OLA-DCF

全局符号

Notations	Description
$\mathbf{R}$	Rating matrix
$\mathbf{U}$	User latent features matrix
$\mathbf{V}$	Item latent features matrix
M	The number of users
N	The number of items
$F(i)$	The set of user i's friends
ϕ_i	User i 's social factor
d	$\mathbf{U} \in \mathbb{R}^{d\times M}$, $\mathbf{V} \in \mathbb{R}^{d\times N}$

模型推导

E-step:

这一部分的推导和结论暂且与OLA论文中的一样。（主要问题在于OLA在这一部分的推导中，用到了 $Lipschitz$ 连续性和 $Hoeffding$ 不等式。我不确定它们能不能直接应用于离散系统。）

M-step:

OLA的目标函数为：

$$\begin{aligned}
\mathcal{L} \quad = \quad & \min \Bigg[\frac{1}{2} \sum\nolimits_{i=1}^{M} \sum\nolimits_{j=1}^{N} I_{ij}^{R} (R_{ij} - \phi_i^T V_j)^2 \\
+ \quad & \frac{\delta_\phi}{2} \sum_{i=1}^{M} \Big(\phi_i - \sum_{u\in F(i)_{k^*}} \alpha_{iu}^{*} U_u\Big)^T \Big(\phi_i - \sum_{u\in F(i)_{k^*}} \alpha_{iu}^{*} U_u\Big) \\
+ \quad & \frac{\delta_\phi}{2} \sum\nolimits_{i=1}^{M} (U_i - \phi_i)^T (U_i - \phi_i) \\
+ \quad & \frac{\delta_U}{2} \sum\nolimits_{i=1}^{M} U_i^T U_i + \frac{\delta_V}{2} \sum\nolimits_{j=1}^{N} V_j^T V_j \Bigg],
\end{aligned}$$

根据DCF的思路我们将其改成：

$$
\begin{aligned}
\underset{\phi,\mathbf{U},\mathbf{V},\mathbf{X},\mathbf{Y},\mathbf{Z}}{argmin} & \sum_{i,j\in\mathcal{V}}\left(R_{ij}-\phi_i^TV_j\right)^2 \\
& +\delta_\phi\sum_{i=1}^{M}\left(\phi_i-\sum_{u\in F(i)_{k^*}}\alpha_{iu}^*U_u\right)^T\left(\phi_i-\sum_{u\in F(i)_{k^*}}\alpha_{iu}^*U_u\right) \\
& -2\delta_\phi tr(\mathbf{U}^T\mathbf{\Phi})-2\beta tr(\mathbf{\Phi}^T\mathbf{X})-2\gamma tr(\mathbf{U}^T\mathbf{Y})-2\eta tr(\mathbf{V}^T\mathbf{Z})
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
s.t.\ & \Phi\in\{\pm1\}^{d\times M},\Phi\in\{\pm1\}^{d\times M},\Phi\in\{\pm1\}^{d\times M}, \\
& \mathbf{X1}=\mathbf{0},\mathbf{Y1}=\mathbf{0},\mathbf{Z1}=\mathbf{0}, \\
& \mathbf{XX}^T=M\mathbf{I},\mathbf{YY}^T=M\mathbf{I},\mathbf{ZZ}^T=N\mathbf{I}
\end{aligned}
$$

其中，$\mathcal{V}$为$\mathbf{R}$中评分的索引集合，$\mathbf{\Phi}=\{\phi_1,\phi_2,\ldots,\phi_M\}$，并假设$R_{ij}\in[-d,d]$.

- **$\mathbf{XYZ}$子问题**

  由于对 $\mathbf{X}$(或$\mathbf{Y},\mathbf{Z}$) 进行更新时，将其它项视为常数，所以对 $\mathbf{X}$(或$\mathbf{Y},\mathbf{Z}$) 的更新与 DCF 中的一致。

- **$\mathbf{\Phi}$子问题**

  目标函数：

$$
\begin{aligned}
\underset{\mathbf{\Phi}}{argmin} & \sum_{i,j\in\mathcal{V}}\left(R_{ij}-\phi_i^TV_j\right)^2 \\
& +\delta_\phi\sum_{i=1}^{M}\left(\phi_i-\sum_{u\in F(i)_{k^*}}\alpha_{iu}^*U_u\right)^T\left(\phi_i-\sum_{u\in F(i)_{k^*}}\alpha_{iu}^*U_u\right) \\
& -2\delta_\phi tr(\mathbf{U}^T\mathbf{\Phi})-2\beta tr(\mathbf{\Phi}^T\mathbf{X})
\end{aligned} \tag{2}
$$

