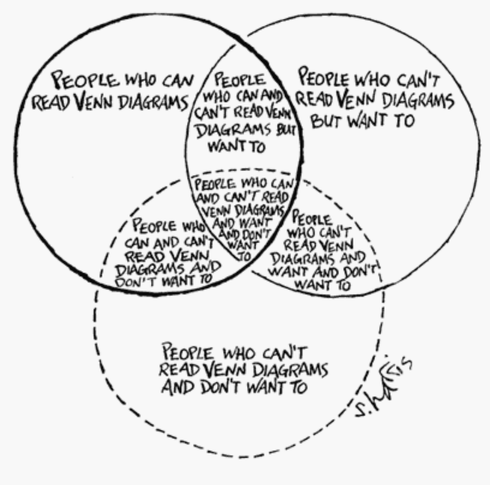
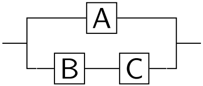
Ch. 2 – Probability



Example – Three-Component System

Suppose a system has three components, and to work either component A must work, *or* both components B and C must work.



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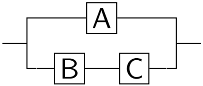

- The probabilities that components A, B, and C work are .7, .4, and .9, respectively.

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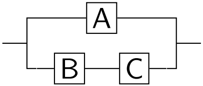
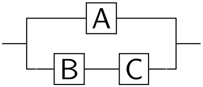


- The probabilities that components A, B, and C work are .7, .4, and .9, respectively.
- What is the probability that the system will work?

Example – Direct but Tedious Solution



$P(\text{component A works}) = .7$

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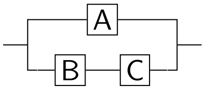

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8 possible outcomes: SSS, SSF, SFS, SFF, FSS, FSF, FFS, FFF
(e.g., FSS means component A fails but B, C successfully work.)

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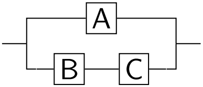

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$$\begin{array}{ll} P(\text{SSS}) = .7 \times .4 \times .9 = .252 & P(\text{FSS}) = .3 \times .4 \times .9 = .108 \\ P(\text{SSF}) = .7 \times .4 \times .1 = .028 & P(\text{FSF}) = .3 \times .4 \times .1 = .012 \\ P(\text{SFS}) = .7 \times .6 \times .9 = .378 & P(\text{FFS}) = .3 \times .6 \times .9 = .162 \\ P(\text{SFF}) = .7 \times .6 \times .1 = .042 & P(\text{FFF}) = .3 \times .6 \times .1 = .018 \end{array}$$

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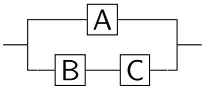

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8 possible outcomes: SSS, SSF, SFS, SFF, FSS, FSF, FFS, FFF
(e.g., FSS means component A fails but B, C successfully work.)

$$P(\text{SSS}) = .7 \times .4 \times .9 = .252 \qquad P(\text{FSS}) = .3 \times .4 \times .9 = .108$$
$$P(\text{SSF}) = .7 \times .4 \times .1 = .028 \qquad P(\text{FSF}) = .3 \times .4 \times .1 = .012$$
$$P(\text{SFS}) = .7 \times .6 \times .9 = .378 \qquad P(\text{FFS}) = .3 \times .6 \times .9 = .162$$
$$P(\text{SFF}) = .7 \times .6 \times .1 = .042 \qquad P(\text{FFF}) = .3 \times .6 \times .1 = .018$$

$$\begin{aligned} P(\{SSS, SSF, SFS, SFF, FSS\}) &= .252 + .028 + .378 + .042 + .108 \\ &= .808 \end{aligned}$$

There is an 80.8% chance the system will still work after four years.

Basic Set Theory

Suppose a random process has a set Ω of possible outcomes.
An **event** is a subset of Ω. Given two events A and B,

- The **intersection** $A \cap B$ consists of outcomes in A *and* B,
- The **union** $A \cup B$ consists of outcomes in A *or* B (or both).
- The **complement** A' consists of outcomes *not* in A.

