Ch. 7 – Confidence Intervals

Standard Normal Probability

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Solving for c gives

$$c = -\Phi^{-1}(.025) = 1.96$$

Problem

Suppose a machine drills holes whose diameters are normally distributed with mean $\mu = 3.0$ mm and standard deviation $\sigma = 0.4$ mm. If a random sample of 16 holes are measured, find an interval $[a, b]$ centered on 3.0 such that the sample mean $\overline{X}$ diameter will be in the interval $[a, b]$ with probability .95.

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Rearranging,

$$.95 = P(2.804 \leq \overline{X} \leq 3.196)$$

Problem – Other Way Around

Suppose a machine drills holes whose diameters are normally distributed with unknown mean μ and known standard deviation $\sigma = 0.4$ mm. Given a random sample of 16 holes, find an interval $[A, B]$ depending on the sample mean $\overline{X}$ such that μ is in the interval $[A, B]$ with probability .95.

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$$\begin{aligned} .95 &= P\left(-1.96 \leq \frac{\overline{X}-\mu}{0.1} \leq 1.96\right) \\ &= P(\overline{X} - .196 \leq \mu \leq \overline{X} + .196) \end{aligned}$$

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So we may take $[A, B] = [\overline{X} - .196, \overline{X} + .196]$. For example, if $\overline{X} = 3.05$, then $[A, B] = [2.854, 3.246]$.

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- In general, a $100(1-\alpha)\%$ **confidence interval** for θ is an interval $[A, B]$ such that $P(A \leq \theta \leq B) = 1 - \alpha$.
- We are usually interested in constructing **equal-tailed** confidence intervals, where $P(\theta < A) = P(\theta > B) = \alpha/2$.

Confidence Interval for Mean of Normal Distribution

If a random sample of size n is taken from a normal distribution with unknown mean μ and known standard deviation σ, then an equal-tailed $100(1-\alpha)\%$ confidence interval for μ is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is a **critical value** given by

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Proof: The sample mean $\overline{X}$ is normal with mean μ and standard deviation $\sigma/\sqrt{n}$, so $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is standard normal, hence

$$\begin{aligned} 1-\alpha &= P\left(-z_{\alpha/2} \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) \\ &= P\left(\overline{X} - \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}\right) \end{aligned}$$

Example

A process produces alginate beads with diameters (in mm) normally distributed with unknown mean μ and standard deviation $\sigma = .7$. A random sample of 9 beads have the following diameters:

$$3.9, 5.1, 5.2, 5.7, 5.8, 6.1, 6.2, 6.3, 6.5$$

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Here $\alpha = 1 - .99 = .01$, so the relevant critical value is

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The sample mean is $\overline{X} = 5.64$, so the confidence interval is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} = 5.64 \pm \frac{2.57 \cdot 0.7}{\sqrt{9}} = 5.64 \pm 0.60$$

Determining Necessary Sample Size

In the previous example, a random sample of 9 beads was used to estimate the mean diameter μ, with a margin of error of 0.60, at a 99% confidence level. What sample size would be required for a margin of error of no more than 0.10, at the 99% confidence level?

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Rounding up to an integer, a sample size of at least $n = 324$ would be required.

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- If $[4.9, 5.1]$ is a 95% confidence interval for the mean diameter μ, we can be 95% confident that μ is between 4.9 and 5.1.
- It does *not* mean that 95% of beads will have diameter between 4.9 and 5.1.

Large-sample Confidence Interval for Mean

If $X_1, \ldots, X_n$ are a random sample from a distribution with unknown mean μ and *unknown* variance σ^2, and if n is sufficiently large, then an approximate $100(1-\alpha)\%$ confidence interval for μ is given by

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- Rule of thumb: The large-sample confidence interval for μ may be used if $n > 40$.

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$$\overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}} = 842 \pm \frac{1.96 \cdot 799}{\sqrt{600}}$$

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&= [778.1, 905.9]
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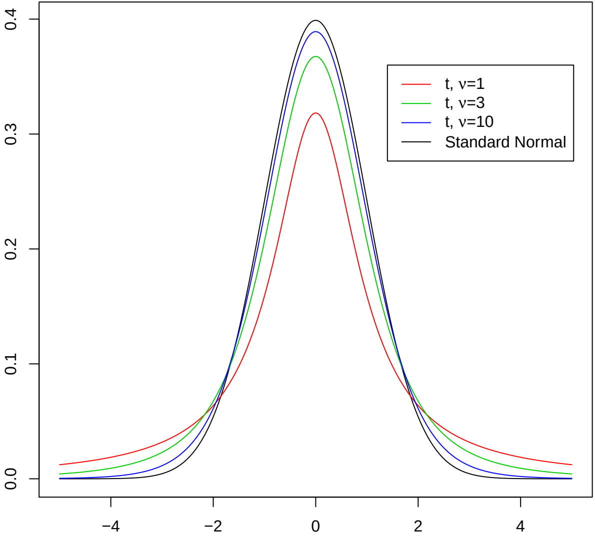
Given a random sample $X_1, \ldots, X_n$ from a normal distribution with unknown mean and variance, A $100(1-\alpha)\%$ confidence interval for the mean μ is

$$\overline{X} \pm \frac{t_{\alpha/2,\nu} \cdot S}{\sqrt{n}}$$

where $t_{\alpha/2,\nu}$ is a critical value from a **t distribution** with $\nu = n-1$ degrees of freedom.

PDF of t Distribution vs. Standard Normal



As $\nu \to \infty$, the t distribution approaches a standard normal.

