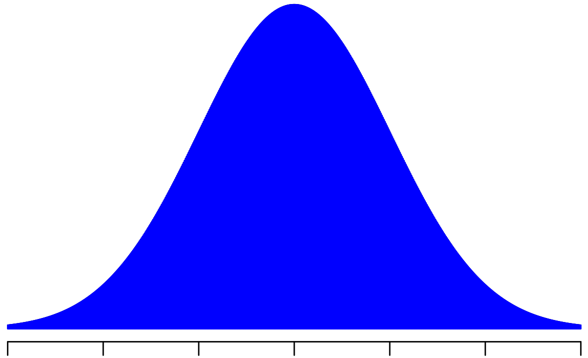
Ch. 4 – Continuous Random Variables



Continuous Random Variables

So far, we have only discussed discrete random variables, which have only a sequence of possible values (usually whole numbers):

- The number of defective widgets in a batch.
- The number of widgets inspected before finding one defective.
- The number of customers who visit a store in an hour.

However, many quantities in real life vary continuously:

- The length of a metal rod.
- The strength of a specimen of concrete.
- The weight of a bottled drink.
- The amount of time until the next customer arrives.

We will need different techniques to deal with continuous random variables.

Uniform Random Variable

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The interval $[0.2, 0.6]$ has length $0.6 - 0.2 = 0.4$, which is 40% of the total length of the interval $[0, 1]$. Therefore, intuitively the probability that X would be in the interval $[0.2, 0.6]$ should be

$$P(0.2 \leq X \leq 0.6) = 0.6 - 0.2 = 0.4$$

In general, for an interval $[a, b]$ inside $[0, 1]$ we should have

$$P(a \leq X \leq b) = b - a$$

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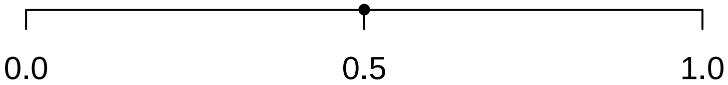

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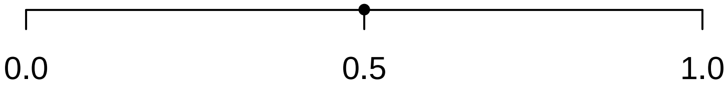

The probability is $P(0.5 \leq X \leq 0.5) = 0.5 - 0.5 = 0$. In fact, for any x in $[0, 1]$ the probability that X is exactly x is 0. And yet, X will always be some number in $[0, 1]$.

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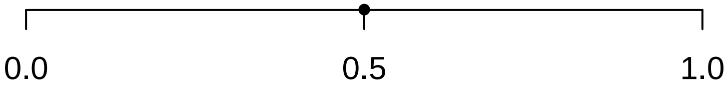

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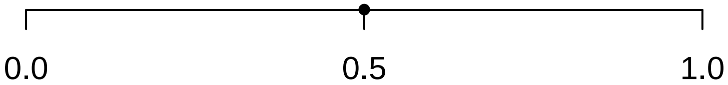

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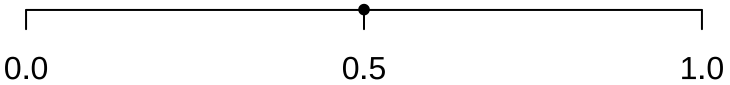

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- Even if an event has probability 0, that doesn't mean it is impossible for it to occur.
- For a continuous random variable, the concept of a probability mass function is useless: every probability $P(X = x)$ is zero.
- We *cannot* find the probability $P(a \leq X \leq b)$ by simply adding up all the probabilities $P(X = x)$ over all x in $[a, b]$.

Continuous Random Variable

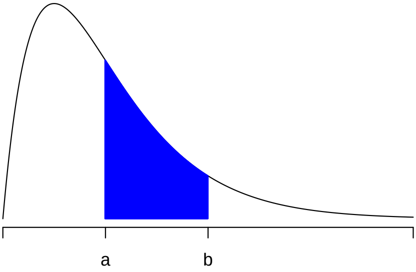
We say that a random variable X is **continuous** if $P(X = x) = 0$ for every x. If there is a function $f(x)$ such that for all $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

then we call $f(x)$ a **probability density function** (pdf) of X.

To be a valid pdf, we must have

1. $f(x) \geq 0$ for all x.
2. $\int_{-\infty}^{\infty} f(x) = 1$.



