On Non-Singular Fractional Derivatives

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List of Symbols

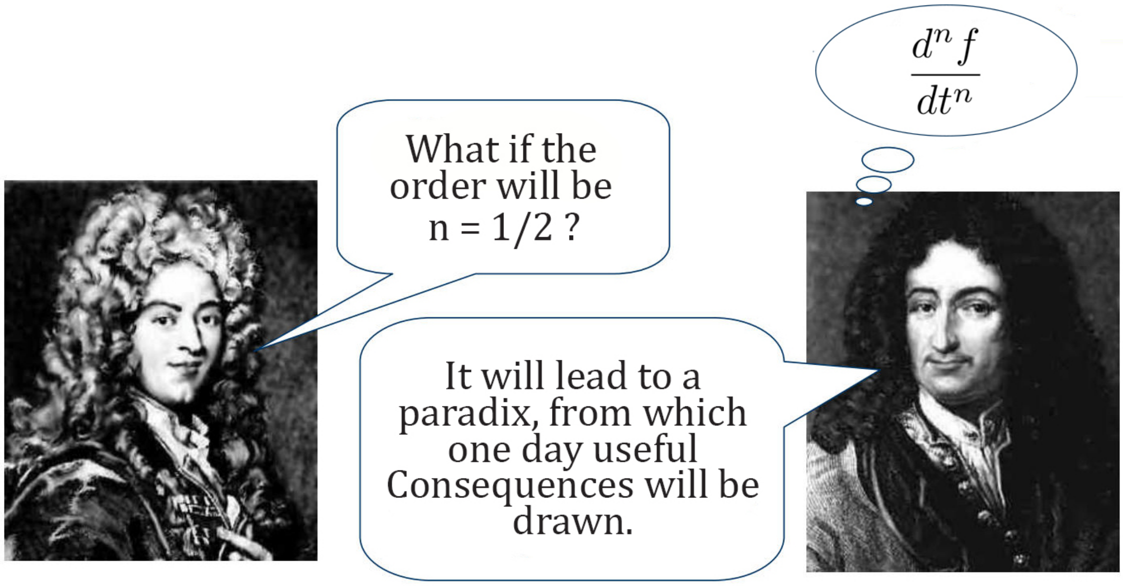
$I^{\alpha}_{a+}f(x) \equiv_a I^{\alpha}_x$	Integral by Riemann-Liouville Left sided
$I^{\alpha}_{b-}f(x) \equiv_x I^{\alpha}_b$	Integral by Riemann-Liouville Right sided
${}^{CF}\mathcal{D}$	New Derivative by Caputo
${}^{ABC}\mathcal{D}$	Derivative by Atangana-Baleanu in Caputo sence
${}^{ABR}\mathcal{D}$	Derivative by Atangana-Baleanu in Riemann sence
$\mathfrak{L}(f(t))$	Laplace transform
$L^p[A]$	Space of all integrable functions such that $(\int_A \lvert f(\xi)\rvert^p d\xi)^{\frac{1}{p}} < \infty$
$H^1(a,b)$	Sobolev space of order 1
$\Gamma(n)$	Gamma function
$E_{\alpha,\beta}(t)$	Mittag-Leffler function

Chapter 1

Introduction

The original question that led to the name *"fractional calculus"* was: Can the meaning of a derivative of integer order $\frac{d^n y}{dx^n}$ be extended to have meaning when n is a fraction? Later the question became: Can n be any number: fractional, irrational, or complex?
Because the latter question was answered affirmatively, the name fractional calculus has become a misnomer and might better be called integration and differentiation to an arbitrary order.

Leibniz invented the notation $\frac{d^n y}{dx^n}$. Perhaps it was a naive play with symbols that prompted **L'Hopital** in 1695 to ask **Leibniz**: "What if n be $\frac{1}{2}$? " Leibniz replied: "You can see by that, sir , that one can be express by an infinite series a quantity such as $d^{\frac{1}{2}}xy$ or $d^{1:2}xy$. Although infinite series and geometry are distant relations, infinite series admits only the use of exponents that are positive and negative integers, and does not, as yet, know the use of fractional exponents."
Later, in the same letter. **Leibniz** continues prophetically: "Thus it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx : x}$. This is as apparent paradox from which, one day, useful consequences will be drawn."



G.F.A. de L'Hôpital (1661-1704)

G.W. Leibniz (1646-1716)

In his correspondence with Johann Bernoulli, Leibniz mentions derivative of "general order." In Leibniz's correspondence with John Wallis, in which Wallis's infinite product for $\frac{1}{2}\pi$ is discussed, Leibniz states that differential calculus might have been used to achieve this result. He uses the notation $d^{1/2}y$ to denote the derivative of order $\frac{1}{2}$.

The subject of fractional calculus did not escape Euler's attention.
In 1730 he wrote: "When n is positive integer, and if p should be a function of x, the ratio $d^n p$ to dx^n can always be expressed algebraically, so that if $n = 2$ and $p = x^3$, then d^2x^3 to dx^2 is $6x$ to 1.
Now it is asked what kind of ratio can then be made if n be a fraction. The difficulty in this case can easily be understood For if n is a positive integer d^n can be found by continued differentiation. Such a way, however, is not evident if n is a fraction. But yet with the help of interpolation which i have already explained in this dissertation, one may br able to expedite the matter"
J. L. Lagrange contributed to fractional calculus indirectly.
In 1772 he developed the law of exponents (indices) for differential operators of integer order and wote:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}.\frac{\mathrm{d}^n}{\mathrm{d}x^n}y = \frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}}y$$

In modern notation the dot is omitted, for it is not a multiplication.
Later, when the theory of fractional calculus developed, mathematicians were interested in knowing what restrictions has to be imposed on $y(x)$ so that an analogous rule held true for m and n arbitrary.
In 1812 P. S. Laplace defined a fractional derivative of arbitrary order appears in a text. S. F. Lacroix devoted less than two pages of his 700-pages text to this topic. He developed a mere mathematical exercise generalizing from a case of integer order. starting with $y = x^m$, m a positive integer, Lacroix easily develops the nth derivative

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n$$

Using Legendre's symbol for the generalized fractional (the gamma function), he generalizations

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \tag{1.1}$$

He then gives the example for $y = x$ and $n = \frac{1}{2}$, and obtains

$$\frac{\mathrm{d}^{1/2} y}{\mathrm{d}x^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

Joseph B. J. Fouriier was the next to mention derivatives of arbitrary order. His definition of fractional operations was obtained from his integral represen-tation of $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\alpha) d\alpha \int_{-\infty}^{\infty} \cos\left(p(x-\alpha)\right) dp$$

Now

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} cos(p(x-\alpha)) = p^n cos[p(x-\alpha) + \frac{1}{2} n\pi]$$

for n an integer. Formally replacing n with u (u arbitrary), he obtains the generalization

$$\frac{\mathrm{d}^u}{\mathrm{d}x^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\alpha) d\alpha \int_{-\infty}^{\infty} p^u cos[p(x-\alpha) + \frac{1}{2} u\pi] dp$$

Fourier states: "The number u that appears in the above will be regarded as any quantity whatsever, positive of negative."

