Exercise 4.9

(a) $$\min \sum_{t=1}^{T} b_t p_t$$

Subject to.

$$p_t \geq \alpha^{t-1} d_t \qquad t=1,\cdots,T$$
$$p_t - f q_t \geq -\alpha^{t-1} c \qquad t=1,\cdots,T$$
$$q_t - q_{t+1} \geq 0 \qquad t=1,\cdots,T-1$$
$$q_T \geq \alpha^T d_{T+1}$$
$$-p_t + q_t \geq 0 \qquad t=1,\cdots,T$$

(b) denote the solution as $q_t^{(0)}, p_t^{(0)}$

$$\begin{cases} q_T^{(0)} = \alpha^T d_{T+1} \\ p_T^{(0)} = \max\{\alpha^{T-1} d_T, f q_T^{(0)} - \alpha^{T-1} c\} \\ q_t^{(0)} = \max\{q_{t+1}^{(0)}, \alpha^{t-1} d_t\} \\ p_t^{(0)} = \max\{\alpha^{t-1} d_t, f q_t^{(0)} - \alpha^{t-1} c\} \end{cases}$$

First we prove that $q_T^{(0)}, p_T^{(0)}$ is feasible:
Substitute into constraints, we have

$$p_t^{(0)} = \max\{\alpha^{t-1} d_t, f q_t^{(0)} - \alpha^{t-1} c\} \geq \alpha^{t-1} d_t \qquad t=1,\cdots,T$$
$$p_t^{(0)} - f q_t^{(0)} = \max\{\alpha^{t-1} d_t, f q_t^{(0)} - \alpha^{t-1} c\} - f q_t^{(0)} \geq -\alpha^{t-1} c \qquad t=1,\cdots,T$$
$$q_t^{(0)} - q_{t+1}^{(0)} = \max\{q_{t+1}^{(0)}, \alpha^{t-1} d_t\} - q_{t+1}^{(0)} \geq 0 \qquad t=1,\cdots,T-1$$
$$q_T^{(0)} = \alpha^T d_{T+1}$$
$$-p_t^{(0)} + q_t^{(0)} = q_t^{(0)} - \max\{\alpha^{t-1} d_t, f q_t^{(0)} - \alpha^{t-1} c\}$$

$= \min \{ q_t^{(0)} - \alpha^{t-1} d_t , (1-f) q_t^{(0)} + \alpha^{t-1} c \}$

because $f<1, \alpha>0, c>0, (1-f) q_t^{(0)} + \alpha^{t-1} c > 0$

So it is feasible. Then we are going to prove that it is optimal. Suppose that there is another feasible solution $\tilde{q}_t, \tilde{p}_t$, we're going to prove that $\tilde{q}_t \geq q_t^{(0)}$, $\tilde{p}_t \geq p_t^{(0)}$

we can see that it holds for $t=T$, then, suppose that it holds for $t=\tau+1$, we can see that it holds for $t=\tau$,

So, $\tilde{q}_t \geq q_t^{(0)}$, $\tilde{p}_t \geq p_t^{(0)}$ holds for every t.

So the solution is optimal.

(c) every optimal dual solution gives a optimal primal solution

we can do like this to obtain optimal primal solution

denote dual problem as $\begin{cases} \max p^T b \\ p^T A \leq c^T \end{cases}$

rearrange optimal p as: $p^T A = \underbrace{[\,c_B}_{\text{binding}} \mid \underbrace{c_N\,]}_{\text{feasible}}$

Correspondingly, A as: $A = [\,B \mid N\,]$

primal solution: $x^T = [\,B^{-1} c_B \mid 0\,]$

primal maximazing value is equal to dual minimizing value

Simple rules for order fullfillment

(a). cost is 23274

220 costomers are fully served by nearest

```
model = Model(Gurobi.Optimizer)
@variable(model, X[1:100, 1:500]);
@objective(model, Min, sum(sum(cost[i,j]*X[i,j] for j in 1:500) for i in 1:100));
@constraint(model,a_constraint[i in 1:100 ,j in 1:500],X[i,j] >= 0 );
@constraint(model,d_constraint[j in 1:500],sum(X[i,j] for i in 1:100) >= demand[j] );
@constraint(model,s_constraint[i in 1:100],sum(X[i,j] for j in 1:500) <= supply[i] );
optimize!(model)
```

