Ch. 6 – Point Estimation

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- Each of the observations $X_1, \ldots, X_n$ will be less than θ, and if n is large we expect one of them to be close to θ. So we may estimate θ using the maximum: $\hat{\theta} = \max\{X_1, \ldots, X_n\}$.

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We gave two possible estimators for the parameter θ of a uniform distribution on $[0, \theta]$:

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Since $E(\hat{\theta}_1) = \theta$, this means that $\hat{\theta}_1$ is an unbiased estimator for θ.

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so $\hat{\theta}_3$ is in fact an unbiased estimator for θ.

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Question: Which of these estimators is better?

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We now have two unbiased estimators for the parameter θ of a uniform distribution on $[0, \theta]$:

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More advanced statistical theory can be used to show that in fact $\hat{\theta}_3$ is a **minimum variance unbiased estimator**: it has a smaller variance than any other unbiased estimator.

Unbiasedness of Sample Variance

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Setting this equal to zero gives $40 - 2\lambda = 0$, so $\lambda = 20$. This is called the **maximum likelihood estimate** of λ.

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$$\hat{\lambda} = \frac{1}{X}$$

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Example: Given an observation of an exponential random variable X with unknown rate λ, we found that the mle for λ was

$$\hat{\lambda} = \frac{1}{X}$$

Therefore, the mle for $\mu = 1/\lambda$ is

$$\hat{\mu} = 1/\hat{\lambda} = \frac{1}{1/X} = X$$

Thus, the mle for μ is simply the observed value X. It can be shown that this in fact is the minimum variance unbiased estimator for μ.

MLE of Several Parameters

There are many families of distributions where we would want to estimate two parameters at once. For example,

- For a normal distribution, we want to estimate μ and σ.
- For a gamma distribution, we want to estimate k and λ.
- For a Weibull distribution, we want to estimate α and β.

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Given a random sample $X_1, \ldots, X_n$ from a distribution with several parameters $\theta_1, \ldots, \theta_m$, the mle of $(\theta_1, \ldots, \theta_m)$ is the combination of values $(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ maximizing the joint pmf or joint pdf $f(x_1, \ldots, x_n; \theta_1, \ldots, \theta_m)$ of $X_1, \ldots, X_n$.

Estimating Parameters of a Normal Distribution

Given a random sample $X_1, \ldots, X_n$ from a normal distribution:

The MLE for μ is the sample mean $\overline{X}$, and this is also the minimum variance unbiased estimator.

This means that alternative estimators for μ, such as the sample median $\tilde{X}$, must have a greater variance. (In fact, $V(\overline{X}) = \frac{\sigma^2}{n}$, while $V(\tilde{X}) \approx \frac{\pi}{2} \cdot \frac{\sigma^2}{n}$ for large n.)

The MLE for σ^2 is $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$. However, this estimator is biased; the minimum variance unbiased estimator is the sample variance $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$.