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Mini-Boson Stars
Analytical and Computational Analysis

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Contents

Abstract

We investigate the equilibrium configurations of spherically symmetric, self-gravitating objects composed of a complex scalar field minimally coupled to gravity, commonly referred to as mini-boson stars. Starting from the classical framework of hydrostatic equilibrium and polytropic equations of state, we outline the derivation of the Lane–Emden equation for modeling compact objects under Newtonian gravity. We then transition to a general relativistic treatment by considering spherically symmetric spacetimes governed by Einstein's field equations. The stress-energy tensor for a complex scalar field is derived from a minimally coupled Lagrangian, and a harmonic ansatz is employed to ensure a static spacetime while preserving the dynamical nature of the field. Assuming a Schwarzschild-like metric, we reduce the Einstein–Klein–Gordon system to a set of coupled, first-order ordinary differential equations governing the scalar field and spacetime geometry. For the case of a free scalar field with a quadratic potential, the system simplifies and permits the exploration of equilibrium solutions characterized by finite mass and radius. This formalism serves as a foundational model for studying the structure and properties of boson stars, providing insight into non-baryonic compact objects within general relativity.

Introduction

The study of compact astrophysical objects has long served as a cornerstone in understanding the interplay between gravity, quantum fields, and high-energy physics. While white dwarfs and neutron stars are well-understood through the frameworks of classical and relativistic hydrostatics, recent interest has turned to more exotic candidates such as boson stars [3, 1]. These are theoretical objects composed of scalar particles bound together by gravity, stabilized by quantum mechanical effects rather than degeneracy pressure.

Boson stars arise from solutions to the Einstein–Klein–Gordon (EKG) system, which couples a complex scalar field to the curvature of spacetime [5, 2]. In the simplest case of a free massive scalar field with no self-interaction, the solutions are known as mini-boson stars. More general models include self-interacting potentials, yielding a rich landscape of equilibrium configurations with diverse physical properties [4].

Boson stars are compelling for several reasons. From a theoretical perspective, they provide an ideal testbed for studying quantum field theory in curved spacetime [3]. Physically, they offer viable candidates for dark matter halos or primordial remnants from the early universe [4]. Observationally, boson stars may mimic the gravitational signatures of black holes, challenging the conventional interpretation of compact object observations in gravitational wave astronomy and electromagnetic surveys [1].

In this work, we focus on constructing static, spherically symmetric boson star solutions in the minimal case of a free scalar field. Beginning with a review of classical hydrostatic equilibrium and the Lane–Emden equation, we transition to the relativistic regime and derive the field equations from the Einstein-Hilbert action coupled to a complex scalar field. A harmonic time dependence is assumed for the scalar field to reduce the system to a set of ordinary differential equations (ODEs), which are then solved numerically. Through this approach, we investigate the mass-radius relation, field profiles, and metric components of boson star configurations, setting the groundwork for further exploration into more complex models and astrophysical implications.

Preliminaries

2.1 Equation of State

The internal structure of compact astrophysical objects can be modeled using the hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{1}$$

This balances gravitational compression with internal pressure. The enclosed mass at radius r is:

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr' \tag{2}$$

A widely used simplification assumes a polytropic equation of state:

$$P = K\rho^{1+\frac{1}{n}} \tag{3}$$

where n is the polytropic index and K is a constant. Substituting this into the hydrostatic equilibrium yields the Lane–Emden equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho}{dr}\right) = -\frac{4\pi G}{(n+1)K}\rho^n \tag{4}$$

Introducing the dimensionless density function $\rho = \rho_c \Theta^n$, the equation becomes:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n \tag{5}$$

with boundary conditions:

$$\Theta(0) = 1, \quad \left.\frac{d\Theta}{d\xi}\right|_{\xi=0} = 0 \tag{6}$$

Solving this provides the radial density profile $\rho(r)$, from which physical properties such as total mass and radius are derived. This framework is essential for modeling self-gravitating systems under simplified but realistic assumptions.

2.2 Metric Tensor and Coordinates

We begin with the most general spherically symmetric line element:

$$ds^2 = -\left(\alpha^2 - a^2\beta^2\right) dt^2 + 2a^2\beta\, dtdr + a^2dr^2 + r^2b^2d\Omega^2 \tag{7}$$

where: - $\alpha(t,r)$ is the lapse function, controlling time slicing, - $\beta(t,r)$ is the radial shift vector component, governing spatial coordinates, - $a(t,r), b(t,r)$ are spatial metric functions, - $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard angular metric.

