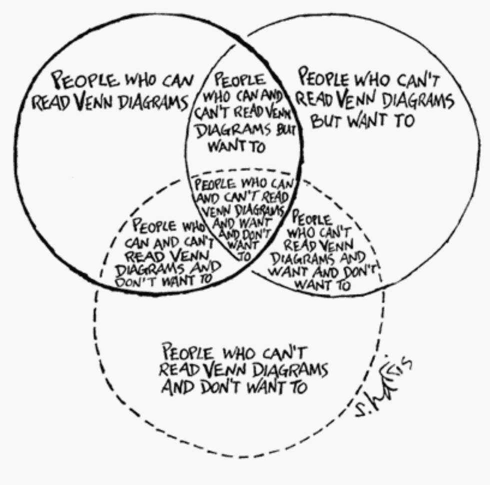
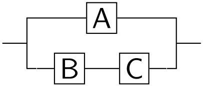
Ch. 2 – Probability



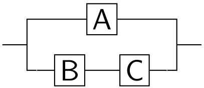
Example – Three-Component System

A system has three components, and to work either component A must work, *or* both components B and C must work.



Example – Three-Component System

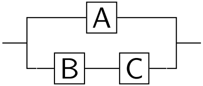
A system has three components, and to work either component A must work, *or* both components B and C must work.



- The probabilities that components A, B, and C work are .7, .4, and .9, respectively.

Example – Three-Component System

A system has three components, and to work either component A must work, *or* both components B and C must work.



- The probabilities that components A, B, and C work are .7, .4, and .9, respectively.
- What is the probability that the system will work?

Basic Set Theory

Suppose a random process has a set Ω of possible outcomes.
An **event** is a subset of Ω. Given two events A and B,

- The **intersection** $A \cap B$ consists of outcomes in A *and* B,
- The **union** $A \cup B$ consists of outcomes in A *or* B (or both).
- The **complement** A' consists of outcomes *not* in A.

Basic Set Theory

Suppose a random process has a set Ω of possible outcomes.
An **event** is a subset of Ω. Given two events A and B,

- The **intersection** $A \cap B$ consists of outcomes in A *and* B,
- The **union** $A \cup B$ consists of outcomes in A *or* B (or both).
- The **complement** A' consists of outcomes *not* in A.

For instance, suppose we roll a six-sided die.

- Let A be the event that we roll an even number.
- Let B be the event that we roll a 4 or higher.

Basic Set Theory

Suppose a random process has a set Ω of possible outcomes.
An **event** is a subset of Ω. Given two events A and B,

- The **intersection** $A \cap B$ consists of outcomes in A *and* B,
- The **union** $A \cup B$ consists of outcomes in A *or* B (or both).
- The **complement** A' consists of outcomes *not* in A.

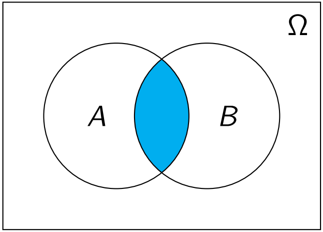
For instance, suppose we roll a six-sided die.

- Let A be the event that we roll an even number.
- Let B be the event that we roll a 4 or higher.

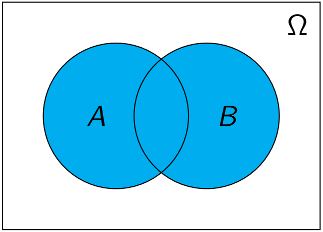
This can be represented as follows:

$\Omega = \{1, 2, 3, 4, 5, 6\}$	We roll any number.
$A = \{2, 4, 6\}$	We roll an even number.
$B = \{4, 5, 6\}$	We roll a number 4 or higher.
$A \cap B = \{4, 6\}$	We roll an even number which is 4 or higher.
$A \cup B = \{2, 4, 5, 6\}$	We roll a number which is even or 4 or higher.
$A' = \{1, 3, 5\}$	We do not roll an even number.

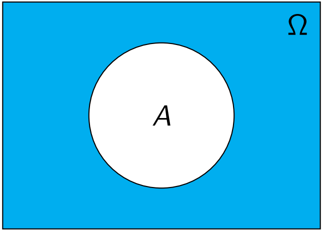
Venn Diagrams



Venn diagram for $A \cap B$

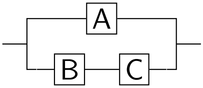


Venn diagram for $A \cup B$



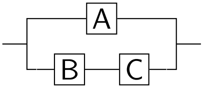
Venn diagram for A'

A system has three components, and to work either component A must work, *or* both components B and C must work.



Example – Three-Component System

A system has three components, and to work either component A must work, *or* both components B and C must work.



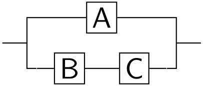
We can use set theory to describe the event that the system works:

$$\text{System works} = A \cup (B \cap C)$$

where A, B, and C represent the events that components A, B, and C work, respectively.

Example – Three-Component System

A system has three components, and to work either component A must work, *or* both components B and C must work.



We can use set theory to describe the event that the system works:

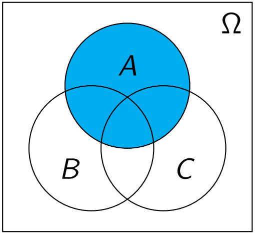
$$\text{System works} = A \cup (B \cap C)$$

where A, B, and C represent the events that components A, B, and C work, respectively.

