Math 3070, Applied Statistics

Section 1

November 11, 2019

Lecture Outline, 11/11

Section 8.1

- Z-Tests for Population Mean

z Test for Mean of Normal Distribution with Known σ

Given a random sample $X_1, \ldots, X_n$ from a normal distribution with known standard deviation σ, the z *test* for the null hypothesis $H_0 : \mu = \mu_0$, based on the test statistic $Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$, is given by the following rejection region, depending on whether a one-tailed or two-tailed test is desired:

Alternative hypothesis		Rejection region
(Upper-tailed test)	$H_a : \mu > \mu_0$	$Z \geq z_\alpha$
(Lower-tailed test)	$H_a : \mu < \mu_0$	$Z \leq -z_\alpha$
(Two-tailed test)	$H_a : \mu \neq \mu_0$	$\lvert Z \rvert \geq z_{\alpha/2}$

Here α is the significance level (Type I error probability), and z_α is a critical value from the standard normal distribution.

Example

A machine is specified to drill holes with diameter 4 mm. We wish to test the null hypothesis $H_0 : \mu = 4$ against the alternative $H_a : \mu \neq 4$. If the diameters are normally distributed with $\sigma = .2$, and we observe $\overline{X} = 3.87$ in a sample of size 10, do we reject the null hypothesis at the significance level $\alpha = .05$?

The test statistic is

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{3.87 - 4}{.2/\sqrt{10}} = -2.06$$

The rejection region is $\{|Z| > z_{\alpha/2}\}$ where $z_{\alpha/2} = z_{.025} = 1.96$.

Since $|Z| = 2.06 > 1.96$, the test statistic is in the rejection region, so we reject the null hypothesis.

Type II Error Probability for z Test

Given a random sample $X_1, \ldots, X_n$ from a normal distribution with known standard deviation σ, the Type II error probability β for the z test is as follows:

Alternative hypothesis	Type II Error Probability β
$H_a : \mu > \mu_0$	$\Phi(z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}})$
$H_a : \mu < \mu_0$	$1 - \Phi(-z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}})$
$H_a : \mu \neq \mu_0$	$\Phi(z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}) - \Phi(-z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}})$

The following formula is the for the sample size so that β is no more than a certain value. Note: β in this equation is β the bound placed on the probability of a type two error.

$$n = \begin{cases} \left[\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu}\right]^2 & \text{for one-tailed tests} \\ \left[\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu}\right]^2 & \text{for two-tailed tests} \end{cases}$$

Example

A machine is specified to drill holes with diameter 4 mm. We wish to test the hypothesis $H_0 : \mu = 4$ against the alternative $H_a : \mu \neq 4$ using a z test with significance level $\alpha = .05$ on a random sample of size 15. If the diameters are normally distributed with $\sigma = .2$ and the true mean is $\mu = 3.95$, what is the Type II error probability?

Here $\mu_0 = 4$, so

$$\begin{aligned}
\beta &= \Phi(z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}) - \Phi(-z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}) \\
&= \Phi(z_{.025} + \frac{4 - 3.95}{.2/\sqrt{15}}) - \Phi(-z_{.025} + \frac{4 - 3.95}{.2/\sqrt{15}}) \\
&= \Phi(1.96 + 0.97) - \Phi(-1.96 + 0.97) \\
&= \Phi(2.93) - \Phi(-0.99) = .837
\end{aligned}$$

Example

Consider the previous problem. How large should the sample size be so that the probability of a type II error is no more than 0.05?

$$z_\beta = -\Phi^{-1}(0.05) \approx 1.645$$

$$n = \left[\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu}\right]^2 \approx \left[\frac{0.2(1.96 + 1.645)}{4 - 3.95}\right]^2 \approx 209$$

P-Values for z Test

With all of the z test procedures, the P-value may be calculated as follows, where z is the observed value of the test statistic Z (assumed to have a standard normal distribution):

Alternative hypothesis		P-value
(Upper-tailed test)	$H_a : \mu > \mu_0$	$P(Z \geq z)$
(Lower-tailed test)	$H_a : \mu < \mu_0$	$P(Z \leq z)$
(Two-tailed test)	$H_a : \mu \neq \mu_0$	$P(\|Z\| \geq \|z\|)$

Example

A type of candy bar is labeled 60 grams. Someone suggests that the candy bars weigh less than specified. To test this, we gather a random sample of size 25 and observe $\overline{X} = 59.7$ and $\sigma = 0.9$ is known. What is the P-value for the test?

We use the z test with the one-tailed alternative $H_a : \mu < 60$. As before, the test statistic is

$$T = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{59.7 - 60}{0.9/\sqrt{25}} \approx -1.667$$

We use a table to find the probability of observing a value for T at least this extreme:

$$P = P(T \leq -1.667) \approx 0.0478$$

So $P = 0.0478$ is the P-value for the test.