Question 1: (below $\Delta x = h$)

$$u_{xx} = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$

A) $$u_x = \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

B) $$u_x = \frac{u_j - u_{j-1}}{\Delta x}$$

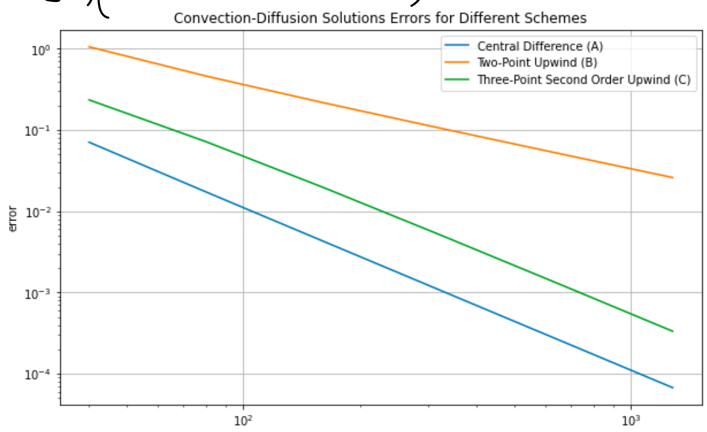
C) $$u_x = \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x}$$, for $j = N-1$, $$u_x = \frac{4u_{j+1} - 2u_j}{2\Delta x}$$

descretize equation $-u_x = \varepsilon u_{xx} + s$ into

$Ax = b$, and solve it by $x = A^{-1}b$

code implementation shows as in Q1.ipynb.

result is, (with max norm)



With linear regression, the slope is the order

order for A is 2

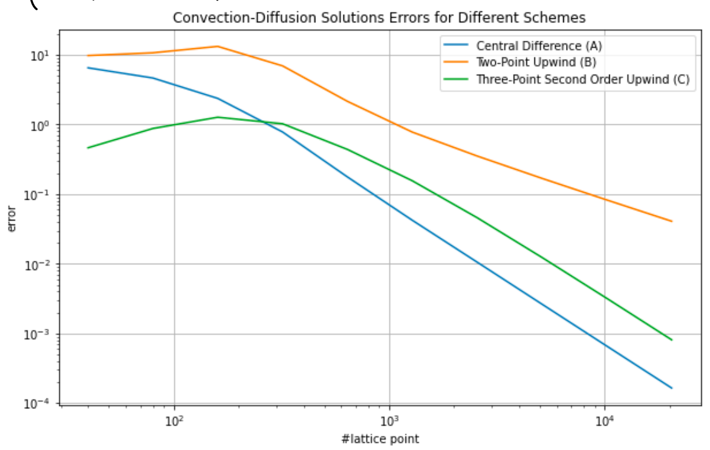
for B is 1

for C is 2.

A is most accurate

Question 2
(below $\Delta x = h$)

(max norm)



code shown in Q2.ipynb

order for A is 2

for B is 1

for C is 2

when lattice is coarse, A is the most accurate,
when lattice is fine, C is the most accurate

I can observe that relative accuracy of A and C exchange at approximately $h \sim 1/300$, and convergence order is the same as Q1

C is best for coarser lattice

Question 3

$$U_x = \frac{U_{j+1} - U_{j-1}}{2\Delta x}$$

$$U_{xx} = \frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta x^2}$$

$$U_{xxx} = \frac{-U_{j+2} + 2U_{j+1} - 2U_{j-1} + U_{j-2}}{2\Delta x^3}$$

$$U_{xxxx} = \frac{U_{j+2} - 4U_{j+1} + 6U_j - 4U_{j-1} + U_{j-2}}{\Delta x^4}$$

extra terms

$$\alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx}$$

$$= \left(-\frac{\beta}{2\Delta x^3} + \frac{\gamma}{\Delta x^4}\right) u_{j+2} + \left(\frac{\alpha}{\Delta x^2} + \frac{2}{2\Delta x^3}\beta - \frac{4}{\Delta x^4}\gamma\right) u_{j+1}$$

$$+ \left(-\frac{2}{\Delta x^2}\alpha + \frac{6}{\Delta x^4}\gamma\right) u_j$$

$$+ \left(\frac{\alpha}{\Delta x^2} - \frac{2\beta}{2\Delta x^3} - \frac{4}{\Delta x^4}\gamma\right) u_{j-1} - \left(\frac{\beta}{2\Delta x^3} + \frac{\gamma}{\Delta x^4}\right) u_{j-2}$$

i) to make it equal to B, we have

$$\begin{cases} -\dfrac{\beta}{2\Delta x^3} + \dfrac{\gamma}{\Delta x^4} = 0 \\ \dfrac{\beta}{2\Delta x^3} + \dfrac{\gamma}{\Delta x^4} = 0 \\ -\dfrac{2\epsilon}{\Delta x^2} - \dfrac{1}{\Delta x} = -\dfrac{2}{\Delta x^2}(\epsilon+\alpha) \\ \dfrac{\epsilon}{\Delta x^2} + \dfrac{1}{\Delta x} = \dfrac{\epsilon+\alpha}{\Delta x^2} + \dfrac{1}{2\Delta x} \\ \dfrac{\epsilon}{\Delta x^2} = \dfrac{\epsilon+\alpha}{\Delta x^2} - \dfrac{1}{2\Delta x} \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = \dfrac{\Delta x}{2} = \dfrac{h}{2} \\ \beta = \gamma = 0 \end{cases}$$

ii) To make it identical to C

$$\begin{cases} -\dfrac{\beta}{2\Delta x^3} + \dfrac{\gamma}{\Delta x^4} = 0 \\ \dfrac{\beta}{2\Delta x^3} + \dfrac{\gamma}{\Delta x^4} = -\dfrac{1}{2\Delta x} \end{cases}$$

$$-\frac{2(\epsilon+\alpha)}{\Delta x^2} + \frac{6\gamma}{\Delta x^4} = -\frac{2\epsilon}{\Delta x^2} - \frac{3}{2\Delta x}$$

$$\Rightarrow \begin{cases} \alpha = 0 \\ \beta = -\frac{\Delta x^2}{2} = -\frac{h^2}{2} \\ \gamma = -\frac{\Delta x^3}{4} = -\frac{h^3}{4} \end{cases}$$

Comment: we can observe that in i), α, β, γ are bounded by $O(h)$, while in ii), α, β, γ are bounded by $O(h^2)$, this in coincide with convergence order of B and C respectively. This implies that the discretization formulism permits an error (which is $\alpha U_{xx} + \beta U_{xxx} + \gamma U_{xxxx}$) bounded by its convergence order