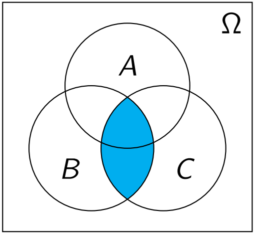
We will come back to this problem again after introducing some probability theory.

Venn Diagrams with Three Events

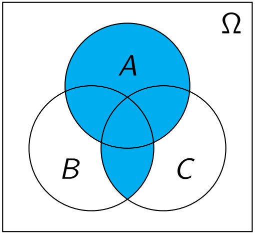
To draw a Venn diagram involving three or more events, it may help to work step-by-step. For example, to draw a Venn diagram for $A \cup (B \cap C)$, first draw Venn diagrams for A and $B \cap C$, then combine them to get the Venn diagram for $A \cup (B \cap C)$:



A



$B \cap C$



$A \cup (B \cap C)$

Disjoint events

- The **null event**, containing no outcomes, is denoted $\emptyset$.

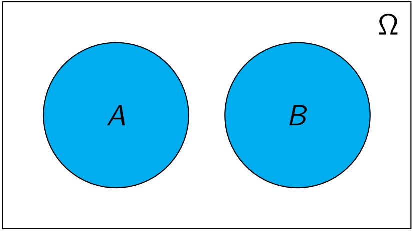
Disjoint events

- The **null event**, containing no outcomes, is denoted $\emptyset$.
- Two events A and B are **disjoint** (or **mutually exclusive**) if $A \cap B = \emptyset$, i.e., if they have no outcomes in common.

Disjoint events

- The **null event**, containing no outcomes, is denoted $\emptyset$.
- Two events A and B are **disjoint** (or **mutually exclusive**) if $A \cap B = \emptyset$, i.e., if they have no outcomes in common.
- For example, if we roll a die, let A be the event that it is a 2 or lower and B be the event that it is a 4 or higher:

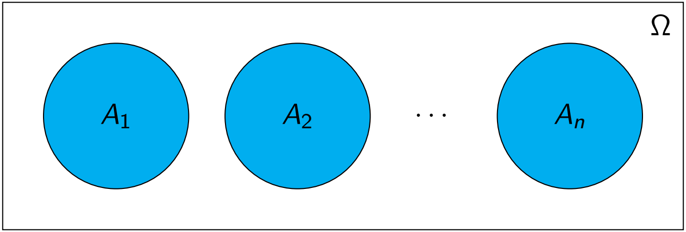
$$A = \{1, 2\}$$
$$B = \{4, 5, 6\}$$



Venn diagram for $A \cup B$ when A and B are disjoint

Several disjoint events

Events $A_1, A_2, \ldots, A_n$ are **disjoint** if A_i and A_j are disjoint for every pair $i \neq j$.



Venn diagram for $A_1 \cup A_2 \cup \cdots \cup A_n$ when $A_1, A_2, \ldots, A_n$ are disjoint

Set-Theoretic Identities

The following identities always hold for any events A, B, and C:

$A \cap A = A$	$A \cup A = A$
$A \cap B = B \cap A$	$A \cup B = B \cup A$
$A \cap (B \cap C) = (A \cap B) \cap C$	$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$(A \cap B)' = A' \cup B'$	$(A \cup B)' = A' \cap B'$
$A \cap A' = \emptyset$	$A \cup A' = \Omega$
$A \cap \emptyset = \emptyset$	$A \cup \Omega = \Omega$
$A \cap \Omega = A$	$A \cup \emptyset = A$
$\emptyset' = \Omega$	$\Omega' = \emptyset$
$A'' = A$	

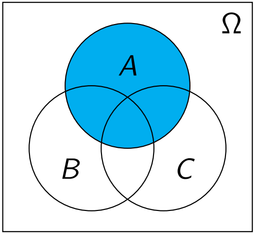
Set-Theoretic Identities

The following identities always hold for any events A, B, and C:

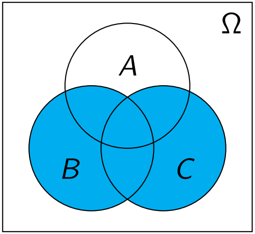
$A \cap A = A$	$A \cup A = A$
$A \cap B = B \cap A$	$A \cup B = B \cup A$
$A \cap (B \cap C) = (A \cap B) \cap C$	$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$(A \cap B)' = A' \cup B'$	$(A \cup B)' = A' \cap B'$
$A \cap A' = \emptyset$	$A \cup A' = \Omega$
$A \cap \emptyset = \emptyset$	$A \cup \Omega = \Omega$
$A \cap \Omega = A$	$A \cup \emptyset = A$
$\emptyset' = \Omega$	$\Omega' = \emptyset$
$A'' = A$	

You don't need to memorize these. We can use Venn diagrams to see why they are true:

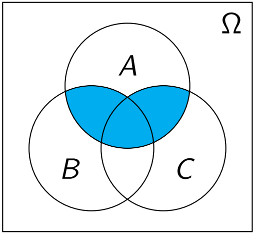
"Proof" that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



