I Definitions

Let $W(x) := \left(-\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3} - \left(\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3}$

and define $Lz := -az - b\frac{\partial z}{\partial x}$, where $a(x) := 1 + \frac{W(x)}{x} + \frac{\partial W}{\partial x}(x)$ and $b(x) := \frac{3x}{2} + W(x)$.

II Computation in Sobolev spaces

Let $\langle f, g\rangle := \langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(x)g(x)\,dx$ be the usual inner product of $L^2(\mathbb{R})$.
Let $w, z \in H^k(\mathbb{R})$ for k large enough. For simplicity, we will denote $\frac{\partial z}{\partial x} := z'$.

II.1 Symetric part in L^2 space

In L^2, the symetric part is computed as follows :

$$\begin{aligned}\langle Lz, w\rangle_{L^2} &= \langle -az - bz', w\rangle = \langle z, -aw\rangle - \langle z, b'w + bw'\rangle \\ &= \langle z, (-a + b')w + bw'\rangle = \langle z, L^*w\rangle\end{aligned}$$

Thus, $\frac{1}{2}(L + L^*)z = \frac{1}{2}(-az - bz' - az + b'z + bz') = -az + \frac{b'}{2}z$ in L^2.

II.2 Quadratic form in H^1 space

In H^1, the quadratic form is computed as follows :

$$\begin{aligned}\langle Lz, z\rangle_{H^1} &= \langle -az - bz', z\rangle + \langle -a'z - az' - b'z' - bz'', z'\rangle \\ &= \langle -az, z\rangle + \langle -bz - a'z, z'\rangle + \langle -az' - b'z', z'\rangle + \langle -bz', z''\rangle \\ &= \langle -az, z\rangle + \langle \frac{1}{2}(b' + a'')z, z\rangle + \langle (-a - b')z', z'\rangle + \langle \frac{1}{2}b'z', z'\rangle \\ &= \langle (-a + \frac{b'}{2} + \frac{a''}{2})z, z\rangle + \langle (-a - \frac{b'}{2})z', z'\rangle\end{aligned}$$

Remarque. The operator $(Lz)'$ is not defined on H^1 as it involves second derivatives of z, but it is a classical fact that the quadratic form of an operator as a larger domain that the operator itself.

II.3 Quadratic form in H^2 space

In H^2, the quadratic form is computed as follows :

$$\begin{aligned}\langle (Lz)'', z''\rangle &= \langle -a''z - a'z' - a'z' - az'' - b''z' - b'z'' - b'z'' - bz^{(3)}, z''\rangle \\ &= \langle -a''z, z''\rangle + \langle (-2a' - b'')z', z''\rangle + \langle (-a - 2b')z'', z''\rangle + \langle -bz^{(3)}, z''\rangle \\ &= \langle a^{(3)}z + a''z', z'\rangle + \langle \frac{1}{2}(2a'' + b^{(3)})z', z'\rangle + \langle (-a - 2b')z'', z''\rangle + \langle \frac{1}{2}b'z'', z''\rangle \\ &= \langle -\frac{1}{2}a^{(4)}z, z\rangle + \langle 2a'' + \frac{1}{2}b^{(3)})z', z'\rangle + \langle (-a - \frac{3}{2}b')z'', z''\rangle\end{aligned}$$

Thus, we have in H^2 :

$$\langle Lz, z\rangle_{H^2} = \langle (-a + \frac{b'}{2} + \frac{a''}{2} - \frac{a^{(4)}}{2})z, z\rangle + \langle (-a - \frac{b'}{2} + 2a'' + \frac{b^{(3)}}{2})z', z'\rangle + \langle (-a - \frac{3}{2}b')z'', z''\rangle$$

