Math 3070, Applied Statistics

Section 1

August 23, 2019

Lecture Outline, 8/23

- 2.1: Sample Spaces and Events
- Set Operations
- 2.2: Axioms, Interpretations, and Properties of Probability

Sample Spaces and Events, Section 2.1

Goal: A mathematical framework which allows one to describe experimental outcomes, extend observations into models and calculate or predict the randomness of outcomes.

Approach:

- Section 2.1: Sample spaces and events are the foundation of most modern probabilistic models and used to describe experimental or random outcomes.
- Section 2.2: Probability employs a function which maps outcomes to their likelihood or chance of occuring. Note, the likelihood is often called the probability of an event.
- Later chapters combine probabilistic models and experimental observations to estimate models which can be used calculate or predict likelihood or probabilities, measuring randomness.

Sample Spaces and Events, Definitions

set: collection of objects. Sets are denoted by curly brackets, $\{\}$. Theory of sets is *topology*, used in higher level probability.

Outcome: values which can be observed.

Sample Space $\mathcal{S}, \Omega$: set all of all possible outcomes of an experiment (or random object).

Events: subsets of the sample space defining and denoting outcomes of experiments (or random objects). Events which have zero likeihood or probability are still events. They only need to be a subset of the sample space. Technically, $\mathcal{S}$ is a subset of itself.

Null Set $\varnothing$: set with no members. Used to denoted when nothing happens or no outcome is observed. Always a subset of any set.

Sample Spaces and Events, Example Coin Flip

Experiment: Flip a coin. Finite discrete, categorical outcomes.

Sample Space $:= \{H, T\}$

All Events (Subsets):

$$\emptyset, \{H\}, \{T\}, \{H, T\}$$

Note, $\{H, T\}$ is $\mathcal{S}$ and refers to a set with two members or observing any outcome in that subset. In this case, it means observing heads or tails.

Note that outcomes and events in this framework have different meanings. The next example will meant to explain why.

Sample Spaces and Events, Example Two Coin Flips

Experiment: Flip 2 coins. Finite discrete, categorical outcomes.

Sample Space $:= \mathcal{S} = \{(H,H),(T,H),(H,T),(T,T)\}$

Note, denoting the outcome of each coin as (coin 1, coin 2).

The space of all events is has an annoying number of members, $2^4 = 16$. Generally, sample spaces with n outcomes have 2^n events.

All Events (Subsets):

$$\varnothing, \{(H,H)\}, \{(H,T)\}, \{(T,H)\}, \{(T,H)\},$$

$$\{(H,H),(H,T)\}, \ldots, \{(H,H),(T,H),(H,T),(T,T)\}$$

Sample Spaces and Events, Example Two Coin Flips

Listing out all events is nothing more than a long excerise. A major benefit provided by distingushing events and outcomes is the ability to describe outcomes written in english in mathematical notation, explained in the following examples.

$$\text{"first coin lands on heads"} = \{(H, T), (H, H)\}$$

$$\text{"coins have different outcomes"} = \{(H, T), (T, H)\}$$

English statements can be accurately translated into events and inserted into mathematical (specifically probabilistic) statements.

$$\text{English} \rightarrow \text{Probability} \rightarrow \text{Calculus} \rightarrow \text{Knowledge} \rightarrow \$$$

Questions?

Set Operations (on Events), Definitions

- **Simple Event**: event with only one outcome. Example: $\{H\}; \{T\}$.
- **Compound Event**: event more than one outcome. Example: $\{H, T\}, \{(H, T), (T, H)\}$; "coins have different outcomes".
- **Complement** of an event A, denoted by A': the set of all outcomes in $\mathcal{S}$, but not the event A.
- **Union** of two events A and B, denoted by $A \cup B$: the set of all outcomes which are in either A, B or both.
- **Intersection** of two events A and B, denoted by $A \cap B$: the set of all outcomes which are in both A and B.
- **Difference** of two events A and B, denoted by $A \setminus B$ or $A - B$: the set of all outcomes which are in A, but not B. Order matters. Note, $A' = \mathcal{S} \setminus A$.
- Sets A and B are **disjoint** if $A \cap B = \varnothing$.