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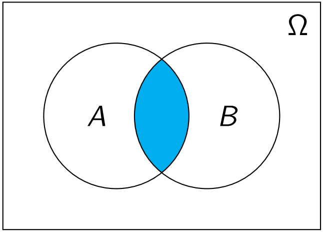
- Let A be the event that we roll an even number.
- Let B be the event that we roll a 4 or higher.

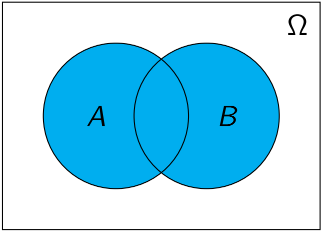
Then

$$\begin{aligned} \Omega &= \{1, 2, 3, 4, 5, 6\} \\ A &= \{2, 4, 6\} \\ B &= \{4, 5, 6\} \\ A \cap B &= \{4, 6\} \\ A \cup B &= \{2, 4, 5, 6\} \\ A' &= \{1, 3, 5\} \end{aligned}$$

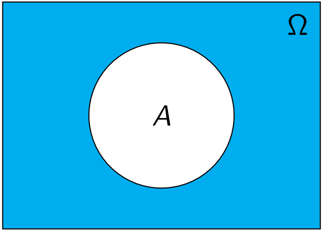
Venn Diagrams



Venn diagram for $A \cap B$



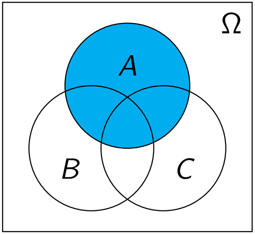
Venn diagram for $A \cup B$



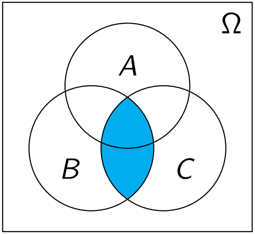
Venn diagram for A'

Venn Diagrams with Three Events

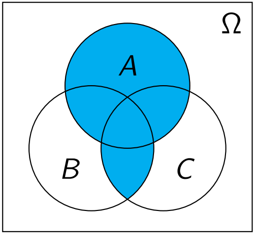
To draw a Venn diagram involving three or more events, it may help to work step-by-step. For example, to draw a Venn diagram for $A \cup (B \cap C)$, first draw Venn diagrams for A and $B \cap C$, then combine them to get the Venn diagram for $A \cup (B \cap C)$:



A



$B \cap C$

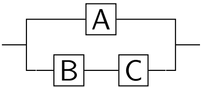


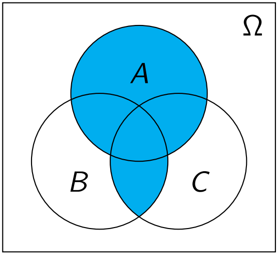
$A \cup (B \cap C)$

Example – In Terms of Set Theory

In the example, $\Omega = \{LLL, LLF, LFL, LFF, FLL, FLF, FFL, FFF\}$, and we have events

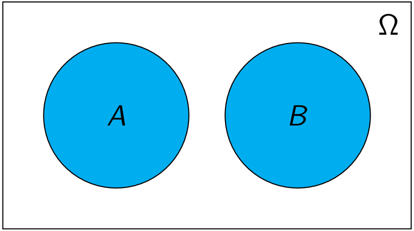
$A = \{LFF, LFL, LLF, LLL\}$	Component A works
$B = \{FLF, FLL, LLF, LLL\}$	Component B works
$C = \{FFL, FLL, LFL, LLL\}$	Component C works
$B \cap C = \{FLL, LLL\}$	Components B,C work
$A \cup (B \cap C) = \{LFF, LFL, LLF, LLL, FLL\}$	System works





Disjoint events

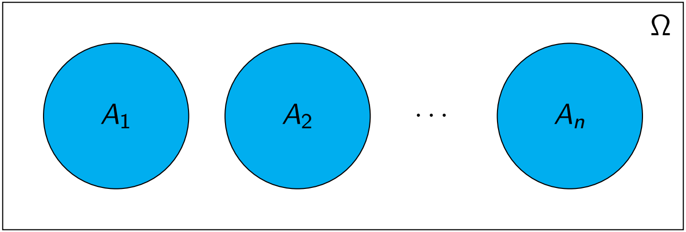
- The **null event**, containing no outcomes, is denoted $\emptyset$.
- Two events A and B are **disjoint** (or **mutually exclusive**)if $A \cap B = \emptyset$, i.e., if they have no outcomes in common.