Leibniz, Euler, Laplace, Lacroix and Fourier made mention of derivatives of arbitrary order, but the first use of fractional operations was made by Niels Henrik Abel in 1823. Abel applied the fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone porblem [1]. If the time of slide is a known constant, then Abel's integral equation is

$$k = \int_0^x (x-t)^{-\frac{1}{2}} f(t) dt \tag{1.2}$$

The integral in (1.2), except for the multiplicative factor $\frac{1}{\Gamma(\frac{1}{2})}$, is a particular case of a definite integral that defines fractional integration of order $\frac{1}{2}$. In integral equations such as (1.2), the function f in the integrand is unknown and is to be determined. Abel wrote the right-hand side of (1.2) as

$$\sqrt{\pi}[d^{-1/2}/dx^{-1/2}]f(x)$$

Then he operatoed on both sides of the equation with $d^{1/2}/dx^{1/2}$ to obatin

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} k = \sqrt{\pi} f(x) \tag{1.3}$$

Because there fractional operators (with suitable condition of f) have the property that $D^{1/2}D^{-1/2}f = D^0 f = f$. Thus when the fractional derivative of order $\frac{1}{2}$ of the constant k in (1.3) is computed, $f(x)$ is determined. This is a remarkable achievement of abel in the fractional calculus. It is important to note that the fractional derivative of a constant is not always equal to zero. It is this curious fact that lies at the center of a mathematical controversy to be discussed shortly.

mathematicians have described Abel's solution as "elegant." Perhaps it was Fourier's integral formula and Abel's solution that has attracted the attention of Liouville, who made the first major study of fractional calculus. He publis-

hed three long memoris in 1832 and several more publications to problems in potential theory

The starting point for his theoretical development was the known result for

[1] The problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve under the action of gravity is independent of the starting point

derivatives of integral order:

$$D^m e^{ax} = a^m e^{ax}$$

which he extended in a natural way to derivatives of arbitrary order

$$D^\nu e^{ax} = a^\nu e^{ax}$$

He assumed that the arbitrary derivative of function $f(x)$ that may be extended in a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \quad \Re\mathrm{e}(a_n) > 0 \tag{1.4}$$

is

$$\mathcal{D}^\nu f(x) = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n \nu} \tag{1.5}$$

Formula (1.5) is known as Liouville's first formula for a fractional derivative. It generalizes in a natural way a derivative of arbitrary order ν, where ν is any number: rational, irrational, or complex. But it has the obvious disadvantage of being applicable only to functions of he formulated a second definition. To obtain his second definition he strated with a definite integral related to the gamma function:

$$\mathfrak{I} = \int_0^\infty u^{a-1} e^{-xu}\, du \quad a > 0 \; x > 0$$

The change of variable $xu = t$ yields

$$\begin{aligned} \mathfrak{I} &= x^{-a} \int_0^\infty t^{a-1} e^{-t}\, dt \\ &= x^{-a} \Gamma(a) \end{aligned}$$

Or

$$x^{-a} = \frac{1}{\Gamma(a)} \mathfrak{I}$$

Then Liouvlle operators with $\mathcal{D}^\nu$ on both sides of the equation above, to obtain, according to Liouville's basic assumption

$$\mathcal{D}^\nu x^{-a} = \frac{(-1)^\nu}{\Gamma(a)} \int_0^\infty u^{a+\nu-1} e^{-xu}\, du$$

Thus Liouville obtains his second definition of a fractionla derivvative:

$$\mathcal{D}^{\nu} x^{-a} = \frac{(-1)^{\nu}\gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}, \quad a > 0 \tag{1.6}$$

But Liouville's definitions were too narrow to last. The first definition os restricted to functions of the class (1.4), and the second definiton os useful only for functions of the type x^{-a} (with $a > 0$). Neither is suitable to be applied to a wide class of functions.

William Center observed that the fractional derivative of a constant, according to the Lacroix-Peacock method, is unequal to zero. Using x^0 to denote unity, Center finds the fractional derivative of unity of irder $\frac{1}{2}$, by letting $m = 0$ and $n = \frac{1}{2}$ in (1.1) (even thought Lacroix assumed that $m \geq n$) to obtain

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} x^0 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} x^{-1/2} = \frac{1}{\sqrt{\pi x}} \tag{1.7}$$

But as Center points out, according to Liouville's "system"[2], by letting $a = 0$ (even though Liouville assumed that $a > 0$), the fractional derivative ofunity equals to zero because $\Gamma(0) = \infty$. He continues: "The whole question in plainly reduces to what is $\frac{d^u x^0}{dx^u}$. For when this is determined we shall determine at the same time which is the correct system "
Augustus De Morgan devoted three pages to fractional calculus: "Both these systems may very possible be parts of a more general system, but at present I incline to the conclusion that neither system has any claim to be considerer as giving the form $D^n x^m$, though eother may be a form."

The state of affairs complained about by De Morgan and Center is now thoroughly cleared up. De Morgan's judgment proved to be correct, for the two systems that Center thought led to irreconcilable results have now been incroporated into a more general system. It is only fair to state that mathematicians at that time were aiming for a plausible defintion of generalized integration and differentiation.

Liouville and Later C. J. Hargreave wrote on the generalixation of Leibniz'

[2] referring to Liouville's second defintion goven in formaule (1.6)

nth derivative of a product when n is not a positive integr. In modern form

$$\mathcal{D}^{\nu} f(x)g(x) = \sum_{n=0}^{\infty} \binom{\nu}{n} \mathcal{D}^{n} f(x) \mathcal{D}^{\nu-n} g(x)$$

Where $\mathcal{D}^n$ is the ordinary differentiation operator of order n, $\mathcal{D}^{\nu-n}$ a fractional operator, and $\binom{\nu}{n}$ the generalized bionomial coefficient $\frac{\Gamma(\nu+1)}{n!\Gamma(\nu-n+1)}$. The general-

ized Leibniz rule may be found in many modern applications H. R. Greer wrote on finite differences of fractional order. Surprisingly, a recent access to a fractional derivative is by means of finite differences. Mention should also be made of a paper by W. Zachartchenxo. He imporoves on the work of Greer, and he ends his paper with an amusing note, which no modern amthematiciam would admit. Concerning his research on a topic: "I konw that Liouville, peacock and Kelland have written on this topci, but i have had no opportunity to read their works." H. Holmgren wrote a long monograph on the application of fractional calculus to the solution of cartain ordinary differential equations. In the introduction to this work as his point of departure, states that his aim in this paper is to find a complete solution not subject to the restrictions on the independent variable that his predecessors has made. He proceeds along formal lines. For example, the index law is used:

$$\mathcal{D}^{\nu} y'' = \mathcal{D}^{\nu} \mathcal{D}^{2} y = \mathcal{D}^{\nu+2} y$$

Although his rule is valid for ν a positive integer, modern mathematicians would seek to justify this rule when ν is arbitrary

Finally, After many developments, By the second half of the twentieth century, the field of fractional calculus had grown to such extent that in 1974 the first conference “The First Conference on Fractional Calculus and its Applications” concerned solely with the theory and applications of fractional calculus was held in New Haven. In the same year, the first book on fractional calculus by Oldham and Spanier was published after a joint collaboration started in 1968

A number of additional books have appeared since then, for example McBride (1979), Nishimoto (1991) , Miller and Ross (1993), etc

In 1998 the first issue of the mathematical journal “Fractional calculus and applied analysis” was printed. This journal is solely concerned with topics on the theory of fractional calculus and its applications. Finally, in 2004 the first conference “Fractional differentiation and its applications” was held in Bordeaux, and it is organized every second year since 2004.

This research is organized as follows.
Chapter 2 on imporant Preliminaries, Chapter 3 Review of fractional derivatives definitions, chapter 4 Disscusion on fractional derivatives with non-singular (bounded) kernels ,their definiton ,propoerties and drawbacks, and finally chapter 5 Introduction to Fractional Differential Equations with non-singular kernel derivatives and on Existence and uniquesness theorems

Chapter 2

Preliminaries

Special functions

Gamma Function

We start by considering the Gamma function which is denoted by $\Gamma(.)$ is defined as:

$$\Gamma(n) := \int_0^\infty e^{-t} t^{n-1} dt$$

Properties of Gamma function

Function $\Gamma(n)$ is convergent for $n > 0$ and consequently is divergent for $n \leq 0$
Gamma function obeys the property:

$$\Gamma(n+1) = n\Gamma(n)$$

For Integer inputs of this function it could be treated as usual factorial function

$$\Gamma(n+1) = n! \quad n \text{ is integer}$$

Mittag-Leffler Function

In this section we introduce the one- and two-parameter Mittag-Leffler functions, denoted as $E_\alpha(.)$ and $E_{\alpha,\beta}(.)$ respectilvely.

