# Math 3070, Applied Statistics

Section 1

August 28, 2019

# Lecture Outline, 8/28

Section 2.4

- Conditional Probability and Multiplication Rule
- Law of Total Probability and Bayes' Theorem
- Questions about Quiz Materials

# Conditional Probability

Goal: Compute probabilities of events when another event has occured.

Notation: $P(A|B)$ represents the conditional probability of event $A$ given that the event $B$ has occured.

Example:

$$A = \text{"six-sided die lands on 2"} = \{2\}$$

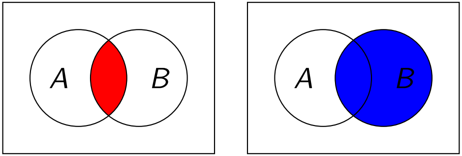
$$B = \text{"six-sided die lands on an even number"} = \{2, 4, 6\}$$

$$P(B|A) = 1$$

$$P(A|B) = \frac{1}{3}$$

There are three even numbers so we expect that the probability that two is one of them is $\frac{1}{3}$ when an even number must land.

# Conditional Probability, Definition




**Definition**

Let $B$ be an event with $P(B) > 0$. The **conditional probability** that an event $A$ occurs, given that $B$ occurs, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule: $P(A \cap B) = P(A|B) \cdot P(B)$

# Conditional Probability, Intuition

Intuition: event $B$ has occured the only outcomes in event $A$ that can occur must also be in event $B$, $A \cap B$. Since event $B$ has occured, the number of possible outcomes is the number of outcomes in $B$.

$$P(A|B) = \frac{\# \text{ of outcomes in } A \cap B}{\# \text{ of outcomes in } \mathcal{S}} \Bigg/ \frac{\# \text{ of outcomes in } B}{\# \text{ of outcomes in } \mathcal{S}}$$

$$= \frac{\# \text{ of outcomes in } A \cap B}{\# \text{ of outcomes in } B}$$

# Conditional Probability, Verification Example

Example: Use the formula to verify the previous example.

$$A = \text{"six-sided die lands on 2"} = \{2\}$$

$$B = \text{"six-sided die lands on an even number"} = \{2, 4, 6\}$$

$$P(A) = \frac{1}{6} \quad P(B) = \frac{3}{6} \quad P(A \cap B) = \frac{1}{6}$$

$$P(A|B) = \frac{P(A \cap B)}{B} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$P(B|A) = \frac{P(A \cap B)}{A} = \frac{1/6}{1/6} = 1$$

# Conditional Probability, Multiplication Rule Example

Problem: The 2010 US Census found that 23% of US residents are undergraduate students. CNBC reports that 73% of undergraduate students have student loans. Determine the probability that a randomly selected person is an undergraduate student and has student loans.

$$\text{event } A = \text{"some one with student loans is selected"}$$

$$\text{event } B = \text{"an undergraduate student is selected"}$$

$$P(A \cap B) = P(A|B) \cdot P(B) = 0.73 \cdot 0.23 = 0.1679$$

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# Conditional Probability, Summary

- Conditional Probability and Multiplication Formula, respectively,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(A \cap B) = P(A|B)P(B)$$

- Used when a condition is given, must happen or always true.
- Homework Hint: event $A$ from the verification example is contained inside event $B$.

Questions?

# Law of Total Probability, Explanation

Let $A_1, \ldots, A_n$ be disjoint events with $\mathcal{S} = A_1 \cup \cdots \cup A_n$ ($A_i$ **partition** $\mathcal{S}$). Then for any event $B$,

$$P(B) = \sum_{i=1}^{n} P(A_i) P(B|A_i)$$

picture on page 80

Proof: By the multiplication rule,

$$\begin{aligned}
\sum_{i=1}^{n} P(A_i)P(B|A_i) &= \sum_{i=1}^{n} P(A_i)\frac{P(A_i \cap B)}{P(A_i)} \\
&= P(A_1 \cap B) + \cdots + P(A_n \cap B) \\
(B \cap A_i \text{ are disjoint}) &= P((A_1 \cap B) \cup \cdots \cup (A_n \cap B)) \\
&= P((A_1 \cup \cdots \cup A_n) \cap B) \\
&= P(\mathcal{S} \cap B) = P(B)
\end{aligned}$$

# Law of Total Probability, Complement Example

Problem: The probability of seeing the sun is 9% when there is rain and 65% when there is no rain. The probability that there is rain today is 60%. What is the probability of seeing the sun?

