Wigner's Semicircle Law

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Universality in Probability Theory

Wigner's Law

Sketch of the proof

What comes next ?

Universality in Probability Theory

First theorems

Two fundamentals theorems :

- **Law of Large Numbers :** $(X_i)_{i\in\mathbb{N}^*}$ an infinite sequence of *i.i.d*, *Lebesgue integrable* random variables where
$\begin{cases} \mu := \mathbb{E}[X_1] \\ \overline{X_n} := \frac{1}{n}(X_1 + \dots X_n) \end{cases}$

$$\overline{X_n} \xrightarrow{\mathbb{P}} \mu$$

First theorems

Two fundamentals theorems :

- **Central Limit Theorem :** $(X_i)_{i\in\mathbb{N}^*}$ an infinite sequence of *i.i.d* random variables with *finite variance* where
$$\begin{cases} \mu := \mathbb{E}[X_1] \\ \sigma := \mathbb{E}[(X_1-\mu)^2] \\ \overline{X_n} := \frac{1}{n}(X_1 + \dots X_n) \end{cases}$$

$$\sqrt{n}(\overline{X_n} - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$$

Universality

In both of the previous theorems, the probability distribution of the $(X_i)_{i\in\mathbb{N}}$ does not affect the limit results. Therefore, these laws are described as **universal**, meaning the can be applied for every random sample of independant experiments.

Wigner's Law

History

First studied by Wishart in the 1920s, then by Wigner in the case of a physics problem : Wigner studied the interactions between atoms, which can be described using linear operators. Therefore, the behavior of the eigenvalues of those operators was crucial (ex: PCA), but almost impossible to compute. Wigner's idea was to consider random matrices that were similar to the ones describing atoms interactions, but whose eigenvalues were easier to study.

Notations

We consider an infinite family of *i.i.d*, random variables$(W_{i,j})_{i,j\in\mathbb{N}}$ that verifies the following properties :

$$\begin{cases} \forall i,j \;\; W_{i,j} = W_{j,i} \\ \mathbb{E}(W_{1,1}) = 0 \\ \mathbb{E}\left(|W_{1,1}|^2\right) = 1 \\ \forall k \geq 3,\, \mathbb{E}(|W_{1,1}|^k) < +\infty \end{cases}$$

Notations

Then, we consider $W_n := (W_{i,j})_{1 \leq i,j \leq n}$. Because W_n is symmetric, it has exactly n eigenvalues. Therefore, we denote : $\mathrm{SP}(\frac{W_n}{\sqrt{n}}) = \{\lambda_i, i \in \{1, \ldots, n\}\}$.

Theorem

Wigner's Law : For every "reasonable" function f :

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i)\right) = \int_{-2}^{2} f(x)\,\mathrm{d}\mu_{SC}^{(\sigma)}(x)$$

where $\mathrm{d}\mu_{SC}$ is the density of the semi-cercle of radius 2σ, centered in 0, given by :

$$\mathrm{d}\mu_{SC}^{(\sigma)}(x) = \frac{1}{2\pi\sigma^2}\mathbb{1}_{[-2;2]}(x)\sqrt{4\sigma^2 - x^2}\mathrm{d}x$$

Ex : f can be a continous and bouded function, or an indicator function.

Explanation

One way of seeing the link with a semi-circle is to apply the theorem with f and indicator function of the set $[-\infty; a]$, where $a \in [-2; 2]$:

$$\lim_{n \to \infty} \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{[-\infty; a]}(\lambda_i) \right) = \int_{-2}^{2} \mathbb{1}_{[-\infty; a]}(x) \, \mathrm{d}\mu_{SC}$$

Thus

$$\lim_{n \to \infty} \mathbb{E} \left(\frac{|\{i, \lambda_i \leq a\}|}{n} \right) = \int_{-\infty}^{a} \mathrm{d}\mu_{SC}$$

Therefore, the limit density of the $(\lambda_i)_{i \in \mathbb{N}}$ is given by the semi-circle density.

