何典峻 12010207

7.9 5、9、13

5.

$$x' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$$|A - \lambda I| = 0$$

$$(2-\lambda)(-2-\lambda)+3=0 \qquad \lambda^2 = 1 \qquad \therefore \lambda = \pm 1$$

$$(A-\lambda I)\xi = 0 \qquad \therefore \xi_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore S = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix} \qquad S^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}$$

let $\vec{x} = S\vec{y}$

$$\therefore (S\vec{y})' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \cdot S\vec{y} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot e^t$$

$$\vec{y}' = S^{-1} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} S\vec{y} + S^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$$\vec{y}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{y} + \left(-\tfrac{1}{2}\right) \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix} e^t$$

$$\begin{cases} y_1' = y_1 + e^t & y_1' - y_1 = e^t \\ y_2' = -y_2 - 2e^t & y_2' + y_2 = -2e^t \end{cases}$$

$y_1 = te^t \qquad y_2 = - e^t$

$$\therefore y = \begin{pmatrix} te^t \\ -e^t \end{pmatrix} \quad \therefore x = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} te^t \\ -e^t \end{pmatrix}$$

$$x_p = \begin{pmatrix} (3t-1)e^t \\ (1-t)e^t \end{pmatrix}$$

$\therefore$ Genaral Solution. $x = x_h + x_p = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 3t-1 \\ 1-t \end{pmatrix} e^t$

9:

$$t x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x + \begin{pmatrix} -2t \\ t^4-1 \end{pmatrix}, \quad x^{(c)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2$$

$$t x^{(c)\prime} = -c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + 2c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 \quad (1)$$

$$\because \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \cdot x^{(c)} = c_1 \cdot t^{-1} \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 t^2 \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= c_1 t^{-1} \cdot \begin{pmatrix} -1 \\ -2 \end{pmatrix} + c_2 t^2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (2)$$

$\because$ (1)式 = (2)式 $\therefore$ $x^{(i)}$是 solution of homogeous system

$$\therefore \psi(t) = \begin{bmatrix} -t^{-1} & 4t^2 \\ -2t^{-1} & 2t^2 \end{bmatrix}$$

令 $v(t) = \psi(t) \cdot \vec{u}(t)$

$$\because t \cdot v'(t) = A v(t) + \begin{pmatrix} -2t \\ t^4 - 1 \end{pmatrix}$$

$$t \psi'(t) \vec{u}(t) + t \psi(t) \cdot u'(t) = A \cdot \psi(t) u(t) + \begin{pmatrix} -2t \\ t^4 - 1 \end{pmatrix}$$

$$t \psi'(t) \cdot \vec{u}(t) = A \cdot \psi(t) \, u(t)$$

($\because \psi(t)$是 homogenous solution)

$$t \cdot \psi(t) u'(t) = \begin{pmatrix} -2t \\ t^4 - 1 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 4t^3 \\ -2 & 2t^3 \end{bmatrix} \cdot \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -2t \\ t^4 - 1 \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{6t^3}\begin{bmatrix} 2t^3 & -4t^3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2t \\ t^4-1 \end{bmatrix}$$

$$= \frac{1}{6t^3}\begin{bmatrix} -4t^4-4t^7+4t^3 \\ -4t-t^4+1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3}t-\frac{2}{3}t^3+\frac{2}{3} \\ -\frac{2}{3}t^{-2}-\frac{1}{6}t+\frac{1}{6}t^{-3} \end{bmatrix}$$

$$\therefore u(t) = \begin{bmatrix} -\frac{1}{3}t^2-\frac{1}{6}t^4+\frac{2}{3}t \\ \frac{2}{3}t^{-1}-\frac{1}{12}t^2-\frac{1}{12}t^{-2} \end{bmatrix}$$

$$\therefore v(t) = \Psi(t)\cdot u(t) = \cdots$$

13.

(a) $x' = Ax + g(t) \qquad x(0) = x^0$

$\Phi(t-s) = \Phi(t)\,\Phi^{-1}(s)$

homo: $x = \Phi(t)\,x^0$

$$\left(\int_0^t \Phi(t-s)\, g(s)\, ds \right)'$$

$''$

$\Phi(0)\, g(t) = g(t) \cdot 1$

$\because x' = Ax + g(x)$

$\therefore x = \Phi(t)x^0 + g(x)$

5. equal to show that.

$$e^{At} = \Phi(t)$$

91: 1. 4. 7 12. 15. 18. 19.