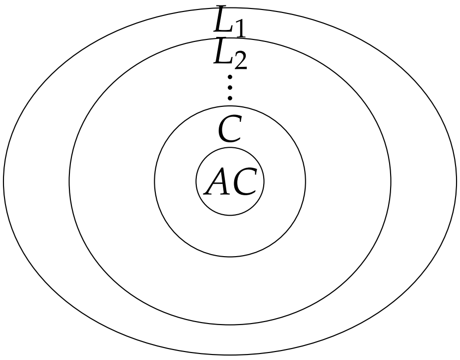
$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad \Re\mathrm{e}(\alpha) > 0$$

$$E_{(\alpha,\beta)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \quad \Re e(\alpha) > 0, \Re e(\beta) > 0$$

One of important properties is that

$$\mathfrak{L}\{E_\alpha\left(-\lambda t^\alpha\right)\} = \frac{p^{\alpha-1}}{s^\alpha + \frac{\alpha}{1-\alpha}}$$

L_p Spaces



We introduce some important definitions

Definition 1. L_p **Spaces** Let $\mathfrak{u} \in \Omega_L$. The $L_p(\mathfrak{u}), p \geq 1$ space consisting of all measurable functions f, with the property that

$$\left(\int_{\mathfrak{u}} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty, p \in [1,\infty) \text{ and } ess\ sup_{x\in\mathfrak{u}}|f(x)| < \infty, p = \infty$$

Definition 2. **Absolute Continuous functions** By $AC^n[\Omega], n \in \mathbb{N} := \{1,2,3,\ldots\}$, we denotes the space of functions f defined on $[0,b]$ which have continuous derivatives up to order $n-1$ on $[0,b]$ with $f^{(n-1)}$ is absolutely continuous $[\Omega]$

$$AC^n(\Omega) = \{f : f^{(i)} \in C(\Omega) \forall i = 1,2,\ldots,(n-1) \text{and} f^{(n-1)}(x) \in AC(\Omega)\}$$

Here is an important theorem about Absolute continuous functions

Theorems 1.
If $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then f is differentiable a.e. in $[a,b]$, $f' \in L^1(a,b)$, and

$$f(x) = f(a) + \int_a^x f'(t)dt \qquad \forall x \in [a,b]$$

Definition 3. Sobolev Space of order 1
The sobolev space is defined easily as follows. The sobolev space of order 1 denoted by H^1

$$H^1(a,b) = \{u \in L^2(a,b) \,|\, u' \in L^2(a,b)\}$$

Chapter 3

How Many Fractional Derivatives Are There ?

In this chapter we consider the most important definitions of fractional derivatives [10,18]

- Liouville derivative

$$\mathcal{D}^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x}(x-\xi)^{-\alpha}f(\xi)d\xi\,, \quad -\infty < x < +\infty \tag{3.1}$$

- Liouville left-sided derivative

$$\mathcal{D}^{\alpha}_{0+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{0}^{x}(x-\xi)^{-\alpha+n-1}f(\xi)d\xi\ , x > 0 \tag{3.2}$$

- Liouville right-sided derivative

$$\mathcal{D}^{\alpha}_{-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{\infty}(x-\xi)^{-\alpha+n-1}f(\xi)d\xi\ , x < \infty \tag{3.3}$$

- Riemann-Liouville left-sided derivative

$$^{Rl}\mathcal{D}^{\alpha}_{a+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{a}^{x}(x-\xi)^{n-\alpha+1}f(\xi)d\xi\ , x \geq a \tag{3.4}$$

- Riemann-Liouville right-sided derivative

$$^{Rl}\mathcal{D}^{\alpha}_{b-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{b}(\xi-x)^{n-\alpha+1}f(\xi)d\xi\ , x \leq b \tag{3.5}$$

- Caputo left-sided derivative

$$_{*}\mathcal{D}^{\alpha}_{a+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\xi)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]d\xi\ , x \geq a \tag{3.6}$$

- Caputo right-sided derivative

$$ {}_{*}\mathcal{D}^{\alpha}_{b-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (\xi - x)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]d\xi \text{ }, x \leq b \qquad (3.7) $$

- Grünwald-Letnikov left-sided derivative

$$ {}^{Gl}\mathcal{D}^{\alpha}_{a+}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \text{ }, nh = x - a \qquad (3.8) $$

- Grünwald-Letnikov right-sided derivative

$$ {}^{Gl}\mathcal{D}^{\alpha}_{b-}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)f(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \text{ }, nh = b - x \qquad (3.9) $$

- Weyl derivative

$$ {}_{x}\mathcal{D}^{\alpha}_{\infty}[f(x)] = \mathcal{D}^{\alpha}_{-}[f(x)] = (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n [{}_{x}W^{\alpha}_{\infty}[f(x)]] \qquad (3.10) $$

- Marchaud derivative

$$ \mathcal{D}^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}} d\xi \qquad (3.11) $$

- Marchaud left-sided derivative

$$ \mathcal{D}^{\alpha}_{+}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x-\xi)}{\xi^{1+\alpha}} d\xi \qquad (3.12) $$

- Marchaud right-sided derivative

$$ \mathcal{D}^{\alpha}_{-}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x+\xi)}{\xi^{1+\alpha}} d\xi \qquad (3.13) $$

- Hadamard derivative

$$ \mathcal{D}^{\alpha}_{+} = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{f(x) - f(\xi)}{[\ln(\frac{x}{\xi})]^{1+\alpha}} \frac{d\xi}{\xi} \qquad (3.14) $$

- Chen left-sided derivative

$$ \mathcal{D}^{\alpha}_{c}[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{c}^{x} (x-\xi)^{-\alpha} f(\xi) d\xi \text{ }, x > c \qquad (3.15) $$

- Chen right-sided derivative

$$\mathcal{D}_c^{\alpha}[f(x)] = \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^c (\xi - x)^{-\alpha} f(\xi) d\xi \ , x < c \tag{3.16}$$

- Davidson-Essex derivative

$$\mathcal{D}_0^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}^{n+1-k}}{\mathrm{d}x^{n+1-k}} \times \int_0^x (x-\xi)^{\alpha} \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}[f(\xi)] d\xi \tag{3.17}$$

- Coimbra derivative

$$\mathcal{D}_0^{\alpha(x)}[f(x)] = \frac{1}{\Gamma(1-\alpha(x))} \times \left(\int_0^x (x-\xi)^{-\alpha(x)} \frac{\mathrm{d}}{\mathrm{d}\xi}[f(x)] d\xi + f(0) x^{-\alpha(x)} \right) \tag{3.18}$$

- Canavati derivative

$${}_a\mathcal{D}_x^{\nu}[f(x)] = \frac{1}{\Gamma(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-\xi)^{\mu} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)] d\xi \ , n = \lfloor \nu \rfloor \ , \mu = n - \nu \tag{3.19}$$

- Jumarie derivative , $n = 1$

$$\mathcal{D}_x^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \times \int_0^x (x-\xi)^{n-\alpha-1} [f(\xi) - f(0)] d\xi \tag{3.20}$$

- Riesz derivative

$$\begin{aligned} \mathcal{D}_x^{\alpha}[f(x)] = &- \frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)} \frac{1}{\Gamma(\alpha)} \\ &\times \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\int_{-\infty}^x (x-\xi)^{n-\alpha-1} f(\xi) d\xi + \int_x^{\infty} (\xi - x)^{n-\alpha-1} f(\xi) d\xi \right) \end{aligned} \tag{3.21}$$

- Cossar derivative

$$\mathcal{D}_{-}^{\alpha}[f(x)] = -\frac{1}{\Gamma(1-\alpha)} \lim_{N\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^N (\xi - x)^{-\alpha} f(\xi) d\xi \tag{3.22}$$

- Local fractional Yang derivative

$$\mathcal{D}_{-}^{\alpha}[f(x)]|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^{\alpha}[f(x) - f(x_0)]}{(x-x_0)^{\alpha}} \tag{3.23}$$

- Left Riemann-Liouville derivative of variable fractional order

$$ {}_a\mathcal{D}_x^{\alpha(.,.)}[f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x (x-\xi)^{-\alpha(\xi,x)} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} \tag{3.24} $$

- Right Riemann-Liouville derivative of variable fractional order

$$ {}_x\mathcal{D}_b^{\alpha(.,.)}[f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}\int_x^b (\xi-x)^{-\alpha(\xi,x)} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} \tag{3.25} $$

- Left Caputo derivative of variable fractional order

$$ {}_a\mathcal{D}_x^{\alpha(.,.)}[f(x)] = \int_a^x (x-\xi)^{-\alpha(\xi,x)} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} \tag{3.26} $$

- Right Caputo derivative of variable fractional order

$$ {}_x\mathcal{D}_b^{\alpha(.,.)}[f(x)] = \int_x^b (\xi-x)^{-\alpha(\xi,x)} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} \tag{3.27} $$

