Math 3070, Applied Statistics

Section 1

September 27, 2019

Lecture Outline, 9/27

Section 4.4

- Exponential Random Variable
- Chi-Squared (χ^2) Random Variable
- Gamma Random Variable

Exponential Random Variable

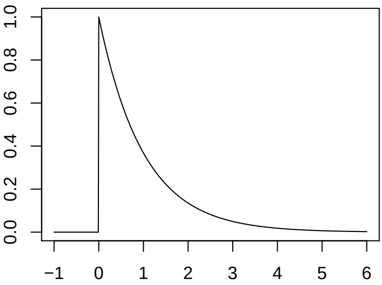
We say that X is an **exponential** random variable with rate $\lambda > 0$ if X has pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$X \sim exp(\lambda)$

We can check that this is a valid pdf:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= \int_0^{\infty} \lambda e^{-\lambda x}\ dx \\ &= \lambda \frac{-1}{\lambda} e^{-\lambda x} \Big|_{x=0}^{\infty} \\ &= 0 - (-1) = 1 \end{aligned}$$



CDF of Exponential Random Variable

Recall the pdf of an exponential random variable x with rate λ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The cdf $F(x)$ is therefore, for $x \geq 0$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\ dt = \int_{0}^{x} \lambda e^{-\lambda t}\ dt \\ &= -e^{-\lambda t}\Big|_{t=0}^{x} = 1 - e^{-\lambda x} \end{aligned}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Exponential Waiting Times

Consider a Poisson Random Variable with rate λ.

- Let Y_t be the number of events occurring in the interval $[0, t]$.
- So Y_t is a Poisson random variable with mean λt.
- Let T_1 be the time of the first event. Find it's PDF.

$$
\begin{aligned}
P(T_1 \leq t) &= P(\text{first event occurs by time } t) \\
&= P(\text{at least one event occurs in the interval } [0, t]) \\
&= P(Y_t \geq 1) \\
&= 1 - P(Y_t = 0) \\
&= 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!} \\
&= 1 - e^{-\lambda t}
\end{aligned}
$$

This is the cdf of an exponential random variable of rate λ. Therefore, in a Poisson process, the waiting time for the first event is an exponential random variable with rate λ.

Mean of Exponential

Using integration by parts, we can find the mean of an exponential random variable X of rate λ:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\ dx \\
&= \int_{0}^{\infty} x\lambda e^{-\lambda x}\ dx \\
&= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\ dx \\
&= 0 - 0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = \frac{1}{\lambda}
\end{aligned}$$

Variance of Exponential Distribution

We can find the variance of an exponential random variable X by using integration by parts twice:

$$
\begin{aligned}
E(X^2) &= \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\ dx \\
&= -x^2 e^{-\lambda x}\Big|_0^\infty - \int_0^\infty -2xe^{-\lambda x}\ dx \\
&= 2\int_0^\infty xe^{-\lambda x}\ dx \\
&= 0 - 0 + 2\left[-\frac{1}{\lambda}xe^{-\lambda x}\Big|_0^\infty - \int_0^\infty -\frac{1}{\lambda}e^{-\lambda x}\ dx\right] \\
&= 2\left[0 - 0 - \frac{1}{\lambda^2}e^{-\lambda x}\Big|_0^\infty\right] = \frac{2}{\lambda^2}
\end{aligned}
$$

So $V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$. Moreover, the standard deviation is $\sigma = 1/\lambda = \mu$.

Memoryless Property of Exponential

The exponential distribution is useful for modeling the lifetime of components where breakdowns are the result of sudden, random failures rather than gradual deterioration.

In general, an exponential random has the "memoryless" property:

$$P(X \geq s + t \mid X \geq s) = P(X \geq t)$$

Proof:
$$\begin{aligned} P(X \geq s + t \mid X \geq s) &= \frac{P(X \geq s + t \cap X \geq s)}{P(X \geq s)} \\ &= \frac{P(X \geq s + t)}{P(X \geq s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} = P(X \geq t) \end{aligned}$$

Due to the memoryless property, the exponential distribution can model times between events of a Poisson Random Variable.

Example

Suppose that lightbulbs failures over 200 days follow a Poisson Distribution and average 2 light bulbs. What is the probability that a lightbulb lifetime is at least 50 days?

