Calculus I

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Contents

Chapter 1

Gamma Functions

Objective

- Definition of gamma function
- understand uses and applications

1.1 Definitions

Gamma function denoted by $\Gamma(x)$. Also known as generalized factorial function. Gamma has several equivalent definitions.

Definition 1.1.1: Euler Representation

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)(x+2)\dots(x+n-1)(x+n)} \tag{1.1}$$

Limit exist if $x \neq 0$ or negative number.

Definition 1.1.2: Integral Representation

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad , x > 0 \tag{1.2}$$

Integral (1.2) is improper integral due to the infinite upper limit of integration and t^{x-1} at $t = 0$ when $0 < x < 1$

Let's take a look on some important properties (using (1.1) and (1.2))

- What is the value of $\Gamma(1)$?

$$\begin{aligned}\Gamma(1) &= \lim_{n\to\infty} \frac{n!\,n}{1.2.3\ldots(n+1)} \\ &= \lim_{n\to\infty} \frac{n!\,n}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{n}{n+1} = 1\end{aligned}$$

Or using the integral representation

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}t^0 dt \\ &= 1\end{aligned}$$

- We will deduce an important formula

$$\begin{aligned}\Gamma(x+1) &= \lim_{n\to\infty} \frac{n!\,n^{x+1}}{(x+1)(x+2)\ldots(x+n)(x+n+1)} \times \frac{x}{x} \\ &= \lim_{n\to\infty} \frac{n!\,n^{x}}{x(x+1)(x+2)\ldots(x+n)} \times \lim_{n\to\infty} \frac{n\,x}{x+n+1} \\ &= \Gamma(x)x\end{aligned}$$

Or from integral representation

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty e^{-t}t^x\,dt \quad \text{By parts} \\ &= \left[-e^{-t}t^x\right]_0^\infty + \int_0^\infty xe^{-t}t^{x-1}\,dt \\ &= x\int_0^\infty e^{-t}t^{x-1}\,dt \\ &= x\Gamma(x)\end{aligned}$$

Hence, An important formula called Recurrence formula

$$\boxed{\Gamma(x+1) = x\Gamma(x)} \tag{1.3}$$

Remark.

If x is integer $\Gamma(x+1) = x!$ or it has another notation by Gauss $\prod(x) = x!$

Example.

- $\Gamma(5) = 4!$
- $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$

Recurrence formula (1.3) can be written as

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

Remark.

> If x is negative or $= 0 \quad \Rightarrow \Gamma(x)$ divergent
>
> $$|\Gamma(n)| = \infty, \quad n \leq 0$$

Example.

> - $\Gamma(0) = \frac{\Gamma(1)}{0} = \infty$
> - $\Gamma(-1) = -\infty$
> - $\Gamma(-2) = \infty$
> - $\vdots$

1.2 Weierstrass' infinite product

Definition 1.2.1: Weierstrass' infinite product

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \tag{1.4}$$

Where γ : Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) \approx 0.577215$$

Proof. We start with reciprical of Euler representation

$$\begin{aligned} \frac{1}{\Gamma(x)} &= \lim_{n\to\infty} \frac{x(x+1)(x+2)\dots(x+n)}{n!\, n^x} \\ &= x \lim_{n\to\infty} n^{-x} \left(\frac{x+1}{1} \times \frac{x+2}{2} \times \frac{x+3}{3} \dots \frac{x+n}{n} \right) \\ &= x \lim_{n\to\infty} e^{-x\ln(n)} \prod_{k=1}^{n} \left(1 + \frac{x}{k}\right) \qquad \text{(I)} \end{aligned}$$

And obviously,

$$e^{x \sum_{k=1}^{n} \frac{1}{k}} = \prod_{k=1}^{n} e^{\frac{x}{k}} \qquad \text{(II)}$$

Multiply L.H.S of (II) by (I)

$$\begin{aligned}
\frac{e^{x\sum\frac{1}{k}}}{\Gamma(x)} &= x \lim_{n\to\infty} e^{-x\ln(n)} \left[\prod_{k=1}^{n}(1+\frac{x}{k})\right] \times e^{x\sum\frac{1}{k}} \\
&= x \lim_{n\to\infty} \exp\left(x(\sum_{k=1}^{n}\frac{1}{k} - \ln(n))\right) \prod_{k=1}^{n}(1+\frac{x}{k}) \\
\frac{1}{\Gamma(x)} &= x \lim_{n\to\infty} \exp\left(x(\sum_{k=1}^{n}\frac{1}{k} - \ln(n))\right) \prod_{k=1}^{n}(1+\frac{x}{k}) \prod_{k=1}^{n} e^{-\frac{x}{k}} \\
&= x \lim_{n\to\infty} \exp\left(x(\sum_{k=1}^{n}\frac{1}{k} - \ln(n))\right) \times \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{x}{k}\right) e^{-\frac{x}{k}} \\
&= xe^{\gamma x} \times \prod_{k=1}^{\infty}(1+\frac{x}{k})e^{-\frac{x}{k}}
\end{aligned}$$

$\square$

Solution. sss □

Theorem 1.2.2: Theorem Name

A theorem.

Lemma 1.2.3: Lemma Name

A lemma.

Objective

A fact.

Corollary 1.2.4

A corollary.

Proposition 1.2.5

A proposition.

Claim

A claim.

Proof for Claim.

A reference to Theorem 1.2.2

Proof. Veniam velit incididunt deserunt est proident consectetur non velit ipsum voluptate nulla quis. Ea ullamco consequat non ad amet cupidatat cupidatat aliquip tempor sint ea nisi elit dolore dolore.

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Example.

> Nostrud esse occaecat Lorem dolore laborum exercitation adipisicing eu sint sunt et. Excepteur voluptate consectetur qui ex amet esse sunt ut nostrud qui proident non. Ipsum nostrud ut elit dolor. Incididunt voluptate esse et est labore cillum proident duis.

Some remark.

Remark.

> Some more remark.

1.3 Pictures

Figure 1.1: Waterloo, ON