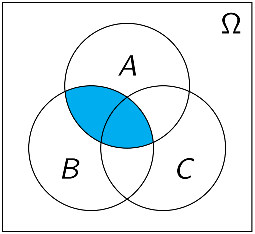
A



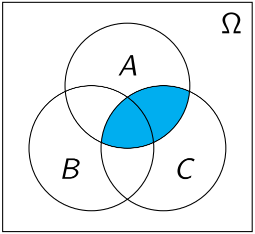
$B \cup C$



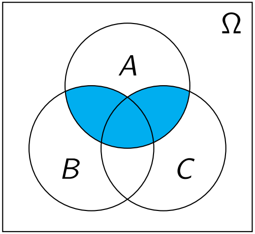
$A \cap (B \cup C)$



$A \cap B$

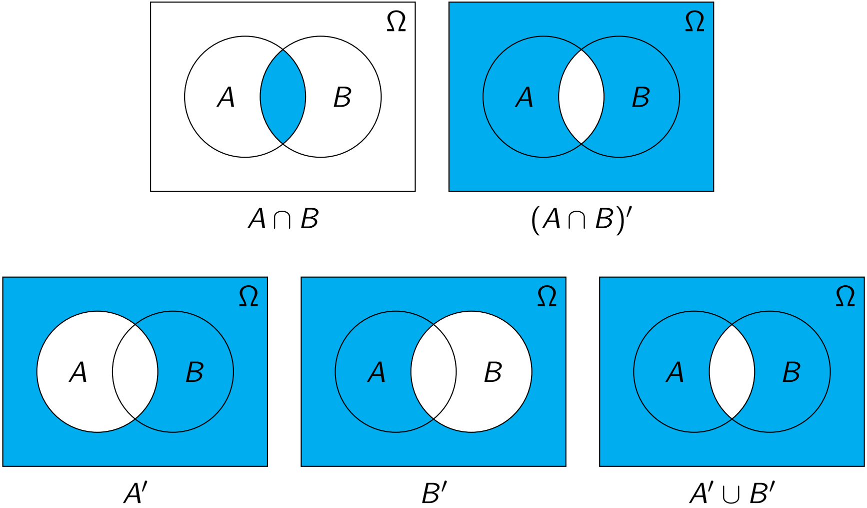


$A \cap C$

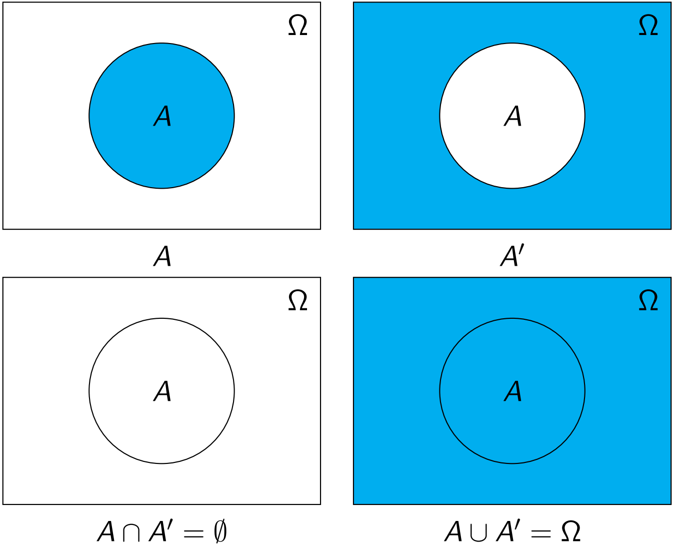


$(A \cap B) \cup (A \cap C)$

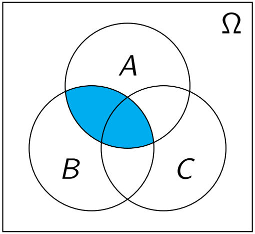
"Proof" that $(A \cap B)' = A' \cup B'$



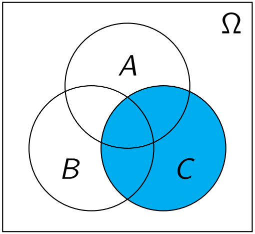
"Proof" that $A \cap A' = \emptyset$ and $A \cup A' = \Omega$



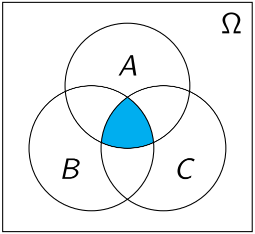
"Proof" that $(A \cap B) \cap C = A \cap (B \cap C)$



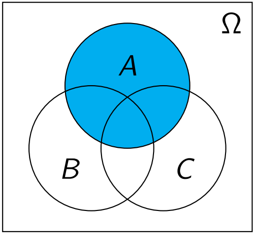
$A \cap B$



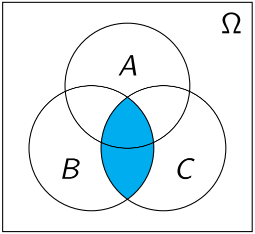
C



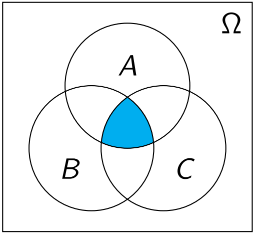
$(A \cap B) \cap C$



A



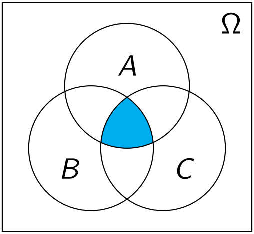
$B \cap C$



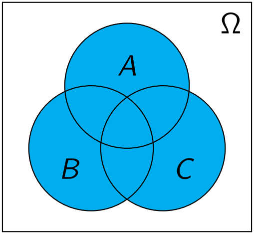
$A \cap (B \cap C)$

Multiple Intersections and Unions

Since $(A \cap B) \cap C = A \cap (B \cap C)$, we don't need to use parentheses when writing the intersection of three or more events; we can simply write $A \cap B \cap C$. A similar statement applies to unions.