II.4 Quadratic form in H^3 space

In H^3, the quadratic form is computed as follows :

$$\begin{aligned}
\langle (Lz)^{(3)}, z^{(3)} \rangle &= \langle -a'''z - 3a''z' - 3a'z'' - az^{(3)} - b'''z' - 3b''z'' - 3b'z^{(3)} - bz^{(4)}, z^{(3)} \rangle \\
&= \langle -a'''z, z^{(3)} \rangle + \langle (-3a'' - b''')z', z^{(3)} \rangle + \langle (-3a' - 3b'')z'', z^{(3)} \rangle + \langle (-a - 3b')z^{(3)}, z^{(3)} \rangle \\
&\quad + \langle -bz^{(4)}, z^{(3)} \rangle \\
&= \langle a^{(4)}z + a'''z', z'' \rangle + \langle (3a''' + b^{(4)})z' + (3a'' + b''')z'', z'' \rangle + \langle \tfrac{3}{2}(a'' + b''')z'', z'' \rangle \\
&\quad + \langle (-a - 3b')z^{(3)}, z^{(3)} \rangle + \langle \tfrac{1}{2}b'z^{(3)}, z^{(3)} \rangle \\
&= \langle -a^{(5)}z - a^{(4)}z', z' \rangle + \langle -\frac{1}{2}a^{(4)}z', z' \rangle + \langle \frac{1}{2}(-3a^{(4)} - b^{(5)})z', z' \rangle + \langle (3a'' + b''')z'', z'' \rangle \\
&\quad + \langle \tfrac{3}{2}(a'' + b''')z'', z'' \rangle + \langle (-a - 3b')z^{(3)}, z^{(3)} \rangle + \langle \tfrac{1}{2}b'z^{(3)}, z^{(3)} \rangle \\
&= \langle \frac{a^{(6)}}{2}z, z \rangle + \langle (-3a^{(4)} - \frac{1}{2}b^{(5)})z', z' \rangle + \langle (\tfrac{9}{2}a'' + \tfrac{5}{2}b^{(3)})z'', z'' \rangle + \langle (-a - \frac{5}{2}b')z^{(3)}, z^{(3)} \rangle
\end{aligned}$$

Thus, we have in H^3 :

$$\begin{aligned}
\langle Lz, z \rangle_{H^3} &= \langle (-a + \frac{b'}{2} + \frac{a''}{2} - \frac{a^{(4)}}{2} + \frac{a^{(6)}}{2})z, z \rangle + \langle (-a - \frac{b'}{2} + 2a'' + \frac{b^{(3)}}{2} - 3a^{(4)} - \frac{1}{2}b^{(5)}))z', z' \rangle \\
&\quad + \langle (-a - \frac{3}{2}b' + \tfrac{9}{2}a'' + \tfrac{5}{2}b^{(3)})z'', z'' \rangle + \langle (-a - \frac{5}{2}b')z^{(3)}, z^{(3)} \rangle
\end{aligned}$$

III Compact part of the quadratic form

We proved in the previous section that the quadratic form associated with L in H^3 is of the form :

$$\langle Lz, z \rangle_{H^3} = \langle \varphi_0 z, z \rangle + \langle \varphi_1 z', z' \rangle + \langle \varphi_2 z'', z'' \rangle + \langle \varphi_3 z^{(3)}, z^{(3)} \rangle$$

In the next section, we will show that φ_3 has a sign and is bounded. This leaves to study the lower order terms, and we will prove that there exists a compact operator M such that

$$\langle Mz, z \rangle_{H^3} = \langle \varphi_0 z, z \rangle + \langle \varphi_1 z', z' \rangle + \langle \varphi_2 z'', z'' \rangle$$

Combining those results yield the following energy estimate :

$$\langle Lz, z \rangle_{H^3} \leqslant -\delta \|z\|_{H^3} + \langle Mz, z \rangle_{H^3}$$

We will use the Fourier transform, with the following convention :

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

and we will denote

$$\mathcal{F}^{-1}(f)(x) := \int_{-\infty}^{\infty} f(\xi)\, e^{2\pi i x \xi}\, d\xi$$

the inverse Fourier transform.

