# I Computation of the kernel

## I.1 Definitions

Let $W(x) := \left(-\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3} - \left(\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3}$, and define $Lz := -az - b\frac{\partial z}{\partial x}$, where $a(x) := 1 + \frac{W(x)}{x} + \frac{\partial W}{\partial x}(x)$ and $b(x) := \frac{3x}{2} + W(x)$.

## I.2 Computation in Sobolev spaces

Let $\langle f, g\rangle := \langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(x)g(x)\,dx$ be the usual inner product of $L^2(\mathbb{R})$.

Let $w, z \in H^k(\mathbb{R})$ for $k$ large enough. For simplicity, we will denote $\frac{\partial z}{\partial x} := z'$.

### I.2.1 Symetric part in $L^2$ space

Let $z$ and $w$ be smooth, we can compute the symetric part of $L$ with respect to the $L^2$ inner product as follows :

$$\begin{aligned}\langle Lz, w\rangle_{L^2} &= \langle -az - bz', w\rangle = \langle z, -aw\rangle + \langle z, b'w + bw'\rangle \\ &= \langle z, (-a + b')w + bw'\rangle = \langle z, L^* w\rangle\end{aligned}$$

Thus, even if $L$ is only defined on $H^1$, its symetric part can be extended to $L^2$:

$$L_\sigma z := \frac{1}{2}(L + L^*)z = \frac{1}{2}(-az - bz' - az + b'z + bz') = -az + \frac{b'}{2}z.$$

### I.2.2 Quadratic form in $H^1$ space

In $H^1$, the quadratic form associated to $L_\sigma$ is computed using integration by parts:

$$\begin{aligned}\langle L_\sigma z, z\rangle_{H^1} &= \langle -az + \frac{b'}{2}z, z\rangle + \langle -a'z - az' + \frac{b''}{2}z + \frac{b'}{2}z', z'\rangle \\ &= \langle (-a + \frac{b'}{2})z, z\rangle + \langle -a'z + \frac{b''}{2}z, z'\rangle + \langle (-a + \frac{b'}{2})z', z'\rangle \\ &= \langle (-a + \frac{b'}{2})z, z\rangle + \langle \frac{2a'' - b^{(3)}}{4}z, z\rangle + \langle (-a + \frac{b'}{2})z', z'\rangle \\ &= \frac{1}{4}\langle (-4a + 2b' + 2a'' - b^{(3)})z, z\rangle + \langle (-a + \frac{b'}{2})z', z'\rangle\end{aligned}$$

### I.2.3 Quadratic form in $H^2$ space

In $H^2$, the quadratic form associated to $L_\sigma$ is computed as follows:

$$\langle (Lz)'', z''\rangle =$$

Thus, we have in $H^2$ :

$$\langle Lz, z\rangle_{H^2} =$$

### I.2.4 Quadratic form in $H^3$ space

## I.3 Compact part of the quadratic form

We proved in the previous section that the quadratic form associated with $L$ in $H^3$ is of the form :

$$\langle Lz, z\rangle_{H^3} = \langle \varphi_0 z, z\rangle + \langle \varphi_1 z', z'\rangle + \langle \varphi_2 z'', z''\rangle + \langle \varphi_3 z^{(3)}, z^{(3)}\rangle$$

In the next section, we will show that $\varphi_3$ has a sign and is bounded. This leaves to study the lower order terms, and we will prove that there exists a compact operator $M$ such that

$$\langle Mz, z\rangle_{H^3} = \langle \varphi_0 z, z\rangle + \langle \varphi_1 z', z'\rangle + \langle \varphi_2 z'', z''\rangle$$

Combining those results yield the following energy estimate :

$$\langle Lz, z\rangle_{H^3} \leq -\delta \|z\|^2_{H^3} + \langle Mz, z\rangle_{H^3} \tag{1.1}$$

We will use the Fourier transform, with the following convention :

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$$

and we will denote

$$\mathcal{F}^{-1}(f)(x) := \int_{-\infty}^{\infty} f(\xi)\, e^{2\pi i x\xi}\, d\xi$$

the inverse Fourier transform.