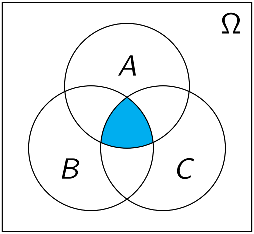
$A \cap B \cap C$



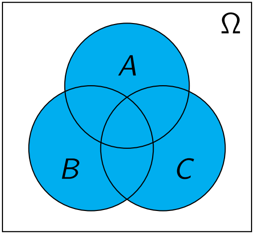
$A \cup B \cup C$

Multiple Intersections and Unions

Since $(A \cap B) \cap C = A \cap (B \cap C)$, we don't need to use parentheses when writing the intersection of three or more events; we can simply write $A \cap B \cap C$. A similar statement applies to unions.



$A \cap B \cap C$



$A \cup B \cup C$

Caution: $A \cap (B \cup C)$ is *not* the same as $(A \cap B) \cup C$. Parentheses must still be used to distinguish these.

DeMorgan's Laws

The following two identities are known as DeMorgan's laws:

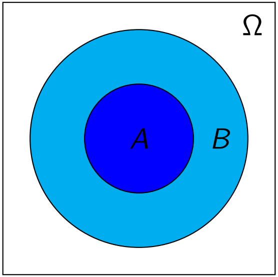
$$(A \cap B)' = A' \cup B'$$
$$(A \cup B)' = A' \cap B'$$

They also extend to three or more events:

$$(A \cap B \cap C)' = A' \cup B' \cup C'$$
$$(A \cup B \cup C)' = A' \cap B' \cap C'$$

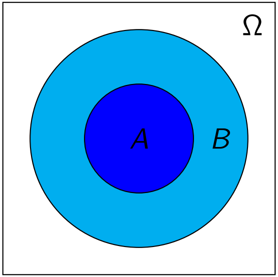
Containment

Given events A and B, if every outcome in A is also in B, then we say that A is **contained** in B, and we write $A \subseteq B$.



Containment

Given events A and B, if every outcome in A is also in B, then we say that A is **contained** in B, and we write $A \subseteq B$.



For example, if we roll a six-sided die, and let A be the event of getting a 5 or higher and B be the event of getting a 3 or higher, then $A \subseteq B$:

$$A = \{5, 6\} \subseteq \{3, 4, 5, 6\} = B$$

Properties of Containment

The following properties hold for any events A, B, and C:

1. $A \subseteq A$.
2. $\emptyset \subseteq A$ and $A \subseteq \Omega$.
3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
4. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
5. If $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$.
6. $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
7. $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
8. If $A \subseteq B$, then $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$.
9. If $A \subseteq B$, then $B' \subseteq A'$.

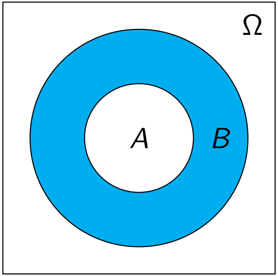
Again, you don't need to memorize these.

Difference of Two Events

Given events A, B with $A \subseteq B$, we define their **difference** $B - A$ as the set of all outcomes of B which are not in A. In other words,

$$B - A = B \cap A'$$

This may be depicted using a Venn diagram:



$B - A$

Interpretations of Probability

What is “probability”?

What is “probability”?

Under the **objective** interpretation, the probability of an event A is the proportion of times that A occurs when the experiment is repeated many times.

Interpretations of Probability

What is "probability"?

Under the **objective** interpretation, the probability of an event A is the proportion of times that A occurs when the experiment is repeated many times.

- For example, in tossing a coin, we say that the probability of getting heads is .5.
- This only makes sense if the experiment can be repeated.

Interpretations of Probability

What is "probability"?

Under the **objective** interpretation, the probability of an event A is the proportion of times that A occurs when the experiment is repeated many times.

- For example, in tossing a coin, we say that the probability of getting heads is .5.
- This only makes sense if the experiment can be repeated.

Under the **subjective** interpretation, probability is a measure of how likely a statement is to be true.

- For example, someone says, "There is a 70% chance the Utes will beat Arizona this year."

Interpretations of Probability

What is "probability"?

Under the **objective** interpretation, the probability of an event A is the proportion of times that A occurs when the experiment is repeated many times.

- For example, in tossing a coin, we say that the probability of getting heads is .5.
- This only makes sense if the experiment can be repeated.

Under the **subjective** interpretation, probability is a measure of how likely a statement is to be true.

- For example, someone says, "There is a 70% chance the Utes will beat Arizona this year."
- Different people may have different opinions and hence may assign different probabilities to the same statements.