This form allows for time-dependent configurations while preserving spherical symmetry, making it suitable for both static and dynamic simulations.

For static, spherically symmetric spacetimes, we can reduce the general form to:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\Omega^2, \tag{8}$$

where $\Phi(r)$ and $\Lambda(r)$ are arbitrary functions of the radial coordinate. The Einstein tensor is hence written as:

$$G_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\rho\lambda} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \left(\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\rho_{\rho\lambda} - \Gamma^\rho_{\alpha\lambda}\Gamma^\lambda_{\beta\rho} \right) \quad (9)$$

The non-zero diagonal components of the Einstein tensor for this metric are:

$$G_{tt} = \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) - \frac{2\Lambda'}{r} e^{-2\Lambda}, \quad (10)$$

$$G_{rr} = \frac{1}{r^2}\left(e^{-2\Lambda} - 1\right) + \frac{2\Phi'}{r} e^{-2\Lambda}, \quad (11)$$

$$G_{\theta\theta} = G_{\phi\phi}/\sin^2\theta = \left[\Phi'' + \Phi'(\Phi' - \Lambda') + \frac{\Phi' - \Lambda'}{r}\right] e^{-2\Lambda}. \quad (12)$$

As a consistency check, in vacuum ($T_{\mu\nu} = 0$), the Einstein equations $G_{\mu\nu} = 0$ must yield the Schwarzschild metric. Consider:

$$G_{tt} = -\frac{1}{r^2}\frac{d}{dr}\left[r\left(1 - e^{-2\Lambda}\right)\right] = 0, \quad (13)$$

which integrates to:

$$e^{-2\Lambda} = 1 - \frac{C}{r}, \quad (14)$$

where C is an integration constant. Substituting into G_{rr} and evaluating $G_{rr} - G_{tt} = 0$ yields:

$$\Phi' = -\Lambda' \quad \Rightarrow \quad \Phi = -\Lambda, \quad (15)$$

implying:

$$e^{2\Phi} = 1 - \frac{C}{r}. \quad (16)$$

This results in the Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{C}{r}\right) dt^2 + \left(1 - \frac{C}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (17)$$

which describes the unique static, spherically symmetric vacuum solution to Einstein's equations. Here, $C = 2GM$ corresponds to the Schwarzschild radius.

2.3 Einstein Equations and Stress-Energy Tensor

The full dynamics of spacetime are governed by the Einstein equations:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \quad (18)$$

For a complex scalar field ϕ minimally coupled to gravity, the Lagrangian is:

$$\mathcal{L} = \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* - V(|\phi|^2)\right] \quad (19)$$

From this, the stress-energy tensor becomes:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi^* + \partial_\nu\phi\partial_\mu\phi^* - g_{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi^* + V(|\phi|^2)\right) \quad (20)$$

This tensor encodes how the scalar field influences the geometry of spacetime.

Mini-Boson Stars: Axioms and Analytical Framework

A boson star is a self-gravitating configuration of a complex scalar field that remains localized and time-independent in spacetime geometry while allowing for a harmonic time dependence in the scalar field itself. This structure arises from a balance between gravitational attraction and quantum pressure or dispersive effects due to the wave nature of the field.

3.1 Harmonic Ansatz

To construct equilibrium configurations of boson stars, we assume a *harmonic ansatz* for the complex scalar field:

$$\phi(r,t) = \phi_0(r)e^{i\omega t} \tag{21}$$

where: - $\phi_0(r)$ is a real-valued function representing the radial profile of the scalar field, - ω is a real constant corresponding to the angular frequency of the field's phase in the complex plane. This ansatz ensures that the stress-energy tensor becomes time-independent, even though the scalar field itself has a harmonic time dependence. This allows for static spacetime geometries while maintaining non-trivial dynamics in the matter field.

3.2 Metric Ansatz

We consider spherically symmetric, static spacetimes described by the Schwarzschild-like metric:

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2 \tag{22}$$

where: - $\alpha(r)$ is the lapse function, - $a(r)$ is the spatial metric function, - $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard angular metric.

