Applications of duality

15.093: Optimization

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Duality as a proof technique: Farkas lemma

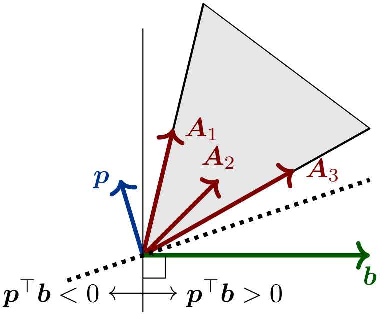
Farkas lemma: geometric view

Theorem

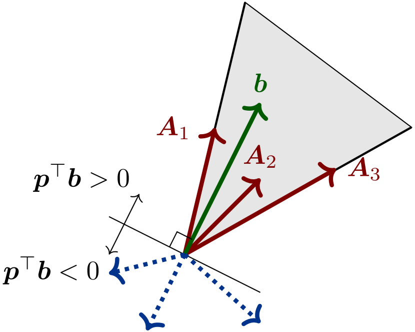
Exactly one of the following alternatives is true:

- *There exists* $\boldsymbol{x} \geq \mathbf{0}$ *such that* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
- *There exists* $\boldsymbol{p}$ *such that* $\boldsymbol{p}^\top \boldsymbol{A} \geq \mathbf{0}^\top$ *and* $\boldsymbol{p}^\top \boldsymbol{b} < 0$

$\exists \boldsymbol{p} : \boldsymbol{p}^\top \boldsymbol{A}_j \geq 0, \ \forall j$ and $\boldsymbol{p}^\top \boldsymbol{b} < 0$



$\exists \boldsymbol{x} \geq \mathbf{0} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$



Farkas lemma: algebraic view

Theorem

Exactly one of the following alternatives is true:

- *There exists* $\boldsymbol{x} \geq \mathbf{0}$ *such that* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
- *There exists* $\boldsymbol{p}$ *such that* $\boldsymbol{p}^\top \boldsymbol{A} \geq \mathbf{0}^\top$ *and* $\boldsymbol{p}^\top \boldsymbol{b} < 0$

$\Longrightarrow$ If $\exists \boldsymbol{x} \geq \mathbf{0} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, and if $\boldsymbol{p}^\top \boldsymbol{A} \geq \mathbf{0}^\top$, then $\boldsymbol{p}^\top \boldsymbol{b} = \boldsymbol{p}^\top \boldsymbol{A}\boldsymbol{x} \geq 0$

$\Longleftarrow$ Otherwise, the following primal problem (P) is infeasible:

Primal problem (P)

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \min \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \geq \mathbf{0}^\top \end{aligned}$$

Hence, the dual problem (D) is either infeasible or unbounded.
(D) is feasible, because $\boldsymbol{p} = \mathbf{0}$ is dual feasible.
Hence, (D) is unbounded: $\exists \boldsymbol{p} : \boldsymbol{p}^\top \boldsymbol{A} \geq \mathbf{0}^\top$ and $\boldsymbol{p}^\top \boldsymbol{b} < 0$.

Farkas lemma: application to asset pricing

- n different assets with m possible outcomes (or scenarios)
- Return matrix $\boldsymbol{R}$: r_{si} = return of asset i in scenario s
- x_i: amount of asset i in portfolio, with price $p_i x_i$ and return $r_{si} x_i$
  - $x_i > 0$: long position
  - $x_i < 0$: short position
- Absence of arbitrage: no investor can get a guaranteed non-negative return $\boldsymbol{Rx} \geq \boldsymbol{0}$ out of a negative investment $\boldsymbol{p}^\top \boldsymbol{x} < 0$, i.e.:

$$\text{If } \boldsymbol{Rx} \geq \boldsymbol{0} \text{ then } \boldsymbol{p}^\top \boldsymbol{x} \geq 0$$

Theorem

No arbitrage if and only if there exists a nonnegative vector $\boldsymbol{q} = (q_1, \cdots, q_m)$ such that $p_i = \sum_{s=1}^m q_s r_{si}$ for all assets $i = 1, \cdots, n$.

- Consequence of Farkas lemma
  - Arbitrage: $\boldsymbol{Rx} \geq \boldsymbol{0}$ and $\boldsymbol{p}^\top \boldsymbol{x} < 0$, that is, $\boldsymbol{x}^\top \boldsymbol{R}^\top \geq \boldsymbol{0}^\top$ and $\boldsymbol{x}^\top \boldsymbol{p} < 0$
  - No arbitrage: There exists $\boldsymbol{q} \geq \boldsymbol{0}$ such that $\boldsymbol{R}^\top \boldsymbol{q} = \boldsymbol{p}$
- Core of asset pricing in finance theory and in practice

The dual simplex algorithm

Feasibility and optimality conditions

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Basic solution:

$$\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N]$$

- Feasibility condition:

$$\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

- Optimality condition:

$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}^\top$$

→ The simplex algorithm maintains primal feasibility and works toward dual feasibility

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \end{aligned}$$

- Dual basic solution:

$$\boldsymbol{p}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$$

- Feasibility condition

$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}^\top$$

- Optimality condition:

$$\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

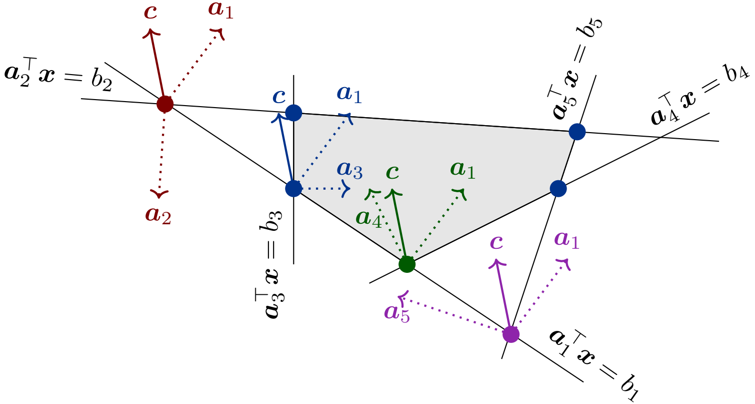
→ The dual simplex algorithm maintains dual feasibility and works toward primal feasibility

Geometry of primal simplex vs. dual simplex

→ Primal simplex: from **solutions that are primal feasible and dual infeasible** to a **solution that is primal feasible and dual feasible**

→ Dual simplex: from **solutions that are primal infeasible and dual feasible** to a **solution that is primal feasible and dual feasible**

- No visit to **solutions that are primal infeasible and dual infeasible**



Primal vs. dual simplex algorithm

$-\boldsymbol{c}_B^\top \boldsymbol{x}_B$	$\bar{c}_1$	$\cdots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_1$	$\cdots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_n$
$x_{B(m)}$	$\vert$		$\vert$

Primal simplex

- $\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N]$
- Requires $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- Cost: $\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{c}_B^\top \boldsymbol{x}_B$
- $\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B \boldsymbol{B}^{-1}\boldsymbol{b}$
- If $\overline{\boldsymbol{c}} \geq \boldsymbol{0}$, `STOP`; else, $\bar{c}_j < 0$, and $\boldsymbol{u}$: j^{th} column of $\boldsymbol{B}^{-1}\boldsymbol{A}$
- If $\boldsymbol{u} \leq \boldsymbol{0}$, cost unbounded
- Else, $l \in \arg\min_{i:u_i>0} \frac{x_{B(i)}}{u_{B(i)}}$
- $\boldsymbol{A}_j$ enters the basis, $\boldsymbol{A}_{B(l)}$ exits.

Dual simplex

- $\boldsymbol{p}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$
- Requires $\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{p}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top$
- Cost: $\boldsymbol{p}^\top \boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{x}_B$
- $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$
- If $\boldsymbol{x}_B \geq \boldsymbol{0}$, `STOP`; else, $x_{B(l)} < 0$, and $\boldsymbol{v}$: l^{th} row of $\boldsymbol{B}^{-1}\boldsymbol{A}$
- If $\boldsymbol{v} \geq \boldsymbol{0}$, cost unbounded
- Else, $j \in \arg\min_{k:v_k<0} \frac{\bar{c}_k}{|v_k|}$
- $\boldsymbol{A}_j$ enters the basis, $\boldsymbol{A}_{B(l)}$ exits.

Example

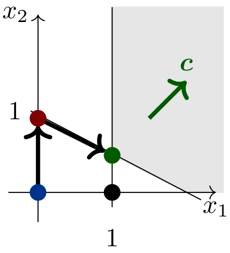
$$\min_{\boldsymbol{x} \geq \mathbf{0}} \quad x_1 + x_2$$
$$\text{s.t.} \quad x_1 + 2x_2 - x_3 = 2$$
$$x_1 - x_4 = 1$$

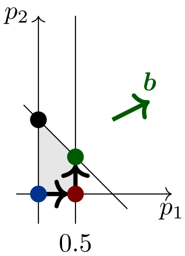
Primal

$$\min_{\boldsymbol{x} \geq \mathbf{0}} \quad x_1 + x_2$$
$$\text{s.t.} \quad x_1 + 2x_2 \geq 2$$
$$x_1 \geq 1$$

Dual

$$\max_{\boldsymbol{p} \geq \mathbf{0}} \quad 2p_1 + p_2$$
$$\text{s.t.} \quad p_1 + p_2 \leq 1$$
$$2p_1 \leq 1$$





Dual simplex

First iteration

		x_1	x_2	x_3	x_4
	0	1	1	0	0
x_3	-2	-1	-2	1	0
x_4	-1	-1	0	0	1

Second iteration

		x_1	x_2	x_3	x_4
	-1	1/2	0	1/2	0
x_2	1	1/2	1	-1/2	0
x_4	-1	-1	0	0	1

Third iteration

		x_1	x_2	x_3	x_4
	-3/2	0	0	1/2	1/2
x_2	1/2	0	1	-1/2	1/2
x_1	1	1	0	0	-1

The dual simplex algorithm

Algorithm

1. *Basis* $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \cdots, \boldsymbol{A}_{B(m)}]$ *with* $\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}^\top$.
   - *If* $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$, *STOP: Return primal optimal solution* $\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N]$ *with* $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$ *and* $\boldsymbol{x}_N = \boldsymbol{0}$; *and dual optimal solution* $\boldsymbol{p}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$
   - *Else select* l *such that* $x_{B(l)} < 0$; *PROCEED.*
2. *Compute* $\boldsymbol{v} \in \mathbb{R}^n$ *as the* l^{th} *row of the tableau*
   - *If* $\boldsymbol{v} \geq \boldsymbol{0}$, *then the dual cost is unbounded; STOP. Else, PROCEED.*
3. $\theta^* = \min_{k=1,\cdots,n, v_k<0} \dfrac{\overline{c}_k}{|v_k|}$. *Let* j *be such that* $\theta^* = \frac{\overline{c}_j}{|v_j|}$
4. *Form new basis:* $[\boldsymbol{A}_{B(1)} \cdots \boldsymbol{A}_{B(l-1)} \boldsymbol{A}_j \boldsymbol{A}_{B(l+1)} \cdots \boldsymbol{A}_{B(m)}]$
5. *Perform pivot operations to update the tableau. In particular, update* $\overline{c}_k \leftarrow \overline{c}_k + \theta^* v_k$ *for* $k = 1, \cdots, n$
6. *Go back to Step 2.*

→ The dual simplex algorithm proceeds iteratively, by increasing the cost and maintaining non-negative reduced costs until primal feasibility

Simplex and dual simplex: summary

- The variables with zero reduced costs ($\bar{c}_j = 0$) correspond to active constraints in the dual ($\boldsymbol{p}^\top \boldsymbol{A}_j = c_j$)
- Every basis $\boldsymbol{B}$ determines a basic primal solution $\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N]$ with $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$ and $\boldsymbol{x}_N = \boldsymbol{0}$; and a basic dual solution $\boldsymbol{p}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$
  - If $\boldsymbol{x}$ is non-degenerate, it is associated with a unique dual variable
- The basis induces primal feasibility and/or dual feasibility
  - $\boldsymbol{x}$ is primal feasible if and only if it is non-negative: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
  - $\boldsymbol{p}$ is dual feasible if and only if all reduced costs are non-negative: $\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}^\top$
- Primal simplex and dual simplex both iterate from one extreme point to an adjacent one
  - Primal simplex maintains primal feasibility, works toward dual feasibility
  - Dual simplex maintains dual feasibility, works toward primal feasibility
- Degeneracy can happen in the primal and the dual
  - Primal degeneracy: some basic variable is equal to zero
  - Dual degeneracy: some non-basic variable has a zero reduced cost

When to use primal simplex vs. dual simplex?

- Primal simplex: when we have a feasible primal solution and we need to work toward an optimal primal solution. Example: change in $\boldsymbol{c}$:

$$(P) \quad \begin{array}{rl} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \longrightarrow \quad (P') \quad \begin{array}{rl} \min & (\boldsymbol{c} + \Delta \boldsymbol{e}_k)^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

  - If Δ is "large", the primal solution $\boldsymbol{x}$ is feasible but no longer optimal

  $\rightarrow$ Use of primal simplex to restore dual feasibility, or primal optimality

- Dual simplex: when we have a feasible dual solution and we need to work toward a feasible primal solution. Example: change in $\boldsymbol{b}$:

$$(P) \quad \begin{array}{rl} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \longrightarrow \quad (P') \quad \begin{array}{rl} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \Delta \boldsymbol{e}_l \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

  - If Δ is "too large", the primal solution $\boldsymbol{x}$ is no longer feasible but satisfies optimality conditions

  $\rightarrow$ Use of dual simplex to restore primal feasibility

Duality toward sensitivity analysis

Intuition

$$(P) \quad \begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \quad \longrightarrow \quad (P') \quad \begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{\varepsilon} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Consider $\boldsymbol{x}^*$ an optimal (non-degenerate) solution of (P)

$$\boldsymbol{x}^* = \boxed{\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} > 0} \boxed{\boldsymbol{x}_N = \boldsymbol{0}}$$

- $\boldsymbol{x}^*$ no longer optimal for (P'), and not even feasible: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \neq \boldsymbol{b} + \boldsymbol{\varepsilon}$
- But $\boldsymbol{B}$ may still be an optimal basis for (P') if $\boldsymbol{\varepsilon}$ is small enough

$$\boxed{\boldsymbol{B}^{-1}(\boldsymbol{b} + \boldsymbol{\varepsilon}) > 0?} \boxed{\boldsymbol{x}_N = \boldsymbol{0}}$$

- What is the impact on the optimal cost?
  - Cost of (P): $\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{x}_B$
  - Cost of (P'): $\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}(\boldsymbol{b} + \boldsymbol{\varepsilon}) = \boldsymbol{c}_B^\top \boldsymbol{x}_B + \boldsymbol{p}^\top \boldsymbol{\varepsilon}$

$\rightarrow$ p_i: marginal cost, or "shadow price", of constraint $i = 1, \cdots, m$

Change in b: constraint perturbation

$$(P) \quad \begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \quad \longrightarrow \quad (P') \quad \begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \Delta \cdot \boldsymbol{e}_l \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Consider an optimal basis $\boldsymbol{B}$ for (P); is it still optimal for (P')?
  - No impact on optimality: $\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} \boldsymbol{A} \geq \boldsymbol{0}^\top$
  - The solution remains feasible if $\boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta \cdot \boldsymbol{e}_l) \geq \boldsymbol{0}$, i.e.:

$$\max_{j:\beta_{jl}>0} \left(-\frac{\bar{b}_j}{\beta_{jl}} \right) \leq \Delta \leq \min_{j:\beta_{jl}<0} \left(-\frac{\bar{b}_j}{\beta_{jl}} \right), \text{ where } \boldsymbol{B}^{-1} = [\beta_{jk}] \text{ and } \boldsymbol{B}^{-1}\boldsymbol{b} = [\bar{b}_j]$$

→ If perturbation is "small enough", the solution remains optimal and the cost increases by $\Delta \cdot p_l$:

  - Cost of (P): $\boldsymbol{c}_B^\top \boldsymbol{B}^{-1} \boldsymbol{b}$
  - Cost of (P'): $\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta \cdot \boldsymbol{e}_l) = \boldsymbol{c}_B^\top \boldsymbol{x}_B + \Delta \cdot p_l$

→ Dual variable: "shadow price" of a constraint

Change in c: cost perturbation

$$
(P)\quad \begin{array}{rl} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \longrightarrow \quad (P')\quad \begin{array}{rl} \min & (\boldsymbol{c} + \Delta \cdot \boldsymbol{e}_k)^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}
$$

- Consider an optimal basis $\boldsymbol{B}$ for (P); is it still optimal for (P')?
  - No impact on feasibility: $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
  - Impact on optimality, if x_k is a non-basic variable:

$$
c_k + \Delta \geq \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} \boldsymbol{A}_k \iff \Delta \geq -\overline{c}_k
$$

  - Impact on optimality, if x_k is a basic variable:

$$
c_j \geq (\boldsymbol{c}_B + \Delta \cdot \boldsymbol{e}_k)^\top \boldsymbol{B}^{-1} \boldsymbol{A}_j, \text{ or } \overline{c}_j \geq \Delta \cdot \alpha_{kj} \ \forall j \neq k
$$

$$
\iff \max_{j:\alpha_{kj}<0} \left(\frac{\overline{c}_j}{\alpha_{kj}} \right) \leq \Delta \leq \min_{j:\alpha_{kj}>0} \left(\frac{\overline{c}_j}{\alpha_{kj}} \right), \text{ where } \boldsymbol{B}^{-1}\boldsymbol{A} = [\alpha_{kj}]
$$

$\rightarrow$ If perturbation is “small enough”, primal solution $\boldsymbol{x}$ remains optimal

Back to manufacturing: formulation

Formulation (Primal problem)

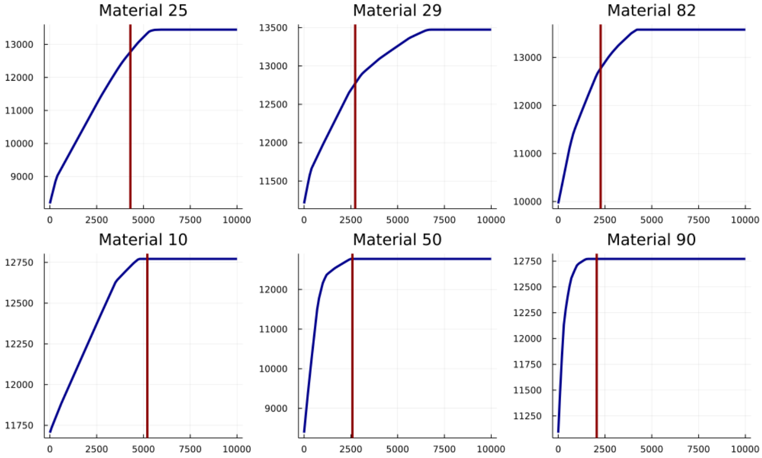
$$\begin{aligned} \max \quad & \pi_1 x_1 + \cdots + \pi_n x_n \\ \textit{s.t.} \quad & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + \cdots + a_{2n}x_n \leq b_2 \\ & \ldots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\ & x_1, \cdots, x_n \geq 0 \end{aligned}$$

Formulation (Dual problem)

$$\begin{aligned} \min \quad & b_1 p_1 + \cdots . + b_m p_m \\ \textit{s.t.} \quad & a_{11}p_1 + \cdots + a_{m1}p_m \geq \pi_1 \\ & a_{12}p_1 + \cdots + a_{m2}p_m \geq \pi_2 \\ & \ldots \\ & a_{1n}p_1 + \cdots + a_{mn}p_m \geq \pi_n \\ & p_1, \cdots, p_m \geq 0 \end{aligned}$$

- n products, m materials
- Decision x_j: quantity of product $j = 1, \cdots, n$ to manufacture
- Formulation: maximizing profit given limited resource availability
- Dual variable p_i: additional profit gained by increasing the supply of material $i = 1, \cdots, m$ by one (small) unit

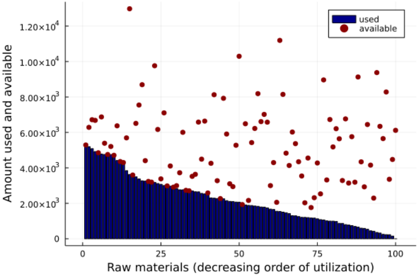
Back to manufacturing: changes in resource availability



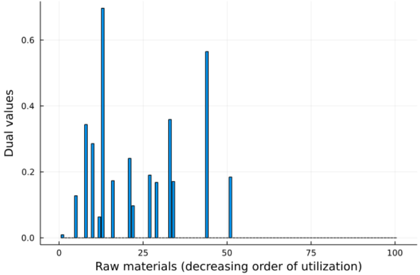
- Top row: additional resources help the company increase its profits
- Bottom row: additional resources do not lead to higher profits

Back to manufacturing: dual variables and shadow prices

Resource utilization decisions



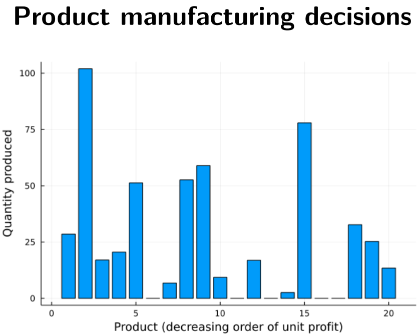
Dual variables

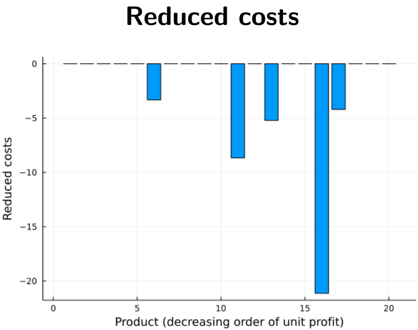


- Complementary slackness: additional resources provide gains only if the corresponding resource has already been depleted
- Dual variable: marginal benefits from added resources

→ Shadow prices guide investments in extra resources, for instance

Back to manufacturing: primal variables and reduced costs





- Optimality conditions: reduced-costs are non-zero only if the variable is not part of the optimal solution
- Reduced cost: loss induced by producing "suboptimal" products

→ Reduced costs guide product design decisions, for instance

Conclusion

Summary

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \end{aligned}$$

Takeaway

Duality as proof technique: Farkas lemma

Takeaway

Duality for algorithms: dual simplex moves from a dual BFS to an adjacent one, maintaining non-negative reduced cost until primal feasibility.

Takeaway

Duality for sensitivity analysis: interpretation of the dual variables as the "shadow price" of the constraints.