Q1 Part A:

The motion of a comet is dictated through the gravitational force between the sun and the comet as shown below:

$$m\,\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{GMm}{r^2}\,\frac{\boldsymbol{r}}{r} \qquad (1)$$

where $r = \sqrt{x^2 + y^2}$ and the bold font indicates a vector quantity. Assuming the comet is orbiting in the $z = 0$ plane, only two dimensions need to be considered. To numerically model the solution, a new variable v (velocity) is introduced to split equation (1) into two simultaneous first-order differential equations as below:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} \quad (2), \qquad \frac{d\boldsymbol{v}}{dt} = -\frac{GM}{r^3}\,\boldsymbol{r} \quad (3)$$

When modelling the orbit, r and v represent the position and velocity vectors respectively. Equations 2 and 3 are then split into the respective x and y components, for example, $r = [x, y]$ and $v = [v_x, v_y]$.

To simplify the initial conditions, the system is orientated so that at $t = 0$, the comet is located on the x-axis. Hence, r only has an x component and v only has a y component. In addition, we shall choose the units specifically so $GM_{solar} = 4\pi^2\ (AU^3 years^{-2})$. Within the code, the initial conditions are calculated and assigned to r and v respectively. The function 'acc' outputs the acceleration using equation (3) and the 'vel' function outputs the velocity. The fourth order Runge-Kutta method (RK4) is then used, total accuracy $O(h^4)$, to numerically model the orbit as seen in figure 1. The RK4 method is used over Euler's method as is more accurate, using four approximations to the slope of the function rather than just one. The variable h is assigned to the timestep.

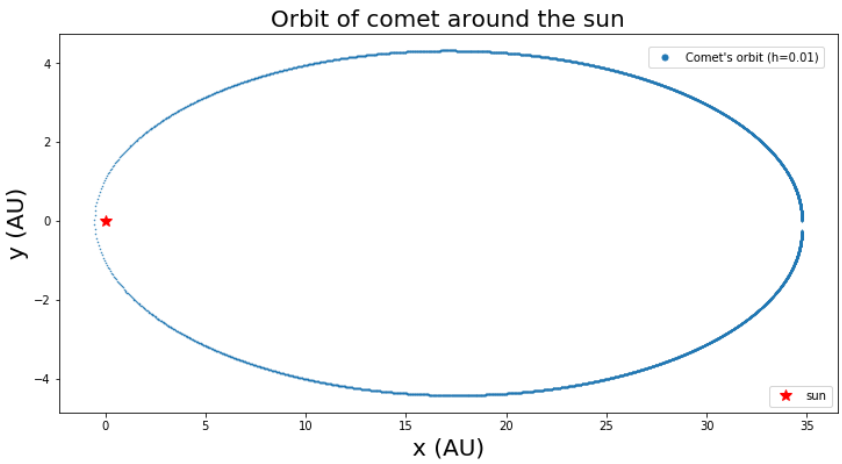

Figure 1: Single orbit of comet with time step $\mathrm{h} = 0.01$ over time interval of 74 years (approximately 1 orbit).

The comet's orbit appears highly elliptical as expected. To investigate the stability of the program, the orbit is then modelled for a range of h values.

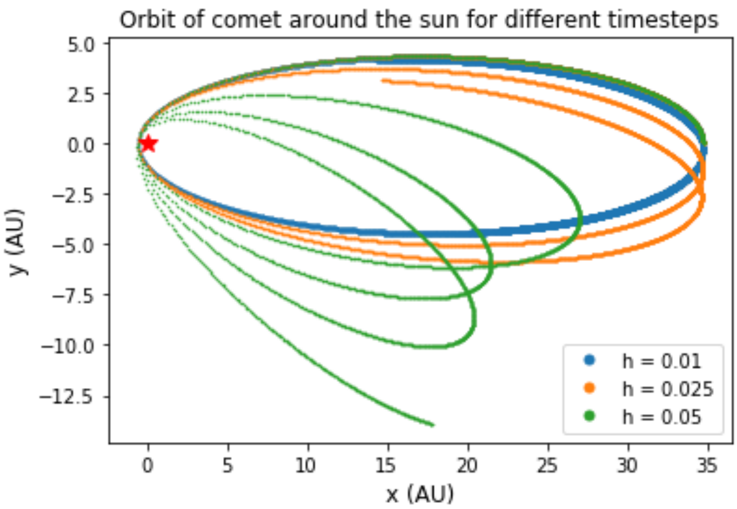

Figure 2: Comet's orbit for varying time step values h over a time interval of 180 years (approximately 2.4 orbits).

