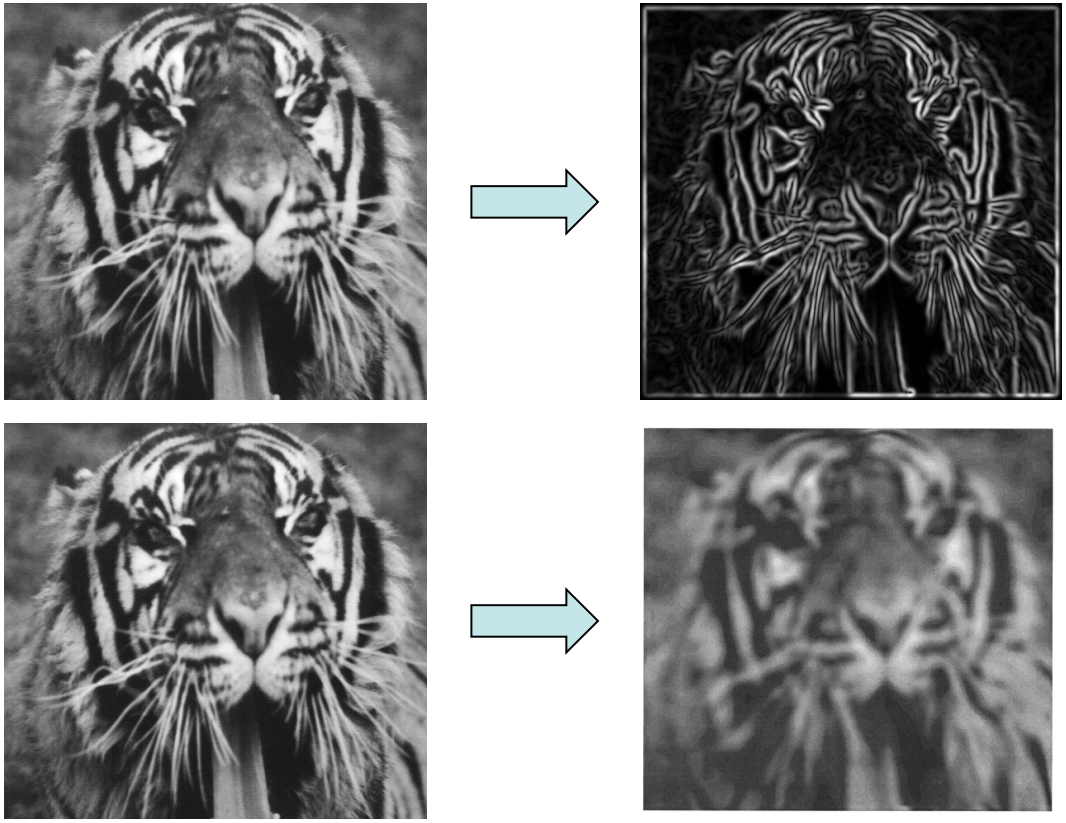
Basic Filters (7)

- Convolution/correlation/Linear filtering
- Gaussian filters
- Smoothing and noise reduction
- First derivatives of Gaussian
- Second derivative of Gaussian: Laplacian
- Oriented Gaussian filters
- Steerability



Convolution

- 1D Formula:

$$(h * f)(x) = \int_u h(x-u) f(u) du$$

$$(h * f)[x] = \sum_i h[x-i] f[i]$$

- 2D Formula:

$$(h * f)(x,y) = \int_u h(x-u, y-v) f(u,v) du$$

$$(h * f)[x,y] = \sum_i h[x-i, y-j] f[i,j]$$

- Example on the web:
  - http://www.jhu.edu/~signals/convolve/


Original Image
50
100
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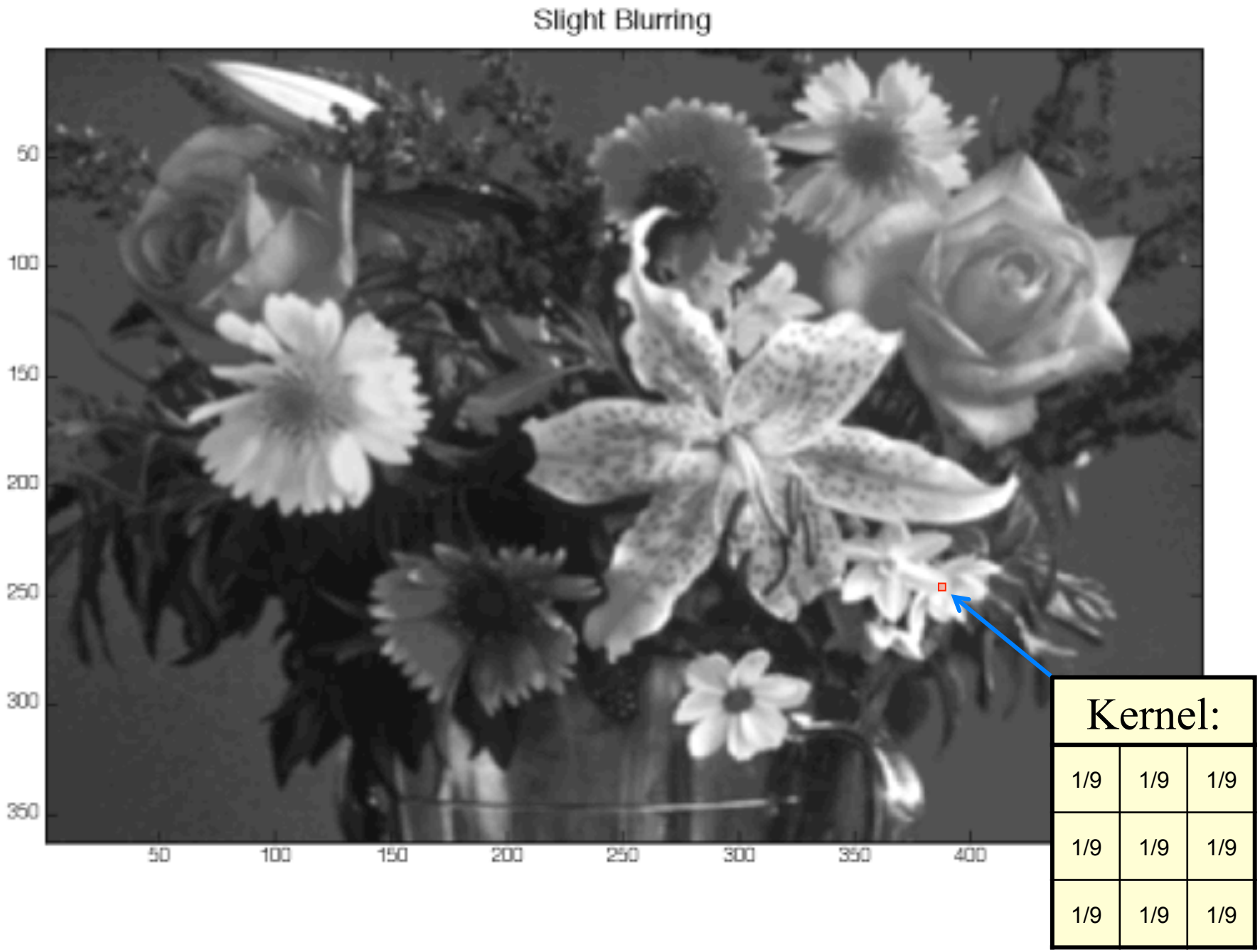
500


Slight Blurring
50
100
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50
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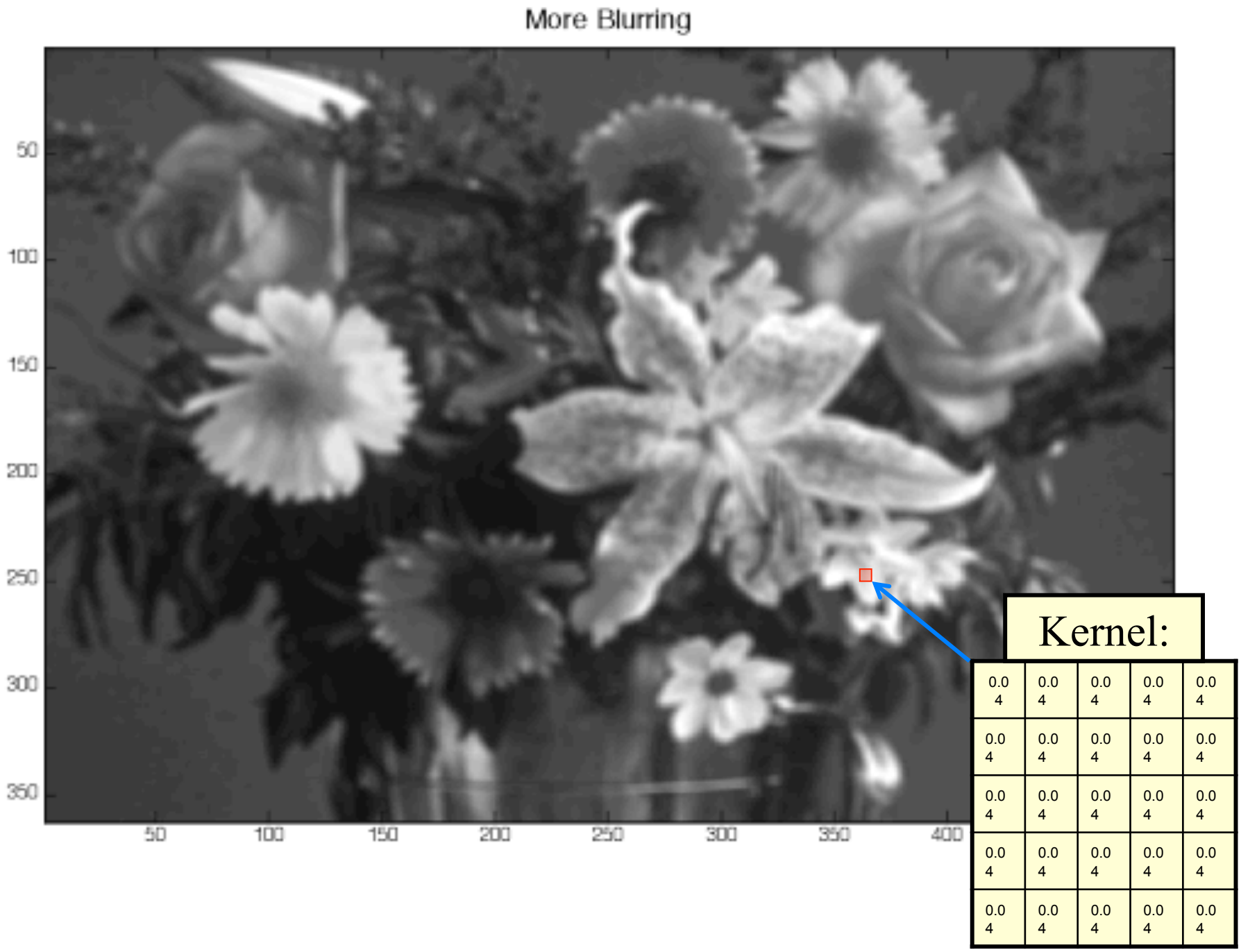
350
400
Kernel:
1/9 1/9 1/9
1/9 1/9 1/9
1/9 1/9 1/9


More Blurring
50
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Kernel:
0.04 0.04 0.04 0.04 0.04
0.04 0.04 0.04 0.04 0.04
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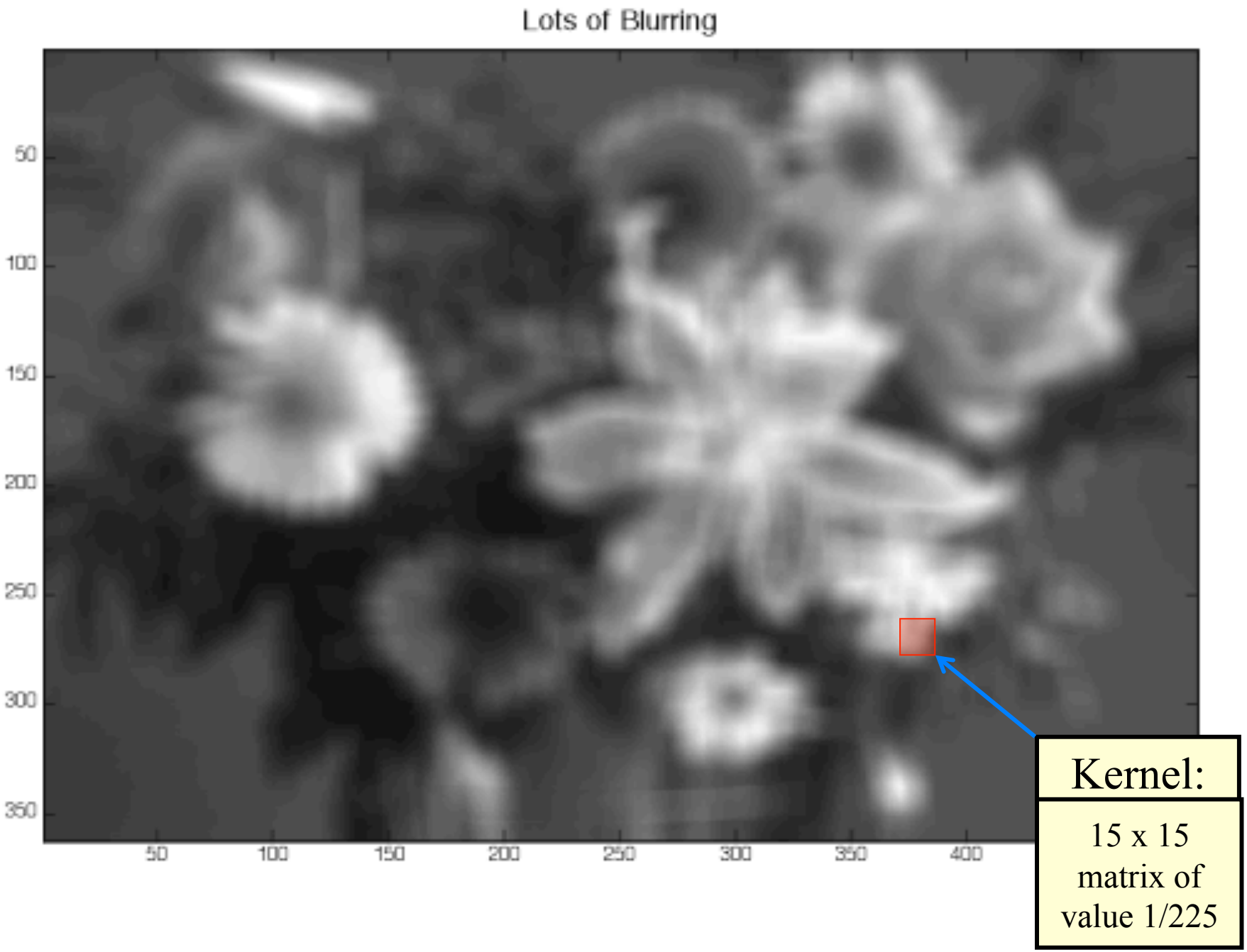
Lots of Blurring
50
100
150
200
250
300
350
50
100
150
200
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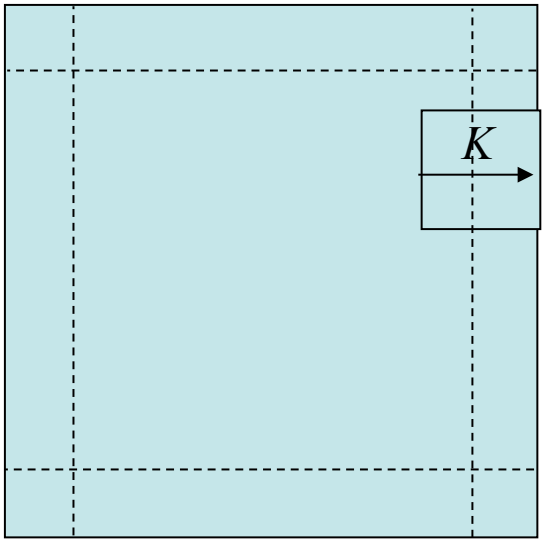
300
350
400
Kernel:
15 x 15 matrix of value 1/225

Basic Properties

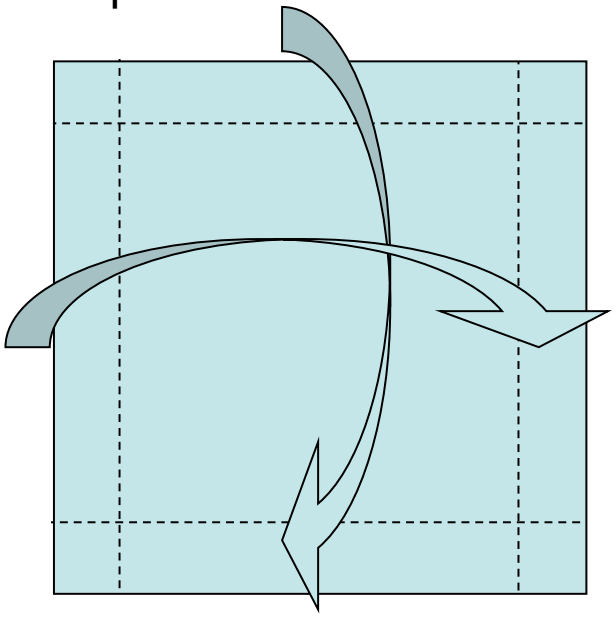
- Commutes: $f * g = g * f$
- Associative: $(f * g) * h = f * (g * h)$
- Linear: $(af + bg) * h = a\, f * h + b\, g * h$
- Shift invariant: $f_t * h = (f * h)_t$
- Only operator both linear and shift invariant
- Differentiation: $$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$$

Practicalities (discrete convolution/correlation)

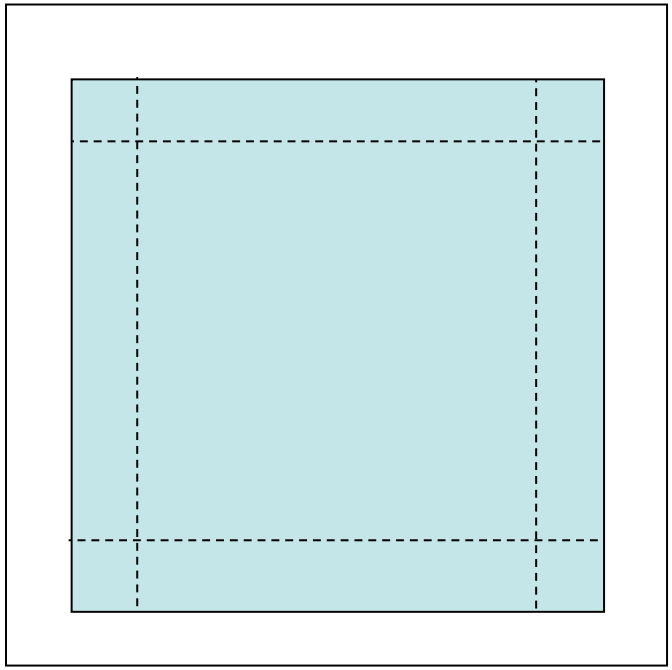
- MATLAB: `conv` (1D) or `conv2` (2D), `corr`
- Border issues:
  - When applying convolution with a KxK kernel, the result is undefined for pixels closer than K pixels from the border of the image
- Options:



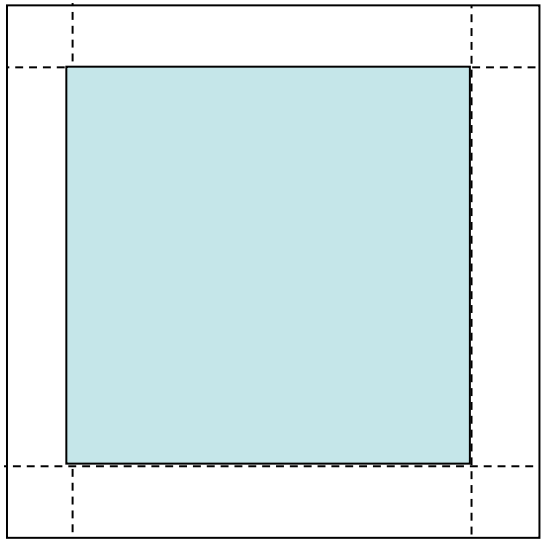
Warp around



Expand/Pad

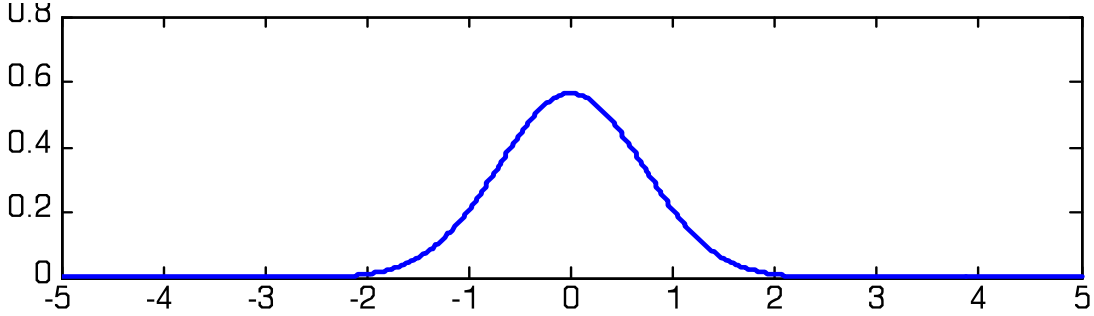


Crop



1-D:

$$g(x) = e^{-\frac{x^2}{2\sigma^2}}$$

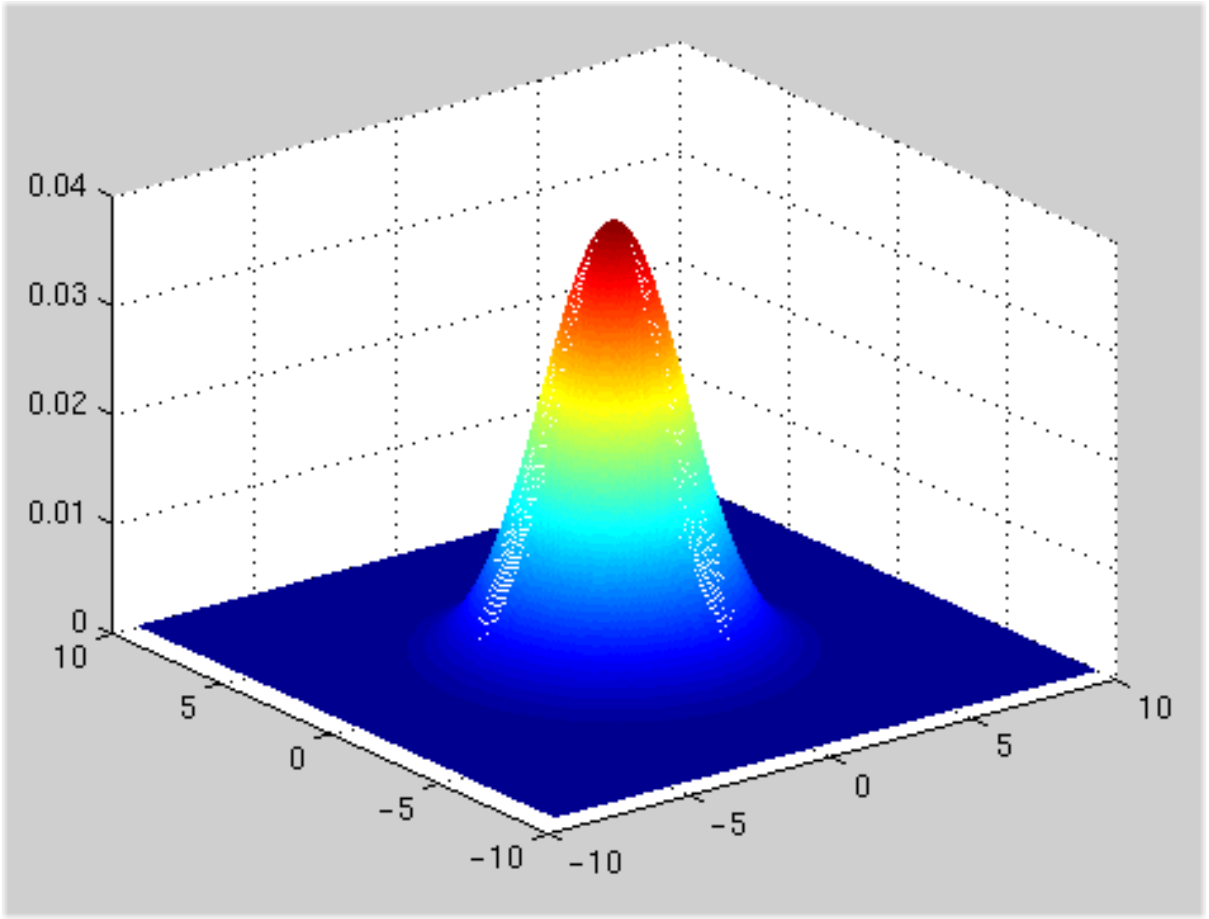


2-D:

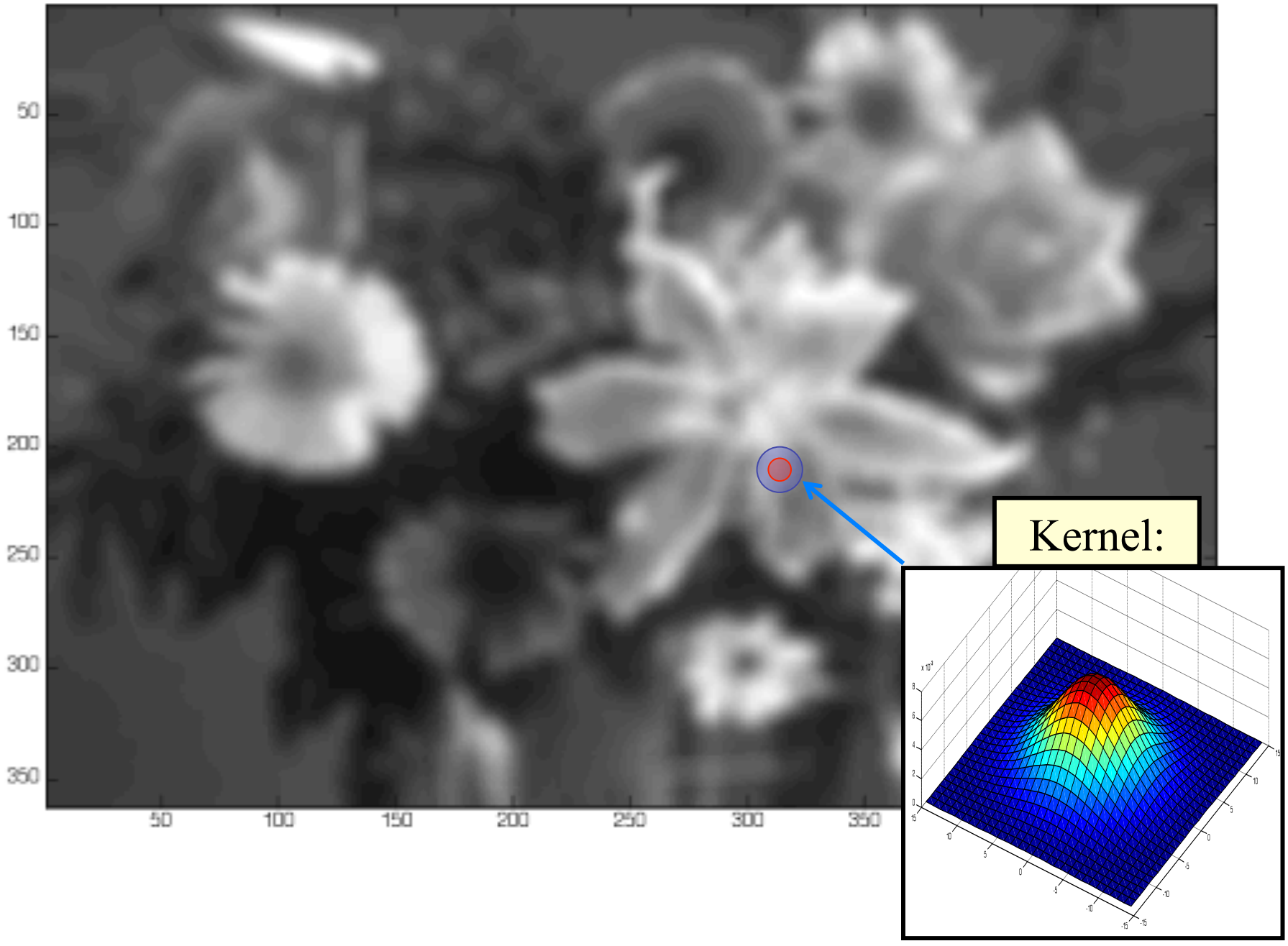
$$G(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Slight abuse of notations:
We ignore the normalization constant such that

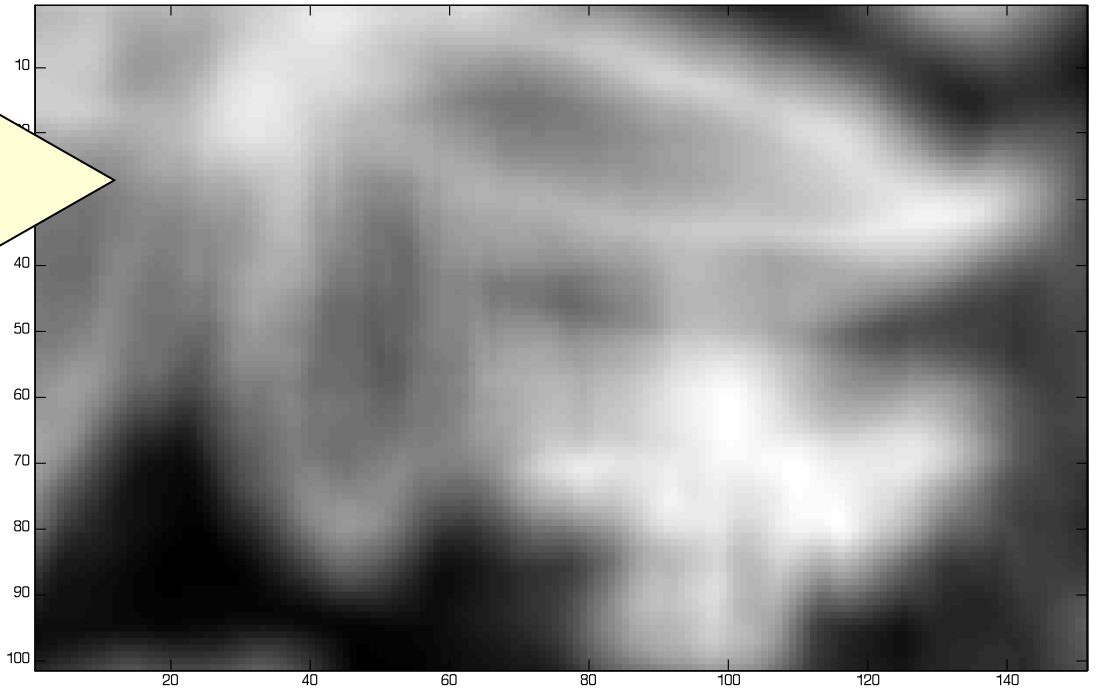
$$\int g(x)dx = 1$$



Gaussian Blurring, $\sigma = 5$



Simple Averaging



Gaussian Smoothing

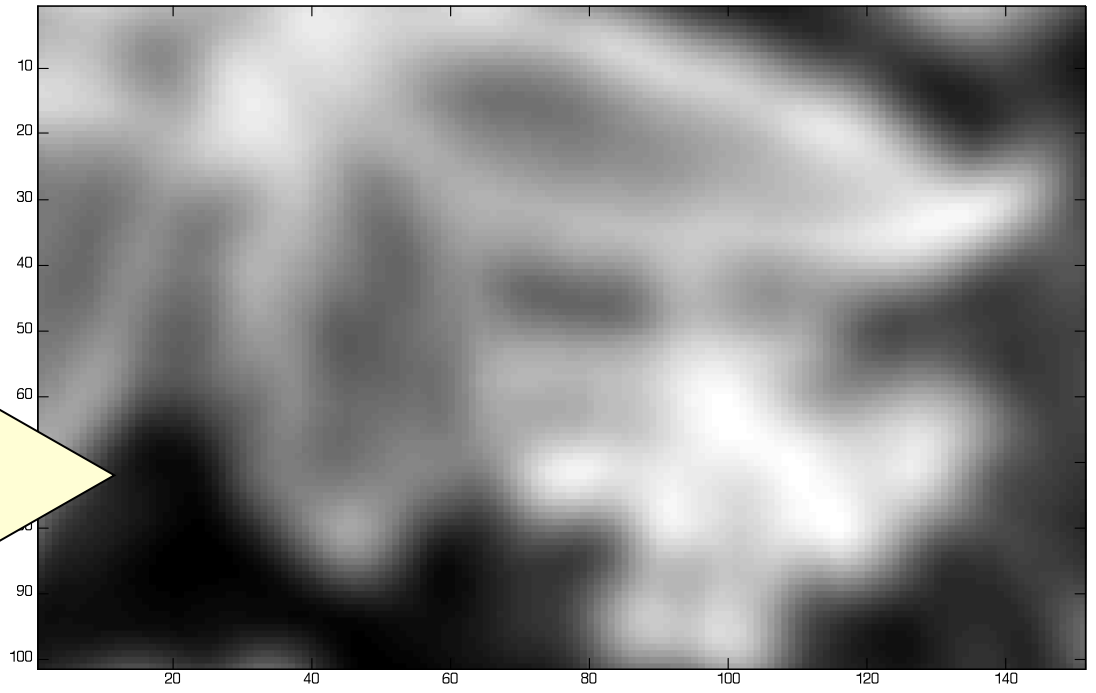
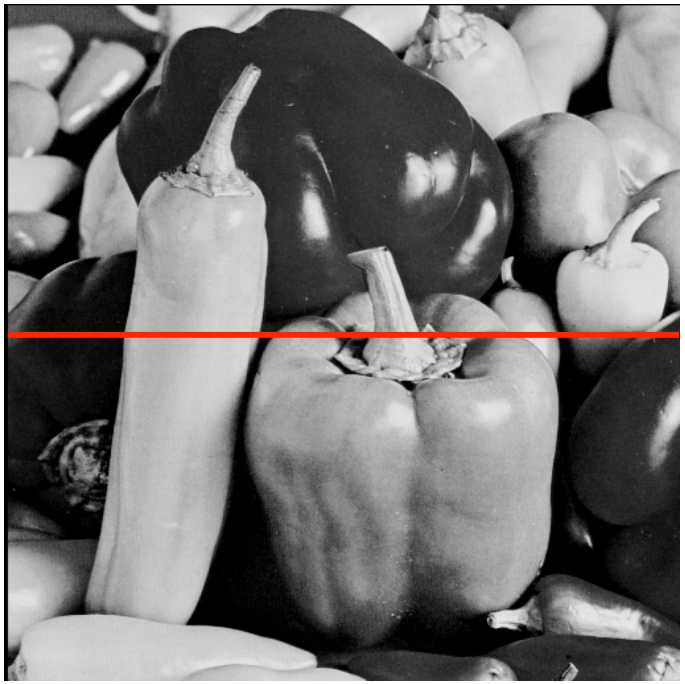
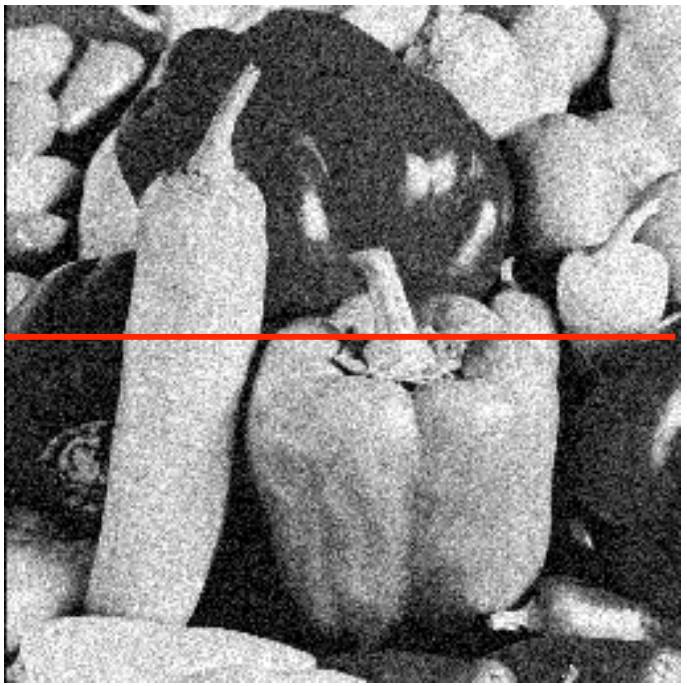
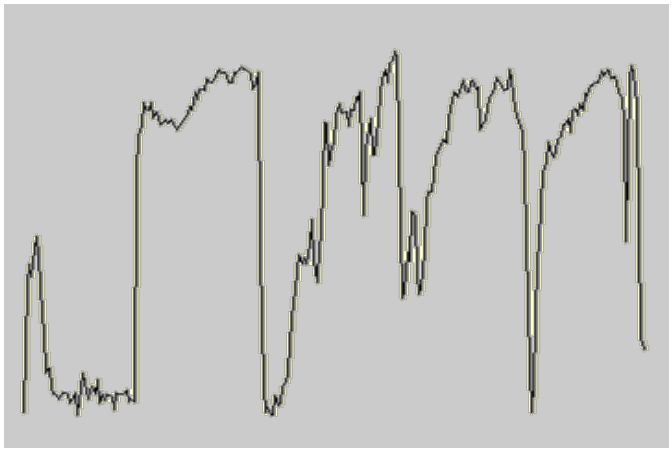
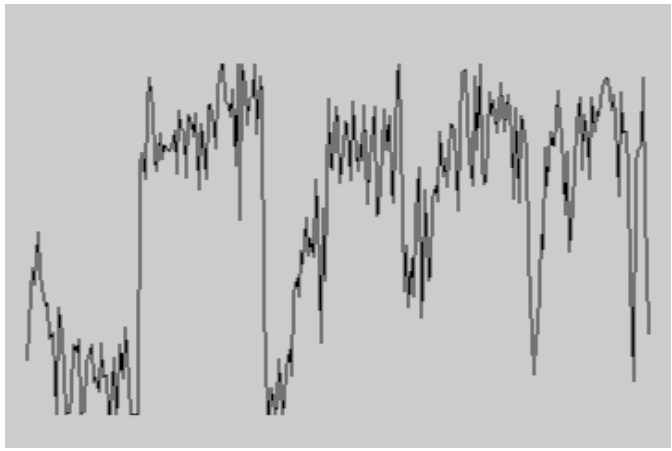