Axioms of Probability

The same mathematical theory applies to various interpretations of probability. Recall that given a set Ω of possible outcomes, subsets of Ω are called **events**. We use the notation $P(A)$ to represent the probability of an event A, and we assume that probabilities satisfy the following axioms:

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. If $A_1, A_2, A_3, \ldots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Equally Likely Outcomes

Given a finite set Ω of outcomes, one of the simplest ways to define a probability measure is to assume that the outcomes are **equally likely**:

$$P(A) = \frac{\#A}{\#\Omega} = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}$$

Equally Likely Outcomes

Given a finite set Ω of outcomes, one of the simplest ways to define a probability measure is to assume that the outcomes are **equally likely**:

$$P(A) = \frac{\#A}{\#\Omega} = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}$$

Example

Suppose we roll a fair six-sided die. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $A = \{2, 4, 6\}$ be the event that we roll an even number. Then

$$P(A) = \frac{\#A}{\#\Omega} = \frac{3}{6} = \frac{1}{2} = .5$$

Equally Likely Outcomes

Given a finite set Ω of outcomes, one of the simplest ways to define a probability measure is to assume that the outcomes are **equally likely**:

$$P(A) = \frac{\#A}{\#\Omega} = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}$$

Example

Suppose we roll a fair six-sided die. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $A = \{2, 4, 6\}$ be the event that we roll an even number. Then

$$P(A) = \frac{\#A}{\#\Omega} = \frac{3}{6} = \frac{1}{2} = .5$$

Caution: It is not always reasonable to assume the outcomes are equally likely.

Example – Sum of Two Dice

Problem

Suppose we roll two fair six-sided dice. What is the probability that they add up to 8?

Example – Sum of Two Dice

Problem

Suppose we roll two fair six-sided dice. What is the probability that they add up to 8?

There are 6 possibilities for the first roll and 6 possibilities for the second roll, so there are $6 \times 6 = 36$ equally likely outcomes:

$$\Omega = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(6,6)\}$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Out of these 36 possibilities, in 5 of them the dice add up 8. Therefore the probability that the dice add up to 8 is

$$P(A) = \frac{\#A}{\#\Omega} = \frac{5}{36} \approx .139$$

Properties of Probability

The following properties of probability can be derived from the axioms:

1. $P(\emptyset) = 0$, $P(\Omega) = 1$.
2. If $A_1, \ldots, A_n$ are disjoint events, then

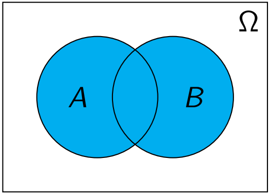
$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

3. If $A \subseteq B$ then $P(B - A) = P(B) - P(A)$.
4. $A \subseteq B$ then $P(A) \leq P(B)$.
5. $0 \leq P(A) \leq 1$, for all events A.
6. $P(A') = 1 - P(A)$ for any event A.

Probability of a Union of Two Events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A, B.

Intuition: To find the probability of $A \cup B$, we add the probabilities of A and B, but then we have double counted the intersection $A \cap B$, so we have to subtract that.



$A \cup B$

Probability of a Union of Two Events

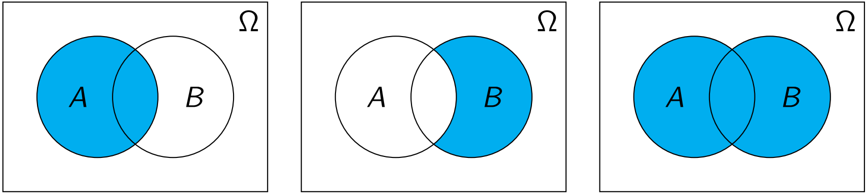
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A, B.

Proof: We may write $A \cup B$ as a union of two disjoint events:

$$A \cup B = A \cup (B - (A \cap B))$$

Therefore,

$$\begin{aligned} P(A \cup B) &= P(A \cup (B - (A \cap B))) \\ &= P(A) + P(B - (A \cap B)) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

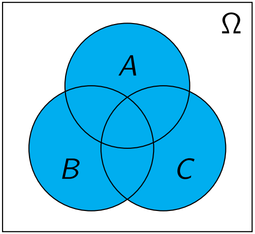


A $\qquad$ $B - (A \cap B)$ $\qquad$ $A \cup B$

Probability of a Union of Three Events

$$
\begin{aligned}
P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\
& - P(A \cap B) - P(A \cap C) - P(B \cap C) \\
& + P(A \cap B \cap C)
\end{aligned}
$$

Intuition: To find the probability of $A \cup B \cup C$, we add the probabilities of the events A, B, and C and subtract the overlap of each pair; but then we've subtracted the three-way overlap $A \cap B \cap C$ one too many times, so we add it back.

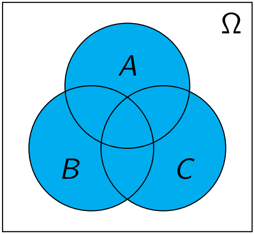


$A \cup B \cup C$

Probability of a Union of Three Events

$$
\begin{aligned}
P(A \cup B \cup C) = P(A) + P(B) + P(C) \\
- P(A \cap B) - P(A \cap C) - P(B \cap C) \\
+ P(A \cap B \cap C)
\end{aligned}
$$

Intuition: To find the probability of $A \cup B \cup C$, we add the probabilities of the events A, B, and C and subtract the overlap of each pair; but then we've subtracted the three-way overlap $A \cap B \cap C$ one too many times, so we add it back.



