Math 3070, Applied Statistics

Section 1

November 1, 2019

Lecture Outline, 11/1

Section 7.1 and 7.2

- Confidence Intervals
- Large-sample Confidence Interval

Confidence Interval for the Sample Mean

If a random sample of size $n > 30$ is taken from a distribution with unknown mean μ and known standard deviation σ, then an equal-tailed $100(1-\alpha)\%$ confidence interval for μ is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is a *critical value* given by

$$z_{\alpha/2} = -\Phi^{-1}(\alpha/2)$$

Confidence Interval for Sample Mean

Proof: The sample mean $\overline{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$, so $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is standard normal, hence

$$\begin{aligned} 1-\alpha &\approx P\left(-z_{\alpha/2} \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) \\ &= P\left(\overline{X} - \frac{z_{\alpha/2}\cdot\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + \frac{z_{\alpha/2}\cdot\sigma}{\sqrt{n}}\right) \end{aligned}$$

The "$\approx$" uses CLT.

Confidence Intervals

- Suppose we are given a random sample $X_1, \ldots, X_n$ from a distribution with an unknown parameter θ.
- A 95% *confidence interval* for θ is an interval $[A, B]$ based on the sample, such that $P(A \leq \theta \leq B) = .95$.
- In general, a $100(1-\alpha)\%$ *confidence interval* for θ is an interval $[A, B]$ such that $P(A \leq \theta \leq B) = 1 - \alpha$.
- We are usually interested in constructing *equal-tailed* confidence intervals, where $P(\theta < A) = P(\theta > B) = \alpha/2$.

Standard Normal Probability

If Z is a standard normal random variable, find a constant c such that $P(-c \leq Z \leq c) = .95$.

Solution: We have

$$.95 = P(-c \leq Z \leq c) = 1 - 2\Phi(-c)$$

Solving for c gives

$$c = -\Phi^{-1}(.025) = 1.96$$

Problem

Suppose a machine drills holes whose diameters are normally distributed with mean $\mu = 3.0$ mm and standard deviation $\sigma = 0.4$ mm. If a random sample of 16 holes are measured, find an interval $[a, b]$ centered on 3.0 such that the sample mean $\overline{X}$ diameter will be in the interval $[a, b]$ with probability .95.

The sample mean $\overline{X}$ is normal with mean $\mu = 3.0$ and standard deviation $\sigma_{\overline{X}} = \sigma/\sqrt{16} = 0.1$. So $(\overline{X} - 3.0)/0.1$ is a standard normal random variable. By the previous slide,

$$E(X) = \mu = 3.0$$
$$V(X) = \frac{\sigma^2}{n} = \frac{0.16}{16} = 0.01$$

$$
\begin{aligned}
.95 &= P(a \le \overline{X} \le b) \\
&= P(3.0 - c \le \overline{X} \le 3.0 + c) \\
&= P\left(\frac{-c}{.1} \le \frac{\overline{X} - 3.0}{0.1} \le \frac{c}{.1}\right) = 1 - 2\Phi(-c/.1)
\end{aligned}
$$

This gives $c = -(.1)\Phi^{-1}(.025) \approx -(.1)(-1.96) = .196$. Therefore $\overline{X}$ will be between 2.804 and 3.196 with probability .95.

$$
.95 = P(-1.96 \le \frac{\overline{X} - 3.0}{0.1} \le 1.96)
$$

Rearranging,

$$
.95 = P(2.804 \le \overline{X} \le 3.196)
$$

Problem – Other Way Around

Suppose a machine drills holes whose diameters are normally distributed with unknown mean μ and known standard deviation $\sigma = 0.4$ mm. Given a random sample of 16 holes, find an interval $[A, B]$ depending on the sample mean $\overline{X}$ such that μ is in the interval $[A, B]$ with probability .95.

$\overline{X}$ is normal with mean μ and standard deviation $\sigma/\sqrt{16} = 0.1$, so $\frac{\overline{X}-\mu}{0.1}$ is a standard normal random variable. Therefore

$$\begin{aligned} .95 &= P\left(-1.96 \le \frac{\overline{X}-\mu}{0.1} \le 1.96\right) \\ &= P(\overline{X} - .196 \le \mu \le \overline{X} + .196) \end{aligned}$$

So we may take $[A, B] = [\overline{X} - .196, \overline{X} + .196]$. For example, if $\overline{X} = 3.05$, then $[A, B] = [2.854, 3.246]$.