Image Noise







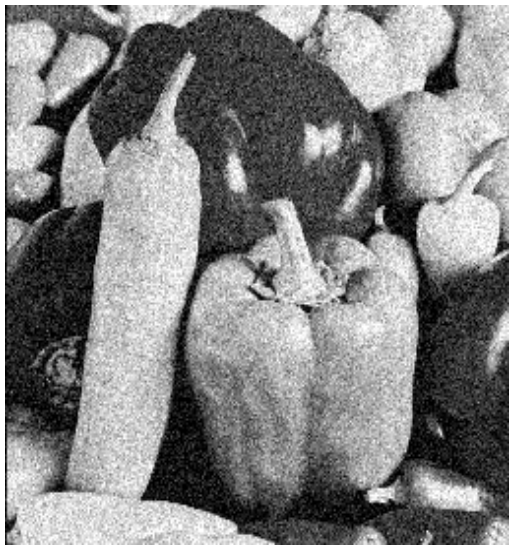


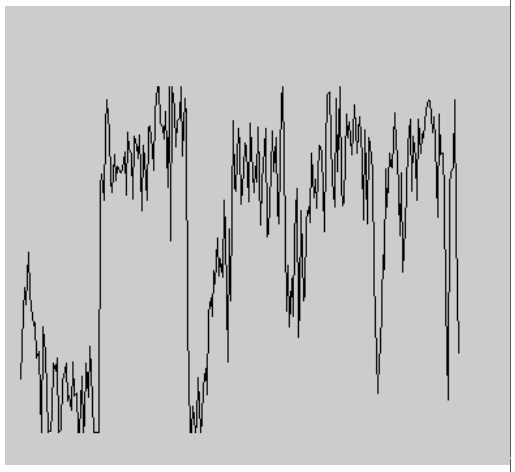
$$f(x,y) = \overbrace{\hat{f}(x,y)}^{\text{Ideal Image}} + \overbrace{\eta(x,y)}^{\text{Noise process}}$$

Gaussian i.i.d. ("white") noise:

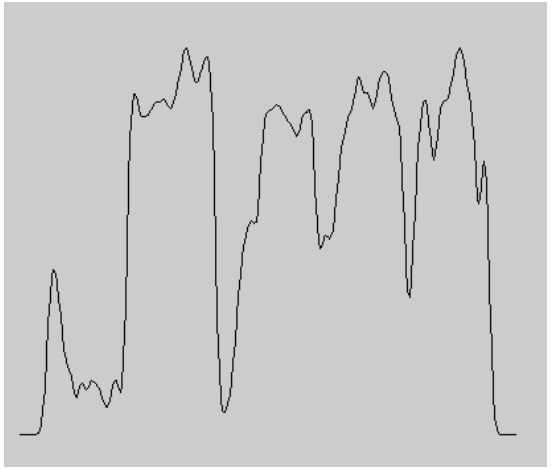
$\eta(x,y) \sim \mathcal{N}(\mu, \sigma)$

Gaussian Smoothing to Remove Noise





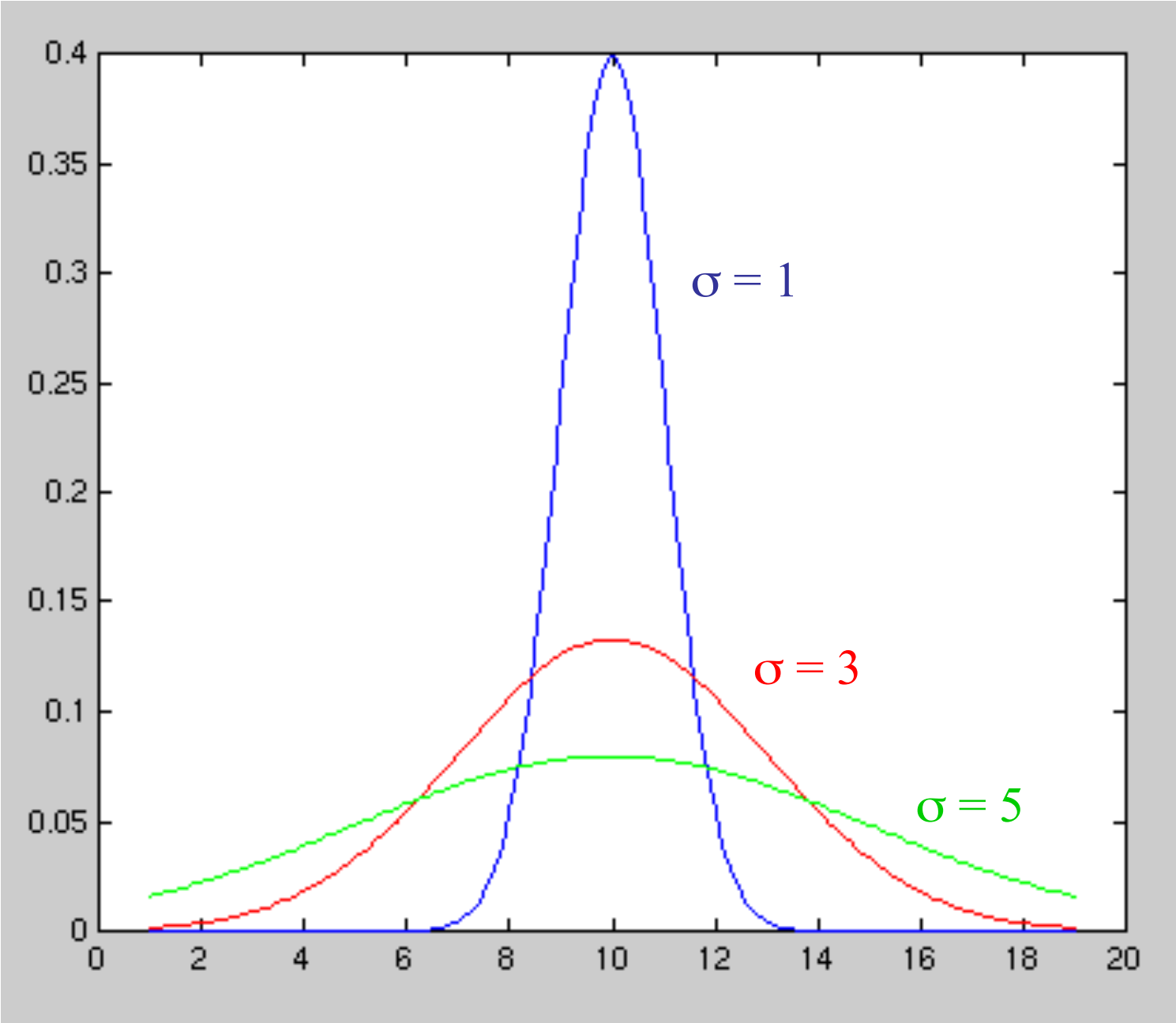
No smoothing



$\sigma = 2$

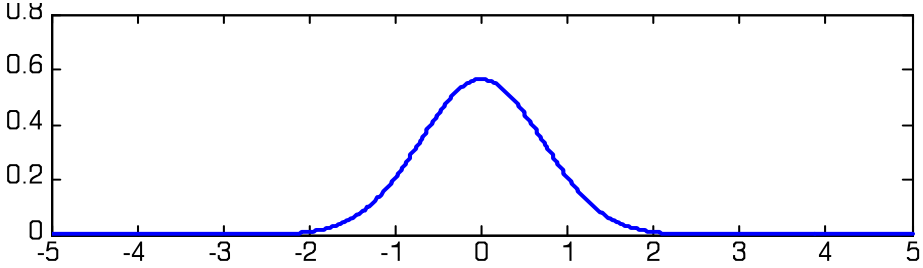
$\sigma = 4$

Shape of Gaussian filter as function of σ

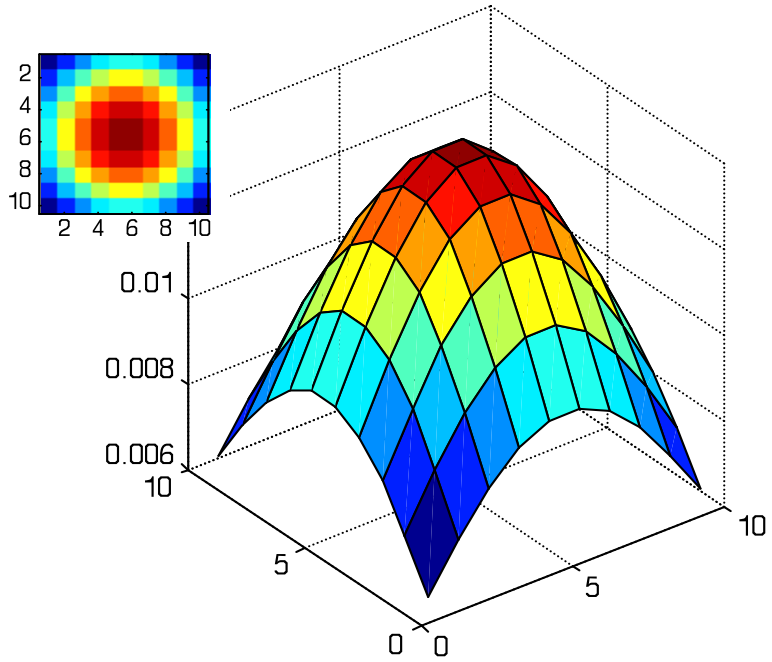


Note about Finite Kernel Support

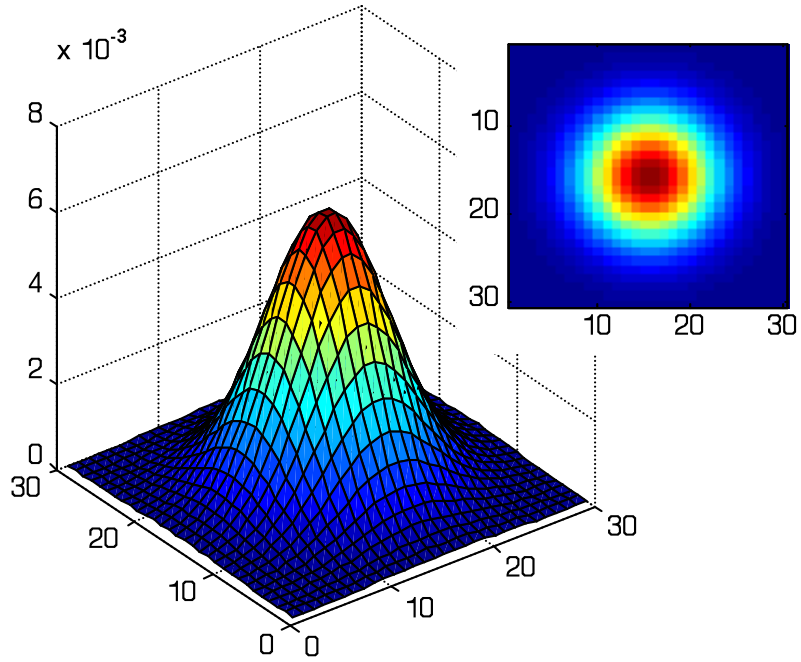
- Gaussian function has infinite support



- In discrete filtering, we have finite kernel



$\sigma = 5$ with 10x10 kernel



$\sigma = 5$ with 30x30 kernel

Increasing σ

Basic Properties

- Gaussian removes “high-frequency” components from the image → “low pass” filter
- Larger σ remove more details
- Combination of 2 Gaussian filters is a Gaussian filter:

$$G_{\sigma_1} * G_{\sigma_2} = G_{\sigma} \quad \sigma^2 = \sigma_1^2 + \sigma_2^2$$

- Separable filter:

$$G_{\sigma} * f = g_{\sigma\rightarrow} * g_{\sigma\uparrow} * f$$

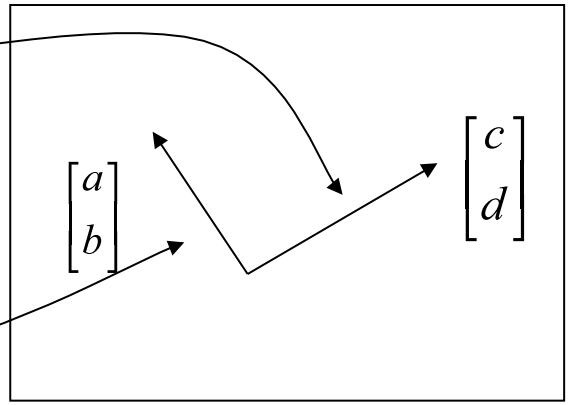
- Critical implication: Filtering with a *N*x*N* Gaussian kernel can be implemented as two convolutions of size *N* → reduction quadratic to linear → *must* be implemented that way

https://gamedev.stackexchange.com/questions/70075/how-can-i-find-the-perpendicular-to-a-2d-vector
So { x, y } becomes { y, -x }.

Oriented Gaussian Filters

- G_σ smoothes the image by the same amount in all directions
- If we have some information about preferred directions, we might want to smooth with some value σ_1 in the direction defined by the unit vector [a b] and by σ_2 in the direction defined by [c d]

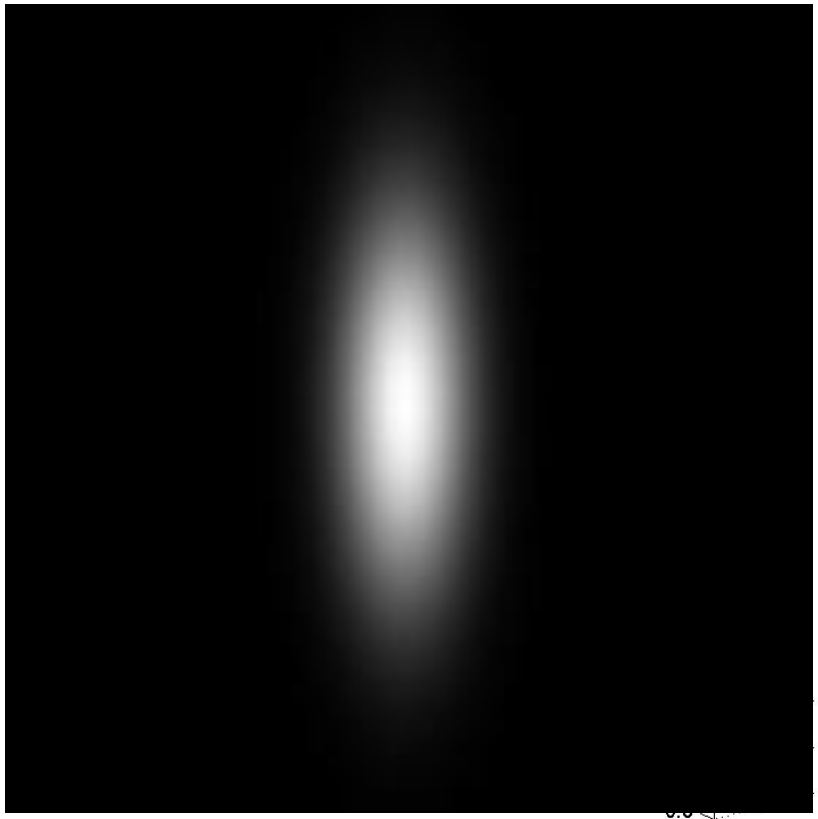
$$G = e^{-\frac{(ax+by)^2}{2\sigma_1^2} - \frac{(cx+dy)^2}{2\sigma_2^2}}$$



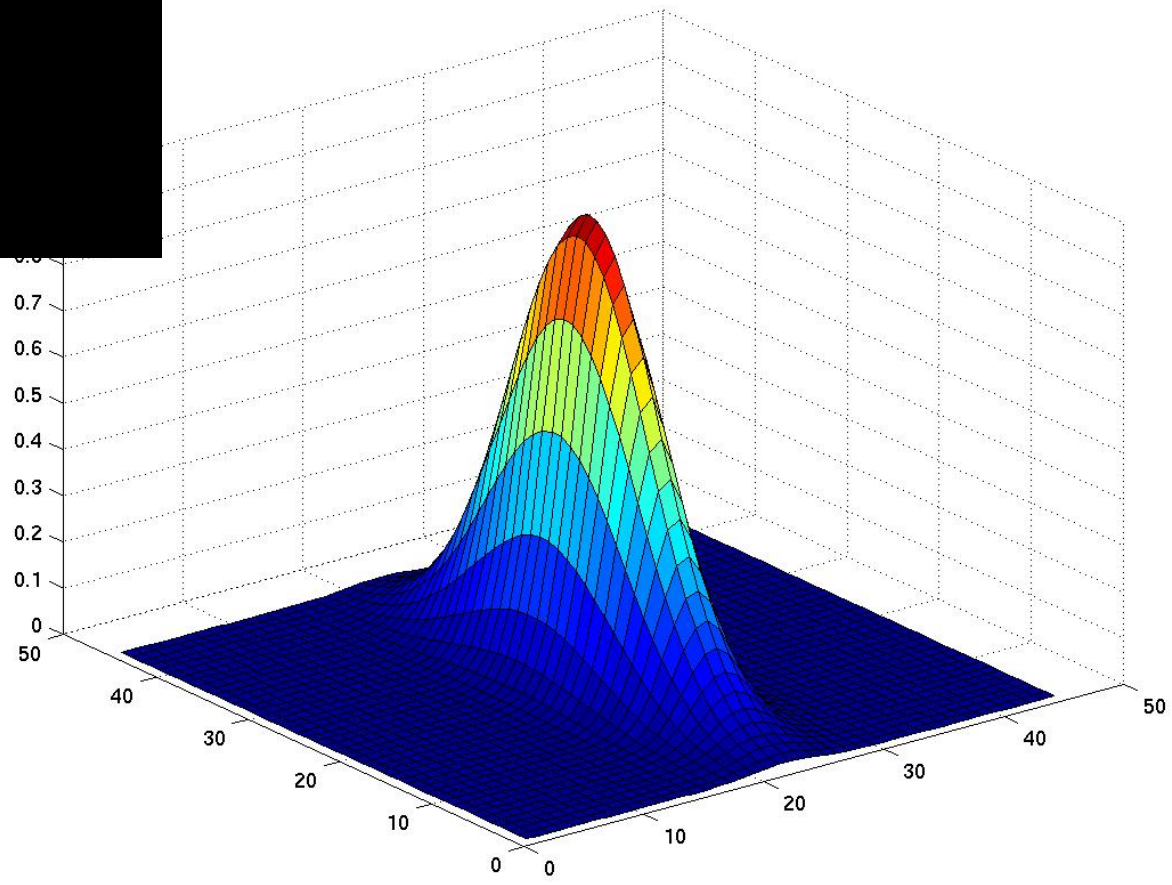
- We can write this in a more compact form by using the standard multivariate Gaussian notation:

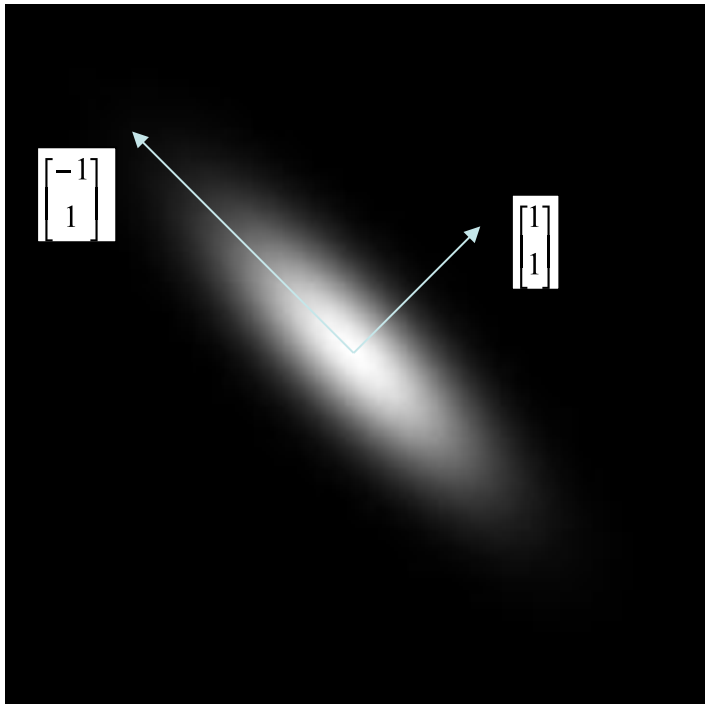
$$G_\Sigma = e^{-\frac{X^T \Sigma^{-1} X}{2}} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

- The two (orthogonal) directions of filtering are given by the eigenvectors of Σ, the amount of smoothing is given by the square root of the corresponding eigenvalues of Σ.



