Duality in linear optimization

15.093: Optimization

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Dual problem: motivation and formulation

Lagrangian relaxation and duality

- Consider a linear optimization problem, and relax the constraints
  - $\rightarrow$ Assign a cost p_i to violating each constraint $\boldsymbol{a}_i^\top \boldsymbol{x} = b_i$

$$
(LO) \quad \begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \qquad \longrightarrow \qquad g(\boldsymbol{p}) = \begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{p}^\top (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}
$$

- The relaxed problem provides a lower bound of the optimal cost

$$
g(\boldsymbol{p}) \leq \boldsymbol{c}^\top \boldsymbol{x}^* + \boldsymbol{p}^\top \underbrace{(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*)}_{=0} = \boldsymbol{c}^\top \boldsymbol{x}^* \quad \text{where } \boldsymbol{x}^* \text{ solves (LO)}
$$

$\rightarrow$ Seek the tightest possible lower bound: $\max g(\boldsymbol{p})$

$$
g(\boldsymbol{p}) = \boldsymbol{p}^\top \boldsymbol{b} + \underbrace{\min_{\boldsymbol{x} \geq \boldsymbol{0}} \left((\boldsymbol{c}^\top - \boldsymbol{p}^\top \boldsymbol{A}) \boldsymbol{x} \right)}_{\begin{aligned} &= 0 && \text{if } c_j - \boldsymbol{p}^\top \boldsymbol{A}_j \geq 0, \ \forall j \\ &= -\infty && \text{otherwise} \end{aligned}} \qquad \longrightarrow \qquad \underbrace{\max_{\boldsymbol{p} \,:\, \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top} \left(\boldsymbol{p}^\top \boldsymbol{b} \right)}_{\textbf{dual problem}}
$$

Dual formulation

Formulation (Dual of standard form problem)

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \textit{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \textit{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \end{aligned}$$

Formulation (General form of the dual for minimization problems)

Primal problem

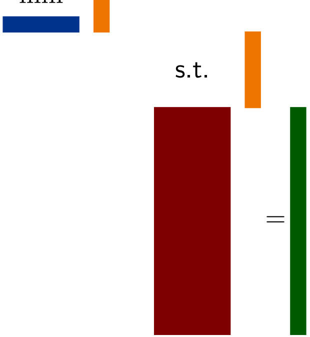
$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \textit{s.t.} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i, \ \forall i \in \mathcal{M}_1 \\ & \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \ \forall i \in \mathcal{M}_2 \\ & \boldsymbol{a}_i^\top \boldsymbol{x} = b_i, \ \forall i \in \mathcal{M}_0 \\ & x_j \geq 0, \ \forall j \in \mathcal{N}_1 \\ & x_j \leq 0, \ \forall j \in \mathcal{N}_2 \\ & x_j \ \textit{free}, \ \forall j \in \mathcal{N}_0 \end{aligned}$$

Dual problem

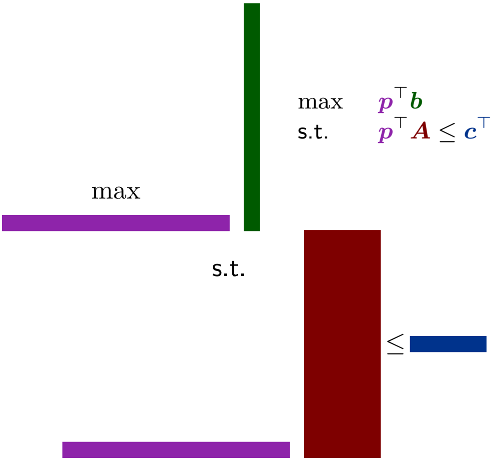
$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \textit{s.t.} \quad & p_i \geq 0, \ \forall i \in \mathcal{M}_1 \\ & p_i \leq 0, \ \forall i \in \mathcal{M}_2 \\ & p_i \ \textit{free}, \ \forall i \in \mathcal{M}_0 \\ & \boldsymbol{p}^\top \boldsymbol{A}_j \leq c_j, \ \forall j \in \mathcal{N}_1 \\ & \boldsymbol{p}^\top \boldsymbol{A}_j \geq c_j, \ \forall j \in \mathcal{N}_2 \\ & \boldsymbol{p}^\top \boldsymbol{A}_j = c_j, \ \forall j \in \mathcal{N}_0 \end{aligned}$$

