Formulation of Type 1 Lowering with Padding and Stride

April 2015

1 Notation

The following notation is used throughout this document:

$\widehat{D} \in \mathbb{R}^{m^2 b \times k^2 d}$, $\widehat{K} \in \mathbb{R}^{k^2 d \times o}$, and $\widehat{R} = \widehat{D} \times \widehat{K} \in \mathbb{R}^{m^2 b \times o}$,

$\widehat{D}, \widehat{K}$ and $\widehat{R}$ are the (type 1) lowered versions of D, K, and R respectively. $D \in \mathbb{R}^{n \times n \times d \times b}$, $K \in \mathbb{R}^{k \times k \times d \times o}$, and $R \in \mathbb{R}^{m \times m \times o \times b}$.

In words, $n \times n$ is the size of an input feature map of D, and d is the depth of a cube of D and K. b is the batch size or number of $n \times n \times d$ cubes in D. o is the number of output feature maps, or the depth of a single cube in R. The number of cubes in R is also b. m is given by the following equation,

$$m = \frac{n + 2p - k}{s} + 1 \tag{1}$$

where p is the padding and s is the stride.

2 Running Example

The formulations presented in this document reference the following example of D, in which $n = 4$, $d = 1$, and $b = 1$.

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

The kernel cube always has $k = 2$ and $o = 1$. These simplifications allow the focus of the examples to be the impact of padding and stride on $\widehat{D}$. Moreover in all cases $\widehat{K}$ and $\widehat{R}$ require trivial lifting, and it is always true that $\widehat{R} = \widehat{D} \times \widehat{K}$, so the examples will focus on the formulation of $\widehat{D}$ only.

3 No padding, stride=1

First consider the case of $p = 0$ and $s = 1$. Then $m = \frac{n+2p-k}{s} + 1 = n - k + 1$. For $r, c \in 0, \ldots, m-1$, and $b_i \in 0, \ldots, b-1$,

$$\widehat{D}[b_i m^2 + rm + c, :] = \mathbf{vec}(D[r : r + k, c : c + k, :, b_i]) \tag{2}$$

Revisiting the example, $\widehat{D} =$

a	b	e	f
b	c	f	g
c	d	g	h
e	f	i	j
f	g	j	k
g	h	k	l
i	j	m	n
j	k	n	o
k	l	o	p

4 Stride ≥ 1

Including arbitrary stride $s \geq 1$ into equation 2, for $r, c \in 0, \ldots, m-1$, and $b_i \in 0, \ldots, b-1$,

$$\widehat{D}[b_i m^2 + rm + c, :] = \mathbf{vec}(D[rs : rs + k, cs : cs + k, :, b_i]) \tag{3}$$

Note that m has also changed according to equation 1. Also notice in the example that this amounts to eliminating certain rows from $\widehat{D}$, shown here for $s = 2$:

a	b	e	f
c	d	g	h
i	j	m	n
k	l	o	p

5 Stride ≥ 1 and Padding ≥ 0

Including arbitrary padding $p \geq 0$ into equation 3, for $r, c \in 0, \ldots, m-1$, and $b_i \in 0, \ldots, b-1$,

$$\widehat{D}[b_i m^2 + rm + c, :] = \mathbf{vec}(D[rs - p : rs - p + k, cs - p : cs - p + k, :, b_i]) \tag{4}$$

Note that this is true only when the check is in-bounds. Otherwise, that element of $\widehat{D}$ is equal to zero.

6 Examples

These examples are generated using `pad_stride_example.py` and checked by hand.

6.1 p=1,s=2

$$\begin{array}{cccc} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & 0 & d & 0 \\ 0 & e & 0 & i \\ f & g & j & k \\ h & 0 & l & 0 \\ 0 & m & 0 & 0 \\ n & o & 0 & 0 \\ p & 0 & 0 & 0 \end{array}$$

6.2 p=1,s=3

$$\begin{array}{cccc} 0 & 0 & 0 & a \\ 0 & 0 & c & d \\ 0 & i & 0 & m \\ k & l & o & p \end{array}$$

6.3 n=5,k=3,d=b=o=1,p=2,s=2

For this example $D =$

$$\begin{array}{ccccc} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{array}$$

The lowered version is $\widehat{D} =$

0	0	0	0	0	0	0	0	a
0	0	0	0	0	0	a	b	c
0	0	0	0	0	0	c	d	e
0	0	0	0	0	0	e	0	0
0	0	a	0	0	f	0	0	k
a	b	c	f	g	h	k	l	m
c	d	e	h	i	j	m	n	o
e	0	0	j	0	0	o	0	0
0	0	k	0	0	p	0	0	u
k	l	m	p	q	r	u	v	w
m	n	o	r	s	t	w	x	y
o	0	0	t	0	0	y	0	0
0	0	u	0	0	0	0	0	0
u	v	w	0	0	0	0	0	0
w	x	y	0	0	0	0	0	0
y	0	0	0	0	0	0	0	0