Trapped ion

I. DERIVATION OF EQ(11)

$$H_i = \frac{\Omega_1\Omega_2}{4\Delta}\exp[-i[\boldsymbol{\Delta k}\cdot\boldsymbol{r_j(t)} - \delta t - \Delta\phi]]|0\rangle_j\langle 1| + h.c. \tag{1}$$

where $\boldsymbol{\Delta k} \equiv \boldsymbol{k_1 - k_2}$, $\Delta\phi \equiv \phi_1 - \phi_2$. In below, we define $\Omega_j^{(eff)} \equiv \Omega_1\Omega_2/2\Delta$ as the effective Rabi frequency on ion j by Ω_j. We apply totally 3 laser as shown in figure 3. When $\omega_{01} \ll \omega_1, \omega_2, \omega_3$, we have

$$H_I = \frac{\Omega_j}{2}\left[\exp\left(-i\Delta k x_j(t) + i\mu t + i\Delta\phi_b\right) + \exp\left(i\Delta k x_j(t) - i\mu t + i\Delta\phi_b\right)\right]|0\rangle_j\langle 1| + h.c. \tag{2}$$

$$= \Omega_j \cos\left[\mu t + \phi_j^{(m)} - \Delta k x_j(t)\right] \times \left(\sigma_j^x \cos\phi_j^{(s)} - \sigma_j^y \sin\phi_j^{(s)}\right) \tag{3}$$

with $\phi_j^{(m)} \equiv (\Delta\phi_b - \Delta\phi_r)/2$ and $\phi_j^{(s)} \equiv (\Delta\phi_b - \Delta\phi_r)/2$. We further define $\sigma_j \equiv \sigma_j^x \cos\phi_j^{(s)} - \sigma_j^y \sin\phi_j^{(s)}$ to get

$$H_I = \Omega_j \cos\left[\mu t + \phi_j^{(m)} - \Delta k x_j(t)\right]\sigma_j. \tag{4}$$

Then, we quantize the position of ion j as

$$x_j(t) = \sum_k b_j^k \sqrt{\frac{1}{2m\omega_k}}\left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \tag{5}$$

where b_j^k is the kth normalized mode vector of the collective oscillation. We define the Lamb-Dicke parameter $\eta_k \equiv \sqrt{1/2m\omega_k}$, and obtain

$$H_I = \Omega_j \cos\left[\mu t + \phi_j^{(m)} - \sum_k \eta_k b_j^k\left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\right]\sigma_j. \tag{6}$$

We then expand the expression according to the power of η_k to the fourth order:

$$H_I = \sum_{j=1,2}\Omega_j(t)\sum_k \Big[\tag{7}$$

$$\cos\left(\mu t + \phi_j^{(m)}\right) \tag{8}$$

$$+\sin\left(\mu t + \phi_j^{(m)}\right) \times \sum_k \eta_k b_j^k\left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \tag{9}$$

$$-\frac{1}{2}\cos(\mu t + \phi_j^{(m)})\sum_{k,l}\eta_k\eta_l b_j^k b_j^l \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right) \tag{10}$$

$$-\frac{1}{6}\sin(\mu t + \phi_j^{(m)})\sum_{k,l,l'}\eta_k\eta_l\eta_{l'} b_j^k b_j^l b_j^{l'} \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right)\left(a_{l'} e^{-i\omega_{l'} t} + a_{l'}^\dagger e^{i\omega_{l'} t}\right) \tag{11}$$

$$+\frac{\cos(\mu t + \phi_j^{(m)})}{24}\sum_{k,l,l',l''}\eta_k\eta_l\eta_{l'}\eta_{l''} b_j^k b_j^l b_j^{l'} b_j^{l''}\left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right)\left(a_{l'} e^{-i\omega_{l'} t} + a_{l'}^\dagger e^{i\omega_{l'} t}\right)\left(a_{l''} e^{-i\omega_{l''} t} + a_{l''}^\dagger e^{i\omega_{l''} t}\right) \tag{12}$$

$$\Big]\sigma_j + O(\eta_k^5). \tag{13}$$

II. MAGNUS EXPANSION

We consider the phonon non-preserving terms as error, and keep it only to the order of $O(\eta)$. Therefore, we can make an approximation to equation above:

$$H_I = \sum_{j=1,2} \Omega_j(t) \sum_k \Big[\tag{14}$$

$$\sin\left(\mu t + \phi_j^{(m)}\right) \times \sum_k \eta_k b_j^k \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \tag{15}$$

