Math 3070, Applied Statistics

Section 1

August 26, 2019

Lecture Outline, 8/26

Section 2.3

- Product Rule for k-tuples
- Permutations and Combinations
- Examples

The probability of equally likely outcomes is known as soon as the number of outcomes is.

Goal: Develop tools to count outcomes, surprisingly hard.

- All of these variables are discrete. Continuous variables are, by nature, uncountable.
- When dealing with equally likely events,

$$P(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in sample space}}$$

Product Rule for k-tuples, Method

Suppose that you have k random objects. Object 1, O_1, has n_1 outcomes. Object 2, O_2, has n_2 outcomes. Generally, object i, O_i, has n_i outcomes.
Idea: O_1 can take n_1 values. O_2 can take n_2 values. So, the 2-tuple can take $n_1 n_2$ values. The argument can be repeated for any k by induction.
The random k-**tuple**

$$(O_1, O_2, \ldots, O_k)$$

has

$$\prod_{i=1}^{k} n_i = n_1 n_2 \ldots n_k \text{ outcomes.}$$

Product Rule for k-tuples, Coin Flip Example

Count the number outcomes from 8 coin flips. What is the probability that the first coin is heads? What is the probability that all coins are the same?

Each random object is a distinct coin flip has two outcomes, $\{H, T\}$. The flips are organized as an 8-tuple, $(O_1, O_2, \ldots, O_7, O_8)$. Keep in mind that each is O_i represents a random event with outcomes.

$$\text{number of outcomes in sample space} = 2^8 = 256$$

Product Rule for k-tuples, Coin Flip Example

Count the number outcomes from 8 coin flips. What is the probability that the first coin is heads? What is the probability that all coins are the same?

One can see that there's a 0.5 probability for the first coin landing on heads and there is no impact from the rest of the coins. To compute this using counting methods, we need to count the outcomes in event $A =$ "first coin lands on heads".

$$A = \{(H, O_2, O_3, \ldots, O_7, O_8)\}$$

or the set of all 8-tuple were the first is H. In A, O_1 has only outcome.

$$\text{number of outcomes in A} = 1 \cdot 2^7 = 128$$

$$P(A) = \frac{128}{256} = 0.5$$

Product Rule for k-tuples, Coin Flip Example

Count the number outcomes from 8 coin flips. What is the probability that the first coin is heads? What is the probability that all coins are the same?

There are only two outcomes in event $C =$"all coins land on the same side",

$$(H, H, \ldots, H, H) \text{ and } (T, T, \ldots, T, T).$$

$$P(C) = \frac{2}{256} = \frac{1}{128}$$

Moreover,

$$P(\text{"At least two coins land on a different side"}) = 1 - P(C) = \frac{127}{128}$$

Questions?

Permutations and Combinations, Definition

Given group of n **distinct** objects, distinguishable by some trait,

Permutations are ordered subsets, and

Combinations are unordered subset.
For example, when the set is $\{A, B, C, D\}$ the subsets

$$(C, B, A) \text{ and } (A, B, C)$$

are different permutations, but the same combination.

Personally, I like to write to use curly brackets, $\{\}$ for combinations since the ordering is irrelevant and parentheses, (), for permutations. For example, I would have written the last combination as $\{A, B, C\}$

We will explore how to determine if objects should be modeled as k-tuples, combinations or permutations.

Permutations, Explanation

Suppose that you have n distinct objects and your ordered subset has a size of $k \leq n$. The 1^{st} ordered element can take any of the n outcomes. The 2^{nd} has one less outcome after the first is observed so it can be one of $n-1$ outcomes. This continues until the k^{th} element can take one of $n-(k-1)$ outcomes. Multiply the number of outcomes for each order element to determine the number of outcomes,

$$P_{k,n} = \text{number of permutations of } n \text{ distinct objects of length } k$$

$$P_{k,n} = n \cdot (n-1) \cdots (n-(k-1)) = \frac{n \cdot (n-1) \cdots (n-(k-1)) \cdot (n-k)!}{(n-k)!}$$

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Recall,

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \text{ and } 0! = 1$$

Permutations, Example

What is the probability that a random string of 5 unique letters spells out "gnome"?

Each letter can only appear once, order matters for the strings and each string is equally likely.

There are 26 letters and 5 are selected. The permutation formula can be used.

$$P_{5,26} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7893600$$

"gnome" or (g,n,o,m,e) is one possible outcome.

$$P(\text{"gnome"}) = \frac{1}{P_{5,26}} = \frac{1}{7893600}$$

Permutations, Comments and Questions

- Used when ordering matters and each object appears once.
- permutations of k out of n objects $= P_{k,n} = \frac{n!}{(n-k)!}$

Questions?

