Simplex algorithm (1/2)

15.093: Optimization

Dimitris Bertsimas
Alexandre Jacquillat

Sloan School of Management
Massachusetts Institute of Technology

Feasibility and optimality conditions

Background and motivation

$$(LO): \quad \min_{\boldsymbol{x} \in \mathcal{P}} \quad \boldsymbol{c}^\top \boldsymbol{x}, \quad \text{where } \mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

- Let us start from a basic feasible solution $\boldsymbol{x} \in \mathcal{P}$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \longrightarrow [\boldsymbol{B} \ \boldsymbol{N}]\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{x}_N = \boldsymbol{0} \qquad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

- What if we moved away in direction $\boldsymbol{d} \in \mathbb{R}^n$, by a small amount $\theta > 0$?

$$\boldsymbol{x} \leftarrow \boldsymbol{x} + \theta \boldsymbol{d}, \ \text{with} \begin{cases} d_j = 1 & \text{for non-basic variable } j \in \mathcal{N} \\ d_i = 0 & \text{for other non-basic variables } i \neq j \in \mathcal{N} \\ \boldsymbol{d}_B = ?? & \text{for basic variables in } \mathcal{B} \end{cases}$$

1. What does it take to retain feasibility: $\boldsymbol{x} + \theta \boldsymbol{d} \in \mathcal{P}$?
2. What does it take to improve the solution: $\boldsymbol{c}^\top(\boldsymbol{x} + \theta \boldsymbol{d}) < \boldsymbol{c}^\top \boldsymbol{x}$?

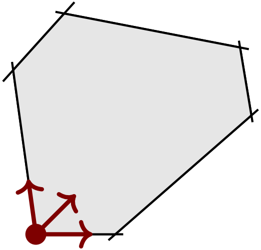
Feasibility conditions

- Polyhedral constraints:

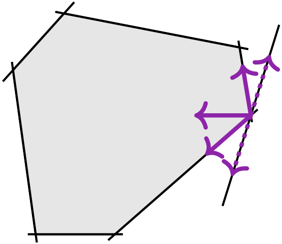
$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x} + \theta \boldsymbol{d}) = \boldsymbol{b} &\implies \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0} \\ &\implies \boldsymbol{B}\boldsymbol{d}_B + \boldsymbol{A}_j = \boldsymbol{0} \\ &\implies \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j \end{aligned}$$

- Non-negativity constraints:
  - If $\boldsymbol{x}$ is nondegenerate, $\boldsymbol{x}_B > \boldsymbol{0}$ so $\boldsymbol{x}_B + \theta \boldsymbol{d}_B > \boldsymbol{0}$ if θ is small
  - If $\boldsymbol{x}$ is degenerate, $\boldsymbol{x} + \theta \boldsymbol{d}$ is not necessarily feasible if $\theta > 0$!

A non-degenerate BFS



A degenerate BFS



Optimality conditions

- What does it take to improve the solution:

$$\begin{aligned} \boldsymbol{c}^\top(\boldsymbol{x}+\theta\boldsymbol{d}) < \boldsymbol{c}^\top\boldsymbol{x} &\implies \boldsymbol{c}^\top\boldsymbol{d} < 0 \\ &\implies \boldsymbol{c}_B^\top\boldsymbol{d}_B + c_j < 0 \\ &\implies c_j - \boldsymbol{c}_B^\top\boldsymbol{B}^{-1}\boldsymbol{A}_j < 0 \end{aligned}$$

Definition

Reduced cost of variable x_j: $\bar{c}_j = c_j - \boldsymbol{c}_B^\top\boldsymbol{B}^{-1}\boldsymbol{A}_j$

- The reduced cost of basic variables is 0

  if $j \in B$, then $\bar{c}_j = c_j - \boldsymbol{c}_B^\top\boldsymbol{B}^{-1}\boldsymbol{A}_j = c_j - \boldsymbol{c}_B^\top\boldsymbol{e}_j = c_j - c_j = 0$