$$-\frac{1}{2}\cos(\mu t + \phi_j^{(m)}) \sum_{k,l} \eta_k \eta_l b_j^k b_j^l \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right) \tag{16}$$

$$-\frac{1}{6}\sin(\mu t + \phi_j^{(m)}) \left(\sum_{k,l\neq k} 3\eta_k(\eta_l)^2 b_j^k (b_j^l)^2 \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right)^2 + \sum_k \eta_k^3 (b_j^k)^3 \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)^3\right) \tag{17}$$

$$+\frac{1}{24}\cos(\mu t + \phi_j^{(m)}) \sum_{k,l>k} 6(\eta_k b_j^k)^2 (\eta_l b_j^l)^2 \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)^2 \left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right)^2 + (\eta_k b_j^k)^4 \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)^4 \tag{18}$$

$$\Big]\sigma_j + O(\eta_k^5). \tag{19}$$

The zero-order term is discarded because it is a single qubit rotation and commute with other terms. The above equation can be further simplified as

$$H_I = \sum_{j=1,2} \Omega_j(t) \sum_k \Big[\tag{20}$$

$$\sin\left(\mu t + \phi_j^{(m)}\right) \times \sum_k \eta_k b_j^k \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \tag{21}$$

$$-\frac{1}{2}\cos(\mu t + \phi_j^{(m)}) \sum_{k,l} \eta_k \eta_l b_j^k b_j^l \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right) \tag{22}$$

$$-\frac{1}{6}\sin(\mu t + \phi_j^{(m)}) \sum_{k,l\neq k} 3\eta_k(\eta_l)^2 b_j^k (b_j^l)^2 \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)(n_l + 1) + \sum_k \eta_k^3 (b_j^k)^3 \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)(n_k + 1) \tag{23}$$

$$+\frac{1}{24}\cos(\mu t + \phi_j^{(m)}) \sum_k (\eta_k b_j^k)^4 \left(6(n_k)^2 + 6n_k + 3\right) \tag{24}$$

$$+\frac{1}{24}\cos(\mu t + \phi_j^{(m)}) \sum_{k,l>k} 6(\eta_k b_j^k)^2 (\eta_l b_j^l)^2 \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)^2 \left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right)^2 \tag{25}$$

$$\Big]\sigma_j + O(\eta_k^5). \tag{26}$$

when $n_l \ll 1$, we have

$$H_I \approx \sum_{j=1,2} \Omega_j(t) \sum_k \Big[\tag{27}$$

$$\sin\left(\mu t + \phi_j^{(m)}\right) \times \sum_k \eta_k b_j^k \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \tag{28}$$

$$-\frac{\eta^2}{2}\cos(\mu t + \phi_j^{(m)}) \sum_{k,l} b_j^k b_j^l \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right) \tag{29}$$

$$-\frac{\eta^3}{6}\sin(\mu t + \phi_j^{(m)}) \left(\sum_{k,l\neq k} 3b_j^k (b_j^l)^2 + \sum_k (b_j^k)^3\right) \times \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \tag{30}$$

$$+\frac{\eta^4}{24}\cos(\mu t + \phi_j^{(m)}) \left(\sum_{k,l>k} 6(b_j^k)^2 (b_j^l)^2 + \sum_k 3(b_j^k)^4\right) \tag{31}$$

$$\Big]\sigma_j + O(\eta_k^5). \tag{32}$$

We rewrite the Hamiltonian in the following form

$$H(t) = H_A(t) + H_B(t) + H_C(t) + H_D(t) \tag{33}$$

$$= \sum_j \Omega_j(t) \left[\sum_k A_j^k(t)\xi_k(t) + \sum_{k,l} B_j^{k,l}(t)\xi_k(t)\xi_l(t) + \sum_{k,l} C_j^{k,l}(t)\xi_k + \sum_{k,l\geqslant k} D_j^{k,l}(t)\right]\sigma_j \tag{34}$$

where $\xi_k(t) = a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}$ and

$$A_j^k(t) = \eta \sin\left(\mu t + \phi_j^{(m)}\right) b_j^k, \tag{35}$$

$$B_j^{k,l}(t) = -\frac{\eta^2}{2}\cos(\mu t + \phi_j^{(m)}) b_j^k b_j^l, \tag{36}$$

$$C_j^{k,l}(t) = -\frac{\eta^3}{6}\sin(\mu t + \phi_j^{(m)}) \left(3b_j^k (b_j^l)^2 + (b_j^k)^3 \times (k == l)\right), \tag{37}$$

$$D_j^{k,l}(t) = \frac{\eta^4}{8}\sin(\mu t + \phi_j^{(m)}) \left(2(b_j^k)^2 (b_j^l)^2 + (b_j^k)^3 \times (k == l)\right). \tag{38}$$