1 $\frac{dx}{dt} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} x$

(a) $|A - \lambda I| = 0$

$(3-\lambda)(-2-\lambda)+4=0$

$\lambda^2 - \lambda - 2 = 0 \quad (\lambda-2)(\lambda+1)=0$

$\therefore \lambda = 2$ 或 -1

$r_1 \qquad r_2$

$(A-\lambda I)\xi = 0$

$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}\xi_1 = 0 \quad \therefore \xi_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix}\xi_2 = 0$

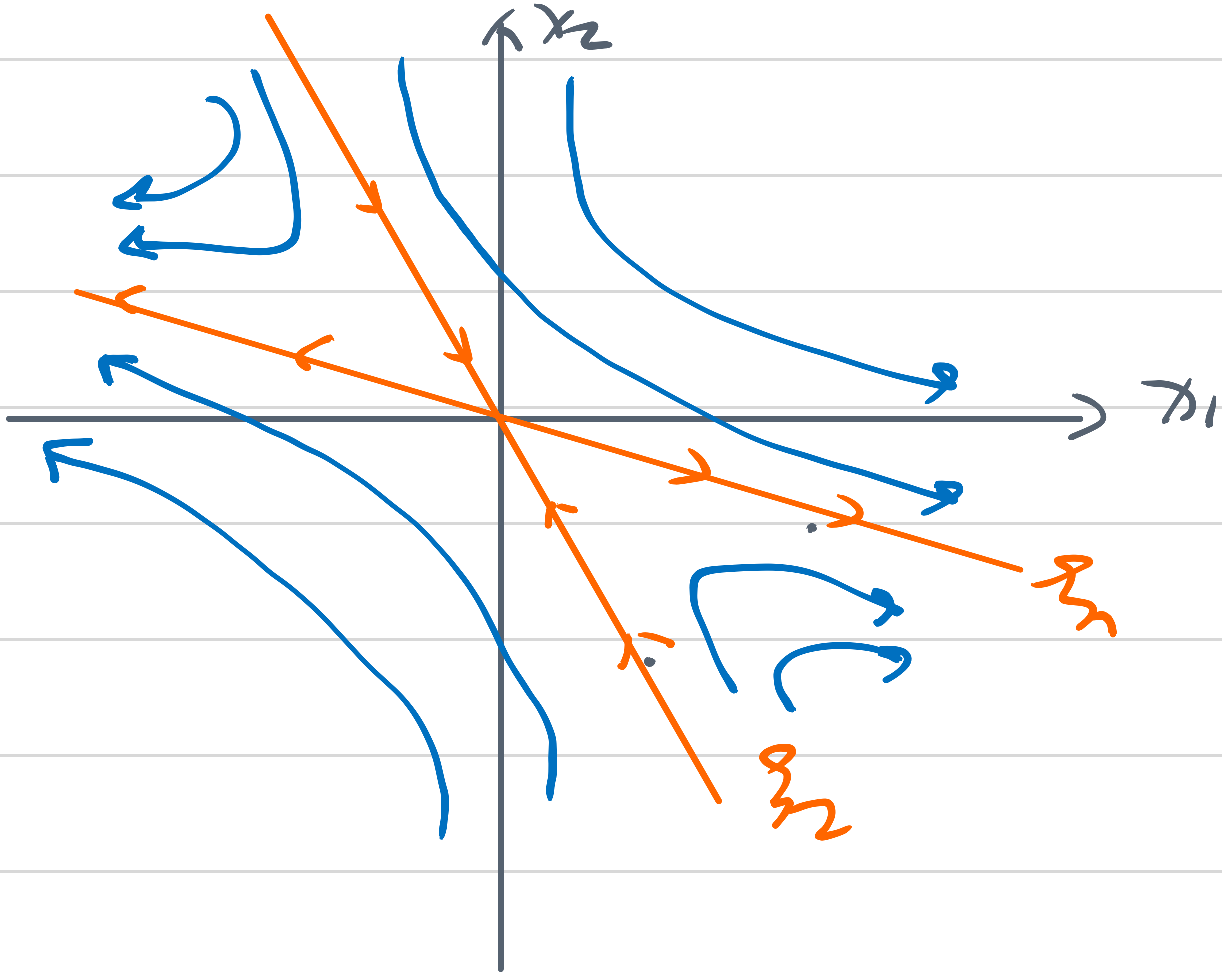
$\therefore \xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(b):

