Math 3070, Applied Statistics

Section 1

September 27, 2019

Lecture Outline, 9/27

Section 4.4

- Exponential Random Variable
- Chi-Squared (χ^2) Random Variable
- Gamma Random Variable

Exponential Random Variable

We say that X is an **exponential** random variable with rate $\lambda > 0$ if X has pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$X \sim exp(\lambda)$

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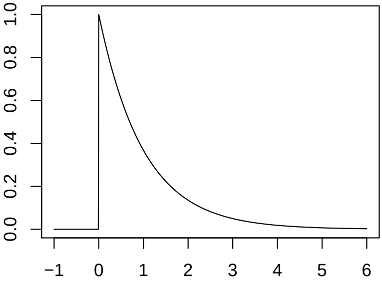
$X \sim exp(\lambda)$

We can check that this is a valid pdf:

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\ dx$$
$$= \lambda \frac{-1}{\lambda} e^{-\lambda x} \Big|_{x=0}^{\infty}$$
$$= 0 - (-1) = 1$$



CDF of Exponential Random Variable

Recall the pdf of an exponential random variable x with rate λ is

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The cdf $F(x)$ is therefore, for $x \geq 0$,

$$F(x) = \int_{-\infty}^{x} f(t)\ dt = \int_{0}^{x} \lambda e^{-\lambda t}\ dt$$
$$= -e^{-\lambda t}\Big|_{t=0}^{x} = 1 - e^{-\lambda x}$$

$$Fx) = \begin{cases} \lambda 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

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Consider a Poisson Random Variable with rate λ.

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This is the cdf of an exponential random variable of rate λ. Therefore, in a Poisson process, the waiting time for the first event is an exponential random variable with rate λ.

Mean of Exponential

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\end{aligned}$$

So $V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$. In other words, the standard deviation is $\sigma = 1/\lambda = \mu$.

Memoryless Property of Exponential

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Proof:
$$\begin{aligned} P(X \geq s + t \mid X \geq s) &= \frac{P(X \geq s + t \cap X \geq s)}{P(X \geq s)} \\ &= \frac{P(X \geq s + t)}{P(X \geq s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} = P(X \geq t) \end{aligned}$$

Due to the memoryless property, the exponential distribution can model times between events of a Poisson Random Variable.

Example

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In general, if X is an exponential random variable with rate λ,

$$P(X \geq t) = e^{-\lambda t}$$

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This is the same as the probability of a new bulb lasting 50 days, as we calculated on the previous slide.

Summary

- $X \sim exp(\lambda)$ means X follows an exponential distribution with parameter λ.
- Poisson models number of events with rate λ. Exponential with the same λ models wait times between them.
- Memoryless property is a subtle modeling issue, but mathematical nice.
- PDF:
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
- CDF:
$$Fx) = \begin{cases} \lambda 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
- $$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

Chi-squared Random Variable

If Z is a standard normal random variable, then Z^2 has a so-called **chi-squared** distribution with one **degree of freedom**.

A **chi-square** random variable with $\nu > 0$ degrees of freedom has pdf

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$X \sim \chi^2(\nu)$

Since Z is important in the central limit theorem, one should expect that χ^2 will appear in many applications.

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx$$

Chi-squared Random Variable, Z^2 Explained

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Last step uses symmetry of the distribution. Differentiate in x to recover the PDF.

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Last step uses symmetry of the distribution. Differentiate in x to recover the PDF.

$$\frac{d}{dx} P(Z^2 < x) = \frac{2}{\sqrt{2\pi}} \exp\left(\frac{-x}{2}\right) \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} \exp\left(\frac{-x}{2}\right)$$

Observe this is the PDF of a $\chi^2(1)$ random variable.

Chi-squared Random Variable, Mean and Variance

Consider $X \sim \chi^2(\nu)$.

$$E[X] = \nu$$

$$V(X) = 2\nu$$

Gamma Distribution

A **gamma** random variable X with parameters $\alpha, \beta > 0$ is given by the pdf

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(k)} x^{\alpha-1} e^{-x/\beta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$X \sim \Gamma(\alpha, \beta)$

$\beta = 1$ is called the standard gamma.

Used to relate other random variables together. For example, a $\Gamma(\nu/2, 2)$ and $\chi^2(\nu)$ have the same PDFs.

These relationships allows us to relate their parameters.

Gamma Distribution, Mean and Variance

Consider $X \sim \Gamma(\alpha, \beta)$

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$$E[X] = \alpha\beta = \frac{\nu}{2}2 = \nu$$

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Summary

$X \sim \chi^2(\nu)$ with $\nu > 0$
PDF:

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

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$X \sim \Gamma(\alpha, \beta)$ with $\alpha, \beta > 0$
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- $Z \sim N(0,1) \rightarrow Z^2 \sim \chi^2(1)$
- $X \sim \Gamma(\nu/2, 2) \iff X \sim \chi^2(\nu)$
- More relations between random variables to come.