Math 3070, Applied Statistics

Section 1

November 11, 2019

Section 8.1

- Z-Tests for Proportion
- T-Tests for Population Mean with unknown σ

Large-Sample z Test for Proportion

Given a random sample from a Bernoulli distribution with unknown parameter p, the z *test* for the null hypothesis $H_0 : p = p_0$ based on the test statistic $Z = \frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}}$ is given by the following rejection region:

Alternative hypothesis		Rejection region
(Upper-tailed test)	$H_a : p > p_0$	$Z \geq z_\alpha$
(Lower-tailed test)	$H_a : p < p_0$	$Z \leq -z_\alpha$
(Two-tailed test)	$H_a : p \neq p_0$	$\lvert Z \rvert \geq z_{\alpha/2}$

Here α is the nominal significance level, and z_α is a critical value from the standard normal distribution.

Example

We are given a coin which someone suggests may give outcomes with unequal proportions when we spin it on a table. We test this by spinning the coin 80 times. If we observe 54 heads, do we reject the null hypothesis at the $\alpha = .01$ significance level?

We will use a large-sample z test for the null hypothesis $H_0 : p = \frac{1}{2}$ against the alternative $H_0 : p \neq \frac{1}{2}$. The test statistic is

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{\frac{54}{80} - \frac{1}{2}}{\sqrt{\frac{1}{2}(1-\frac{1}{2})/80}} = 3.13$$

The rejection region is $\{|Z| > z_{\alpha/2}\}$ where $z_{\alpha/2} = z_{.005} = 2.58$. Since $|Z| = 3.13 > 2.58$, we reject the null hypothesis.

In other words, the test provides strong evidence that the coin indeed gives heads more often than tails when spun.

Example

We are given a coin which someone suggests may give outcomes with unequal proportions when we spin it on a table. We test this by spinning the coin 80 times. If we observe 54 heads, what is the P-value of the test?

Again, we are using a large-sample z test for the null hypothesis $H_0 : p = \frac{1}{2}$ against the alternative $H_0 : p \neq \frac{1}{2}$. We already calculated the test statistic:

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{\frac{54}{80} - \frac{1}{2}}{\sqrt{\frac{1}{2}(1-\frac{1}{2})/80}} = 3.13$$

The P-value is the probability that we would observe a value of Z this extreme (i.e., a value of Z with $|Z| \geq 3.13$):

$$P = P(|Z| \geq 3.13) = 2\Phi(-3.13) \approx 2(.0009) = .0018$$

t Test for Mean of Normal Distribution with Unknown σ

Given a random sample $X_1, \ldots, X_n$ from a normal distribution with unknown standard deviation, the *t test* for the null hypothesis $H_0 : \mu = \mu_0$, based on the test statistic $T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$, is given by the following rejection region:

Alternative hypothesis		Rejection region
(Upper-tailed test)	$H_a : \mu > \mu_0$	$T \geq t_{\alpha,n-1}$
(Lower-tailed test)	$H_a : \mu < \mu_0$	$T \leq -t_{\alpha,n-1}$
(Two-tailed test)	$H_a : \mu \neq \mu_0$	$\lvert T \rvert \geq t_{\alpha/2,n-1}$

Here α is the significance level (Type I error probability), and $t_{\alpha,n-1}$ is a critical value from the t distribution with $n-1$ degrees of freedom.

Example

A type of candy bar is labeled 60 grams. Someone suggests that the candy bars weigh less than specified. To test this, we gather a random sample of size 5 and observe $\overline{X} = 58.8$ and $S = 0.9$. Do we reject the null hypothesis $H_0 : \mu = 60$ at the significance level $\alpha = .01$?

We use the t test with the one-tailed alternative $H_a : \mu < 60$. The test statistic is

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{58.8 - 60}{0.9/\sqrt{5}} = -2.98$$

The critical value is $t_{\alpha,n-1} = t_{.01,4} = 3.747$. The rejection region is $\{T < -3.747\}$, whereas in our sample $T = -2.98 > -3.747$, so we do not reject the null hypothesis.

In other words, the data does *not* allow us to conclude that the average weight of the candy bars is less than specified.

Example

A type of candy bar is labeled 60 grams. Someone suggests that the candy bars weigh less than specified. To test this, we gather a random sample of size 5 and observe $\overline{X} = 58.8$ and $S = 0.9$. What is the P-value for the test?

We use the t test with the one-tailed alternative $H_a : \mu < 60$. As before, the test statistic is

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{58.9 - 60}{0.9/\sqrt{5}} = -2.98$$

We use a table to find the probability of observing a value for T at least this extreme:

$$P = P(T \leq -2.98) \approx .020$$

So $P = .020$ is the P-value for the test.

Summary

$\alpha =$ probability of Type I error, that H_0 is true but is rejected

$\beta =$ probability of a Type II error, that H_0 is false but is not rejected

$P =$ P-value $=$ smallest α for which the test would reject H_0

Test	Null Hypothesis	Test Statistic
z test	$H_0 : \mu = \mu_0$	$Z = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$
t test	$H_0 : \mu = \mu_0$	$T = \frac{\overline{X}-\mu_0}{S/\sqrt{n}}$
z test for a proportion	$H_0 : p = p_0$	$Z = \frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}}$

Alternative hypothesis		P-value for z test
(Upper-tailed test)	$H_a : \mu > \mu_0$	$P(Z \geq z)$
(Lower-tailed test)	$H_a : \mu < \mu_0$	$P(Z \leq z)$
(Two-tailed test)	$H_a : \mu \neq \mu_0$	$P(\lvert Z\rvert \geq \lvert z\rvert)$

Given a fixed significance level α, we reject H_0 if and only if $P \leq \alpha$. Here, $z \sim N(0, 1)$.

Alternative hypothesis		P-value for z test
(Upper-tailed test)	$H_a : \mu > \mu_0$	$P(T \geq t)$
(Lower-tailed test)	$H_a : \mu < \mu_0$	$P(T \leq t)$
(Two-tailed test)	$H_a : \mu \neq \mu_0$	$P(\lvert T\rvert \geq \lvert t\rvert)$

Given a fixed significance level α, we reject H_0 if and only if $P \leq \alpha$. Here, $t \sim t(n - 1)$.