Instruction: N *Roots Theorem, Non-Real Roots Theorem, & Descartes' Rule of Signs*

The discussion in this section and the remaining sections will rely on the following rule and three theorems.

> ***N* Roots Theorem**: *Every polynomial* P(x) *of degree* n > 0 *can be expressed as the product of* n *linear factors. Hence, P(x) has exactly* n *roots–not necessarily distinct.*

The *N* Roots Theorem says that the degree of the polynomial determines how many linear factors (counting factors containing complex numbers that are not real) and how many roots the polynomial possesses. The roots may not be distinct, meaning the roots can be repeated (that is, there may be a multiplicity of a particular root).

> **Non-Real Roots Theorem:** *If a polynomial with real coefficients has non-real roots, then the non-real roots occur in conjugate pairs*.

Since this course deals only with polynomials with real coefficients, the Non-Real Roots Theorem tells us that complex roots with an imaginary part occur in conjugate pairs. If a polynomial has a complex root $a + bi$ where $b \neq 0$, it will have a complex root $a - bi$.

> **Descartes' Rule of Sign:** *If* P(x) *defines a polynomial function with real coefficients and with terms in descending powers of* x, *the number of positive real roots of* P(x) *either equals the number of variations in sign occurring in the coefficients of* P(x) *or is less than the number of variations by a multiple of two. Moreover, the number of negative real roots of* P(x) *either equals the number of variations in sign occurring in the coefficients of* P(–x) *or is less than the number of variations by a multiple of two*.

Descartes' Rule of Sign tells us, "If a polynomial with a non-zero constant has k sign changes when written in proper descending order, the number of positive roots of the polynomial equals k (the number of sign changes) or $k - 2p$ where p is the number of pairs of non-real roots possessed by the polynomial."

> **Turning Points Theorem:** *Every polynomial* P(x) *of degree* n > 0 *has at most* n – 1 *turning points--that is, the polynomial will change from increasing to decreasing or from decreasing to increasing behavior at most* n – 1 *times.*

The Turning Points Theorem tells us that the graph of a n-degree polynomial will change from one type of behavior to another (from increasing to decreasing or vice versa) at most $n-1$ times.

Instruction: R*udimentary Sketches of Polynomial Functions with All Roots Real*

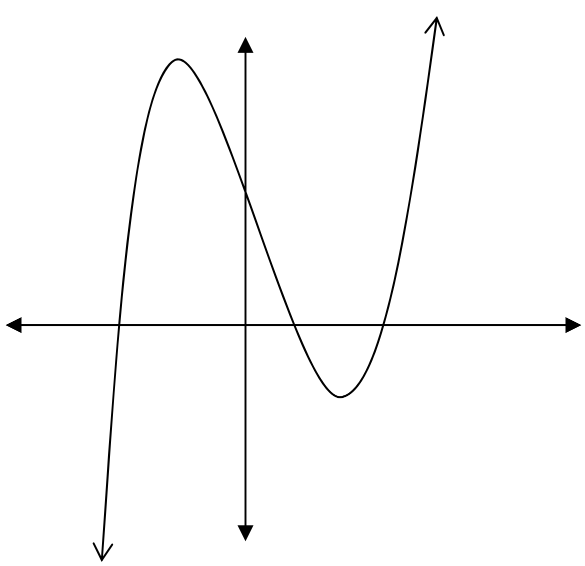
The degree of a polynomial determines the number of roots. Consider $f(x) = x^3 - x^2 - 9x + 9$. Since $f(x)$ is a cubic, it has three roots. Consequently, $f(x)$ will cross the x-axis three times barring repeated roots and non-real roots.

Assuming all roots are real, the number of sign changes in the polynomial (when it is written in descending order) determines the number of positive roots. Since $f(x)$ has two sign changes (from positive x^3 to negative x^2 and from $-9x$ to positive 9), it has two positive real roots. This number indicates the number of positive real roots if all roots are real. If some roots are non-real, the number might be reduced by multiples of two, so normally we would conclude that $f(x)$ has 2 or 0 positive real roots. Since we are told, however, that $f(x)$ has <u>all</u> real roots, we know it has 2 positive real roots and crosses the x-axis twice on the positive side.

To determine the number of negative roots, count the sign changes in $f(-x)$. If the polynomial has a non-zero constant, we can simply subtract the number of positive roots from the total number of roots to quickly obtain a maximum number of negative roots. Note that $f(x)$ has a non-zero constant ($a_0 = 9$). Since $f(x)$ has three roots and two are positive, it has one negative root.

The leading coefficient determines the right end behavior. If the leading coefficient is positive the function increases as the x-values increase. If the leading coefficient is negative, the function decreases as the x-values increase. Since the leading coefficient of $f(x)$ is positive, the function rises on the right.

The degree of the polynomial determines whether the left-end behavior and the right-end behavior will both rise, both fall, or rise on one side while falling on the other. If the degree is even, the polynomial will do the same thing, fall or rise, on both ends. If the degree is odd, the polynomial has opposite behavior on the two ends, one end rising while the other falls. Since $f(x)$ is a third-degree polynomial, it has opposite end behavior. Thus, $f(x)$, which rises on the right, will fall on the left (meaning the function will decrease as the x-values decrease).