- Caputo derivative of variable fractional order

$$ {}_*\mathcal{D}_x^{\alpha(x)}[f(x)] = \frac{1}{\Gamma(1-\alpha(x))}\int_0^x (x-\xi)^{-\alpha(\xi,x)} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) d\xi \tag{3.28} $$

- Modified Riemann-Liouville fractional derivative

$$ \mathcal{D}^\alpha[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^\alpha [f(\xi)-f(0)] d\xi \tag{3.29} $$

- Osler fractional derivative

$$ {}_a\mathcal{D}_z^\alpha[f(z)] = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{\mathcal{C}(a,z^+)} \frac{f(z)}{(\xi-z)^{1+\alpha}} d\xi \tag{3.30} $$

- K-fractional Hilfer derivative

$$ {}^k\mathcal{D}^{\mu,\nu} f(x) = I_k^{\nu(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}x} I_k^{(1-\mu)(1-\nu)} f(x) \tag{3.31} $$

- Gohar fractional derivative

$$ G_\alpha f(x) = \lim_{h\to 0}\frac{f\left(x\left[1+\ln\left(1+h\frac{\Gamma(n)}{\Gamma(n-\alpha+1)}\right)x^\alpha\right]\right)-f(x)}{h} \tag{3.32} $$

Chapter 4

Derivatives with non-singular kernels

Now, we introduce a simple discussion on derivatives with non-singular kernels, properties and their drawbacks

4.1 Caputo-Fabrizio Type

Let us recall the usual Caputo fractional derivative of order α, given by

$$ {}^{C}\mathcal{D}^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\xi)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) d\xi \tag{4.1} $$

with $\alpha \in [0,1)$ and $a \in (-\infty, t)\,, f \in H^1(a,b)\,, b > a$. By changing the kernel $(t-\xi)^{-\alpha}$ with the function $\exp\left(-\frac{\alpha}{1-\alpha}(t-\xi)\right)$ and $\frac{1}{\Gamma(1-\alpha)}$ by $\frac{M(\alpha)}{1-\alpha}$, we obtain the following new definition of fractional derivative

Definition 4.
Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$, then, the Caputo-Fabrizio derivative with order α is defined as:

$$ {}^{CF}\mathcal{D}^{\alpha}_{a+}f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha(t-\xi)}{1-\alpha}\right) \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} d\xi \tag{4.2} $$

where $M(\alpha)$ is a normalization function such taht $M(0) = M(1) = 1$. According to the new definition (4.2) The new fractional derivative is zero when $f(t)$ is constant, as in the usual definiton (4.1), but contrary to the usual definition, the kernel does not have singularity for $t = \xi$
The new definition can also be applied to functions that do not belong to

$H^1(a,b)$. Indeed, the definition (4.2) can be formulated also for $f \in L^1(a,b)$

$$ {}^{CF}\mathcal{D}^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(\xi)) \exp\left(-\frac{\alpha(t-\xi)}{1-\alpha}\right) d\xi \tag{4.3} $$

Now, it is worth to observe that if we put

$$ \sigma = \frac{1-\alpha}{\alpha} \in [0,\infty] \ , \ \alpha = \frac{1}{1+\sigma} \in [0,1] $$

The definition (4.2) can be written as

$$ {}^{CF}\tilde{\mathcal{D}}^{\sigma} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} \exp\left(-\frac{(t-\xi)}{\sigma}\right) d\xi \tag{4.4} $$

Where $\sigma \in [0,\infty]$ and $N(\sigma)$ is the corresponding normalization term of $M(\alpha)$, such that $N(0) = N(\infty) = 1$. Moreover, because

$$ \lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left(-\frac{(t-\xi)}{\sigma}\right) = \delta(t-\xi) \tag{4.5} $$

and for $\alpha \to 1$, we have $\sigma \to 0$. Then

$$ \begin{aligned} \lim_{\alpha \to 1} {}^{CF}\mathcal{D}^{\alpha} f(t) &= \lim_{\alpha \to 1} \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} \exp\left(-\frac{\alpha(t-\xi)}{1-\alpha}\right) d\xi \\ &= \lim_{\sigma \to 0} \frac{N(\sigma)}{\sigma} \int_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} \exp\left(-\frac{(t-\xi)}{\sigma}\right) d\xi = \frac{\mathrm{d}f(t)}{\mathrm{d}t} \end{aligned} $$

Otherwise, when $\alpha \to 0$, then $\sigma \to +\infty$. Hence,

$$ \begin{aligned} \lim_{\alpha \to 0} {}^{CF}\mathcal{D}^{\alpha} f(t) &= \lim_{\alpha \to 0} \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} \exp\left(-\frac{\alpha(t-\xi)}{1-\alpha}\right) d\xi \\ &= \lim_{\sigma \to +\infty} \frac{N(\sigma)}{\sigma} \int_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} \exp\left(-\frac{(t-\xi)}{\sigma}\right) d\xi = f(t) - f(a) \end{aligned} $$

Remark 1.

$$ \int_{-\infty}^{\infty} f(\theta)\delta(t-\theta) d\theta = f(t) $$

for any $f(\theta)$ continuous at t

Also, If $n \geq 1$, and $\alpha \in [0,1]$ the fractional derivative $\mathcal{D}^{(\alpha+n)} f(t)$ of order

$(n + \alpha)$ is defined by

$$\mathcal{D}^{(\alpha+n)} f(t) := \mathcal{D}^{\alpha}(\mathcal{D}^{n} f(t)) \tag{4.6}$$

Theorems 2.

For new fractional derivative, if the function $f(t)$ is such that

$$f^{(s)}(a) = 0, \qquad s = 1, 2, \ldots, n$$

Then we have

$$\mathcal{D}^{n}(\mathcal{D}^{\alpha} f(t)) = \mathcal{D}^{\alpha}(\mathcal{D}^{n} f(t)) \tag{4.7}$$

Proof. We bagin considering $n = 1$, then from defintion (4.6) of $\mathcal{D}^{\alpha+1} f(t)$ we obtain

$$\mathcal{D}^{\alpha}(\mathcal{D}^{1} f(t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t \ddot{f}(\xi) \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] d\xi \tag{4.8}$$

Hence, after an integration by parts and assuming $f'(a) = 0$, we hhave

$$\begin{aligned}
\mathcal{D}^{\alpha}(\mathcal{D}^{1} f(t)) &= \frac{M(\alpha)}{1-\alpha} \int_a^t \ddot{f}(\xi) \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] d\xi \\
&= \frac{M(\alpha)}{1-\alpha} \left[\exp\left(-\frac{\alpha(t-\xi)}{1-\alpha}\right) \dot{f}(t)|_a^t + \frac{\alpha}{1-\alpha} \int_a^t (\dot{f}(\xi) \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] d\xi\right] \\
&= \frac{M(\alpha)}{1-\alpha} \left[\dot{f}(t) + \frac{\alpha}{1-\alpha} \mathcal{D}^{\alpha} f(t)\right]
\end{aligned}$$

Similarly

$$\mathcal{D}^{1}(\mathcal{D}^{\alpha} f(t)) = \frac{M(\alpha)}{1-\alpha} \left[\dot{f}(t) + \frac{\alpha}{1-\alpha} \mathcal{D}^{\alpha} f(t)\right]$$

It is easy to generalize the proof for any $n > 1$ □

Now after the introduction of a new derivative, the associate anti-derivative becomes important, the associated integral of the new Caputo derivative with fractional order was proposed by Losada and Nieto

Definition 5.
Let $0 < \alpha < 1$. the fractional integral of order α of a function f is defined By

$$\mathcal{I}^{\alpha} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(s)ds, \ t \geq 0 \tag{4.9}$$

On the other hand, we discuss Caputo-Fabrizio in Riemann-Liouville Sense

Definition 6.
Let f be a function not necessarily differential, Let α be a real number such that $0 \leq \alpha \leq 1$, then the Caputo-Fabrizio derivative in Riemann sense with order α is given as:

$$^{CR}\mathcal{D}^{\alpha}_{a+} f(t) = \frac{1}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(\xi) \exp\left[-\alpha \frac{t-\xi}{1-\alpha}\right] d\xi \tag{4.10}$$

If α is zero, we have the following

$$^{CR}\mathcal{D}^{0} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(\xi) d\xi = f(t)$$

Using the argument by Caputo and Fabrizio, we also have that when $\alpha \to 1$, we recover the first derivative.