To calculate the evolution under $H(t)$, we perform Magnus expansion to the fourth order

$$U(\tau) \approx \exp\Big[\tag{39}$$

$$-i\int_0^\tau dt_1 H(t_1) \tag{40}$$

$$-\frac{1}{2}\int_0^\tau dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)] \tag{41}$$

$$+\frac{i}{6}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 [H(t_1), [H(t_2), H(t_3)]] + [H(t_3), [H(t_2), H(t_1)]] \tag{42}$$

$$+\frac{1}{12}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 [[[[H(t_1), H(t_2)], H(t_3)]H(t_4)] + [H(t_1), [[H(t_2), H(t_3)]H(t_4)]] \tag{43}$$

$$+ [H(t_1), [H(t_2), [H(t_3), H(t_4)]]] + [H(t_2), [H(t_3), [H(t_1), H(t_4)]]] \tag{44}$$

$$\Big] \tag{45}$$

Let

$$U(\tau) = \exp\left[\Upsilon_0(\tau) + \Upsilon'(\tau)\right], \tag{46}$$

where $\Upsilon_0(\tau)$ contains phonon-conserving terms to the order of $O(\eta^2)$ and non-conserving terms to the order of $O(\eta)$, and Υ' contains only phonon-conserving terms to the order of $O(\eta^4)$. We have

$$\Upsilon_0(\tau) = -i\int_0^\tau dt_1\,(H_A(t_1)+H_B(t_2)) - \frac{1}{2}\int_0^\tau dt_1\int_0^{t_1} dt_2[H_A(t_1),H_A(t_2)] \tag{47}$$

$$= i\sum_{i<j}\Theta_{ij}(\tau)\sigma_i\sigma_j + i\sum_j[\phi_j(\tau)+\psi_j(\tau)]\sigma_j \tag{48}$$

where

$$\phi_j = -i\sum_k[\alpha_j^k(\tau)a_k^\dagger - \alpha_j^{k^*}(\tau)a_k], \tag{49}$$

$$\alpha_j^k(\tau) = -i\eta_k b_j^k\int_0^\tau \chi_j(t)e^{i\omega_k t}dt, \tag{50}$$

$$\psi_j(\tau) = \sum_k \lambda_j^k(\tau)(n_k+1/2), \tag{51}$$

$$\lambda_j^k(\tau) = (\eta_k b_j^k)^2\int_0^\tau \theta_j(t)dt. \tag{52}$$

and

$$\Theta_{ij}(\tau) = \sum_k \eta_k^2 b_i^k b_j^k\int_0^\tau dt_1\int_0^{t_1} dt_2[\chi_i(t_1)\chi_j(t_2)+\chi_j(t_1)\chi_i(t_2)]\sin[\omega_k(t_1-t_2)]. \tag{53}$$

A. Error terms and error scaling estimation

After some calculation, we find that the errors contains three parts: single-, double- and triple interaction terms. More specifically

$$\Upsilon' \approx \Upsilon_1' + \Upsilon_2' + \Upsilon_3' \tag{54}$$

where

$$\Upsilon_1' \propto \sum_j \Omega_j\sigma_j\left(2\sum_{k,l>k}(b_j^k)^2(b_j^l)^2 + \sum_k(b_j^k)^4\right) \tag{55}$$

$$\Upsilon_2' \propto \sum_j \Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\left(2\sum_{k,l>k}(b_j^k b_j^l)(b_{j'}^k b_{j'}^l) + \sum_k(b_j^k)^2(b_{j'}^k)^2\right) \tag{56}$$

$$\Upsilon_3' \propto \sum_j \Omega_j\Omega_{j'}\Omega_{j''}\sigma_j\sigma_{j'}\sigma_{j''}\sum_k M_{k,j,j',j''}(A,A,B). \tag{57}$$

The expression of $M_{k,j,j',j''}(A,A,B)$ is complicated and can be found in Appendix B. Because $\sum_k(b_j^k)^2 = 1$, when the Rabi frequency Ω_i, operation time τ and detuning μ are fixed, we have

$$\|\Upsilon_{1,2,3}\| = O(1). \tag{58}$$

In other words, the total error will not increase with the number of ions in the trap.