Figure 2 shows a maximum h value (approximately $h = 0.02$), above which the stability of the program decreases, and the orbit begins to wildly drift. Decreasing the timestep sufficiently removes the orbit drift. Modelling approximately 4 orbits, using a timestep of $h = 0.001$ the orbit drift is negligible. As a result, it is important that there are sufficient time steps used for the Runge-Kutta method to work successfully.

At the perihelion, where the comet is moving the fastest, the points are spaced further apart as seen in figure 1. To improve the program, we could use an adaptive step size which will be discussed in part B.

Q1 Part B:

The program from Part A is modified to use an adaptive timestep. An adaptive time step can be beneficial where there are specific areas where the function varies more rapidly, hence smaller timesteps are required. Also, there are areas where the function varies slowly and timesteps can be more spread out without significantly increasing the error. Figure 1 indicates that it would be useful to have more points nearer the perihelion, and fewer near the aphelion. The overall number of steps can be reduced, making the program more efficient, despite requiring more evaluations for each step.

The functions 'RK_step_r' and 'RK_step_v' have been introduced to simplify the code and make it easier to input a range of h values.

Starting at a certain value of t, the value of r is calculated at $r(t + 2h)$ first using two separate steps of size h. The value of $r(t + 2h)$ is then calculated using a single step of $2h$. The difference between these values can represent some error, ϵ. This error ϵ is then compared to some tolerated error level. If ϵ is larger than the tolerated error, then the step size h will be halved, if ϵ is less than the tolerated error, then the step size h will be doubled. Limits to the value of h (h_max and h_min), are required to prevent the timestep h from becoming too small (wasting computational time) or too large (leading to in correct orbit).

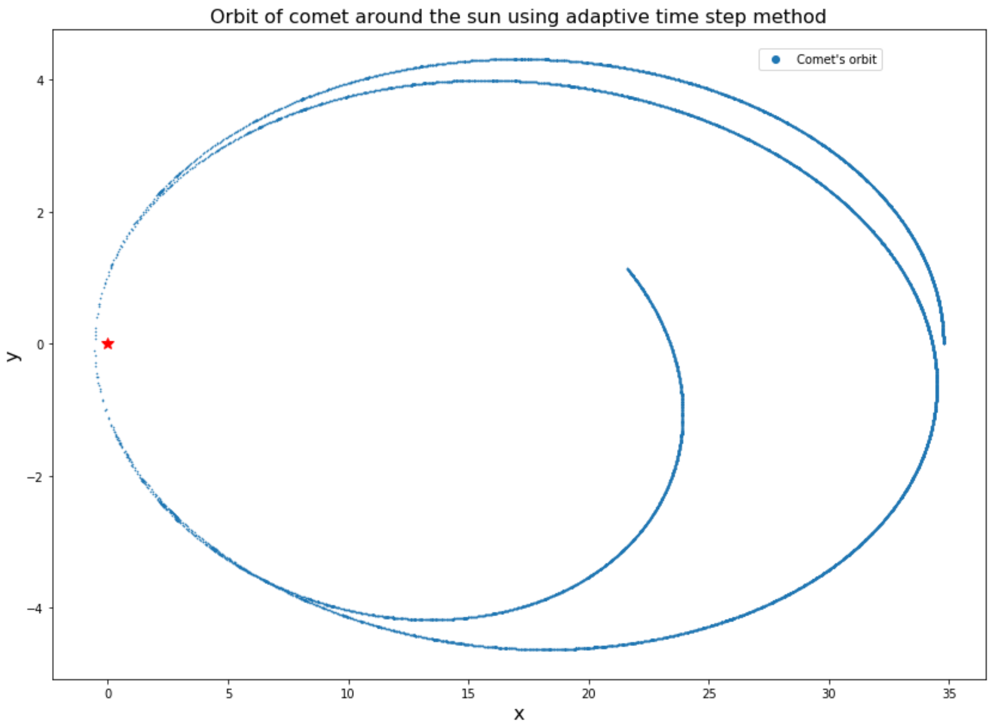

Figure 3: Orbit of comet around the sun using an adaptive time step method, over a time interval of 140 years.