Instruction: *Rudimentary Sketches of Polynomial Functions*

Example 1
Determining the Number of Roots

Consider $Y(x) = a_9 x^9 + a_8 x^8 + \cdots + ax + a_0$. I. How many roots does $Y(x)$ possess? II. What is the maximum number of complex roots that $Y(x)$ can possess?

I. Note the degree and quote the *N* Roots Theorem.

$Y(x)$ has a degree of 9, $n = 9$, so it has exactly 9 roots, not necessarily distinct.

II. Note the degree and quote the Complex Roots Theorem.

If complex roots exist, they exist in pairs. $9 \div 2 = 4.5$. $Y(x)$ has at most 4 complex roots.

Example 2
Determining the Number of Turning Points

Consider $H(x) = a_6 x^6 + a_5 x^5 + \cdots + ax + a_0$. What is the maximum number of times that $H(x)$ can change from increasing to decreasing or from decreasing to increasing behavior?

Note the degree and quote the Turning Points Theorem.

$H(x)$ has a degree of 6, $n = 6$, so it has at most 6 – 1 turning points. $H(x)$ changes behavior a maximum number of 5 times.

Example 3
Determining the Number of Complex Roots

Consider $f(x) = x^7 - x^6 - 2x^5 - 4x^3 + 4x^2 + 8x$. Determine the minimum number of complex (non-real) roots.

The function $f(x)$ has a degree of 7, so it has 7 roots. The function has zero as its constant. Since x is the greatest common factor, zero is one of the 7 roots. Noting the sign changes of the coefficients with the polynomial written in expanded form with descending order, we see two sign changes (from a_7 to a_6 and from a_3 to a_2). By Descartes' Rule of Signs, the polynomial has at most two positive roots. Examining $f(-x) = -x^7 - x^6 + 2x^5 + 4x^3 + 4x^2 - 8x$ we note two sign changes. By Descartes' Rule of Signs, the polynomial has at most two negative roots. Since $f(x)$ has at most 2 positive roots, at most 2 negative roots, and exactly 1 root at zero, we know that $f(x)$ must have at least two complex roots (7 - 5 = 2).

Example 4
Using Descartes' Rule of Signs

Consider $D(x) = 2x^4 - 4x^5 + 7x^3 - 3x^2 + 5x + 1$. What is the maximum number and minimum number of positive and negative roots of $D(x)$?

Rewrite the polynomial in descending order.

$$D(x) = -4x^5 + 2x^4 + 7x^3 - 3x^2 + 5x + 1$$

Count the changes in sign from one coefficient to the next moving from left to right. The polynomial has three sign changes: from –4 to 2, from 7 to –3, and from –3 to 5. According to Descartes' Rule of Signs, the polynomial has at most three positive roots. Since complex roots come in pairs, the polynomial may have only one positive root (with the presence of complex roots).

Maximum number of positive roots of $D(x)$: 3
Minimum number of positive roots of $D(x)$: 1

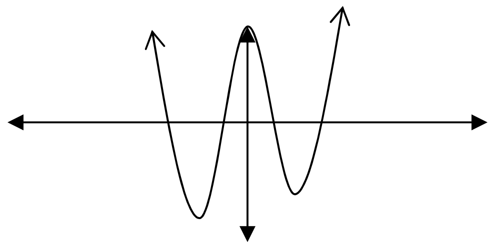
Since $D(x)$ is a 5-degree polynomial, it has 5 roots. Since $D(x)$ has at most 5 sign changes, it will have at most 2 negative roots (because 5 – 3 = 2) and as few as zero negative roots (with the presence of complex roots).

Maximum number of negative roots of $D(x)$: 2
Minimum number of negative roots of $D(x)$: 0

Example 5
Using Rules for End Behavior

Consider $E(x) = 9x^4 + \cdots + 3$. Show the proper end behavior with a sketch of the graph of $E(x)$ assuming all the roots are real and distinct (meaning the graph crosses the x-axis a number of times equal to its degree).

Use the rules for end behavior to determine whether the graph rises or falls on the right and left. The leading coefficient, 9, is positive, so the graph rises on the right. The degree, $n = 4$, is even, so the graph has the same end-behavior on the left, so it will rise on the left. Since the polynomial is a four-degree polynomial, it will cross the x-axis four times assuming all roots are real and distinct. Sketch the polynomial rising at both ends with four x-intercepts.



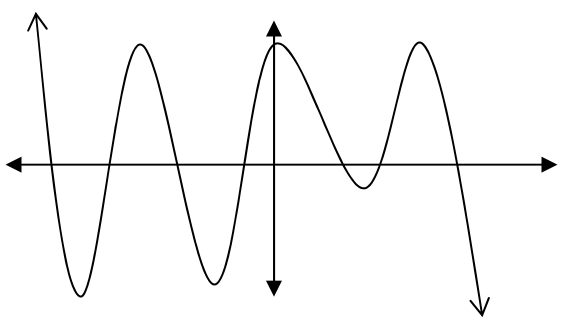
Example 6
Sketch the Graph Assuming all Roots are Real and Distinct

Consider $S(x) = a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. If $a_7 < 0$, $a_3 < 0$, and all the other coefficients (a_6, a_5, a_4, a_2, a_1) as well as the constant (a_0) are greater than zero, sketch a graph of $S(x)$ showing the proper end behavior and the proper number of positive roots, negative roots assuming that all the roots of $S(x)$ are real and distinct.