The last two definitions Caputo Fabrizio and Caputo in Riemann sense can be connected as follows

Theorems 3.
The Caputo-Fabrizio derivative with fractional order is connected to the CR derivative as follows:

$$\theta(\alpha)^{CF}\mathcal{D}^{\alpha}(f(t)) = \theta(\alpha)^{CR}\mathcal{D}^{\alpha}(f(t)) + f(0)\exp(-f(\alpha)x) \tag{4.11}$$

Where

$$\theta(\alpha) = \frac{1-\alpha}{M(\alpha)} \qquad f(\alpha) = \frac{\alpha}{1-\alpha}$$

Proof. By definition, we have the following:

$$\begin{aligned}
\theta(\alpha)^{CF}\mathcal{D}^{\alpha}(f(t)) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^t h(y)\exp(-f(\alpha)(t-y))dy \\
&= h(t) - f(\alpha)\int_0^t h(y)\exp(-f(\alpha)(t-y))dy \\
&= h(y) - f(\alpha)\left[\frac{h(t)}{f(\alpha)} - \frac{h(0)}{f(\alpha)}\exp(-f(\alpha)t)\right. \\
&\left. -\frac{1}{f(\alpha)}\int_0^t \exp(-f(\alpha)(t-y))dy\right] \\
&= \frac{h(0)}{f(\alpha)}\exp(-f(\alpha)t) + \int_0^t \frac{\mathrm{d}h(y)}{\mathrm{d}t}\exp(-f(\alpha)(t-y))dy \\
&= \theta(\alpha)^{CR}\mathcal{D}^{\alpha}f(t) + f(0)\exp(-f(\alpha)x)
\end{aligned}$$

This completes the proof. □

4.2 Atangana-Baleanu

The derivative introduced by Caputo cannot produce the original function when α is zero. How ever this issue was, so far, independently solved by Atangana with Goufo and Caputo with Fabrizio, Respectively
Some issues [9] were pointed out against the Caputo-Fabrizio derivative with fractional order:

- The kernal used in the Caputo-Fabrizio derivative is Local
- The anti-derivative associated to their derivatives is not a fractional integral but the average of the function and its integral
- When the fractional order is zero we do not recover the initial (original) function

It appears that such a used kernel cannot be used for many physical problems. Therefore, for a given data we ask the following question: What is the most accurate kernel which better describes it? Then Atangana and Baleanu suggested a possible answer, presented in the following discussion.

The exponential function is the solution of the following ordinary differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\,y \tag{4.12}$$

However, the Mittag-Leffler function os a solution of the following fractional ordinary differential equation:

$$\mathcal{D}^{\alpha}(y) = a\,y \quad 0 < \alpha < 1 \tag{4.13}$$

The mittag-Leffler function and ots generalized versions are therefore consided as non-local functions. let us consider the following generalized Mittag-Leffler function:

$$E_{\alpha}(-t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-t)^{\alpha k}}{\Gamma(\alpha k + 1)} \tag{4.14}$$

The above function has the following properties

$$E_1(-t) = \sum_{k=0}^{\infty} \frac{(-t)^{k}}{\Gamma(k + 1)} = \exp(-t) \tag{4.15}$$

The Taylor series of $\exp(-a(t-y))$ is given by:

$$\exp(-a(t-y)) = \sum_{k=0}^{\infty} \frac{(-a(t-y))^{k}}{k!} \tag{4.16}$$

Chosing $a = \frac{\alpha}{1-\alpha}$ and replacing the above expression into Caputo-Fabrizio derivative, the following formula is obtained:

$${}^{C}\mathcal{D}^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_b^t f'(\xi) \sum_{k=0}^{\infty} \frac{(-a(t-\xi))^{k}}{k!} d\xi \tag{4.17}$$

Rearranging, they obtained

$${}^{C}\mathcal{D}^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-a)^{k}}{k!} \int_b^t f'(\xi)(t-\xi)^{k} d\xi \tag{4.18}$$

To solve the problem of non-locality, they derived the following expression. in equation (4.18) they replaced $k!$ by $\Gamma(\alpha k + 1)$ and $(t-\xi)^k$ by $(t-\xi)^{\alpha k}$ to obatin

$${}^{AB}\mathcal{D}^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-a)^{k}}{\Gamma(\alpha k + 1)} \int_b^t f'(\xi)(t-\xi)^{\alpha k} d\xi \tag{4.19}$$

Then the following derivative proposed

Definition 7.
Let $f \in H^1(a,b)\,, b > a\,, \alpha \in [0,1]$, Then the definition of the Atangana-Baleanu derivative in caputo sense is given as:

$$ {}_{a}^{ABC}\mathcal{D}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} E_\alpha \left[-\alpha \frac{(t-\xi)^\alpha}{1-\alpha} \right] d\xi \tag{4.20} $$

Of course $B(\alpha)$ has the same properties as in Caputo and Fabrizio case. The above definition will be helpful to real-world problems and also will have a great advantage when usgine Laplace Transform to solve some physical problems with initial conditions. However, when α is 0 we do not recover the originl function except when at the origin the function vanished.

To avoid this issue, we propose the following definiton such that researchers in the field of fractional calculus have a choice when dealing with a given problem

Definition 8.
Let $f \in H^1(a,b)\,, b > a\,, \alpha \in [0,1]$, Then the definition of the Atangana-Baleanu derivative in Riemann-Liouville sense is given as:

$$ {}_{b}^{ABR}\mathcal{D}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_b^t f(\xi) E_\alpha \left[-\alpha \frac{(t-\xi)^\alpha}{1-\alpha} \right] d\xi \tag{4.21} $$

Equations (5.5) and (5.6) have non-local kernels
Also equation (5.5) when the function is constant the fractional derivative produces zero. Now, we discuss some important Properties

1. $$ \mathfrak{L}\{{}_{a}^{ABC}\mathcal{D}^{\alpha} x(t)\} = \frac{B(\alpha)}{1-\alpha} \frac{p^\alpha \mathfrak{L}\{x(t)\} - p^{\alpha-1}x(a)}{p^\alpha + \frac{\alpha}{1-\alpha}} \tag{4.22} $$

2. $$ \mathfrak{L}\{{}_{a}^{ABR}\mathcal{D}^{\alpha} x(t)\} = \frac{B(\alpha)}{1-\alpha} \frac{p^\alpha \mathfrak{L}\{x(t)\}}{p^\alpha + \frac{\alpha}{1-\alpha}} \tag{4.23} $$

Definition 9.
The associated fractional integral is defined by

$$ {}^{AB}_{a}\mathfrak{J}^{\alpha}x(t) = \frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}({}_{a}\mathfrak{J}^{\alpha}x(t)) $$

Where $({}_{a}\mathfrak{J}^{\alpha})$ is the left Riemann-Liouville fractional integral

4.3 Drawbacks of non-singular kernel

At first sight, fractional derivatives defined using non-sigular (i.e bounded) kernels may appear very attractive since they avoid several difficulties that are caused by the singular nature of the RL and Dzhrbashyan-Caputo kernel.[11]

Caputo type derivatives that are defined using nonsingular kernels must fails to satisfy the fundamental theorem of fractional calculus. In other words they don't allow the existence of a corresponding convolution integral for which the derivative is left-inverse

Theorems 4.(The fundamental theorem of fractional calculus)
If f is continous and $\alpha \geq 0$, Then

$$ \mathcal{D}_{a}^{\alpha} I_{a}^{\alpha} f = f \tag{4.24} $$

First, we state the following well-known technical result.

Lemma 1.
If f is integrable on a set A, Then, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$ \left| \int_{E} f(x) d\mu \right| < \epsilon $$

for every measurable set $E \subset A$ of measure less that δ

Proof see [15]

In this discussion we investiagte what conditions a fundamental theorem of fractional calculus imposes on the kernels of the differential and integral operators.