Although the time step h is successfully varied, the results as seen in figure 3 are not desired. Near the perihelion the timestep is still larger than required. Some difficulty may arise due to the fact that near the perihelion the values of x and y are very small, hence the error is small. One attempt to overcome this issue was to use a fixed time step of h_min around these small r values. From figure 3 it appears that after approximately 1.5 orbital periods the orbit seems to collapse inwards towards the sun which is incorrect. Another method that hasn't been explored, would be to shift the orbit so that the sun isn't situated at the origin, hence removing the very small x and y values near the perihelion.

Q2 Part A:

This question compromises a 3-body problem, with the sun as the central body positioned at the origin, and Saturn and Jupiter as two orbiting bodies. Interactions of 3 or more bodies due to gravity cannot be solved analytically, hence a numerical solution is required. Jupiter is labelled as body 1 and Saturn as body 2. Firstly, the respective properties of the orbits are calculated, as seen in table 1.

Table 1: Properties of the orbits of Saturn and Jupiter

Planet	Mass (M_{solar})	a (AU)	e	Initial v (AU/yr)	Time period (yrs)
Body 1: Jupiter	9.547×10^{-4}	5.204	0.049	2.622	11.87
Body 2: Saturn	2.859×10^{-4}	9.583	0.057	1.891	30.93

The function within the code 'v_aph(a, e)' inputs the semi-major axis, a and eccentricity, e of the orbit and returns the velocity of the planet at the aphelion (the largest distance from the sun). The function applies equation (4), Kepler's Third Law and equation (5) to calculate this velocity

$$P^2 = \frac{4\pi^2}{GM}\, a^{\,3} \qquad (4) \qquad\qquad v_{aph} = \left(\frac{2\pi a}{P}\right)\left[\frac{(1-e)}{(1+e)}\right]^{\frac{1}{2}} \qquad (5)$$

where P is the time period of the orbit and $\frac{4\pi^2}{GM} = 1$ due to the choice of units.

Similarly, to Question 1, we align both planets at $t = 0$ along the x-axis to simplify the initial conditions (no y component in position or x component in velocity of planet).

The coupled first order ODE's equations (6) and (7) are solved numerically using the 4th order Runge-Kutta method for orbiting body 1 (Jupiter) and orbiting body 2 (Saturn).

$$\boldsymbol{v_1} = \frac{d\boldsymbol{r_1}}{dt}, \qquad \frac{d\boldsymbol{v_1}}{dt} = -\frac{GM}{{r_1}^3}\boldsymbol{r_1} + \frac{Gm_2}{{r_{21}}^3}\boldsymbol{r_{21}} \qquad (6)$$

$$\boldsymbol{v_2} = \frac{d\boldsymbol{r_2}}{dt}, \qquad \frac{d\boldsymbol{v_2}}{dt} = -\frac{GM}{{r_2}^3}\boldsymbol{r_2} - \frac{Gm_1}{{r_{21}}^3}\boldsymbol{r_{21}} \qquad (7)$$

Within the code, the position and velocity of both planets are stored within the same vector r and v respectively. Within these vectors the position/velocity is split into x and y components. For example, $r = [x1,\ x2,\ y1,\ y2]$ and $v = [vx1,\ vx2,\ vy1,\ vy2]$.

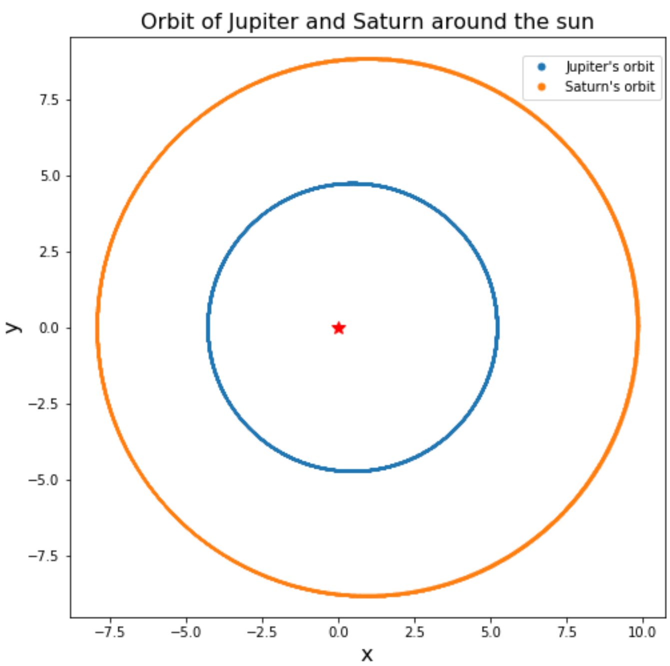

Figure 4: The orbits of Jupiter and Saturn around the sun over a time interval of 155 years (approximately 5 orbits for Saturn) with timestep $h = 0.01$.