Standard Uniform Random Variable

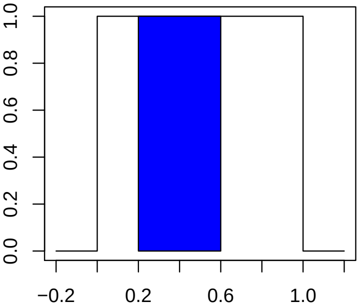
Define a pdf by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The continuous random variable X with this pdf is called a **standard uniform** random variable; it takes values uniformly on the interval [0,1].

For example, the probability that X is between .2 and .6 is

$$\begin{aligned} & P(.2 \leq X \leq .6) \\ &= \int_{.2}^{.6} 1 \; dx \\ &= x\Big|_{.2}^{.6} \\ &= .6 - .2 \\ &= .4 \end{aligned}$$



Uniform Random Variable

We say that X is a **uniform** random variable on the interval $[a, b]$ if X has pdf

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$$\begin{aligned} P(X \geq 10) &= \int_{10}^{\infty} f(x)\ dx \\ &= \int_{10}^{15} \frac{1}{15-0}\ dx \\ &= \frac{1}{15} x \Big|_{10}^{15} \\ &= \frac{15-10}{15} = 1/3 \end{aligned}$$

Exponential Random Variable

We say that X is an **exponential** random variable with rate $\lambda > 0$ if X has pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

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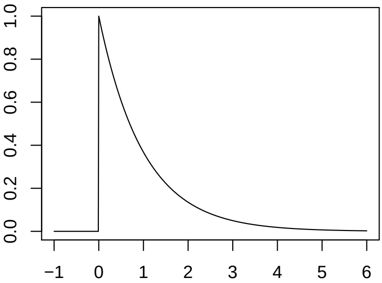
We can check that this is a valid pdf:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$
$$= \lambda \frac{-1}{\lambda} e^{-\lambda x} \Big|_{x=0}^{\infty}$$
$$= 0 - (-1) = 1$$



CDF of Continuous Random Variable

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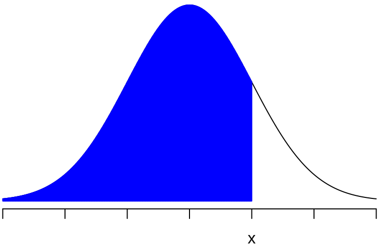
By the Fundamental Theorem of Calculus, $F'(x) = f(x)$, if f is continuous at x .



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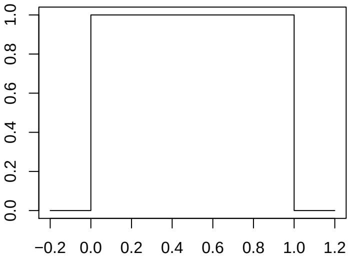
For $x \le 0$, clearly $F(x) = 0$, while for $x \ge 1$, $F(x) = 1$.



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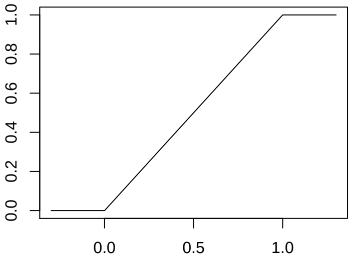

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The cdf $F(x)$ is therefore, for $x \geq 0$,

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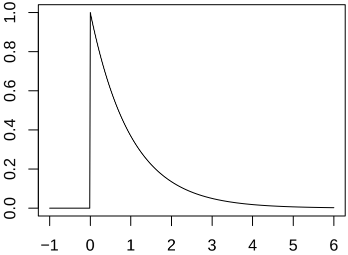

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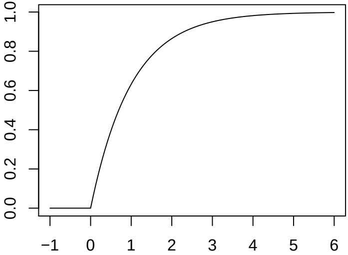

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In general, if X is an exponential random variable with rate λ,

$$P(X \geq t) = e^{-\lambda t}$$

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This is the same as the probability of a new bulb lasting 50 days, as we calculated on the previous slide.

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Proof:
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Exponential Waiting Times

Consider a Poisson process with rate λ.

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This is the cdf of an exponential random variable of rate λ. Therefore, in a Poisson process, the waiting time for the first event is an exponential random variable with rate λ.

Variance and Standard Deviation

The **variance** of a continuous random variable X with pdf $f(x)$ is

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So $V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$. In other words, the standard deviation is $\sigma = 1/\lambda = \mu$.

