# Ch. 7 – Confidence Intervals

# Standard Normal Probability

If $Z$ is a standard normal random variable, find a constant $c$ such that $P(-c \leq Z \leq c) = .95$.

Solution: We have

$$.95 = P(-c \leq Z \leq c) = 1 - 2\Phi(-c)$$

Solving for $c$ gives

$$c = -\Phi^{-1}(.025) = 1.96$$

# Problem

Suppose a machine drills holes whose diameters are normally distributed with mean $\mu = 3.0$ mm and standard deviation $\sigma = 0.4$ mm. If a random sample of 16 holes are measured, find an interval $[a, b]$ centered on 3.0 such that the sample mean $\overline{X}$ diameter will be in the interval $[a, b]$ with probability .95.

The sample mean $\overline{X}$ is normal with mean $\mu = 3.0$ and standard deviation $\sigma_{\overline{X}} = \sigma/\sqrt{16} = 0.1$. So $(\overline{X} - 3.0)/0.1$ is a standard normal random variable. By the previous slide,

$$.95 = P(-1.96 \leq \frac{\overline{X} - 3.0}{0.1} \leq 1.96)$$

Rearranging,

$$.95 = P(2.804 \leq \overline{X} \leq 3.196)$$

# Problem – Other Way Around

Suppose a machine drills holes whose diameters are normally distributed with unknown mean $\mu$ and known standard deviation $\sigma = 0.4$ mm. Given a random sample of 16 holes, find an interval $[A, B]$ depending on the sample mean $\overline{X}$ such that $\mu$ is in the interval $[A, B]$ with probability .95.

$\overline{X}$ is normal with mean $\mu$ and standard deviation $\sigma/\sqrt{16} = 0.1$, so $\frac{\overline{X}-\mu}{0.1}$ is a standard normal random variable. Therefore

$$\begin{aligned} .95 &= P\left(-1.96 \le \frac{\overline{X}-\mu}{0.1} \le 1.96\right) \\ &= P(\overline{X} - .196 \le \mu \le \overline{X} + .196) \end{aligned}$$

So we may take $[A, B] = [\overline{X} - .196, \overline{X} + .196]$. For example, if $\overline{X} = 3.05$, then $[A, B] = [2.854, 3.246]$.

# Confidence Intervals

- Suppose we are given a random sample $X_1, \ldots, X_n$ from a distribution with an unknown parameter $\theta$.
- A 95% **confidence interval** for $\theta$ is an interval $[A, B]$ based on the sample, such that $P(A \leq \theta \leq B) = .95$.
- In general, a $100(1-\alpha)\%$ **confidence interval** for $\theta$ is an interval $[A, B]$ such that $P(A \leq \theta \leq B) = 1 - \alpha$.
- We are usually interested in constructing **equal-tailed** confidence intervals, where $P(\theta < A) = P(\theta > B) = \alpha/2$.

# Confidence Interval for Mean of Normal Distribution

If a random sample of size $n$ is taken from a normal distribution with unknown mean $\mu$ and known standard deviation $\sigma$, then an equal-tailed $100(1-\alpha)\%$ confidence interval for $\mu$ is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is a **critical value** given by

$$z_{\alpha/2} = -\Phi^{-1}(\alpha/2)$$

Proof: The sample mean $\overline{X}$ is normal with mean $\mu$ and standard deviation $\sigma/\sqrt{n}$, so $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is standard normal, hence

$$\begin{aligned} 1-\alpha &= P\left(-z_{\alpha/2} \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) \\ &= P\left(\overline{X} - \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}\right) \end{aligned}$$

# Example

A process produces alginate beads with diameters (in mm) normally distributed with unknown mean $\mu$ and standard deviation $\sigma = .7$. A random sample of 9 beads have the following diameters:

$$3.9, 5.1, 5.2, 5.7, 5.8, 6.1, 6.2, 6.3, 6.5$$

Find a 99% confidence interval for the mean diameter $\mu$.