$$\text{event } A = \text{"seeing the sun"}$$

$$\text{event } B = \text{"there is rain today"}$$

1. $B$ and $B'$ partition $\mathcal{S}$
2. $P(B') = 1 - P(B) = 1 - 0.6 = 0.4$
3. 

$$\begin{aligned} P(A) &= P(B)P(A|B) + P(B')P(A|B') \\ &= 0.6 \cdot 0.09 + 0.4 \cdot 0.65 \\ &= 0.314 \end{aligned}$$

# Law of Total Probability, Example

Problem: Select flipping 1,2 or 3 fair coins, equally likely. What is the probability that at all coins land on heads?

$$\text{event } A_i = \text{i ''coins selected''}$$

$$\text{event } B = \text{''all coins land on heads''}$$

1. $P(A_i) = \frac{1}{3}$
2. $P(B|A_1) = \frac{1}{2}$
3. $P(B|A_2) = \frac{1}{4}$
4. $P(B|A_2) = \frac{1}{8}$
5. $$\begin{aligned} P(B) &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) \\ &= \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{4} + \frac{1}{3}\frac{1}{8} = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\ &\approx 0.291666667 \end{aligned}$$

# Law of Total Probability, Summary

- Useful if you can **partition** your sample space into disjoint events, $A_i$. Remeber that for $\{A_i\}$ to be a partition, $\cup_{i=1}^{n} A_i = \mathcal{S}$ and $A_i \cap A_j = \varnothing$ for any two $i$ and $j$.
- Can be useful with $A$ and $A'$.
- $P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$

Questions?

# Bayes' Theorem (Simpler Version), Explanation

How are $P(A|B)$ and $P(B|A)$ related?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \qquad \text{(assuming } P(A) > 0, P(B) > 0\text{)}$$

Proof: Apply the definition of conditional probability, then the multiplication rule.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

# Bayes' Theorem, Frequent False Positives Example

A rare disease affects 1 in 1000 people. A person with the disease tests positive 99% of the time, whereas a person without the disease tests positive only 2% of the time. If a randomly selected person tests positive, what is the probability that they have the disease?

The given information may be expressed,

$$P(D) = .001, P(T|D) = .99, P(T|D') = .02$$

The law of total probability implies

$$\begin{aligned} P(T) &= P(T|D)P(D) + P(T|D')P(D') \\ &= (.99)(.001) + (.02)(.999) = .02097 \end{aligned}$$

Bayes' theorem then implies

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{(.99)(.001)}{.02097} \approx .047$$

# Bayes' Theorem (Book Version), Explanation

Recall that $B$ and $B'$ partition $\mathcal{S}$. Apply to Bayes' Theorem.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B')P(B')}$$

assuming $(P(A) > 0, P(B) > 0)$.

Now, apply the idea with a partition into $k$ set $A_i$, $i = 1, \ldots, k$.

Let $A_1, \ldots, A_k$ partition $\mathcal{S}$. Then for any other event $B$ for which $P(B) > 0$,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_j) \cdot P(A_i)} \quad j = 1, \ldots, k$$

$P(A_j)$ are the *prior* probabilities of $A_j$ and $P(A_j|B)$*posterior* probabilities of $A_j$ given that $B$ has occured.

# Bayes' Theorem (Book Version), Reversal Example

Problem: At a shoe store, 40% of online shoppers make a purchase. In-store shoppers make a purchase 95% of the time. Sales data shows that 70% of customers shopped online. What is the probability that a purchase comes from an online shopper?

event $A =$ "made purchase", event $B =$ "shopped online"

- $P(B) = 0.7,\ P(B') = 1 - P(B) = 0.3$
- $P(A|B) = 0.4, P(A|B') = 0.95$

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B')P(B')} \\ &= \frac{0.4 \cdot 0.7}{0.4 \cdot 0.7 + 0.95 \cdot 0.3} \approx 0.49557522123 \end{aligned}$$

All conditioning was related to events $B$ or $B'$, but we got a probabilty conditioned on $A$ without collecting data conditioned on $A$. Some would call this free information.

# Bayes' Theorem, Summary

- Used when the "given" or conditioning events are switched.
- Simple version:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Several version with different partitions: $A, A'$ yields

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B')P(B')}$$

- and $\{A_i\}$ yields

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_j) \cdot P(A_i)} \quad j = 1, \ldots, k$$

- Used in Bayesian Statistics. Will present if time permits. Think of $B$ as data and $A_j$ as a system state.

# Closing

- These formulas are used frequently in probability.
- Questions about the quiz?