automatically the undergo boundary and continuity requirements into the governing equations of motion, we proceed as follows:

(i) We define a rectangular window function $\pi(kR_0, kR_1)$:

$$\pi(kR_0, kR_1) = \begin{cases} 1 & kR_0 \le q_1 \le kR_1 \\ 0 & elsewhere \end{cases} \quad (5)$$

(ii) As the multilayer FGM structure imply a gradual change of elastic c_{ij} constants and mass density along the radial direction, we need to express such properties variation as a function of q_1 using Einstein summation convention:

$$c_{ij}^{(M)}(q_1) = c_{ij}^{(l)}\left(\frac{q_1}{kH}\right)^l \quad (6)$$

$$\rho_{ij}^{(M)}(q_1) = \rho_{ij}^{(l)}\left(\frac{q_1}{kH}\right)^l \quad (7)$$

Where: $l = 0, 1, 2, \ldots L$

$c_{ij}^{(M)}$ and $\rho_{ij}^{(M)}$ indicate the physical properties at a specific point $M(q_1)$. L, $c_{ij}^{(l)}$ and $\rho_{ij}^{(l)}$ are coefficients to be determined in order to fit the polynomials in Eq.6 and Eq.7 to the original stiffness modulo and mass density of the elastic medium inside the cylinder [11].

From (i) and (ii), the boundary and continuity conditions are established:

$$\begin{cases} C_{ij}^{(M)}(q_1) = C_{ij}^{(l)}\left(\frac{q_1}{kH}\right)^l \pi(kR_0,\ kR_1) \\ \rho_{ij}^{(M)}(q_1) = \rho_{ij}^{(l)}\left(\frac{q_1}{kH}\right)^l \pi(kR_0,\ kR_1) \end{cases},\ l = 1, 2, \dots L \quad (8)$$

We need to extend the Legendre orthogonal polynomial method to solve the coupled wave equation of motion for the multilayered FGM cylinder. The idea is to expand the mechanical displacement components u, v and w into 3 orthogonal polynomial series in an orthonomal basis:

$$\mathrm{u}(q_1, q_2, q_3, t) = \frac{1}{\sqrt{2\pi}} e^{ikq_2} e^{i(\omega t - q_3)} \sum_{m=0}^{\infty} p_m^1 Q_m(q_1) \quad (9a)$$

$$\mathrm{v}(q_1, q_2, q_3, t) = \frac{1}{\sqrt{2\pi}} e^{ikq_2} e^{i(\omega t - q_3)} \sum_{m=0}^{\infty} p_m^2 Q_m(q_1) \quad (9b)$$

$$\mathrm{w}(q_1, q_2, q_3, t) = \frac{1}{\sqrt{2\pi}} e^{ikq_2} e^{i(\omega t - q_3)} \sum_{m=0}^{\infty} p_m^3 Q_m(q_1) \quad (9c)$$

where:

p_m^i $(i = 1, 2\ and\ 3)$ are the expansion coefficients;

ω is the pulsation.

The polynomials $Q_m(q_1)$ are given by:

$$Q_m(q_1) = \sqrt{\frac{2m+1}{kH}} P_m\left(\frac{2q_1 - (kR_1 + kR_0)}{kH}\right) \quad (10)$$

P_m is the Legendre polynomial of m order. $Q_m(q_1)$ is the complete orthonormal set in a range of $kR_0 \le q_1 \le kR_1$ and can represent any continuous function.

The stresses components expressed in Eq.3 and the mechanical displacements expressed in Eq.9 can be integrated into the equation of motion given by Eq.4. Taking into account the boundary conditions expressed using the rectangle window function's derivatives $\pi(kR_0, kR_1)$ resulting in two terms $\delta(q_1 = kR_0)$ and $\delta(q_1 = kR_1)$. Then, each member of Eq.4 is multiplied by $1/\sqrt{2\pi}\, Q_j^*(q_1) e^{-inq_2}$, with j in range of 0 to infinity. Integrating obtained equation over j from 0 to infinity, over q_1 from kR_0 to kR_1 and over q_2 from 0 to 2π. The following is the system of equations that have been deduced:

$$\begin{bmatrix} {}_j^m\boldsymbol{B}_{11}^l & {}_j^m\boldsymbol{B}_{12}^l & {}_j^m\boldsymbol{B}_{13}^l \\ {}_j^m\boldsymbol{B}_{21}^l & {}_j^m\boldsymbol{B}_{22}^l & {}_j^m\boldsymbol{B}_{23}^l \\ {}_j^m\boldsymbol{B}_{31}^l & {}_j^m\boldsymbol{B}_{32}^l & {}_j^m\boldsymbol{B}_{33}^l \end{bmatrix} \begin{Bmatrix} p_m^1 \\ p_m^2 \\ p_m^3 \end{Bmatrix} = -\eta^2 \begin{bmatrix} \boldsymbol{M}_{m,j}^l & 0 & 0 \\ 0 & \boldsymbol{M}_{m,j}^l & 0 \\ 0 & 0 & \boldsymbol{M}_{m,j}^l \end{bmatrix} \begin{Bmatrix} p_m^1 \\ p_m^2 \\ p_m^3 \end{Bmatrix} (10)$$

This equation yield an eigenvalue problem in which the eigenvalue η^2 allows the calculation of angular frequencies of diffrents modes. The eigenvectors $p_m^{(1,2,3)}$ gives associated field quantities.

III. NUMERICAL RESULTS

Based on the above formulation, a numerical program is developed using Matlab software to calculate normalized frequencies of different propagation modes and associated filed profiles.

The structure is composed of three layers with dissimilar material properties. Different physical parameters and material properties of Silicon nitride and Stainless steel are listed in Tables I and II. The gradient index along this paper is fixed at $s = 4$.

TABLE I. MATERIAL PRPERTIES OF STUDIED MATERIALS [11]

Properties	$E(GPa)$	v	$\rho(kg/m^3)$
Silicon nitride	322.4	0.24	2370
Stainless steel	207.82	0.317	8166

TABLE II. POWER SERIES EXPANSION COEFFICIENTS [19]

$c_{11}^{(0)}$	$c_{11}^{(1)}$	$c_{11}^{(2)}$	$c_{11}^{(3)}$
35.9189	3.0960	31.6281	-3.1436
$c_{12}^{(0)}$	$c_{12}^{(1)}$	$c_{12}^{(2)}$	$c_{12}^{(3)}$
12.5007	-0.7422	13.3911	0.5262
$c_{44}^{(0)}$	$c_{44}^{(1)}$	$c_{44}^{(2)}$	$c_{44}^{(3)}$
11.7091	1.9191	9.1185	-1.8349
$\rho^{(0)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
2.1266	6.7364	-2.1266	3.7996

We have calculated and represented the frequency parameter $\Omega = \omega H / c^{(\alpha)}$, with $c^{(\alpha)} = (c_{44}/\rho)^{1/2}$, of the multilayered FGM cylinder as a function of kH for different propagating modes. Figures 2, 3 and 4 gives, respectively, obtained results for Longitudinal L(0,m) and Torsional T(0,m) modes, Flexural modes F(1,m) and F(2,m).

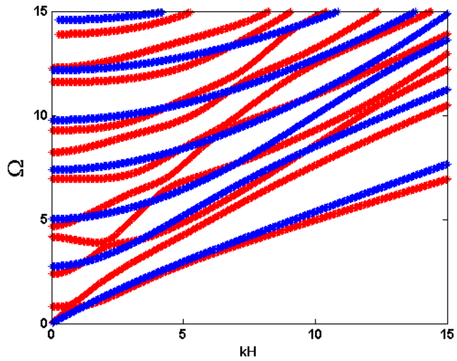

Figure 2. Normalized frequency parameter of a multilayered FGM cylinder as a function of kH, Longitudinal L(0,m) and Torsional T(0,m) modes.

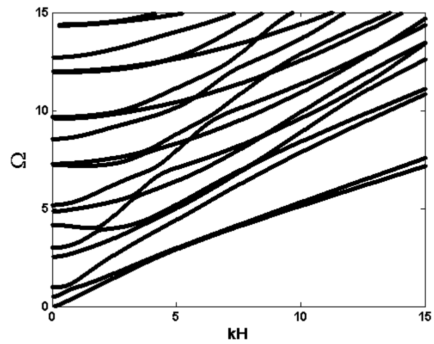

Figure 3. Normalized frequency parameter of a multilayered FGM cylinder as a function of kH, Flexural mode F(1,m).