✓ 1.8s Julia

```
cnt = 0
for j in 1:500
    # find the nearest
    cj = cost[:,j]
    mVal,i = findmin(cj)
    cost_tmp = abs(value(X[i,j]) - demand[j])
    if cost_tmp < 1e-6
        cnt+=1
    end
end
print(cnt)
```

✓ 0.0s Julia

```
220
```

comment: about half costomers are not fully served by nearest, the distribution is not very efficient

(b). dual problem

$$\max_{p,q} \sum_{j=1}^{n} p_j d_j - \sum_{i=1}^{m} q_i S_i$$

$$p_j \geq 0 \quad j=1,\cdots,n$$

$$q_i \geq 0 \quad i=1,\cdots,m$$

$$p_j - q_i \leq C_{ij} \quad i=1,\cdots,m \quad j=1,\cdots,n$$

I propose β_{ij} to be the reduced cost $\beta_{ij} = C_{ij} - p_j + q_i$

because this quantity represents how much the cost function in primal problem will increase if a unit flow is sent from i to j, so the less β_{ij} is, the more likely $i \to j$ will be included in the optimal solution.

(c)

$$\min \sum_{i=1}^{m} \sum_{k=1}^{K} \sum_{j=1}^{n} C^1_{ik} C^2_{kj} x_{ikj}$$

$$\text{s.t.} \quad \sum_{k=1}^{K} \sum_{j=1}^{n} x_{ikj} \le S_i \qquad i=1,\cdots,m$$

$$\sum_{i=1}^{m} \sum_{k=1}^{K} x_{ikj} \ge d_j \qquad j=1,\cdots,n$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ikj} \le C_k \qquad k=1,\cdots,K$$

(d) the dual problem

$$\max_{p,q,r} \sum_{j=1}^{n} d_j p_j - \sum_{i=1}^{m} S_i q_i - \sum_{k=1}^{K} C_k r_k$$

$$\text{s.t.} \quad p_j \ge 0 \qquad j=1,\cdots,n$$

$$q_i \ge 0 \qquad i=1,\cdots,m$$

$$r_k \ge 0 \qquad k=1,\cdots,K$$

$$p_j - q_i - r_k \le C^1_{ik} C^2_{kj}$$

I propose γ_{ikj} to be reduced cost,

$$Y_{ijk} = C^1_{ik} C^2_{kj} - (p_j - q_i - r_k)$$

because this quantify represents how much the cost function in primal problem will increase if a unit flow is sent from i to k to j, so the less Y_{ijk} is, the more likely $i \to k \to j$ will be included in the optimal solution.

Excercise 4.5

(a) True. If the x^* is optimal, because

x^* optimal $\Leftrightarrow$ one of its corresponding dual is feasible,

so one of the dual is feasible, contradicts.

(b) because the auxiliary problem is always feasible and bounded, so it has a optimal solution. By strong duality, the dual problem has a optimal solution, so it's feasible

(c) True. without loss of generality, we assume the first constraint is removed, i.e. $\sum_j a_{ij} x_j = b_i \ (i=1,2,\cdots,n) \longrightarrow \sum_j a_{ij} x_j = b_i \ (i=2,\cdots,n)$

correspondingly $\quad \sum_{i=1}^{n} p_i a_{ij} \le c_j \ (j=1,\cdots,m) \longrightarrow \sum_{i=2}^{n} p_i a_{ij} \le c_j \ (j=1,\cdots,m)$

It is equivalent to set $p_1 = 0$

(d) True. primal unbounded means the cost is $-\infty$, by weak duality, the dual can be only infeasible