- The reduced cost of non-basic variables can be $<$0, 0 or $>$0
  - If there exists a non-basic variable $j \in N$ such that $\bar{c}_j < 0$, we might be able to improve the solution: $\boldsymbol{c}^\top(\boldsymbol{x}+\theta\boldsymbol{d}) < \boldsymbol{c}^\top\boldsymbol{x}$
  - Otherwise, we have found an optimal solution

Characterization of optimal solutions

Theorem

Let $\boldsymbol{x}$ be a basic feasible solution associated with basis $\boldsymbol{B}$ and let $\overline{\mathbf{c}}$ be the vector of reduced costs. Then:

1. *If $\overline{\boldsymbol{c}} \geq \mathbf{0}$, then $\boldsymbol{x}$ is an optimal solution of (LO)*
2. *If $\boldsymbol{x}$ is an optimal solution of (LO) and is nondegenerate, then $\overline{\boldsymbol{c}} \geq \mathbf{0}$*

- Proof of 1: proving optimality
  - Consider a BFS $\boldsymbol{x}$ associated with basis $\boldsymbol{B}$ such that $\overline{\boldsymbol{c}} \geq \mathbf{0}$
  - Consider an arbitrary feasible solution $\boldsymbol{y} \in \mathcal{P}$ and denote $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x}$
  - Feasibility condition: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b}$ so $\boldsymbol{A}\boldsymbol{d} = \mathbf{0}$, i.e.,

$$\boldsymbol{B}\boldsymbol{d}_B + \sum_{i \in \mathcal{N}} \boldsymbol{A}_i d_i = \mathbf{0}, \text{ or } \boldsymbol{d}_B = -\sum_{i \in \mathcal{N}} \boldsymbol{B}^{-1}\boldsymbol{A}_i d_i$$

  - Feasibility condition: $\boldsymbol{x}_N = \mathbf{0}$ and $\boldsymbol{y} \geq \mathbf{0}$ so $\boldsymbol{d}_N \geq \mathbf{0}$
  - Optimality check:

$$\boldsymbol{c}^\top \boldsymbol{y} - \boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{c}^\top \boldsymbol{d} = \boldsymbol{c}_B^\top \boldsymbol{d}_B + \sum_{i \in \mathcal{N}} c_i d_i = \sum_{i \in \mathcal{N}} \overline{c}_i d_i \geq 0$$

$\rightarrow$ $\boldsymbol{x}$ is an optimal solution

Characterization of optimal solutions

Theorem

Let $\boldsymbol{x}$ be a basic feasible solution associated with basis $\boldsymbol{B}$ and let $\overline{\boldsymbol{c}}$ be the vector of reduced costs. Then:

1. *If $\overline{\boldsymbol{c}} \geq \mathbf{0}$, then $\boldsymbol{x}$ is an optimal solution of (LO)*
2. *If $\boldsymbol{x}$ is an optimal solution of (LO) and is nondegenerate, then $\overline{\boldsymbol{c}} \geq \mathbf{0}$*

- Proof of 2: improving the cost
  - Assume by contradiction that $\overline{c}_j = c_j - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} \boldsymbol{A}_j < 0$
  - Consider $\theta > 0$ and define $\boldsymbol{y} = \boldsymbol{x} + \theta \boldsymbol{d}$ with

$$\begin{cases} d_j = 1 & \\ d_i = 0 & \text{for other non-basic variables } i \neq j \in \mathcal{N} \\ \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j & \text{for basic variables in } \mathcal{B} \end{cases}$$

  - Cost reduction: $\boldsymbol{c}^\top \boldsymbol{y} - \boldsymbol{c}^\top \boldsymbol{x} = \theta(c_j + \boldsymbol{c}_B^\top \boldsymbol{d}_B) = \theta \overline{c}_j < 0$
  - Feasibility: $\boldsymbol{x} + \theta \boldsymbol{d}$ is feasible for $\theta > 0$ small enough
    - By construction, $\boldsymbol{A}\boldsymbol{d} = \mathbf{0}$, so $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b}$
    - Since $\boldsymbol{x}$ is nondegenerate, $\boldsymbol{x}_B > \mathbf{0}$ so $\boldsymbol{x}_B + \theta \boldsymbol{d}_B > \mathbf{0}$ for θ small enough

$\rightarrow$ $\boldsymbol{x}$ is not an optimal solution

Moving to a new basic feasible solution

- Consider a non-optimal basic feasible solution $\boldsymbol{x}$
- Assume that you have found a non-basic variable j such that $\bar{c}_j < 0$

$$\boldsymbol{c}^\top(\boldsymbol{x} + \theta \boldsymbol{d}) = \boldsymbol{c}^\top \boldsymbol{x} + \theta \bar{c}_j < \boldsymbol{c}^\top \boldsymbol{x}$$

- In fact, the larger θ, the larger the cost reduction

$\rightarrow$ Question: By how much can we improve the solution?