Venn diagram for $A \cup B$ when A and B are disjoint

Several disjoint events

Events $A_1, A_2, \ldots, A_n$ are **disjoint** if A_i and A_j are disjoint for every pair $i \neq j$.



Venn diagram for $A_1 \cup A_2 \cup \cdots \cup A_n$ when $A_1, A_2, \ldots, A_n$ are disjoint

Set-Theoretic Identities

The following identities always hold for any events A, B, and C:

$A \cap A = A$	$A \cup A = A$
$A \cap B = B \cap A$	$A \cup B = B \cup A$
$A \cap (B \cap C) = (A \cap B) \cap C$	$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$(A \cap B)' = A' \cup B'$	$(A \cup B)' = A' \cap B'$
$A \cap A' = \emptyset$	$A \cup A' = \Omega$
$A \cap \emptyset = \emptyset$	$A \cup \Omega = \Omega$
$A \cap \Omega = A$	$A \cup \emptyset = A$
$\emptyset' = \Omega$	$\Omega' = \emptyset$
$A'' = A$	

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$(A \cap B)' = A' \cup B'$	$(A \cup B)' = A' \cap B'$
$A \cap A' = \emptyset$	$A \cup A' = \Omega$
$A \cap \emptyset = \emptyset$	$A \cup \Omega = \Omega$
$A \cap \Omega = A$	$A \cup \emptyset = A$
$\emptyset' = \Omega$	$\Omega' = \emptyset$
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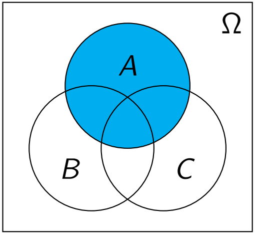
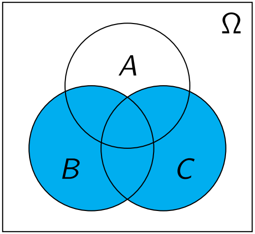
Although this list may appear unfriendly at first, these identities are all just common sense. If some of them are not obvious, we can use a Venn diagram to see why they are true:

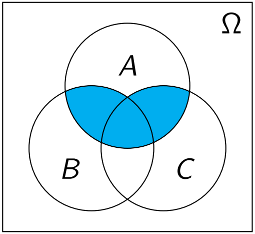
"Proof" that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