III. NUMERICAL ESTIMATION

Recall that

$$U(\tau) = \exp\left[\Upsilon_0(\tau)+\Upsilon'(\tau)\right], \tag{59}$$

where $\Upsilon'(\tau) \approx \Upsilon'_1 + \Upsilon'_2$. The phonon mode frequency is approximated as the average of all phonons, which is $2.11 \times 2\pi$MHz. Other parameters are set as laser wave length $\lambda = 355$nm, Rabi frequency $\Omega_i = 1 \times 2\pi$MHz, evolution time $\tau = 200\mu s - 400\mu s$, detuning $\mu = 3 \times 2\pi$MHz, and mass of ions $m = 171 \times 1.673 \times 10^{-27}$kg. We obtain $\eta = 2k\sqrt{\hbar/2m\omega} \approx 0.134$.

$$\begin{aligned}
\Upsilon'_1 =& -i\frac{\eta^4\left[1-\cos\left(\mu\tau\right)\right]}{8\mu}\sum_j \Omega_j\sigma_j\left(2\sum_{k,l>k}(b_j^k b_j^l)^2 + \sum_k (b_j^k)^4\right) && (60)\\
\approx& -i\frac{\eta^4\left[1-\cos\left(\mu\tau\right)\right]}{8\mu}\sum_j \Omega_j\sigma_j && (61)\\
\approx& -i1.35\times 10^{-5}\left[1-\cos\left(\mu\tau\right)\right]\sum_j \sigma_j && (62)
\end{aligned}$$

as a comparison, the parameters ϕ_j in Eq.(51) is about 10^{-3}.

$$\begin{aligned}
\Upsilon'_2 &\approx -\frac{1}{2}\int_0^\tau dt_1 \int_0^{t_1} dt_2 [H_B(t_1), H_B(t_2)] && (63)\\
&\approx -\frac{\eta^4 i}{2}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \cos(\mu t_1)\cos(\mu t_2)\sin 2\omega(t_2-t_1)\sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\left(2\sum_{k,l>k}(b_j^k b_j^l)(b_{j'}^k b_{j'}^l) + \sum_k (b_j^k)^2(b_{j'}^k)^2\right) && (64)\\
&\approx -i0.000163\times(-0.04)\int_0^\tau dt_1 \int_0^{t_1} dt_2 \cos(\mu t_1)\cos(\mu t_2)\sin 2\omega(t_2-t_1)\sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'} && (65)\\
&\approx i6.52\times 10^{-5}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \cos(\mu t_1)\cos(\mu t_2)\sin 2\omega(t_2-t_1)\sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'} && (66)\\
&\approx -i0.001\sum_{j,j'}\sigma_j\sigma_{j'} && (67)
\end{aligned}$$

As a comparison, $\Theta_{i,j}(\tau)$ is expected to be at the order of 1. So the error rate is about 10^{-3}.

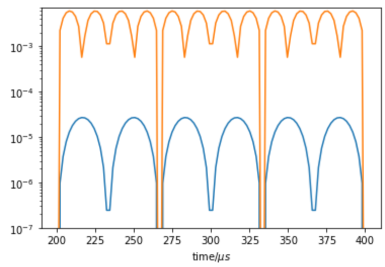

FIG. 1: Comparison of local field terms of η^2 and η^4 order.

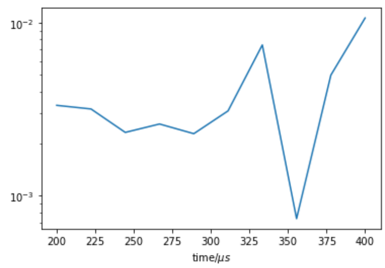

FIG. 2: Comparison of coupling strength of η^2 and η^4 order.