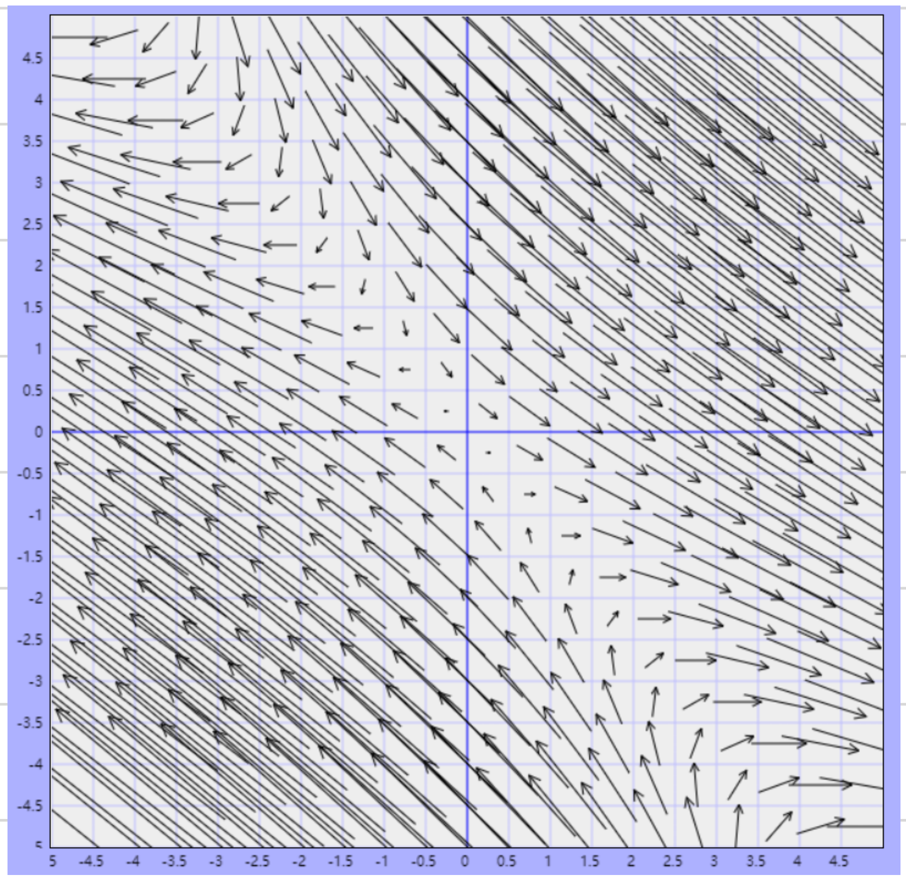
$\because \quad r_2 < 0 < r_1$

$\therefore$ Critical Point $\Longrightarrow$ Saddle Point $\Longrightarrow$ Unstable

(c). $\xi_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad x = c_1 \cdot e^{2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$



(d).



4: $\frac{dx}{dt} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \cdot x$

(a).

$|A - \lambda I| = 0$ $\quad$ $(1-\lambda)(-7-\lambda)+16=0$

$\lambda^2 + 6\lambda + 9 = 0$

$\lambda = -3, -3$

$(A - \lambda I)\vec{\xi} = 0$

$\begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} \vec{\xi} = 0$ $\quad$ $\therefore \vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} \vec{\eta} = \vec{\xi}$

$$\begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} \vec{\eta} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