- By construction, the new solution satisfies the polyhedral constraints:

$$\boldsymbol{A}(\boldsymbol{x} + \theta \boldsymbol{d}) = \underbrace{\boldsymbol{A}\boldsymbol{x}}_{=\boldsymbol{b}} + \theta \underbrace{\boldsymbol{A}\boldsymbol{d}}_{=\boldsymbol{0}} = \boldsymbol{b}$$

- What about non-negativity constraints: $\boldsymbol{x} + \theta \boldsymbol{d} \geq \boldsymbol{0}$?
  - If $\boldsymbol{d} \geq \boldsymbol{0}$, then $\boldsymbol{x} + \theta \boldsymbol{d} \geq \boldsymbol{0}$ for all $\theta \geq 0$ and the problem is unbounded
  - If there exists j with $d_j < 0$, then $\boldsymbol{x} + \theta \boldsymbol{d} \geq \boldsymbol{0}$ requires $\theta \leq -\frac{x_j}{d_j}$

$\Longrightarrow$ minimum ratio rule: $\quad \theta^* = \min\limits_{i=1,\cdots,m: d_{B(i)}<0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right)$

Example: constructing a basic feasible solution

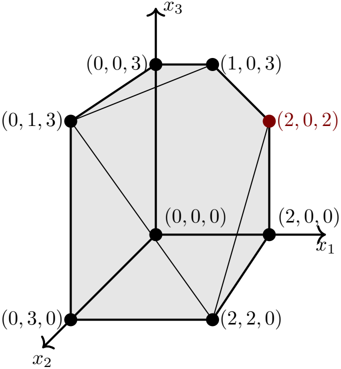
$$\min_{\boldsymbol{x},\boldsymbol{s}\geq \mathbf{0}} \quad x_1 + 5x_2 - 2x_3$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + s_1 = 4$$

$$x_1 + s_2 = 2$$

$$x_3 + s_3 = 3$$

$$3x_2 + x_3 + s_4 = 6$$



$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

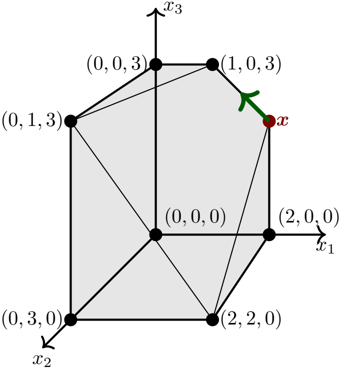
$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$\implies \boldsymbol{x} = (2, 0, 2, 0, 0, 1, 4)^\top$$

- $\boldsymbol{x} \geq \mathbf{0}$ so it is a basic feasible solution
- Is there a better basic feasible solution?

Example: finding a direction of cost improvement

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{s}\geq \mathbf{0}} \quad & x_1 + 5x_2 - 2x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + s_1 = 4 \\ & x_1 + s_2 = 2 \\ & x_3 + s_3 = 3 \\ & 3x_2 + x_3 + s_4 = 6 \end{aligned}$$