Appendix A: commutation relations for $\xi_k(t)$

The commutators with two ξ:

$$[\xi_k(t_1), \xi_{k'}(t_2)] = \delta_{k,k'}([a_k e^{-i\omega_k t_1}, a_k^\dagger e^{i\omega_k t_2}] + [a_k^\dagger e^{i\omega_k t_1}, a_k e^{-i\omega_k t_2}]) \tag{A1}$$
$$= \delta_{k,k'}(e^{-i\omega_k(t_1-t_2)}[a_k, a_k^\dagger] + e^{i\omega_k(t_1-t_2)}[a_k^\dagger, a_k]) \tag{A2}$$
$$= \delta_{k,k'}(e^{-i\omega_k(t_1-t_2)} - e^{i\omega_k(t_1-t_2)}) \tag{A3}$$
$$= 2i\sin\omega(t_2-t_1)\delta_{k,k'} \tag{A4}$$

with three ξ, we have

$$[\xi_k(t_1), \xi_k(t_2)\xi_k(t_2)] \tag{A5}$$
$$= \xi_k(t_1)\xi_k(t_2)\xi_k(t_2) - \xi_k(t_2)\xi_k(t_2)\xi_k(t_1) \tag{A6}$$
$$= \xi_k(t_1)\xi_k(t_2)\xi_k(t_2) - \xi_k(t_2)[\xi_k(t_1)\xi_k(t_2) - 2i\sin\omega_k(t_2-t_1)] \tag{A7}$$
$$= \xi_k(t_1)\xi_k(t_2)\xi_k(t_2) - \xi_k(t_2)\xi_k(t_1)\xi_k(t_2) + \xi_k(t_2)2i\sin\omega_k(t_2-t_1) \tag{A8}$$
$$= 2i\sin\omega_k(t_2-t_1)\xi_k(t_2) + \xi_k(t_2)2i\sin\omega_k(t_2-t_1) \tag{A9}$$
$$= 4i\sin\omega_k(t_2-t_1)\xi_k(t_2) \tag{A10}$$

so when $k \neq l$, we have

$$[\xi_k(t_1), \xi_k(t_2)\xi_k(t_2)] = 4\xi_k(t_2)i\sin\omega_k(t_2-t_1) \tag{A11a}$$
$$[\xi_k(t_1), \xi_k(t_2)\xi_l(t_2)] = \xi_l(t_2)[\xi_k(t_1), \xi_k(t_2)] = \xi_l(t_2)2i\sin\omega_k(t_2-t_1) \tag{A11b}$$
$$[\xi_k(t_1), \xi_l(t_2)\xi_k(t_2)] = \xi_l(t_2)2i\sin\omega_k(t_2-t_1) \tag{A11c}$$
$$[\xi_l(t_1), \xi_k(t_2)\xi_k(t_2)] = 0 \tag{A11d}$$

other terms are similar. With four ξ, we have

$$[\xi_k(t_1)\xi_k(t_1), \xi_k(t_2)\xi_k(t_2)] = \xi_k(t_1)\xi_k(t_1)\xi_k(t_2)\xi_k(t_2) - \xi_k(t_2)\xi_k(t_2)\xi_k(t_1)\xi_k(t_1) \tag{A12}$$
$$= [\xi_k(t_1)\xi_k(t_2) + \xi_k(t_2)\xi_k(t_1)]4i\sin\omega_k(t_2-t_1) \tag{A13}$$
$$= [(a_k e^{-i\omega_k t_1} + a_k^\dagger e^{i\omega_k t_1})(a_k e^{-i\omega_k t_2} + a_k^\dagger e^{i\omega_k t_2}) + (a_k e^{-i\omega_k t_2} + a_k^\dagger e^{i\omega_k t_2})(a_k e^{-i\omega_k t_1} + a_k^\dagger e^{i\omega_k t_1})]4i\sin\omega(t_2-t_1) \tag{A14}$$
$$\approx [a_k a_k^\dagger e^{-i\omega_k(t_1-t_2)} + a_k^\dagger a_k e^{i\omega_k(t_1-t_2)} + a_k a_k^\dagger e^{i\omega_k(t_1-t_2)} + a_k^\dagger a_k e^{-i\omega_k(t_1-t_2)}]4i\sin\omega(t_2-t_1) \tag{A15}$$
$$= [2a_k a_k^\dagger\cos\omega_k(t_2-t_1) + 2a_k^\dagger a_k\cos\omega_k(t_2-t_1)]4i\sin\omega(t_2-t_1) \tag{A16}$$
$$= [a_k a_k^\dagger + a_k^\dagger a_k]8i\cos\omega_k(t_2-t_1)\sin\omega_k(t_2-t_1) \tag{A17}$$
$$= 8i\sin 2\omega_k(t_2-t_1)[n_k + \frac{1}{2}] \tag{A18}$$
$$\approx 4i\sin 2\omega_k(t_2-t_1) \tag{A19}$$
$$\tag{A20}$$

