Exercise 5

1. The probability density of the random variable Y is given by

$$f(y) = \begin{cases} \frac{1}{8}(y+1), & \text{for } 2 < y < 4 \\ 0, & \text{elsewhere} \end{cases}$$

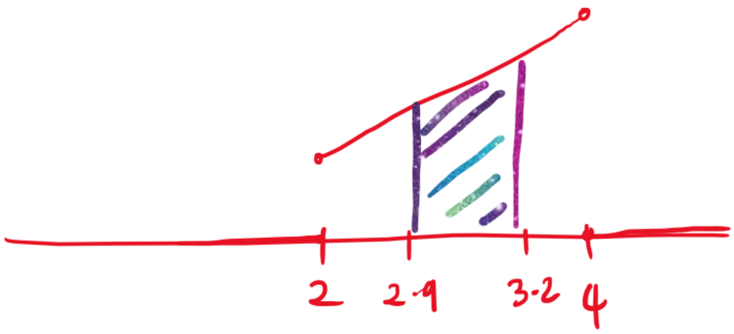
   a. Show that the function $f(y)$ represents a density function.

   $f(y) \geq 0$

   $$\int_{-\infty}^{\infty} f(y)\ d(y) = \int_2^4 \frac{1}{8}(y+1)\,dy = \left[\frac{y^2}{16} + \frac{y}{8}\right]_2^4 = \left(\frac{16}{16} + \frac{4}{8}\right) - \left(\frac{4}{16} + \frac{2}{8}\right) = 1$$

   The two properties are met. So, it is a probability density function.

   b. Sketch a graph of this function, and indicate the area associated with the probability that $2.9 < Y < 3.2$.



   c. Find $P(2.9 \leq Y < 3.2)$.

$$P(2.9 \leq Y < 3.2) = \int_{2.9}^{3.2} f(y)\,dy = \int_{2.9}^{3.2} \frac{1}{8}(y+1)\,dy = \left[\frac{y^2}{16} + \frac{y}{8}\right]_{2.9}^{3.2}$$

$$= \left(\frac{3.2^2}{16} + \frac{3.2}{8}\right) - \left(\frac{2.9^2}{16} + \frac{2.9}{8}\right) = 0.1519$$

   d. Find the mean of this distribution.

$$\mu = \int_2^4 y f(y)\,dy = \int_2^4 \left(\frac{y^2}{8} + \frac{y}{8}\right) dy = \left[\frac{y^3}{24} + \frac{y^2}{16}\right]_2^4 = \frac{11}{3} - \frac{7}{12} = \frac{37}{12}$$

2. The distribution function of the random variable X is given by

$$F(x) = \begin{cases} 0, & \text{for } x < -1 \\ \frac{x+1}{2}, & \text{for } -1 \leq x < 1 \\ 1, & \text{for } x \geq 1 \end{cases}$$

   a. Find $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$.

$$P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = 0.75 - 0.25 = 0.5$$

b. Find $P(2 < X < 3)$.

$P(2 < X < 3) = F(3) - F(2) = 1 - 1 = 0$

c. Write down the probability density distribution $f(x)$.

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x+1}{2}\right) = \frac{1}{2}, \qquad \text{for } -1 \le x < 1$$

3. The random variable X is normally distributed with mean $\mu = 18$ and standard deviation $\sigma = 7.6$.

a. Find the probability

i. $P(0 < X \le 5)$

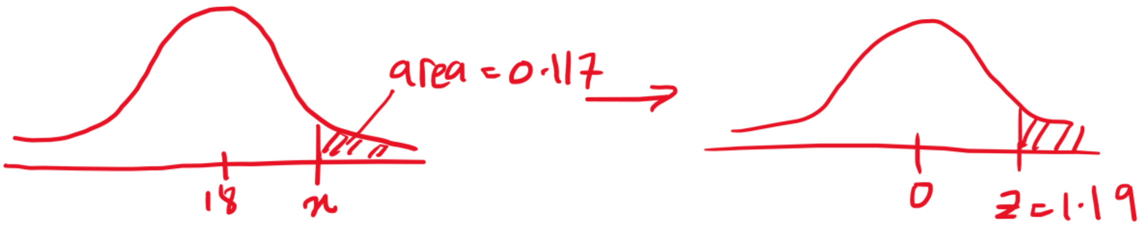
$$P(0 < X \le 5) = P\left(\frac{0-18}{7.6} < Z < \frac{5-18}{7.6}\right) = P(-2.37 < Z < -1.71)$$
$$= P(Z < -1.71) - P(Z < -2.37) = P(Z > 1.71) - P(Z > 2.37)$$
$$= 0.0436 - 0.00889 = 0.03471$$