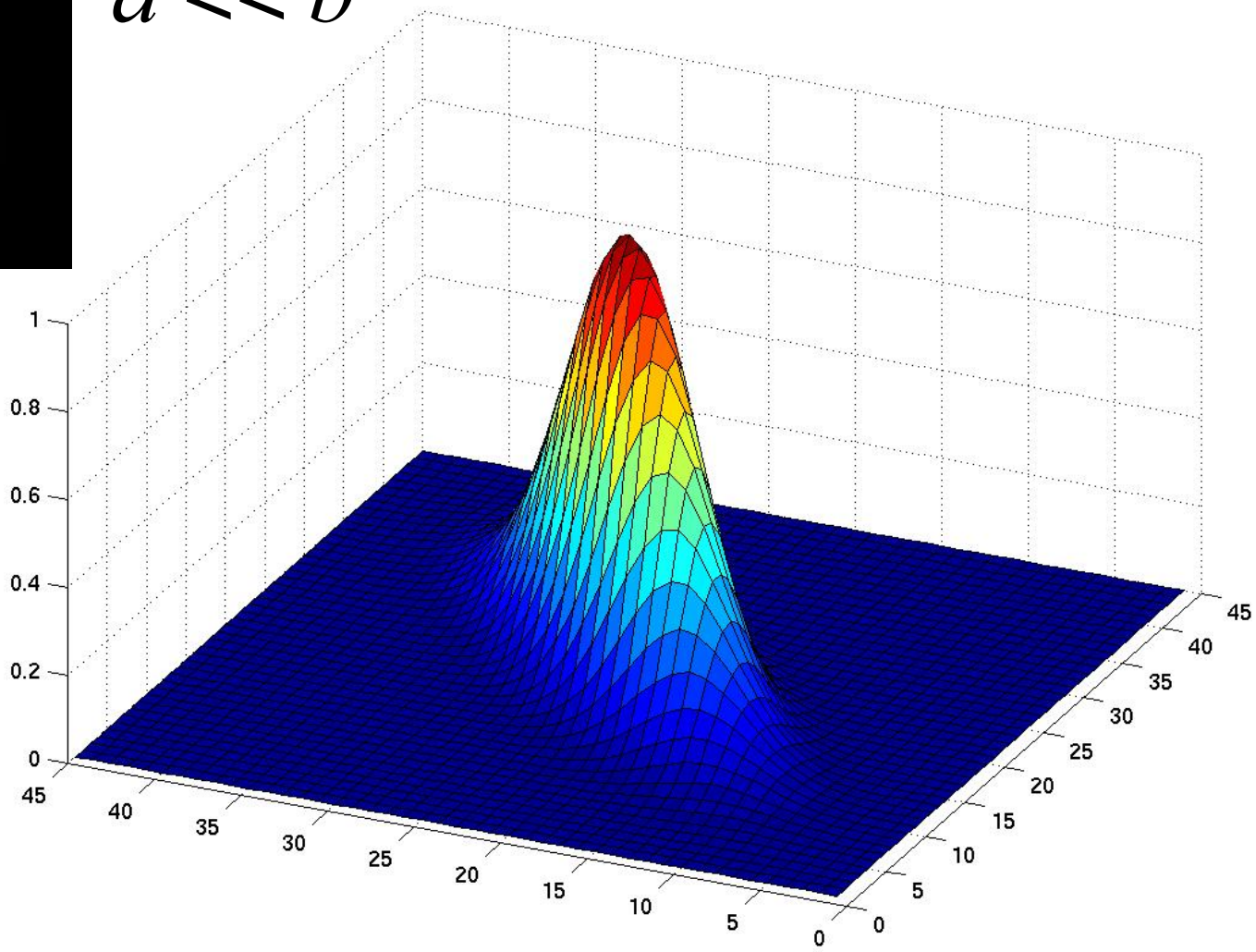
$$\Sigma = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad a << b$$





$$\Sigma = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$a << b$$



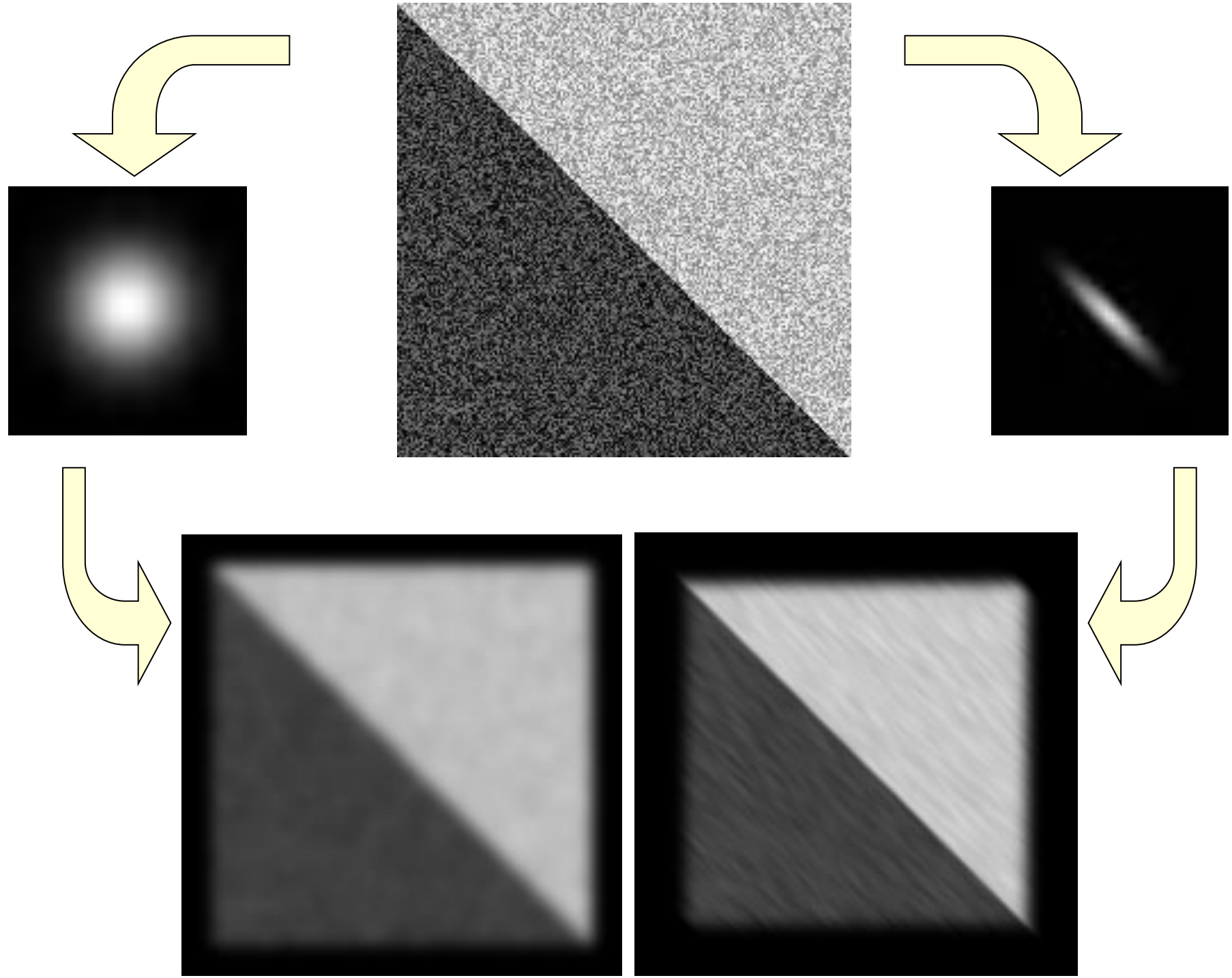


Image Derivatives

- Image Derivatives
- Derivatives increase noise
- Derivative of Gaussian
- Laplacian of Gaussian (LOG)

Image Derivatives

- We want to compute, at each pixel (x,y) the derivatives:
- In the discrete case we could take the difference between the left and right pixels:

$$\frac{\partial I}{\partial x} \approx I(i+1, j) - I(i-1, j)$$

- Convolution of the image by

$$\partial_x = \boxed{-1 \mid 0 \mid 1}$$

- Problem: Increases noise

$$\underbrace{I(i+1, j) - I(i-1, j)}_{\text{Difference between Actual image values}} = \underbrace{\hat{I}(i+1, j) - \hat{I}(i-1, j)}_{\text{True difference (derivative)}} \underbrace{+ n_+ + n_-}_{\text{Twice the amount of noise as in the original image}}$$

Original Image

Noise Added

Derivative in the horizontal direction

Smooth Derivatives

- Solution: First smooth the image by a Gaussian G_σ and then take derivatives:

$$\frac{\partial f}{\partial x} \approx \frac{\partial \left(G_\sigma * f\right)}{\partial x}$$

- Applying the differentiation property of the convolution:

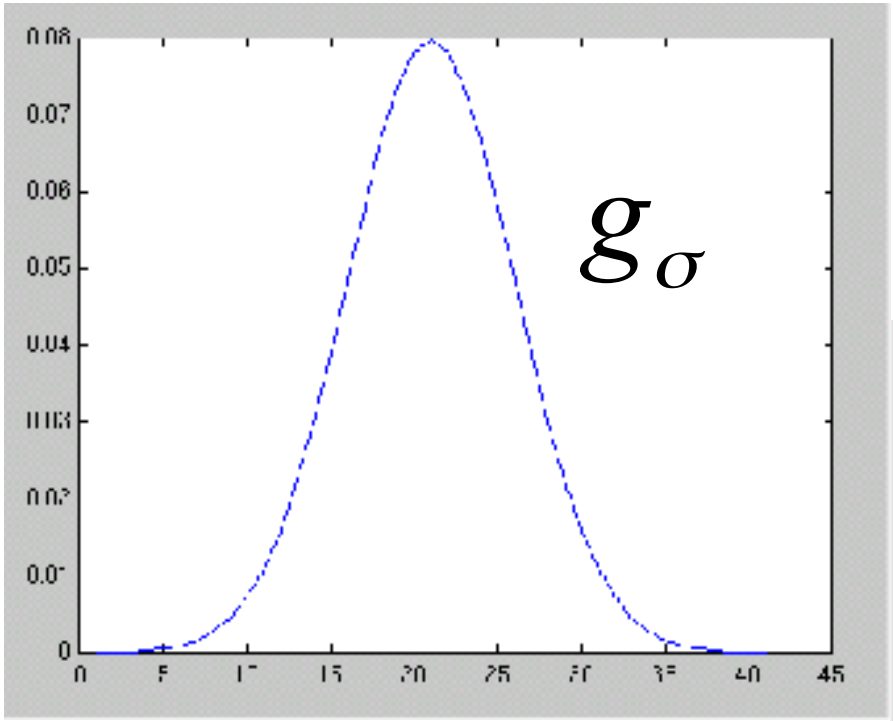
$$\frac{\partial f}{\partial x} \approx \frac{\partial G_\sigma}{\partial x} * f$$

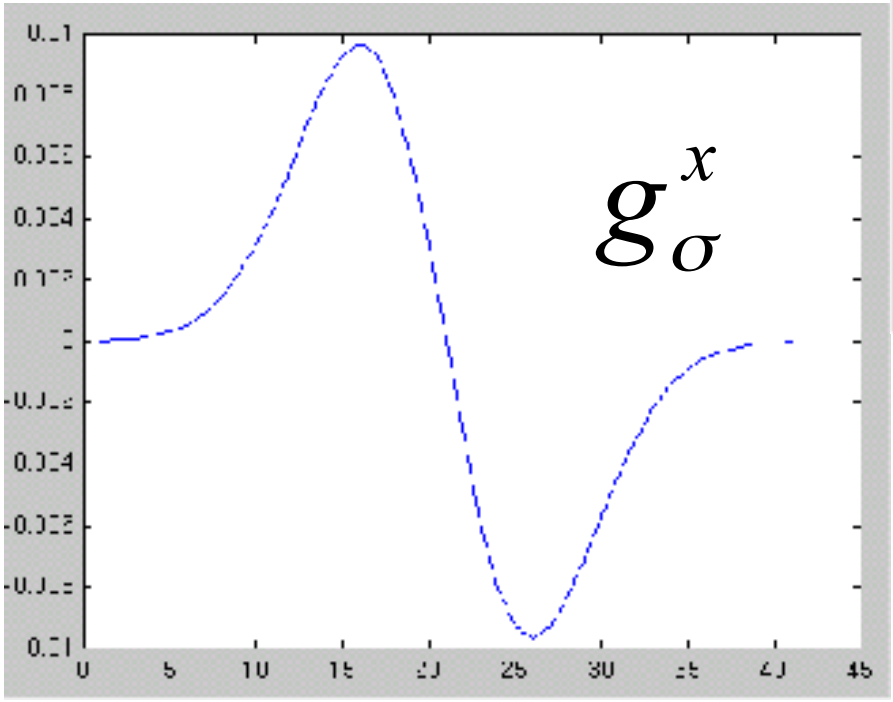
- Therefore, taking the derivative in x of the image can be done by convolution with the derivative of a Gaussian:

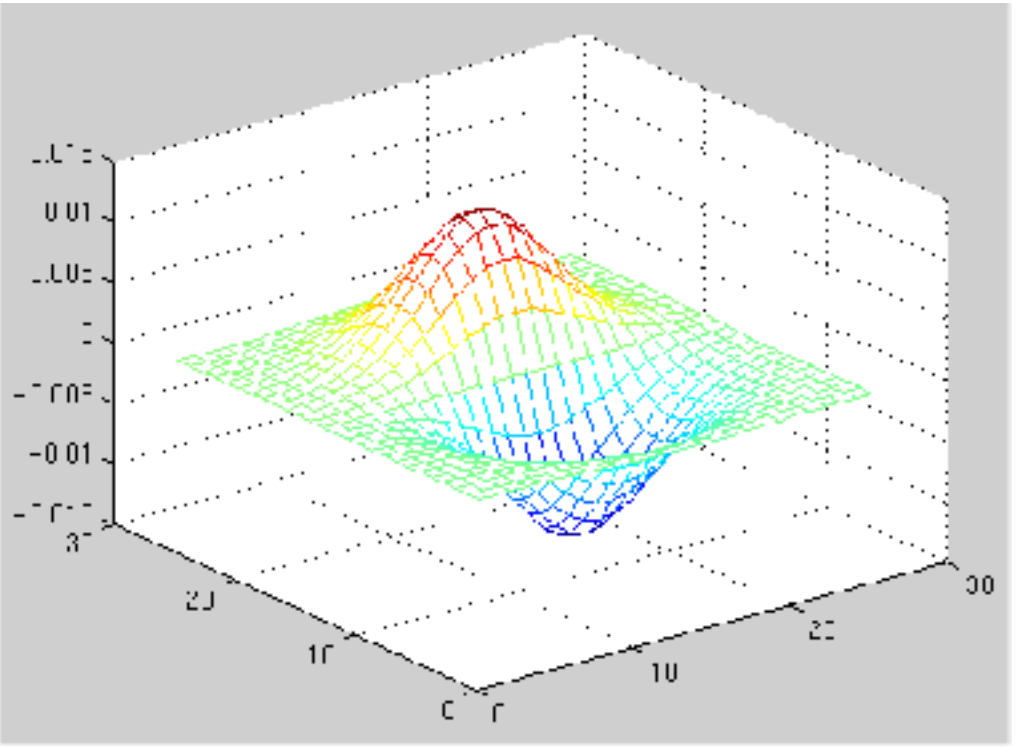
$$G_\sigma^x = \frac{\partial G_\sigma}{\partial x} = xe^{-\frac{x^2+y^2}{2\sigma^2}}$$

- Crucial property: The Gaussian derivative is also separable:

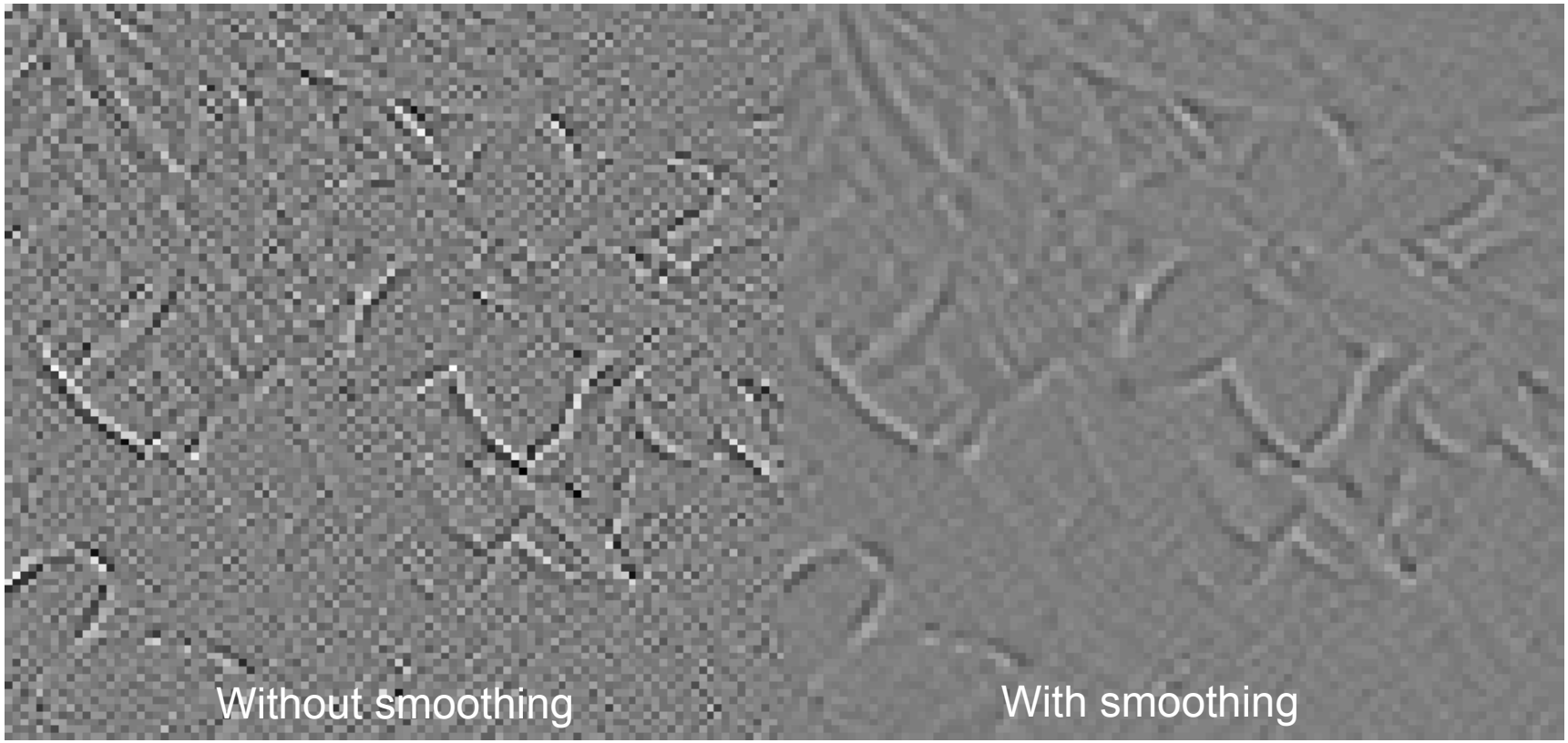
$$G_\sigma^x * f = g_\sigma^x * g_{\sigma\uparrow} * f$$


g_σ


g_σ^x



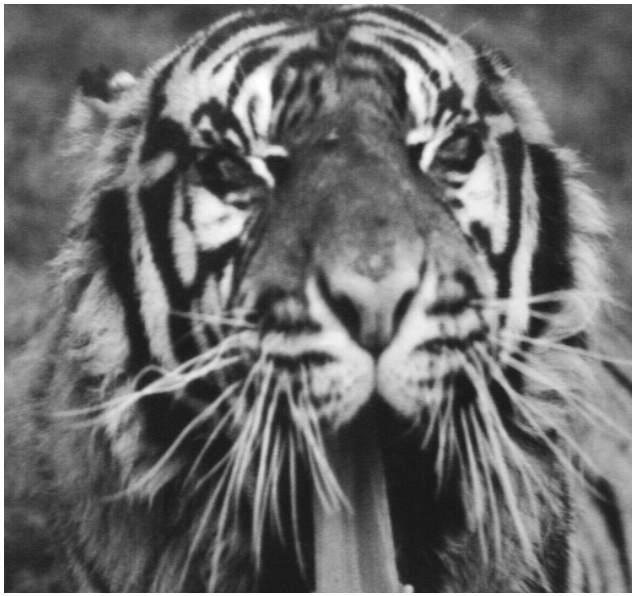
Derivative + Smoothing



Better but still blurs away edge information

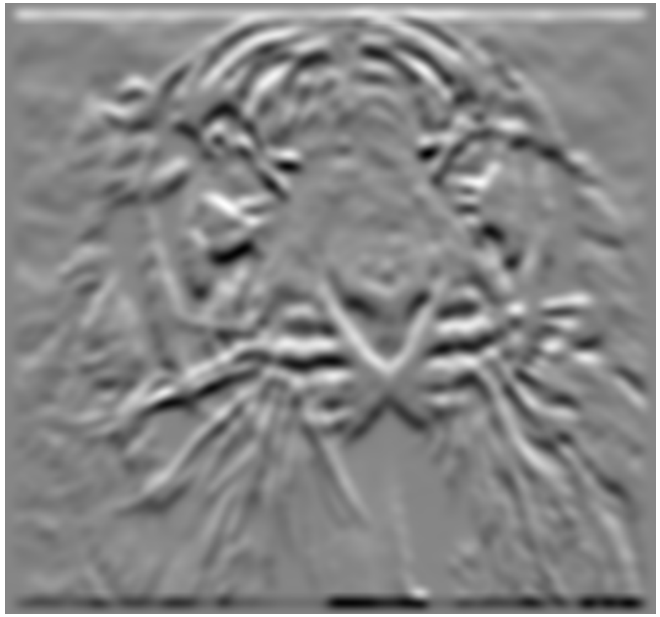
Applying the first derivative of Gaussian

I



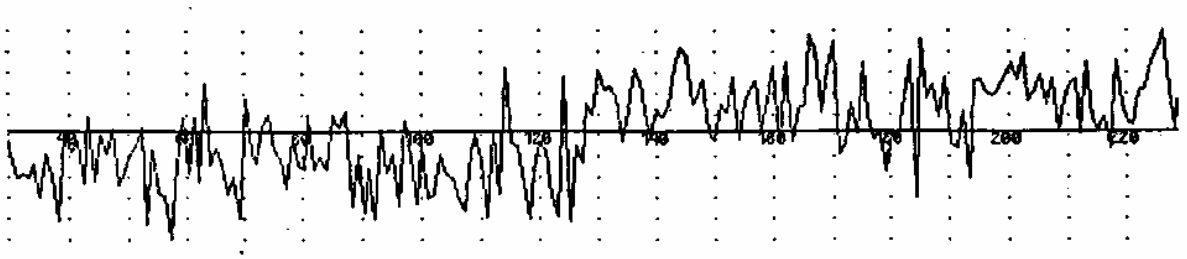
$$\left|\nabla I\right| = \sqrt{\frac{\partial I}{\partial x}^2 + \frac{\partial I}{\partial y}^2}$$

$\frac{\partial I}{\partial x}$

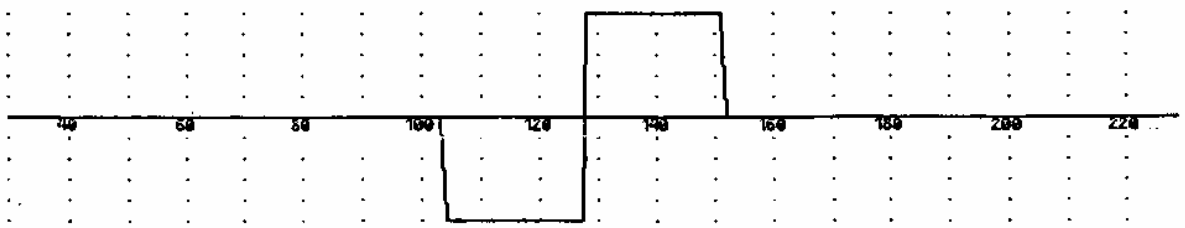


$\frac{\partial I}{\partial y}$

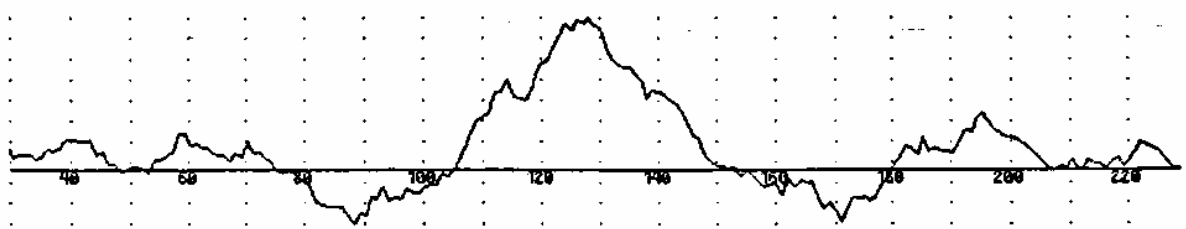
Input



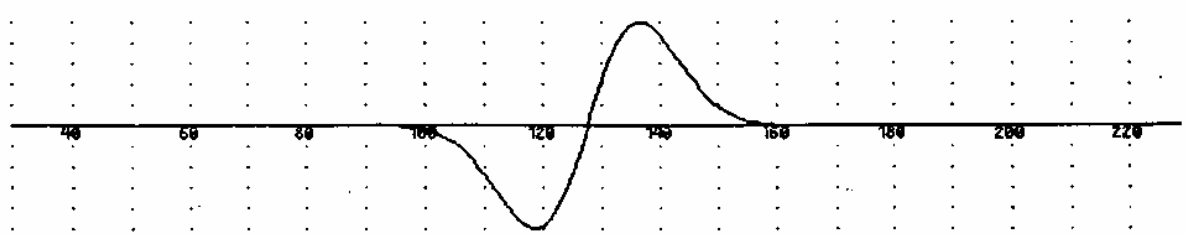
Difference operator



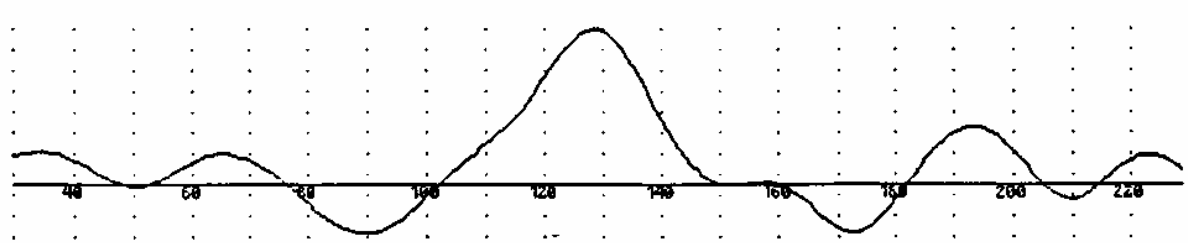
Derivative from difference operator



Gaussian derivative operator

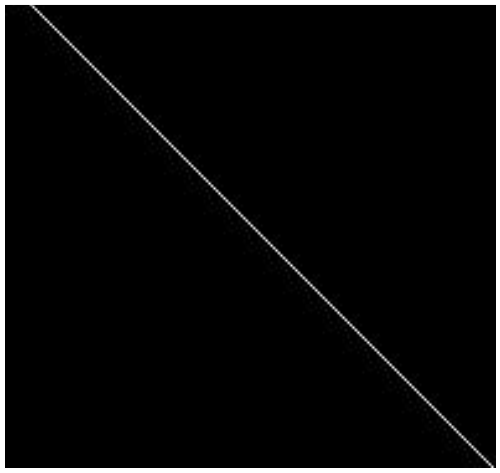


Derivative from Gaussian derivative



There is ALWAYS a tradeoff between smoothing and good edge localization!

Image with Edge



Edge Location

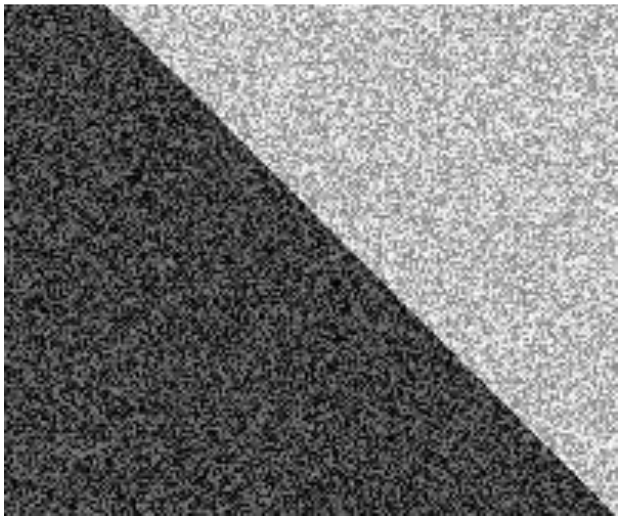
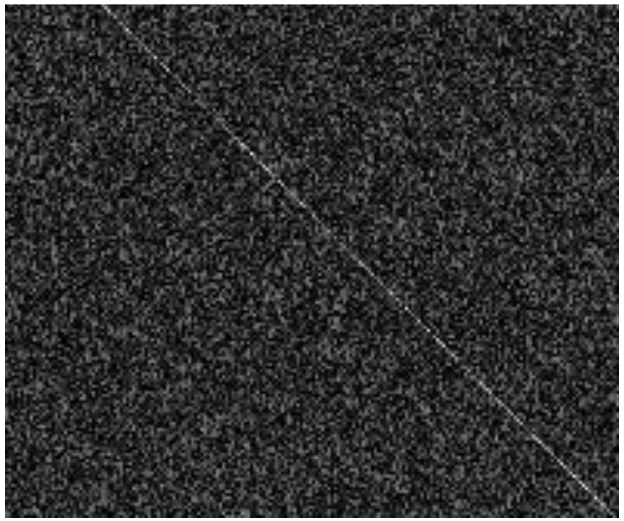
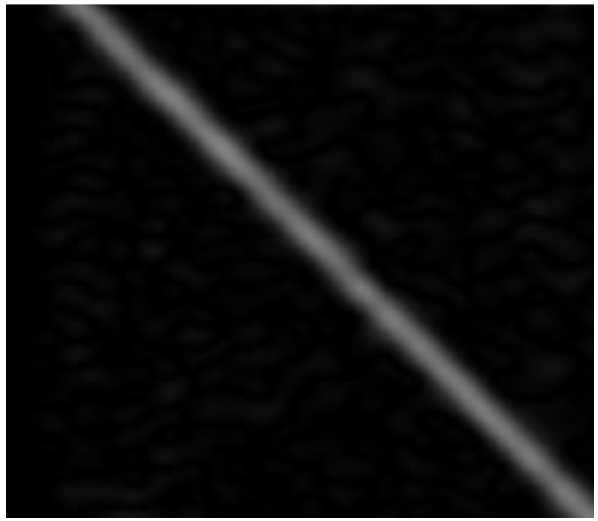