Set Operations (on Events), Example Two Coin Flips

Consider the two coin flip example.

$$\mathcal{S} = \{(H, H), (T, H), (H, T), (T, T)\}$$

$$A = \text{"first coin lands on heads"} = \{(H, T), (H, H)\}$$

$$B = \text{"coins have different outcomes"} = \{(H, T), (T, H)\}$$

$$A' = \{(T, H), (T, T)\} = \text{"first coin lands on tails"}$$

$$A \cap B = \{(H, T)\} =$$

"first coin lands on heads and the coins have different outcomes"

$$A \cup B = \{(H, T), (H, H), (T, H)\} =$$

"first coin lands on heads or the coins have different outcomes"

Set Operations (on Events), Comments and Questions

- Translating english statements into probabilistic statements is often the most difficult part of this class, but is one of the most used skills.
- "not event A" translates into A'.
- "event A or event B" translates into $A \cup B$.
- "event A and event B" translates into $A \cap B$.
- $(A')' = A$
- Sometimes it's easier to consider A' instead of A.

Questions?

DeMorgan's Laws

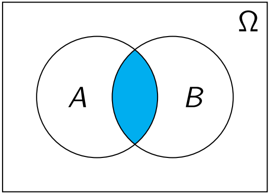
The following two identities are known as DeMorgan's laws:

$$(A \cap B)' = A' \cup B'$$
$$(A \cup B)' = A' \cap B'$$

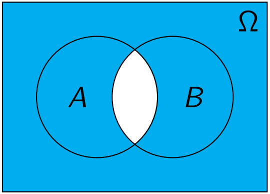
They also extend to three or more events:

$$(A \cap B \cap C)' = A' \cup B' \cup C'$$
$$(A \cup B \cup C)' = A' \cap B' \cap C'$$

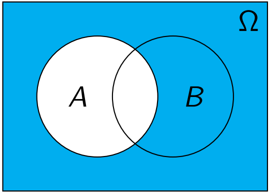
DeMorgan's Laws, Explanation



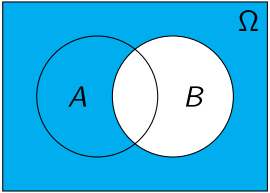
$A \cap B$



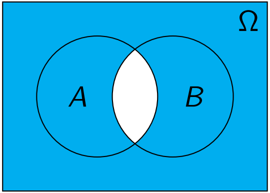
$(A \cap B)'$



A'



B'



$A' \cup B'$

DeMorgan's Law, questions

DeMorgan's Law can be useful when simplifying calculations.

Questions?

Interpretation of Probability, Section 2.2

In practical settings, the probability of an event is the relative frequency as the sample size become arbitrarily large, the *frequentist* perspective. This is different from $\hat{p}$ which depends on the sample. Typically, the probability of a real event is hard or impossible to measure, but can be decently modeled.

Axioms

Given a Sample Space $\mathcal{S}$, P is a function whose inputs are events of $\mathcal{S}$, and outputs are numbers between 0 and 1. These axioms aim to model the likelihood or chance of any event occuring. $P(A)$ is the probability of event A. P must follow the following axioms.

1. For any event A, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$.
3. If $A_1, A_2, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i).$$

Caution: Axiom 3 only works with disjoint sets.

Properties of Probability

Proposition: $P(\varnothing) = 0$.

1. $\varnothing \cap \varnothing = \varnothing$. Disjoint.
2. $\varnothing \cup \varnothing \cup \ldots = \varnothing$.
3. Axiom 3: $P(\varnothing \cup \varnothing \cup \ldots) = P(\varnothing) + P(\varnothing) + P(\varnothing) + \ldots$
4. If $P(\varnothing) > 0$, then Axiom 2 is false. Observe, $\mathcal{S} = \mathcal{S} \cup \varnothing \cup \varnothing \ldots$ and $\varnothing \cap \mathcal{S} = \varnothing$, disjoint. Axiom 3 imples $P(\mathcal{S}) = P(\mathcal{S}) + P(\varnothing) + P(\varnothing) + \ldots > 1$.
5. It must be true that $P(\varnothing) = 0$ uphold axiom 2.