3.3 Equilibrium Equations

The Einstein–Klein–Gordon (EKG) system governs the dynamics of a complex scalar field minimally coupled to gravity. The system consists of: - The Einstein equations:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

- The Klein–Gordon equation:

$$\nabla^\mu \nabla_\mu \phi = \frac{dV}{d|\phi|^2}\phi$$

where $V(|\phi|^2)$ is the self-interaction potential of the scalar field.

For a spherically symmetric, static spacetime and a harmonic scalar field ansatz:

$$\phi(r,t) = \phi_0(r)e^{i\omega t}$$

the stress-energy tensor becomes time-independent, allowing for a static metric:

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2$$

Under these assumptions, the EKG system reduces to a set of three coupled first-order ordinary differential equations:

$$\partial_r a = \frac{a}{2r}\left(1 - a^2\right) + 4\pi G r a^3 \rho \tag{23}$$
$$\partial_r \alpha = \frac{\alpha}{2r}\left(a^2 - 1\right) + 4\pi G r \alpha a^2 P \tag{24}$$
$$\partial_r \Phi = -\left[\frac{1 + a^2 + 4\pi G r^2 a^2 (P - \rho)}{r} + \left(\frac{\omega^2}{\alpha^2} - \frac{dV}{d|\phi|^2}\right) a^2 \phi_0\right]\Phi \tag{25}$$

where: - $\Phi = \partial_r \phi_0$ - $\rho = T^t_t = \frac{\omega^2}{\alpha^2}\phi_0^2 + \Phi^2 + m_b^2\phi_0^2$ is the energy density, - $P = \frac{1}{3}(T^r_r + T^\theta_\theta + T^\phi_\phi)$ is the isotropic pressure.

For the simplest case of a free scalar field with mass m, the potential is:

$$V(|\phi|^2) = m_b^2 |\phi|^2 \tag{26}$$

leading to the simplified system:

$$\partial_r a = \frac{a}{2}\left[-\frac{a^2 - 1}{r} + 4\pi G r \left(\frac{\omega^2}{\alpha^2} + m_b^2\right) a^2 \phi_0^2 + \Phi^2\right] \tag{27}$$
$$\partial_r \alpha = \frac{\alpha}{2}\left[\frac{a^2 - 1}{r} + 4\pi G r \left(\frac{\omega^2}{\alpha^2} - m_b^2\right) a^2 \phi_0^2 + \Phi^2\right] \tag{28}$$
$$\partial_r \Phi = -\left[\frac{1 + a^2 - 4\pi G r^2 a^2 m_b^2 \phi_0^2}{r}\right]\Phi - \left(\frac{\omega^2}{\alpha^2} - m_b^2\right) a^2 \phi_0 \tag{29}$$

These equations form a boundary value problem that must be solved numerically, subject to appropriate regularity and asymptotic flatness conditions at the origin and spatial infinity.

3.4 Derivation from Schwarzschild solutions

To derive the equations governing the structure of mini-boson stars, we start with the Einstein–Klein–Gordon (EKG) system under the assumptions:

- Spherical symmetry,
- Static spacetime,
- A complex scalar field of the form $\phi(r,t) = \phi_0(r) e^{i\omega t}$,
- No self-interaction potential (free field): $V(|\phi|^2) = m_b^2 |\phi|^2$.

We adopt the spherically symmetric metric in Schwarzschild-like coordinates:

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2$ is the metric on the unit 2-sphere.
The stress-energy tensor for the scalar field is time-independent due to the harmonic ansatz. Its nonzero components are:

$$T^t_t = -\rho = -\left(\omega^2 e^{-2\Phi}\phi_0^2 + (\phi_0')^2 e^{-2\Lambda} + m_b^2 \phi_0^2\right),$$
$$T^r_r = P_r = \omega^2 e^{-2\Phi}\phi_0^2 + (\phi_0')^2 e^{-2\Lambda} - m_b^2 \phi_0^2,$$

$$T^\theta_\theta = T^\phi_\phi = P_\perp = \omega^2 e^{-2\Phi}\phi_0^2 - (\phi_0')^2 e^{-2\Lambda} - m_b^2\phi_0^2.$$

From the Einstein equation component $G^t_t = 8\pi G T^t_t$, we obtain:

$$\frac{1}{r^2}\frac{d}{dr}\left[r\left(1 - e^{-2\Lambda}\right)\right] = 8\pi G\rho.$$