$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A}$$

$$\boldsymbol{c}^\top = (\mathbf{1}, \mathbf{5}, \mathbf{\text{-}2}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$$

$$\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} = (\mathbf{1}, \mathbf{\text{-}2}, \mathbf{0}, \mathbf{0}) \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{\text{-}1} & \mathbf{0} & \mathbf{0} \\ \mathbf{\text{-}1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{\text{-}1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \boldsymbol{A}$$

$$\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} = (\mathbf{\text{-}2}, \mathbf{3}, \mathbf{0}, \mathbf{0})\boldsymbol{A}$$

$$\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} = (\mathbf{1}, \mathbf{\text{-}2}, \mathbf{\text{-}2}, \mathbf{\text{-}2}, \mathbf{3}, \mathbf{0}, \mathbf{0})^\top$$

$$\implies \overline{\boldsymbol{c}} = (\mathbf{0}, \mathbf{7}, \mathbf{0}, \mathbf{2}, \mathbf{\text{-}3}, \mathbf{0}, \mathbf{0})^\top$$

$$\implies \begin{cases} d_5 = \mathbf{1} \\ d_2 = d_4 = \mathbf{0} \\ \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_5 = (\mathbf{\text{-}1}, \mathbf{1}, \mathbf{\text{-}1}, \mathbf{\text{-}1})^\top \end{cases}$$

$$\implies \boldsymbol{y} = (\mathbf{2} - \boldsymbol{\theta}, \mathbf{0}, \mathbf{2} + \boldsymbol{\theta}, \mathbf{0}, \boldsymbol{\theta}, \mathbf{1} - \boldsymbol{\theta}, \mathbf{4} - \boldsymbol{\theta})^\top$$

Example: moving to a new basic feasible solution

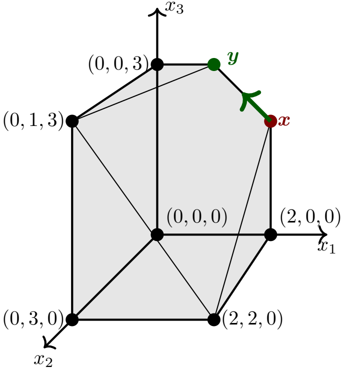
$$\min_{\boldsymbol{x},\boldsymbol{s}\geq \mathbf{0}} \quad x_1 + 5x_2 - 2x_3$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + s_1 = 4$$

$$x_1 + s_2 = 2$$

$$x_3 + s_3 = 3$$

$$3x_2 + x_3 + s_4 = 6$$



$$\boldsymbol{y} = (\mathbf{2-\theta}, \mathbf{0}, \mathbf{2+\theta}, \mathbf{0}, \boldsymbol{\theta}, \mathbf{1-\theta}, \mathbf{4-\theta})^\top$$

- We have found a direction that satisfies $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b}$ and such that $\boldsymbol{c}^\top \boldsymbol{y} < \boldsymbol{c}^\top \boldsymbol{x}$
- How far can we go?

$$\begin{cases} \mathbf{2} - \boldsymbol{\theta} \geq \mathbf{0} \\ \mathbf{2} + \boldsymbol{\theta} \geq \mathbf{0} \\ \mathbf{1} - \boldsymbol{\theta} \geq \mathbf{0} \\ \mathbf{4} - \boldsymbol{\theta} \geq \mathbf{0} \end{cases} \implies \theta = 1$$

$$\implies \boldsymbol{y} = (\mathbf{1}, \mathbf{0}, \mathbf{3}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{3})^\top$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$ x_6 left the basis and x_5 entered the basis

$\rightarrow$ Move from one extreme point to another

Example: moving to another basic feasible solution

$$\min_{\boldsymbol{x},\boldsymbol{s}\geq \mathbf{0}} \quad x_1 + 5x_2 - 2x_3$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + s_1 = 4$$

$$x_1 + s_2 = 2$$

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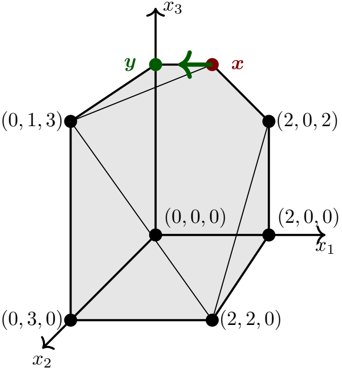


$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

$$\boldsymbol{x} = (\mathbf{1}, \mathbf{0}, \mathbf{3}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{3})^\top$$