Dual transformations

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$



$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \end{aligned}$$



Theorem

The dual of the dual is the primal

Example: manufacturing problem

Formulation (Primal problem)

$$\begin{aligned} \max \quad & \pi_1 x_1 + \cdots + \pi_n x_n \\ \textit{s.t.} \quad & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + \cdots + a_{2n}x_n \leq b_2 \\ & \ldots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\ & x_1, \cdots, x_n \geq 0 \end{aligned}$$

Formulation (Dual problem)

$$\begin{aligned} \min \quad & b_1 p_1 + \cdots . + b_m p_m \\ \textit{s.t.} \quad & a_{11}p_1 + \cdots + a_{m1}p_m \geq \pi_1 \\ & a_{12}p_1 + \cdots + a_{m2}p_m \geq \pi_2 \\ & \ldots \\ & a_{1n}p_1 + \cdots + a_{mn}p_m \geq \pi_n \\ & p_1, \cdots, p_m \geq 0 \end{aligned}$$

- Reminder on the manufacturing problem
  - Decisions: quantities of each of n products to manufacture
  - Maximize profit, subject to limited availability of m resources
- Dual formulation: perspective of a procurement division
  - Decision: amount paid for acquiring each resource
  - Minimize total spend, while ensuring that the supplier does not have incentives to sell products directly to your customers

Example: transportation problem

Formulation (Primal problem)

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\textit{s.t.} \quad & \sum_{i=1}^{m} x_{ij} \geq d_j, && \forall j = 1, \cdots, n \\
& \sum_{j=1}^{n} x_{ij} \leq s_i, && \forall i = 1, \cdots, m \\
& x_{ij} \geq 0, && \forall i, j
\end{aligned}
$$

Formulation (Dual problem)

$$
\begin{aligned}
\max \quad & -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
\textit{s.t.} \quad & v_j - u_i \leq c_{ij}, \ \forall i, j \\
& u_i \geq 0, \ \forall i \\
& v_j \geq 0, \ \forall j
\end{aligned}
$$

- Reminder on the transportation problem
  - Decisions: shipments from each of n factories to each of m markets
  - Minimize cost, subject to capacity restrictions and demand requirements
- Dual formulation: perspective of a third-party provider
  - Decision: price collected at destination (v_j), amount paid at origin (u_i)
  - Maximize profit, as long as the company will not internalize shipping

Duality theory

Weak duality

Theorem (Weak duality)

If $\boldsymbol{x}$ is primal feasible and $\boldsymbol{p}$ is dual feasible, then $\boldsymbol{p}^\top \boldsymbol{b} \leq \boldsymbol{c}^\top \boldsymbol{x}$

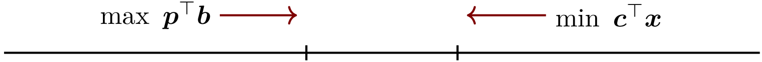
- Proof: $\boldsymbol{p}^\top \boldsymbol{b} = \boldsymbol{p}^\top(\boldsymbol{A}\boldsymbol{x}) = (\boldsymbol{p}^\top \boldsymbol{A})\boldsymbol{x} \leq \boldsymbol{c}^\top \boldsymbol{x}$ since $\boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top$ and $\boldsymbol{x} \geq \boldsymbol{0}$

Corollary

The optimal dual value is less than the optimal primal cost.

Corollary

If $\boldsymbol{x}$ is primal feasible, $\boldsymbol{p}$ is dual feasible, and $\boldsymbol{p}^\top \boldsymbol{b} = \boldsymbol{c}^\top \boldsymbol{x}$, then $\boldsymbol{x}$ is optimal in the primal and $\boldsymbol{p}$ is optimal in the dual.