Important use: the empty set is disjoint with all sets. This allows Axiom 3 to used with finite disjoint set by unioning infinitely many $\varnothing$ to them.

Properties of Probability

Proposition: $P(A') = 1 - P(A)$

1. $A \cap A' = \varnothing$. Disjoint.
2. $A \cup A' = \mathcal{S}$.
3. Axioms 2 and 3:

$$1 = P(\mathcal{S}) = P(A \cup A') = P(A) + P(A')$$

Useful when simplifying calculations.

Proposition: $P(A) \leq 1$

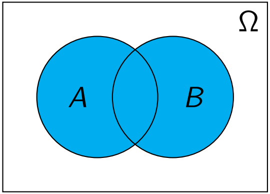
1. By Axiom 1, $P(A), P(A') \geq 0$.
2. $\mathcal{S} = A \cup A'$
3. $A \cap A' = \varnothing$
4. $1 = P(\mathcal{S}) = P(A) + P(A') \geq P(A)$.

All probabilities of events are between 0 and 1.

Probability of a Union of Two Events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A, B.

Intuition: To find the probability of $A \cup B$, we add the probabilities of A and B, but then we have double counted the intersection $A \cap B$, so we have to subtract that. In this analogy, we want to measure the



$A \cup B$

Probability of a Union of Two Events

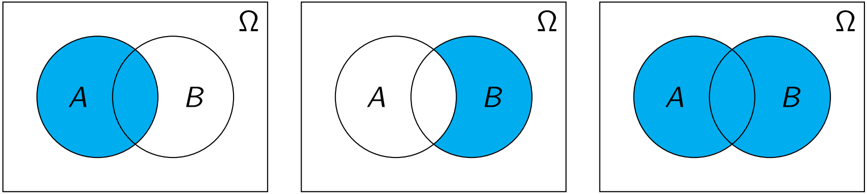
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A, B.

Proof: We may write $A \cup B$ as a union of two disjoint events:

$$A \cup B = A \cup (B - (A \cap B))$$

Therefore,

$$\begin{aligned} P(A \cup B) &= P(A \cup (B - (A \cap B))) \\ &= P(A) + P(B - (A \cap B)) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$



A $\qquad$ $B - (A \cap B)$ $\qquad$ $A \cup B$

Probability of a Union of Three Events

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

Intuition: To find the probability of $A \cup B \cup C$, we add the probabilities of the events A, B, and C and subtract the overlap of each pair; but then we've subtracted the three-way overlap $A \cap B \cap C$ one too many times, so we add it back.

Nice to know, but rarely used.

Properties of Probability, Summary

Axioms

1. For any event A, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$.
3. If $A_1, A_2, \ldots$ is an infinite collection of disjoint events, then $P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$.

Propositions

1. $P(\varnothing) = 0$
2. If $A_1, A_2, \ldots, A_n$ is an finite collection of disjoint events, then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$.
3. $P(A') = 1 - P(A)$
4. For any events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Interpretation of Probability

Under this model of the probability of an event is the proportion of times it should occur in a very large sample, the *frequentist* perspective. This is different from $\hat{p}$ which depends on the sample. Typically, the probability of a real event is hard or impossible to measure, but can be decently modeled.

Equally Likely Outcomes

Given a finite sample space Ω, one of the simplest ways to define a probability measure is to assume that the outcomes are **equally likely**:

$$P(A) = \frac{\#A}{\#\Omega} = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}$$

Example

Suppose we roll a fair six-sided die. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $A = \{2, 4, 6\}$ be the event that we roll an even number. Then

$$P(A) = \frac{\#A}{\#\Omega} = \frac{3}{6} = \frac{1}{2} = .5$$

It is not always reasonable to assume the outcomes are equally likely.

Questions?

Disjoint Sum Example, Rolling Two Dice

Problem: What is the probability that a roll of two fair die lands on two exactly once?