Multiplying both sides by r^2 and integrating yields the mass function:

$$m'(r) = 4\pi G r^2 \rho = 4\pi G r^2 \left(\omega^2 e^{-2\Phi}\phi_0^2 + (\phi_0')^2 e^{-2\Lambda} + m_b^2\phi_0^2\right).$$

The gravitational mass enclosed within radius r is thus

$$m(r) = \int_0^r 4\pi G r'^2 \rho(r')\, dr',$$

and the radial metric function satisfies

$$e^{-2\Lambda(r)} = 1 - \frac{2Gm(r)}{r}.$$

Using the $G^r_r = 8\pi G T^r_r$ component, we find the derivative of the temporal metric function:

$$\Phi'(r) = \frac{Gm(r) + 4\pi G^2 r^3 P_r(r)}{r\left(1 - \frac{2Gm(r)}{r}\right)} = \frac{Gm(r) + 4\pi G^2 r^3 P_r(r)}{r\left(1 - 2Gm(r)/r\right)}.$$

Finally, the Klein–Gordon equation in this geometry reduces to:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 e^{-\Lambda-\Phi}\frac{d\phi_0}{dr}\right) = e^{-2\Lambda}\left(\omega^2 e^{-2\Phi} - m_b^2\right)\phi_0.$$

Expanding, this yields the second-order ODE for the scalar field profile:

$$\phi_0'' + \left(\frac{2}{r} + \Lambda' - \Phi'\right)\phi_0' + \left(\omega^2 e^{2(\Lambda-\Phi)} - m_b^2 e^{2\Lambda}\right)\phi_0 = 0.$$

These three equations:

1. The mass function $m(r)$,
2. The metric function $\Lambda(r)$,
3. The scalar field equation for $\phi_0(r)$,

form the core system describing equilibrium configurations of mini-boson stars.

Coordinate-Independent Mass and ADM Mass:
The gravitational mass function $m(r)$ can equivalently be expressed in a coordinate-independent manner using the areal radius $R \equiv r$ and the covariant gradient:

$$m(r) = \frac{R}{2G}\left(1 - \nabla_\mu R \nabla^\mu R\right).$$

In Schwarzschild coordinates, since $\nabla_\mu R \nabla^\mu R = e^{-2\Lambda}$, this reduces to

$$m(r) = \frac{r}{2G}\left(1 - e^{-2\Lambda}\right),$$

which matches the previous definition.
The total ADM mass M_{ADM}, representing the total mass-energy as measured by a distant observer, is given by the asymptotic value

$$M_{\text{ADM}} = \lim_{r\to\infty} m(r).$$

This ADM mass serves as a fundamental global parameter characterizing the gravitational field of mini-boson stars in both analytical and numerical studies.

3.5 Boundary Conditions

To obtain physical solutions, we impose the following boundary conditions at the origin and at infinity:

At $r = 0$:

$$\phi_0(0) = \phi_c \tag{30}$$

$$\Phi(0) = 0 \tag{31}$$

$$a(0) = 1 \tag{32}$$

At $r \to \infty$:

$$\lim_{r\to\infty} \phi_0(r) = 0 \tag{33}$$

$$\lim_{r\to\infty} \alpha(r) = \lim_{r\to\infty} \frac{1}{a(r)} \tag{34}$$

These ensure regularity at the center and asymptotic flatness of the spacetime.

Results

4.1 The Einstein–Klein–Gordon System

The Einstein–Klein–Gordon system describes a massive complex scalar field interacting gravitationally in a spherically symmetric spacetime. The system consists of:

- The metric ansatz:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \tag{35}$$

- The scalar field ansatz:

$$\phi(r,t) = \phi_0(r)\, e^{i\omega t} \tag{36}$$

- The stress-energy tensor components:

- **Energy Density** $\rho(r) = -T^t{}_t$:

$$\rho(r) = \omega^2 e^{-2\Phi}\phi_0^2 + e^{-2\Lambda}(\phi_0')^2 + m_b^2\phi_0^2 \tag{37}$$

- **Radial Pressure** $p_r(r) = T^r{}_r$:

$$p_r(r) = e^{-2\Lambda}(\phi_0')^2 + \omega^2 e^{-2\Phi}\phi_0^2 - m_b^2\phi_0^2 \tag{38}$$

- **Tangential Pressure** $p_\perp(r) = T^\theta{}_\theta = T^\varphi{}_\varphi$:

$$p_\perp(r) = -e^{-2\Lambda}(\phi_0')^2 + m_b^2\phi_0^2 - \omega^2 e^{-2\Phi}\phi_0^2 = -p_r \tag{39}$$

