Robot Mapping

Extended Kalman Filter

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SLAM is a State Estimation Problem

- Estimate the map and robot's pose
- Bayes filter is one tool for state estimation
- **Prediction**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\; bel(x_{t-1})\, dx_{t-1}$$

- **Correction**

$$bel(x_t) = \eta\, p(z_t \mid x_t)\; \overline{bel}(x_t)$$

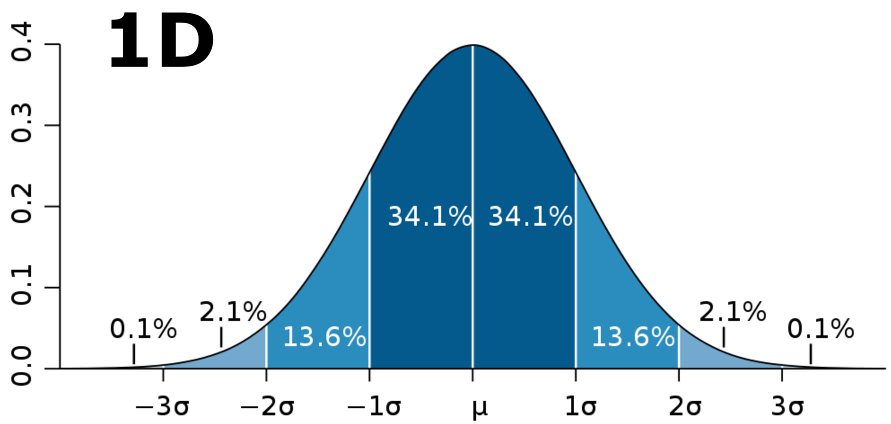
Kalman Filter

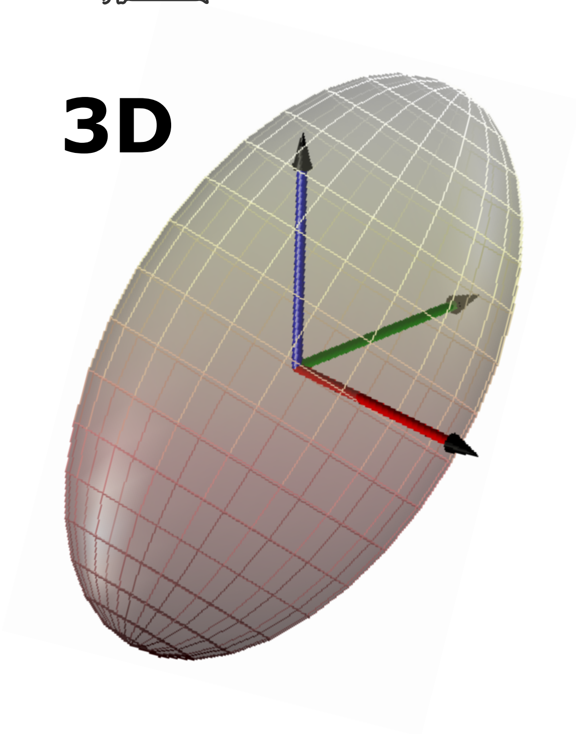
- It is a Bayes filter
- Estimator for the linear Gaussian case
- Optimal solution for linear models and Gaussian distributions

Kalman Filter Distribution

- Everything is Gaussian

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$





Properties: Marginalization and Conditioning

- Given

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix} \qquad p(x) = \mathcal{N}$$

- The marginals are Gaussians

$$p(x_a) = \mathcal{N} \qquad p(x_b) = \mathcal{N}$$

- as well as the conditionals

$$p(x_a \mid x_b) = \mathcal{N} \qquad p(x_b \mid x_a) = \mathcal{N}$$

Marginalization

- Given $p(x) = p(x_a, x_b) = \mathcal{N}(\mu, \Sigma)$

  with $\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$

- The marginal distribution is

$$p(x_a) = \int p(x_a, x_b)\, dx_b = \mathcal{N}(\mu, \Sigma)$$

  with $\mu = \mu_a \qquad \Sigma = \Sigma_{aa}$

Conditioning

- Given $p(x) = p(x_a, x_b) = \mathcal{N}(\mu, \Sigma)$

  with $\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$

- The conditional distribution is

$$p(x_a \mid x_b) = \frac{p(x_a, x_b)}{p(x_b)} = \mathcal{N}(\mu, \Sigma)$$

with $\mu = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(b - \mu_b)$

$$\Sigma = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$$

Linear Model

- The Kalman filter assumes a linear transition and observation model
- Zero mean Gaussian noise

$$x_t = A_t x_{t-1} + B_t u_t + \epsilon_t$$

$$z_t = C_t x_t + \delta_t$$

Components of a Kalman Filter

A_t Matrix $(n \times n)$ that describes how the state evolves from $t-1$ to t without controls or noise.

B_t Matrix $(n \times l)$ that describes how the control u_t changes the state from $t-1$ to t.

C_t Matrix $(k \times n)$ that describes how to map the state x_t to an observation z_t.

ϵ_t

δ_t Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t respectively.

Linear Motion Model

- Motion under Gaussian noise leads to

$$p(x_t \mid u_t, x_{t-1}) = ?$$

Linear Motion Model

- Motion under Gaussian noise leads to

$$p(x_t \mid u_t, x_{t-1}) = \det(2\pi R_t)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right)$$

- R_t describes the noise of the motion

Linear Observation Model

- Measuring under Gaussian noise leads to

$$p(z_t \mid x_t) = ?$$

Linear Observation Model

- Measuring under Gaussian noise leads to

$$p(z_t \mid x_t) = \det(2\pi Q_t)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right)$$

- Q_t describes the measurement noise

Everything stays Gaussian

- Given an initial Gaussian belief, the belief is always Gaussian

$$\overline{bel}(x_t) = \int \underline{p(x_t \mid u_t, x_{t-1})}\ \underline{bel(x_{t-1})}\, dx_{t-1}$$

$$bel(x_t) = \eta\ \underline{p(z_t \mid x_t)}\ \underline{\overline{bel}(x_t)}$$