Confidence Intervals

Recall the confidence interval for the mean μ of a normal population:

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$$

The margin of error may be decreased in any of the following ways:

- Decrease the confidence level (e.g., from 99% to 90%).
- Decrease the standard deviation σ of the population.
- Increase the sample size n.

Caution: Confidence intervals are used to estimate parameters, not to predict future observations:

- If $[4.9, 5.1]$ is a 95% confidence interval for the mean diameter μ, we can be 95% confident that μ is between 4.9 and 5.1.
- It does *not* mean that 95% of beads will have diameter between 4.9 and 5.1.

Constructing a Confidence Interval

- A common method for constructing confidence intervals is to use a *pivotal statistic*, a quantity defined in terms of the random sample and the parameters but whose distribution is known.
- For a normal random sample with unknown mean μ and known standard deviation σ, a pivotal statistic is given by $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$, which has a standard normal distribution regardless of the parameters μ and σ.

Example

A process produces alginate beads with diameters (in mm) normally distributed with unknown mean μ and standard deviation $\sigma = .7$. A random sample of 9 beads have the following diameters:

$$3.9, 5.1, 5.2, 5.7, 5.8, 6.1, 6.2, 6.3, 6.5$$

Find a 99% confidence interval for the mean diameter μ.

Here $\alpha = 1 - .99 = .01$, so the relevant critical value is

$$z_{\alpha/2} = z_{.005} = -\Phi^{-1}(.005) = 2.57$$

The sample mean is $\overline{X} = 5.64$, so the confidence interval is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} = 5.64 \pm \frac{2.57 \cdot 0.7}{\sqrt{9}} = 5.64 \pm 0.60$$

Note: we are ignoring the need for $n > 30$. This is usually not okay, but this is an excerise for learning.

Determining Necessary Sample Size

In the previous example, a random sample of 9 beads was used to estimate the mean diameter μ, with a margin of error of 0.60, at a 99% confidence level. What sample size would be required for a margin of error of no more than 0.10, at the 99% confidence level?

We were given that the standard deviation of the bead diameters is $\sigma = .7$. Setting the margin of error $\frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}$ equal to 0.10 gives an equation

$$\frac{2.57 \cdot 0.7}{\sqrt{n}} = 0.10$$

Solving for n gives

$$n = \left(\frac{2.57 \cdot 0.7}{0.10}\right)^2 = 323.64$$

Rounding up to an integer, a sample size of at least $n = 324$ would be required.

Large-sample Confidence Interval for Mean

If $X_1, \ldots, X_n$ are a random sample from a distribution with unknown mean μ and *unknown* variance σ^2, and if n is sufficiently large, then an approximate $100(1-\alpha)\%$ confidence interval for μ is given by

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}}$$

- This is based on the fact that if n is large then $S \approx \sigma$.
- Here S is the sample standard deviation

$$S = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

- Rule of thumb: The large-sample confidence interval for μ may be used if $n > 40$.

Example

Suppose that a random sample of 600 light bulbs of a certain type had a mean lifetime $\overline{X} = 842$ hours, with sample standard deviation $S = 799$. Find the large-sample approximate 95% confidence interval for the mean lifetime μ.

$$\begin{aligned}\overline{X} \pm \frac{z_{\alpha/2} \cdot S}{\sqrt{n}} &= 842 \pm \frac{1.96 \cdot 799}{\sqrt{600}} \\ &= 842 \pm 63.9 \\ &= [778.1, 905.9]\end{aligned}$$

If we had used our earlier method for constructing a confidence interval for the mean of an exponential distribution, we would have gotten $[778.5, 913.7]$.
One advantage of the large-sample method is that it does not require a specific assumption about the underlying distribution, e.g. the bulbs could have a Weibull distribution instead of exponential.

Estimating a Proportion

Suppose X is a binomial random variable counting the number of successes in n trials where each trial has probability p of success. If n is sufficiently large an approximate $100(1-\alpha)\%$ confidence interval is given by

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $\hat{p} = X/n$ is the sample proportion, and $z_{\alpha/2}$ is a critical value from the standard normal distribution.

Rule of thumb: This may be used if the number of successes X and the number of failures $n - X$ are both at least 10.

Example

A quality control team for a manufacturer tests 200 randomly selected devices, out of which 15 are defective. Assume that defective devices occur independently of one another. Find an approximate 95% confidence interval for the proportion defective.

Here the sample proportion is $\hat{p} = 15/200 = .075$, and the critical value is $z_{.025} = 1.96$, so the approximate 95% confidence interval is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = .075 \pm 1.96\sqrt{\frac{.075(1-.075)}{200}}$$
$$= .075 \pm .037$$