I Introduction

TODO

II Analysis of shock and self-similar singularities in a fluid

In this section, I will present the concepts of shock formation and self-similar blow-up for a nonlinear partial differential equation, illustrated by the fundamental example of Burgers' equation.

III Energy inequality for the linearized operator

b

IV Bootstrap method

c

A Computation of the kernel

A.1 Definitions

Let $W(x) := \left(-\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3} - \left(\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3}$

and define $Lz := -az - b\frac{\partial z}{\partial x}$, where $a(x) := 1 + \frac{W(x)}{x} + \frac{\partial W}{\partial x}(x)$ and $b(x) := \frac{3x}{2} + W(x)$.

A.2 Computation in Sobolev spaces

Let $\langle f, g\rangle := \langle f, g\rangle_{L^2} = \displaystyle\int_{\mathbb{R}} f(x)g(x)\,dx$ be the usual inner product of $L^2(\mathbb{R})$.

Let $w, z \in H^k(\mathbb{R})$ for k large enough. For simplicity, we will denote $\frac{\partial z}{\partial x} := z'$.

A.2.1 Symetric part in L^2 space

In L^2, the symetric part is computed as follows :

$$\begin{aligned}\langle Lz, w\rangle_{L^2} &= \langle -az - bz', w\rangle = \langle z, -aw\rangle - \langle z, b'w + bw'\rangle \\ &= \langle z, (-a + b')w + bw'\rangle = \langle z, L^*w\rangle\end{aligned}$$

Thus, $\frac{1}{2}(L + L^*)z = \frac{1}{2}(-az - bz' - az + b'z + bz') = -az + \dfrac{b'}{2}z$ in L^2.

A.2.2 Quadratic form in H^1 space

In H^1, the quadratic form is computed as follows :

$$\begin{aligned}\langle Lz, z\rangle_{H^1} &= \langle -az - bz', z\rangle + \langle -a'z - az' - b'z' - bz'', z'\rangle \\ &= \langle -az, z\rangle + \langle -bz - a'z, z'\rangle + \langle -az' - b'z', z'\rangle + \langle -bz', z''\rangle \\ &= \langle -az, z\rangle + \langle \frac{1}{2}(b' + a'')z, z\rangle + \langle (-a - b')z', z'\rangle + \langle \frac{1}{2}b'z', z'\rangle \\ &= \langle (-a + \frac{b'}{2} + \frac{a''}{2})z, z\rangle + \langle (-a - \frac{b'}{2})z', z'\rangle\end{aligned}$$

Remark. The operator $(Lz)'$ is not defined on H^1 as it involves second derivatives of z, but it is a classical fact that the quadratic form of an operator as a larger domain than the operator itself.

A.2.3 Quadratic form in H^2 space

In H^2, the quadratic form is computed as follows :

$$\begin{aligned}\langle (Lz)'', z''\rangle &= \langle -a''z - a'z' - a'z' - az'' - b''z' - b'z'' - b'z'' - bz^{(3)}, z''\rangle \\ &= \langle -a''z, z''\rangle + \langle (-2a' - b'')z', z''\rangle + \langle (-a - 2b')z'', z''\rangle + \langle -bz^{(3)}, z''\rangle \\ &= \langle a^{(3)}z + a''z', z'\rangle + \langle \frac{1}{2}(2a'' + b^{(3)})z', z'\rangle + \langle (-a - 2b')z'', z''\rangle + \langle \frac{1}{2}b'z'', z''\rangle \\ &= \langle -\frac{1}{2}a^{(4)}z, z\rangle + \langle 2a'' + \frac{1}{2}b^{(3)})z', z'\rangle + \langle (-a - \frac{3}{2}b')z'', z''\rangle\end{aligned}$$

Thus, we have in H^2 :

$$\langle Lz, z\rangle_{H^2} = \langle (-a + \frac{b'}{2} + \frac{a''}{2} - \frac{a^{(4)}}{2})z, z\rangle + \langle (-a - \frac{b'}{2} + 2a'' + \frac{b^{(3)}}{2})z', z'\rangle + \langle (-a - \frac{3}{2}b')z'', z''\rangle$$