Explanation

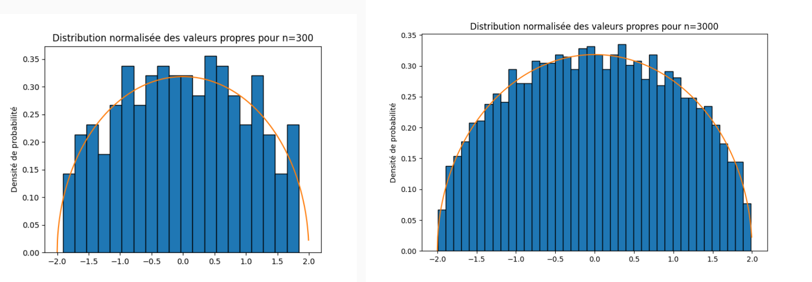


Figure 1: Exemple where $W_{1,1}$ follows a Rademacher law

Sketch of the proof

Idea

The main idea of the proof is to reduce the problem to the exponent functions : $x \mapsto x^k$. Then, using density theorems such as Weierstrass theorem, we could easily generalize to continous bounded function on intervals. Finally, limiting the maximum eigenvalue will give the general case.

Proof

Our goal is to show that : $\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^k\right) = \int_{-2}^{2} x^k \,\mathrm{d}\mu_{SC}$

Proof

We need to study the following quantity :

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^k\right) &= \mathbb{E}\left(\frac{1}{n}\mathrm{Tr}\left(\left(\frac{W_n}{\sqrt{n}}\right)^k\right)\right) \\
&= \sum_{i_1,\dots,i_k=1}^{n} n^{-\frac{k}{2}-1}\mathbb{E}[(W_n)_{i_1,i_2}\dots(W_n)_{i_k,i_1}] \\
&= \sum_{I\in\{1,\dots,n\}^k} n^{-\frac{k}{2}-1}P(I)
\end{aligned}$$

where if $I=(i_1,\dots,i_k)\in\{1,\dots,n\}^k$ then
$P(I) := \mathbb{E}[(W_n)_{i_1,i_2}\dots(W_n)_{i_k,i_1}]$.

Proof

The technique to use is the following : if there is too much different values in I, because $(W_{i,j})_{i,j \in \mathbb{N}}$ are independant, we will be able to extract one $\mathbb{E}[(W_n)_{i,j}]$, making $P(I)$ equal to 0 because of the assumptions. Next, the set of I that contains a 'small' amout of values has a negligible size, therefore it will not contribute to the final sum. Therefore, the only terms contributing to the final value are the ones between these two cases.

Proof

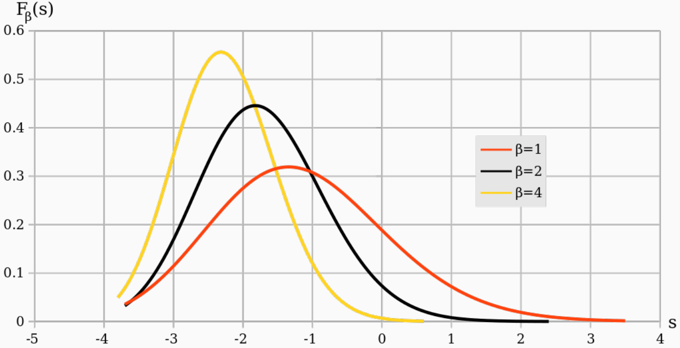
Lemma 1 : The important cases is reached for the l such that the number of distinct values in I is $\frac{k}{2}$.

Lemma 2 : $\begin{cases} \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^{2q}\right) = \mathrm{Cat}(q) \\ \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^{2q+1}\right) = 0 \end{cases}$

What comes next ?

More precision

As seen in the first slides, we can now wonder what is the error of our theorem by looking at the extreme eigenvalue. It's evolution is given by the Tracy-Widom law :



More precision

In the "bulk", the behavior of the eigenvalues is well-known, with an average space between two consecutive eigenvalues of $\frac{1}{n}$

Questions

Any questions ?