- Proof is non-trivial
(see Probabilistic Robotics, Sec. 3.2.4)

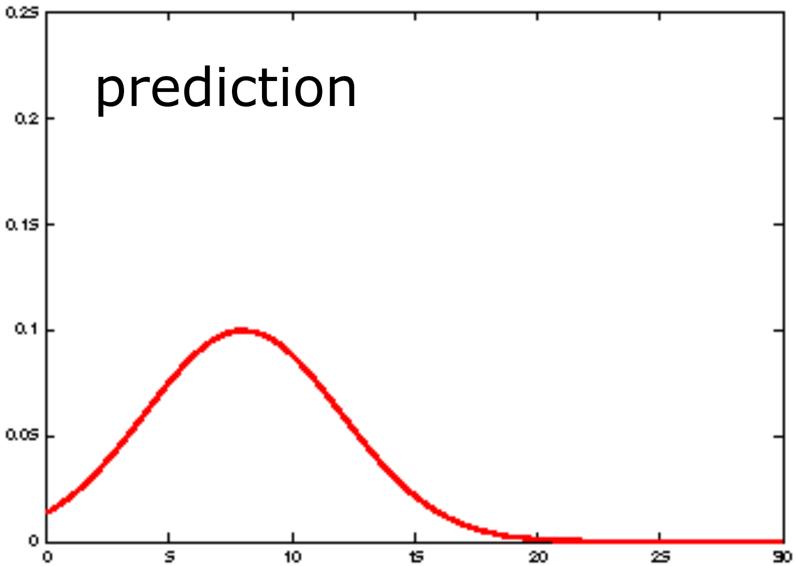
Kalman Filter Algorithm

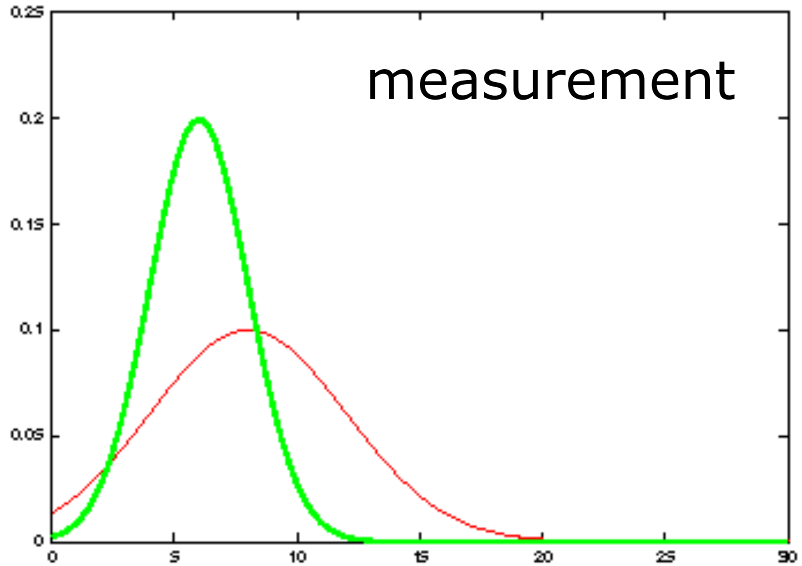
Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

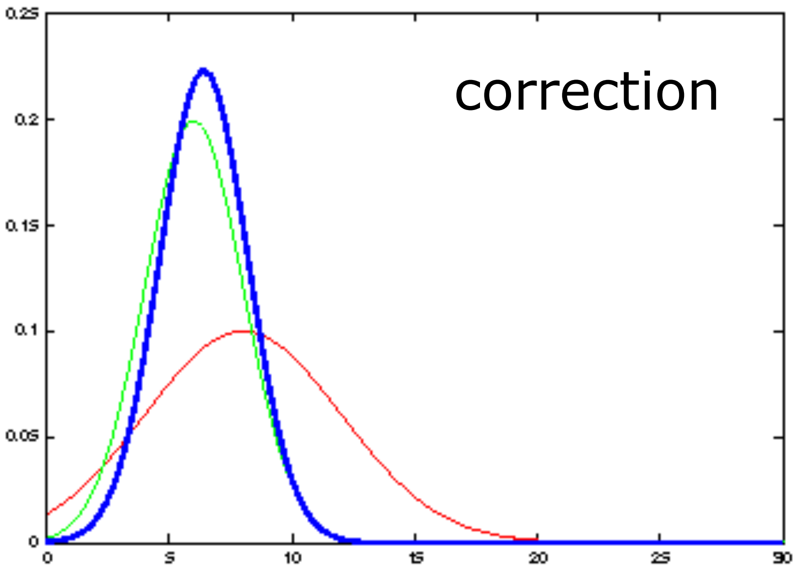
$$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$$
$$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$$

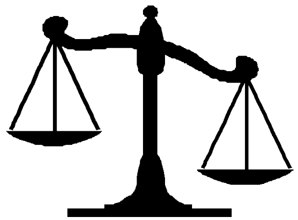
$$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\, \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$$
$$return\ \mu_t, \Sigma_t$$

1D Kalman Filter Example (1)