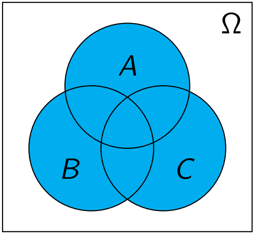
$A \cup B \cup C$

Extra credit 1: Prove this identity using only algebra, set-theoretic identities, and the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Probability of a Union of Three Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Intuition: To find the probability of $A \cup B \cup C$, we add the probabilities of the events A, B, and C and subtract the overlap of each pair; but then we've subtracted the three-way overlap $A \cap B \cap C$ one too many times, so we add it back.



$A \cup B \cup C$

Extra credit 1: Prove this identity using only algebra, set-theoretic identities, and the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Extra credit 2: Use this identity to solve problem 2.25 from the textbook.

Examples

Problems 2.12, 2.13, and 2.14.

Independence of Two Events

Definition

Events A and B are **independent** if $P(A \cap B) = P(A)P(B)$.

Independence of Two Events

Definition

Events A and B are **independent** if $P(A \cap B) = P(A)P(B)$.

Example

If we roll two fair six-sided dice and define events

$$A = \text{first die is a six}$$
$$B = \text{second die is a six}$$

Then A and B are independent, $P(A) = 1/6$, $P(B) = 1/6$, so

$$P(\text{both dice are sixes}) = P(A \cap B) = P(A)P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Dependence of Two Events

Example

If we roll a fair six-sided dice and define events

$$A = \text{the roll is a 4 or lower}$$
$$B = \text{the roll is a 3 or higher}$$

Then $P(A) = P(B) = \frac{4}{6} = \frac{2}{3}$ but

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3} \neq \frac{2}{3} \cdot \frac{2}{3} = P(A)P(B)$$

So these two events are *not* independent. We say that they are **dependent**.

Extra credit 3: Based only on the outcome of a single roll of a fair six-sided die, find two events which are independent.

Example

Problem

If we roll two fair six-sided dice, what is the probability that at least one of them is a six?

Define events

$$A = \text{first die is a 6}$$
$$B = \text{second die is a 6}$$

Example

Problem

If we roll two fair six-sided dice, what is the probability that at least one of them is a six?

Define events

$$A = \text{first die is a 6}$$
$$B = \text{second die is a 6}$$

Then

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \\ &= \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \\ &= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36} \approx .305 \end{aligned}$$

Independence of Several Events

Definition

Events $A_1, \ldots, A_n$ are **independent** if

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

for all choices of indices $1 \le i_1 < i_2 < \cdots < i_k \le n$

Independence of Several Events

Definition

Events $A_1, \ldots, A_n$ are **independent** if

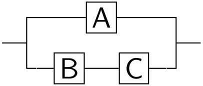
$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

for all choices of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

For example, saying three events A, B, C are independent means

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B)P(C) \\ P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

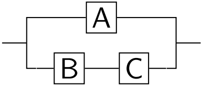
Example – Three-Component System



$P(\mathrm{A}) = .7$
$P(\mathrm{B}) = .4$
$P(\mathrm{C}) = .9$

Assumed that failures in three components are independent of each other. Then, letting A, B, C be the events that components A, B, and C work, respectively, we can solve the problem using the identity $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$:

Example – Three-Component System

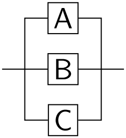


$P(\text{A}) = .7$
$P(\text{B}) = .4$
$P(\text{C}) = .9$

Assumed that failures in three components are independent of each other. Then, letting A, B, C be the events that components A, B, and C work, respectively, we can solve the problem using the identity $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$:

$$\begin{aligned} P(\text{system works}) &= P(A \cup (B \cap C)) \\ &= P(A) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A) + P(B)P(C) - P(A)P(B)P(C) \\ &= .7 + (.4)(.9) - (.7)(.4)(.9) = .808 \end{aligned}$$

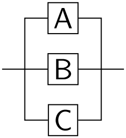
A Different Three-Component System



$P(\mathrm{A}) = .7$
$P(\mathrm{B}) = .4$
$P(\mathrm{C}) = .9$

Now consider a new system where all three components are arranged in parallel, so that the system works as long as at least one of the three components does.

A Different Three-Component System



$P(\text{A}) = .7$
$P(\text{B}) = .4$
$P(\text{C}) = .9$

Now consider a new system where all three components are arranged in parallel, so that the system works as long as at least one of the three components does.

$$\begin{aligned} P(\text{system works}) &= P(A \cup B \cup C) \\ &= 1 - P((A \cup B \cup C)') \\ &= 1 - P(A' \cap B' \cap C') \\ &= 1 - P(A')P(B')P(C') = 1 - (.3)(.6)(.1) = .982 \end{aligned}$$

This again assumes that the components work or fail independently of each other.

Conditional Probability

Definition

Let B be an event with $P(B) > 0$. The **conditional probability** that an event A occurs, given that B occurs, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For example, if we roll a fair six-sided die and let

$$A = \text{we roll an even number} = \{2, 4, 6\}$$
$$B = \text{we roll a 4 or higher} = \{4, 5, 6\}$$

Then the conditional probability that we roll an even number, *given* that we roll a 4 or higher, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3} \approx .667$$

Conditional Probability

Problem

Suppose we roll two fair six-sided dice. What is the probability that one of the dice is a 6, given that the two dice sum to 8?