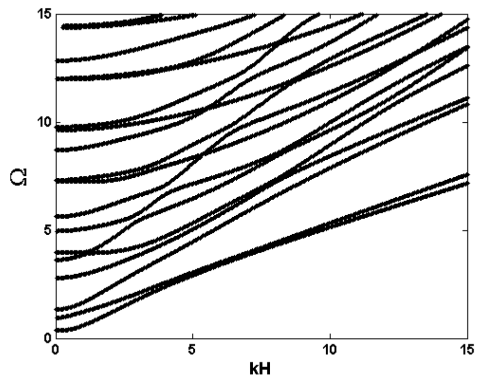

Figure 4. Normalized frequency parameter of a multilayered FGM cylinder as a function of kH, Flexural mode F(2,m).

After determining the normalized frequencies successfully so far, and to be complete and dependable, the method must retrieve associated dispersion curves and fields profiles with the full satisfaction of the imposed boundary and continuity conditions of the multilayered FGM cylinder. Figures 5, 6 and 7 gives the normal stress profiles for 3 first modes, respectively. As illustrated, the boundary conditions are satisfied, the normal stress σ_{rr} , σ_{rz} and $\sigma_{r\varphi}$ are both nil in the interfaces.

Figures 8, 9 and 10 depict the distribution along the thicknesses direction of the mechanical displacements components for the first Longitudinal modes $(u, w \neq 0, v = 0)$, respectively. As it clear from these figures, the mode shape is dominated by the circumferential component for the first and second longitudinal modes. In all longitudinal modes, the axial component remains equal zero.

Figures 11, 12 and 13 gives phase velocity dispersion curves along thicknesses direction of Longitudinal mode L(0,m), Flexural modes L(1,m) and L(2,m), respectively.

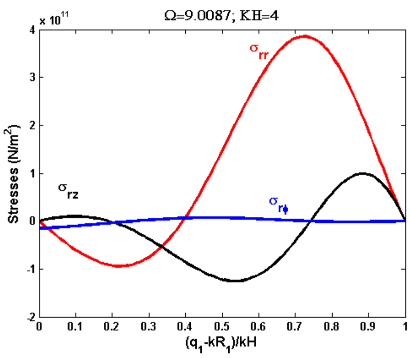

Figure 5. Normal stress profile of the multilayered FGM cylinder, contour mode 1.

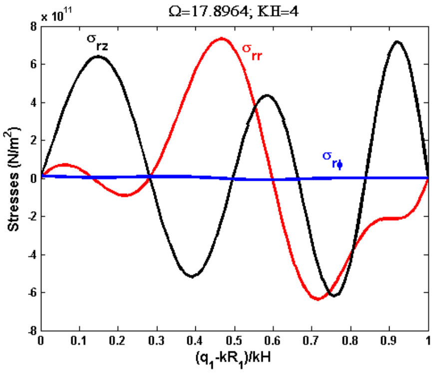

Figure 6. Normal stress profile of the multilayered FGM cylinder, contour mode 2

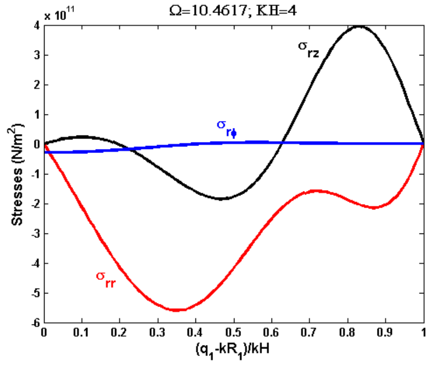

Figure 7. Normal stress profile of the multilayered FGM cylinder, contour mode 3

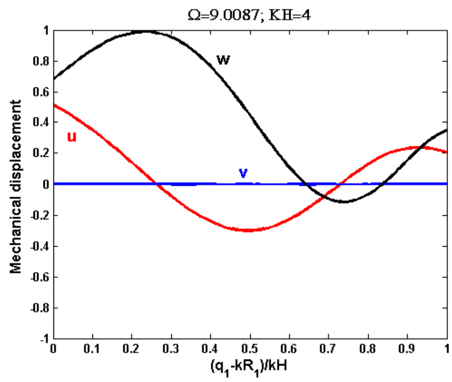

Figure 8. Mechanical diplacements along thicknesses direction, first Longitudinal mode.

