Math 3070, Applied Statistics

Section 1

September 4, 2019

Lecture Outline, 9/4

Section 3.1

- Random Variables
- Probability Mass Functions
- Cumulative Distribution Function
- Examples

Random Variables

defintion

A **random variable** is any rule that associates a number with each outcome in of a sample space, $\mathcal{S}$.

notation

Most of the time random variables will be denoted with as capital case letters, X, Y, Z, U, V.

When statement such as $P(X = 1)$ are writen this means the probability that the event $X = 1$ occurs. To find the probability, one could find the probability of the event which yields $X = 1$.

foreshadowing

Later we will discuss random variables without explictly referring to an associated sample space, $\mathcal{S}$. The sample space is the outcomes of the random variable.

Random Variables, Discrete Versus Continuous

discrete random variables

Random variables are **discrete** if they can take finitely many or countably many values. **Countably many** means the values can be listed one by one. For example, the integers and whole numbers are countable.

continuous random variables

A random variables is **continuous** when

1. its possible values contain on any interval
2. and the probability that it take on exactly one value is one, $P(X = c) = 0$ for all c.

Random variables which take values on anywhere on the number line, an interval or half line, and satisfy property 2 are continuous random variables.

Discrete Random Variable Example, Die Roll

Example: Suppose we roll one fair six-sided die. The outcome of the roll is event that it lands on $1, 2, 3, 4, 5$ or 6. Naturally, the number associated with the outcome is a random variable.

Discrete Random Variable Example, Coin Flips

Example: Suppose we toss a fair coin 3 times. The set of outcomes is

$$\Omega = \{(TTT), (TTH), (THT), (THH), (HTT), (HTH), (HHT), (HHH)\}$$

Let X be the number of heads. Then X is a random variable:

$$\begin{aligned} TTT &: X = 0 & \qquad HTT &: X = 1 \\ TTH &: X = 1 & \qquad HTH &: X = 2 \\ THT &: X = 1 & \qquad HHT &: X = 2 \\ THH &: X = 2 & \qquad HHH &: X = 3 \end{aligned}$$

The possible values of X are 0, 1, 2, and 3.
X is a discrete random variable.

Random Variables, Summary

- Random variables are function that associate numbers to outcomes.
- Know the difference between discrete and continuous random variables. They will require different calculations.
- Mathematical expressions such as $P(X = 5)$ and $P(X < 3)$ read as the probability that the random variable X takes the value 5 or that probability that the random variable X takes a value less than 3. These are still probabilities of <u>events</u>. This is why the previous rules of probability still apply.

Questions?

Probability Mass Function, Definition

defintion

The **probability mass function** (PMF) of a discrete random variable is $f(x) = P(X = x)$.

$f(x)$ is a function.

x is a varible, meaning that one may set it equal to numbers.

$P(X = x)$ is the probability that the event $X = x$ occurs.

PMFs only pertain to discrete random variables.

Probability Mass Function, Coin Flip Example

Consider the flip of a fair coin. The random variable X takes 1 if the coin lands on heads and 0 if the coin lands on tails. Find the probability mass function of X.

$$f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

definition

A **Bernoulli** random variable is one which only takes 0 or 1 as values. Note, the probabilities do not need to be 1/2.

Probability Mass Function, Die Roll Example

Consider X the value yielded from a single roll of a fair six-sided die. Determine the probabilty mass function.
There is a $1/6$ probabilty of observing any side.

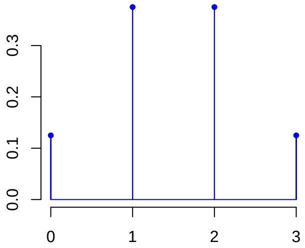
$$f(x) = \begin{cases} \frac{1}{6}, & x = 1, 2, 3, 4, 5, 6 \\ 0, & \text{otherwise} \end{cases}$$

In the previous example, we can calculate the probability of the number of heads out of 3 coins flips X being each of the values 0, 1, 2, and 3:

$$P(X=0) = P(\{TTT\}) = 1/8$$
$$P(X=1) = P(\{HTT, THT, TTH\}) = 3/8$$
$$P(X=2) = P(\{HHT, HTH, THH\}) = 3/8$$
$$P(X=3) = P(\{HHH\}) = 1/8$$

The PMF, $f(x) = P(X = x)$, describes the probability of each possible value:

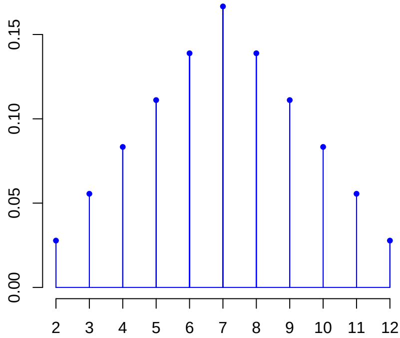
$$f(x) = \begin{cases} 1/8, & \text{for } x = 0 \\ 3/8, & \text{for } x = 1 \\ 3/8, & \text{for } x = 2 \\ 1/8, & \text{for } x = 3 \\ 0, & \text{otherwise} \end{cases}$$



Sum of Two Random Variables

Suppose we roll two six-sided dice, and let X and Y be the results. Their sum is a random variable $X + Y$ with values 2, 3, ..., 12. What is the probability mass function of $X + Y$?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12



x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Sum of Two Random Variables

Suppose we roll two six-sided dice, and let X and Y be the results. Their sum is a random variable $X+Y$ with values 2, 3, . . . , 12. What is the probability that $X+Y$ is even? What is the probability that $X+Y$ is less than 5.