  令

$$
\mathbf{G}=\left\{\sum_{u\in F(1)_{k^*}}\alpha_{1u}^*U_u,\sum_{u\in F(2)_{k^*}}\alpha_{2u}^*U_u,\ldots,\sum_{u\in F(M)_{k^*}}\alpha_{Mu}^*U_u\right\}
$$

$$
\mathbf{E}=\delta_\phi\mathbf{G}+\delta_\phi\mathbf{U}+\beta\mathbf{X}
$$

  (2) 可化简为

$$
\begin{aligned}
& \underset{\mathbf{\Phi}}{argmin}\sum_{i,j\in\mathcal{V}}\left(R_{ij}-\phi_i^TV_j\right)^2-2\delta_\phi tr(\mathbf{\Phi}^T\mathbf{G})-2\delta_\phi tr(\mathbf{\Phi}^T\mathbf{U})-2\beta tr(\mathbf{\Phi}^T\mathbf{X}) \\
= & \underset{\mathbf{\Phi}}{argmin}\sum_{i,j\in\mathcal{V}}\left(R_{ij}-\phi_i^TV_j\right)^2-2tr\left(\mathbf{\Phi}^T(\delta_\phi\mathbf{G}+\delta_\phi\mathbf{U}+\beta\mathbf{X})\right) \\
= & \underset{\mathbf{\Phi}}{argmin}\sum_{i,j\in\mathcal{V}}\left(R_{ij}-\phi_i^TV_j\right)^2-2tr\left(\mathbf{\Phi}^T\mathbf{E}\right)
\end{aligned} \tag{3}
$$

  考虑 ϕ_i：

$$
\begin{aligned}
& \underset{\phi_i}{argmin}\sum_{j\in\mathcal{V}_i}\left(R_{ij}-\phi_i^TV_j\right)^2-2\phi_i^TE_i \\
= & \underset{\phi_i}{argmin}\sum_{j\in\mathcal{V}_i}\left((\phi_i^TV_j)^2-2R_{ij}\phi_i^TV_j\right)-2\phi_i^TE_i \\
= & \underset{\phi_i}{argmin}\sum_{j\in\mathcal{V}_i}(\phi_i^TV_j)^2-2(\sum_{j\in\mathcal{V}_i}R_{ij}V_j^T)\phi_i^T-2\phi_i^TE_i
\end{aligned} \tag{4}
$$

  其中 $\mathcal{V}_i=\{j|(i,j)\in\mathcal{V}\}$.

  考虑 ϕ_{ik}：

$$\underset{\phi_{ik}\in\pm 1}{argmin}\sum_{j\in\mathcal{V}_i}(\phi_i^T V_j)^2 - 2(\sum_{j\in\mathcal{V}_i} R_{ij}V_j^T)\phi_i^T - 2\phi_i^T E_i \tag{5}$$

其中

$$\begin{aligned}\sum_{j\in\mathcal{V}_i}\left(\phi_i^T V_j\right)^2 =& 2\phi_{ik}(\sum_{j\in\mathcal{V}_i} V_{jk}\phi_{i\bar{k}}^T V_{j\bar{k}}) \\ &\underbrace{+\sum_{j\in\mathcal{V}_i}\left((\phi_{ik}V_{jk})^2 + (\phi_{i\bar{k}}^T V_{j\bar{k}})^2\right)}_{constant} \\ -2(\sum_{j\in\mathcal{V}_i} R_{ij}V_j^T)\phi_i^T - 2\phi_i^T E_i =& -2\phi_{ik}\sum_{j\in\mathcal{V}_i} R_{ij}V_{jk} - 2\phi_{ik}E_{ik} \\ &\underbrace{-2(\sum_{j\in\mathcal{V}_i} R_{ij}V_{j\bar{k}}^T)\phi_{i\bar{k}} - 2\phi_{i\bar{k}}E_{i\bar{k}}}_{constant}\end{aligned}$$

故(5)等价于:

$$\begin{aligned}&\underset{\phi_{ij}\in\pm 1}{argmin}\,\phi_{ik}\hat{\phi}_{ik} \\ &\hat{\phi}_{ik} = \sum_{j\in\mathcal{V}_i} V_{jk}(\phi_{i\bar{k}}^T V_{j\bar{k}} - R_{ij}) - E_{ik}\end{aligned} \tag{6}$$

ϕ_{ik} 取与 $\hat{\phi}_{ik}$ 异号即可。

- $\mathbf{U}$ 子问题

  目标函数:

$$\underset{\mathbf{U}}{argmin}\,\delta_\phi\sum_{i=1}^{M}\left(\phi_i - \sum_{u\in F(i)_{k^*}}\alpha_{iu}^* U_u\right)^T\left(\phi_i - \sum_{u\in F(i)_{k^*}}\alpha_{iu}^* U_u\right) - 2\delta_\phi tr(\mathbf{U}^T\mathbf{\Phi}) - 2\gamma tr(\mathbf{U}^T\mathbf{Y}) \tag{7}$$

  设 $\mathbf{F} = \delta_\phi\mathbf{\Phi} + \gamma\mathbf{Y}$，则 (7) 等价于:

$$\begin{aligned}&\underset{\mathbf{U}}{argmin}\,\delta_\phi\sum_{i=1}^{M}\left(\phi_i - \sum_{u\in F(i)_{k^*}}\alpha_{iu}^* U_u\right)^T\left(\phi_i - \sum_{u\in F(i)_{k^*}}\alpha_{iu}^* U_u\right) - 2tr\left(\mathbf{U}^T(\delta_\phi\mathbf{\Phi} + \gamma\mathbf{Y})\right) \\ =\;&\underset{\mathbf{U}}{argmin}\,\delta_\phi\sum_{i=1}^{M}\left(\phi_i - \sum_{u\in F(i)_{k^*}}\alpha_{iu}^* U_u\right)^T\left(\phi_i - \sum_{u\in F(i)_{k^*}}\alpha_{iu}^* U_u\right) - 2tr\left(\mathbf{U}^T\mathbf{F}\right)\end{aligned} \tag{8}$$