$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} = (\mathbf{0}, \mathbf{13}, \mathbf{0}, \mathbf{-1}, \mathbf{0}, \mathbf{3}, \mathbf{0})^\top$$

$$\implies \begin{cases} d_4 = \mathbf{1} \\ d_2 = d_6 = \mathbf{0} \\ \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_4 = (\mathbf{-1}, \mathbf{0}, \mathbf{1}, \mathbf{0})^\top \end{cases}$$

$$\implies \boldsymbol{y} = (\mathbf{1} - \boldsymbol{\theta}, \mathbf{0}, \mathbf{3}, \boldsymbol{\theta}, \mathbf{1} + \boldsymbol{\theta}, \mathbf{0}, \mathbf{3})^\top$$

$$\implies \boldsymbol{y} = (\mathbf{0}, \mathbf{0}, \mathbf{3}, \mathbf{1}, \mathbf{2}, \mathbf{0}, \mathbf{3})^\top$$

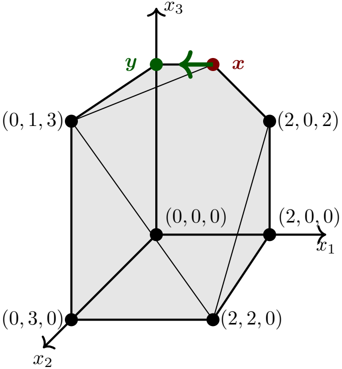
→ x_1 left the basis and x_4 entered the basis

→ Move to yet another extreme point

Example: proving optimality

$$\min_{\boldsymbol{x},\boldsymbol{s}\geq 0} \quad x_1 + 5x_2 - 2x_3$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_2 + x_3 + s_1 = 4 \\ & x_1 + s_2 = 2 \\ & x_3 + s_3 = 3 \\ & 3x_2 + x_3 + s_4 = 6 \end{aligned}$$



$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

$$\boldsymbol{x} = (0,0,3,1,2,0,3)^\top$$

$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} = (1,11,0,0,0,2,0)^\top$$

$\rightarrow$ The reduced costs are all positive

$\rightarrow$ Optimal solution:
$\boldsymbol{x} = (0,0,3,1,2,0,3)^\top$

$\rightarrow$ Optimal cost: -6

The simplex algorithm

One iteration

- Iterative moves from one extreme point to an adjacent one

$\rightarrow$ One non-basic variable enters the basis and one basic variable leaves

  - Start with basis $\boldsymbol{B} = [\boldsymbol{A}_{B(1)} \cdots \boldsymbol{A}_{B(m)}]$
  - Variable $x_{B(l)}$ leaves the basis; variable x_j enters the basis
  - Move to new basis $\overline{\boldsymbol{B}} = [\boldsymbol{A}_{B(1)} \cdots \boldsymbol{A}_{B(l-1)} \boldsymbol{A}_j \boldsymbol{A}_{B(l+1)} \cdots \boldsymbol{A}_{B(m)}]$

$$\begin{cases} d_j = 1 & \\ d_i = 0 & \text{for other non-basic variables } i \neq j \in \mathcal{N} \\ \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j & \text{for basic variables in } \mathcal{B} \end{cases}$$

$$\theta^* = \min_{i=1,\cdots,m: d_{B(i)}<0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right) = -\frac{x_{B(l)}}{d_{B(l)}}$$

Theorem

- $\overline{\boldsymbol{B}} = [\boldsymbol{A}_{B(1)} \cdots \boldsymbol{A}_{B(l-1)} \boldsymbol{A}_j \boldsymbol{A}_{B(l+1)} \cdots \boldsymbol{A}_{B(m)}]$ *is a basis*
- $\boldsymbol{y} = \boldsymbol{x} + \theta^* \boldsymbol{d}$ *is a BFS associated with basis* $\overline{\boldsymbol{B}}$.