Strong duality

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \end{aligned}$$

Theorem (Strong duality)

If (LO) has an optimal solution, so does the dual, and the optimal dual value is equal to the optimal primal cost.

- Proof of strong duality
  - Basic feasible solution: $\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N]$ with $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$, $\boldsymbol{x}_N = \boldsymbol{0}$
  - Optimality conditions from the simplex: $\bar{c}_j = c_j - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A}_j \geq 0, \ \forall j$
  - Define $\boldsymbol{p}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$
  - $\boldsymbol{p}$ is a feasible solution of the dual problem: $\boldsymbol{p}^\top \boldsymbol{A}_j = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A}_j \leq c_j$
  - Moreover, $\boldsymbol{x}$ and $\boldsymbol{p}$ achieve the same objective value:
$$\boldsymbol{p}^\top \boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{x}_B = \boldsymbol{c}^\top \boldsymbol{x}$$

$\rightarrow$ By weak duality, $\boldsymbol{x}$ and $\boldsymbol{p}$ optimal

Four possible outcomes in linear optimization

1. The primal problem is infeasible and the dual problem is infeasible

$$\max\ \boldsymbol{p}^\top \boldsymbol{b} = -\infty \qquad\qquad \min\ \boldsymbol{c}^\top \boldsymbol{x} = +\infty$$

2. The primal problem is unbounded and the dual problem is infeasible

$$\max\ \boldsymbol{p}^\top \boldsymbol{b} = -\infty \longleftarrow \min\ \boldsymbol{c}^\top \boldsymbol{x}$$

3. The primal problem and the dual problem have the same optimal values

$$\max\ \boldsymbol{p}^\top \boldsymbol{b} \longrightarrow \longleftarrow \min\ \boldsymbol{c}^\top \boldsymbol{x}$$

4. The primal problem is infeasible and the dual problem is unbounded

$$\max\ \boldsymbol{p}^\top \boldsymbol{b} \longrightarrow \min\ \boldsymbol{c}^\top \boldsymbol{x} = +\infty$$

Complementary slackness

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{aligned}$$

Theorem

Let $\boldsymbol{x}$ primal feasible and $\boldsymbol{p}$ dual feasible. $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal if and only if

$$x_j(c_j - \boldsymbol{p}^\top \boldsymbol{A}_j) = 0, \ \forall j = 1, \cdots, n$$

$$p_i(\boldsymbol{a}_i^\top \boldsymbol{x} - b_i) = 0, \ \forall i = 1, \cdots, m$$

- Interpretation
  - If a solution is non-zero, then its reduced cost is zero; vice versa, if a reduced cost is non-zero, then the solution is zero
  - If a dual price is non-zero, then the constraint is active (or binding); if a constraint is not active (or not binding), then the dual price is zero

Primal feasibility, dual feasibility, and optimality

Standard form: primal and dual feasibility

Primal BFS

$$A = \begin{bmatrix} B & N \end{bmatrix}$$

$$x^\top = \begin{bmatrix} \underbrace{x_B = B^{-1}b}_{\geq 0} & x_N = \mathbf{0} \end{bmatrix}$$

$$c^\top = \begin{bmatrix} c_B & c_N \end{bmatrix}$$

$$\bar{c}^\top = \begin{bmatrix} \mathbf{0} & \underbrace{c_j - c_B^\top B^{-1} A_j}_{\geq 0?} \end{bmatrix}$$

$$p^\top = \begin{bmatrix} c_B \end{bmatrix} \begin{bmatrix} B^{-1} \end{bmatrix}$$

$$p^\top A = \begin{bmatrix} \underbrace{c_B}_{\text{binding}} & \underbrace{c_B^\top B^{-1} A_j}_{\leq c_j?} \end{bmatrix}$$