- The Einstein equations:

$$\begin{aligned} G_{tt} &= 8\pi G T_{tt} \\ G_{rr} &= 8\pi G T_{rr} \\ G_{\theta\theta} &= 8\pi G T_{\theta\theta} \end{aligned} \tag{40}$$

- The Klein–Gordon equation:

$$\nabla_\mu \nabla^\mu \phi + m_b^2 \phi = 0 \tag{41}$$

4.2 The Resulting System of ODEs

From the Einstein–Klein–Gordon system, we derive the following closed system of main ordinary differential equations:

$$m'(r) = 4\pi G r^2 \left[\omega^2\phi_0^2 + e^{2(\Phi-\Lambda)}(\phi_0')^2 + m_b^2 e^{2\Phi}\phi_0^2\right] \tag{42}$$

$$\Phi'(r) = \frac{m(r) + 4\pi G r^3 \left[(\phi_0')^2 + e^{2(\Lambda-\Phi)}\omega^2\phi_0^2 - m_b^2 e^{2\Lambda}\phi_0^2\right]}{r(r-2Gm(r))} \tag{43}$$

$$\phi_0'' + \left(\frac{2}{r} + \Lambda' - \Phi'\right)\phi_0' + \left(\omega^2 e^{2(\Lambda-\Phi)} - m_b^2 e^{2\Lambda}\right)\phi_0 = 0 \tag{44}$$

$$\Lambda(r) = -\frac{1}{2}\ln\left(1 - \frac{2Gm(r)}{r}\right) \tag{45}$$

This system must be solved simultaneously with appropriate boundary conditions to yield physical configurations of the mini-boson stars.

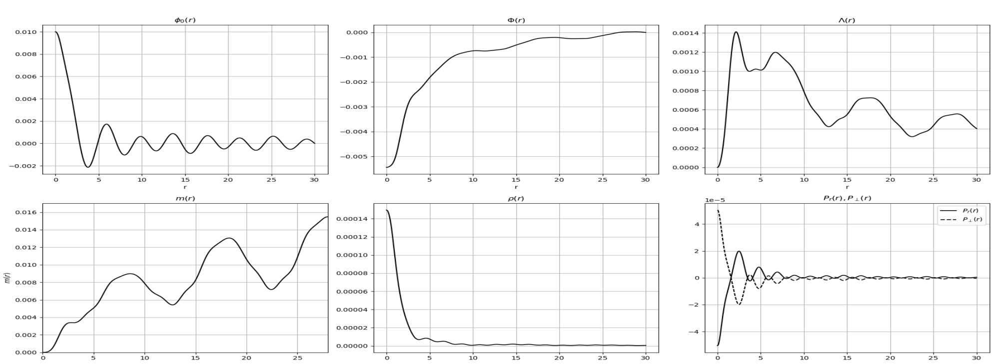

Figure 1: Basic profiles for $\omega = 0.7 m_b\, m_b$

4.3 Effect of the Frequency ω on the System

The frequency ω plays a crucial role in determining the structure and stability of mini-boson stars. Since the scalar field evolves as $\phi(r,t) = \phi_0(r)e^{i\omega t}$, ω must satisfy specific physical constraints:

- **Bound state requirement:** For the scalar field to remain localized (i.e., bound), ω must satisfy $\omega < m_b$, where m_b is the boson mass. If $\omega \geq m_b$, the scalar field becomes oscillatory at infinity, leading to non-normalizable, non-physical solutions.

- **Numerical instability at low ω:** For low values such as $\omega/m_b \lesssim 0.8$, the mass function $m(r)$ may exhibit pathological behavior, such as decreasing with radius or becoming negative. This typically indicates numerical instability or that the solution lies in an unstable branch of the mass–frequency relation. Such configurations are unphysical and should be discarded.

- **Stable regime:** Intermediate values, such as $\omega/m_b \approx 0.85$–0.95, yield well-behaved, localized solutions with monotonic $m(r)$ and exponential decay of the scalar field. These correspond to stable or metastable mini-boson stars. The maximum stable mass is reached near $\omega \approx 0.9$, as observed in the M_{ADM} vs. ω plot.