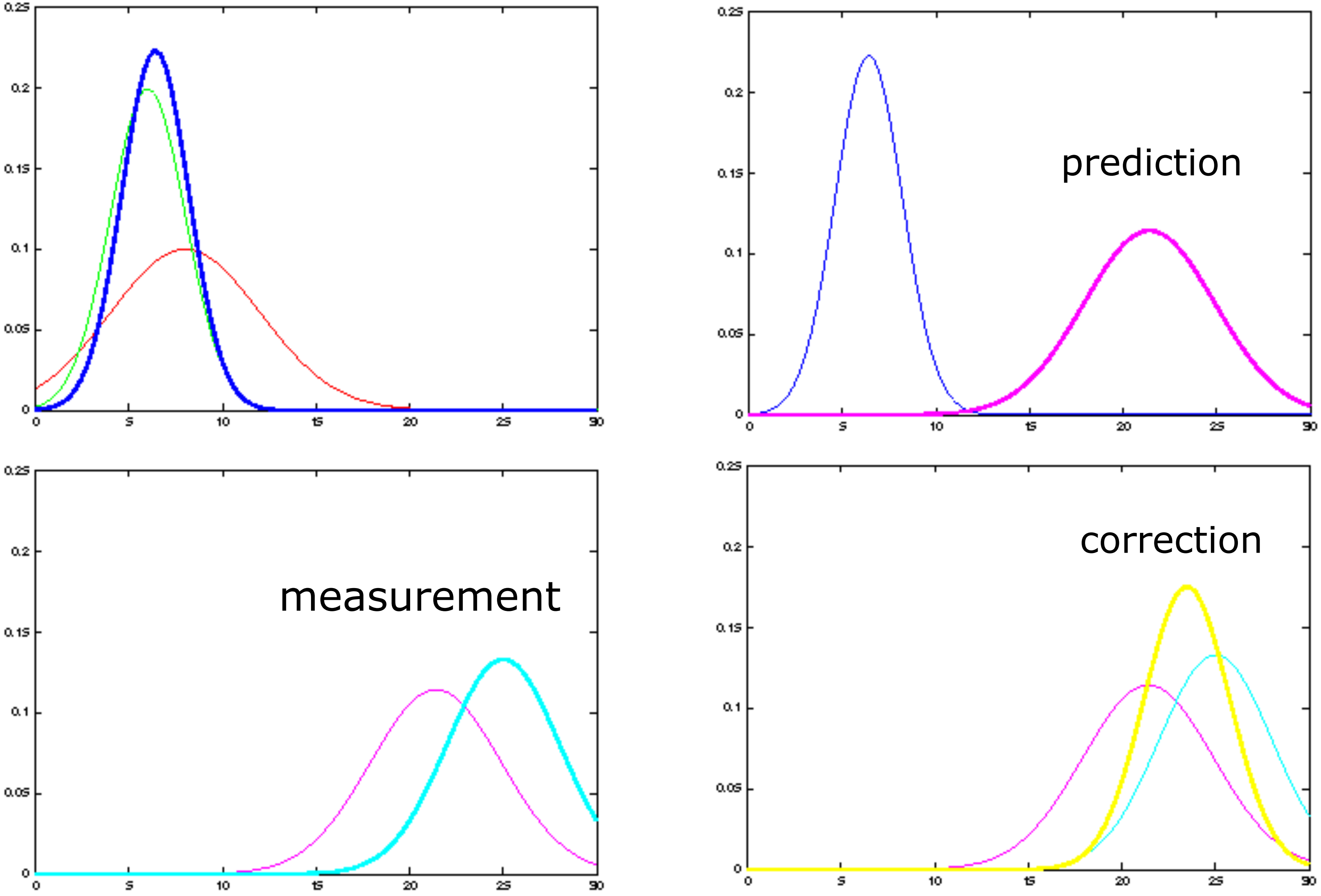






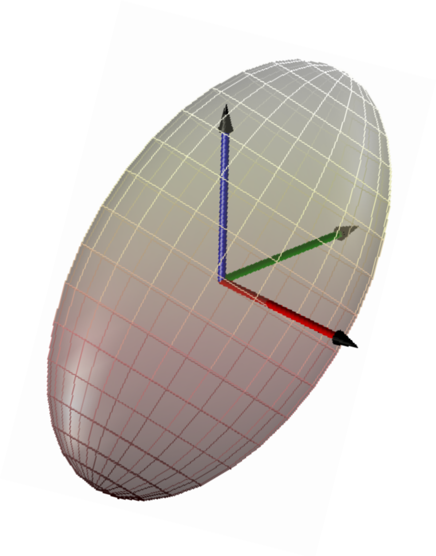
It's a weighted mean!

1D Kalman Filter Example (2)



Kalman Filter Assumptions

- Gaussian distributions and noise
- Linear motion and observation model



$$x_t = A_t x_{t-1} + B_t u_t + \epsilon_t$$

$$z_t = C_t x_t + \delta_t$$

What if this is not the case?

Non-linear Dynamic Systems

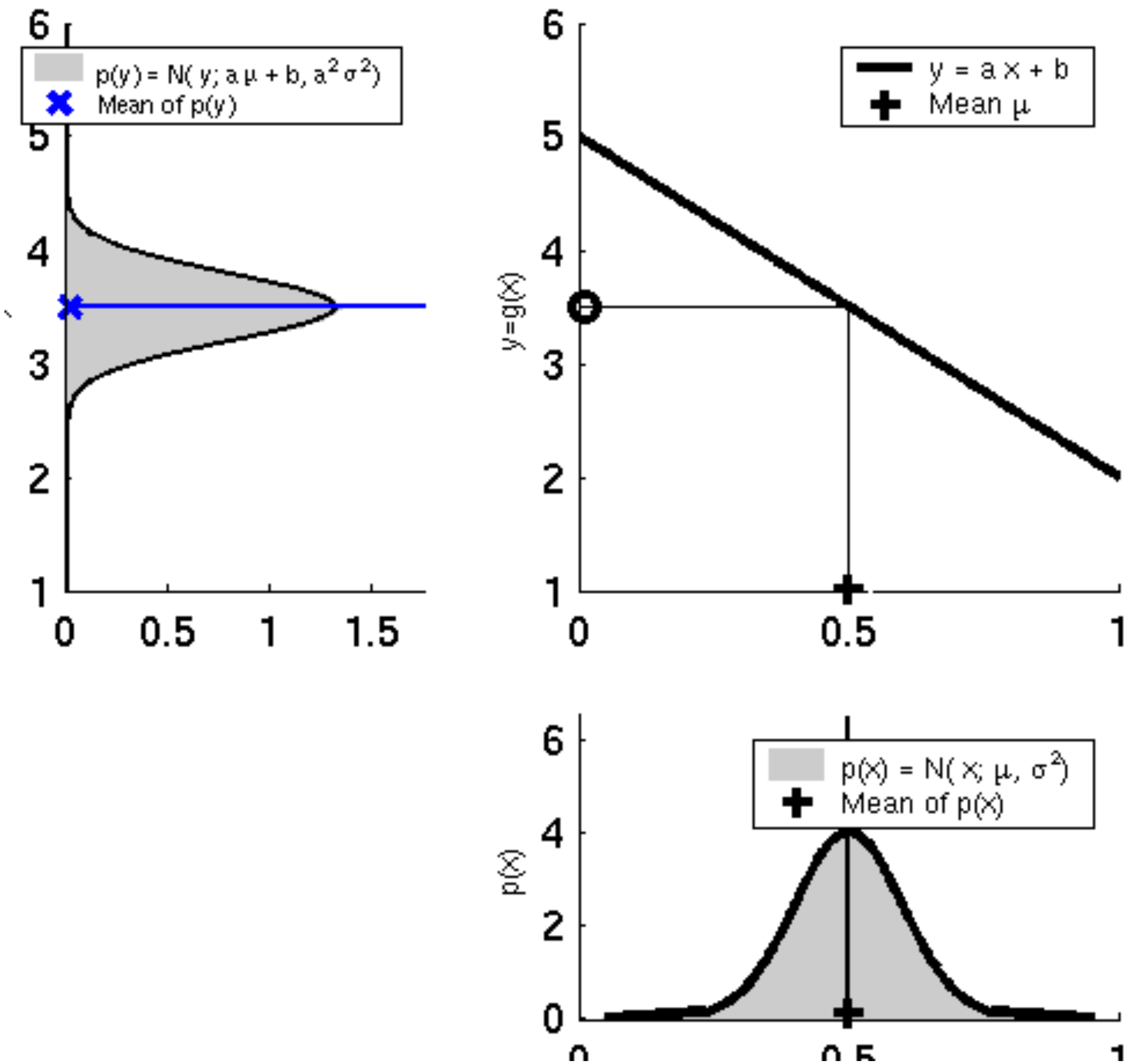
- Most realistic problems (in robotics) involve nonlinear functions

$$\cancel{x_t = A_t x_{t-1} + B_t u_t + \epsilon_t} \qquad \cancel{z_t = C_t x_t + \delta_t}$$

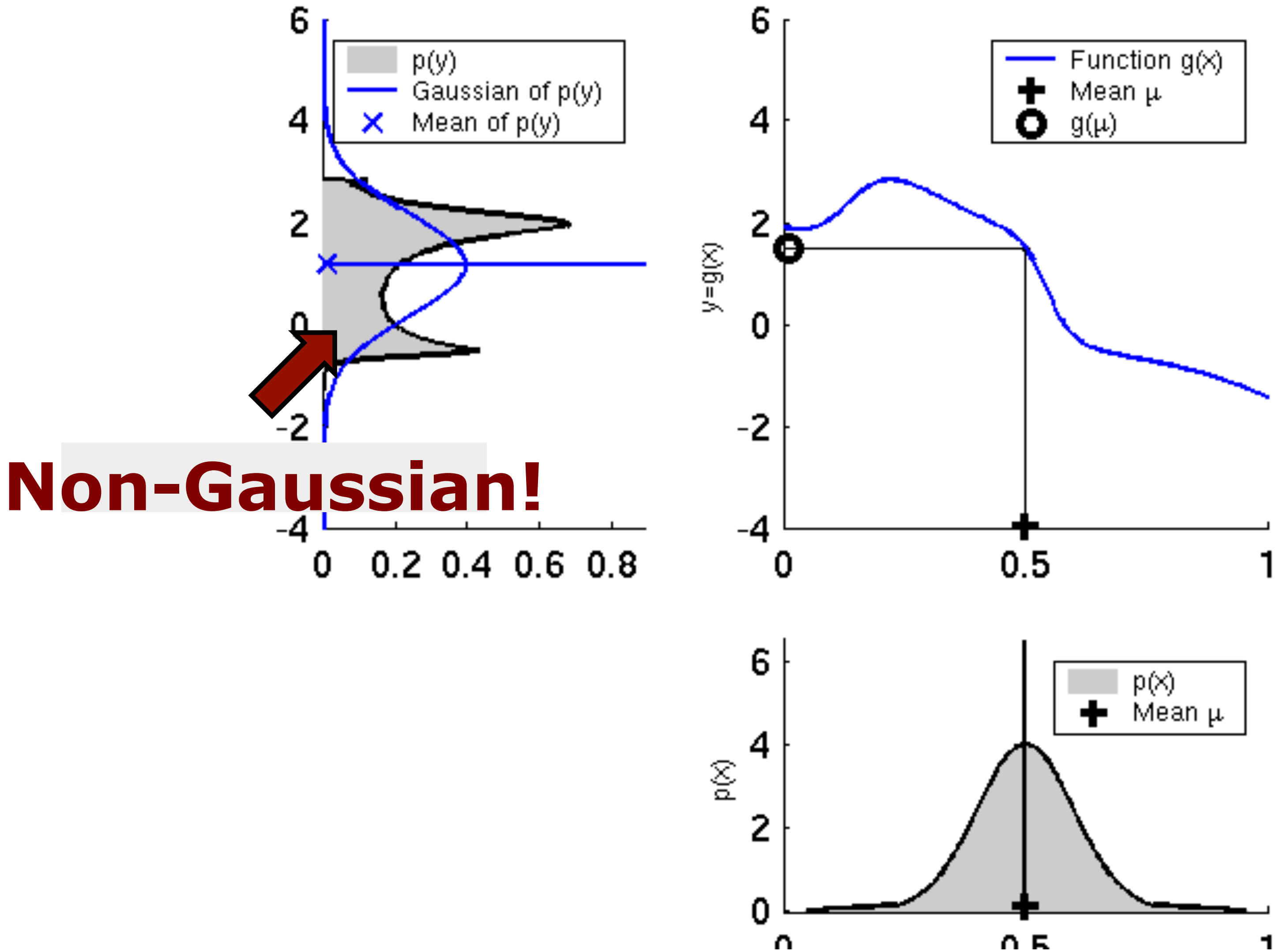
$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$x_t = g(u_t, x_{t-1}) + \epsilon_t \qquad z_t = h(x_t) + \delta_t$$

Linearity Assumption Revisited



Non-Linear Function



Non-Gaussian Distributions

- The non-linear functions lead to non-Gaussian distributions
- Kalman filter is not applicable anymore!