Properties of Expected Value and Variance

The same properties of expected value and variance which we used for discrete random variables also work for continuous random variables:

1. $E(c) = c$
2. $E(cX) = cE(X)$
3. $E(X + Y) = E(X) + E(Y)$

1. $V(c) = 0$
2. $V(cX) = c^2 V(X)$
3. $V(X + c) = V(X)$
4. If X and Y are independent, $V(X + Y) = V(X) + V(Y)$.

Shifting and Scaling a Uniform Distribution

Starting with a uniform random variable X on $[a, b]$, if we add or multiply by a constant c, then we obtain a new uniform random variable:

Namely, $X + c$ is a uniform random variable on $[a + c, b + c]$, while cX is a uniform random variable on $[ca, cb]$.

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Starting from a standard uniform random variable U on $[0, 1]$, a uniform random variable X on $[a, b]$ may be obtained by scaling and shifting:

$$X = (b - a)U + a$$

- First notice that $(b - a)U$ is uniform on $[0, b - a]$.
- Therefore $(b - a)U + a$ is uniform on $[0 + a, (b - a) + a] = [a, b]$.

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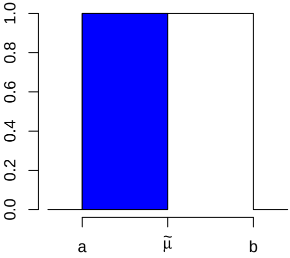

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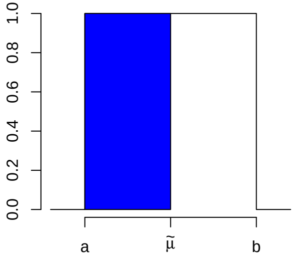

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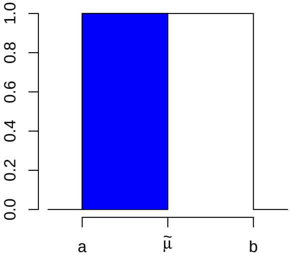

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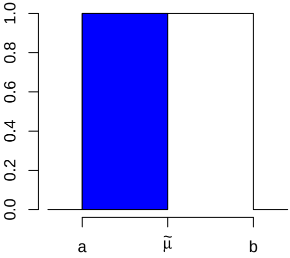

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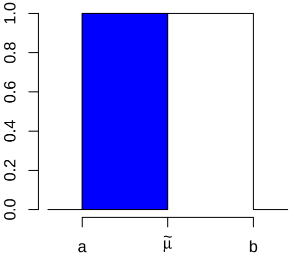

In this case, the median $\tilde{\mu}$ is the same as the mean μ.

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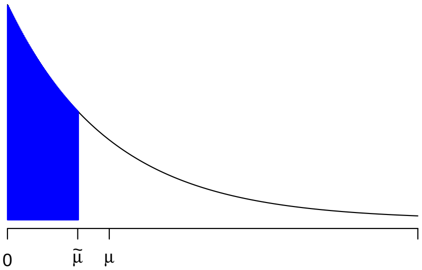

So the median is $\tilde{\mu} = \mu \ln 2 \approx .693\mu$.

Percentiles of a Continuous Random Variable

Given a continuous random variable X and $0 \leq p \leq 1$, the $100p$th **percentile** of X is the value x such that $P(X \leq x) = p$.

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So the 90th percentile of X is 37.

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Differentiating the cdf, we find the pdf of X:

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Gamma Distribution

Given a Poisson process with rate λ, the waiting time for k events has a **gamma distribution**, with pdf

$$f(x) = \begin{cases} \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

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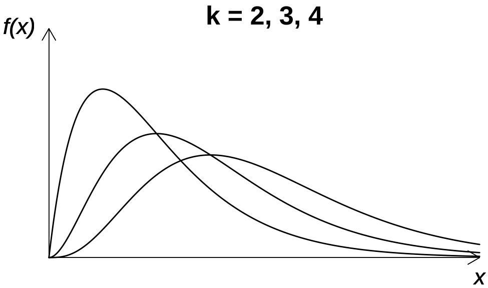

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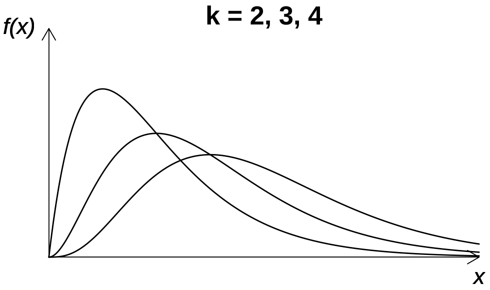
When $k = 1$ this is an exponential distribution: $f(x) = \lambda e^{-\lambda x}$



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In other words, a gamma random variable with parameters k and λ may be expressed as a sum of k independent exponential random variables with rate λ.