Image + Noise

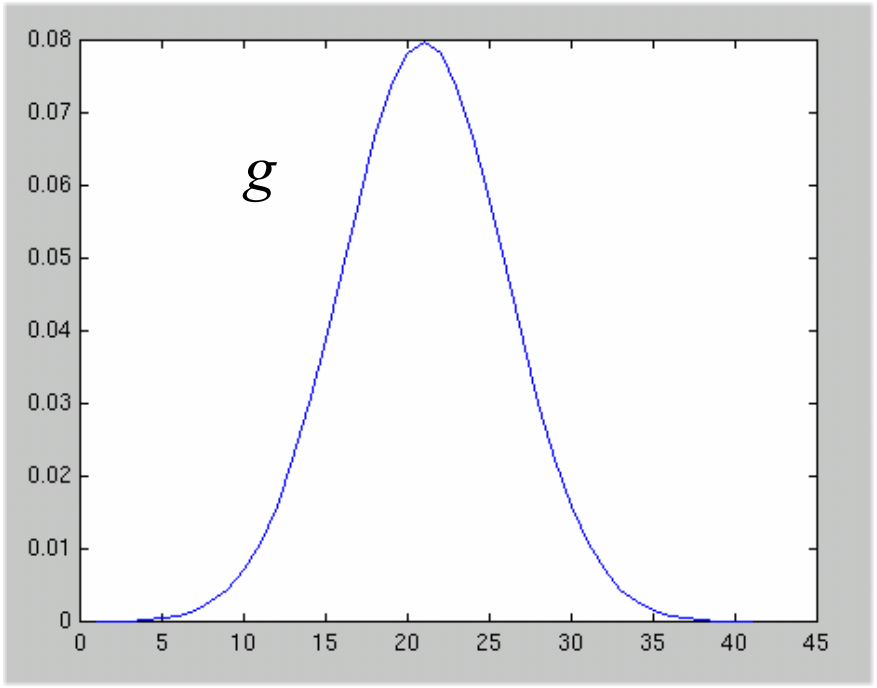


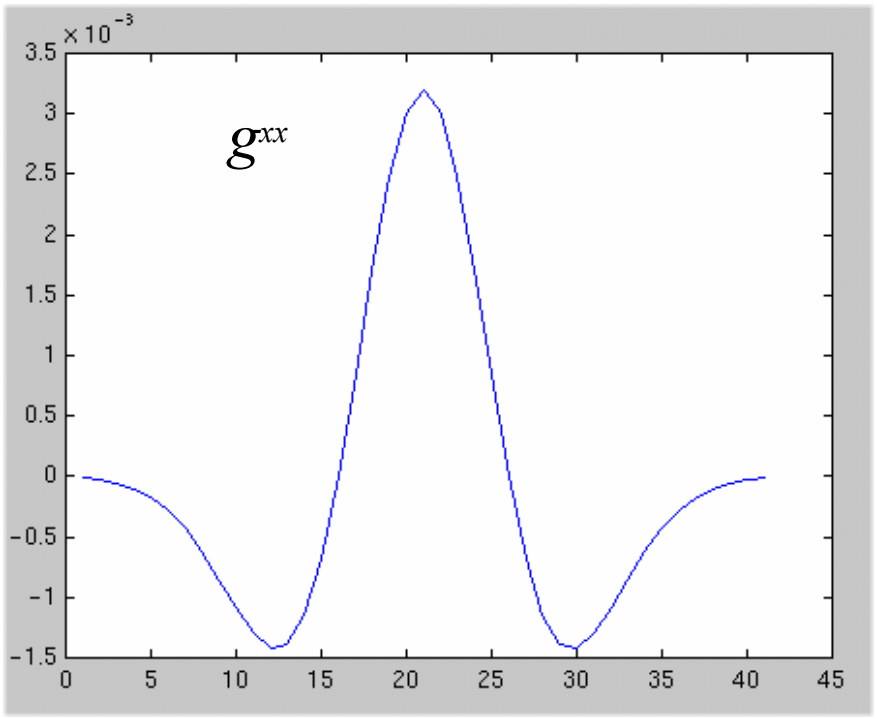
Derivatives detect edge *and* noise

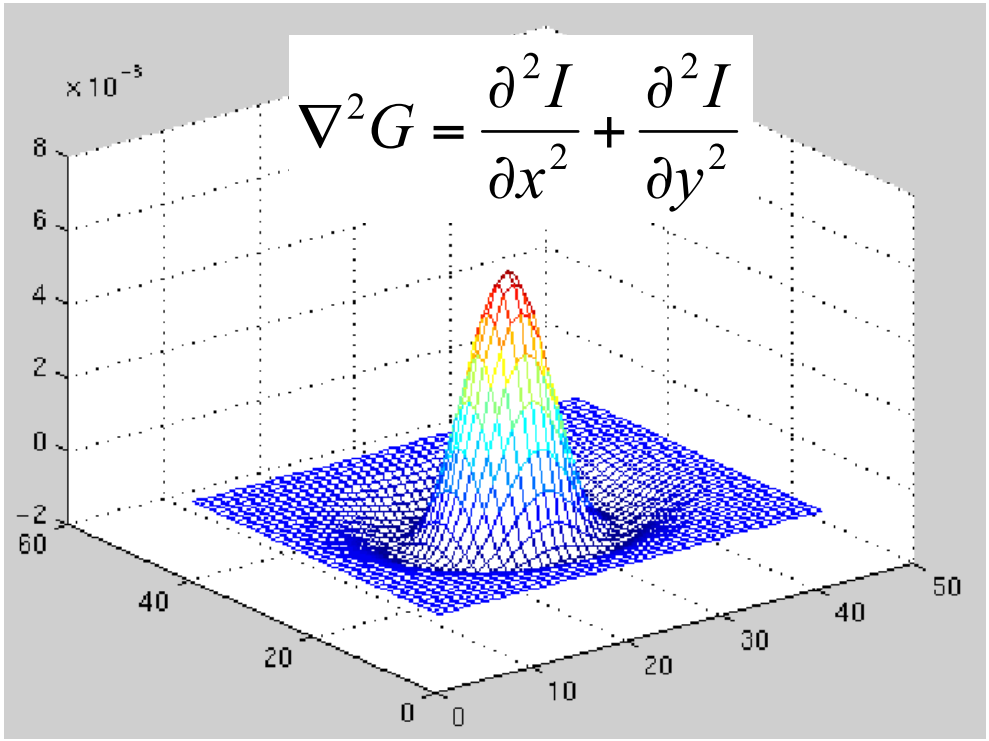


Smoothed derivative removes noise, but blurs edge

Second derivatives: Laplacian

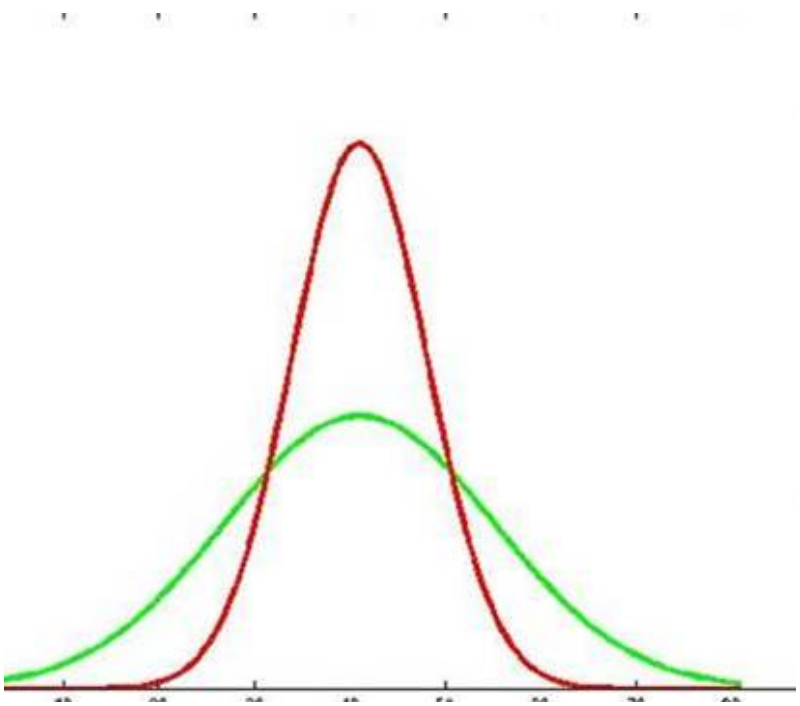


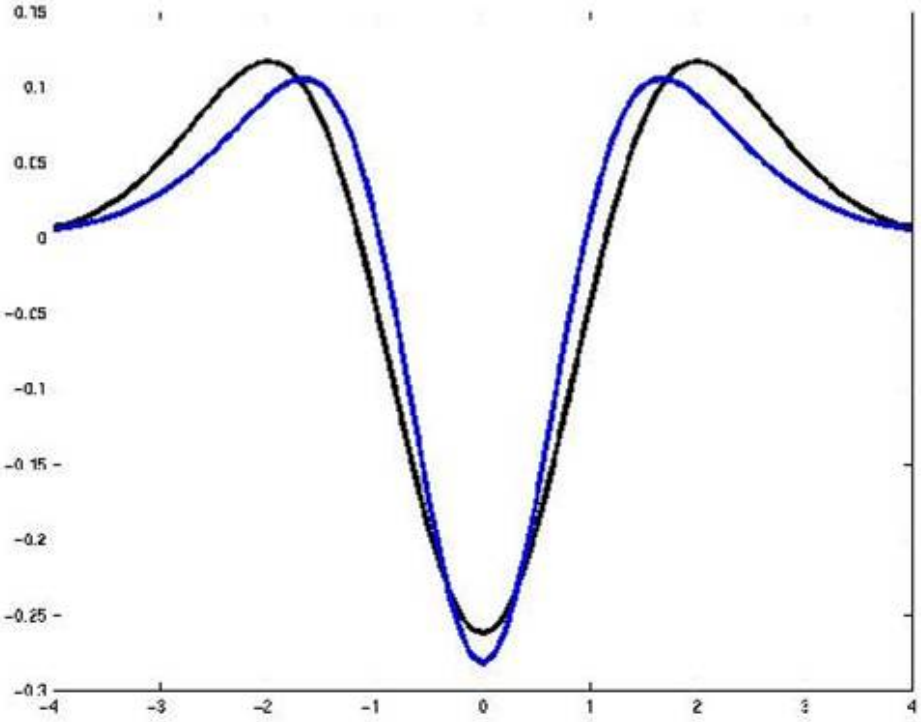




DOG Approximation to LOG

$$\nabla^2 G_\sigma \approx G_{\sigma_1} - G_{\sigma_2}$$





Gaussian

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Separable, low-pass filter

Derivatives of Gaussian

$$\frac{\partial G_\sigma(x,y)}{\partial x} \propto x e^{-\frac{x^2+y^2}{2\sigma^2}} \quad \frac{\partial G_\sigma(x,y)}{\partial y} \propto y e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\nabla G_\sigma = \left[\frac{\partial G_\sigma}{\partial x} \quad \frac{\partial G_\sigma}{\partial y} \right]^t$$

Separable, output of convolution is gradient at scale σ: $\nabla I = I * \nabla G_\sigma$

Laplacian

$$\nabla^2 G_\sigma(x,y) = \frac{\partial^2 G_\sigma(x,y)}{\partial x^2} + \frac{\partial^2 G_\sigma(x,y)}{\partial y^2}$$

Not-separable, approximated by A difference of Gaussians. Output of convolution is Laplacian of image: Zero-crossings correspond to edges

---

Directional Derivatives

$$\cos\theta \frac{\partial G_\sigma}{\partial x} + \sin\theta \frac{\partial G_\sigma}{\partial y}$$

Output of convolution is magnitude of derivative in direction θ. Filter is linear combination of derivatives in x and y

---

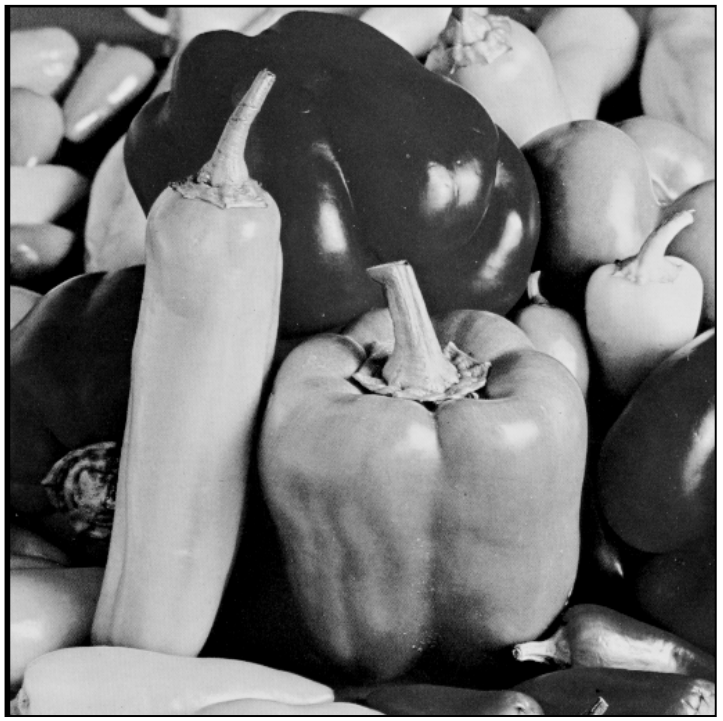
Oriented Gaussian

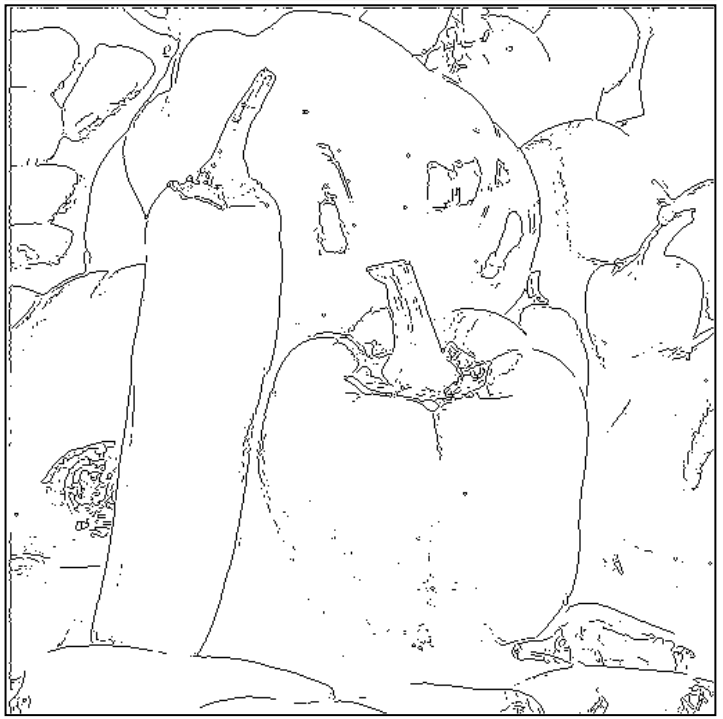
$$e^{-\frac{(a_1 x + b_1 y)^2}{2\sigma_1^2} - \frac{(a_2 x + b_2 y)^2}{2\sigma_2^2}}$$

Smooth with different scales in orthogonal directions

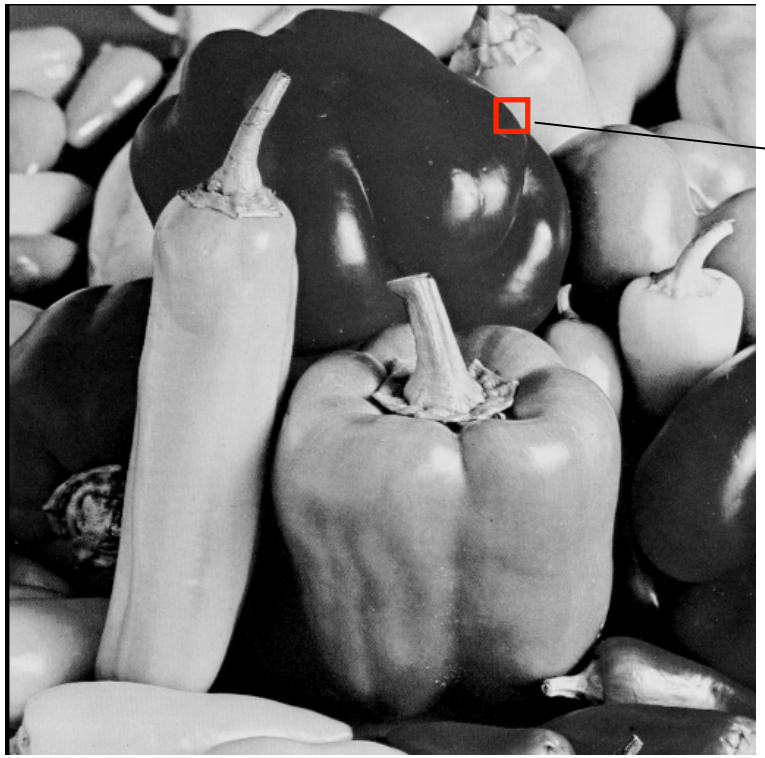
Edge Detection

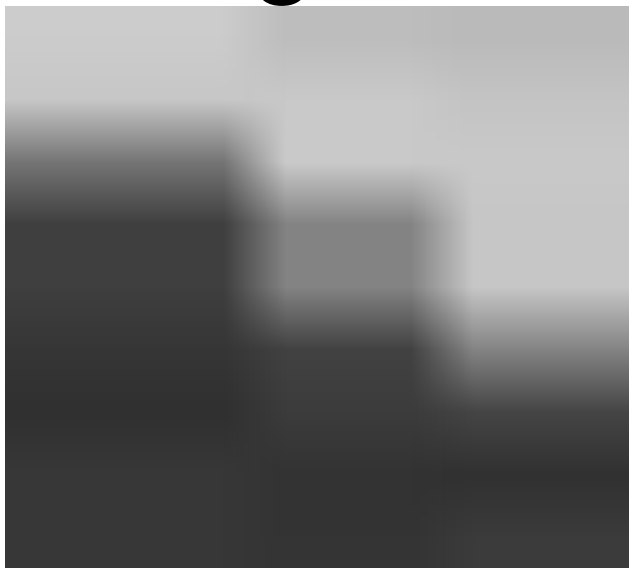
- Edge Detection
  - Gradient operators
  - Canny edge detectors
  - Laplacian detectors

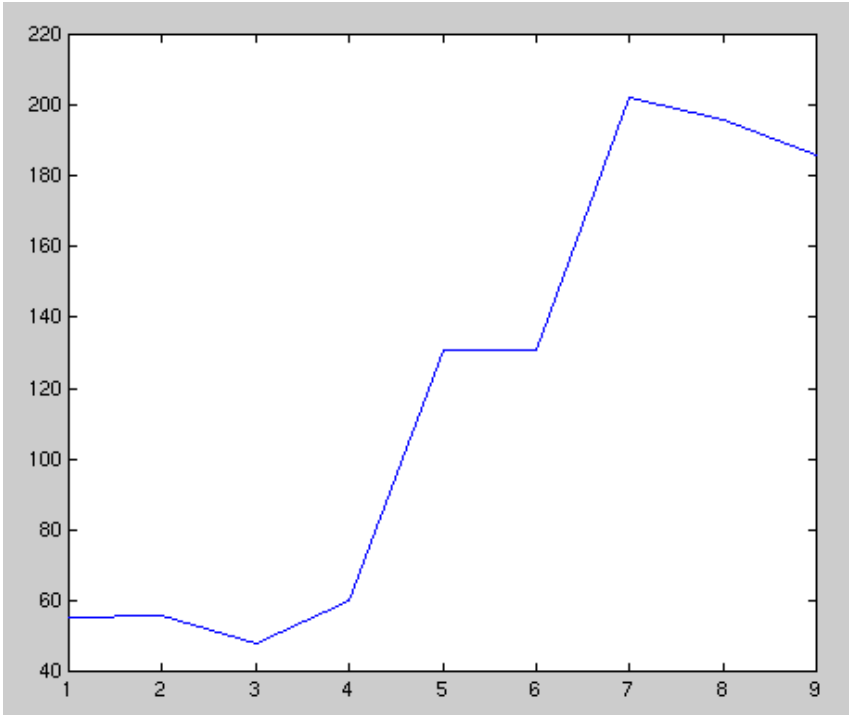




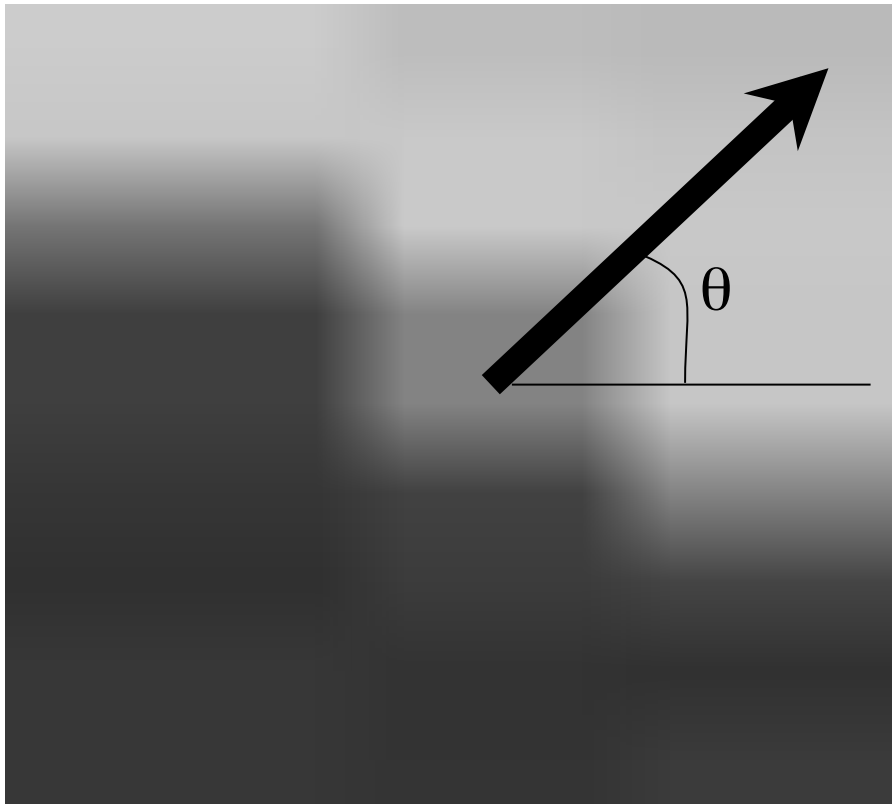
What is an edge?