Suppose that, for $0 < \alpha < 1$, we define for a function $f \in AC[0,T]$ a Caputo-type derivative $\mathcal{D}_\phi$ By

$$\mathcal{D}_\phi f(t) := \int_0^t \phi(t-\tau) f'(\tau) d\tau, \quad 0 < t \leq T \tag{4.25}$$

Where the kernel function ϕ is yet unspecified, except that we require $\phi \in L^1[0,T]$ to ensure that $\mathcal{D}_\phi f(t)$ is defined almost everywhere. (it is well known that the comvolution of two functions in $L_1[0,T]$ also lies in $L_1[0,T]$)

Such operators based on non-singular kernels usually have a normalization factor that multiplies the integral, depends on α , and ensures that $\mathcal{D}_\phi$ approaches the classical first-order derivative when $\alpha \to 1$. For simplicity we do not write this factor explicitly; instead it is absorbed into the kernal ϕ

In order to have a fundamental theorem of fractional calculus for our derivative $\mathcal{D}_\phi$, we need to define a corresponding integral I_ψ in a similar way

$$I_\psi g(t) = \int_0^t \psi(t-\tau) g(\tau) d\tau \quad 0 < t \leq T \tag{4.26}$$

Where $\psi \in L^1[0,T]$ is yet to be chosen in such a way that $\mathcal{D}_\phi[I_\psi f(t)] = f(t)$ for all $f(t) \in AC[0,T]$ and $0 < t \leq T$. writing out this identity in detial, we have

$$\begin{aligned} f(t) &= \int_0^t \phi(t-\tau)(I_\psi f)'(\tau) d\tau \\ &= \int_0^t \phi(\tau)(I_\psi f)'(t-\tau) d\tau \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \phi(\tau)(I_\psi)(t-\tau) d\tau\right) \end{aligned}$$

The third equation follows from Leibniz's Rule for differentiating integrals, combined with $\lim_{t\to 0} I_\psi f(t) = 0$ (which follows from Lemma 1 since $\psi \in$

$L^1[0,T]$ and f bounded impliies that the integrand of $\mathcal{I}_\psi f$ lies in $L^1[0,T]$). Now make another change of variables, then recall the definition of $\mathcal{I}_\psi$ to get

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \phi(t-\tau)(I_\psi f)(\tau)d\tau\right)$$
$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \phi(t-\tau)\left[\int_0^\tau \psi(\tau-s)f(s)ds\right]d\tau\right)$$

Next, apply Fubini's theorem to interchange order of integration, Then apply Leibniz's Rule

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t f(s)\left[\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau\right]ds\right)$$
$$f(t) = f(t)\lim_{s\to t}\left[\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau\right] + \int_0^t f(s)\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau\right]ds$$

We want this equation to hold true for all $f \in AC[0,T]$ and $0 < t \le T$. This is possible only if

$$\lim_{s\to t}\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau = 1 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau = 0$$

The change of variables $\tau - s = r$ shows that each integral here equals

$$\int_0^{t-s} \phi(t-s-r)\psi(r)dr$$

Thus, the value of integral depends on the length $t-s$ of the interval of integration but not separately on t and s. Consequently one can rewrite $\lim_{s\to t}$ in the first condition as $\lim_{t\to s}$. But the second condition says that

$$\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau$$

is a constant as t varies; then the first consition forces

$$\int_s^t \phi(t-\tau)\psi(\tau-s)d\tau = 1 \quad 0 \le s < t \le T \tag{4.27}$$

Suppose that one of these functions is bounded on $[0,T]$ say, $|\phi(t)| \le M$ for

$0 \leq t \leq T$, Then

$$\left| \int_s^t \phi(t-\tau)\psi(\tau - s)d\tau \right| \leq M \int_s^t |\psi(\tau - s)|\, d\tau$$

By Lemma 2.1 The right-hand side will go to zero if $s \to t$. But this implies that equation (4.27) cannot be satisfied when s is close to t. Thus we cannot have ϕ bounded on $[0, T]$ (and likewise ψ)

We can summarize the previous disscusion in the following Theorem.

Theorems 5.
Given a Caputo-type fractional derivative of the form (4.25) whose kernel $\phi : [0, T] \to \mathbb{R}$ is bounded (i.e non singular), one cannot define a corresponding integral operator such that the fundamental theorem of fractional calculus is valid.

Derivatives with non-singular kernel impose restrictive and unnatural assumptions

CF and ABC derivatives are not the left-inverse of the coreesponding integrals.

A CF integral ${}^{CF}\mathfrak{I}_0^{\alpha} f(t)$ has been proposed in the literature, defines By

$$ {}^{CF}\mathfrak{I}_0^{\alpha} f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau)d\tau \quad t \geq 0 \tag{4.28}$$

It has the property that ${}^{CF}\mathfrak{I}_0^{\alpha}[{}^{CF}\mathcal{D}_0^{\alpha} f(t)] = f(t) - f(0)$; that is, the differential operator is the right-inverse of the integral operator on the space of functions $\{f \in AC[0, T] : f(0) = 0\}$. This property is similar to the identity $\int_0^t f'(s)ds = f(t) - f(0)$ enjoyed by classical first order derivative and the standard integral operator. But first order derivatives also have the left-inverse property

$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(s)ds = f(t)$ whereas for the CF integral and derivative we have the following result.

Proposition 1.
Let $f \in AC[0,T]$. The CF derivative and integral satisfy the Relation

$$ {}^{CF}\mathcal{D}_0^\alpha[{}^{CF}\mathfrak{I}_0^\alpha f(t)] = f(t) - \exp\left(-\frac{\alpha}{1-\alpha}t\right) f(0) \tag{4.29} $$

This unfavourable result says that the CF derivative ${}^{CF}\mathcal{D}_0^\alpha$ is the left-inverse of ${}^{CF}\mathfrak{I}_0^\alpha$ only on the restricted space $\{f \in AC[0,T] : f(0) = 0\}$ and not on the full space $AC[0,T]$, as one would except.
The constraint $f(0) = 0$ on functions for which ${}^{CF}\mathcal{D}_0^\alpha$ is the left inverse of ${}^{CF}\mathfrak{I}_0^\alpha$ has serious consequences if ${}^{CF}\mathfrak{I}_0^\alpha$ is employed to solve an initial-value problem such as

$$ \begin{cases} {}^{CF}\mathcal{D}_0^\alpha y(t) = g(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{4.30} $$

For applying ${}^{CF}\mathfrak{I}_0^\alpha$ to both sides of the differential equations, we obtain

$$ y(t) = y_0 + \frac{1-\alpha}{M(\alpha)} g(t, y(t)) + \frac{\alpha}{M(\alpha)} \int_0^t g(\tau, y(\tau)) d\tau \tag{4.31} $$

But, replacing $f(t)$ in (4.29) by $g(t, y(t))$, we see immediately that

$$ \begin{aligned} {}^{CF}\mathcal{D}_0^\alpha[{}^{CF}\mathfrak{I}_0^\alpha g(t, y(t)] &= g(t, y(t)) - \exp\left(-\frac{\alpha}{1-\alpha}t\right) g(0, y(0)) \\ {}^{CF}\mathcal{D}_0^\alpha y(t) &= g(t, y(t)) - \exp\left(-\frac{\alpha}{1-\alpha}t\right) g(0, y(0)) \end{aligned} $$

Hence ${}^{CF}\mathcal{D}y(t) \neq g(t, y(t))$ if $g(0, y(0)) \neq 0$. that is, although one might believe erroneously that $y(t)$ in (4.31) is the solution of the Differential equation, this is not true unless $g(0, y_0) = 0$.
Situation is similar for the AB integral

$$ {}^{AB}\mathfrak{I}_0^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{4.32} $$

Here again ${}^{AB}\mathfrak{I}_0^\alpha[{}^{ABC}\mathcal{D}_0^\alpha f(t)] = f(t) - f(0)$, but ${}^{ABC}\mathcal{D}_0^\alpha$ is not the left inverse of ${}^{AB}\mathfrak{I}_0^\alpha$

Proposition 2.
Let $f \in AC[0,T]$. The ABC derivative and the AB integral satisfy the Relation

$$ {}^{ABC}\mathcal{D}^{\alpha}_{0}[{}^{AB}\mathfrak{I}^{\alpha}_{0} f(t)] = f(t) - E_{\alpha}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right) f(0) \tag{4.33} $$

Similar to the CF derivative, the ABC derivative is the left-inverse of the AB integral only on the restricted space $\{f \in AC[0,T] : f(0) = 0\}$
The use of ${}^{AB}\mathfrak{I}^{\alpha}_{0}$ to solve the same differential equation (4.30) but with ABC derivative will thus produce a function $y(t) = y_0 +^{AB} \mathfrak{I}^{\alpha}_{0} g(t, y(t))$ that is not a solution of the equation since

$$ {}^{ABC}\mathcal{D}^{\alpha}_{0} y(t) = g(t, y(t)) - E_{\alpha}(-\frac{\alpha}{1-\alpha}t^{\alpha})g(0, y_0) $$