Use Descartes' Rule of Signs to determine the number of positive roots. $S(x)$ has three sign changes (from a_7 to a_6, from a_4 to a_3, and from a_3 to a_2), so it has three positive roots (assuming all roots are real), which means it crosses the x-axis at three different x-values (assuming all roots are distinct).

Determine the number of negative roots. The polynomial is a 7-degree polynomial, which means it has 7 roots. If three roots are positive and the constant is not zero, then four roots must be negative (assuming all roots are real), which means the polynomial crosses the x-axis at four different x-values (assuming all roots are distinct.)

Use the rules for end behavior to determine whether the graph rises or falls on the right and left. The leading coefficient, a_7, is negative, so the graph falls on the right. The degree, n = 7, is odd, so the graph does the opposite on the left, so it will rise on the left.



Example 7
Determining the Number of Complex Roots

Consider $\vartheta(x) = x^4 + x^2 + 5$. Determine the number of complex (non-real) roots of $\vartheta(x)$.

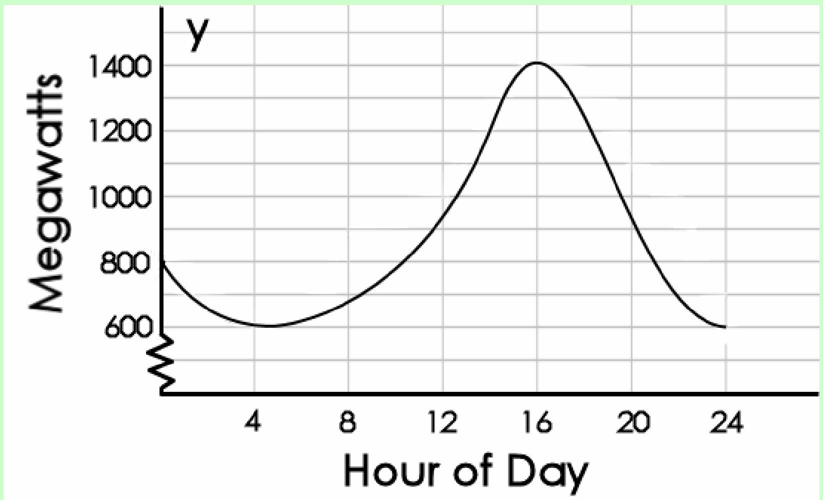
The function $\vartheta(x)$ has a degree of 4, so it has 4 roots, but it does not have any sign changes, so none of the roots are positive. Examining $\vartheta(-x) = x^4 + x^2 + 5$ we note that there are no negative roots. The constant is not zero, so zero is not a root. Hence, all of the roots must be complex. The function $\vartheta(x)$ has 4 complex roots.

Suggested Homework from Blitzer

Section 3.2: #41-61 odd
Section 3.3: #33-37 odd

Application Exercise

The graph shows the megawatts of electricity used on a given day by residents and businesses of a particular town.



What is the minimum degree of the polynomial that could be used to model this information? Why?

Practice Problems

In the following polynomial functions, letters of the English alphabet represent the coefficients. **Consider consonants as positive integers and vowels as negative integers**, and determine the number of positive, real roots for each polynomial function according to Descartes' rule of signs.

#1 $f(x) = ax^3 + ex^2 + fx + g$

#2 $h(x) = rx^3 + sx^2 + ux + v$

#3 $p(x) = ax^8 + cx^7 + ex^6 + gx^5 + ix^4 + mx^3 + ox^2 + sx + u$

Sketch the graphs of the following polynomial functions. Assume all roots are real and unique (not repeated). Making this assumption, a fifth-degree polynomial would cross the *x*-axis five times. Graphs should show the correct end behavior, the correct number of positive and negative roots, and the correct number of total roots **making the assumption that *p* represents a positive coefficient and *n* represents a negative coefficient**.

#4 $Y(x) = nx^6 + px^5 + nx^4 + nx^3 + px^2 + px + n$

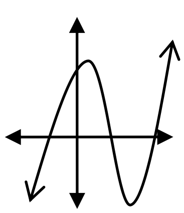
#5 $f(x) = px^3 + nx^2 + px + p$

#6 $g(x) = nx^2 + px + p$

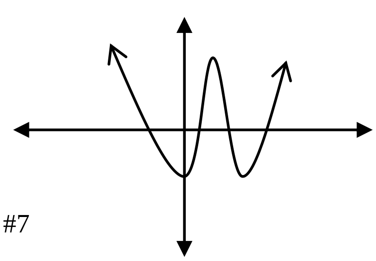
#7 $F(x) = px^4 + nx^3 + nx^2 + px + n$

#1 one

#3 eight or six or four or two or none



#5



#7