16.920J/6.339J/2.097J - Numerical Methods for Partial Differential Equations

Massachusetts Institute of Technology

Homework 1 version 1.0

Due: Monday, October 25, 2023

Problem

We consider the steady state convection diffusion problem

$$-u_x = \varepsilon u_{xx} + s \qquad \text{on} \quad x \in (0,1)$$

with $u(0) = 1$ and $u(1) = 0$. This problem has a solution

$$u(x) = 1 - sx + (s-1)\left(\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)$$

we set $s = 10$ and consider the numerical solution on a uniform grid using a second order three-point central difference approximation for the second derivate term and the following approximation for the convective term:

A) three-point central difference.

B) two-point upwind (i.e. using points j and $j+1$ to approximate $u_x(x_j)$.

C) three-point second order upwind (i.e. using points j, $j+1$ and $j+2$ to approximate $u_x(x_j))$). For the last interior point on the right boundary you should write a modified stencil utilizing linear extrapolation. That is $\hat{u}_{J+1} = 2\hat{u}_J - \hat{u}_{J-1} = -\hat{u}_{J-1}$, since $\hat{u}_J = 0$ from the boundary condition.

Question 1 (30 points)

Perform a convergence analysis for the above schemes with $\varepsilon = 1/20$ using an appropriate norm of your choice, and a sequence of at least 6 uniformly refined grids starting with $h = 1/20$. What are the orders of convergence? Which is the most accurate scheme?

Question 2 (30 points)

Repeat question 1 for the case $\varepsilon = 1/500$. Describe what you observe. Here you may need to consider finer meshes to obtain the asymptotic orders of convergence. What would the best scheme be if you want qualitatively accurate solutions on coarse meshes?

Question 3 (40 points)

Consider now the modified equation

$$-u_x = (\varepsilon + \alpha)u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} + s \qquad \text{on} \quad x \in (0,1)$$

discretized with central difference approximations on a uniform grid for all derivatives (use three points for the first and second derivatives and five points for the third and fourth derivatives).

Determine α, β and γ so that the discretization of the modified equation is identical to: i) B and ii) C. Comment on your results.