A.2.4 Quadratic form in H^3 space

In H^3, the quadratic form is computed as follows :

$$\begin{aligned}
\langle (Lz)^{(3)}, z^{(3)} \rangle &= \langle -a'''z - 3a''z' - 3a'z'' - az^{(3)} - b'''z' - 3b''z'' - 3b'z^{(3)} - bz^{(4)}, z^{(3)} \rangle \\
&= \langle -a'''z, z^{(3)} \rangle + \langle (-3a'' - b''')z', z^{(3)} \rangle + \langle (-3a' - 3b'')z'', z^{(3)} \rangle + \langle (-a - 3b')z^{(3)}, z^{(3)} \rangle \\
&\quad + \langle -bz^{(4)}, z^{(3)} \rangle \\
&= \langle a^{(4)}z + a'''z', z'' \rangle + \langle (3a''' + b^{(4)})z' + (3a'' + b''')z'', z'' \rangle + \langle \tfrac{3}{2}(a'' + b''')z'', z'' \rangle \\
&\quad + \langle (-a - 3b')z^{(3)}, z^{(3)} \rangle + \langle \tfrac{1}{2}b'z^{(3)}, z^{(3)} \rangle \\
&= \langle -a^{(5)}z - a^{(4)}z', z' \rangle + \langle -\frac{1}{2}a^{(4)}z', z' \rangle + \langle \frac{1}{2}(-3a^{(4)} - b^{(5)})z', z' \rangle + \langle (3a'' + b''')z'', z'' \rangle \\
&\quad + \langle \tfrac{3}{2}(a'' + b''')z'', z'' \rangle + \langle (-a - 3b')z^{(3)}, z^{(3)} \rangle + \langle \tfrac{1}{2}b'z^{(3)}, z^{(3)} \rangle \\
&= \langle \frac{a^{(6)}}{2}z, z \rangle + \langle (-3a^{(4)} - \frac{1}{2}b^{(5)})z', z' \rangle + \langle (\tfrac{9}{2}a'' + \tfrac{5}{2}b^{(3)})z'', z'' \rangle + \langle (-a - \frac{5}{2}b')z^{(3)}, z^{(3)} \rangle
\end{aligned}$$

Thus, we have in H^3 :

$$\begin{aligned}
\langle Lz, z \rangle_{H^3} &= \langle (-a + \frac{b'}{2} + \frac{a''}{2} - \frac{a^{(4)}}{2} + \frac{a^{(6)}}{2})z, z \rangle + \langle (-a - \frac{b'}{2} + 2a'' + \frac{b^{(3)}}{2} - 3a^{(4)} - \frac{1}{2}b^{(5)}))z', z' \rangle \\
&\quad + \langle (-a - \frac{3}{2}b' + \tfrac{9}{2}a'' + \tfrac{5}{2}b^{(3)})z'', z'' \rangle + \langle (-a - \frac{5}{2}b')z^{(3)}, z^{(3)} \rangle
\end{aligned}$$

A.3 Compact part of the quadratic form

We proved in the previous section that the quadratic form associated with L in H^3 is of the form :

$$\langle Lz, z \rangle_{H^3} = \langle \varphi_0 z, z \rangle + \langle \varphi_1 z', z' \rangle + \langle \varphi_2 z'', z'' \rangle + \langle \varphi_3 z^{(3)}, z^{(3)} \rangle$$

In the next section, we will show that φ_3 has a sign and is bounded. This leaves to study the lower order terms, and we will prove that there exists a compact operator M such that

$$\langle Mz, z \rangle_{H^3} = \langle \varphi_0 z, z \rangle + \langle \varphi_1 z', z' \rangle + \langle \varphi_2 z'', z'' \rangle$$

Combining those results yield the following energy estimate :

$$\langle Lz, z \rangle_{H^3} \leq -\delta \|z\|^2_{H^3} + \langle Mz, z \rangle_{H^3} \tag{1.1}$$

We will use the Fourier transform, with the following convention :

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

and we will denote

$$\mathcal{F}^{-1}(f)(x) := \int_{-\infty}^{\infty} f(\xi)\, e^{2\pi i x \xi}\, d\xi$$

the inverse Fourier transform.