Conditional Probability

Problem

Suppose we roll two fair six-sided dice. What is the probability that one of the dice is a 6, given that the two dice sum to 8?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$A =$ one of the dice is a 6

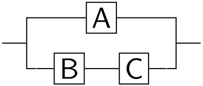
$B =$ dice sum to 8

$$P(B) = \frac{5}{36}$$

$$P(A \cap B) = \frac{2}{36}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$$

Example – Three-Component System



$P(\text{A}) = .7$

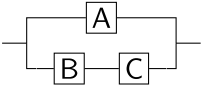
$P(\text{B}) = .4$

$P(\text{C}) = .9$

Problem

Suppose the above three-component system has failed. What is the probability that component B has failed?

Example – Three-Component System



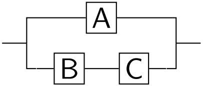
$P(\text{A}) = .7$
$P(\text{B}) = .4$
$P(\text{C}) = .9$

Problem

Suppose the above three-component system has failed. What is the probability that component B has failed?

$$P(\text{system fails}) = 1 - P(\text{system works}) = 1 - .808 = .192$$

Example – Three-Component System



$$P(\mathrm{A}) = .7$$
$$P(\mathrm{B}) = .4$$
$$P(\mathrm{C}) = .9$$

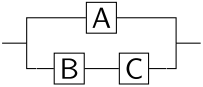
Problem

Suppose the above three-component system has failed. What is the probability that component B has failed?

$$P(\text{system fails}) = 1 - P(\text{system works}) = 1 - .808 = .192$$

$$\begin{aligned} P(B' \cap \text{system fails}) &= P(B' \cap (A \cup (B \cap C))') \\ &= P(B' \cap A' \cap (B' \cup C')) \\ &= P(B' \cap A') = P(B')P(A') = (.6)(.3) = .18 \end{aligned}$$

Example – Three-Component System



$$P(\text{A}) = .7$$
$$P(\text{B}) = .4$$
$$P(\text{C}) = .9$$

Problem

Suppose the above three-component system has failed. What is the probability that component B has failed?

$$P(\text{system fails}) = 1 - P(\text{system works}) = 1 - .808 = .192$$

$$\begin{aligned} P(B' \cap \text{system fails}) &= P(B' \cap (A \cup (B \cap C))') \\ &= P(B' \cap A' \cap (B' \cup C')) \\ &= P(B' \cap A') = P(B')P(A') = (.6)(.3) = .18 \end{aligned}$$

$$P(B'|\text{system fails}) = \frac{P(B' \cap \text{system fails})}{P(\text{system fails})} = \frac{.18}{.192} = .9375$$

Independence and Conditional Probability

Assume A and B are events with $P(B) > 0$.

A and B are independent if and only if $P(A|B) = P(A)$.

Independence and Conditional Probability

Assume A and B are events with $P(B) > 0$.

A and B are independent if and only if $P(A|B) = P(A)$.

In other words, saying A and B are independent means that knowing B occurs doesn't change the probability that A occurs.

Independence and Conditional Probability

Assume A and B are events with $P(B) > 0$.

A and B are independent if and only if $P(A|B) = P(A)$.

In other words, saying A and B are independent means that knowing B occurs doesn't change the probability that A occurs.

Proof: By the definition of conditional probability,

$$\begin{aligned} P(A|B) = P(A) &\iff \frac{P(A \cap B)}{P(B)} = P(A) \\ &\iff P(A \cap B) = P(A)P(B) \\ &\iff A \text{ and } B \text{ are independent} \end{aligned}$$

Multiplication Rule

Assume A is an event with $P(A) > 0$.

$$P(A \cap B) = P(A)P(B|A)$$

Multiplication Rule

Assume A is an event with $P(A) > 0$.

$$P(A \cap B) = P(A)P(B|A)$$

Proof: By the definition of conditional probability,

$$P(A)P(B|A) = P(A)\frac{P(A \cap B)}{P(A)} = P(A \cap B)$$

Example

A standard deck of 52 cards contains 4 aces. If you draw two cards at random, what is the probability that both will be aces?

Example

A standard deck of 52 cards contains 4 aces. If you draw two cards at random, what is the probability that both will be aces?

Define events

$$A = \text{first card is an ace}$$
$$B = \text{second card is an ace}$$

Clearly $P(A) = 4/52$. Then, given that the first card is an ace, there are only 3 aces among the 51 remaining cards, so $P(B|A) = 3/51$. Therefore,

$$P(A \cap B) = P(A)P(B|A) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221} \approx .0045$$

Multiplication Rule for Three Events

The multiplication rule may be extended to three events:

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Multiplication Rule for Three Events

The multiplication rule may be extended to three events:

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Proof: By the multiplication rule for two events,

$$\begin{aligned} P(A \cap B \cap C) &= P((A \cap B) \cap C) \\ &= P(A \cap B)P(C|A \cap B) \\ &= P(A)P(B|A)P(C|A \cap B) \end{aligned}$$

Multiplication Rule for Three Events

The multiplication rule may be extended to three events:

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Proof: By the multiplication rule for two events,

$$\begin{aligned} P(A \cap B \cap C) &= P((A \cap B) \cap C) \\ &= P(A \cap B)P(C|A \cap B) \\ &= P(A)P(B|A)P(C|A \cap B) \end{aligned}$$