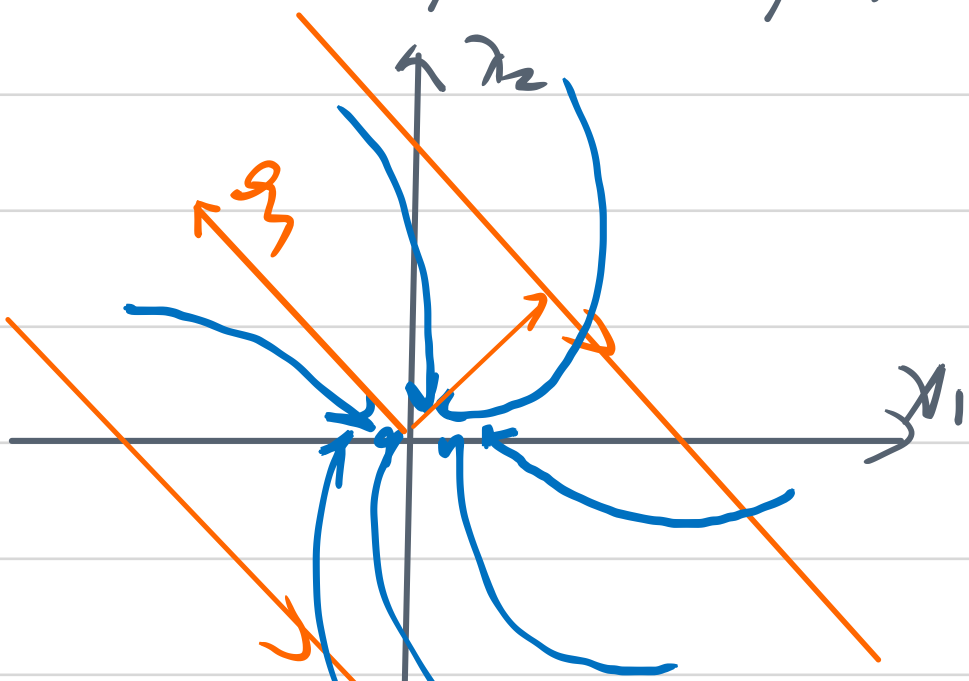
$$\therefore \vec{\eta} = \begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \end{pmatrix}$$

$\therefore \vec{\xi}, \vec{\eta}$ independent

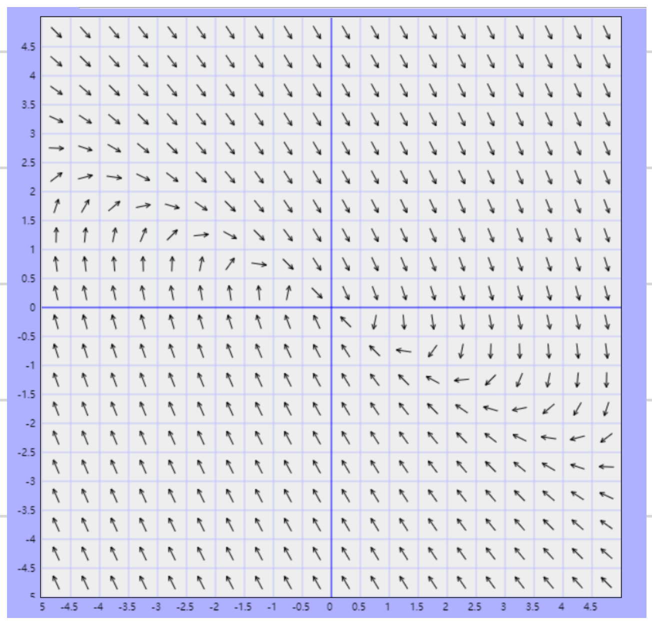
(b).

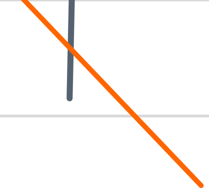
critical point $\Rightarrow$ proper node $\Rightarrow$ Asymptotically stable

(c)



(d).





7: $\frac{dx}{dt} = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} x$

(a)