In general the CF and AB integrals cannot be used to solve differential equations with the corresponding fractional derivatives, unless one imposes the additional and restrictive condition $g(0, y_0) = 0$ to have the identities ${}^{CF}\mathcal{D}^{\alpha}_{0}[{}^{CF}\mathfrak{I}^{\alpha}_{0} g(t, y(t$ $g(t, y(y))$ and ${}^{ABC}\mathcal{D}^{\alpha}_{0}[{}^{AB}\mathfrak{I}^{\alpha}_{0} g(t, y(t))] = g(t, y(y))$

To appreciate how unnatural the condition $g(0, y_0) = 0$ is, consider the simple linear problem where $g(t, y(t)) = \lambda y(t)$ with a Caputo-Fabrizio or Atangana-Baleanu in Caputo sense derivative. Imposing the condition $g(0, y_0) = 0$, so that the CF or AB integral solves the problem correctly, requires either $\lambda = 0$ or $y_0 = 0$; but then the problem has only the trivial constant solution $y(t) \equiv y_0$ for all $t \geq 0$; but then the problem has only to describle constant solution is not worthwhile.

• Non-Singular kernel derivatives are always zero at zero. The restriction on the initial condition of differential equations with CF and ABC derivatives is consequence of the fact that these derivatives are zero at the origin. For instance, taking the power function $f(t) = t^{\gamma}$ for constant $\gamma > 0$, one can compute

$$ {}^{ABC}\mathcal{D}^{\alpha}_{0} f(t) = \frac{B(\alpha)}{1-\alpha}\Gamma(\gamma+1)t^{\gamma}E_{\alpha,\gamma+1}(-\frac{\alpha}{1-\alpha}t^{\alpha}) $$

and consequently ${}^{ABC}\mathcal{D}^{\alpha}_{0} f(t)|_{t=0} = 0$ (similarly for ${}^{CF}\mathcal{D}^{\alpha}_{0} f(t)$). We call this the zero-zero property (namely, the derivative at zero is always zero). It holds true not only for CF and ABC derivatives, and not only for the function $f(t) = t^\gamma$, but much more generally, as we now show

Theorems 6. Zero-zero property
Let ϕ be bounded on $[0, T]$, $\mathcal{D}_\phi$ the operator defined by (4.25) and $f \in AC[0, T]$. then

$$\lim_{t\to 0^+} \mathcal{D}_\phi f(t) = 0$$

Proof. Since ϕ is bounded on $[0, T]$, for ant $t \in (0, T]$ one has

$$|\mathcal{D}_\phi f(t)| = \left| \int_0^t \phi(t-\xi) f'(\xi) d\xi \right| \leq (\sup_{t\in[0,T]} |\phi(t)|) \int_0^t |f'(\xi)| d\xi$$

But $f \in AC[0, T]$ means that $f' \in L^1[0, T]$, so Lemma 2.1 implies the desired result.

□

Consider now a general differential equation, with a non-singular (i.e bounded) kernel derivative $\mathcal{D}_\phi$, of the form

$$\begin{cases} \mathcal{D}_\phi y(t) = g(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{4.34}$$

for which (zero-zero property) gives $0 = \mathcal{D}_\phi y(t)|_{t=0^+} = g(0, y_0)$. Hence (4.34) can have a solution only if $g(0, y_0) = 0$
Thus in (4.34) one is forces to choose the initial data y_0 such that $g(0, y_0) = 0$. This is of course restrictive - and may even be impossible in some cases (or in some real life applications).

Chapter 5

Fractional Differential Equations

5.1 Abstract

In this chapter, We study the existence and uniqueness [13] of Cauchy problems of the from

$$\begin{cases} {}_{a}^{ABC}\mathcal{D}^{\alpha}x(t) = f(t, x(t)) \quad 0 < \alpha < 1 \\ x(a) = x_0 \end{cases} \tag{5.1}$$

Where ${}_{a}^{ABC}\mathcal{D}^{\alpha}$ is the Atangana-Baleanu in Caputo sense fractional derivative under certain conditions in space of continuous functions and sobolev space H^1.

5.2 Preliminaries

In what follows, we recall some basic defintions and tools about classical fractional calculus

For $\alpha > 0$ the left Riemann-Liouville fractional integral of order α starting at a has the following form

$$ {}_{a}\mathfrak{I}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\xi)^{\alpha-1}x(\xi)d\xi \tag{5.2}$$

And recall fractional derivatives definitions of Riemann and Caputo Respectively with $0 < \alpha < 1$

$$ {}_{a}\mathcal{D}^{\alpha}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}({}_{a}\mathfrak{I}^{1-\alpha})x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t}(t-\xi)^{-\alpha}x(\xi)d\xi \tag{5.3}$$

$$ {}_{a}^{C}\mathcal{D}^{\alpha}x(t) = ({}_{a}\mathcal{J}^{1-\alpha})\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}(t-\xi)^{-\alpha}x'(\xi)d\xi \tag{5.4} $$

Finally recall Caputo Atangana-Baleanu and Riemann Atangana-Baleanu Respectively

$$ {}^{ABC}\mathcal{D}^{\alpha}x(t) = \frac{B(\alpha)}{1-\alpha}\int_{b}^{t}\frac{\mathrm{d}x(\xi)}{\mathrm{d}\xi}E_{\alpha}\left[-\alpha\frac{(t-\xi)^{\alpha}}{1-\alpha}\right]d\xi \tag{5.5} $$

$$ {}^{ABR}\mathcal{D}^{\alpha}x(t) = \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{b}^{t}x(\xi)E_{\alpha}\left[-\alpha\frac{(t-\xi)^{\alpha}}{1-\alpha}\right]d\xi \tag{5.6} $$

The associated fractional integral is defined By

$$ {}_{a}^{AB}\mathcal{J}^{\alpha}x(t) = \frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}({}_{a}\mathcal{J}^{\alpha}x(t)) \tag{5.7} $$

Where ${}_{a}\mathcal{J}^{\alpha}$ is the left Riemann-Liouville fractional integral defined above.

The relation between Caputo Atangana-Baleanu and Riemann-Liouville Atangana-Baleanu fractional derivatives is presented in the following theorem

Theorems 7.
Let $x \in H^{1}(a,b)$ and $\alpha \in [0,1]$. Then the following relation holds

$$ {}_{a}^{ABC}\mathcal{D}^{\alpha}x(t) = {}_{a}^{ABR}\mathcal{D}^{\alpha}x(t) - \frac{B(\alpha)}{1-\alpha}x(a)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-a)^{\alpha}\right] \tag{5.8} $$

Proof. See [9]

5.3 Existence and uniqueness of Cauchy problem

First we consider Banach fixed point theorem (Contraction mapping)

Theorems 8. Banach fixed point
Let $X = (X, d)$ be a nonempty, complete metric space and $T : X \to X$ be a contraction mapping on X. Then T has only one fixed point (i.e Unique solution)

Where the contraction mapping is defined as

Definition 10. Contraction mapping
Let $X = (X, d)$ be a metric space, the mapping $T : X \to X$ is said top be contraction on X if there is a nonnegative number α less that 1 such that

$$d(Tx, Ty) \leq \alpha d(x, y) \qquad , 0 \leq \alpha < 1 \forall x, y \in X$$

5.3.1 In Space of continuous functions

Now, we consider the Existence and Uniqueness theorem for our Cauchy problem in space of continuous functions.

Theorems 9.
Let $x(t) \in C[a, b]$ such that ${}^{ABC}_{a}\mathcal{D}^{\alpha} x(t) \in C[a, b]$. suppose that $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq k|x - y|$$

Then. if $f(a, x(a)) = 0$ and $k\left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) < 1$, the Cauchy problem (5.1) has a unique solution

Proof. First we have to prove that $x(t)$ satisfy (5.1) if and only if $x(t)$ satisfies the integral equation

$$x(t) = x_0 + {}^{AB}_{a}\mathfrak{I}^{\alpha} f(t, x(t)) \tag{5.9}$$

Let $x(t)$ satisfy the differential equation in (5.1). Applying the Atangana-Baleanu fractional integral to both sides, we get

$$ {}_{a}^{AB}\mathfrak{I}^{\alpha}\, {}_{a}^{ABC}\mathcal{D}^{\alpha}x(t) = {}_{a}^{AB}\mathfrak{I}^{\alpha}f(t,x(t)) \tag{5.10}$$

we obtain

$$x(t) - x(a) = {}_{a}^{AB}\mathfrak{I}^{\alpha}f(t,x(t))$$

Since $x(a) = x_0$ from initial condition in (5.1) and $f(a,x(a)) = 0$, (5.9) is satisfied.