A.3.1 Base case

We want to find M_0 such that

$$\langle M_0 z, z \rangle_{H^3} = \langle \varphi_0 z, z \rangle \tag{1.2}$$

The Parseval identity gives :

$$\int \hat{z}(\xi) \widehat{M_0 z}(\xi)(1 + \xi^2)^3 \mathrm{d}\xi = \int \hat{z}(\xi) \widehat{\varphi_0 z}(\xi) \mathrm{d}\xi$$

Thus, choosing M_0 such that $\widehat{M_0 z}(\xi) = \frac{1}{(1+\xi^2)^3}\widehat{\varphi_0 z}(\xi)$ would give the equality.

Defining $\lambda_0(\xi) := \frac{1}{(1+\xi^2)^3}$, this condition is equivalent to :

$$\widehat{M_0 z} = \widehat{\mathcal{F}^{-1}(\lambda_0)}\widehat{\varphi_0 z} = \widehat{\mathcal{F}^{-1}(\lambda_0) * \varphi_0 z}$$

i.e. $M_0 z = \mathcal{F}^{-1}(\lambda_0) * \varphi_0 z$ satisfies eq. (7.2).

A.3.2 First order case

We want to find M_1 such that

$$\langle M_1 z, z\rangle_{H^3} = \langle \varphi_1 z', z'\rangle \tag{1.3}$$

Integrating by parts and applying the Parseval identity, we have the equivalence

$$\begin{aligned}
\langle M_1 z, z\rangle_{H^3} &= -\langle \varphi_1' z' + \varphi_1 z'', z\rangle = -\langle \varphi_1' z, z'\rangle - \langle \varphi_1 z, z''\rangle \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_1 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= -\int (2\pi i\xi)\hat{z}(\xi)\widehat{\varphi_1' z}(\xi)\mathrm{d}\xi + \int (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_1 z}(\xi)\mathrm{d}\xi \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_1 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= \int \hat{z}\left[-(2\pi i\xi)\widehat{\varphi_1' z}(\xi) + (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_1 z}(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

Defining $\lambda_1(\xi) := -\frac{2\pi i\xi}{(1+\xi^2)^3}$ and $\lambda_2(\xi) := \frac{4\pi^2\xi^2}{(1+\xi^2)^3}$, we have that

$$M_1 z := \left(\mathcal{F}^{-1}(\lambda_1) * \varphi_1' z\right) + \left(\mathcal{F}^{-1}(\lambda_2) * \varphi_1 z\right)$$

satisfies eq. (7.3).

A.3.3 Second order case

We want to find M_2 such that

$$\langle M_2 z, z\rangle_{H^3} = \langle \varphi_2 z'', z''\rangle \tag{1.4}$$

Integrating by parts twice and applying the Parseval identity, we have the equivalence

$$\begin{aligned}
\langle M_2 z, z\rangle_{H^3} &= \langle \varphi_2'' z'' + 2\varphi_2' z^{(3)} + \varphi_2 z^{(4)}, z\rangle = \langle \varphi_2'' z, z''\rangle + \langle 2\varphi_2' z, z^{(3)}\rangle + \langle \varphi_2 z, z^{(4)}\rangle \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_2 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= -\int (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_2'' z}(\xi)\mathrm{d}\xi - \int (i16\pi^3\xi^3)\hat{z}(\xi)\widehat{\varphi_2' z}(\xi)\mathrm{d}\xi + \int (16\pi^4\xi^4)\hat{z}(\xi)\widehat{\varphi_2 z}(\xi)\mathrm{d}\xi \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_2 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= \int \hat{z}\left[-(4\pi^2\xi^2)\widehat{\varphi_2'' z}(\xi) - (i16\pi^3\xi^3)\hat{z}(\xi)\widehat{\varphi_2' z}(\xi) + (16\pi^4\xi^4)\widehat{\varphi_2 z}(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

Defining $\lambda_3(\xi) := -\frac{i16\pi^3\xi^3}{(1+\xi^2)^3}$ and $\lambda_4(\xi) := \frac{16\pi^4\xi^4}{(1+\xi^2)^3}$, we have that

$$M_2 z := \left(-\mathcal{F}^{-1}(\lambda_2) * \varphi_2'' z\right) + \left(\mathcal{F}^{-1}(\lambda_3) * \varphi_2' z\right) + \left(\mathcal{F}^{-1}(\lambda_4) * \varphi_2 z\right)$$

satisfies eq. (7.4).