Example: Suppose we draw three cards at random. Let A, B, C be the events that the first, second, and third cards are aces, respectively. Then

$$P(A) = 4/52, P(B|A) = 3/51, P(C|A \cap B) = 2/50$$

Therefore, by the multiplication rule,

$$P(A \cap B \cap C) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{1}{5525}$$

Law of Total Probability

Let $A_1, \ldots, A_n$ be disjoint events with $\Omega = A_1 \cup \cdots \cup A_n$. Then for any event B,

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

Law of Total Probability

Let $A_1, \ldots, A_n$ be disjoint events with $\Omega = A_1 \cup \cdots \cup A_n$. Then for any event B,

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

Proof: By the multiplication rule,

$$\begin{aligned}
\sum_{i=1}^{n} P(A_i)P(B|A_i) &= \sum_{i=1}^{n} P(A_i \cap B) \\
&= P(A_1 \cap B) + \cdots + P(A_n \cap B) \\
&= P((A_1 \cap B) \cup \cdots \cup (A_n \cap B)) \\
&= P((A_1 \cup \cdots \cup A_n) \cap B) \\
&= P(\Omega \cap B) = P(B)
\end{aligned}$$

Example

A manufacturer produces widgets using two machines in parallel: 70% of widgets are produced by machine A; the rest are produced by machine B. Of the widgets produced by machine A, 3% are defective; of those produced by machine B, 8% are defective. If a random widget is chosen, what is the probability it is defective?

Example

A manufacturer produces widgets using two machines in parallel: 70% of widgets are produced by machine A; the rest are produced by machine B. Of the widgets produced by machine A, 3% are defective; of those produced by machine B, 8% are defective. If a random widget is chosen, what is the probability it is defective?

Solution: Define events

$$\begin{aligned} A &= \text{the widget is produced by machine A} \\ B &= \text{the widget is produced by machine B} \\ D &= \text{the widget is defective} \end{aligned}$$

Then the given information may be expressed,

$$P(A) = .7, P(B) = .3, P(D|A) = .03, P(D|B) = .08$$

Example

A manufacturer produces widgets using two machines in parallel: 70% of widgets are produced by machine A; the rest are produced by machine B. Of the widgets produced by machine A, 3% are defective; of those produced by machine B, 8% are defective. If a random widget is chosen, what is the probability it is defective?

Solution: Define events

$$\begin{aligned} A &= \text{the widget is produced by machine A} \\ B &= \text{the widget is produced by machine B} \\ D &= \text{the widget is defective} \end{aligned}$$

Then the given information may be expressed,

$$P(A) = .7, P(B) = .3, P(D|A) = .03, P(D|B) = .08$$

The law of total probability gives

$$P(D) = P(A)P(D|A) + P(B)P(D|B) = (.7)(.03) + (.3)(.08) = .045$$

Bayes' Theorem

How are $P(A|B)$ and $P(B|A)$ related?

Bayes' Theorem

How are $P(A|B)$ and $P(B|A)$ related?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \qquad \text{(assuming } P(A) > 0, P(B) > 0\text{)}$$

Bayes' Theorem

How are $P(A|B)$ and $P(B|A)$ related?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \qquad \text{(assuming } P(A) > 0, P(B) > 0\text{)}$$

Proof: By the multiplication rule,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' Theorem

How are $P(A|B)$ and $P(B|A)$ related?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \qquad \text{(assuming } P(A) > 0, P(B) > 0\text{)}$$

Proof: By the multiplication rule,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Example: In the previous problem, if a randomly selected widget is defective, what is the probability it was produced by machine A?

Bayes' Theorem

How are $P(A|B)$ and $P(B|A)$ related?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \qquad \text{(assuming } P(A) > 0, P(B) > 0\text{)}$$

Proof: By the multiplication rule,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Example: In the previous problem, if a randomly selected widget is defective, what is the probability it was produced by machine A?

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{(.03)(.7)}{.045} \approx .467$$

Example – Medical Testing

A rare disease affects 1 in 1000 people. A person with the disease tests positive 99% of the time, whereas a person without the disease tests positive only 2% of the time.

Problem

If a randomly selected person tests positive, what is the probability that they have the disease?

Example – Medical Testing

A rare disease affects 1 in 1000 people. A person with the disease tests positive 99% of the time, whereas a person without the disease tests positive only 2% of the time.

Problem

If a randomly selected person tests positive, what is the probability that they have the disease?

The given information may be expressed,

$$P(D) = .001, P(T|D) = .99, P(T|D') = .02$$

The law of total probability implies

$$\begin{aligned} P(T) &= P(T|D)P(D) + P(T|D')P(D') \\ &= (.99)(.001) + (.02)(.999) = .02097 \end{aligned}$$

Bayes' theorem then implies

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{(.99)(.001)}{.02097} \approx .047$$

Summary

- Set-theory notation: $A \cap B$, $A \cup B$, A'
- Venn diagrams
- For any A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$.
- If A and B are independent, $P(A \cap B) = P(A)P(B)$.
- Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
- Multiplication rule: $P(A \cap B) = P(A)P(B|A)$
- Bayes' Theorem: $P(B|A) = \dfrac{P(A|B)P(B)}{P(A)}$
- Law of Total Probability: If $A_1, \ldots, A_n$ are disjoint with $\Omega = A_1 \cup \cdots \cup A_n$, then $P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$.