The simplex algorithm

Algorithm

1. *Start with basis* $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \cdots, \boldsymbol{A}_{B(m)}]$ *and BFS* $\boldsymbol{x}$.
2. *Compute* $\bar{c}_j = c_j - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} \boldsymbol{A}_j$
   - *If* $\bar{c}_j \geq 0$ *for all non-basic variables* j, *then* $\boldsymbol{x}$ *is optimal;* `STOP`.
   - *Else select non-basic variable* j *such that* $\bar{c}_j < 0$; `PROCEED`.
3. *Compute* $\boldsymbol{u} = \boldsymbol{B}^{-1} \boldsymbol{A}_j$.
   - *If* $\boldsymbol{u} \leq \boldsymbol{0}$, *then the cost is unbounded;* `STOP`. *Else,* `PROCEED`.
4. $\theta^* = \min\limits_{i=1,\cdots,m,u_i>0} \dfrac{x_{B(i)}}{u_{B(i)}}$. *Let* l *be such that* $\theta^* = \frac{x_{B(l)}}{u_{B(l)}}$
5. *Form new basis:* $[\boldsymbol{A}_{B(1)} \cdots \boldsymbol{A}_{B(l-1)} \boldsymbol{A}_j \boldsymbol{A}_{B(l+1)} \cdots \boldsymbol{A}_{B(m)}]$
6. *Get new BFS:* $y_j = \theta^*,\ \ y_{B(i)} = x_{B(i)} - \theta^* u_i$ *for all* i
7. *Go back to Step 2.*

$\rightarrow$ The simplex algorithm proceeds iteratively, moving from one extreme point to an adjacent one until all reduced costs become non-negative

Algorithm convergence

$$(LO): \quad \min_{\boldsymbol{x} \in \mathcal{P}} \quad \boldsymbol{c}^\top \boldsymbol{x}, \quad \text{where } \mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

Theorem

If $\mathcal{P} \neq \emptyset$ and every BFS is nondegenerate, the simplex algorithm terminates after a finite number of iterations. At termination:

- *either we have found a basis $\boldsymbol{B}^*$, with an optimal BFS $\boldsymbol{x}^*$*
- *or we have found a direction $\boldsymbol{d}$ such that $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$, $\boldsymbol{d} \geq \boldsymbol{0}$, $\boldsymbol{c}^\top \boldsymbol{d} < \boldsymbol{0}$; the problem is unbounded (i.e., the optimal cost is $-\infty$)*

- If the algorithm stops with $\overline{\boldsymbol{c}} \geq \boldsymbol{0}$ (Step 2), the solution is optimal
- If the algorithm stops with $\boldsymbol{u} \leq \boldsymbol{0}$ (Step 3), the problem is unbounded
- Otherwise, the cost strictly decreases at each iteration ($\boldsymbol{c}^\top \boldsymbol{y} < \boldsymbol{c}^\top \boldsymbol{x}$), so the algorithm never revisits the same solution twice

$\rightarrow$ Finite convergence due to the finite number of extreme points

Simplex method by example (1/3)

$$
\begin{array}{lrcrcrcrcrcrcl}
\min\limits_{\boldsymbol{x}\geq\mathbf{0}} & \mathbf{-10}x_1 & \mathbf{-} & \mathbf{12}x_2 & \mathbf{-} & \mathbf{12}x_3 & & & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & & & = & 20 \\
 & 2x_1 & + & x_2 & + & 2x_3 & & & + & x_5 & & & = & 20 \\
 & 2x_1 & + & 2x_2 & + & x_3 & & & & & + & x_6 & = & 20
\end{array}
$$

- Start from basic variable: $\boldsymbol{x} = (0, 0, 0, 20, 20, 20)$, with objective of 0
- **Increase x_1, as much as possible**

$$
\begin{cases}
x_1 = 20 - 2x_2 - 2x_3 - x_4 \\
x_1 = 10 - 0.5x_2 - x_3 - 0.5x_5 \\
x_1 = 10 - x_2 - 0.5x_3 - 0.5x_6
\end{cases}
$$

$$
\Longrightarrow \; x_1 = 10 - 0.5x_2 - x_3 - 0.5x_5
$$

- Plug into the problem formulation:

$$
\begin{array}{lrcrcrcrcrcrcl}
\min\limits_{\boldsymbol{x}\geq\mathbf{0}} & -100 & & & \mathbf{-} & \mathbf{7}x_2 & \mathbf{-} & \mathbf{2}x_3 & & & \mathbf{+} & \mathbf{5}x_5 & & & & \\
\text{s.t.} & & & & & 1.5x_2 & + & x_3 & + & x_4 & - & 0.5x_5 & & & = & 10 \\
 & & x_1 & & + & 0.5x_2 & + & x_3 & & & + & 0.5x_5 & & & = & 10 \\
 & & & & & x_2 & - & x_3 & & & - & x_5 & + & x_6 & = & 0
\end{array}
$$