A.4 Quality of the approximation

The compact operators that we are studying are of the form:

$$K(f)(x) = \int_{\mathbb{R}} K(x,y)f(y)\mathrm{d}y$$

Assuming that we can bound the growth of the kernel at infinity, a natural choice of a finite rank approximation on a bounded domain $[-A, A]$ would be to use a Riemann sum :

$$K_n(f)(x) := \sum_{i=0}^{n} \delta K(x, y_i) f(y_i)$$

where $\delta := \frac{2A}{n}$ is the integration step and $\begin{cases} y_0 = -A \\ y_{i+1} = y_i + \delta \end{cases}$ are the sample points.

Now, we want to get a precise bound on the quality of this approximation in order to compute a relevant upper bound in the energy estimate 7.1.
Assuming that $f \in H^3(\mathbb{R})$ is also in $L^1(\mathbb{R})$, we can use the inversion formula and work in Fourier space :

$$K_n(f)(x) = \sum_{i=1}^{n} \delta K(x, y_i) f(y_i) = \int_{\mathbb{R}} \hat{f}(\xi) \sum_{i=0}^{n} \delta K(x, y_i) e^{2i\pi\xi y_i} \mathrm{d}\xi \tag{1.5}$$

On the other hand, the operator can be written as :

$$\begin{aligned} K(f)(x) &= \int_{\mathbb{R}} K(x,y) f(y)\mathrm{d}y = \int_{\mathbb{R}} \hat{K}(x,\xi)\hat{f}(\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \hat{f}(\xi) \int_{\mathbb{R}} K(x,y) e^{-2i\pi\xi y} \mathrm{d}y\mathrm{d}\xi \end{aligned}$$

There is a sign problem in the phase of the exponential, but if we can assume that the function f is even, then we can also reverse the sign in the approximate kernel formula, and now the difference between the kernel and its approximation is easily bounded using the mean value theorem on $[-A, A]$, and by giving an explicit bound on the decay of the kernel away from zero.

Now, taking the difference of the two expressions, we have :

$$\begin{aligned} |K(f)(x) - K_n(f)(x)| &\leq \left| \int_{\mathbb{R}} \hat{f}(\xi) \left(\int_{-A}^{A} K(x,y) e^{-2i\pi\xi y} \mathrm{d}y - \sum_{i=0}^{n} \delta K(x,y_i) e^{2i\pi\xi y_i} \right) \mathrm{d}\xi \right| \\ &+ \left| \int_{\mathbb{R}} \hat{f}(\xi) \int_{\mathbb{R}\setminus[-A,A]} K(x,y) e^{2i\pi\xi y} \mathrm{d}y\mathrm{d}\xi \right| \\ &\leq (I) + (II) \end{aligned}$$

Let's first bound the difference on the compact domain $[-A, A]$ by applying the mean value theorem:

$$\begin{aligned} (I) &\leq \left| \int_{\mathbb{R}} \hat{f}(\xi) \sum_{i=0}^{n} \left(\int_{y_i}^{y_{i+1}} K(x,y) e^{2i\pi\xi y} - K(x,y_i) e^{2i\pi\xi y_i} \mathrm{d}y \right) \mathrm{d}\xi \right| \\ &\leq \int_{\mathbb{R}} \left| \hat{f}(\xi) \right| \sum_{i=0}^{n} \left(\int_{y_i}^{y_{i+1}} \delta \sup_{y \in [-A,A]} \left| \partial_y K(x,y) e^{2i\pi\xi y} + 2i\pi\xi K(x,y) e^{2i\pi\xi y} \right| \mathrm{d}y \right) \mathrm{d}\xi \\ &\leq n\delta^2 \int_{\mathbb{R}} \left| \hat{f}(\xi) \right| \sup_{y \in [-A,A]} \Big(|\partial_y K(x,y)| + |2\pi\xi K(x,y)| \Big) \mathrm{d}\xi \\ &\leq n\delta^2 \Big(\|\hat{f}\|_{L^1} \sup_{y \in [-A,A]} |\partial_y K(x,y)| + \|\xi \hat{f}(\xi)\|_{L^1} \sup_{y \in [-A,A]} 2\pi |K(x,y)| \Big) \\ &\leq C \frac{4A^2}{n} \Big(\sup_{y \in [-A,A]} |\partial_y K(x,y)| + \sup_{y \in [-A,A]} |2\pi K(x,y)| \Big) \|f\|_{H^2} \end{aligned}$$