  设 $E(u) = \{i|u\in F(i)_{k^*}\}$，考虑 $\mathbf{U}_u$:

$$\underset{\mathbf{U}_u}{argmin}\,\delta_\phi\sum_{i\in E(u)}\left(\phi_i - \sum_{p\in F(i)_{k^*}}\alpha_{ip}^* U_p\right)^T\left(\phi_i - \sum_{p\in F(i)_{k^*}}\alpha_{ip}^* U_p\right) - 2U_u^T F_u \tag{9}$$

  其中

$$
\begin{aligned}
& argmin\ \left(\phi_i - \sum_{p\in F(i)_{k^*}} \alpha^*_{ip} U_p\right)^T \left(\phi_i - \sum_{p\in F(i)_{k^*}} \alpha^*_{ip} U_p\right) \\
= & \ argmin\ \phi_i^T\phi_i - 2\phi_i^T \sum_{p\in F(i)_{k^*}} \alpha^*_{ip} U_p + \sum_{m,n\in F(i)_{k^*}} \alpha^*_{im}\alpha^*_{in} U_p^T U_q \\
= & \ argmin\ \underbrace{\phi_i^T\phi_i}_{constant} - 2\alpha^*_{iu}\phi_i^T U_u \underbrace{- 2\phi_i^T \sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip} U_p}_{constant} + \sum_{m,n\in F(i)_{k^*}} \alpha^*_{im}\alpha^*_{in} U_p^T U_q \\
= & \ argmin\ -2\alpha^*_{iu}\phi_i^T U_u + \underbrace{(\alpha^*_{iu})^2 U_u^T U_u}_{constant} + 2\sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip}\alpha^*_{iu} U_p^T U_u \\
= & \ argmin\ -\alpha^*_{iu}\phi_i^T U_u + \sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip}\alpha^*_{iu} U_p^T U_u
\end{aligned}
$$

故 (9) 等价于:

$$
\begin{aligned}
& \underset{\mathbf{U}_u}{argmin}\ \delta_\phi \sum_{i\in E(u)} \left(-\alpha^*_{iu}\phi_i^T U_u + \sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip}\alpha^*_{iu} U_p^T U_u\right) - 2U_u^T F_u \\
= & \ \underset{U_u}{argmin}\ \alpha^*_{iu}\delta_\phi U_u^T \sum_{i\in E(u)} \left(-\phi_i + \sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip} U_p\right) - 2U_u^T F_u \\
= & \ \underset{U_u}{argmin}\ U_u^T \left(\alpha^*_{iu}\delta_\phi \sum_{i\in E(u)} \left(-\phi_i + \sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip} U_p\right) - 2F_u\right) \\
= & \ \underset{U_u}{argmin}\ U_u^T \hat{U}_u
\end{aligned}
\tag{10}
$$

考虑 U_{uk}:

$$
\begin{aligned}
& \underset{U_{uk}}{argmin}\ U_{uk}\hat{U}_{uk} \underbrace{+ U_{u\bar{k}}^T \hat{U}_{u\bar{k}}}_{constant} \\
= & \ \underset{U_{uk}}{argmin}\ U_{uk}\hat{U}_{uk} \\
& \hat{U}_{uk} = \alpha^*_{iu}\delta_\phi \sum_{i\in E(u)} \left(-\phi_{ik} + \sum_{\substack{p\in F(i)_{k^*} \\ \wedge p\neq u}} \alpha^*_{ip} U_{pk}\right) - 2F_{uk}
\end{aligned}
\tag{11}
$$

U_{uk} 取与 $\hat{U}_{uk}$ 异号。

- **V** 子问题

目标函数：

$$\underset{\mathbf{V}}{argmin} \sum_{i,j \in \mathcal{V}} \left(R_{ij} - \phi_i^T V_j\right)^2 - 2\eta tr(\mathbf{V}^T \mathbf{Z}) \tag{12}$$

这与 DCF 的 $\mathbf{B}$ 子问题是完全一样的，最后得出的$\hat{V}_j$为：

$$\hat{V}_{jk} = \sum_{i \in \mathcal{V}_j} \phi_{ik} (\phi_{i\bar{k}}^T V_{j\bar{k}} - R_{ij}) - \eta Z_{ik} \tag{13}$$

其中 $\mathcal{V}_j = \{i|(i,j) \in \mathcal{V}\}$.

遗留问题

- 在 E-step 中， $Lipschitz$ 连续性和 $Hoeffding$ 不等式能否直接用于离散系统
- 感觉计算复杂度比较大