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V(X) &= V(Y_1 + \cdots + Y_k) \\
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&= 1/\lambda^2 + \cdots + 1/\lambda^2 \\
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Example

Cars pass a certain point on a road according to a Poisson process with rate $\lambda = 20$ per hour. If we wait until 100 cars have passed, what are the mean and standard deviation of the amount of time we will have to wait?

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Solution: Let X be the amount of time until 100 cars have passed. X is a gamma random variable with parameters $k = 100$ and $\lambda = 20$. We find the mean and standard deviation of X (in hours) using the formulas on the previous slide:

$$\mu = E(X) = k/\lambda = 5$$

$$\sigma = \sqrt{V(X)} = \sqrt{k/\lambda^2} = \sqrt{1/4} = 1/2$$

Bernoulli Process vs. Poisson Process

A sequence of independent Bernoulli random variables $Y_1, Y_2, \ldots$ each with parameter p is called a **Bernoulli process** with rate p. We interpret this as a process where events occur only at discrete times, 1, 2, 3, . . . , as opposed to a Poisson process where the time of occurence of an event may be any positive real number.

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	Bernoulli process	Poisson process
# of events in a unit time period	Bernoulli(p)	Poisson(λ)
# of events in a period of length n	Binomial(n, p)	Poisson($n\lambda$)
Waiting time for first event	Geometric(p)	Exponential(λ)
Waiting time for r events	Negative Binomial (r, p)	Gamma(r, λ)

Standard Gamma Distribution

Recall: the pdf of a gamma random variable with parameters k and λ is

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$$f(x) = \begin{cases} \frac{1}{(k-1)!} x^{k-1} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Since $f(x)$ is a valid pdf, $\int_0^\infty \frac{1}{(k-1)!} x^{k-1} e^{-x} \, dx = 1$. In other words,

$$\int_0^\infty x^{k-1} e^{-x} \, dx = (k-1)!$$

We showed that for any integer $k \geq 1$,

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For integers $k \geq 1$, then, $\Gamma(k) = (k-1)!$.

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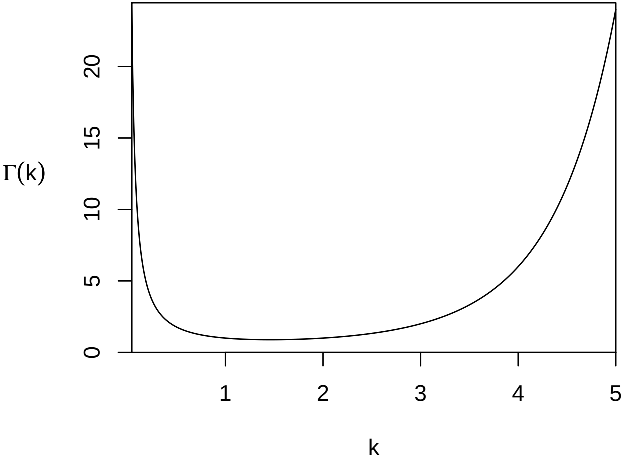

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Using the gamma function, we may extend the definition of gamma random variables to include cases where k may not be an integer:

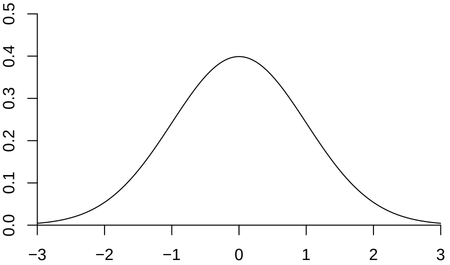
A **gamma** random variable X with parameters $k, \lambda > 0$ is given by the pdf

$$f(x) = \begin{cases} \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Standard Normal Distribution

We say a random variable X has the **standard normal distribution** if it has pdf

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

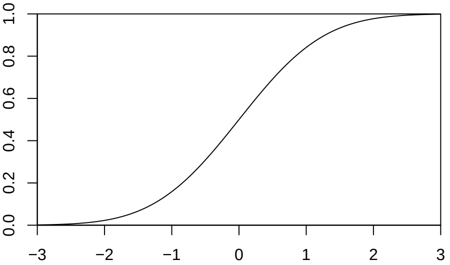


CDF of Standard Normal Distribution

The cdf of the standard normal distribution is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

There is no simple formula for evaluating this integral. However, it can easily be evaluated numerically by a computer.