Where $C := \max\Big(\int_{\mathbb{R}} \frac{1}{(1+\xi^2)}\mathrm{d}\xi, \int_{\mathbb{R}} \frac{\xi^2}{(1+\xi^2)^2}\mathrm{d}\xi\Big)$ and we used the fact that, because $f \in H^3(\mathbb{R})$, we have:

$$\|\hat{f}(\xi)\|_{L^1} = \|\frac{1}{(1+\xi^2)^{1/2}}(1+\xi^2)^{1/2}\hat{f}(\xi)\|_{L^1} \leq \|\frac{1}{(1+\xi^2)^{1/2}}\|_{L^2}\|(1+\xi^2)^{1/2}\hat{f}(\xi)\|_{L^2} = \|\frac{1}{(1+\xi^2)^{1/2}}\|_{L^2}\|f\|_{H^1}$$

and

$$\|\xi\hat{f}(\xi)\|_{L^1} = \|\frac{\xi}{(1+\xi^2)}(1+\xi^2)\hat{f}(\xi)\|_{L^1} \leq \|\frac{\xi}{(1+\xi^2)}\|_{L^2}\|(1+\xi^2)\hat{f}(\xi)\|_{L^2} = \|\frac{\xi}{(1+\xi^2)}\|_{L^2}\|f\|_{H^2}$$

Then for the second part, by Fubini's theorem:

$$\begin{aligned}(II) &= \left|\int_{\mathbb{R}\setminus[-A,A]} K(x,y)\int_{\mathbb{R}} \hat{f}(\xi)e^{2i\pi\xi y}\mathrm{d}\xi\mathrm{d}y\right| = \left|\int_{\mathbb{R}\setminus[-A,A]} K(x,y)f(y)\mathrm{d}y\right| \\ &= \left|\int_{\mathbb{R}\setminus[-A,A]} \lambda(x-y)\varphi(y)f(y)\mathrm{d}y\right|\end{aligned}$$

We can compute that $\lambda(x) = \frac{\pi}{4}e^{-2\pi|x|}\left(2\pi^2x^2+3\pi|x|+\frac{3}{2}\right)$, and thus defining $P(x) := 2\pi^2x^2+3\pi|x|+\frac{3}{2}$ and recalling that $|x+y| \leq |x|+|y|$ and $(x+y)^2 \leq 2x^2+2y^2$, we can bound the lambda term:

$$P(x+y) \leq 2\pi^2(2x^2+2y^2)+3\pi(|x|+|y|)+\tfrac{3}{2} = 2P(x)+2P(y).$$

Hence

$$\begin{aligned}\lambda(x+y) = \tfrac{\pi}{4}e^{-2\pi|x+y|}P(x+y) &\leq \tfrac{\pi}{4}e^{-2\pi|x+y|}(2P(x)+2P(y)) \\ &\leq 2e^{-2\pi|y|}\lambda(x)+2e^{-2\pi|x|}\lambda(y)\end{aligned}$$

Assuming we can show that φ is bounded, we now have :

$$\begin{aligned}(II) &\leq \|\varphi\|_{L^\infty}\int_{\mathbb{R}\setminus[-A,A]}\Big(2e^{-2\pi|y|}\lambda(x)+2e^{-2\pi|x|}\lambda(y)\Big)|f(y)|\mathrm{d}y \\ &\leq \|\varphi\|_{L^\infty}\left(\lambda(x)\int_{\mathbb{R}\setminus[-A,A]}2e^{-2\pi|y|}|f(y)|\mathrm{d}y+2e^{-2\pi|x|}\int_{\mathbb{R}\setminus[-A,A]}\lambda(y)|f(y)|\mathrm{d}y\right) \\ &\leq \|\varphi\|_{L^\infty}\left(\lambda(x)\left(\int_{\mathbb{R}\setminus[-A,A]}2e^{-4\pi|y|}\mathrm{d}y\right)^{\frac{1}{2}}\|f\|_{L^2}+2e^{-2\pi|x|}\int_{\mathbb{R}\setminus[-A,A]}\lambda(y)^2\mathrm{d}y\|f\|_{L^2}\right)\end{aligned}$$