ii. $P(X < 9 \text{ or } X > 27)$

$$P(X < 9 \text{ or } X > 27) = P(X < 9) + P(X > 27)$$
$$= P\left(Z < \frac{9-18}{7.6}\right) + P\left(Z > \frac{27-18}{7.6}\right)$$
$$= P(Z < -1.18) + P(Z > 1.18)$$
$$= 2P(Z > 1.18) = 2 \times 0.1190 = 0.238$$

b. Find the value of x that has 88.3% of the distribution's area to its left.

Want to find x such that $P(X < x) = 0.883$, or equivalently, $P(X > x) = 1 - 0.883 = 0.117$.



From the table $P(Z > 1.19) = 0.117$. So, choose $z = 1.19$

$x = \mu + z\sigma = 18 + (1.19)(7.6) = 27.04$

c. Find the value of x that has 64.8% of the distribution's area to its right.

Want to find x such that $P(X > x) = 0.648$, or equivalently, $P(X < x) = 1 - 0.648 = 0.352$.

From the table $P(Z > 0.38) = 0.352 = P(Z < -0.38)$. So, choose $z = -0.38$

$x = \mu + z\sigma = 18 + (-0.38)(7.6) = 15.11$

4. The amounts of time Facebook users spend on the website each week are normally distributed, with a mean of 6.7 hours and a standard deviation of 1.8 hours.

   a. Find the probability that a Facebook user spends less than four hours on the website in a week.

   X = time a Facebook user spend on website in hours in a week

   $X \sim N(6.7, 1.8^2)$

   $$P(X < 4) = P\left(Z < \frac{4 - 6.7}{1.8}\right) = P(Z < -1.5) = P(Z > 1.5) = 0.0668$$

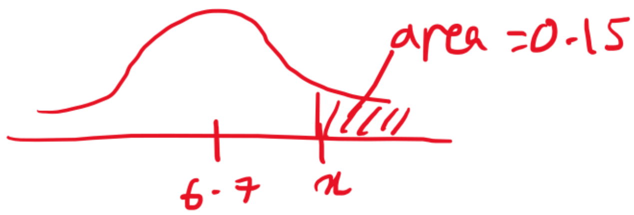
   b. Out of 800 Facebook users, about how many would you expect to spend between 2 and 3 hours on the website in a week?

   $$P(2 < X < 3) = P\left(\frac{2 - 6.7}{1.8} < Z < \frac{3 - 6.7}{1.8}\right) = P(-2.61 < Z < -2.06)$$
   $$= P(Z < -2.06) - P(Z < -2.61) = P(Z > 2.06) - P(Z > 2.61)$$
   $$= 0.0197 - 0.00453 = 0.01517$$

   Number of users = $0.01517 \times 800 \approx 12$

   c. What is the lowest amount of time spent on Facebook in a week that would still place a user in the top 15% of times?

   Want to find x such that $P(X > x) = 0.15$.



   From the table, $P(Z > 1.04) = 0.1492$ gives the closes probability. Choose $z = 1.04$.

   $x = \mu + z\sigma = 6.7 + (1.04)(1.8) = 8.572$

5. A recent study of the life span of a wireless sound system found the average to be 3.7 years with a standard deviation of 0.6 year.

   a. A random sample of 16 people who own the wireless sound system is selected.

i. Find the mean and standard deviation of the sampling distribution of the sample mean.

$\mu_{\bar{X}} = \mu = 3.7$

$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{0.6}{\sqrt{16}} = 0.15$

ii. In this situation, can the distribution of the sample mean be approximated by a normal distribution using the Central Limit Theorem? Explain why or why not.

Nope. Because the sample size $n = 16$ is too small.

b. If a random sample of 32 people who own the wireless sound system is selected, find the probability that the mean lifetime of the sample will be less than 3.4 years.

$\mu_{\bar{X}} = \mu = 3.7$

$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{0.6}{\sqrt{32}} = 0.1061$

$\bar{X} \sim N\left(3.7, 0.1061^2\right)$ by the Central Limit Theorem

$P\left(\bar{X} < 3.4\right) = P\left(Z < \frac{3.4 - 3.7}{0.1061}\right) = P(Z < -2.83) = P(Z > 2.83) = 0.00233$