Math 3070, Applied Statistics

Section 1

November 18, 2019

Lecture Outline, 11/18

Section 9.1 and 9.2

- Z-Test for the difference between two means.
- T-Test for the difference between two means.

Two-sample z Test

Suppose we are given two independent normal random samples:

- $X_1, \dots, X_m$
- $Y_1, \dots, Y_n$

If we know the variances σ_1^2 and σ_2^2, we may use a *two-sample z test* to test the null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$:

Test Statistic	Alternative hypothesis	Rejection region
$Z = \dfrac{\overline{X} - \overline{Y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$	$H_a : \mu_1 - \mu_2 > \Delta_0$	$Z > z_\alpha$
	$H_a : \mu_1 - \mu_2 < \Delta_0$	$Z < -z_\alpha$
	$H_a : \mu_1 - \mu_2 \neq \Delta_0$	$\lvert Z \rvert > z_{\alpha/2}$

If m and n are large (say, $m > 40$ and $n > 40$), then we may use sample variances S_1^2 and S_2^2 in place of σ_1^2 and σ_2^2 and may drop the assumption that the distributions are normal.

Example

A random sample of 20 specimens of cold-rolled steel had an average yield strength of 29.8 ksi. For a random sample of 25 two-sided galvanized steel specimens the average was 34.7 ksi. Assuming that the two yield-strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$, does the data provide significance evidence (at the $\alpha = .01$ level) for a difference between the mean yield strength of the two types of specimens?

We want to test the null hypothesis $H_0 : \mu_1 - \mu_2 = 0$ against the alternative $H_a : \mu_1 - \mu_2 \neq 0$. We calculate the test statistic:

$$Z = \frac{\overline{X} - \overline{Y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} = \frac{29.8 - 34.7 - 0}{\sqrt{\frac{(4.0)^2}{20} + \frac{(5.0)^2}{25}}} = -3.65$$

The P-value for the test is $P(|Z| > 3.65) = 2\Phi(-3.65) = .00026$. This provides strong evidence for a difference in the mean yield strengths of the two types of specimens.

z Confidence Interval for Difference of Two Means

Suppose we are given two independent normal random samples:

- $X_1, \ldots, X_m$ from a $N(\mu_1, \sigma_1^2)$ distribution
- $Y_1, \ldots, Y_n$ from a $N(\mu_2, \sigma_2^2)$ distribution

Assume we know the variances σ_1^2 and σ_2^2.

A $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$\overline{X} - \overline{Y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$$

If m and n are large (say, $m > 40$ and $n > 40$), then we may use sample variances S_1^2 and S_2^2 in place of σ_1^2 and σ_2^2 and may drop the assumption that the distributions are normal.

Example

A random sample of 20 specimens of cold-rolled steel had an average yield strength of 29.8 ksi. For a random sample of 25 two-sided galvanized steel specimens the average was 34.7 ksi. Assuming that the two yield-strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$, find a 95% confidence interval for the difference in mean yield strength between the two types of specimens?

$$
\begin{aligned}
&\overline{X} - \overline{Y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}} \\
&= 29.8 - 34.7 \pm 1.96\sqrt{\frac{(4.0)^2}{20} + \frac{(5.0)^2}{25}} \\
&= -4.9 \pm 2.63
\end{aligned}
$$

Two-Sample t Test (Welch's t Test)

Suppose we are given two independent normal random samples:

- $X_1, \ldots, X_m$
- $Y_1, \ldots, Y_n$

If we don't know the variances σ_1^2 and σ_2^2, we may use a *two-sample t test* to test the null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$:

Test Statistic	Alternative hypothesis	Rejection region
$T = \dfrac{\overline{X} - \overline{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$	$H_a : \mu_1 - \mu_2 > \Delta_0$	$T > t_{\alpha,\nu}$
	$H_a : \mu_1 - \mu_2 < \Delta_0$	$T < -t_{\alpha,\nu}$
	$H_a : \mu_1 - \mu_2 \neq \Delta_0$	$\lvert T \rvert > t_{\alpha/2,\nu}$

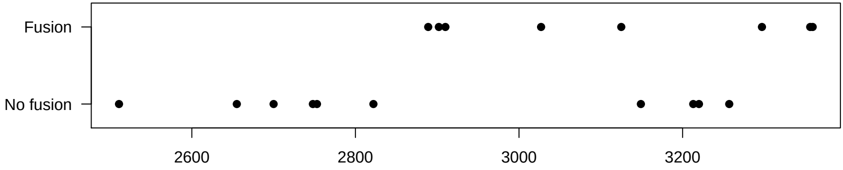
Here the degrees of freedom ν is estimated by

$$\nu = \frac{\left(\dfrac{S_1^2}{m} + \dfrac{S_2^2}{n}\right)^2}{\dfrac{(S_1^2/m)^2}{m-1} + \dfrac{(S_2^2/n)^2}{n-1}}$$

Example

The deterioration of many municipal pipeline networks across the country is a growing concern. One technology proposed for pipeline rehabilitation uses a flexible liner threaded through existing pipe. An article reported the following data on tensile strength (psi) of liner specimens both when a certain fusion process was used and when this process was not used:

No fusion	2748	3149	2700	2655	2822	2511	3257	3213	3220	2753
Fusion	3027	3356	3359	3297	3125	2910	2889	2902		



Example

Does the data provide significant evidence for a difference in the mean tensile strength of the two types of specimens?

