Math 3070, Applied Statistics

Section 1

August 30, 2019

Lecture Outline, 8/30

Section 2.5

- Independence
- Examples

Independence, Definition

Definition

Two events A and B are **independent** if $P(A|B) = P(A)$ and are **dependent** otherwise.

A is independent of B implies that B is independent of A.
Begin by assuming $P(A|B) = P(A)$

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} \\ \text{(Multi. Rule for Cond. Prob.)} &= \frac{P(A|B)P(B)}{P(A)} \\ \text{((A is independent of B))} &= \frac{P(A)P(B)}{P(A)} \\ &= P(B) \end{aligned}$$

Independence, Definition

Definition

Two events A and B are **independent** if $P(A|B) = P(A)$ and are **dependent** otherwise.

Intuition: The occurance of event B does not impact the probability of event A.

Multiplication Rule

Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

Proof: Use the definition of multiplication rule of conditional probability

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

This also means that $P(A|B) = P(A)$ implies $P(B|A) = P(B)$.

Independence, Coin Flip Example

Suppose two fair coins are flipped. Show that the first coin landing on heads independent of the second coin landing on heads.

The expected answer is yes. I will now show this in set notation.

$$\mathcal{S} = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$\text{event } A = \text{"first coin lands on heads"} = \{(H, H), (H, T)\}$$

$$\text{event } B = \text{"second coin lands on heads"} = \{(H, H), (T, H)\}$$

$$P(A) = \frac{1}{2}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2} = P(A)$$

Independence, Comments

If A and B are independent so are A' and B'.

Proof:

$$\begin{aligned}
P(A' \cap B') &= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
\text{(indep. of A and B)} &= 1 - (P(A) + P(B) - P(A)P(B))
\end{aligned}$$

$$\begin{aligned}
P(A')P(B') &= (1 - P(A))(1 - P(B)) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= 1 - (P(A) + P(B) - P(A)P(B))
\end{aligned}$$

$$P(A' \cap B') = P(A')P(B')$$

Independence, Comments

If A and B are independent so are A and B'.

Proof:

$$
\begin{aligned}
P(A) &= P(A \cap B) + P(A \cap B') \\
&= P(A|B)P(B) + P(A|B')P(B') \\
\text{(A and B are indep.)} &= P(A)P(B) + P(A|B')P(B') \\
\text{rearrange} & \\
P(A|B')P(B') &= P(A) - P(A)P(B) \\
&= P(A)(1 - P(B)) \\
&= P(A)(P(B'))
\end{aligned}
$$

$$P(A|B') = P(A)$$

Mutual Independence, Definition

Definition

Events $A_1, \ldots, A_n$ are **mutually independent** if

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

for all choices of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

For example, saying three events A, B, C are independent means

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$
$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C)$$

If any two event are independent, then the events are **Pairwise independent**. Example, the last three statements show pairwise Independence of A, B and C.

Mutual Independence, Example

Suppose you flip 3 fair coins. Event A is that coin 1 lands on heads, event B is that coin 2 lands on heads and event C is that coin 3 lands on heads. Are the events mutually independent?

You can verify by applying the reasoning from the two coin example.

$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap B \cap C) = P(\{(H, H, H)\}) = \frac{1}{8} = \frac{1}{2}\frac{1}{2}\frac{1}{2}$$

Mutual Independence, Nonexample

Suppose you flip 2 fair coins. Event A is that coin 1 lands on heads, event B is that coin 2 lands on tails and event C is that the coins land on the same side. Are the events mutually independent?

$$A = \{(H, T), (H, H)\}, B = \{(H, T), (T, T)\}, C = \{(T, T), (H, H)\}$$

$$P(A \cap B \cap C) = P(\varnothing) = 0$$

nope, but they are pairwise independent.

$$P(A \cap B) = P(\{(H, T)\}) = \frac{1}{4} = P(A)P(B)$$

$$P(A \cap C) = P(\{(H, H)\}) = \frac{1}{4} = P(A)P(C)$$

$$P(B \cap C) = P(\{(T, T)\}) = \frac{1}{4} = P(B)P(C)$$

Independence, Summary

- Independence of A and B:

$$P(A|B) = P(A), \quad P(A \cap B) = P(A)P(B)$$

- A is independent of B implies the converse.
- If A and B are independent then so are A' and B', and A and B'.
- Be careful not to assume events are independent.
- Pairwise Independence does not imply mutual Independence.
- The idea of Independence will come in handy later.

Multiplication Rule and Containment Example

A manufacturing process uses two machines, A and B, in serial. The probability that machine B fails is 1%. The probability that machine A fails is 5%. Assuming that the machines fail independently, what is the probability that machine A fails given that at least one machine fails?

$$\text{event } A = \text{"machine A fails"}$$

$$\text{event } B = \text{"machine B fails"} \ ,$$

$$\text{event } C = \text{"at least one machine fails"} \ ,$$

$$\begin{aligned} P(C) &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \\ &= 0.05 + 0.01 - 0.05(0.01) \\ &= 0.0595 \end{aligned}$$

Multiplication Rule and Containment Example

A manufacturing process uses two machines, A and B. The probability that machine B fails is 1%. The probability that machine A fails is 5%. Assuming that the machines fail independently, what is the probability that machine A fails given that at least one machine fails?

$$\text{event } A = \text{"machine A fails"}$$

$$\text{event } B = \text{"machine B fails"}\ ,$$

$$\text{event } C = \text{"at least one machine fails"}\ ,$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

$$(\text{A is contained in C}) = \frac{P(A)}{P(C)} = \frac{0.05}{0.0595} \approx 0.840336134$$

Containment shown on next slide.

Multiplication Rule and Containment Example

(M_A, M_B) denotes the failure of each machine.
$M_A = F$ denotes that machine A fails.
$M_A = N$ denotes that machine A does not fails.
$M_B = N$ uses the same notation.

$$A = \{(F, F), (F, N)\}$$

$$C = \{(F, F), (F, N), (N, F)\}$$

Serial Example

A company uses 17 machines for a manufacturing process. Each fail mutually independently of each other. Each has a 7% probability of failing. What is the probability that any are failing?

$$\text{event } A_i = \text{machine } i \text{ fails}$$

$$\begin{aligned} P(\text{"any are failing"}) &= 1 - P(\text{"all are working"}) \\ &= 1 - P(A_1' \cap \ldots \cap A_{17}') \\ (\text{indep}) &= 1 - [P(A_1') \cdot \ldots \cdot P(A_{17}')] \end{aligned}$$

$$P(A_i') = 1 - P(A_i) = 1 - 0.07 = 0.93$$

$$P(\text{"any are failing"}) = 1 - 0.93^{17} \approx 0.70878742441$$

Serial Example

If there are n events A_i, same probability and mutually independent, then

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdot \ldots \cdot P(A_n) = P(A_1)^n$$