Geometry of linear optimization

15.093: Optimization

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Reminder: linear optimization

Formulation (General form)

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \textit{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

Formulation (Standard form)

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \textit{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- A linear optimization problem minimizes or maximizes a linear objective function over a feasible region defined by linear constraints
- Notation: $\boldsymbol{a}_i$ is the i^{th} row of the matrix $\boldsymbol{A}$ and $\boldsymbol{A}_j$ is its j^{th} column

$$\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \iff \boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i, \forall i = 1, \cdots, m \iff \sum_{j=1}^{n} \boldsymbol{A}_j x_j \geq \boldsymbol{b}$$

- Every linear optimization problem can be written in standard form
- The structure of linear optimization problems involves the geometry of the **polyhedron** defining the feasible region
  - Characterization of a "corner" in a polyhedron via a **basis**
  - Critical role of "corners" of a polyhedron in linear optimization

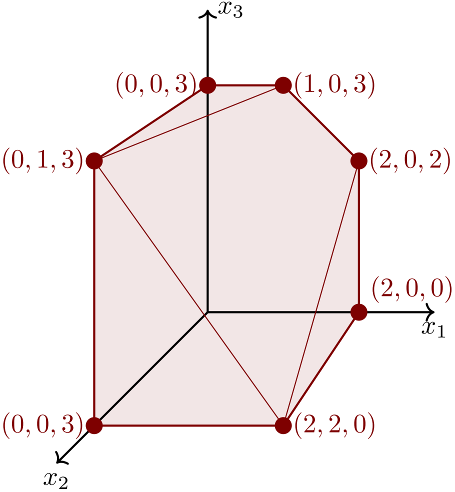
Linear optimization example

General form

$$\begin{array}{lllll} \min & x_1 + 5x_2 - 2x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_3 & \leq & 4 \\ & x_1 & \leq & 2 \\ & x_3 & \leq & 3 \\ & 3x_2 + x_3 & \leq & 6 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Standard form

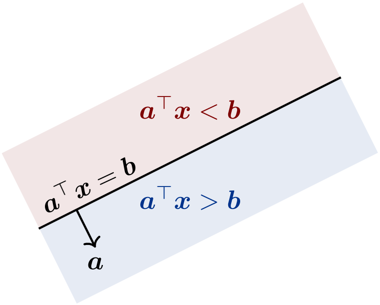
$$\begin{array}{llll} \min & x_1 + 5x_2 - 2x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_3 + s_1 & = & 4 \\ & x_1 + s_2 & = & 2 \\ & x_3 + s_3 & = & 3 \\ & 3x_2 + x_3 + s_4 & = & 6 \\ & x_1, x_2, x_3, s_1, \cdots, s_4 & \geq & 0 \end{array}$$

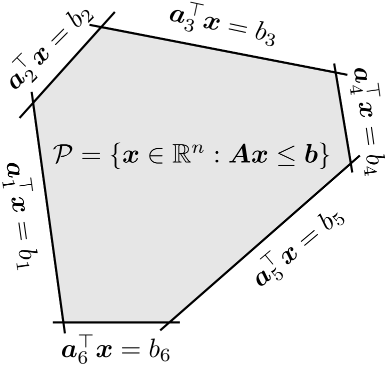


Geometry of polyhedron

Polyhedron: definitions

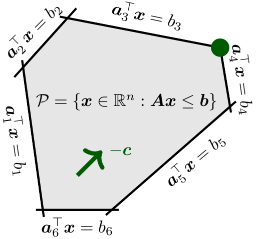
- A linear constraint creates a *hyperplane*: $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^\top \boldsymbol{x} = \boldsymbol{b}\}$
- A hyperplane defines a *halfspace*: $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^\top \boldsymbol{x} \leq \boldsymbol{b}\}$
- Finite intersection of halfspaces is a *polyhedron*: $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$
- A bounded polyhedron is called a *polytope*