Now, if $x(t)$ satisfies (5.9), then by using that $f(a,x(a)) = 0$ it is obvious that $x(a) = x_0$. Applied the Riemann-Liouville Atangana-Baleanu fractional derivative to both sides of (5.9) and using that

$$({}_{a}^{ABR}\mathcal{D}^{\alpha}({}_{a}^{AB}\mathfrak{I}^{\alpha}x))(t) = x(t)$$

we obtain

$$({}_{a}^{ABR}\mathcal{D}^{\alpha}x)(t) = x_0({}_{a}^{ABR}\mathcal{D}^{\alpha}1)(t) + ({}_{a}^{ABR}\mathcal{D}^{\alpha}({}_{a}^{AB}\mathfrak{I}x))(t) \tag{5.11}$$

Thus, we have

$$({}_{a}^{ABR}\mathcal{D}^{\alpha}x)(t) = x_0\frac{B(\alpha)}{1-\alpha}E_{\alpha}(-\frac{\alpha}{1-\alpha}(t-a)^{\alpha}) + f(t,x(t)) \tag{5.12}$$

Then, the result is obatained by befefiting from Theorem 7

Now, define the operator

$$Tx(t) = x_0 + {}_{a}^{AB}\mathfrak{I}^{\alpha}f(t,x(t))$$

Then, we have

$$
\begin{aligned}
|T(x)-T(y)| &= |{}_a^{AB}\mathfrak{I}^{\alpha} f(t,x(t)) -{}_a^{AB}\mathfrak{I}^{\alpha} f(t,x(t))| \\
&= |\frac{1-\alpha}{B(\alpha)} f(t,x(t)) - \frac{1-\alpha}{B(\alpha)} f(t,y(y)) \\
&\quad + \frac{\alpha}{B(\alpha)}{}_a\mathfrak{I}^{\alpha} f(t,x(t)) - \frac{\alpha}{B(\alpha)}{}_a\mathfrak{I}^{\alpha} f(t,y(t))| \\
&\leq \frac{1-\alpha}{B(\alpha)} |f(t,x(t)) - f(t,y(t))| + \frac{\alpha}{B(\alpha)} \\
&\quad {}_a\mathfrak{I}^{\alpha} |f(t,x(t)) - f(t,y(t))| \\
&\leq \frac{1-\alpha}{B(\alpha)} k|x-y| + \frac{\alpha}{B(\alpha)} k|x-y|({}_a\mathfrak{I}^{\alpha}1)(t) \\
&= \frac{1-\alpha}{B(\alpha)} k|x-y| + \frac{\alpha}{B(\alpha)} k|x-y| \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \\
&= k\left(\frac{1-\alpha}{B(\alpha)} + \frac{\alpha(t-a)^{\alpha}}{B(\alpha)\Gamma(\alpha+1)}\right)|x-y| \\
&< k\left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right)|x-y| \\
&< |x-y| \\
&\leq ||x-y||
\end{aligned}
$$

Where $||.||$ is the max-norm in the space $C[a,b]$. Therefore. T is a contraction mapping and thus is has a unique fixed point □

5.3.2 In Sobolev space $H^1(a,b)$

Recall the Differential equation (5.1)

Let $x(t) \in H^1(a,b)$, $x(t)$ solution of differential equation $\iff$ solution of integral equation

$$x(t) = x_0 +^{AB} \mathfrak{I} f(t,x(t))$$

Or

$$x(t) = x_0 + \frac{1-\alpha}{B(\alpha)} f(t,x(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(\xi,x(\xi))(t-\xi)^{\alpha-1} d\xi$$

Now, we consider the Existence and Uniqueness theorem for our Cauchy problem in space of $H^1(a,b)$. [17]

Theorems 10.
Let $f(t, x(t))$ is smooth function, If $f(t, x(t))$ satisfies Lipschitz function

$$|f(t, x(t)) - f(t, y(t))| \leq K|x(t) - y(t)|$$

with Lipschitz constant

$$K < \frac{\Gamma(\alpha)B(\alpha)}{(1-\alpha)\Gamma(\alpha)+1}$$

Then DE has a unique solution in $H^1(a, b)$

Proof. $\forall x, y \in H^1(a, b)$, $t \in [a, b]$
Define a metric function $d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|$

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[a,b]} |Tx - Ty| \\
&= \sup_{t\in[a,b]} |x_0 + \frac{1-\alpha}{B(\alpha)} f(t, x(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(\xi, x(\xi))(t-\xi)^{\alpha-1} d\xi \\
&\quad - x_0 - \frac{1-\alpha}{B(\alpha)} f(t, y(t)) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(\xi, y(\xi))(t-\xi)^{\alpha-1} d\xi| \\
&= \sup_{t\in[a,b]} |\frac{1-\alpha}{B(\alpha)} (f(t, x(t)) - f(t, y(t))) \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t (f(\xi, x(\xi)) - f(\xi, y(\xi)))(t-\xi)^{\alpha-1} d\xi| \\
&\leq \frac{1-\alpha}{B(\alpha)} \sup_{t\in[a,b]} |f(t, x(t)) - f(t, y(t))| \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sup_{t\in[a,b]} |\int_a^t (t-\xi)^{\alpha-1} (f(\xi, x(\xi)) - f(\xi, y(\xi))) d\xi| \\
&\leq \frac{1-\alpha}{B(\alpha)} K d(x, y) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} K d(x, y) \frac{t^\alpha}{\alpha} \\
&= (\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{t^\alpha}{\alpha}) K d(x, y) \leq \left(\frac{(1-\alpha)\Gamma(\alpha)+1}{B(\alpha)\Gamma(\alpha)} \right) K d(x, y)
\end{aligned}$$

Since, $(\frac{(1-\alpha)\Gamma(\alpha)+1}{B(\alpha)\Gamma(\alpha)})K < 1$, Then T is a contraction mapping, and By Banach fixed point Theorem, T has a unique Solution □

Example 5.1. *Consider the following Cauchy problem*

$$\begin{cases} {}_{0}^{ABC}\mathcal{D}^{\alpha}x(t) = \frac{tx(t)}{200} \quad t \in [0,1], B(\alpha) = 1 \\ x(0) = 1 \end{cases} \tag{5.13}$$

Notice that $f(0, x(0)) = 0$. since $f(t,x) = \frac{tx(t)}{200}$, we have

$$|f(t,x) - f(t,y)| = \frac{t}{200}|x-y| \leq \frac{1}{200}|x-y| \tag{5.14}$$

Thus $k = \frac{1}{200}$. Now,

$$k\left(\frac{1-\alpha}{B(\alpha)} + \frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) = \frac{1}{200}\left(1 - \alpha + \frac{1}{\Gamma(\alpha)}\right) < 1 \tag{5.15}$$

By Existence and Uniqueness Theorem, (5.13) has a unique solution.

Conclusion

This research delved into the realm of fractional derivatives with non-singular kernels. We began by introducing the concept of fractional derivatives and their various definitions, highlighting their unique properties compared to integer-order derivatives. Subsequently, we explored the concept of non-singular kernels, emphasizing their distinct characteristics compared to the commonly used singular kernels in fractional calculus.

In conclusion, while non-singular kernels offer certain advantages in terms of avoiding some mathematical complexities associated with singular kernels, recent research indicates potential drawbacks. Notably, these derivatives may not satisfy the fundamental theorem of fractional calculus and can lead to unnatural restrictions in the initial conditions of differential equations.

Therefore, further research is necessary to fully understand the implications and limitations of utilizing non-singular kernels in fractional calculus. This includes exploring alternative definitions, analyzing their properties in greater depth, and investigating Existence and uniqueness theorems for fractional differential equations involving fractional derivative with non singular kernels

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