Notation: (X_1, X_2),
$X_1 =$ value of first die,
$X_2 =$ value of second die.

$$\mathcal{S} = \{(1,1),(1,2),(1,3)\ldots(6,5),(6,6)\},\ \text{36 pairs}$$

Since the dice are fair, each pair is equally likely, all have probability of $1/36$.

Disjoint Sum Example, Rolling Two Dice

1. Event $A =$ "two rolled exactly once".
2. Event $B =$ "first die is two, second is not".
3. $B = \{(2,1),(2,3),(2,4),(2,5),(2,6)\}$, $P(B) = 5/36$
4. Event $C =$ "second die is two, first is not".
5. $C = \{(1,2),(3,2),(4,2),(5,2),(6,2)\}$, $P(C) = 5/36$
6. $A = B \cup C$
7. $C \cap B = \varnothing$, disjoint.
8. Sum of disjoint sets, $P(A) = P(B) + P(C) = 10/36 = 5/18$

Complement Example, Rolling Two Dice

Problem: What is the probability that two fair dice do not land on the same number?

1. Event $A =$ "dice land on different numbers".
2. Event $A' =$ "dice land the same number".
3. $A' = \{(1,1), (2,2), \ldots, (6,6)\}$.
4. $P(A') = 6/36 = 1/6$.
5. $P(A) = 1 - P(A') = 1 - 1/6 = 5/6$.

Nondisjoint Example, Rolling Two Dice

Problem: What is the probability that two fair dice show either two exactly once or two even numbers?

1. event $A =$ "two rolled exactly once"
2. $P(A) = 10/36$ from before.
3. event $B =$ "rolled two even numbers"
4. $B = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$
5. $P(B) = 9/36$, 9 pairs
6. $B \cap A = \{(2,4),(2,6),(4,2),(6,2)\}$
7. $P(A \cap B) = 4/36$
8. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 10/36 + 9/36 - 4/36 = 15/36$

DeMorgan's Law Example, Flipping Six Coins

Problem: Suppose a fair coin is flipped six times. What is probability that at least one head and one tail is observed?

1. event $A =$ "at least one head and tail is observed"
2. $A =$ "at least one head" $\cap$ "at least one tail"
3. DeMorgan's Law
   $A' =$ ("at least one head")$'$ $\cup$ ("at least one tail")$'$
4. event $B =$ ("at least one tail")$'$ $=$ "six heads observed" $= \{(H, H, H, H, H, H)\}$
5. event $C =$ ("at least one head")$'$ $=$ "six tails observed" $= \{(T, T, T, T, T, T)\}$
6. There are $2^6 = 64$ outcomes, equally likely. Events B and C are one outcome each so $P(B) = P(C) = 1/64$.
7. B and C are disjoint, $P(B \cup C) = P(B) + P(C) = 1/32$.
8. $P(A) = 1 - P(A') = 1 - P(B \cup C) = 31/32$.

Probability Table Example

At an intersection, cars either turn right, turn left or go straight with the following probabilities.

action	left	right	straight
probability	0.14	0.16	0.7

What is the probability that a car turns? What is the probability that a car does not turn right?

- events $\{$"turn right"$\}$ and $\{$"turn left"$\}$ are disjoint.
- $P(\text{"car turns"}) = P(\text{"turn right"} \cup \text{"turn left"}) =$
  $P(\text{"turn right"}) + P(\text{"turn left"}) = 0.14 + 0.16 = 0.3$

$P(\text{"does not turn right"}) = 1 - P(\text{"turn right"}) = 1 - 0.16 = 0.84$

Example, Census Proportions

Census data informs you that 67% of households have more than one car, 63% of households have more than one car and more than one person and 31% of households have one person. What is the probability that a randomly selected household has more than one person or more than one car?

event A = "household has more than one car"

event B = "household has more than one person"

B' = "household has one person"

$$P(B) = 1 - P(B') = 1 - 0.31 = 0.69$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.67 \quad + 0.69 - 0.63 \\ &= 0.73 \end{aligned}$$