EXERCISE 4

Data from a bivariate process are reported in `ESE8_ex4.csv`. Assume we know both the mean and the standard deviation of the quality characteristics x_1 and x_2:

- $\mu_1 = 10$; $\mu_2 = 20$;
- $\sigma_1 = 1$; $\sigma_2 = 2$.

Assume we also know the correlation coefficient: $\rho_{12} = 0.8$.

Design a control chart for the mean of the process.

> ## Solution

In [ ]:

```
# Import the necessary libraries
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats
import pandas as pd

# Import the data
data = pd.read_csv('ESE8_ex4.csv')

# Define the parameters
mu_1 = 10
mu_2 = 20

std_1 = 1
std_2 = 2

corr = 0.8

data.head()
```

Out[ ]:

	Sample	x1	x2
0	1	11.09	20.28
1	2	9.33	19.01
2	3	10.95	20.96
3	4	10.53	21.33
4	5	11.95	22.51

> From the correlation coefficient we can estimate the covariance. Thus, the mean vector and the variance/covariance matrix are:

In [ ]:

```
# Create the mean vector
mu = pd.Series({'x1': mu_1, 'x2': mu_2})

# Calculate the variance/covariance matrix
covariance = corr * std_1 * std_2
```

```python
var_1 = std_1**2
var_2 = std_2**2

# define the covariance matrix as a pandas dataframe
S = pd.DataFrame([[var_1, covariance],
                  [covariance, var_2]],
                  columns=['x1', 'x2'],
                  index=['x1', 'x2'])

print("The mean vector is: \n", mu)
print("\nThe variance/covariance matrix is: \n", S)
```

```
The mean vector is:
 x1    10
x2    20
dtype: int64

The variance/covariance matrix is:
      x1   x2
x1  1.0  1.6
x2  1.6  4.0
```

> The quality charcateristics are monitored by using individual measurements so $n = 1$. We can implement a χ^2 Control Chart (known parameters) with $\alpha = 0.0027$.
>
> EXAMPLE (using the first sample):

In [ ]:

```python
alpha = 0.0027 # significance level
n = 1          # sample size
m = len(data)  # number of samples
p = 2          # number of variables

# drop the sample column
data = data.drop('Sample', axis=1)
data.head()
```

Out[ ]:

	x1	x2
0	11.09	20.28
1	9.33	19.01
2	10.95	20.96
3	10.53	21.33
4	11.95	22.51

In [ ]:

```python
S_inv = np.linalg.inv(S) # inverse of the variance/covariance matrix

# Calculate Chi2 statistic
Chi2_1 = n * (data.iloc[0] - mu).transpose().dot(S_inv).dot(data.iloc[0] - mu)

# Calculate the upper control limit
UCL = stats.chi2.ppf(1 - alpha, df = p)

print("The Chi2 statistic of the sample 1 is: %.3f" % Chi2_1)
print("The UCL is: %.3f" % UCL)
```

```
The Chi2 statistic of the sample 1 is: 2.676
The UCL is: 11.829
```

> The first sample is in control. Now we can extend the process to the other stamples and create the CC.