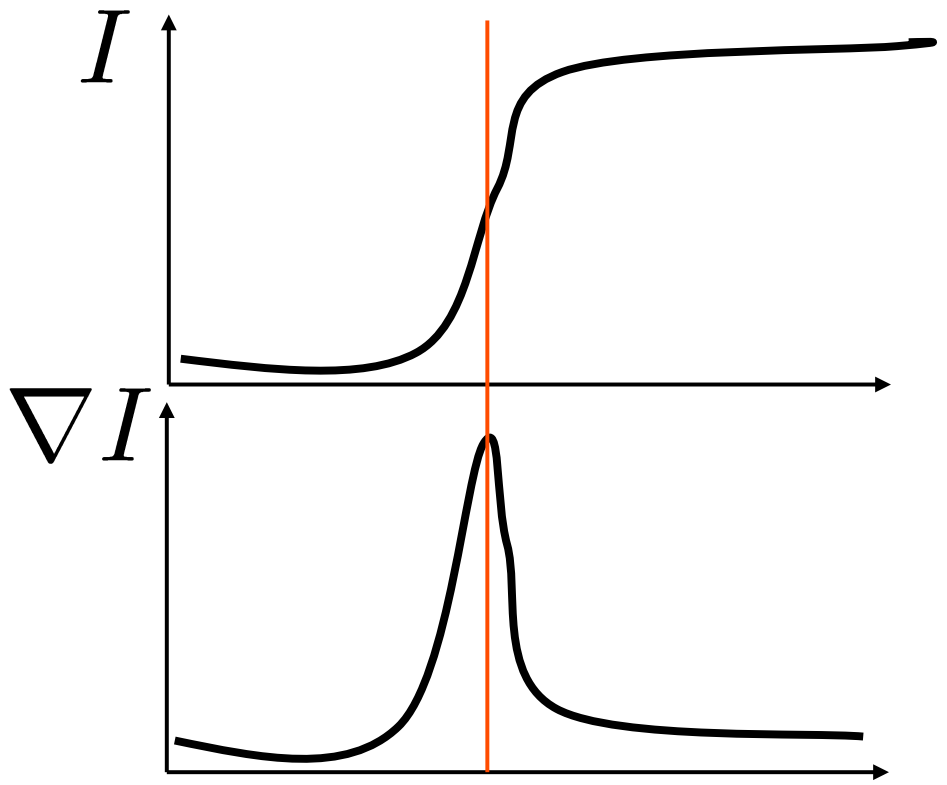






Edge = discontinuity of intensity in some direction.
Could be detected by looking for places where the derivatives of the image have large values.

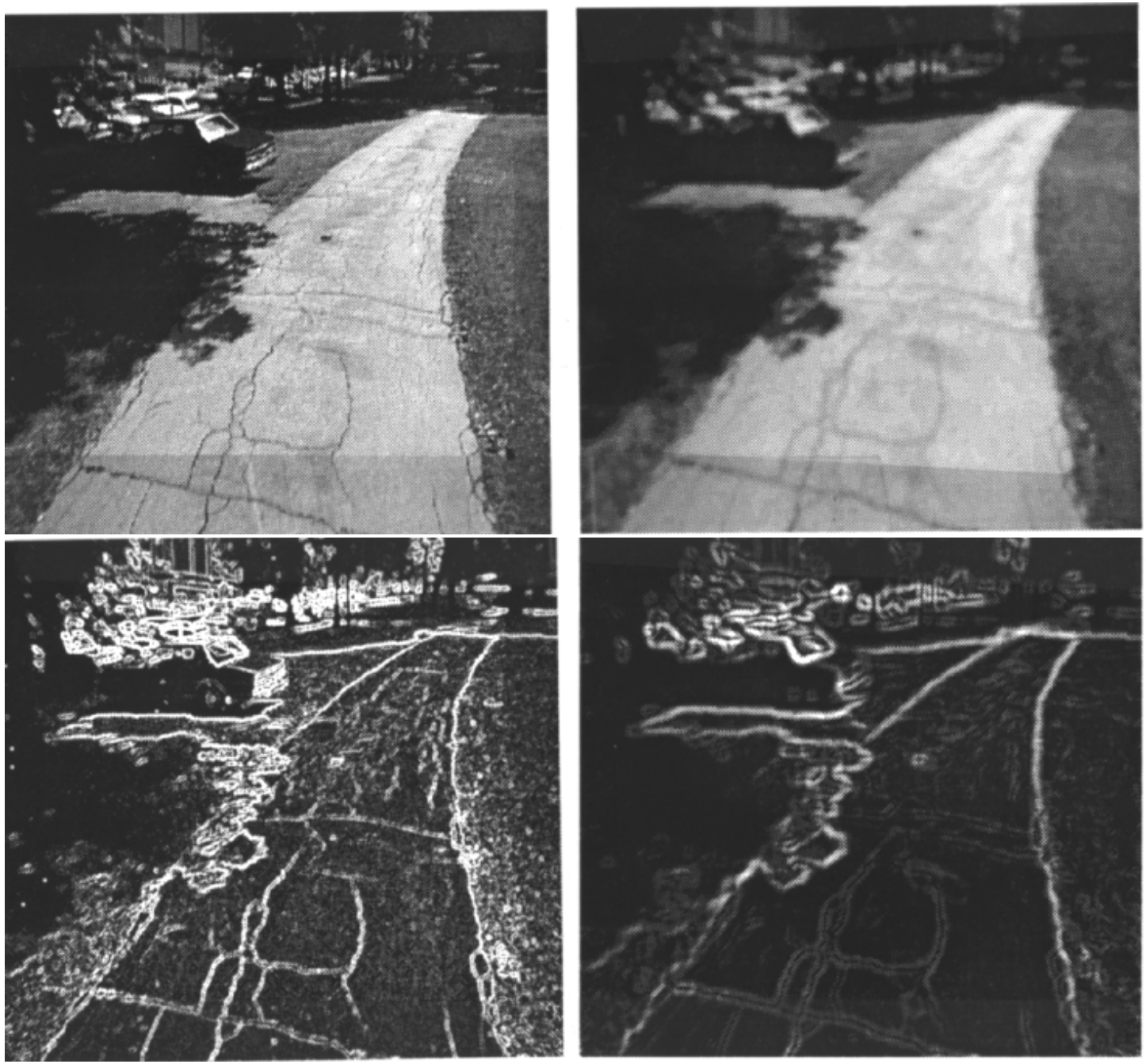




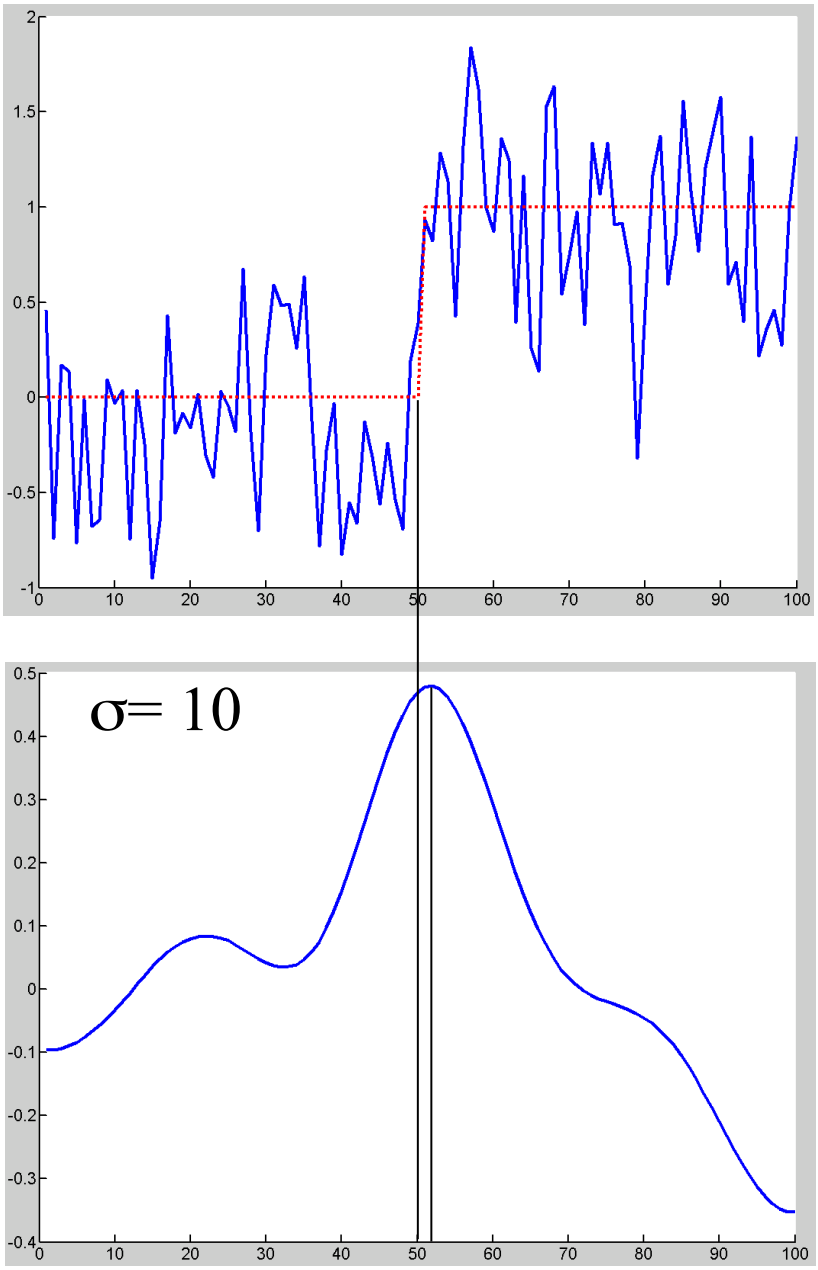
Edge pixels are at local maxima of gradient magnitude
Gradient computed by convolution with Gaussian derivatives
Gradient direction is always perpendicular to edge direction

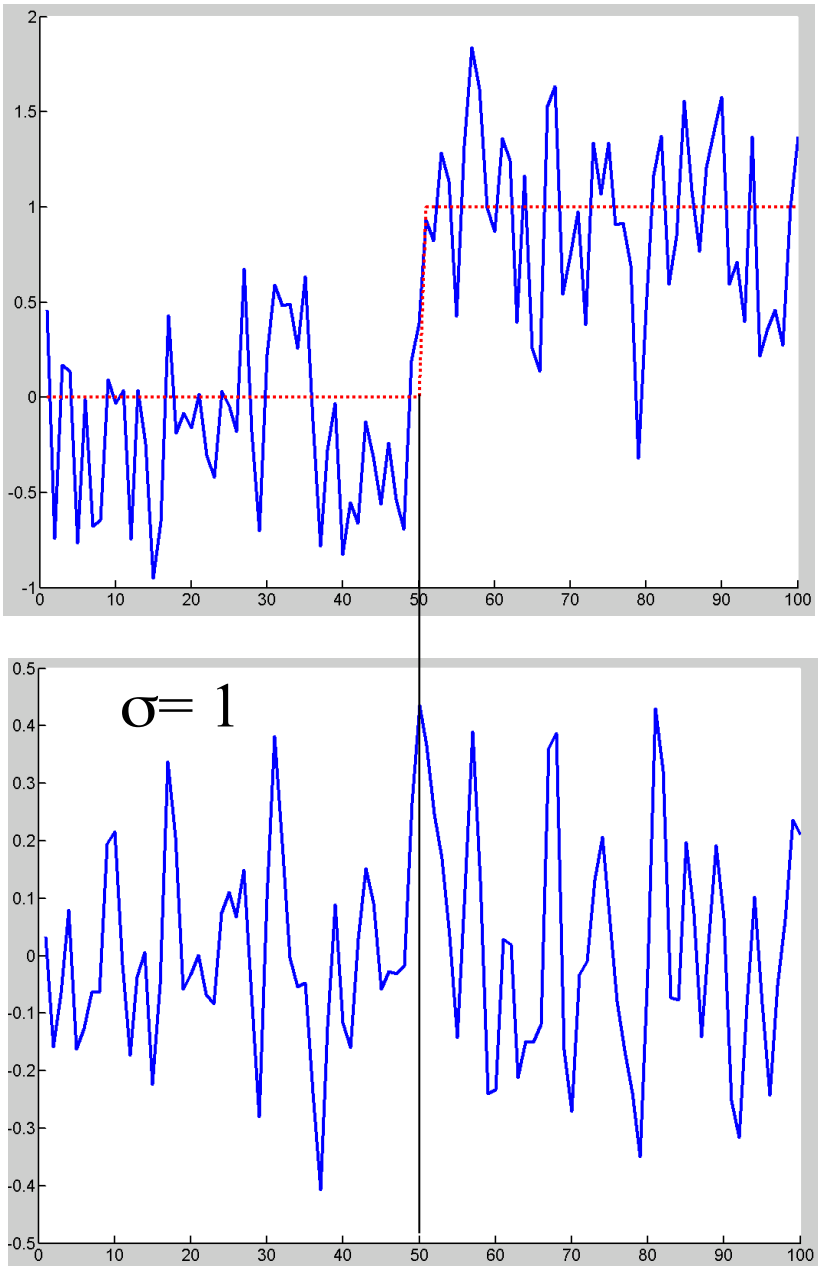
$$\frac{\partial I}{\partial x} = G_\sigma^x * I \qquad \frac{\partial I}{\partial y} = G_\sigma^y * I$$

$$|\nabla I| = \sqrt{\left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2} \qquad \theta = atan2(\frac{\partial I}{\partial y}, \frac{\partial I}{\partial x})$$



Small sigma Large sigma





Large σ → Good detection (high SNR)
Poor localization

Small σ → Poor detection (low SNR)
Good localization

Canny's Result

- Given a filter *f*, define the two objective functions:

  $\Lambda(f)$ large if *f* produces good localization

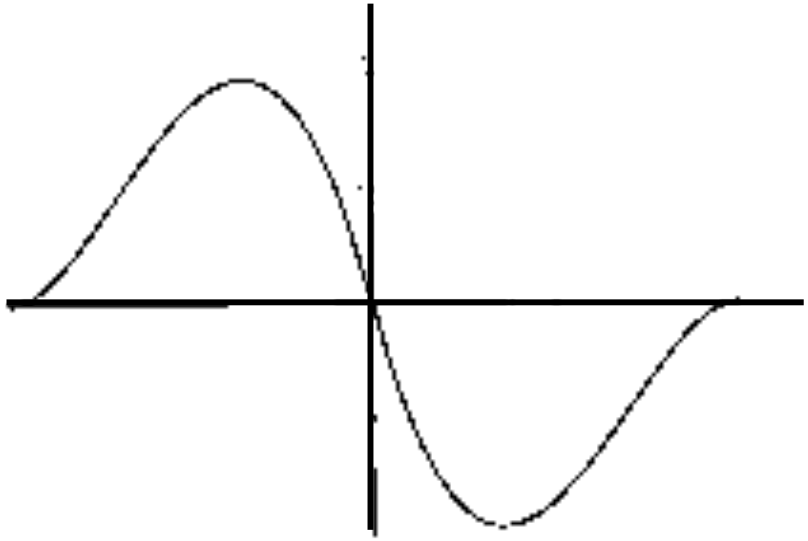
  $\Sigma(f)$ large if *f* produces good detection (high SNR)

- Problem: Find a family of filters *f* that maximizes the compromise criterion

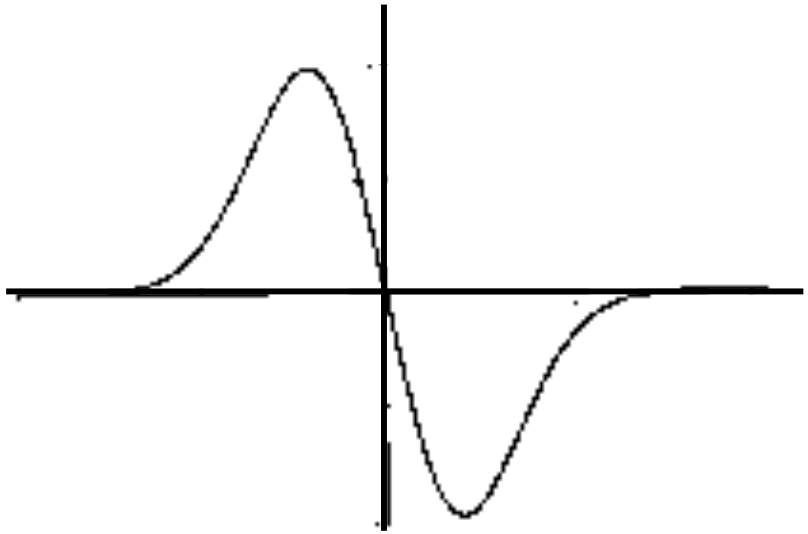
  $$\Lambda(f)\Sigma(f)$$

  under the constraint that a single peak is generated by a step edge

- Solution: Unique solution, a close approximation is the Gaussian derivative filter!