What can be done to resolve this?

Non-Gaussian Distributions

- The non-linear functions lead to non-Gaussian distributions
- Kalman filter is not applicable anymore!

What can be done to resolve this?

Local linearization!

EKF Linearization: First Order Taylor Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \underbrace{\frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}}_{=:\, G_t} (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \underbrace{\frac{\partial h(\bar{\mu}_t)}{\partial x_t}}_{=:\, H_t} (x_t - \bar{\mu}_t)$$

Jacobian matrices

Reminder: Jacobian Matrix

- It is a **non-square matrix** $m \times n$ in general
- Given a vector-valued function

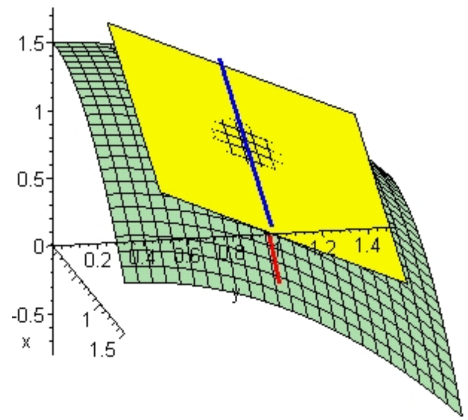
$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{pmatrix}$$

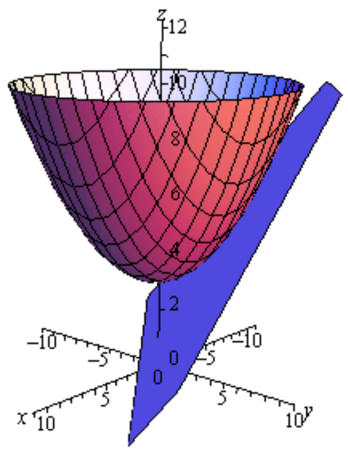
- The **Jacobian matrix** is defined as

$$G_x = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

Reminder: Jacobian Matrix

- It is the orientation of the tangent plane to the vector-valued function at a given point





- Generalizes the gradient of a scalar valued function

EKF Linearization: First Order Taylor Expansion

- Prediction:

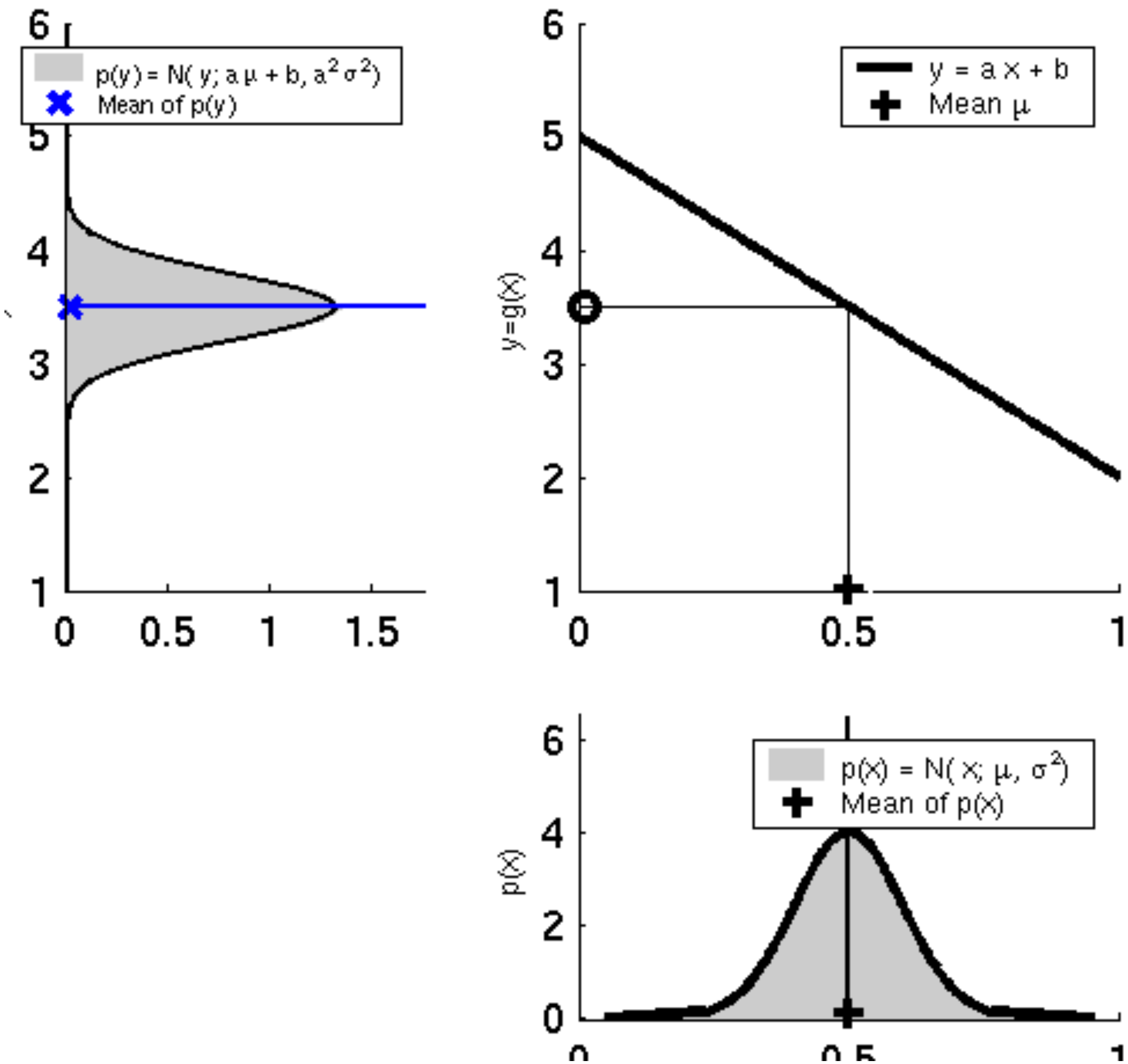
$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \underbrace{\frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}}_{=: G_t} (x_{t-1} - \mu_{t-1})$$