Solution: The rate of failure is $\lambda = \mu/t = 2/200 = 1/100$ per day. Therefore,

$$\begin{aligned} P(X \geq 50) &= \int_{50}^{\infty} \lambda e^{-\lambda x}\ dx \\ &= -e^{-\lambda x}\Big|_{50}^{\infty} \\ &= e^{-50\lambda} \\ &= e^{-50/100} = e^{-1/2} \approx .607 \end{aligned}$$

In general, if X is an exponential random variable with rate λ,

$$P(X \geq t) = e^{-\lambda t}$$

Example

Again suppose that the lifetime X of a lightbulb follows an exponential distribution with mean $\mu = 100$ days. Given that the bulb has survived for 30 days, what is the probability that it will last for at least 50 more?

$$\begin{aligned} P(X \geq 80 \mid X \geq 30) &= \frac{P(X \geq 80 \cap X \geq 30)}{P(X \geq 30)} \\ &= \frac{P(X \geq 80)}{P(X \geq 30)} \\ &= \frac{e^{-80\lambda}}{e^{-30\lambda}} \\ &= e^{-50\lambda} = e^{-50/100} = e^{-1/2} \approx .607 \end{aligned}$$

This is the same as the probability of a new bulb lasting 50 days, as we calculated on the previous slide.

Summary

- $X \sim exp(\lambda)$ means X follows an exponential distribution with parameter λ. On a given interval, $\mu = \lambda t$. μ from Poisson.
- Poisson models number of events with rate λ. Exponential with the same λ models wait times between them.
- Memoryless property is a subtle modeling issue, but mathematical nice. $P(X > t + s | X > s) = P(X > t)$.
- PDF:
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
- CDF:
$$Fx) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
- $$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2} \qquad \sigma_X = \frac{1}{\lambda}$$

Chi-squared Random Variable

If Z is a standard normal random variable, then Z^2 has a so-called **chi-squared** distribution with one **degree of freedom**.

A **chi-square** random variable with $\nu > 0$ degrees of freedom has pdf

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$X \sim \chi^2(\nu)$

Since Z is important in the central limit theorem, one should expect that χ^2 will appear in many applications.

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx$$

Chi-squared Random Variable, Z^2 Explained

Consider $Z \sim N(0,1)$. Compute the pdf of $X = Z^2$. For $x < 0$, $P(Z^2 < 0) = 0$. Consider $x > 0$.

$$
\begin{aligned}
P(Z^2 < x) &= P(-\sqrt{x} < Z\sqrt{x}) \\
&= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) dy \\
&= 2\int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) dy
\end{aligned}
$$

Last step uses symmetry of the distribution. Differentiate in x to recover the PDF.

$$
\frac{d}{dx} P(Z^2 < x) = \frac{2}{\sqrt{2\pi}} \exp\left(\frac{-x}{2}\right) \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} \exp\left(\frac{-x}{2}\right)
$$

Observe this is the PDF of a $\chi^2(1)$ random variable.

Chi-squared Random Variable, Mean and Variance

Consider $X \sim \chi^2(\nu)$.

$$E[X] = \nu$$

$$V(X) = 2\nu$$

Gamma Distribution

A **gamma** random variable X with parameters $\alpha, \beta > 0$ is given by the pdf

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(k)} x^{\alpha-1} e^{-x/\beta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$X \sim \Gamma(\alpha, \beta)$

$\beta = 1$ is called the standard gamma.

Used to relate other random variables together. For example, a $\Gamma(\nu/2, 2)$ and $\chi^2(\nu)$ have the same PDFs.

These relationships allows us to relate their parameters.

Gamma Distribution, Mean and Variance

Consider $X \sim \Gamma(\alpha, \beta)$

$$E[X] = \alpha\beta$$

$$V(X) = \alpha\beta^2$$

Check to see that the mean and standard deviation of the χ^2 are reproduced with $\alpha = \nu/2$ and $\beta = 2$.

$$E[X] = \alpha\beta = \frac{\nu}{2}2 = \nu$$

$$V(X) = \alpha\beta^2 = \frac{\nu}{2}2^2 = 2\nu$$

Summary

$X \sim \chi^2(\nu)$ with $\nu > 0$
PDF:

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E[X] = \nu \qquad V(X) = 2\nu$$

$X \sim \Gamma(\alpha, \beta)$ with $\alpha, \beta > 0$
PDF:

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(k)} x^{\alpha-1} e^{-x/\beta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E[X] = \alpha\beta \qquad V(X) = \alpha\beta^2$$

- $Z \sim N(0,1) \rightarrow Z^2 \sim \chi^2(1)$
- $X \sim \Gamma(\nu/2, 2) \iff X \sim \chi^2(\nu)$
- More relations between random variables to come.