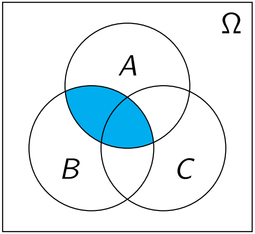
A



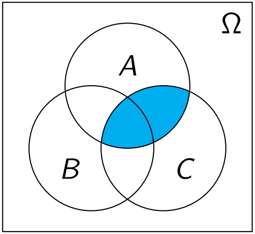
$B \cup C$



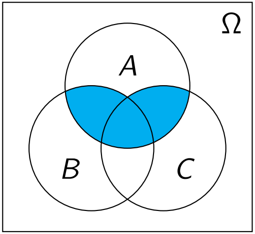
$A \cap (B \cup C)$



$A \cap B$

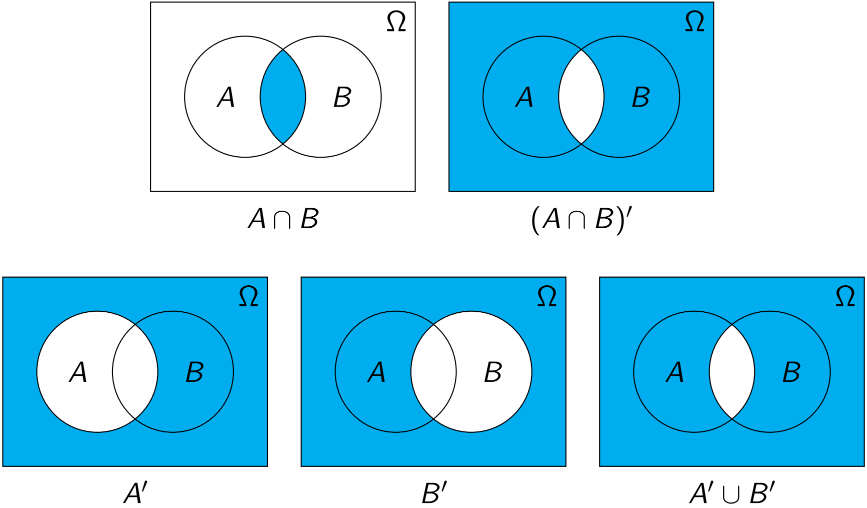


$A \cap C$

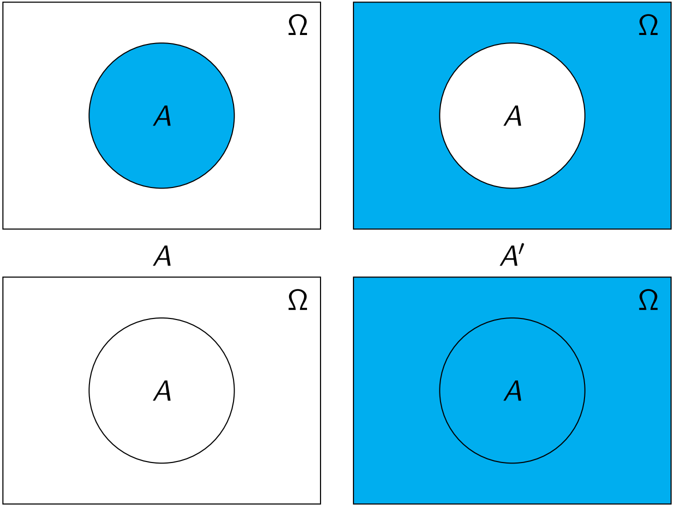


$(A \cap B) \cup (A \cap C)$

"Proof" that $(A \cap B)' = A' \cup B'$



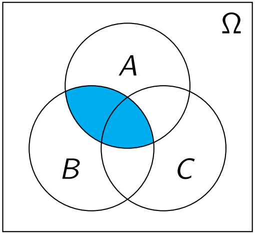
"Proof" that $A \cap A' = \emptyset$ and $A \cup A' = \Omega$



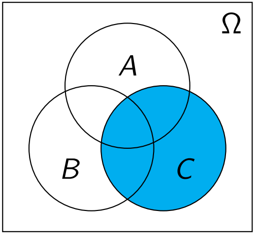
A $\quad$ A'

$A \cap A' = \emptyset$ $\quad$ $A \cup A' = \Omega$

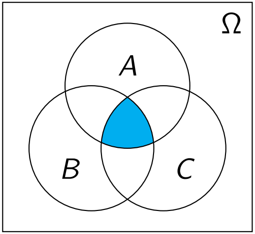
"Proof" that $(A \cap B) \cap C = A \cap (B \cap C)$



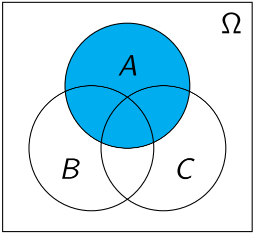
$A \cap B$



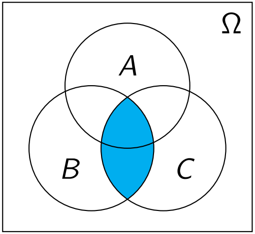
C



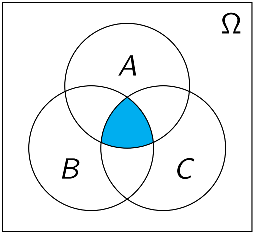
$(A \cap B) \cap C$



A



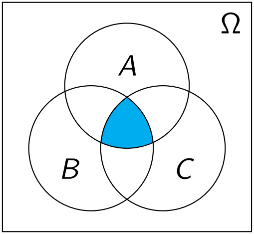
$B \cap C$



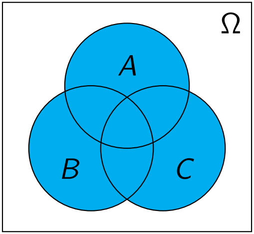
$A \cap (B \cap C)$

Multiple Intersections and Unions

Since $(A \cap B) \cap C = A \cap (B \cap C)$, we don't need to use parentheses when writing the intersection of three or more events; we can simply write $A \cap B \cap C$. A similar statement applies to unions.