- Correction:

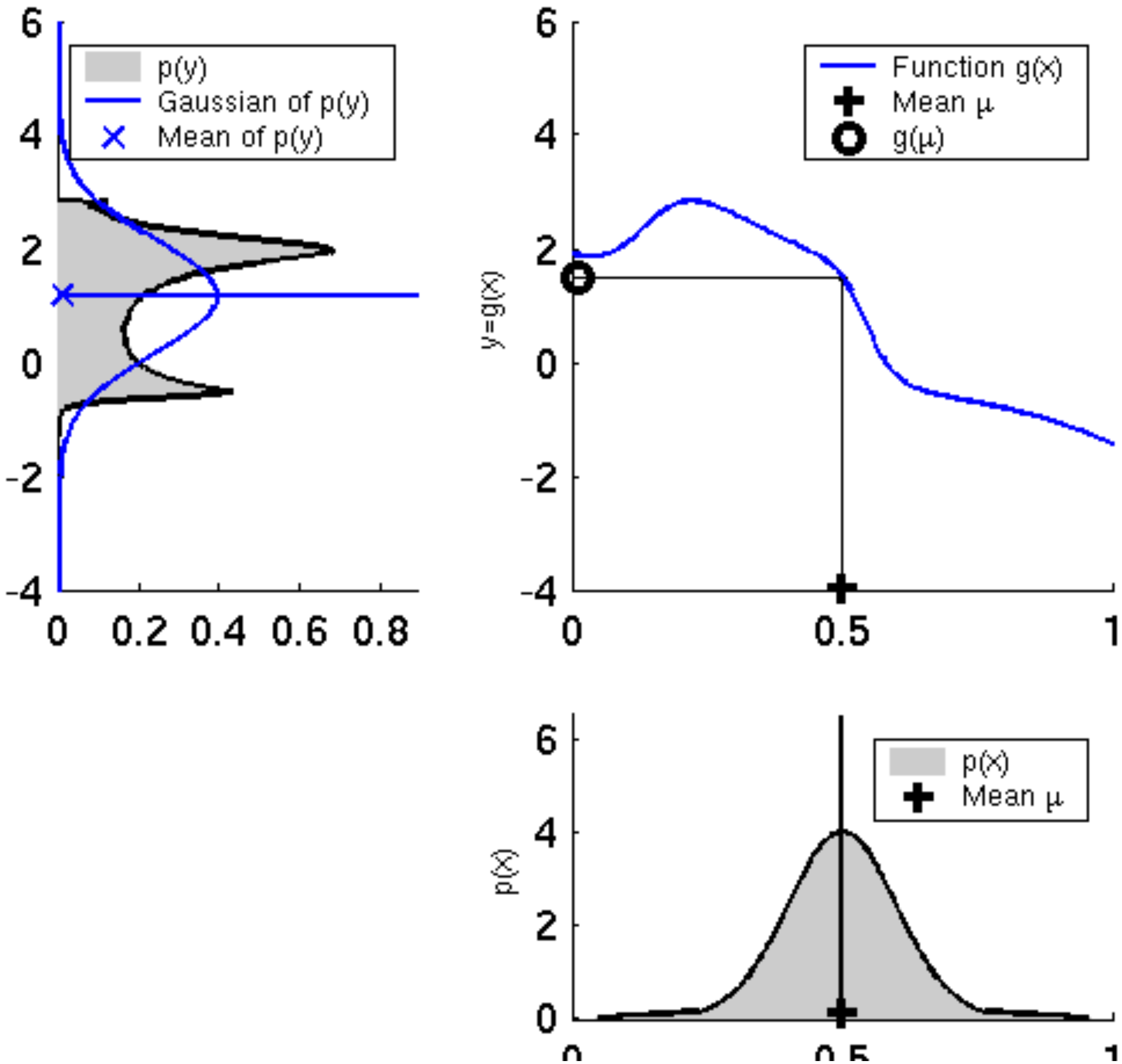
$$h(x_t) \approx h(\bar{\mu}_t) + \underbrace{\frac{\partial h(\bar{\mu}_t)}{\partial x_t}}_{=: H_t} (x_t - \bar{\mu}_t)$$

Linear functions!