$$p^\top b = c_B^\top x_B$$

Dual BFS

$$p^\top A = \begin{bmatrix} \underbrace{c_j}_{\text{binding}} & \underbrace{\leq c_j}_{\text{feasible}} \end{bmatrix}$$

$$c^\top = \begin{bmatrix} c_B & c_N \end{bmatrix}$$

$$A = \begin{bmatrix} B & N \end{bmatrix}$$

$$x_B = \begin{bmatrix} B^{-1} \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$$

$$x^\top = \begin{bmatrix} \underbrace{x_B = B^{-1}b}_{\geq 0?} & x_N = \mathbf{0} \end{bmatrix}$$

$$\bar{c}^\top = \begin{bmatrix} \mathbf{0} & \underbrace{c_j - c_B^\top B^{-1} A_j}_{\geq 0} \end{bmatrix}$$

$$c^\top x = c_B^\top B^{-1} b = p^\top b$$

Standard form: feasibility and optimality conditions

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Basic solution:

$$\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N], \quad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

- Feasibility condition

$$\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

- Optimality condition:

$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}^\top$$

- A primal BFS $\boldsymbol{x}$ defines a dual basic solution $\boldsymbol{p}$
- If $\boldsymbol{p}$ is dual feasible, $\boldsymbol{x}$ is primal optimal and $\boldsymbol{p}$ is dual optimal

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \end{aligned}$$

- Dual basic solution:

$$\boldsymbol{p}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$$

- Feasibility condition

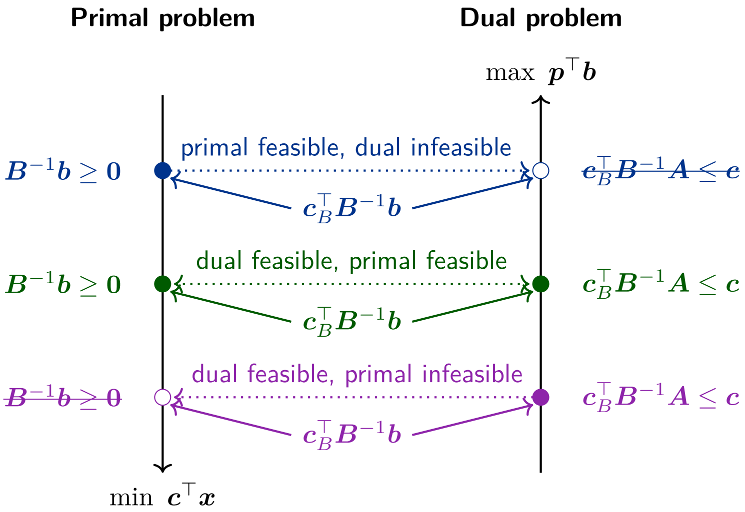
$$\overline{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}^\top$$

- Optimality condition:

$$\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

- A dual BFS $\boldsymbol{p}$ defines a primal basic solution $\boldsymbol{x}$
- If $\boldsymbol{x}$ is primal feasible, $\boldsymbol{x}$ is primal optimal and $\boldsymbol{p}$ is dual optimal

Summary and visualization



Problem in general form

Primal problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i, \ \forall i = 1, \cdots, m \end{aligned}$$

Dual problem

$$\begin{aligned} \max \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \sum_{i=1}^{m} p_i \boldsymbol{a}_i = \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{aligned}$$

- Consider a non-degenerate basic feasible solution $\boldsymbol{x}^I$:

$$\boldsymbol{a}_i^\top \boldsymbol{x}^I = b_i, i \in I, \quad \text{with } |I| = n \text{ and } \{\boldsymbol{a}_i : i \in I\} \text{ linearly independent}$$

- Let $\boldsymbol{p} \in \mathbb{R}^m$; what does it take for $\boldsymbol{x}^I$ and $\boldsymbol{p}$ to be optimal?