- **Unstable high-ω regime:** For $\omega \gtrsim 0.95$, M_{ADM} increases sharply, but this region is likely unphysical because $\omega \geq m_b$ leads to unbound states. Such configurations are not viable for mini-boson stars.

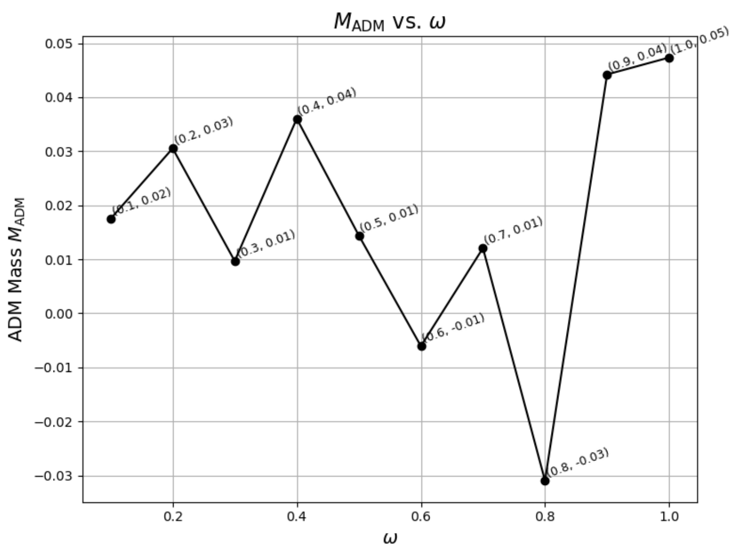

Figure 2: M_{ADM} vs. ω. The peak at $\omega \approx 0.9$ corresponds to the maximum stable mass.

4.4 Boson Polytrope Equation

There is no direct analytical expression $P = f(\rho)$ for the equation of state in the case of mini-boson stars. Instead, one can *numerically extract* the relationship between the radial pressure p_r and the energy density ρ along the radial profile. Therefore, in this context, the equation of state (EoS) is defined implicitly through the computed profiles of $p_r(r)$ and $\rho(r)$, and must be obtained numerically.

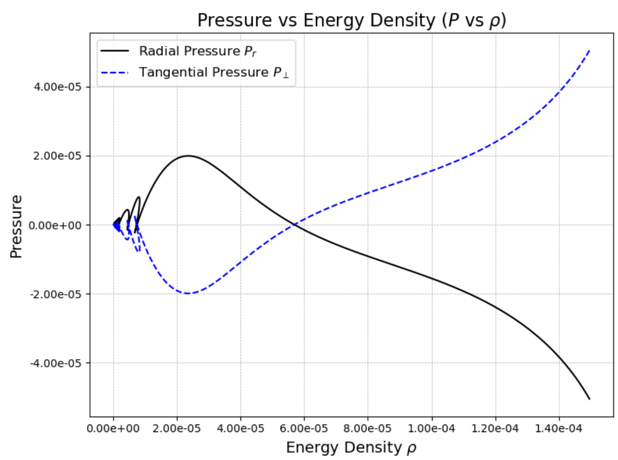

Figure 3: P vs ρ for $\omega = 0.7\,m_b$

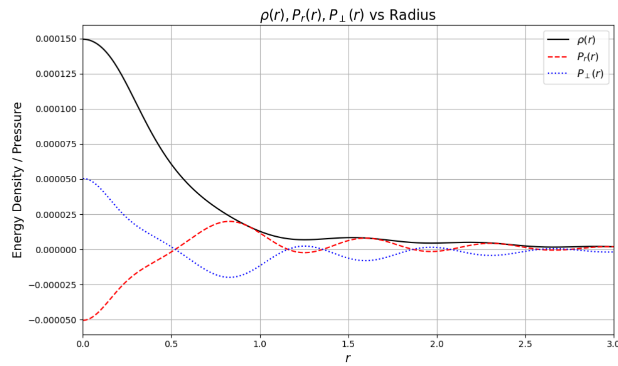

Figure 4: P and ρ for $\omega = 0.7\,m_b$ vs r

Conclusion and Discussion

In this study, we have investigated the equilibrium configurations of mini-boson stars—compact, self-gravitating systems composed of a complex scalar field coupled to gravity. Beginning with the classical Lane–Emden formulation of hydrostatic equilibrium, we transitioned into the general relativistic framework through the Einstein–Klein–Gordon system, employing a harmonic ansatz for the scalar field and a static, spherically symmetric spacetime metric.