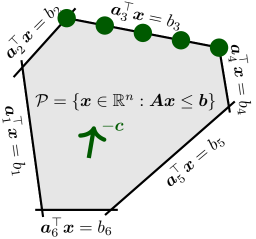


Possible outcomes in linear optimization

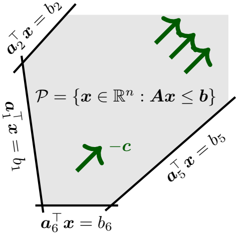
Unique optimal solution



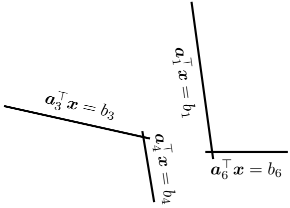
Multiple optimal solutions



Unboundedness: no optimum



Infeasibility: no feasible solution



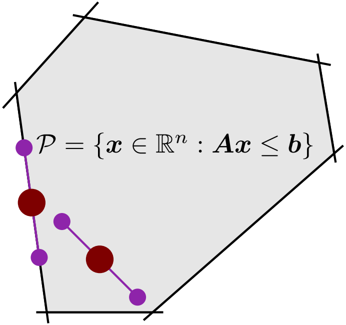
Corner of polyhedron: extreme point

Definition

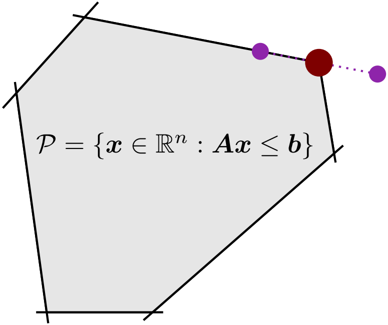
$\boldsymbol{x}$ is an extreme point of polyhedron $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ if:

$$\nexists \boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}, \lambda \in (0,1) : \boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z}.$$

Not extreme points



Extreme point

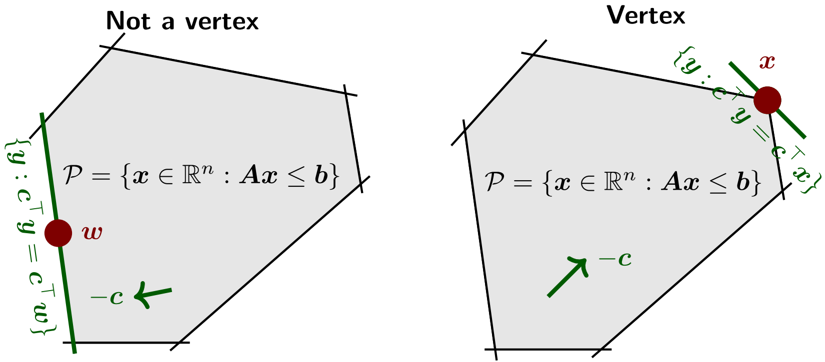


Corner of polyhedron: vertices

Definition

$\boldsymbol{x}$ is a vertex of polyhedron $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ if:

$$\exists \boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{x} \text{ is the unique optimum of } \min_{\boldsymbol{x} \in \mathcal{P}} \boldsymbol{c}^\top \boldsymbol{x}.$$



Corner of polyhedron: basic feasible solutions (BFS)

- Consider a polyhedron $\mathcal{P}$ with equality and inequality constraints:

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\top \boldsymbol{x} = b_i, \forall i \in \mathcal{M}_0,\ \boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i, \forall i \in \mathcal{M}_1\}$$

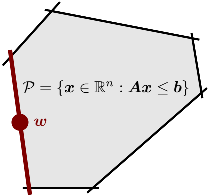
Definition (active constraints, or tight constraints)

The active constraints at $\boldsymbol{x} \in \mathbb{R}^n$ are $\mathcal{I}(\boldsymbol{x}) = \{i \in \mathcal{M}_0 \cup \mathcal{M}_1 : \boldsymbol{a}_i^\top \boldsymbol{x} = b_i\}$.

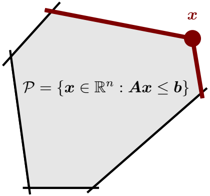
Definition (basic feasible solution)

$\boldsymbol{x}$ is a basic feasible solution if $\boldsymbol{x} \in \mathcal{P}$ and $\{\boldsymbol{a}_i, i \in \mathcal{I}(\boldsymbol{x})\}$ span $\mathbb{R}^n$.

Not a BFS



BFS



Corner of polyhedron: equivalence

Theorem

Let $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ *be a polyhedron and* $\boldsymbol{x} \in \mathcal{P}$.