Example

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$$t_{\alpha/2,\nu} = t_{.005,8} = 3.355$$

The sample mean is $\overline{X} = 5.64$ and the sample standard deviation is $S = .809$, so the confidence interval is given by

$$\overline{X} \pm \frac{t_{\alpha/2} \cdot S}{\sqrt{n}} = 5.64 \pm \frac{3.355 \cdot 0.809}{\sqrt{9}} = 5.64 \pm 0.90$$

Prediction Interval for a Normal Population

Given a random sample $X_1, \ldots, X_n$ from a normal distribution, suppose we want to construct an interval $[A, B]$ which we can be 95% confident will contain a future observation X_{n+1}. Such an interval is called a **prediction interval**.

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The statistic $T = \frac{\overline{X} - X_{n+1}}{S\sqrt{1+\frac{1}{n}}}$ has a t distribution with $\nu = n - 1$ degrees of freedom.

Given a random sample $X_1, \ldots, X_n$ from a normal distribution, a $100(1-\alpha)\%$ prediction interval for an independent observation X_{n+1} is

$$\overline{X} \pm t_{\alpha/2, n-1} \cdot S\sqrt{1 + \frac{1}{n}}$$

Example

An article reports the following data on the breakdown voltage of electrically stressed circuits, assumed to be normally distributed:

$$1470, 1510, 1690, 1740, 1900, 2000, 2030, 2100, 2200,$$
$$2290, 2380, 2390, 2480, 2500, 2580, 2190, 2700$$

Find a 95% confidence interval for the mean μ. Then find a 95% prediction interval for a future observation.

We found $\overline{X} = 2126.5$ and $S^2 = 137324.3$. The critical value is $t_{\alpha/2,\nu} = t_{.025,16} = 2.120$, giving the confidence interval for μ:

$$\overline{X} \pm \frac{t_{\alpha/2,\nu} \cdot S}{\sqrt{n}} = 2126.5 \pm 190.5$$

Likewise we get the prediction interval:

$$\overline{X} \pm t_{\alpha/2,\nu} \cdot S\sqrt{1 + \frac{1}{n}} = 2126.5 \pm 808.4$$

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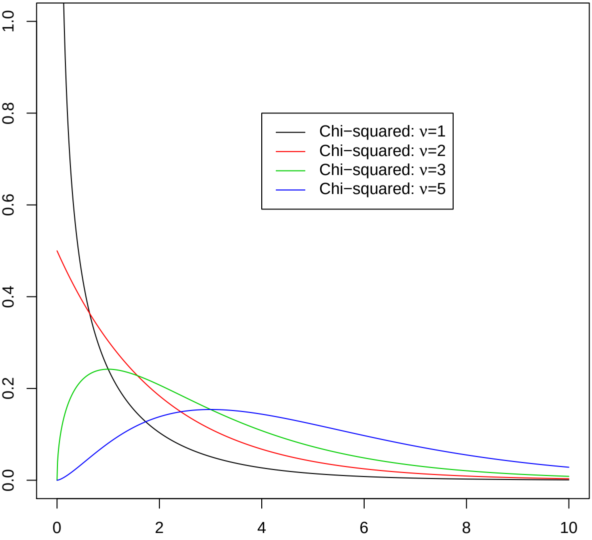
Given a random sample $X_1, \ldots, X_n$ from a normal distribution with unknown mean μ and variance σ^2, A $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right]$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are critical values from a χ^2 **distribution** with $\nu = n-1$ degrees of freedom.

Chi-squared distribution



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The observed sample mean and sample variance are $\overline{X} = 2126.5$ and $S^2 = 137324.3$. The critical values are $\chi^2_{.025,16} = 28.845$ and $\chi^2_{.975,16} = 6.908$,

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The observed sample mean and sample variance are $\overline{X} = 2126.5$ and $S^2 = 137324.3$. The critical values are $\chi^2_{.025,16} = 28.845$ and $\chi^2_{.975,16} = 6.908$, which gives a 95% confidence interval of

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for σ^2.

Example

Recall the breakdown voltage data, assumed to be normal:

$$1470, 1510, 1690, 1740, 1900, 2000, 2030, 2100, 2200,$$
$$2290, 2380, 2390, 2480, 2500, 2580, 2190, 2700$$

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for σ^2. The corresponding confidence interval for σ is

$$[\sqrt{76172.3}, \sqrt{318064.4}] = [276.0, 564.0]$$

Estimating a Proportion

Suppose we have a sequence of n Bernoulli trials, where the probability p of success is unknown. If we observe X successes, we know that the maximum likelihood estimator of p is the sample proportion $\hat{p} = X/n$.

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Suppose X is a binomial random variable counting the number of successes in n trials where each trial has probability p of success. If n is sufficiently large an approximate $100(1-\alpha)\%$ confidence interval is given by

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Rule of thumb: This may be used if the number of successes X and the number of failures $n - X$ are both at least 10.

Example

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Here the sample proportion is $\hat{p} = 15/200 = .075$, and the critical value is $z_{.025} = 1.96$, so the approximate 95% confidence interval is

$$\begin{aligned} \hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= .075 \pm 1.96\sqrt{\frac{.075(1-.075)}{200}} \\ &= .075 \pm .037 \end{aligned}$$

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$$n \approx \frac{1.96^2(.075)(1-.075)}{.01^2} = 2665.11 \approx 2666$$

Summary

Confidence interval for mean μ of normal, σ known	$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$
Large-sample approximate confidence interval for mean μ	$\overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}}$
Confidence interval for mean μ of normal, σ unknown	$\overline{X} \pm \frac{t_{\alpha/2,n-1} \cdot S}{\sqrt{n}}$
Prediction interval for normal observation	$\overline{X} \pm t_{\alpha/2,n-1} \cdot S\sqrt{1 + \frac{1}{n}}$
Confidence interval for variance σ^2 of normal	$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right]$
Approximate confidence interval for proportion p	$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$