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$P("X+Y \text{ is even"}) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{18}{36}$$

Sum of Two Random Variables

Suppose we roll two six-sided dice, and let X and Y be the results. Their sum is a random variable $X+Y$ with values 2, 3, . . . , 12. What is the probability that $X+Y$ is less than 6?

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$P(X+Y<6)=\frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{4}{36}=\frac{10}{36}$$

Note:

$$P(X+Y<6)=P(X+Y\leq 5)$$

Probability Mass Function, Summary and Comments

- The Probability Mass Function $f(x)$ is a function whose input are numbers x and whose output is the probability of its associate discrete random variable X taking that number.
- Only relevant for discrete random variables.
- A Bernoulli random variable is one which only takes 0 or 1 as values.
- Take care to distingush $<$ and $\leq$ in events.
- The sum of the PMF over all possible values of the random variable must be 1, $P(\mathcal{S}) = 1$.
- If the PMF is known, the behavior of the discrete random varible is known.

Cumulative Distribution Function, Definition

definition

The **cumulative distribution function** (CDF) $F(x)$ of a discrete random variable X with PMF $f(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y:y \leq x} p(y).$$

$F(x)$ is also the probability that X is at most x.

Compute $F(x)$ by computing the running total.

Cumulative Distribution Function, Coin Flips Example

Suppose flip 3 coins. Calculate the CDF of X the number of heads out of 3 coin flips.

$$f(x) = \begin{cases} 1/8, & \text{for } x = 0 \\ 3/8, & \text{for } x = 1 \\ 3/8, & \text{for } x = 2 \\ 1/8, & \text{for } x = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1/8, & \text{for } 0 \leq x < 1 \\ 4/8, & \text{for } 1 \leq x < 2 \\ 7/8, & \text{for } 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}$$

Cumulative Distribution Function, Summary

- $F(x) = P(X \leq x) = \sum_{y:y\leq x} p(y)$
- Probability that X is at least x, this includes x.
- For x strictly less than any observable value of X, the CDF is zero.
- For x strictly greater than any observable value of X, the CDF is one.
- The "jumps" include the left endpoints.

Examples, Random Digits

A password generator samples numerical passwords of length of at most 4, equally likely. Consider X is the length of the password. Compute the PDF and CDF of X.

Count the total number of outcomes. Each digit can be repeated and order matters. k-tuple formula applies. Add number of passwords at each length.

$$\text{total number of passwords} = 10 + 10^2 + 10^3 + 10^4 = 11110$$

$$f(x) = \begin{cases} 10/11110, & x = 1 \\ 100/11110, & x = 2 \\ 1000/11110, & x = 3 \\ 10000/11110, & x = 4 \\ 0, & \text{otherwise} \end{cases}$$

Examples, Random Digits

A password generator samples numerical passwords of length at most 4, equally likely. Consider X is the length of the password. Compute the PDF and CDF of X.

$$f(x) = \begin{cases} 10/11110, & x = 1 \\ 100/11110, & x = 2 \\ 1000/11110, & x = 3 \\ 10000/11110, & x = 4 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 1 \\ 10/11110, & 1 \le x < 2 \\ 110/11110, & 2 \le x < 3 \\ 1110/11110, & 3 \le x < 2 \\ 1, & 4 \le x \end{cases}$$

Examples, Nonexample

Is $f(x)$ a PMF?

$$f(x) = \begin{cases} \dfrac{x!}{3!}, & x = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

nope, $f(3) = 1$ so we expect the sum over observable values to be greater than 1.

$$\begin{aligned} \sum_{x=0}^{3} f(x) &= \frac{0!}{3!} + \frac{1!}{3!} + \frac{2!}{3!} + \frac{3!}{3!} \\ &= \frac{1}{6} + \frac{1}{6} + \frac{2}{6} + \frac{6}{6} \\ &= \frac{10}{6} = \frac{5}{3} \end{aligned}$$

Examples, Nonexample

Determine what value of k ensures that $f(x)$ is a PMF?

$$f(x) = \begin{cases} k\frac{x!}{3!}, & x = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

$$k = \frac{3}{5}$$

. We need the sum over all observable values to be 1.

$$\sum_{x=0}^{3} f(x) = k\frac{0!}{3!} + k\frac{1!}{3!} + k\frac{2!}{3!} + k\frac{3!}{3!} = k\frac{5}{3} = 1$$

Relation to Previous "Distributions"

The Probability Mass Function is also called the **probability distribution**. Recall that the distribution of data describes how frequently outcomes or value are observed and a frequentist model describes relative frequency. When we try to model experimental outcomes with a random variable, comparing the PMF to the histogram helps judge the validity of the the probabilistic model.

Moreover, modeling the connections between random variables extends probabilistic models into new settings. For example, you may not want the random variable itself, but a sum or product of them. We have already encountered a weight sum of random variables, the sample mean and variance.

More on that in the next lecture...