$\boldsymbol{x}$ *is a vertex* $\iff$ $\boldsymbol{x}$ *is an extreme point* $\iff$ $\boldsymbol{x}$ *is a basic feasible solution*

- Extreme point: geometric representation of a "corner" of polyhedron
- Vertex: there exists an objective function for which a corner point is the optimal solution of a linear optimization problem
  - Conversely, whenever a linear optimization problem admits an optimal solution, one optimal solution lies on a vertex (established later on)
- Basic feasible solution: algebraic representation, useful for algorithms

→ Equivalent viewpoints: we can use algebra (BFS) to characterize optimal solutions (vertices) via geometric properties (extreme points)

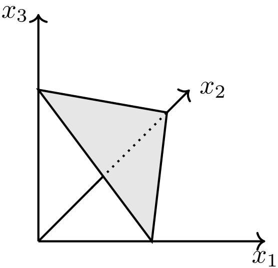
Construction of basic feasible solutions

Geometric vs. standard representation

Geometric representation

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}\}$$

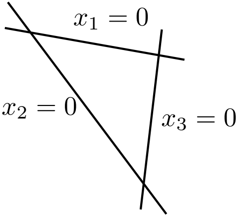
- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$
- $\mathcal{P}$ lives in $\mathbb{R}^n$
- Easier to visualize
- Harder to work with algebraically



Algebraic representation

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$; full row rank $m \le n$
- $\mathcal{P}$ lives in $\mathbb{R}^{n-m}$
- Harder to visualize
- Easier to work with algebraically



BFS for standard form polyhedra: characterization

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}, \text{ with } \boldsymbol{A} \in \mathbb{R}^{m \times n} \text{ of full row rank } (m \leq n)$$

Theorem

$\boldsymbol{x}$ *is a basic feasible solution if and only if:*

- $\boldsymbol{a}_i^\top \boldsymbol{x} = b_i$ *for all* $i = 1, \cdots, m$
- *There exist indices* $B(1), \cdots, B(m)$ *such that* $\boldsymbol{A}_{B(1)}, \cdots, \boldsymbol{A}_{B(m)}$ *are linearly independent and such that* $x_j = 0, \forall j \neq B(1), \cdots, B(m)$
- $x_j \geq 0$ *for all* $j = 1, \cdots, n$

- Reminder: $\boldsymbol{x}$ is a basic feasible solution if $\boldsymbol{x} \in \mathcal{P}$ and there exist n active constraints defined by n linearly independent vectors.
  - We have m active constraints from $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
  - We impose $n - m$ additional ones from the non-negativity constraints
  - By linear independence, the resulting n active constrains span $\mathbb{R}^n$
  - Feasibility question: Does the solution satisfy $\boldsymbol{x} \geq \boldsymbol{0}$?

BFS for standard form polyhedra: construction

→ Procedure for constructing basic feasible solutions

1. Separate $\boldsymbol{A}$ into $\boldsymbol{B} = [\boldsymbol{A}_{B(1)} \cdots \boldsymbol{A}_{B(m)}]$ that comprises m linearly independent columns and $\boldsymbol{N}$ that comprises the other $n - m$ columns
2. Set $x_j = 0$ for all $j \notin B(1), \cdots, B(m)$
3. Solve $\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{b}$ for $x_{B(1)}, \cdots, x_{B(m)}$
4. Check whether $\boldsymbol{x}_B \geq \boldsymbol{0}$:
   - $\boldsymbol{x}$ is a basic feasible solution if $\boldsymbol{x}_B \geq \boldsymbol{0}$
   - $\boldsymbol{x}$ is not a basic feasible solution otherwise

- The m indices $B(1), \cdots, B(m)$ form a **basis**
  - Basic variables $x_{B(1)}, \cdots, x_{B(m)}$ can be non-zero
  - Non-basic variables must be zero

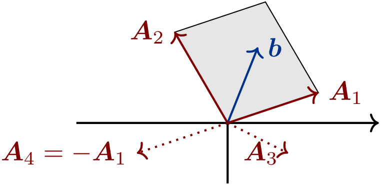
Algebraic representation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \longrightarrow [\boldsymbol{B}\ \boldsymbol{N}]\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{x}_N = \boldsymbol{0} \qquad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}???$$