$A \cap B \cap C$



$A \cup B \cup C$

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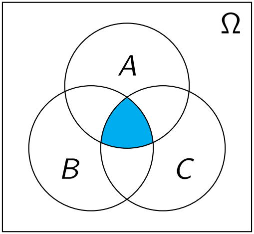
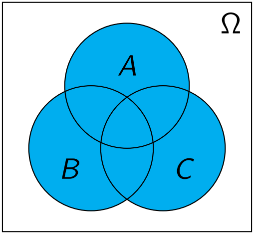


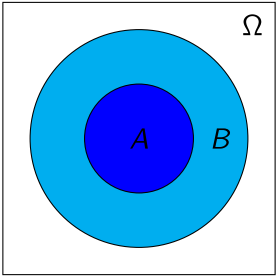
$A \cap B \cap C$



$A \cup B \cup C$

Caution: $A \cap (B \cup C)$ is *not* the same as $(A \cap B) \cup C$. Parentheses must still be used to distinguish these.

Given events A and B, if every outcome in A is also in B, then we say that A is **contained** in B, and we write $A \subseteq B$.



Containment

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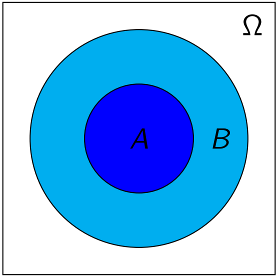


For example, if we roll a six-sided die, and let A be the event of getting a 5 or higher and B be the event of getting a 3 or higher, then $A \subseteq B$:

$$A = \{5, 6\} \subseteq \{3, 4, 5, 6\} = B$$

Properties of Containment

With a little thought, all of the following properties should be clear:

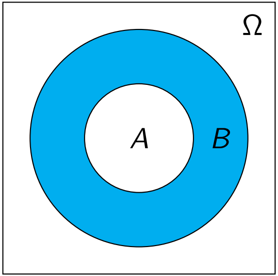
1. $A \subseteq A$ for all events A.
2. $\emptyset \subseteq A$ and $A \subseteq \Omega$ for all events A.
3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
4. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
5. $A \subseteq B$ if and only if $A \cap B = A$.
6. $A \subseteq B$ if and only if $A \cup B = B$.
7. $A \cup B \subseteq C$ if and only if $A \subseteq C$ and $B \subseteq C$.
8. $A \subseteq B \cap C$ if and only if $A \subseteq B$ and $A \subseteq C$.
9. $A \subseteq A \cup B$ for all events A.
10. $A \cap B \subseteq A$ for all events A.
11. If $A \subseteq B$, then $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$.
12. If $A \subseteq B$, then $B' \subseteq A'$.

Set Difference

Given events A, B with $A \subseteq B$, we define their **difference**,

$$B - A = B \cap A'$$

That is, $B - A$ consists of all outcomes of B which are not in A.



$B - A$

We may use identities to simplify expressions involving events. For example,

$$\begin{aligned}(A' \cap B)' \cap B &= (A'' \cup B') \cap B \\ &= (A \cup B') \cap B \\ &= (A \cap B) \cup (B' \cap B) \\ &= (A \cap B) \cup \emptyset \\ &= A \cap B\end{aligned}$$

$$\begin{aligned}B' \cap (A \cup (A \cup B)') &= B' \cap (A \cup (A' \cap B')) \\ &= B' \cap ((A \cup A') \cap (A \cup B')) \\ &= B' \cap (\Omega \cap (A \cup B')) \\ &= B' \cap (A \cup B') = B'\end{aligned}$$

Quiz

Next class we'll have a quiz with three parts:

1. Set-theoretic identities: I'll give you the left-hand sides; you give me the right-hand sides.
2. Venn diagrams: I'll ask you to use Venn diagrams to prove a certain identity (one in our list but not necessarily one proven in the slides).
3. Set-theoretic algebra: I'll give you an expression, and you'll simplify it.

See the Practice Quiz posted on Canvas.