Math 3070, Applied Statistics

Section 1

November 1, 2019

Section 7.3

- Confidence Intervals with unknown μ and σ

Small-Sample Confidence Interval for Mean of Normal

Suppose we have a random sample $X_1, \ldots, X_n$ from a normal distribution, where the mean μ and variance σ^2 are both unknown.

We have seen that if n is large, then the *pivotal* statistic $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ is approximately a standard normal random variable. However, if n is small then T instead has a so-called **t distribution with** $\nu = n-1$ **degrees of freedom** or $T \sim t(\nu)$.

In this class, if $n < 40$, we'll use the $T \sim t(\nu)$. Otherwise, $T \sim N(0,1)$ can be used. In practice, a method should be carefully selected for the data.

T-Distribution

We have not studied the t distribution. It has tables and used similiarly to the normal distribution for the purpose computing of finding confidence intervals.

Useful Properties:

1. symmetric about 0
2. identified by ν
3. as $\nu \to \infty$, the distribution becomes $N(0, 1)$
4. bell shaped, but flatten

Important to these calculations are critical t values, $t_{\alpha/2,\nu}$:

$$1 - \alpha = P(-t_{\alpha/2,\nu} < T < t_{\alpha/2,\nu})$$

where $\nu = n - 1$ and $T \sim t(\nu)$.

Note:

$$\alpha/2 = P(-t_{\alpha/2,\nu} < T)$$

Confidence Interval for Mean of Normal

Suppose we have a random sample $X_1, \ldots, X_n$ from a normal distribution, where the mean μ and variance σ^2 are both unknown.

Previously, we have used the fact that n is large, then the statistic $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ is approximately equal to $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$, which is a standard normal random variable. However, if n is small then this is not a good approximation. Instead, T has a so-called *t distribution*.

Given a random sample $X_1, \ldots, X_n$ from a normal distribution with unknown mean and variance, A $100(1-\alpha)\%$ confidence interval for the mean μ is

$$\overline{X} \pm \frac{t_{\alpha/2,\nu} \cdot S}{\sqrt{n}}$$

where $t_{\alpha/2,\nu}$ is a critical value from a *t distribution* with $\nu = n-1$ degrees of freedom.

Example

A process produces alginate beads with diameters (in mm) normally distributed with unknown mean μ and unknown standard deviation σ. A random sample of 9 beads have the following diameters:

$$3.9, 5.1, 5.2, 5.7, 5.8, 6.1, 6.2, 6.3, 6.5$$

Find a 99% confidence interval for the mean diameter μ.

Here $\alpha = 1 - .99 = .01$, so the relevant critical value is

$$t_{\alpha/2,\nu} = t_{.005,8} = 3.355$$

The sample mean is $\overline{X} = 5.64$ and the sample standard deviation is $S = .809$, so the confidence interval is given by

$$\overline{X} \pm \frac{t_{\alpha/2} \cdot S}{\sqrt{n}} = 5.64 \pm \frac{3.355 \cdot 0.809}{\sqrt{9}} = 5.64 \pm 0.90$$

Prediction Interval for a Normal Population

Given a random sample $X_1, \ldots, X_n$ from a normal distribution, suppose we want to construct an interval $[A, B]$ which we can be 95% confident will contain a future observation X_{n+1}. Such an interval is called a *prediction interval*.

The statistic $T = \frac{\overline{X} - X_{n+1}}{S\sqrt{1+\frac{1}{n}}}$ has a t distribution with $\nu = n - 1$ degrees of freedom.

Given a random sample $X_1, \ldots, X_n$ from a normal distribution, a $100(1-\alpha)\%$ prediction interval for an independent observation X_{n+1} is

$$\overline{X} \pm t_{\alpha/2,n-1} \cdot S\sqrt{1 + \frac{1}{n}}$$

Example

An article reports the following data on the breakdown voltage of electrically stressed circuits, assumed to be normally distributed:

$$1470, 1510, 1690, 1740, 1900, 2000, 2030, 2100, 2200,$$
$$2290, 2380, 2390, 2480, 2500, 2580, 2190, 2700$$

Find a 95% confidence interval for the mean μ. Then find a 95% prediction interval for a future observation.

We found $\overline{X} = 2126.5$ and $S^2 = 137324.3$. The critical value is $t_{\alpha/2,\nu} = t_{.025,16} = 2.120$, giving the confidence interval for μ:

$$\overline{X} \pm \frac{t_{\alpha/2,\nu} \cdot S}{\sqrt{n}} = 2126.5 \pm 190.5$$

Likewise we get the prediction interval:

$$\overline{X} \pm t_{\alpha/2,\nu} \cdot S\sqrt{1 + \frac{1}{n}} = 2126.5 \pm 808.4$$