Geometric intuition



Algebraic example

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{x} = \underbrace{\begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}}_{\boldsymbol{b}}$$

$$\boldsymbol{A} = [\boldsymbol{B}\ \boldsymbol{N}]$$

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{6} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{x} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{8}, \mathbf{12}, \mathbf{4}, \mathbf{6})^\top$$

$\Longrightarrow$ $\boldsymbol{x}$ basic feasible solution

$$\boldsymbol{A} = [\boldsymbol{B}\ \boldsymbol{N}]$$

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{6} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{x} = (\mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{-12}, \mathbf{4}, \mathbf{6})^\top$$

$\Longrightarrow$ $\boldsymbol{x}$ not a basic feasible solution

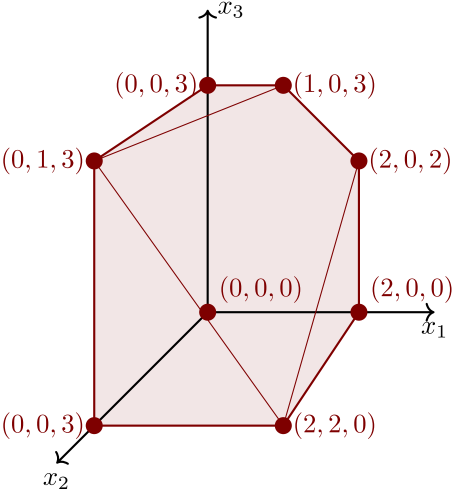
Back to our original example

General form

$$\begin{array}{lll} \min & x_1 + 5x_2 - 2x_3 & \\ \text{s.t.} & x_1 + x_2 + x_3 & \leq & 4 \\ & x_1 & \leq & 2 \\ & x_3 & \leq & 3 \\ & 3x_2 + x_3 & \leq & 6 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Standard form

$$\begin{array}{llll} \min & x_1 + 5x_2 - 2x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_3 + s_1 & = & 4 \\ & x_1 + s_2 & = & 2 \\ & x_3 + s_3 & = & 3 \\ & 3x_2 + x_3 + s_4 & = & 6 \\ & x_1, x_2, x_3, s_1, \cdots, s_4 & \geq & 0 \end{array}$$



Back to our original example: finding BFS

General form

- BFS: 3 linearly independent constraints active

$\rightarrow$ Choose 3 active constraints

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_3 = 3 \\ x_2 = 0 \end{cases}$$

- Check that $(1,1,1)$, $(0,0,1)$ and $(0,1,0)$ span $\mathbb{R}^3$
- Solve $\boldsymbol{x} = (1,0,3)$

Standard form

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$\implies \boldsymbol{x} = (\mathbf{1}, \mathbf{0}, \mathbf{3}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{3})^\top$$

$\implies$ $\boldsymbol{x}$ basic feasible solution

Degeneracy

Degeneracy of basic solutions

General form

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}\}$$

$$\mathcal{I}(\boldsymbol{x}) = \{i : \boldsymbol{a}_i^\top \boldsymbol{x} = b_i\}$$

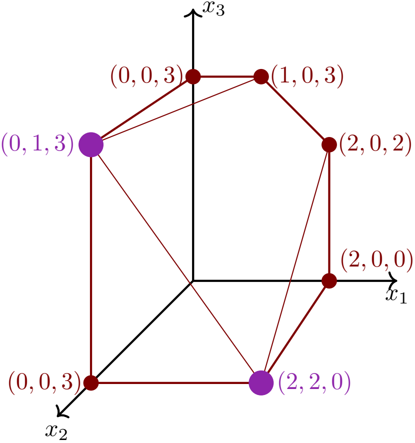
- $\boldsymbol{x}$ BFS: $\{\boldsymbol{a}_i, i \in \mathcal{I}(\boldsymbol{x})\}$ span $\mathbb{R}^n$
- $\boldsymbol{x}$ is nondegenerate if $|\mathcal{I}(\boldsymbol{x})| = n$
- $\boldsymbol{x}$ is degenerate if $|\mathcal{I}(\boldsymbol{x})| > n$

Standard form

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

$$\boldsymbol{x} = [\boldsymbol{x}_B \ \boldsymbol{x}_N], \ \boldsymbol{x}_N = \boldsymbol{0}, \ \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