Here $\alpha = 1 - .99 = .01$, so the relevant critical value is

$$z_{\alpha/2} = z_{.005} = -\Phi^{-1}(.005) = 2.57$$

The sample mean is $\overline{X} = 5.64$, so the confidence interval is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} = 5.64 \pm \frac{2.57 \cdot 0.7}{\sqrt{9}} = 5.64 \pm 0.60$$

# Determining Necessary Sample Size

In the previous example, a random sample of 9 beads was used to estimate the mean diameter $\mu$, with a margin of error of 0.60, at a 99% confidence level. What sample size would be required for a margin of error of no more than 0.10, at the 99% confidence level?

We were given that the standard deviation of the bead diameters is $\sigma = .7$. Setting the margin of error $\frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$ equal to 0.10 gives an equation

$$\frac{2.57 \cdot 0.7}{\sqrt{n}} = 0.10$$

Solving for $n$ gives

$$n = \left( \frac{2.57 \cdot 0.7}{0.10} \right)^2 = 323.64$$

Rounding up to an integer, a sample size of at least $n = 324$ would be required.

# Confidence Intervals

Recall the confidence interval for the mean $\mu$ of a normal population:

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$$

The margin of error may be decreased in any of the following ways:

- Decrease the confidence level (e.g., from 99% to 90%).
- Decrease the standard deviation $\sigma$ of the population.
- Increase the sample size $n$.

*Caution*: Confidence intervals are used to estimate parameters, not to predict future observations:

- If $[4.9, 5.1]$ is a 95% confidence interval for the mean diameter $\mu$, we can be 95% confident that $\mu$ is between 4.9 and 5.1.
- It does *not* mean that 95% of beads will have diameter between 4.9 and 5.1.

# Large-sample Confidence Interval for Mean

If $X_1, \ldots, X_n$ are a random sample from a distribution with unknown mean $\mu$ and *unknown* variance $\sigma^2$, and if $n$ is sufficiently large, then an approximate $100(1-\alpha)\%$ confidence interval for $\mu$ is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}}$$

- This is based on the fact that if $n$ is large then $S \approx \sigma$.
- Here $S$ is the sample standard deviation

$$S = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

- Rule of thumb: The large-sample confidence interval for $\mu$ may be used if $n > 40$.

# Example

Suppose that a random sample of 600 light bulbs of a certain type had a mean lifetime $\overline{X} = 842$ hours, with sample standard deviation $S = 799$. Find the large-sample approximate 95% confidence interval for the mean lifetime $\mu$.

$$\begin{aligned} \overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}} &= 842 \pm \frac{1.96 \cdot 799}{\sqrt{600}} \\ &= 842 \pm 63.9 \\ &= [778.1, 905.9] \end{aligned}$$

# Small-Sample Confidence Interval for Mean of Normal

Suppose we have a random sample $X_1, \ldots, X_n$ from a normal distribution, where the mean $\mu$ and variance $\sigma^2$ are both unknown. We have seen that if $n$ is large, then the pivotal statistic $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ is approximately a standard normal random variable. However, if $n$ is small then $T$ instead has a so-called **t distribution with** $\nu = n - 1$ **degrees of freedom**.

# Confidence Interval for Mean of Normal

Suppose we have a random sample $X_1, \ldots, X_n$ from a normal distribution, where the mean $\mu$ and variance $\sigma^2$ are both unknown.

Previously, we have used the fact that $n$ is large, then the statistic $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ is approximately equal to $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$, which is a standard normal random variable. However, if $n$ is small then this is not a good approximation. Instead, $T$ has a so-called **t distribution**.

Given a random sample $X_1, \ldots, X_n$ from a normal distribution with unknown mean and variance, A $100(1-\alpha)\%$ confidence interval for the mean $\mu$ is

$$\overline{X} \pm \frac{t_{\alpha/2,\nu} \cdot S}{\sqrt{n}}$$

where $t_{\alpha/2,\nu}$ is a critical value from a **t distribution** with $\nu = n-1$ degrees of freedom.