III.1 Base case

We want to find M_0 such that

$$\langle M_0 z, z \rangle_{H^3} = \langle \varphi_0 z, z \rangle \tag{3.1}$$

The Parseval identity gives :

$$\int \hat{z}(\xi) \widehat{M_0 z}(\xi)(1 + \xi^2)^3 \mathrm{d}\xi = \int \hat{z}(\xi) \widehat{\varphi_0 z}(\xi) \mathrm{d}\xi$$

Thus, choosing M_0 such that $\widehat{M_0 z}(\xi) = \frac{1}{(1+\xi^2)^3}\widehat{\varphi_0 z}(\xi)$ would give the equality.
Defining $\theta(\xi) := \frac{1}{(1+\xi^2)^3}$, this condition is equivalent to :

$$\widehat{M_0 z} = \widehat{\mathcal{F}^{-1}(\theta)}\widehat{\varphi_0 z} = \widehat{\mathcal{F}^{-1}(\theta) * \varphi_0 z}$$

i.e. $M_0 z = \mathcal{F}^{-1}(\theta) * \varphi_0 z$ satisfies eq. (3.1).

III.2 First order case

We want to find M_1 such that

$$\langle M_1 z, z\rangle_{H^3} = \langle \varphi_1 z', z'\rangle \tag{3.2}$$

Integrating by parts and applying the Parseval identity, we have the equivalence

$$\begin{aligned}
\langle M_1 z, z\rangle_{H^3} &= -\langle \varphi_1' z' + \varphi_1 z'', z\rangle = -\langle \varphi_1' z, z'\rangle - \langle \varphi_1 z, z''\rangle \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_1 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= -\int (2\pi i\xi)\hat{z}(\xi)\widehat{\varphi_1' z}(\xi)\mathrm{d}\xi + \int (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_1 z}(\xi)\mathrm{d}\xi \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_1 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= \int \hat{z}\left[-(2\pi i\xi)\widehat{\varphi_1' z}(\xi) + (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_1 z}(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

Defining $\lambda_1(\xi) := -\frac{2\pi i\xi}{(1+\xi^2)^3}$ and $\lambda_2(\xi) := \frac{4\pi^2\xi^2}{(1+\xi^2)^3}$, we have that

$$M_1 z := \left(\mathcal{F}^{-1}(\lambda_1) * \varphi_1' z\right) + \left(\mathcal{F}^{-1}(\lambda_2) * \varphi_1 z\right)$$

satisfies eq. (3.2).

III.3 Second order case

We want to find M_2 such that

$$\langle M_2 z, z\rangle_{H^3} = \langle \varphi_2 z'', z''\rangle \tag{3.3}$$

Integrating by parts twice and applying the Parseval identity, we have the equivalence

$$\begin{aligned}
\langle M_2 z, z\rangle_{H^3} &= \langle \varphi_2'' z'' + 2\varphi_2' z^{(3)} + \varphi_2 z^{(4)}, z\rangle = \langle \varphi_2'' z, z''\rangle + \langle 2\varphi_2' z, z^{(3)}\rangle + \langle \varphi_2 z, z^{(4)}\rangle \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_2 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= -\int (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_2'' z}(\xi)\mathrm{d}\xi - \int (i16\pi^3\xi^3)\hat{z}(\xi)\widehat{\varphi_2' z}(\xi)\mathrm{d}\xi + \int (16\pi^4\xi^4)\hat{z}(\xi)\widehat{\varphi_2 z}(\xi)\mathrm{d}\xi \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_2 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= \int \hat{z}\left[-(4\pi^2\xi^2)\widehat{\varphi_2'' z}(\xi) - (i16\pi^3\xi^3)\hat{z}(\xi)\widehat{\varphi_2' z}(\xi) + (16\pi^4\xi^4)\widehat{\varphi_2 z}(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

Defining $\mu_1(\xi) := -\frac{4\pi^2\xi^2}{(1+\xi^2)^3}$, $\mu_2(\xi) := -\frac{i16\pi^3\xi^3}{(1+\xi^2)^3}$ and $\mu_3(\xi) := \frac{16\pi^4\xi^4}{(1+\xi^2)^3}$, we have that

$$M_2 z := \left(\mathcal{F}^{-1}(\mu_1) * \varphi_2'' z\right) + \left(\mathcal{F}^{-1}(\mu_2) * \varphi_2' z\right) + \left(\mathcal{F}^{-1}(\mu_3) * \varphi_2 z\right)$$

satisfies eq. (3.3).