We will test the null hypothesis $H_0 : \mu_1 - \mu_2 = 0$ against the alternative $H_a : \mu_1 - \mu_2 \neq 0$.
The specimens with no fusion have $\overline{X} = 2902.8$ and $S_1 = 277.3$, while those with fusion have $\overline{Y} = 3108.1$ and $S_2 = 205.9$.

$$T = \frac{\overline{X} - \overline{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}} = \frac{2902.8 - 3108.1 - 0}{\sqrt{\frac{(277.3)^2}{10} + \frac{(205.9)^2}{8}}} \approx -1.8$$

$$\nu = \frac{\left(\frac{S_1^2}{m} + \frac{S_2^2}{n}\right)^2}{\frac{(S_1^2/m)^2}{m-1} + \frac{(S_2^2/n)^2}{n-1}} = \frac{(7689.5 + 5299.3)^2}{\frac{(7689.5)^2}{9} + \frac{(5299.3)^2}{7}} = 15.9 \approx 16$$

The P-value for the test is

$$P(|T| > 1.8) = 2P(T > 1.8) = 2 \cdot .045 = .090$$

t Confidence Interval for Difference of Two Means

Suppose we are given two independent normal random samples:

- $X_1, \ldots, X_m$ from a $N(\mu_1, \sigma_1^2)$ distribution
- $Y_1, \ldots, Y_n$ from a $N(\mu_2, \sigma_2^2)$ distribution

Assume we *do not* know the variances σ_1^2 and σ_2^2.

A $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$\overline{X} - \overline{Y} \pm t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}$$

Here, as before the degrees of freedom ν is estimated by

$$\nu = \frac{\left(\dfrac{S_1^2}{m} + \dfrac{S_2^2}{n}\right)^2}{\dfrac{(S_1^2/m)^2}{m-1} + \dfrac{(S_2^2/n)^2}{n-1}}$$

Example

Based on the pipeline liner data, find a 95% confidence interval for the difference in mean tensile strength between the two types of specimens (no fusion vs. fusion).

The specimens with no fusion had $\overline{X} = 2902.8$ and $S_1 = 277.3$, while those with fusion had $\overline{Y} = 3108.1$ and $S_2 = 205.9$. We calculated that the appropriate degrees of freedom was $\nu \approx 16$. This leads to a critical value of $t_{.025,16} = 2.120$. A 95% confidence interval for the difference $\mu_1 - \mu_2$ is then given by

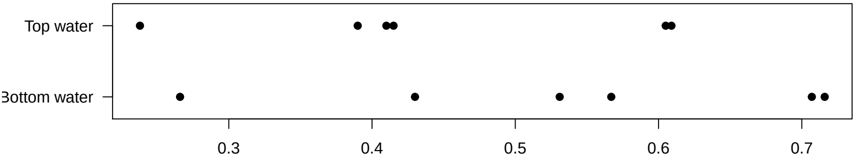
$$\begin{aligned}
&\overline{X} - \overline{Y} \pm t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}} \\
&= 2902.8 - 3108.1 \pm 2.120\sqrt{\frac{(277.3)^2}{10} + \frac{(205.9)^2}{8}} \\
&= -205.3 \pm 241.6
\end{aligned}$$

Problem

An article [1] reports on a study in which six river locations were selected and the zinc concentration (mg/L) determined for both surface water and bottom water at each location:

Location	1	2	3	4	5	6
Bottom water	.430	.266	.567	.531	.707	.716
Surface water	.415	.238	.390	.410	.605	.609

Does the data provide significant evidence the mean zinc concentration in bottom water exceeds that of surface water?



[1] "Trace Metals of South Indian River" (Envir. Studies, 1982: 62–66)

Problem

We want to test the null hypothesis $H_0 : \mu_1 - \mu_2 = 0$ against an alternative $H_0 : \mu_1 - \mu_2 > 0$ based on the data:

Location	1	2	3	4	5	6
Bottom water (X_i)	.430	.266	.567	.531	.707	.716
Surface water (Y_i)	.415	.238	.390	.410	.605	.609

It may seem natural to treat this as a two-sample problem: We could calculate $\overline{X} = .536$, $S_1 = .171$, $\overline{Y} = .444$, and $S_2 = .142$:

$$T = \frac{\overline{X} - \overline{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}} = \frac{.536 - .444 - 0}{\sqrt{\frac{(.171)^2}{6} + \frac{(.142)^2}{6}}} \approx 1.0$$

$$\nu = \left(\frac{S_1^2}{m} + \frac{S_2^2}{n}\right)^2 \div \left(\frac{(S_1^2/m)^2}{m-1} + \frac{(S_2^2/n)^2}{n-1}\right) = 9.7 \approx 10$$

$$P = P(T > 1.0) = .170$$

However, this method would be *incorrect* because the two samples are not independent of each other!