$|A - \lambda I| = 0 \Rightarrow (3-\lambda)(-1-\lambda)+8=0$

$\lambda^2 - 2\lambda + 5 = 0$

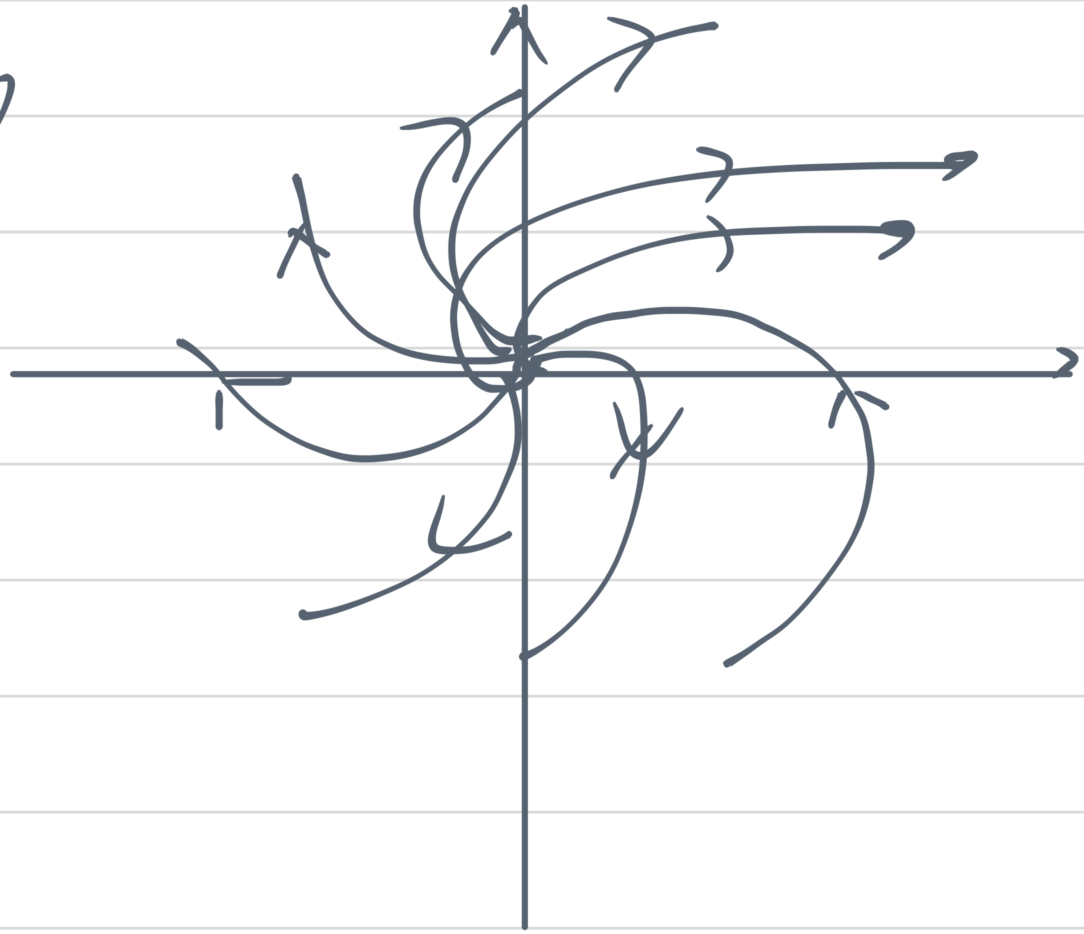
$\lambda = \frac{2 \pm 2i}{2}$

$= 1 \pm i$

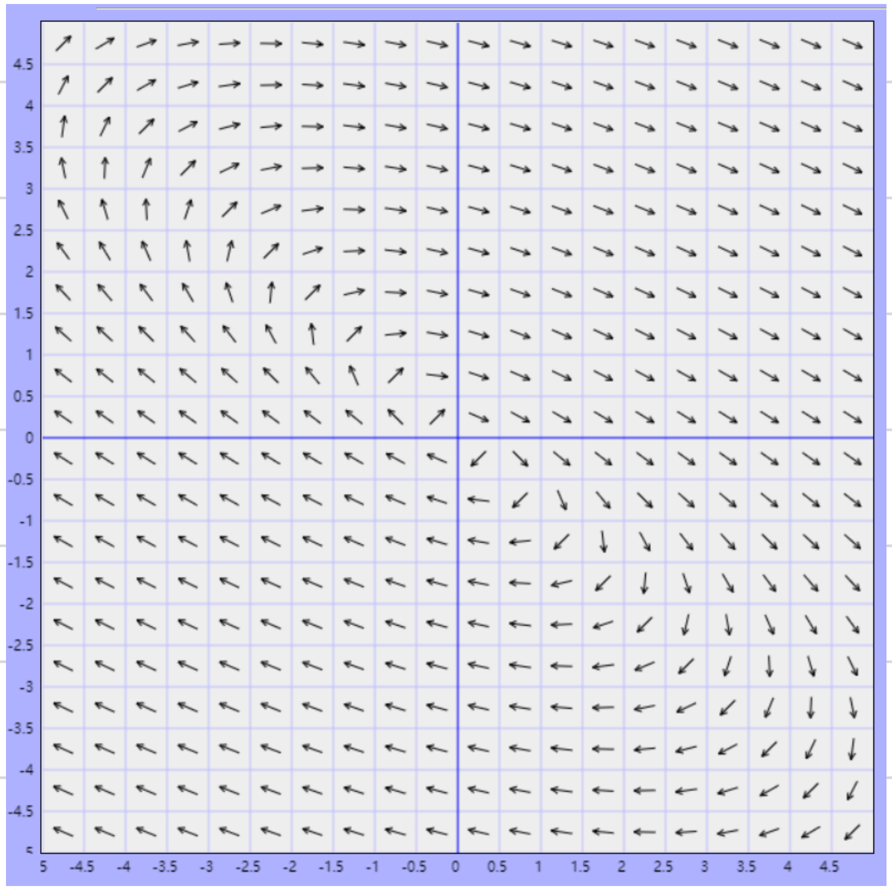
(b). $r_1, r_2 = 1 \pm i$

$\because \lambda > 0$ $\therefore$ spiral point $\Rightarrow$ Unstable

(c)



(d).