### I.3.1 Base case

We want to find $M_0$ such that

$$\langle M_0 z, z\rangle_{H^3} = \langle \varphi_0 z, z\rangle \tag{1.2}$$

The Parseval identity gives :

$$\int \hat{z}(\xi)\widehat{M_0 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi = \int \hat{z}(\xi)\widehat{\varphi_0 z}(\xi)\mathrm{d}\xi$$

Thus, choosing $M_0$ such that $\widehat{M_0 z}(\xi) = \frac{1}{(1+\xi^2)^3}\widehat{\varphi_0 z}(\xi)$ would give the equality.

Defining $\lambda_0(\xi) := \frac{1}{(1+\xi^2)^3}$, this condition is equivalent to :

$$\widehat{M_0 z} = \widehat{\mathcal{F}^{-1}(\lambda_0)}\widehat{\varphi_0 z} = \widehat{\mathcal{F}^{-1}(\lambda_0) * \varphi_0 z}$$

i.e. $M_0 z = \mathcal{F}^{-1}(\lambda_0) * \varphi_0 z$ satisfies eq. (1.2).

### I.3.2 First order case

We want to find $M_1$ such that

$$\langle M_1 z, z\rangle_{H^3} = \langle \varphi_1 z', z'\rangle \tag{1.3}$$

Integrating by parts and applying the Parseval identity, we have the equivalence

$$\begin{aligned}
\langle M_1 z, z\rangle_{H^3} &= -\langle \varphi_1' z' + \varphi_1 z'', z\rangle = -\langle \varphi_1' z, z'\rangle - \langle \varphi_1 z, z''\rangle \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_1 z}(\xi)(1+\xi^2)^3\mathrm{d}\xi &= -\int (2\pi i\xi)\hat{z}(\xi)\widehat{\varphi_1' z}(\xi)\mathrm{d}\xi + \int (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_1 z}(\xi)\mathrm{d}\xi \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_1 z}(\xi)(1+\xi^2)^3\mathrm{d}\xi &= \int \hat{z}\left[-(2\pi i\xi)\widehat{\varphi_1' z}(\xi) + (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_1 z}(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

Defining $\lambda_1(\xi) := -\frac{2\pi i\xi}{(1+\xi^2)^3}$ and $\lambda_2(\xi) := \frac{4\pi^2\xi^2}{(1+\xi^2)^3}$, we have that

$$M_1 z := \left(\mathcal{F}^{-1}(\lambda_1) * \varphi_1' z\right) + \left(\mathcal{F}^{-1}(\lambda_2) * \varphi_1 z\right)$$

satisfies eq. (1.3).

### I.3.3 Second order case

We want to find $M_2$ such that

$$\langle M_2 z, z\rangle_{H^3} = \langle \varphi_2 z'', z''\rangle \tag{1.4}$$

Integrating by parts twice and applying the Parseval identity, we have the equivalence

$$\begin{aligned}
\langle M_2 z, z\rangle_{H^3} &= \langle \varphi_2'' z'' + 2\varphi_2' z^{(3)} + \varphi_2 z^{(4)}, z\rangle = \langle \varphi_2'' z, z''\rangle + \langle 2\varphi_2' z, z^{(3)}\rangle + \langle \varphi_2 z, z^{(4)}\rangle \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_2 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= -\int (4\pi^2\xi^2)\hat{z}(\xi)\widehat{\varphi_2'' z}(\xi)\mathrm{d}\xi - \int (i16\pi^3\xi^3)\hat{z}(\xi)\widehat{\varphi_2' z}(\xi)\mathrm{d}\xi + \int (16\pi^4\xi^4)\hat{z}(\xi)\widehat{\varphi_2 z}(\xi)\mathrm{d}\xi \\
\Leftrightarrow \int \hat{z}(\xi)\widehat{M_2 z}(\xi)(1+\xi^2)^3 \mathrm{d}\xi &= \int \hat{z}\left[-(4\pi^2\xi^2)\widehat{\varphi_2'' z}(\xi) - (i16\pi^3\xi^3)\hat{z}(\xi)\widehat{\varphi_2' z}(\xi) + (16\pi^4\xi^4)\widehat{\varphi_2 z}(\xi)\right]\mathrm{d}\xi
\end{aligned}$$

