Proof of Theorem 1 : Concavity of the Log-Likelihood function

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Abstract

This supplementary material contains the proof that the Log-Likelihood function for the Multi-armed bandit model is concave.

Theorem 1 (Concavity of log likelihood function)**.** *The log likelihood function* $\log \mathcal{L}$ *for* $\mathcal{L}$ *defined in Eq. 3 is concave.*

Proof. Required to prove that

$$\log \mathcal{L}(\alpha\vec{q} + (1-\alpha)\vec{r}) \geq \alpha \log \mathcal{L}(\vec{q}) + (1-\alpha)\log \mathcal{L}(\vec{r}).$$

We begin with a simplification of the notation used in the Likelihood function by converting to vector notation. Let

$$g(\vec{Q}) = \sum_{j=1}^{m+1} N_j \log Q_j$$

where $N_{m+1} = N - \sum_{j=1}^{m} N_j$ and $Q_{m+1} = 1 - \sum_{j=1}^{m} Q_j$. As before (Eq. 3), $Q_j = \sum_{i=1}^{n} q_i p_{i,\sigma_i(k_j)}$ and therefore

$$Q_{m+1} = 1 - \sum_{j=1}^{m}\sum_{i=1}^{n} q_i p_{i,\sigma_i(k_j)} = \sum_{i=1}^{n} q_i (1 - \sum_{j=1}^{m} p_{i,\sigma_i(k_j)}).$$

Observe, we can define a matrix P so $\vec{Q} = P\vec{q}$ and

$$\log \mathcal{L}(\vec{q}) = \sum_{j=1}^{m+1} N_j \log Q_j = g(\vec{Q}).$$

Therefore, we can now rewrite

$$\begin{aligned}
\log \mathcal{L}(\alpha\vec{q} + (1-\alpha)\vec{r}) &= g\left(P(\alpha\vec{q} + (1-\alpha)\vec{r})\right) \\
&= g\left(\alpha P\vec{q} + (1-\alpha)P\vec{r}\right) \\
&= \sum_{j=1}^{m+1} N_j \log(\alpha(P\vec{q})_j + (1-\alpha)(P\vec{r})_j) \\
&\geq \sum_{j=1}^{m+1} N_j (\alpha \log(P\vec{q})_j + (1-\alpha)\log(P\vec{r})_j) \\
&= \alpha \sum_{j=1}^{m+1} N_j \log(P\vec{q})_j + (1-\alpha) \sum_{j=1}^{m+1} N_j \log(P\vec{r})_j \\
&= \alpha g(\vec{Q}) + (1-\alpha) g(\vec{R}) \\
&= \alpha \log \mathcal{L}(\vec{q}) + (1-\alpha) \log \mathcal{L}(\vec{r}).
\end{aligned}$$

$\square$