hw 3.

Problem 1.

(a) for $\forall v \in X$, and $u \in X$ satisfying the strong formulation, suppose that $w = u + v$

then $\frac{1}{2} a(w,w) - l(w) = \frac{1}{2} a(u,u) - l(u)$ (use linear and bilinear)

$$+ a(u,v) - l(v)$$
$$+ \frac{1}{2} a(v,v)$$

where
$$a(u,v) - l(v) = \int_0^{1/2} k^L u_x v_x - f^L v \, dx + \int_{1/2}^1 k^R u_x v_x - f^R v \, dx$$
$$= k^L u_x^L v \Big|_0^{1/2} + \int_0^{1/2} -k^L u_{xx}^L v - f^L v \, dx$$
$$+ k^R u_x^R v \Big|_{1/2}^1 + \int_{1/2}^1 (-k^R u_{xx}^R v - f^R v) \, dx$$

with $v(0) = v(1) = 0$, $k^L u_x(\frac{1}{2}) = k^R u_x(\frac{1}{2})$, we have $k^L u_x^L v\big|_0^{1/2} + k^R u_x v\big|_{1/2}^1 = 0$

and $-k^L u_{xx}^L - f^L = -k^R u_{xx}^R - f^R = 0$, so we have

$a(u,v) - l(v) = 0$ ①

Furthermore, $a(v,v) = \int_0^{1/2} k^L v_x v_x dx + \int_{1/2}^1 k^R v_x v_x dx \geq 0$

so
$$\frac{1}{2} a(w,w) - l(w) = \frac{1}{2} a(u,u) - l(u) + \frac{1}{2} a(v,v)$$
$$\geq \frac{1}{2} a(u,u) - l(u)$$

which means that $u = \arg\min_{w \in X} \frac{1}{2} a(w,w) - l(w)$

from ①, we know that

$$a(u,v) = l(v)$$

(b) $u^L(0) = 0$, $u^R(1) = 0$, $u^L(\frac{1}{2}) = u^R(\frac{1}{2})$ are essential boundary / interface conditions

$-k^L u_x^L(\frac{1}{2}) = -k^R u_x^R(\frac{1}{2})$ is natural boundary / interface condition.

(c) because $k^L u_x^L(\frac{1}{2}) = k^R u_x^R(\frac{1}{2})$ leads to a delta function in $x = \frac{1}{2}$

$u_{xx}$ contains $(u_x(\frac{1}{2}+0) - u_x(\frac{1}{2}-0))\,\delta(x-\frac{1}{2})$, because

$\int_0^1 u_{xx}^2 dx$ = bounded terms + $\int_0^1 (\delta(x))^2 dx$ which is unbounded, so $u \notin H^2(\Omega)$

However, $\int_0^1 u_x^2 dx$ is bounded. Although there is a discontinuous point at $\frac{1}{2}$, but it will not cause any problem. so $u \in H^1(\Omega)$

Problem 2.

(a) $\int_V dV(\nabla^2 u + f)v = \int_V dV \left(\nabla\cdot((\nabla u)v) - (\nabla u)(\nabla v) + fv\right)$

$= \int_S ds\,(\hat{n}\cdot\nabla u)v + \int_V \left(-(\nabla u)\cdot(\nabla v) + fv\right) dV$

$= \int_S ds \frac{\partial u}{\partial n}\cdot v\Big|_{\Gamma} + \int_V dv\, -(\nabla u)\cdot(\nabla v) + fv$

because $-\frac{\partial u}{\partial n}\Big|_{\Gamma R} = h_c u\Big|_{\Gamma R}$ $\quad v\Big|_{\Gamma D} = 0$

$\int_V dV\,(\nabla^2 u + f)v = -\int_{\Gamma R} ds\, h_c uv + \int_V dV\,(-\nabla u\cdot\nabla v + fv)$

when $u$ is the solution of strong formulation, LHS $=0$, so RHS $=0$.

Suppose that $a(u,v) = \int_\Omega dV\, \nabla u\cdot\nabla v + \int_{\Gamma R} ds\, h_c uv$

$l(v) = \int_\Omega dV\, fv$

This satisfies that $a(u,v) = l(v)$ for $\forall v \in X$

functional $J(w) = \frac{1}{2}a(w,w) - l(w)$

$= \frac{1}{2}\int_\Omega dV\,(\nabla w)^2 + \frac{1}{2}\int_{\Gamma R} ds\, h_c w^2 - \int_\Omega dV\, fw$

(b) $u\Big|_{\Gamma D} = 0$ is essential, $-\frac{\partial u}{\partial n}\Big|_{\Gamma R} = h_c u\Big|_{\Gamma R}$ is natural

Problem 3

(a) $\int_0^1 (u_{xxxx} - f)v\,dx = u_{xxx}v\Big|_0^1 - \int_0^1 u_{xxx}v_x\,dx - \int_0^1 fv\,dx$

$= u_{xxx}v\Big|_0^1 - u_{xx}v_x\Big|_0^1 + \int_0^1 u_{xx}v_{xx}\,dx - \int_0^1 fv\,dx$

we can define $a(u,v) = \int_0^1 u_{xx}v_{xx}\,dx$

$$\ell(v) = \int_0^1 f\, v\, dx$$

$$X = \{ v \in H^2(\Omega) \mid v(0) = v(1) = 0,\ v_x(0) = v(1) = 0 \}$$

So that $\int_0^1 (u_{xxxx} - f) v\, dx = a(u,v) - \ell(v)$ for $\forall v \in X$

if $u$ is strong formulation solution, then $a(u,v) = \ell(v)$ for $\forall v \in X$

$$J(w) = \int_0^1 (w_{xx})^2 dx - \int_0^1 f\, w\, dx$$

(b) $X = \{ v \in H^2(\Omega) \mid v(0) = v(1) = 0,\ v_x(0) = v_x(1) = 0 \}$

here we use $H^2(\Omega)$ because $J(w)$ contains $\int_0^1 w_{xx}^2 dx$, which should be bounded

(c) $|\ell(v)| = v_x(\frac{1}{2})$

$= \int_0^{1/2} v_{xx}(x)\, dx$ (here we use $v_x(0) = 0$ for $v \in X$)

$\leq \sqrt{\int_0^{1/2} 1\, dx \int_0^{1/2} v_{xx}^2(x)\, dx}$

$= \frac{1}{\sqrt{2}} \sqrt{\int_0^{1/2} v_{xx}^2\, dx}$

$\leq \frac{1}{\sqrt{2}} \sqrt{\int_0^1 v_{xx}^2\, dx}$

$\|v\|_X = \sqrt{\int_0^1 v^2 + v_x^2 + v_{xx}^2\, dx}$

$\geq \sqrt{\int_0^1 v_{xx}^2\, dx}$

So we have $|\ell(v)| \leq \frac{1}{2} \|v\|_X$, i.e. $C = \sqrt{1/2}$

Problem 4.

(a) for $\forall v \in X$

$$\int_0^1 (u_{xxxx} - f) v\, dx = u_{xxx} v \Big|_0^1 - \int_0^1 u_{xxx} v_x\, dx - \int_0^1 f v\, dx$$

$$= u_{xxx} v \Big|_0^1 - u_{xx} v_x \Big|_0^1 + \int_0^1 u_{xx} v_{xx}\, dx - \int_0^1 f v\, dx$$

we can define $a(u,v) = \int_0^1 u_{xx} v_{xx} \, dx$

$l(v) = \int_0^1 f v \, dx$

because $u_{xx}(0) = u_{xx}(1) = 0$, so $u_{xx} v_x \Big|_0^1 = 0$

$v \in X$, so $v(0) = v(1) = 0$, so $u_{xxx} v \Big|_0^1 = 0$

So

$$\int_0^1 (u_{xxxx} - f) v \, dx = a(u,v) - l(v)$$

so we can obtain the same $a(u,v)$ and $l(v)$, but using only $v(0) = v(1) = 0$

c) $u(0) = 0$, $u(1) = 0$ are essential $u_{xx}(0) = 0$ $u_{xx}(1) = 0$ are natural