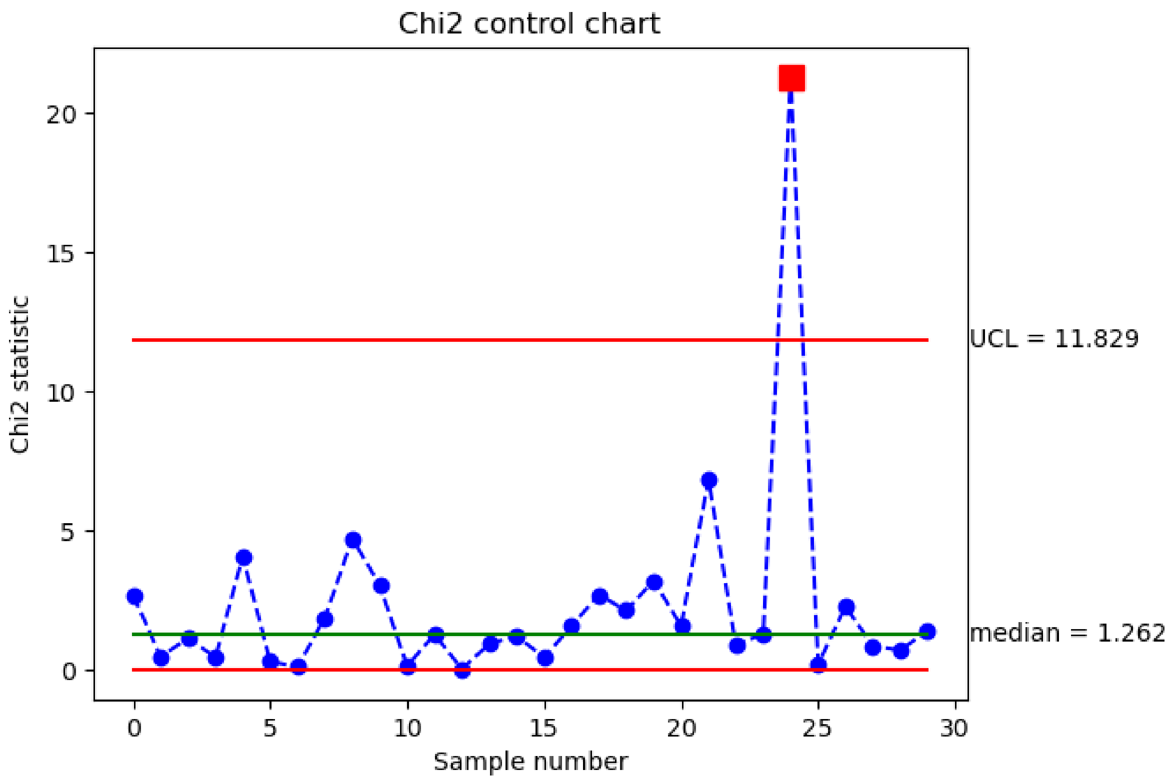
In [ ]:
```
# Add an empty column to the dataframe to store the Chi2 statistic
data_CC = data.copy()
data_CC['Chi2'] = np.nan

for i in range(m):
    data_CC['Chi2'].iloc[i] = n * (data.iloc[i] - mu).transpose().dot(S_inv).dot(d

# Now we can add the UCL, CL and LCL to the dataframe
data_CC['Chi2_UCL'] = UCL
data_CC['Chi2_CL'] = data_CC['Chi2'].median()
data_CC['Chi2_LCL'] = 0

# Add one column to test if the sample is out of control
data_CC['Chi2_TEST'] = np.where((data_CC['Chi2'] > data_CC['Chi2_UCL']), data_CC['

# Inspect the dataset
data_CC.head()
```

Out[ ]:

	x1	x2	Chi2	Chi2_UCL	Chi2_CL	Chi2_LCL	Chi2_TEST
0	11.09	20.28	2.676500	11.829007	1.262424	0	NaN
1	9.33	19.01	0.453569	11.829007	1.262424	0	NaN
2	10.95	20.96	1.120278	11.829007	1.262424	0	NaN
3	10.53	21.33	0.442236	11.829007	1.262424	0	NaN
4	11.95	22.51	4.060903	11.829007	1.262424	0	NaN

> We can now plot the χ^2 Control Chart

In [ ]:
```
# Plot the Chi2 control chart
plt.title('Chi2 control chart')
plt.plot(data_CC['Chi2'], color='b', linestyle='--', marker='o')
plt.plot(data_CC['Chi2_UCL'], color='r')
plt.plot(data_CC['Chi2_CL'], color='g')
plt.plot(data_CC['Chi2_LCL'], color='r')
plt.ylabel('Chi2 statistic')
plt.xlabel('Sample number')
# add the values of the control limits on the right side of the plot
plt.text(len(data_CC)+.5, data_CC['Chi2_UCL'].iloc[0], 'UCL = {:.3f}'.format(data_
plt.text(len(data_CC)+.5, data_CC['Chi2_CL'].iloc[0], 'median = {:.3f}'.format(dat
# highlight the points that violate the alarm rules
plt.plot(data_CC['Chi2_TEST'], linestyle='none', marker='s', color='r', markersize
plt.show()
```