Our numerical integration of the resulting set of coupled ODEs confirmed the existence of localized, regular solutions characterized by a finite ADM mass and compact support. These solutions depend sensitively on the frequency ω of the scalar field oscillation. We found that:

- Stable configurations exist only for frequencies $\omega < m_b$, satisfying the bound-state requirement.
- The mass of the star reaches a maximum at an intermediate frequency $\omega \approx 0.9\,m_b$, beyond which solutions become unstable.
- The pressure-density relation cannot be expressed analytically but is recoverable numerically, suggesting an effective equation of state unique to each solution.

These results reinforce the interpretation of mini-boson stars as viable theoretical models for non-baryonic compact objects. Their capacity to mimic black hole gravitational signatures without event horizons opens intriguing avenues for astrophysical applications, including dark matter modeling and gravitational wave signal analysis.

Future work may extend this model to include scalar self-interactions, explore rotating boson stars, or investigate perturbative stability. Additionally, coupling these models to realistic observational frameworks could enhance the prospects for detecting boson stars in the universe.

Appendix

6.1 The Schwarzchild solution

For a spherically symmetric , static mass distribution , the metric tensor in spherical coordinates is given by :

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \frac{1}{1 - \frac{2m}{r}} dr \otimes dr + r^2 d\Omega \otimes d\Omega$$

Now , the standard derivation for the TOV equation goes as follows:
A) Set up a spherically symmetric ansatz of the form :
[1]

$$g = -e^{(2\Phi)} dt \otimes dt + e^{2\lambda} dr \otimes dr + r^2 d\Omega \otimes d\Omega \tag{46}$$

we leave the temporal and radial equations arbitrary because we don't know if the metric inside will behave like it behaves outside the spherically symmetric distribution .

B)we then impose the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{47}$$

and the normalisation condition for the four velocity :[2]

$$g(u^\mu u^\nu) = -1 \tag{48}$$

C)we get a relationship between p , ρ and r which is the hydrostatic equation of state , so the derivation procedure crudely will go as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \tag{49}$$

for a perfect fluid :

$$T_{\mu\nu} = -(\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{50}$$

in the rest frame of the fluid, in normal coordinates, the stress momentum tensor simplifies significantly to[3] :

$$T_{\mu\nu} = \text{Diag}\{-\rho, p, p, p\} \tag{51}$$

[1] 2Φ to simplify the equations when imposing the normalisation condition

[2] we keep it -1 to match the spacetime units , if you want to make it -1/68.5 or your favourite fraction , you will have to rescale the units

[3] I will reference a section in wald regarding the tetrad formalism , and the frame bundle construction , for now we can stick just to the rest frame

As an exercise, we can list a few steps in the analysis:[4]

A)Christoffel symbols:

The metric compatible connection coefficients are called the christoffel symbols.The metric compatibility condition[5]:

$$\nabla g = 0 \tag{52}$$

The metric induced connection coefficients are given by :

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right),$$

And hence we get :

$$\begin{aligned}
\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \Phi', \quad \Gamma^{r}_{rr} = \Lambda', \quad \Gamma^{r}_{tt} = \Phi' e^{2(\Phi-\Lambda)}, \\
\Gamma^{r}_{\theta\theta} = \Gamma^{r}_{\phi\phi} = -\frac{1}{r}, \quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \\
\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \quad \Gamma^{r}_{\theta\theta} = -r e^{-2\Lambda}, \quad \Gamma^{r}_{\phi\phi} = -r \sin^2\theta e^{-2\Lambda}, \\
\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta.
\end{aligned}$$

The Einstein Tensor is hence written as:

$$G_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \left(\partial_\lambda \Gamma^{\lambda}_{\alpha\beta} - \partial_\beta \Gamma^{\lambda}_{\alpha\lambda} + \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\rho}_{\rho\lambda} - \Gamma^{\rho}_{\alpha\lambda}\Gamma^{\lambda}_{\beta\rho} \right).$$

The 4 diagonal components are given as follows:

$$\begin{aligned}
G^{t}_{t} &= \frac{1}{r^2}\left(e^{-2\Lambda} - 1\right) - \frac{2\Lambda'}{r} e^{-2\Lambda}, \\
G^{r}_{r} &= \frac{1}{r^2}\left(e^{-2\Lambda} - 1\right) + \frac{2\Phi'}{r} e^{-2\Lambda}, \\
G^{\theta}_{\theta} = G^{\phi}_{\phi} &= \left[\Phi'' + \Phi'(\Phi' - \Lambda') + \frac{\Phi' - \Lambda'}{r} \right] e^{-2\Lambda}.
\end{aligned}$$