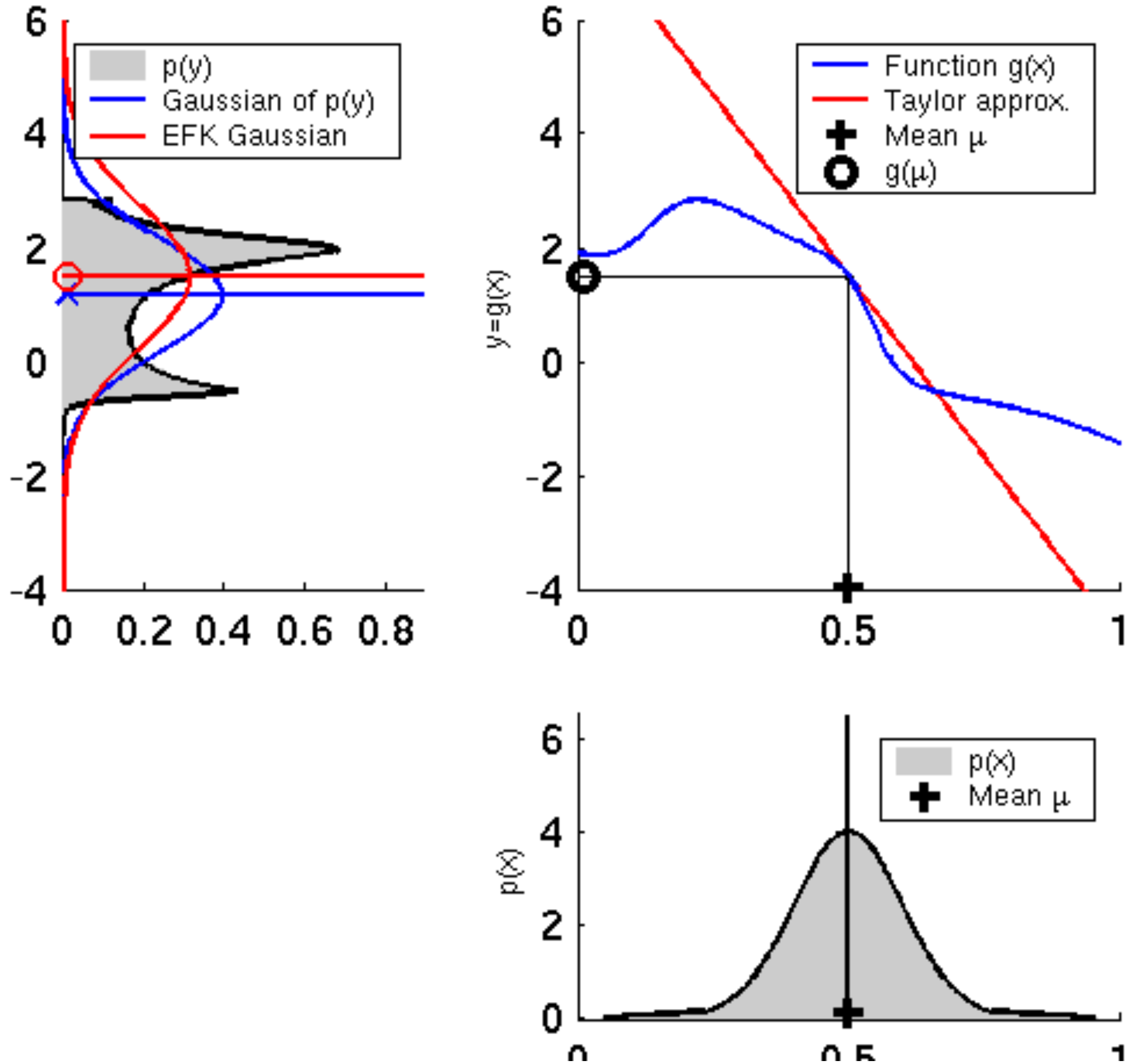
Linearity Assumption Revisited



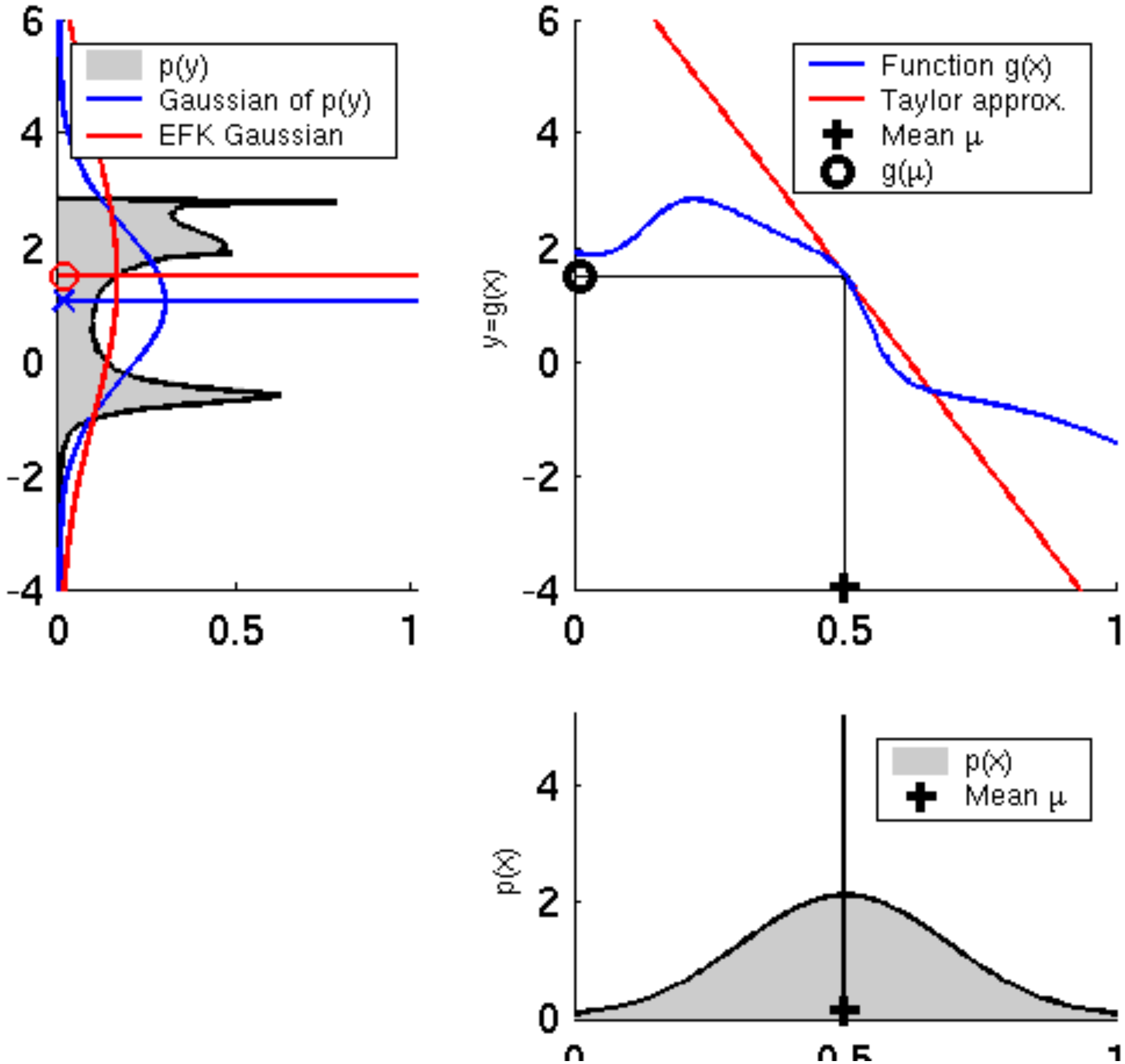
Non-Linear Function



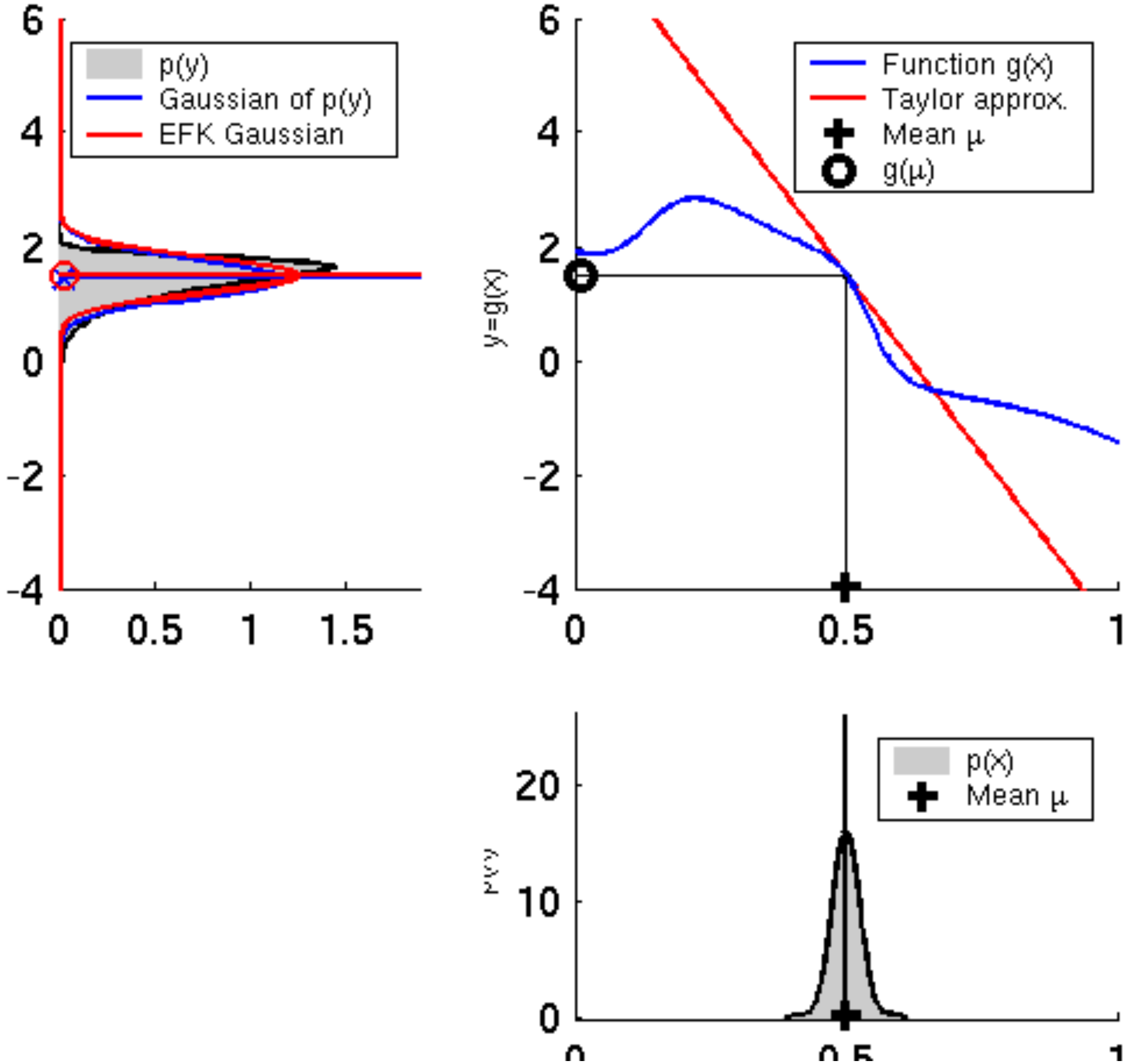
EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)



Linearized Motion Model

- The linearized model leads to

$$
\begin{aligned}
p(x_t \mid u_t, x_{t-1}) \approx & \det\left(2\pi R_t\right)^{-\frac{1}{2}} \\
& \exp\left(-\frac{1}{2} \left(x_t - g(u_t, \mu_{t-1}) - G_t\left(x_{t-1} - \mu_{t-1}\right)\right)^T \right. \\
& \left. R_t^{-1} \left(x_t - \underbrace{g(u_t, \mu_{t-1}) - G_t\left(x_{t-1} - \mu_{t-1}\right)}_{\text{linearized model}}\right)\right)
\end{aligned}
$$

- R_t describes the noise of the motion

Linearized Observation Model

- The linearized model leads to

$$p(z_t \mid x_t) = \det\left(2\pi Q_t\right)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{1}{2}\left(z_t - h(\bar{\mu}_t) - H_t\left(x_t - \bar{\mu}_t\right)\right)^T\right.$$
$$\left.Q_t^{-1}\left(z_t - \underbrace{h(\bar{\mu}_t) - H_t\left(x_t - \bar{\mu}_t\right)}_{\text{linearized model}}\right)\right)$$

- Q_t describes the measurement noise

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\bar{\mu}_t = \underline{g(u_t, \mu_{t-1})}$

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - \underline{h(\bar{\mu}_t)})$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: return μ_t, Σ_t

$$\boxed{\begin{array}{c} A_t \leftrightarrow G_t \\ C_t \leftrightarrow H_t \end{array}}$$

KF vs. EKF

Extended Kalman Filter Summary

- Extension of the Kalman filter
- One way to handle the non-linearities
- Performs local linearizations
- Works well in practice for moderate non-linearities
- Large uncertainty leads to increased approximation error error

Literature

Kalman Filter and EKF

- Thrun et al.: “Probabilistic Robotics”, Chapter 3
- Schön and Lindsten: “Manipulating the Multivariate Gaussian Density”
- Welch and Bishop: “Kalman Filter Tutorial”