# PDF of t Distribution vs. Standard Normal

norm_t.pdf

As $\nu \to \infty$, the t distribution approaches a standard normal.

# Example

A process produces alginate beads with diameters (in mm) normally distributed with unknown mean $\mu$ and unknown standard deviation $\sigma$. A random sample of 9 beads have the following diameters:

$$3.9, 5.1, 5.2, 5.7, 5.8, 6.1, 6.2, 6.3, 6.5$$

Find a 99% confidence interval for the mean diameter $\mu$.

Here $\alpha = 1 - .99 = .01$, so the relevant critical value is

$$t_{\alpha/2,\nu} = t_{.005,8} = 3.355$$

The sample mean is $\overline{X} = 5.64$ and the sample standard deviation is $S = .809$, so the confidence interval is given by

$$\overline{X} \pm \frac{t_{\alpha/2} \cdot S}{\sqrt{n}} = 5.64 \pm \frac{3.355 \cdot 0.809}{\sqrt{9}} = 5.64 \pm 0.90$$

# Prediction Interval for a Normal Population

Given a random sample $X_1, \ldots, X_n$ from a normal distribution, suppose we want to construct an interval $[A, B]$ which we can be 95% confident will contain a future observation $X_{n+1}$. Such an interval is called a **prediction interval**.

The statistic $T = \frac{\overline{X} - X_{n+1}}{S\sqrt{1+\frac{1}{n}}}$ has a t distribution with $\nu = n - 1$ degrees of freedom.

Given a random sample $X_1, \ldots, X_n$ from a normal distribution, a $100(1-\alpha)\%$ prediction interval for an independent observation $X_{n+1}$ is

$$\overline{X} \pm t_{\alpha/2,n-1} \cdot S\sqrt{1 + \frac{1}{n}}$$

# Example

An article reports the following data on the breakdown voltage of electrically stressed circuits, assumed to be normally distributed:

$$1470, 1510, 1690, 1740, 1900, 2000, 2030, 2100, 2200,$$
$$2290, 2380, 2390, 2480, 2500, 2580, 2190, 2700$$

Find a 95% confidence interval for the mean $\mu$. Then find a 95% prediction interval for a future observation.

We found $\overline{X} = 2126.5$ and $S^2 = 137324.3$. The critical value is $t_{\alpha/2,\nu} = t_{.025,16} = 2.120$, giving the confidence interval for $\mu$:

$$\overline{X} \pm \frac{t_{\alpha/2,\nu} \cdot S}{\sqrt{n}} = 2126.5 \pm 190.5$$

Likewise we get the prediction interval:

$$\overline{X} \pm t_{\alpha/2,\nu} \cdot S\sqrt{1 + \frac{1}{n}} = 2126.5 \pm 808.4$$

# Confidence Interval for Variance of Normal

Suppose we want to find a confidence interval for the variance $\sigma^2$ of a normal distribution based on a random sample $X_1, \ldots, X_n$.

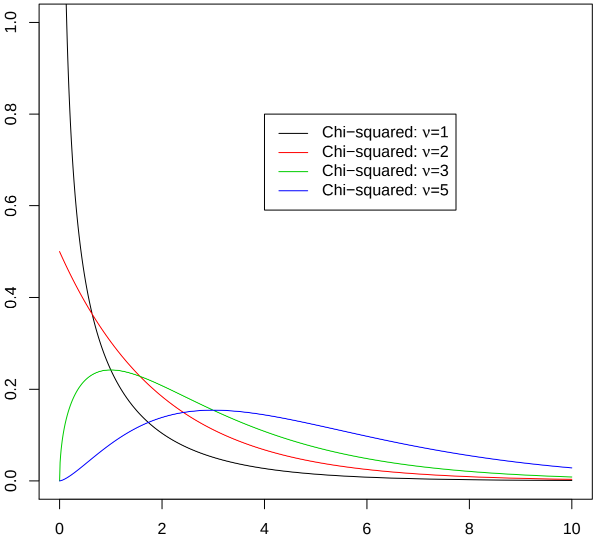
The statistic $(n-1)S^2/\sigma^2$ has a so-called $\chi^2$ distribution with $\nu = n-1$ degrees of freedom.