Primal feasibility

$$\boldsymbol{a}_i^\top \boldsymbol{x}^I \geq b_i, \forall i \in I$$

Comp. slackness

$$p_i = 0, \ \forall i \notin I$$

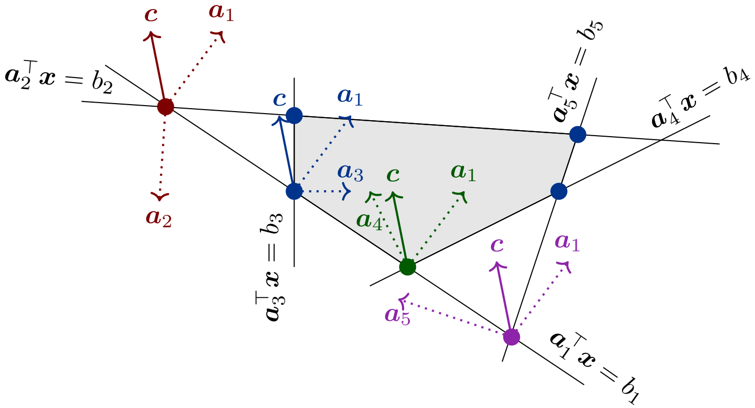
Dual feasibility

$$\sum_{i \in I} p_i \boldsymbol{a}_i = \boldsymbol{c}, \ \boldsymbol{p} \geq \boldsymbol{0}$$

→ Primal feasibility: $\boldsymbol{x}^I$ on the "correct side" of all hyperplanes $\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i$

→ Dual feasibility: the cost vector $\boldsymbol{c}$ can be written as a non-negative linear combination of vectors defining active constraints

Geometry of duality



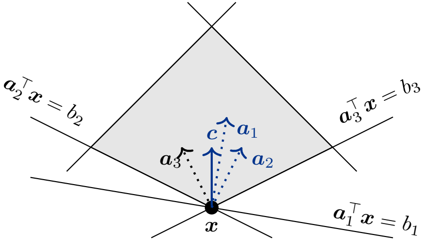
- Solution that is primal infeasible and dual infeasible
- Solution that is primal feasible and dual infeasible
- Solution that is primal infeasible and dual feasible
- Solution that is primal feasible and dual feasible, hence optimal

Duality and degeneracy

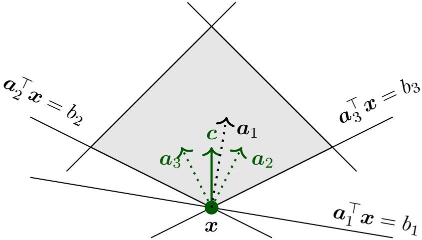
- A non-degenerate primal solution defines a unique dual solution
  - $\boldsymbol{x}$ is defined by a unique subset I such that $\boldsymbol{a}_i^\top \boldsymbol{x} = b_i, \ \forall i \in I$

  → $\boldsymbol{x}$ is associated with a unique dual solution $\boldsymbol{p}$, such that $\sum_{i \in I} p_i \boldsymbol{a}_i = \boldsymbol{c}$
- Now, what if $\boldsymbol{x}$ is a degenerate solution?
  - There exist several subsets I such that $\boldsymbol{a}_i^\top \boldsymbol{x} = b_i, \ \forall i \in I$
  - Each subset I induces a dual solution $\boldsymbol{p}^I$, such that $\sum_{i \in I} p_i^I \boldsymbol{a}_i = \boldsymbol{c}$
  - $\boldsymbol{x}$ is optimal as long as *one* of the $\boldsymbol{p}^I$ solutions is dual feasible

Dual infeasible solution



Dual feasible solution



Conclusion

Summary

Takeaway

Dual formulation: one dual variable per primal constraint, one dual constraint per primal variable, with appropriate dual transformations

Takeaway

A feasible dual solution yields a lower bound to the primal problem.

Takeaway

There is no "duality gap" in linear optimization: if the problem admits an optimal solution, so does its dual and both optimal values coincide.

Takeaway

Each basis defines a basic primal solution and a basic dual solution. If both solutions are feasible, then they are both optimal.