- $\boldsymbol{x}$ is nondegenerate if it contains exactly $n - m$ zeros
- $\boldsymbol{x}$ is degenerate if it contains more than $n - m$ zeros



Degeneracy: example

General form

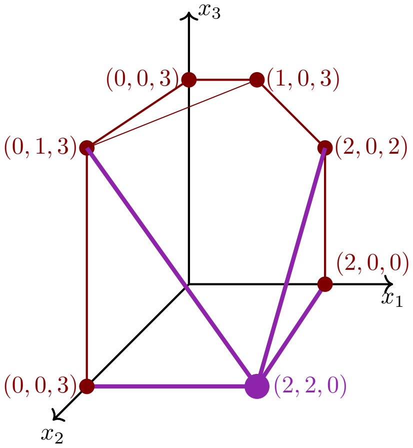
$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 = 2 \\ 3x_2 + x_3 = 6 \\ x_3 = 0 \end{cases}$$

Standard form

$$\boldsymbol{x} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} ?$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} ?$$



Remarks on degeneracy

- Degeneracy is not a purely geometric property, but depends on the representation of a polyhedron
  - $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 - x_2 = 0, x_1 + x_2 + 2x_3 = 2, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$: $(0, 0, 1)$ is degenerate
  - $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 - x_2 = 0, x_1 + x_2 + 2x_3 = 2, x_1 \geq 0, x_3 \geq 0\}$: $(0, 0, 1)$ is nondegenerate
- A basic feasible solution corresponds to a unique extreme point
- The converse, however, is not true in the presence of degeneracy
  - Under non-degeneracy, an extreme point corresponds to a single basis
  - Under degeneracy, an extreme point corresponds to possibly many bases

$\rightarrow$ Degeneracy creates "confusion" regarding the definition of the basis

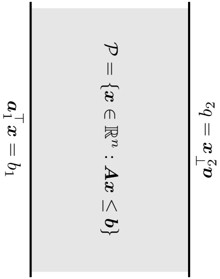
- General form: which active constraints define an extreme point?
- Standard form: which basic columns to consider?

$\rightarrow$ Algorithmic challenges in presence of degeneracy

Existence and optimality of extreme points

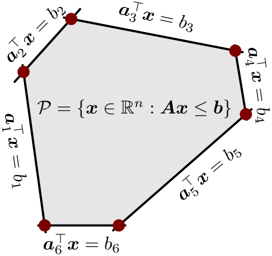
Geometric intuition and visualization

No extreme points



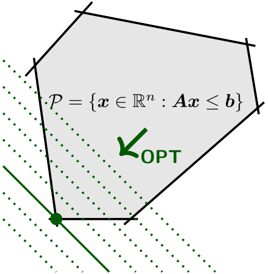
- $\rightarrow$ The polyhedron contains a line
- $\rightarrow$ The polyhedron has no extreme points

Extreme points



- $\rightarrow$ The polyhedron contains no line
- $\rightarrow$ The polyhedron has extreme points

Optimality



- $\rightarrow$ LO solution lies at an extreme point of the polyhedron

Existence of extreme points

Definition

A polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ contains a line if:

$$\exists \boldsymbol{x} \in \mathcal{P}, \boldsymbol{d} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} : \boldsymbol{x} + \lambda \boldsymbol{d} \in \mathcal{P} \ \forall \lambda \in \mathbb{R}$$

Theorem

Let $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$. The following statements are equivalent:

1. *The polyhedron $\mathcal{P}$ has at least one extreme point.*
2. *The polyhedron $\mathcal{P}$ does not contain a line.*
3. *There exist n vectors out of $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ that are linearly independent.*

Corollary

Nonempty polyhedra in standard form contain an extreme point.

Corollary

Nonempty bounded polyhedra contain an extreme point.

Optimality of extreme points: statement and proof

$$(LO): \quad \min_{\boldsymbol{x} \in \mathcal{P}} \quad \boldsymbol{c}^\top \boldsymbol{x}, \quad \text{where } \mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$$

Theorem

If $\mathcal{P}$ does not contain a line and (LO) admits an optimal solution, then there exists an optimal solution that is an extreme point of $\mathcal{P}$.