Combinations, Explanation

Consider the last example. What if the ordering of (g,n,o,m,e) did not matter? In this case, (g,e,o,m,n), (g,o,m,m,e) and all other permutations of those five letters would considered to be the same combination. To find the number of combinations, I can divide the number of permutations 5 letters out of 26, by permutations of 5 letter out of 5.

$$\text{number of permutations of } \{\text{g,n,o,m,e}\} = P_{5,5} = \frac{5!}{0!} = 5!$$

This argument can be applied any collection of 5 unique letters, each unique combination produces 5! permutations of the original 26 letters. Use dimensional analysis.

$$P_{5,26} \text{ Permutations} \frac{\text{Combination}}{P_{5,5} \text{ Permutations}} = \frac{P_{5,26}}{5!} \text{ Combinations}$$

Combinations, Explanation

The argument can be repeated when determining the number of combinations of k out of n objects.

$$\text{combinations of } k \text{ out of } n \text{ objects} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

This operation is called "n choose k" and denoted as $\binom{n}{k}$.

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Combinations, Example

A single person is dealt 5 cards from a standard 52 card deck. What is the probability that there are 4 aces?

The order of of cards is irrelevant and each card may only appear once. Each hand is equally likely to appear. This implies the number of possible hands is 5 choose 52.

$$\binom{52}{5} = \frac{52!}{5!47!} = 2598960$$

The event $A =$ "4 aces dealt" can be written as

$$\{AH, AC, AD, AS, C\},$$

AH is the ace of hearts, AC is the ace of clubs, etc. and C is the unobserved card. C can take $52 - 4 = 48$ values. There are 48 outcomes in A.

$$P(A) = \frac{48}{2598960} = 0.00001846892$$

Summary and Questions

k-tuples

- Used when there are k random objects and object i has n_i outcomes for $i = 1, \dots, k$.
- number of possible k-tuples $= \prod_{i=1}^{k} n_i$.

Permutations

- Used when k distinct objects are randomly selected from a set of n objects, order matters and without duplication.
- $P_{k,n} = \frac{n!}{(n-k)!}$

Combinations

- Used when k distinct objects are randomly selected from a set of n objects, order does not matters and without duplication.
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Example, 2-tuple of Combinations

You run a car dealership and own 26 cars, 16 new and 10 used. There are 8 parking spots in each of your 2 display rows. You want to fill the front row with new cars and the back row with used cars. How ways can the cars be arranged?

Two combinations, one for each row, are being selected. Each number of combinations needs to be found and multiplied, as expressed in the k-tuple formula with $k = 2$.
The first row can have 16 choose 8 combinations and the second row can have 10 choose 8 combinations.

$$\binom{16}{8}\binom{10}{8} = \frac{16!}{8!(16-8)!}\frac{10!}{8!(10-8)!} = \frac{16!10!}{(8!)^3 2!}$$

Example, Permutation in a Combination

Five people get on an elevator that stops at five floors. Assuming that each has an equal probability of going to any one floor, find the probability that they all get off at different floors.

Each person can get off on any floor 1–5, 5 outcomes and 5 random objects.

$$\text{number of ways for 5 people to exit } = 5^5 = 3125$$

If each of the 5 people gets off on a different floor out the 5 floors, then the order matters (person 1 goes to floor blah, person 2 ...) and there is no duplication.

$$P(\text{"exit on different floors"}) = \frac{P_{5,5}}{5^5} = \frac{5!}{5^5}$$

Example, Variable Number of Variables

A computing center has 3 processors that receive n jobs, with the jobs assigned to the processors purely at random. What is the probability that one of them handles all of the jobs?

Each job is a random object that can take one of three values: processor 1, processor 2 and processor 3. In total, there are

$$3^n \text{ job configurations}.$$

Event $A =$ "one processor handle all jobs".

$$A = \{(1, 1, \ldots, 1, 1), (2, 2, \ldots, 2, 2), (3, 3, \ldots, 3, 3)\}$$

$$P(A) = \frac{3}{3^n} = 3^{1-n}$$

Example, Drawing from Multiple Sets

You have a bag of 100 marbles, all numbered. Half are red and 30 are blue and 20 are green. You draw 10 marbles. What is the probability that they are all the same? What is the probability that at least two are different.

Order does not matter, all draws are equally likely and there are no duplicate marbles.

The total number of marble draws is 100 choose 5, $\binom{100}{10}$.

If only red marbles are drawn, then it is a combination of 5 marbles out of 50, $\binom{50}{10}$. Similiarly, $\binom{30}{10}$ for blue marbles and $\binom{20}{10}$ for green marbles.

Each event is disjoint and comprise the event $A =$ "all marbles are the same color",

$$\#\text{ of outcomes in } A = \binom{50}{10} + \binom{30}{10} + \binom{20}{10} = 10302507941$$

Example, Drawing from Multiple Sets

You have a bag of 100 marbles all numbered. Half are red and 30 are blue and 20 are green. You draw 10 marbles. What is the probability that they are all the same? What is the probability that at least two are different.

$$P(A) = \frac{\binom{50}{10} + \binom{30}{10} + \binom{20}{10}}{\binom{100}{10}} = 0.000595166018$$

$$P(\text{"at least two are different"}) = 1 - P(A) = 1 - 0.000595166018$$

Note: if the marbles were not distingushable (unnumbered in our case), this would be a different problem and require a different counting method.