Now as a sanity check , the vaccum solutions must reduce to the standard schwarzchild metric components:

$$G^{t}_{t} = -\frac{1}{r^2}\frac{d}{dr}\left[r\left(1 - e^{-2\Lambda}\right) \right]. = 0 \implies e^{-2\lambda} = 1 - \frac{C}{R} \implies \phi = -\lambda \implies e^{2\lambda} = \frac{1}{1 - \frac{c}{R}} \tag{53}$$

Furthermore, subtracting G^t_t from G^r_r yields :

$$\phi' = -\lambda' \implies \phi = -\lambda \implies e^{2\Phi} = 1 - C/R \tag{54}$$

[4]I will attach a mathematica file for the analysis because i am sane enough not to do this by hand

[5]imposing a zero non metricity tensor

6.2 The TOV equation

Now for the TOV equation, the stress momentum tensor is :

$$\frac{1}{r^2}\left(e^{-2\Lambda}-1\right)-\frac{2\Lambda'}{r}e^{-2\Lambda}=-8\pi\rho, \tag{55}$$

$$\frac{1}{r^2}\left(e^{-2\Lambda}-1\right)+\frac{2\Phi'}{r}e^{-2\Lambda}=8\pi p, \tag{56}$$

$$\left[\Phi''+\Phi'(\Phi'-\Lambda')+\frac{\Phi'-\Lambda'}{r}\right]e^{-2\Lambda}=8\pi p. \tag{57}$$

where we have set c=G=1.
The temporal equation can be manipulated to be written down as :

$$\frac{1}{r^2}\frac{d}{dr}\left[r\left(1-e^{-2\Lambda}\right)\right]=8\pi\rho. \tag{58}$$

Integrating (14) we get :

$$e^{-2\Lambda(r)}=1-\frac{8\pi}{r}\int_0^r \rho(r')r'^2\,dr', \tag{59}$$

We recognize this integral:

$$\int_0^r 4\pi\rho(r')r'^2\,dr'=m(r) \tag{60}$$

Now for the following 2 equations, we do the same manipulation as we did with the vaccum solution.
By plugging back $e^{-2\lambda(r)}=1-\frac{2m(r)}{r}$ to (12), we get:

$$\Phi'(r)=\frac{m(r)+4\pi r^3p(r)}{r\big(r-2m(r)\big)}. \tag{61}$$

Almost there ,now to get an equation involving the derivatives of the pressure , I will use the conservation equation, recall equation (3)[6]:

$$\nabla_\mu T^{\mu\nu}=0$$

We get :

$$P'=-(P+\rho)\Phi' \tag{62}$$

Finally arriving at the TOV equation:

$$P'(r)=-\frac{\big(\rho(r)+P(r)\big)\big(m(r)+4\pi r^3P(r)\big)}{r\big(r-2m(r)\big)}. \tag{63}$$

[6]Again I am sane enough not to plug (17) back into the angular equation

6.3 klein gordon equation

6.3.1 minimal coupling

There is a very fun trick of trade that physicists have been (successfully) abusing for the past 80 years to construct a field action , the trick goes as follows:

1. find a kinetic term , construct a possible scalar function from your field or its derivatives
2. couple the field with a candidate from matter
3. apply euler lagrange
4. go to the lab

the simplest action you can construct from phi is as follows:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \, \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2, \tag{64}$$

now we apply euler lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi)} \right) = 0 \tag{65}$$

For scalar fields, the covariant derivative $\nabla_\mu \phi$ reduces to the ordinary partial derivative $\partial_\mu \phi$, so the equation becomes:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0 \tag{66}$$

- $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$
- $\dfrac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi)} = g^{\mu\nu} \nabla_\nu \phi$
  thus:

$$-m\phi^2 - \nabla_\mu (g^{\mu\nu} \nabla_\nu \phi) = -m\phi^2 \nabla(g)\phi - g\nabla\nabla\phi \qquad = -m\phi^2 - g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \tag{67}$$

$$\implies \nabla^\nu \nabla_\nu \phi + m\phi^2 = 0 \tag{68}$$

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