Defining $\lambda_3(\xi) := -\frac{i16\pi^3\xi^3}{(1+\xi^2)^3}$ and $\lambda_4(\xi) := \frac{16\pi^4\xi^4}{(1+\xi^2)^3}$ , we have that

$$M_2 z := \left(-\mathcal{F}^{-1}(\lambda_2) * \varphi_2'' z\right) + \left(\mathcal{F}^{-1}(\lambda_3) * \varphi_2' z\right) + \left(\mathcal{F}^{-1}(\lambda_4) * \varphi_2 z\right)$$

satisfies eq. (1.4).

### I.4 Quality of the approximation

The compact operators that we are studying are of the form:

$$K(f)(x) = \int_{\mathbb{R}} K(x,y)f(y)\mathrm{d}y$$

Assuming that we can bound the growth of the kernel at infinity, a natural choice of a finite rank approximation on a bounded domain $[-A, A]$ would be to use a Riemann sum :

$$K_n(f)(x) := \sum_{i=0}^{n} \delta K(x, y_i) f(y_i)$$

where $\delta := \frac{2A}{n}$ is the integration step and $\begin{cases} y_0 = -A \\ y_{i+1} = y_i + \delta \end{cases}$ are the sample points.

Now, we want to get a precise bound on the quality of this approximation in order to compute a relevant upper bound in the energy estimate 1.1.
We want to use the following result to get a precise bound on the convergence in the operator norm.

LEMMA I-1 (Schur test). — *Let $K : \mathbb{R}^2 \mapsto \mathbb{R}$ be a square integrable kernel, and $T$ be the operator defined by*

$$T : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \qquad \big(Tf\big)(x) := \int_{\mathbb{R}} K(x,y)f(y)\mathrm{d}y.$$

*Then*

$$\|T\|_{L^2 \mapsto L^2} \leq \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |K(x,y)|\mathrm{d}y \times \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |K(x,y)|\mathrm{d}x.$$

Let's now rewrite the operators to apply this lemma. Recall that the Fourier multiplier $(1-\Delta)^{\frac{3}{2}}$ defines an isometric isomorphism from $H^3(\mathbb{R})$ to $L^2(\mathbb{R})$, and also that the Dirac function that evaluates a function at a given point $x$ has a representation in $H^3(\mathbb{R})$, that we denote $\eta_x$. Now, for a function $f$ in $H^3(\mathbb{R})$, defining $g := (1-\Delta)^{\frac{3}{2}} f$, we have:

$$K(f)(x) = \int_{\mathbb{R}} K(x,y)f(y)\mathrm{d}y = \int_{\mathbb{R}} (1-\Delta_y)^{\frac{-3}{2}} K(x,y)g(y)\mathrm{d}y$$

and

$$\begin{aligned} K_n(f)(x) &= \sum_{i=0}^{n} \delta K(x,y_i)f(y_i) = \sum_{i=0}^{n} \delta K(x,y_i)\langle f, \eta_{y_i}\rangle_{H^3} = \sum_{i=0}^{n} \delta K(x,y_i)\langle g, (1-\Delta_y)^{\frac{3}{2}}\eta_{y_i}\rangle_{L^2} \\ &= \int_{\mathbb{R}} \sum_{i=0}^{n} \delta K(x,y_i)(1-\Delta_y)^{\frac{3}{2}}\eta_{y_i}(y)g(y)\mathrm{d}y \end{aligned}$$

## II Numerical approximation by Gaussians