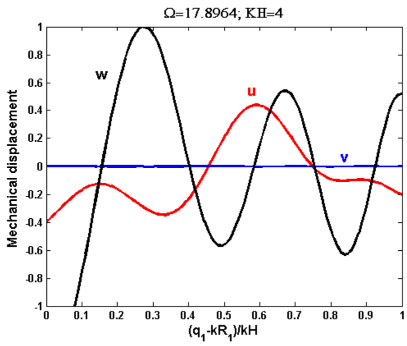

Figure 9. Mechanical diplacements along thicknesses direction, second Lonitudinal mode.

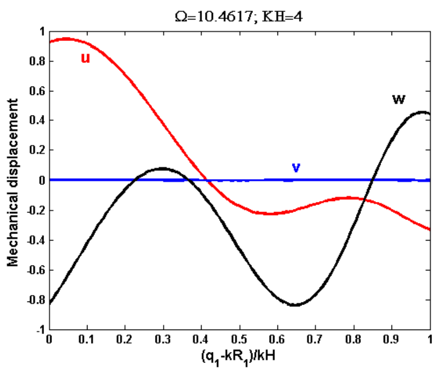

Figure 10. Mechanical diplacements along thicknesses direction, thirth Lonitudinal mode.

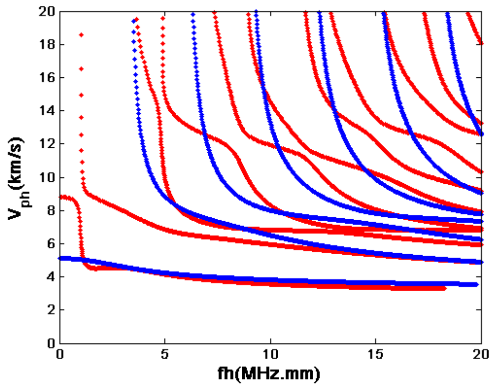

Figure 11. Dispersion profile of phase velocity for the multilayered FGM cylinder, Lonitudinal mode L(0,m)

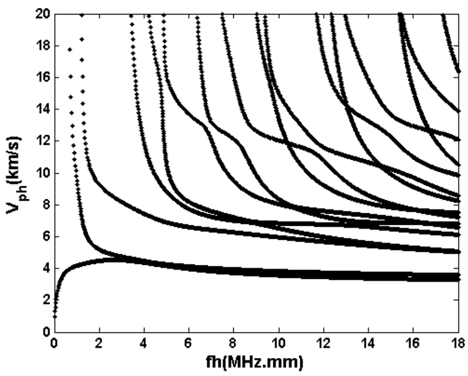

Figure 12. Dispersion profile of phase velocity for the multilayered FGM cylinder, Flexural mode F(1,m).

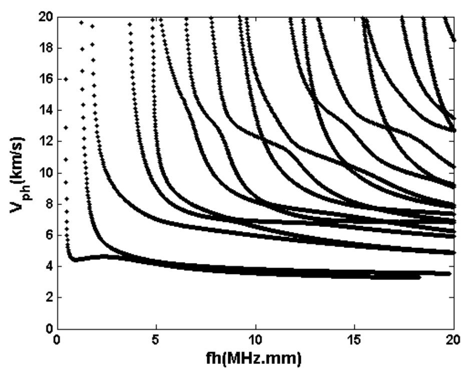

Figure 13. Dispersion profile of phase velocity for the multilayered FGM cylinder, Flexural mode F(2,m).

From these findings, it's clear that orthogonal polynomial method is an accurate and efficient tool of modeling guided wave propagation behavior of a multilayered inhomogeneous cylinder composed of three layers.

IV. Conclusion

As an efficient modeling tool, the Legendre orthogonal polynomial approach has been extended to study the guided wave propagation in a multilayered FGM cylinder composed of silicon nitride and stainless steel. The presented work has shown that, whatever the dissimilarities of physical properties inside the structure, the proposed method is quite accurate and able to perform the semi-variational determination of the frequencies of longitudinal, torsion and flexural modes and restitute associated dispersion curves and field profiles as well.

The formulation is simple to implement and ensure a direct incorporation of boundary and continuity conditions into the governing equation of motion leading to an eigenvalue problem.

Based on the promising findings reported in this work, we intend to extend the Legendre orthogonal polynomial approach to study further FGM structures including piezoelectric materials. Such future research project is under progress.

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