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Mean of Standard Normal

The standard normal distribution is symmetric in the sense that the pdf $\phi(x)$ is an even function, i.e., $\phi(-x) = \phi(x)$:

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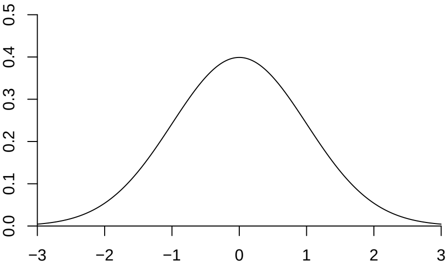
Therefore the mean of a standard normal random variable is

$$E(X) = \int_{-\infty}^{\infty} x\phi(x)\ dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx = 0$$

since the integrand is an odd function.



Variance of Standard Normal

Substituting $u = x^2/2$, note that

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A standard normal random variable has mean 0 and variance 1.

Normal Distributions

If Z is a standard normal random variable, then given numbers μ and $\sigma > 0$, we can define a new scaled, shifted random variable $X = \sigma Z + \mu$:

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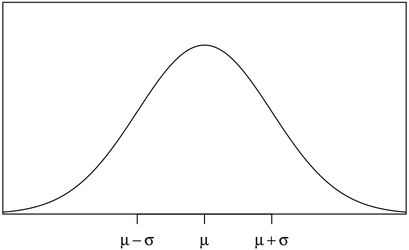
We call X a **normal** random variable with mean μ and standard deviation σ, and we write $X \sim N(\mu, \sigma^2)$.



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The cdf of a normal random variable $X \sim N(\mu, \sigma^2)$ is

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$$\begin{aligned} F(x) &= P(X \leq x) = P(\sigma Z + \mu \leq x) \\ &= P\left(Z \leq \frac{x-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \end{aligned}$$

By taking the derivative and applying the chain rule, we find the pdf of X:

$$\begin{aligned} f(x) = F'(x) &= \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

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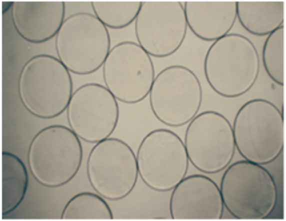
Example

A process produces alginate beads with diameters normally distributed with mean 1 mm and standard deviation .05 mm. Given a random bead, what is the probability that its diameter is between .95 mm and 1.1 mm?



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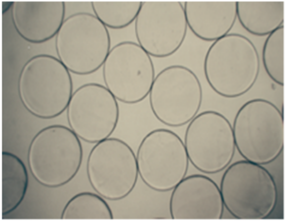

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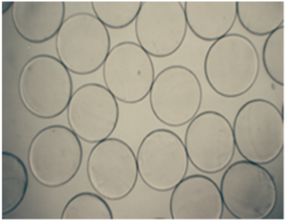

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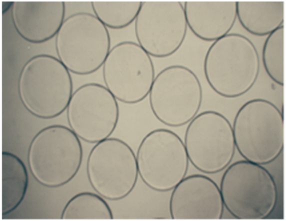

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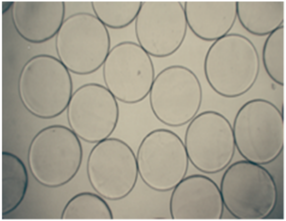

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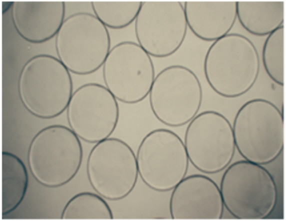
A process produces alginate beads with diameters normally distributed with mean 1 mm and standard deviation .05 mm. Find the 90th percentile of bead diameters.