> As we can see the sample at position 24 (therefore sample 25) is OOC

> # Exercise 4 (continued)
>
> If we don't know the true values of the parameters, we should design the chart (Phase I) with $n = 1$. One way to estimate **S** (the best when $n = 1$) is:

$$\mathbf{v}_i = \mathbf{x}_{i+1} - \mathbf{x}_i \qquad i = 1, 2, \ldots, m-1 \qquad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1' \\ \mathbf{v}_2' \\ \vdots \\ \mathbf{v}_{m-1}' \end{bmatrix}$$

The resulting (*short range*) estimator is: $\mathbf{S}_2 = \frac{1}{2}\frac{\mathbf{V}'\mathbf{V}}{(m-1)}$

Phase I limits: $\begin{cases} \mathrm{UCL} = \frac{(m-1)^2}{m}\beta_{\alpha, p/2, (m-p-1)/2} \\ \mathrm{LCL} = 0 \end{cases}$

Phase II limits: $\begin{cases} \mathrm{UCL} = \frac{p(m+1)(m-1)}{m^2 - mp}F_{\alpha,p,m-p} \\ \mathrm{LCL} = 0 \end{cases}$

In [ ]:

```
# Create the V matrix
V = data.diff().dropna()

# Calculate the short range estimator S2
```

```
S2 = 1/2 * V.transpose().dot(V) / (m-1)

# Display the short range estimator
print("The short range estimator is: \n", S2)
```

```
The short range estimator is:
           x1        x2
x1  1.232222  2.001583
x2  2.001583  4.860412
```

> With the estimator $\mathbf{S_2}$ we can now compute the Hotelling's T^2 statistic of the samples

In [ ]:

```
# Calculate the Xbar from the data
Xbar = data.mean()

S2_inv = np.linalg.inv(S2)

# Calculate the Hotelling T2 statistic
data_CC['T2'] = np.nan
for i in range(m):
    data_CC['T2'].iloc[i] = n * (data.iloc[i] - Xbar).transpose().dot(S2_inv).dot(

# Now we can add the UCL, CL and LCL to the dataframe
data_CC['T2_UCL'] = ((m-1)**2)/m*stats.beta.ppf(1 - alpha, p/2, (m-p-1)/2)
data_CC['T2_CL'] = data_CC['T2'].median()
data_CC['T2_LCL'] = 0

# Add one column to test if the sample is out of control
data_CC['T2_TEST'] = np.where((data_CC['T2'] > data_CC['T2_UCL']), data_CC['T2'],

# Inspect the dataset
data_CC.head()
```

```
c:\Users\matte\anaconda3\envs\qda\lib\site-packages\pandas\core\indexing.py:1732:
SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/stabl
e/user_guide/indexing.html#returning-a-view-versus-a-copy
  self._setitem_single_block(indexer, value, name)
```

Out[ ]:

	x1	x2	Chi2	Chi2_UCL	Chi2_CL	Chi2_LCL	Chi2_TEST	T2	T2_UCL	T2_CL
0	11.09	20.28	2.676500	11.829007	1.262424	0	NaN	1.731331	9.944715	1.124725
1	9.33	19.01	0.453569	11.829007	1.262424	0	NaN	0.596517	9.944715	1.124725
2	10.95	20.96	1.120278	11.829007	1.262424	0	NaN	0.613175	9.944715	1.124725
3	10.53	21.33	0.442236	11.829007	1.262424	0	NaN	0.420886	9.944715	1.124725
4	11.95	22.51	4.060903	11.829007	1.262424	0	NaN	2.776166	9.944715	1.124725

> Let's plot the T^2 CC

In [ ]:

```
# Plot the T2 control chart
plt.title('T2 control chart')
plt.plot(data_CC['T2'], color='b', linestyle='--', marker='o')
```

```
plt.plot(data_CC['T2_UCL'], color='r')
plt.plot(data_CC['T2_CL'], color='g')
plt.plot(data_CC['T2_LCL'], color='r')
plt.ylabel('T2 statistic')
plt.xlabel('Sample number')
# add the values of the control limits on the right side of the plot
plt.text(len(data_CC)+.5, data_CC['T2_UCL'].iloc[0], 'UCL = {:.3f}'.format(data_CC
plt.text(len(data_CC)+.5, data_CC['T2_CL'].iloc[0], 'median = {:.3f}'.format(data_(
# highlight the points that violate the alarm rules
plt.plot(data_CC['T2_TEST'], linestyle='none', marker='s', color='r', markersize=1(
plt.show()
```