Problem 1

1) set B and B' as original point

$$\begin{cases} x = b\xi \\ y = (c + (h-c)\eta)\xi + h\eta(1-\xi) \end{cases}$$

$$\begin{cases} \frac{1}{J^2}(a u_{\xi\xi} - 2b u_{\xi\eta} + c u_{\eta\eta} + d u_\xi + e u_\eta) = -1 \\ u = 0 \quad \text{for } \xi = 1 \text{ or } \eta = 0 \\ y_\eta u_\xi - y_\xi u_\eta = 0 \quad \text{for } \xi = 0 \\ -x_\eta u_\xi + x_\xi u_\eta = 0 \quad \text{for } \eta = 1 \end{cases}$$

where $J = x_\xi y_\eta - x_\eta y_\xi$

$$a = x_\eta^2 + y_\eta^2 \qquad d = \frac{x_\eta \beta - y_\eta \alpha}{J} \qquad \alpha = a x_{\xi\xi} - 2b x_{\xi\eta} + c x_{\eta\eta}$$

$$b = x_\xi x_\eta + y_\xi y_\eta$$

$$c = x_\xi^2 + y_\xi^2 \qquad e = \frac{y_\xi \alpha - x_\xi \beta}{J} \qquad \beta = a y_{\xi\xi} - 2b y_{\xi\eta} + c y_{\eta\eta}$$

Substitute the transformation into this expression, we have

$$-\frac{1}{b^2(h-c\xi)^2}\left[(h-c\xi)^2 u_{\xi\xi} - 2c(h-c\xi)(1-\eta)u_{\xi\eta} + (b^2 + c^2(1-\eta)^2)u_{\eta\eta} - 2c^2(1-\eta)u_\eta\right] = 1$$

$$u|_{\xi=1} = u|_{\eta=0} = 0$$

$$\frac{1}{b} u_\xi - \frac{c(1-\eta)}{b(h-c\xi)} u_\eta \Big|_{\xi=0} = 0$$

$$u_\eta \Big|_{\eta=1} = 0$$

2)

Substitute into previous expression, we have

$$-\frac{1}{b^2} \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta\xi^2} + \frac{2c(1-\eta_j)}{b^2(h-c\xi)} \frac{u_{i+1\,j+1} + u_{i-1\,j-1} - u_{i+1\,j-1} - u_{i-1\,j+1}}{4\Delta\xi\Delta\eta}$$

$$- \frac{b^2 + c^2(1-\eta)^2}{b^2(h-c\xi_i)^2} \frac{u_{i\,j+1} - 2u_{ij} + u_{i\,j-1}}{\Delta\eta^2} + \frac{2c^2(1-\eta)}{b^2(h-c\xi_i)^2} \frac{u_{i\,j+1} - u_{i\,j-1}}{2\Delta\eta}$$

$$= 1$$

$$u_{i0} = 0 \qquad u_{N-1\,j} = 0$$

$$\frac{u_{iN} - u_{iN-2}}{2\Delta\eta} = 0, \qquad \frac{1}{b} \frac{u_{1j} - u_{-1j}}{2\Delta\xi} - \frac{c(1-\eta_j)}{bh} \frac{u_{0\,j+1} - u_{0\,j-1}}{2\Delta\eta}$$

$$= 0$$

for Q, we have

$$Q = 2\int_0^1\int_0^1 u(\xi,\eta) J \, d\xi \, d\eta$$

$$= 2b \, \Delta\xi \, \Delta\eta \left[\sum_{i=1}^{N-1}\sum_{j=0}^{N-2} u_{ij}\left(h - c\frac{i}{N-1}\right) + \frac{h}{2}\sum_{j=0}^{N-2} u_{0j} + \frac{1}{2}\sum_{i=0}^{N-1} u_{i\,N-1}\left(h - c\frac{i}{N-1}\right)\right]$$

3)

To further simplify, we can substitute coefficients into it,

we have, for interior points

$$(\mathcal{L}u)_{ij} = A_{i-1\,j-1}\,U_{i-1\,j-1} + A_{i+1,j+1}\,U_{i+1,j+1} + A_{i+1,j-1}\,U_{i+1,j-1} + A_{i-1\,j+1}\,U_{i-1\,j+1}$$

$$+ A_{i-1\,j}\,U_{i-1\,j} + A_{i+1\,j}\,U_{i+1\,j} + A_{i\,j-1}\,U_{i\,j-1} + A_{i\,j+1}\,U_{i\,j+1}$$

$$+ A_{ij} U_{ij}$$

where $A_{i-1\,j-1} = A_{i+1\,j+1} = -A_{i-1\,j+1} = -A_{i+1\,j-1} = \dfrac{2c(1-\eta_j)}{b^2(h-c\xi_i)} \dfrac{1}{4\Delta\xi\Delta\eta}$

$$A_{i-1\,j} = A_{i+1\,j} = \frac{-1}{b^2 \Delta\xi^2}$$

$$A_{i\,j\pm1} = -\frac{b^2+c^2(1-\eta_j)^2}{b^2(h-c\xi_i)^2} \frac{1}{\Delta\eta^2} \pm \frac{2c^2(1-\eta)}{b^2(h-c\xi_i)} \frac{1}{2\Delta\eta}$$

$$A_{ij} = \frac{2}{b^2\Delta\xi^2} + 2\frac{b^2+c^2(1-\eta)^2}{b^2(h-c\xi_i)^2} \frac{1}{\Delta\eta^2}$$

$$(\mathcal{L}u)_{i0} = U_{i0}$$

$$(\mathcal{L}u)_{N-1,j} = U_{N-1,j}$$

$$(\mathcal{L}u)_{i,N} = \frac{1}{2\Delta\eta}(U_{iN} - U_{iN-2})$$

$$(\mathcal{L}u)_{N-1,j} = \frac{1}{2b\Delta\xi}(U_{1j} - U_{N-1\,j}) - \frac{c(1-\eta_j)}{2bh\Delta\eta}(U_{0,j+1} - U_{0,j-1})$$