The orbits appear almost circular which is expected due to the very low eccentricities. When the timestep h is varied and the time interval is increased, the program appears to be stable. When $h > 1$, Jupiter's orbit shifts around and the points are very spread out, not giving a precise model of the orbit. However, the orbit doesn't wildly drift at higher time steps as seen in the highly elliptical comet's orbit.

<u>Q2 Part B:</u>

To transform the 3-body system of Jupiter and Saturn orbiting the sun from a coordinate system to a frame centred upon the centre of mass, the position and velocity vectors for the centre of mass are required.

$$\boldsymbol{r_{CM}} = \sum_i \frac{m_i}{M_{total}} \boldsymbol{r_i} \quad (8) \qquad \text{and} \qquad \boldsymbol{v_{CM}} = \sum_i \frac{m_i}{M_{total}} \boldsymbol{v_i} \quad (9)$$

Equation (10) and (11) show the position and velocity vectors transformed from the original coordinate system to the centre of mass coordinate frame.

$$\boldsymbol{v_i'} = \boldsymbol{v_i} - \boldsymbol{v_{CM}} \quad (10) \qquad \text{and} \qquad \boldsymbol{r_i'} = \boldsymbol{r_i} - \boldsymbol{r_{CM}} \quad (11)$$

Within the code the functions 'COM_R(r)'and 'COM_V(v)' are introduced to calculate the position and velocity vectors of the centre of mass. Later, $\boldsymbol{r_{CM}}$ and $\boldsymbol{v_{CM}}$ are then subtracted from the velocity and position vectors, switching to the centre of mass reference frame.

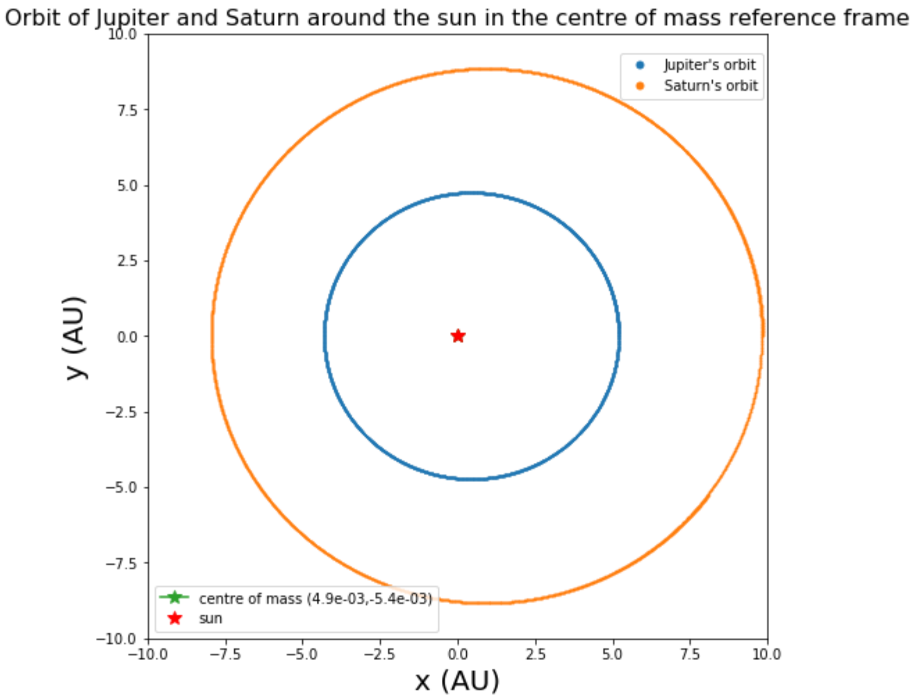

Figure 5: The orbits of Jupiter and Saturn around the sun over a time interval of 50 years (approximately 5 orbits for Saturn) with timestep $h = 0.01$, from the centre of mass reference frame. The centre of mass coordinates are shown at $t = 50$ years for a reference, as the point of the sun and centre of mass are overlapping.