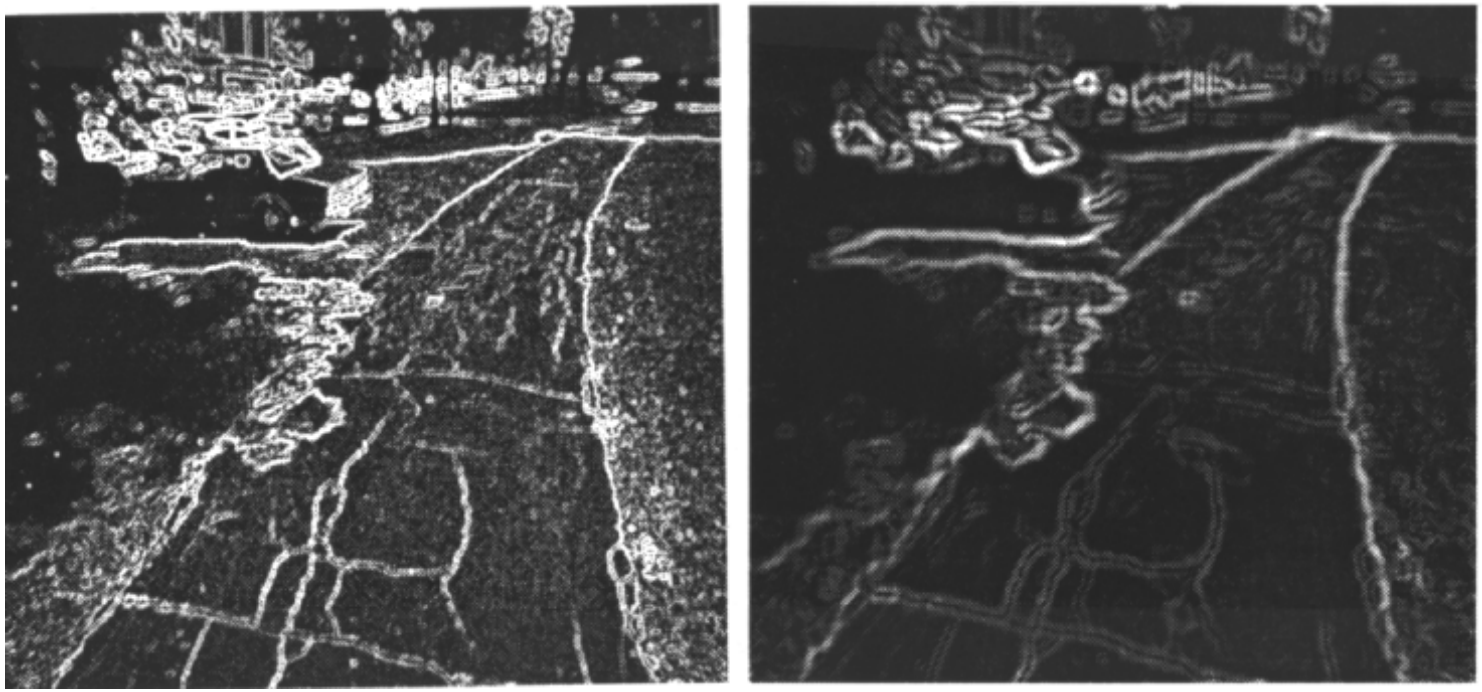
Canny



Derivative of Gaussian

Next Steps

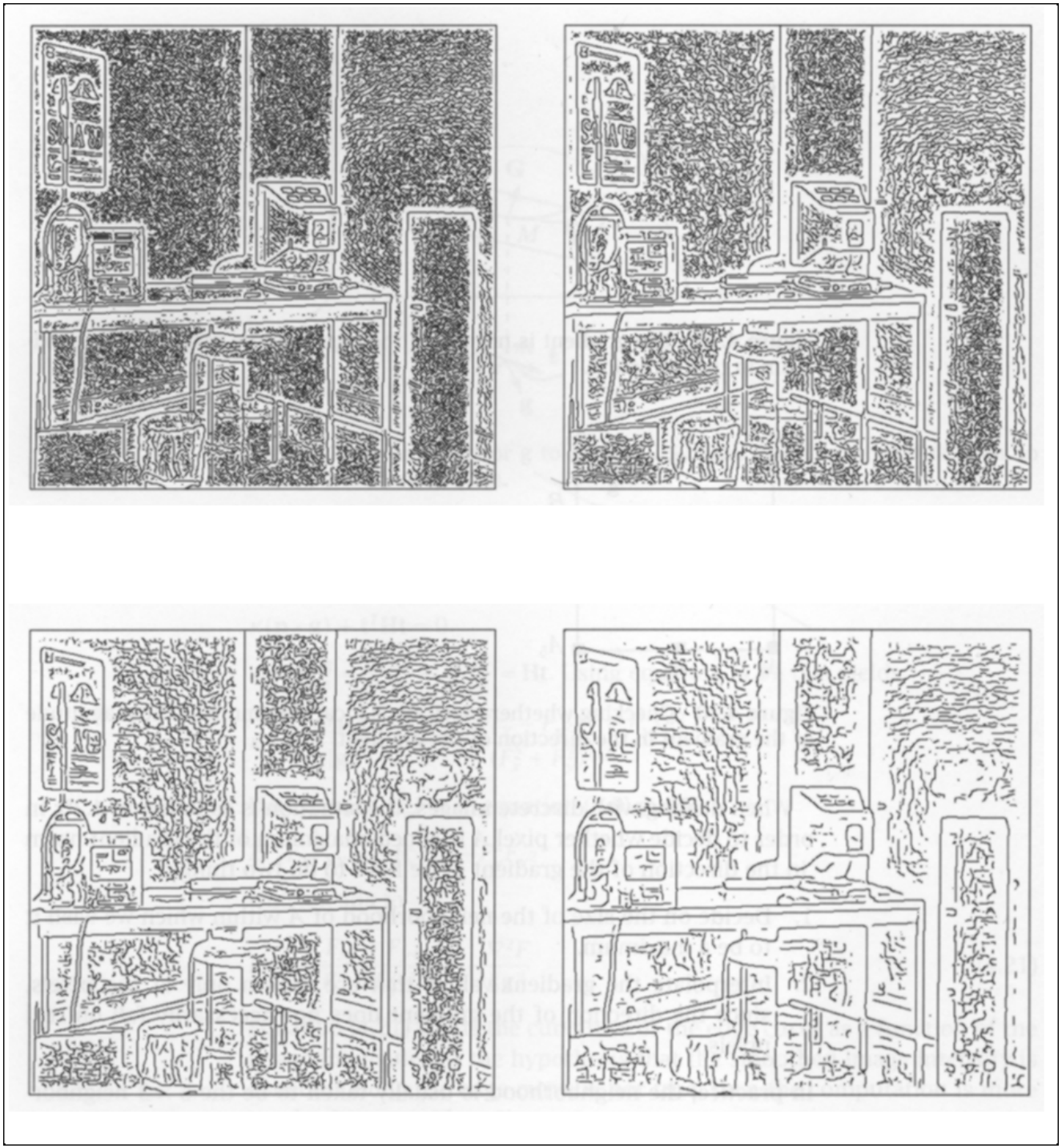
- The gradient magnitude enhances the edges but 2 problems remain:
  - What threshold should we use to retain only the “real” edges?
  - Even if we had a perfect threshold, we would still have poorly localized edges. How to extract optimally localize contours?
- Solution: Two standard tools:
  - Non-local maxima suppression
  - Hysteresis thresholding



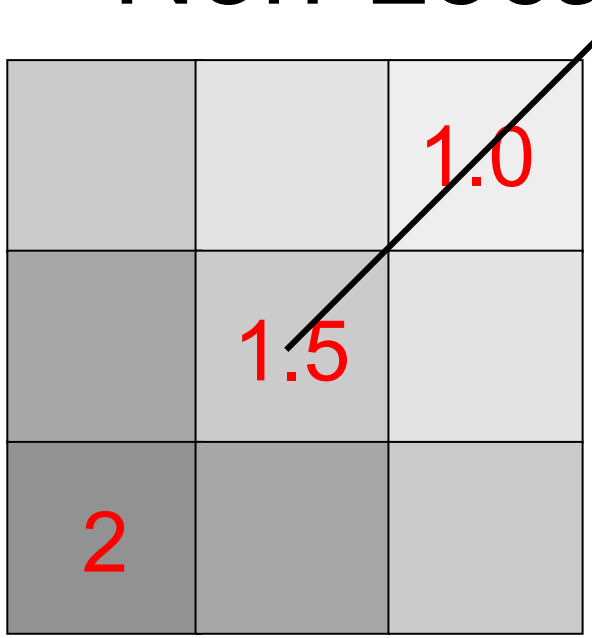
Different thresholds applied to gradient magnitude

Input image

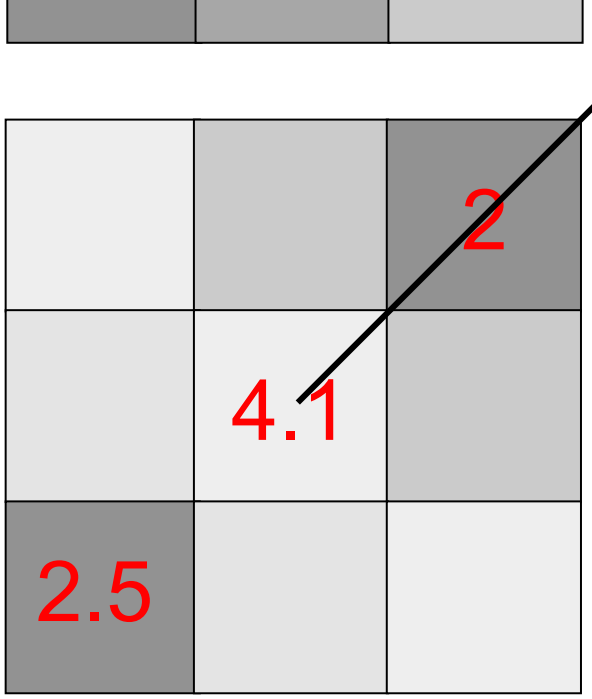
Different thresholds applied to gradient magnitude



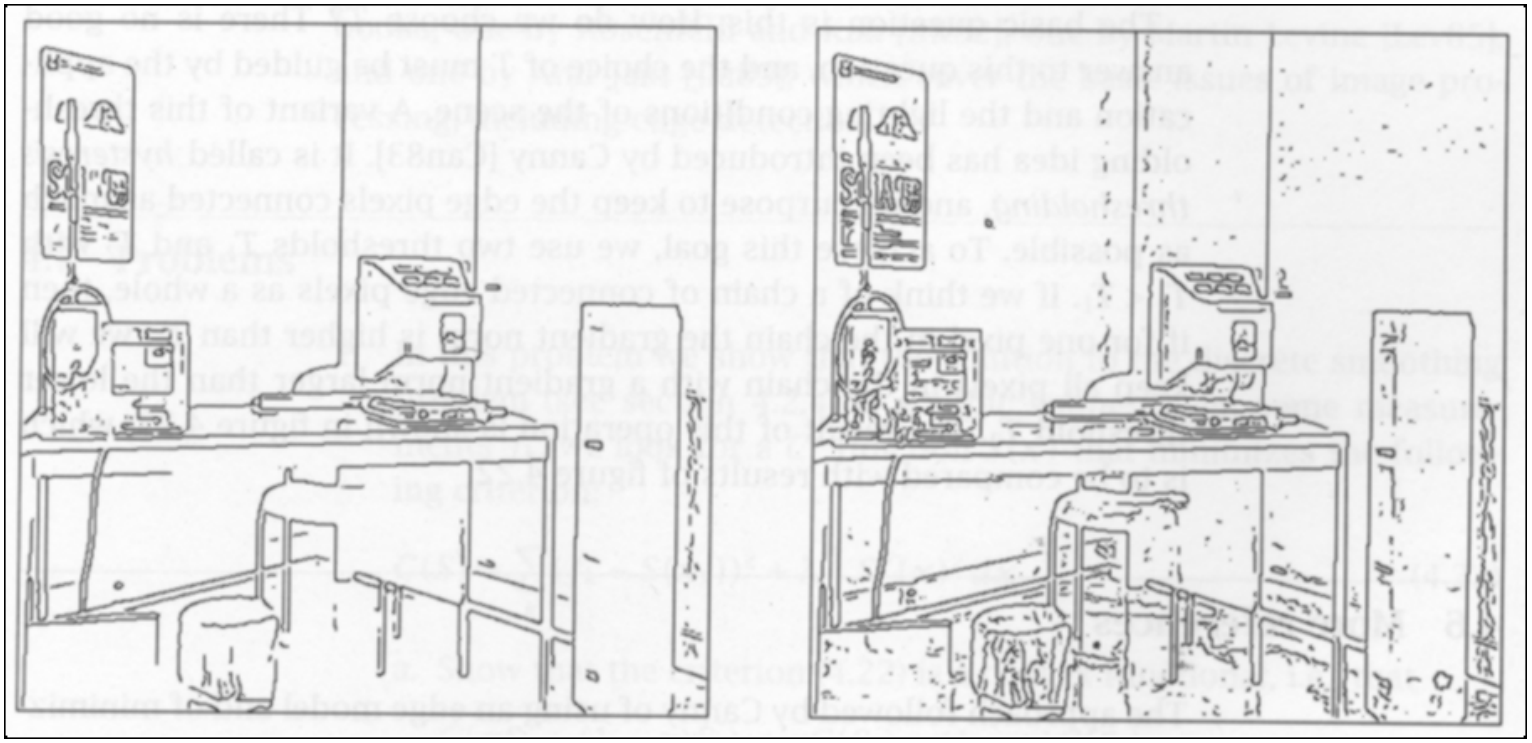
Non-Local Maxima Suppression



Gradient magnitude at center pixel
is lower than the gradient magnitude
of a neighbor *in the direction of the gradient*
→ Discard center pixel (set magnitude to 0)



Gradient magnitude at center pixel
is greater than gradient magnitude
of all the neighbors *in the direction*
of the gradient
→ Keep center pixel unchanged

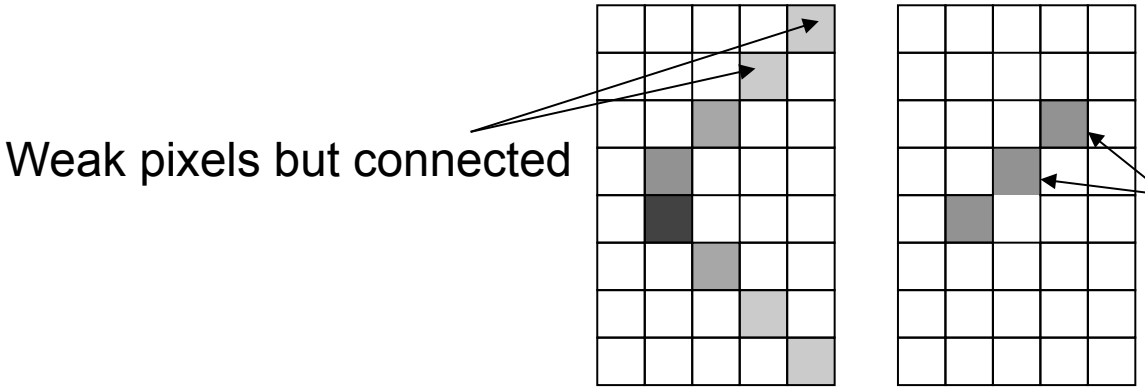


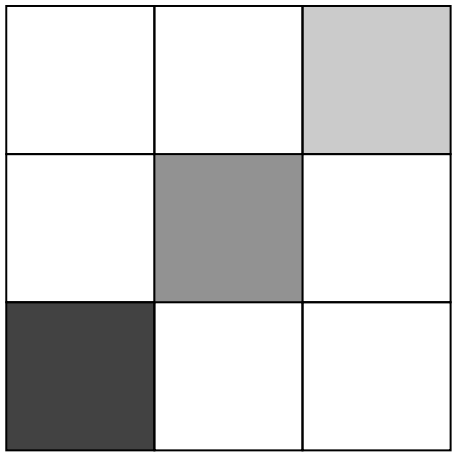
T = 15

T = 5

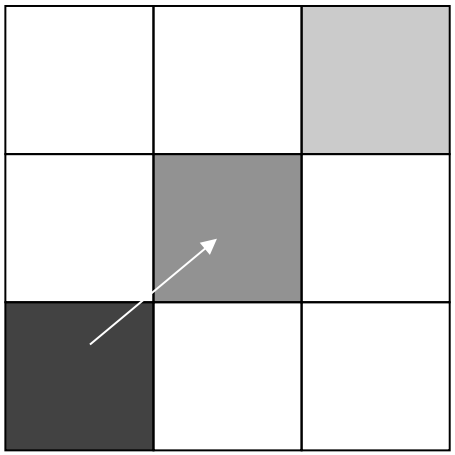
Two thresholds applied to gradient magnitude

Hysteresis Thresholding

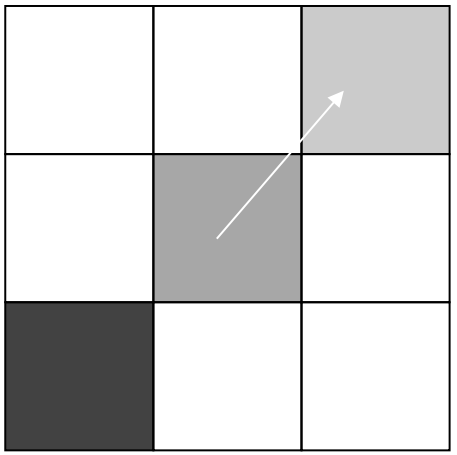




Very strong edge response. Let's start here



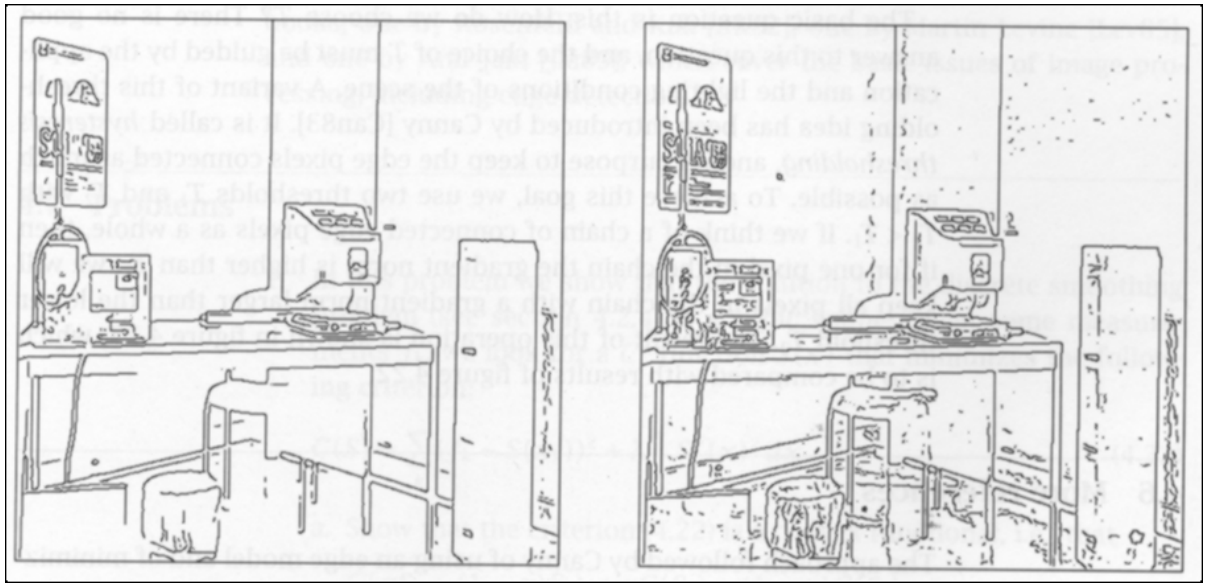
Weaker response but it is connected to a confirmed edge point. Let's keep it.



Continue….

Note: Darker squares illustrate stronger edge response (larger *M*)

T=15



T=5

Hysteresis
T_h=15 T_l = 5

Hysteresis thresholding

1. Compute x and y derivatives of image

$$I_x = G_\sigma^x * I \quad I_y = G_\sigma^y * I$$

2. Compute magnitude of gradient at every pixel

$$M(x, y) = |\nabla I| = \sqrt{I_x^2 + I_y^2}$$

3. Eliminate those pixels that are not local maxima of the magnitude in the direction of the gradient

4. Hysteresis Thresholding

   - Select the pixels such that $M > T_h$ (high threshold)

   - Collect the pixels such that $M > T_l$ (low threshold) that are neighbors of already collected edge points

Summary

- Edges are discontinuities of intensity in images
- Correspond to local maxima of image gradient
- Gradient computed by convolution with derivatives of Gaussian
- General principle applies:
  - Large σ: Poor localization, good detection
  - Small σ: Good localization, poor detection
- Canny showed that Gaussian derivatives yield good compromise between localization and detection
- Edges correspond to zero-crossings of the second derivative (Laplacian in 2-D)