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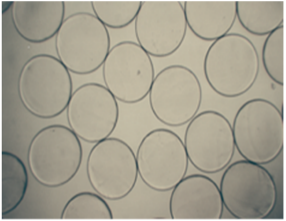

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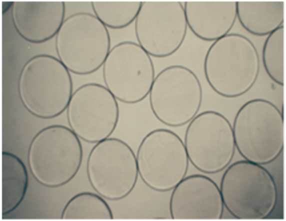

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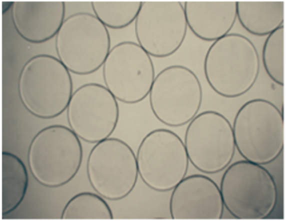

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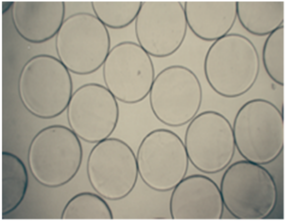

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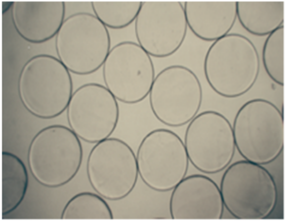

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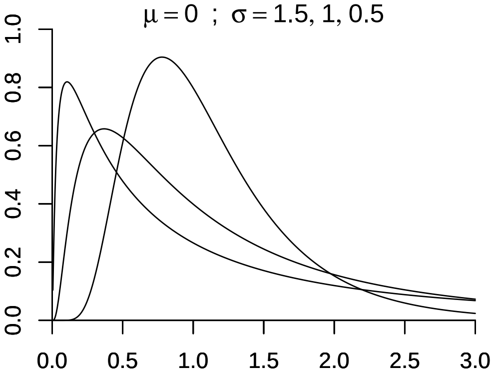
Solving for x,

$$x = .05\Phi^{-1}(.9) + 1 \approx (.05)(1.28) + 1 \approx 1.064 \text{ mm}$$

Log-normal Distribution

A random variable X is said to have a **log-normal** distribution if $\ln(X)$ is a normal random variable.

A log-normal random variable X may be written in the form $X = e^{\sigma Z+\mu}$, where Z is a standard normal random variable.



Example

A certain aerosol spray contains particles whose diameters (in microns) have a log-normal distribution with $\mu = 5$, $\sigma = 0.5$. What proportion of the particles are larger than 300 microns?

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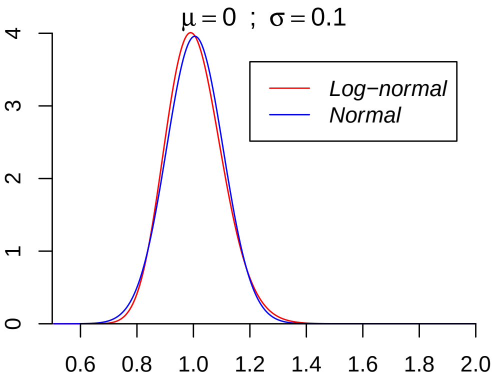
&\approx P(Z > 1.41) \\
&\approx .0793
\end{aligned}
$$

Mean and Variance of Log-normal

If X is log-normal with parameters μ and σ, then

$$E(X) = e^{\mu+\sigma^2/2}$$
$$V(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

If σ is small, then X is approximately normal:



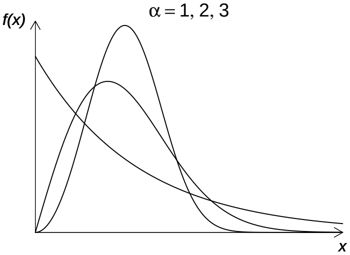
Weibull Distribution

A **Weibull** random variable X with shape $\alpha > 0$ and scale $\beta > 0$ has cdf

$$F(x) = \begin{cases} 1 - e^{-(x/\beta)^\alpha}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Its pdf is

$$f(x) = \begin{cases} \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



Example

The amount X of NO_x emission (g/gal) from a randomly selected engine of a certain type may be modeled as a Weibull random variable with $\alpha = 2$ and $\beta = 10$. What is the probability that a randomly selected engine has $X \geq 20$?

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Therefore,

$$\begin{aligned} P(X \geq 20) &= 1 - P(X \leq 20) = 1 - F(20) \\ &= e^{-(20/10)^2} = e^{-4} \approx .0183 \end{aligned}$$

Summary

Distribution	PDF	Mean	Variance
Uniform $a \leq x \leq b$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential $x \geq 0$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Gamma $x \geq 0$	$\frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}$	k/λ	k/λ^2
Normal $-\infty < x < \infty$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{x-\mu}{\sigma})^2/2}$	μ	σ^2
Weibull $x \geq 0$	$\frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}$	$\beta\Gamma(\frac{\alpha+1}{\alpha})$	$\beta^2\Gamma(\frac{\alpha+2}{\alpha}) - \mu^2$
Log-normal $x \geq 0$	$\frac{1}{x\sigma\sqrt{2\pi}} e^{-(\frac{\ln x-\mu}{\sigma})^2/2}$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$