12:
$$\frac{dx}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 0$$

$$x = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = x^0 \quad ?$$

15:

$$m\frac{du^2}{dt^2}+c\frac{du}{dt}+ku=0$$

$$\begin{cases} my'+cy+kx=0 & (1) \\ my'+cx'+kx=0 \end{cases} \Rightarrow cx'-cy=0 \quad (2)$$

$(1)+(2)$

$$\begin{cases} x'=0x+y \\ y'=-\frac{k}{m}x-\frac{c}{m}y \end{cases}$$

$$\vec{x}'=\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\vec{x} \qquad \vec{x}=\begin{pmatrix} x \\ y \end{pmatrix}$$

when. $x=0, y=0$, $\vec{x}=0 \Rightarrow$ critical point

$$(-\lambda)\left(-\frac{c}{m}-\lambda\right)+\frac{k}{m}=0$$

$$\lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0$$

1. $\left(\frac{c}{m}\right)^2-\frac{4k}{m}>0$ $\qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$-\frac{c}{m}+\sqrt{\left(\frac{c}{m}\right)^2-\frac{4k}{m}}>0 \Rightarrow \lambda_1<0<\lambda_2 \Rightarrow$ Saddle point. Unstable

$-\frac{c}{m}+\sqrt{\left(\frac{c}{m}\right)^2-\frac{4k}{m}}<0 \Rightarrow \lambda_1<\lambda_2<0 \Rightarrow$ Node, Asymptotically stable

2: $\left(\frac{c}{m}\right)^2 - \frac{4k}{m} = 0 \quad \because c, m > 0$

$\therefore \lambda_1 = \lambda_2 < 0 \quad = -\frac{c}{2m}$

$$\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \qquad \begin{bmatrix} \frac{c}{2m} & 1 \\ -\frac{k}{m} & -\frac{c}{2m} \end{bmatrix} \xi_\lambda = 0$$

① $-\frac{k}{m} = \frac{c^2}{4m^2}$

Proper $\xrightarrow{\text{mode}}$ Asympatically seable

② $-\frac{k}{m} \neq \frac{c}{4m^2}$

improper mode $\Rightarrow$ Asympotially seable

$\}: \left(\frac{c^2}{m^2}\right)^2 - \frac{4k}{m} < 0$

$\because \frac{-c}{2m} < 0$

$\therefore$ Spiral point $\Rightarrow$ Asymptotically Stable

18

a:

$(a_{11}-\lambda)(a_{22}-\lambda) - a_{21} \cdot a_{12} = 0$

$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{21} \cdot a_{12} = 0$

$\lambda = \frac{(a_{11}+a_{22}) \pm \sqrt{(_{11}+_{22})^2 - 4[(_{11}\,_{22}) - (_{21}\,_{12})]}}{2}$

if $\begin{cases} a_{11}+a_{22} = 0 \\ a_{11}+a_{22} - (a_{21}+a_{12}) > 0 \end{cases}$ then $\lambda =$ pure imaginary

b) (23) : $a_{11}+a_{22}=0$

(24) : $(a_{11}x+a_{12}y)\cdot y'+[(a_{21}x+a_{22}y)\cdot x]'=0$

if $$\frac{d(a_{11}x+a_{12}y)}{dx}=\frac{d[(-a_{21}x-a_{22}y)x]}{dy} \quad (1)$$

$\longrightarrow$ exact

(1) $\Rightarrow$ $a_{11}=-a_{22}$ (2)

∵已知 : $a_{11}+a_{22}=0$

∴ (2)式得证

(6):

(24). $\dfrac{dy}{dx} = \dfrac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}$

$$a_{21}x^2 + 2a_{22}xy - a_{12}y^2 = k$$

$$\begin{cases} a_{11} + a_{22} = 0 \\ a_{11}a_{22} - a_{12}a_{21} > 0 \end{cases}$$