Simplex method by example (2/3)

$$
\begin{array}{llcccccccccc}
\min_{\boldsymbol{x}\geq\mathbf{0}} & -100 & - & 7x_2 & - & 2x_3 & & & + & 5x_5 & & & & \\
\text{s.t.} & & & 1.5x_2 & + & x_3 & + & x_4 & - & 0.5x_5 & & & = & 10 \\
& x_1 & + & 0.5x_2 & + & x_3 & & & + & 0.5x_5 & & & = & 10 \\
& & & x_2 & - & x_3 & & & - & x_5 & + & x_6 & = & 0
\end{array}
$$

- New basic variable: $\boldsymbol{x} = (10, 0, 0, 10, 0, 0)$, with objective of -100
- **Increase x_3, as much as possible**

$$
\begin{cases}
x_3 = 10 - 1.5x_2 - x_4 + 0.5x_5 \\
x_3 = 10 - x_1 - 0.5x_2 - x_4 - 0.5x_5 \\
x_3 = x_2 + x_5 - x_6
\end{cases}
$$

$$
\implies x_3 = 10 - 1.5x_2 - x_4 + 0.5x_5
$$

- Plug into the problem formulation:

$$
\begin{array}{llcccccccccc}
\min_{\boldsymbol{x}\geq\mathbf{0}} & -120 & - & 4x_2 & & & + & 2x_4 & + & 4x_5 & & & & \\
\text{s.t.} & & & 1.5x_2 & + & x_3 & + & x_4 & - & 0.5x_5 & & & = & 10 \\
& x_1 & - & x_2 & & & - & x_4 & + & x_5 & & & = & 0 \\
& & & 2.5x_2 & & & + & x_4 & - & 1.5x_5 & + & x_6 & = & 10
\end{array}
$$

Simplex method by example (3/3)

$$\begin{array}{llcccccccccccc}
\min_{\boldsymbol{x}\geq \mathbf{0}} & -120 & & - & \mathbf{4}x_2 & & & + & \mathbf{2}x_4 & + & \mathbf{4}x_5 & & & \\
\text{s.t.} & & & & 1.5x_2 & + & x_3 & + & x_4 & - & 0.5x_5 & & = & 10 \\
& & x_1 & - & x_2 & & & - & x_4 & + & x_5 & & = & 0 \\
& & & & 2.5x_2 & & & + & x_4 & - & 1.5x_5 & + \; x_6 & = & 10
\end{array}$$

- New basic variable: $\boldsymbol{x} = (0, 0, 10, 0, 0, 10)$, with objective of -120
- **Increase x_2, as much as possible**

$$\begin{cases} x_2 = 6.67 - 0.67x_3 - 0.67x_4 + 0.33x_5 \\ x_2 = x_1 - x_4 + x_5 \\ x_2 = 4 - 0.4x_4 + 0.6x_5 - 0.4x_6 \end{cases}$$

$$\implies x_2 = 4 - 0.4x_4 + 0.6x_5 - 0.4x_6$$

$$\begin{array}{llccccccccccc}
\min_{\boldsymbol{x}\geq \mathbf{0}} & -136 & & & & + & \mathbf{3.6}x_4 & + & \mathbf{1.6}x_5 & + & \mathbf{1.6}x_6 & & \\
\text{s.t.} & & & & x_3 & + & 0.4x_4 & + & 0.4x_5 & - & 0.6x_6 & = & 4 \\
& & x_1 & & & - & 0.6x_4 & + & 0.4x_5 & + & 0.4x_6 & = & 4 \\
& & & x_2 & & + & 0.4x_4 & - & 0.6x_5 & + & 0.4x_6 & = & 4
\end{array}$$

- New basic variable: $\boldsymbol{x} = (4, 4, 4, 0, 0, 0)$, with objective of -136
- **No more possible improvement**, hence optimal solution