Given a random sample $X_1, \ldots, X_n$ from a normal distribution with unknown mean $\mu$ and variance $\sigma^2$, A $100(1-\alpha)\%$ confidence interval for $\sigma^2$ is

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right]$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are critical values from a $\chi^2$ **distribution** with $\nu = n-1$ degrees of freedom.

# Chi-squared distribution



# Example

Recall the breakdown voltage data, assumed to be normal:

$$1470, 1510, 1690, 1740, 1900, 2000, 2030, 2100, 2200,$$
$$2290, 2380, 2390, 2480, 2500, 2580, 2190, 2700$$

Find a 95% confidence interval for the standard deviation $\sigma$.

The observed sample mean and sample variance are $\overline{X} = 2126.5$ and $S^2 = 137324.3$. The critical values are $\chi^2_{.025,16} = 28.845$ and $\chi^2_{.975,16} = 6.908$, which gives a 95% confidence interval of

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right] = [76172.3, 318064.4]$$

for $\sigma^2$. The corresponding confidence interval for $\sigma$ is

$$[\sqrt{76172.3}, \sqrt{318064.4}] = [276.0, 564.0]$$

# Estimating a Proportion

Suppose we have a sequence of $n$ Bernoulli trials, where the probability $p$ of success is unknown. If we observe $X$ successes, we know that the maximum likelihood estimator of $p$ is the sample proportion $\hat{p} = X/n$.

How do we construct a confidence interval for $p$ based on $\hat{p}$?

Suppose $X$ is a binomial random variable counting the number of successes in $n$ trials where each trial has probability $p$ of success. If $n$ is sufficiently large an approximate $100(1-\alpha)\%$ confidence interval is given by

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $\hat{p} = X/n$ is the sample proportion, and $z_{\alpha/2}$ is a critical value from the standard normal distribution.

Rule of thumb: This may be used if the number of successes $X$ and the number of failures $n - X$ are both at least 10.

# Example

A quality control team for a manufacturer tests 200 randomly selected devices, out of which 15 are defective. Assume that defective devices occur independently of one another. Find an approximate 95% confidence interval for the proportion defective.

Here the sample proportion is $\hat{p} = 15/200 = .075$, and the critical value is $z_{.025} = 1.96$, so the approximate 95% confidence interval is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = .075 \pm 1.96\sqrt{\frac{.075(1-.075)}{200}}$$
$$= .075 \pm .037$$

# Necessary Sample Size for Estimating Proportion

In the previous example, a 95% confidence interval for the proportion was $.075 \pm .037$. Estimate the required sample size to achieve a margin of error of .01.

Setting the margin of error equal to .01 gives an equation

$$.01 = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Solving for $n$ gives

$$n = \frac{z_{\alpha/2}^2\hat{p}(1-\hat{p})}{.01^2}$$

Unfortunately, this depends on $\hat{p}$, which is unknown until *after* the new sample is taken. However, we can estimate the required $n$ by using our previous sample proportion $\hat{p} = .075$:

$$n \approx \frac{1.96^2(.075)(1-.075)}{.01^2} = 2665.11 \approx 2666$$

# Summary

| | |
|---|---|
| Confidence interval for mean $\mu$ of normal, $\sigma$ known | $\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$ |
| Large-sample approximate confidence interval for mean $\mu$ | $\overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}}$ |
| Confidence interval for mean $\mu$ of normal, $\sigma$ unknown | $\overline{X} \pm \frac{t_{\alpha/2,n-1} \cdot S}{\sqrt{n}}$ |
| Prediction interval for normal observation | $\overline{X} \pm t_{\alpha/2,n-1} \cdot S\sqrt{1 + \frac{1}{n}}$ |
| Confidence interval for variance $\sigma^2$ of normal | $\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right]$ |
| Approximate confidence interval for proportion $p$ | $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ |