16.920J/6.339J/2.097J - Numerical Methods for Partial Differential Equations

Massachusetts Institute of Technology

Homework 3 version 1.0

Due: November 1, 2023

Problem 1 (50 points) (Exercise 9 from FEM-1 notes)

Consider a problem with a discontinuous jump in conductivities

$$
\begin{aligned}
-\kappa^{\mathrm{L}} u^{\mathrm{L}}_{xx} &= f^{\mathrm{L}} \qquad 0 < x < \tfrac{1}{2} \ , \\
-\kappa^{\mathrm{R}} u^{\mathrm{R}}_{xx} &= f^{\mathrm{R}} \qquad \tfrac{1}{2} < x < 1 \ ,
\end{aligned}
$$

with boundary conditions

$$u^{\mathrm{L}}(0) = 0, \qquad u^{\mathrm{R}}(1) = 0 \ ,$$

$$u^{\mathrm{L}}(\tfrac{1}{2}) = u^{\mathrm{R}}(\tfrac{1}{2}) \qquad \text{(continuity of solution)} \ ,$$

$$-\kappa^{\mathrm{L}} u^{\mathrm{L}}_{x}(\tfrac{1}{2}) = -\kappa^{\mathrm{R}} u^{\mathrm{R}}_{x}(\tfrac{1}{2}) \qquad \text{(continuity of flux)} \ ;$$

here κ^{L} and κ^{R} are strictly positive.

(a) For $X = \{v \in H^1((0,1)) \mid v(0) = 0,\ v(1) = 0\}$, show that

$$u = \arg\min_{w \in X} \ \tfrac{1}{2}\, a(w,w) - \ell(w) \ ,$$

and

$$a(u,v) = \ell(v), \qquad \forall\, v \in X \ ,$$

where

$$
\begin{aligned}
a(w,v) &= \int_0^{1/2} \kappa^{\mathrm{L}}\, w_x\, v_x\, dx + \int_{1/2}^{1} \kappa^{\mathrm{R}}\, w_x\, v_x\, dx \ , \\
\ell(w) &= \int_0^{1/2} f^{\mathrm{L}}\, v\, dx + \int_{1/2}^{1} f^{\mathrm{R}}\, v\, dx \ .
\end{aligned}
$$

(b) In this problem, which boundary/interface conditions are essential, and which are natural?

(c) Is the solution to this problem in $H^2(\Omega)$? in $H^1(\Omega)$?

Problem 2 (50 points) (Exercise 10 from FEM-1 notes)

Consider the Robin problem

$$
\begin{aligned}
-\nabla^2 u &= f \qquad && \text{in } \Omega \\
u &= 0 && \text{on } \Gamma^D \\
-\frac{\partial u}{\partial n} &= h_c u && \text{on } \Gamma^R \quad (\overline{\Gamma} = \overline{\Gamma^D} \cup \overline{\Gamma^R})
\end{aligned}
$$

where $h_c > 0$ (recall that $\frac{\partial u}{\partial n}$ refers to the outward normal on Γ).

(a) Find the functional J (and hence a and ℓ) such that

$$u = \arg\min_{w \in X} \; J(w) = \tfrac{1}{2}\, a(w,w) - \ell(w) \; ,$$

and

$$a(u,v) = \ell(v), \qquad \forall\, v \in X \; ,$$

where $X = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = 0\}$. *Hint*: multiply the equation by v, integrate by parts, and substitute $-h_c u$ for $\frac{\partial u}{\partial n}$ on the boundary; identify a and ℓ.

(b) In this problem, which boundary conditions are essential, and which are natural?

Problem 3 (OPTIONAL for an additional 10 points) (Exercise 7 from FEM-1 notes)

Consider the fourth-order problem

$$u_{xxxx} = f \qquad \text{in } \Omega = (0,1) \; ,$$
$$u(0) = u_x(0) = u(1) = u_x(1) = 0 \; ;$$

this "biharmonic" equation is relevant to, amongst other applications, the bending of beams.

(a) Find an SPD bilinear form a over X and a linear form ℓ such that

$$u = \arg\min_{w \in X} \; J(w) = \tfrac{1}{2}\, a(w,w) - \ell(w)$$
$$\Updownarrow$$
$$a(u,v) = \ell(v), \qquad \forall\, v \in X \; ,$$

where $w \in X$ are sufficiently smooth and satisfy $w(0) = w_x(0) = w(1) = w_x(1) = 0$. (*Hint*: work backwards, multiplying the strong form by v, and integrating by parts and applying the boundary conditions until symmetry "appears".)

(b) How should X be defined — which Hilbert space $H^m(\Omega)$ do you think is appropriate?

(c) Do you think that $\ell(v) = v_x(\frac{1}{2})$ is an admissible linear functional, in the sense that $\ell \in X'$, that is, $|\ell(v)| \le C\|v\|_X$, $\forall\, v \in X$? (*Hint*: see note 7 in FEM-1)

Problem 4 (OPTIONAL for an additional 10 points after successfully completing Probelm 3) (Exercise 11 from FEM-1 notes)

Consider the fourth-order problem

$$u_{xxxx} = f \qquad \text{in } \Omega = (0,1) \; ,$$
$$u(0) = u_{xx}(0) = u(1) = u_{xx}(1) = 0 \; .$$

(a) Show that the minimization statement of Problem 1 still applies, but that now members v of X need only satisfy $v(0) = v(1) = 0$ (not $v_x(0) = v_x(1) = 0$ as before, or $v_{xx}(0) = v_{xx}(1) = 0$).

(b) Which boundary conditions are essential, and which are natural?