- Let $z^* = \min_{\boldsymbol{x} \in \mathcal{P}} \boldsymbol{c}^\top \boldsymbol{x}$ be the optimal objective value, and $\mathcal{Q} = \{\boldsymbol{x} \in \mathcal{P} : \boldsymbol{c}^\top \boldsymbol{x} = z^*\}$ be the set of optimal solutions
- $\mathcal{Q} \subseteq \mathcal{P}$, so $\mathcal{Q}$ does not contain a line. Thus, $\mathcal{Q}$ admits an extreme point, which we denote by $\boldsymbol{x}^*$
- Claim: $\boldsymbol{x}^*$ is an extreme point of $\mathcal{P}$.
  - By contradiction, $\boldsymbol{x}^* = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{w}$, $\boldsymbol{y}, \boldsymbol{w} \in \mathcal{P}, 0 < \lambda < 1$. Then:

$$\underbrace{\boldsymbol{c}^\top \boldsymbol{x}}_{=z^*} = \lambda \underbrace{\boldsymbol{c}^\top \boldsymbol{y}}_{\geq z^*} + (1-\lambda) \underbrace{\boldsymbol{c}^\top \boldsymbol{w}}_{\geq z^*} \implies \boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{c}^\top \boldsymbol{w} = z^*$$

  $\rightarrow$ $\boldsymbol{y}, \boldsymbol{w} \in \mathcal{Q}$ and $\boldsymbol{x}^*$ is not an extreme point of $\mathcal{Q}$. Contradiction.

$\rightarrow$ $\boldsymbol{x}^*$ is an extreme point of $\mathcal{P}$ and an optimal solution of (LO)

Optimality of extreme points: significance

$$(LO): \quad \min_{\boldsymbol{x} \in \mathcal{P}} \quad \boldsymbol{c}^\top \boldsymbol{x}, \quad \text{where } \mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

Theorem

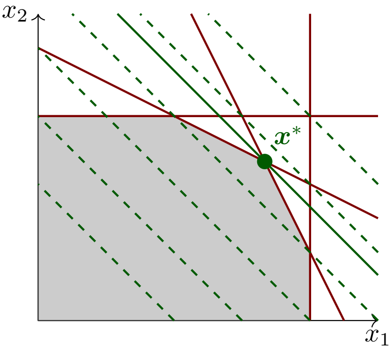
If $\mathcal{P}$ does not contain a line and (LO) admits an optimal solution, then there exists an optimal solution that is an extreme point of $\mathcal{P}$.

- A linear optimization problem has three possible outcomes:
    1. There exists no feasible solution
    2. The problem is unbounded and there exists no optimal solution
    3. An extreme point of the polyhedron is an optimal solution
- Fundamental theorem: we can solve a linear optimization problem by restricting the search to the extreme points of the polyhedron
    - $\rightarrow$ Transformation of an infinite solution space into a finite one
- Still, a polyhedron admits an exponential number of extreme points
    - $\rightarrow$ Need for efficient solution algorithms

Conclusion

Geometry of linear optimization

$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t. } & 0.5x_1 + x_2 \leq 4 \\ & x_1 + 0.5x_2 \leq 4.5 \\ & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 3 \end{aligned}$$



- The feasible region in an LO problem defines a polyhedron
- A constraint is active/binding in $\boldsymbol{x}$ if it is satisfied with equality
- A "corner" $\boldsymbol{x}$ of a polyhedron can be defined in three equivalent ways
  - Extreme point: not on the line between two other points
  - Vertex: $\boldsymbol{x}$ is the unique optimum for some linear objective function
  - BFS: there are n linearly independent active constraints in $\boldsymbol{x}$
- Under mild conditions, a polyhedron admits (many) extreme points
- Then, if an LO problem admits an optimal solution, there exists an extreme point that is optimal

Summary

Takeaway

Three equivalent ways to describe the "corner" of a polyhedron: vertex, extreme point, and basic feasible solution.

Takeaway

If a linear optimization problem admits an optimal solution, one optimal solution lies at an extreme point of the polyhedron.

Takeaway

Problem in standard form: non-negative variables $\boldsymbol{x} \geq \mathbf{0}$ *and equality constraints* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$*, which is useful to construct basic feasible solutions.*

Takeaway

Degenerate problems: extreme points with more active constraints than "necessary", so a BFS can be associated with several possible bases.