and

$$
\begin{aligned}
&[\xi_k(t_1)\xi_l(t_1), \xi_k(t_2)\xi_l(t_2)] = [\xi_l(t_1)\xi_k(t_1), \xi_k(t_2)\xi_l(t_2)] = [\xi_k(t_1)\xi_l(t_1), \xi_l(t_2)\xi_k(t_2)] = [\xi_l(t_1)\xi_k(t_1), \xi_l(t_2)\xi_k(t_2)] && \text{(A21)}\\
=&\xi_k(t_1)\xi_l(t_1)\xi_k(t_2)\xi_l(t_2) - \xi_k(t_2)\xi_l(t_2)\xi_k(t_1)\xi_l(t_1) && \text{(A22)}\\
=&\xi_k(t_1)\xi_k(t_2)\xi_l(t_1)\xi_l(t_2) - (\xi_k(t_1)\xi_k(t_2) + 2i\sin\omega_k(t_1-t_2))(\xi_l(t_1)\xi_l(t_2) + 2i\sin\omega_l(t_1-t_2)) && \text{(A23)}\\
=&2i\sin\omega_l(t_2-t_1)\xi_k(t_1)\xi_k(t_2) + 2i\sin\omega_k(t_2-t_1)\xi_l(t_1)\xi_l(t_2) + 4\sin\omega_l(t_2-t_1)\sin\omega_k(t_2-t_1) && \text{(A24)}\\
\approx&2i\sin\omega_l(t_2-t_1)(a_k^\dagger a_k e^{i\omega_k(t_1-t_2)} + a_k a_k^\dagger e^{-i\omega_k(t_1-t_2)}) + 2i\sin\omega_k(t_2-t_1)(a_l^\dagger a_l e^{i\omega_l(t_1-t_2)} + a_l a_l^\dagger e^{-i\omega_l(t_1-t_2)}) && \text{(A25)}\\
&+ 4\sin\omega_l(t_2-t_1)\sin\omega_k(t_2-t_1) && \text{(A26)}\\
\approx&2i\sin\omega_l(t_2-t_1)(a_k^\dagger a_k e^{i\omega_k(t_1-t_2)} + (a_k^\dagger a_k + 1)e^{-i\omega_k(t_1-t_2)}) + 2i\sin\omega_k(t_2-t_1)(a_l^\dagger a_l e^{i\omega_l(t_1-t_2)}(a_l^\dagger a_l + 1)e^{-i\omega_l(t_1-t_2)}) && \text{(A27)}\\
&+ 4\sin\omega_k(t_2-t_1)\sin\omega_l(t_2-t_1) && \text{(A28)}\\
\approx&4i\sin\omega(t_2-t_1)e^{-i\omega(t_1-t_2)} + 4\sin^2\omega(t_2-t_1) && \text{(A29)}\\
=&4\sin\omega(t_2-t_1)(ie^{-i\omega(t_1-t_2)} + \sin\omega(t_2-t_1)) && \text{(A30)}\\
=&4i\sin\omega(t_2-t_1)(\cos\omega(t_2-t_1)) && \text{(A31)}\\
=&2i\sin 2\omega(t_2-t_1) && \text{(A32)}
\end{aligned}
$$

where only phonon-preserving terms are kept. Other terms has no contribution to the phonon-preserving terms. For example:

$$
[\xi_k(t_1)\xi_l(t_1), \xi_{k'}(t_2)\xi_{l'}(t_2)] = 0 \tag{A33}
$$

Appendix B: Derivation of error terms

We rewrite the error term as

$$
\Upsilon'(\tau) = \Upsilon_1'(\tau) + \Upsilon_2'(\tau) + \Upsilon_3'(\tau) + \Upsilon_4'(\tau). \tag{B1}
$$

where

$$
\begin{aligned}
\Upsilon_1'(\tau) &= -i\int_0^\tau dt_1 H_D(t) && \text{(B2)}\\
\Upsilon_2'(\tau) &= -\frac{1}{2}\int_0^\tau dt_1 \int_0^{t_1} dt_2\,([H_B(t_1), H_B(t_2)] + [H_A(t_1), H_C(t_2)] + [H_C(t_1), H_A(t_2)]) && \text{(B3)}\\
\Upsilon_3'(\tau) &= \frac{1}{6}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \Big([H_A(t_1), [H_A(t_2), H_B(t_3)]] + [H_A(t_1), [H_B(t_2), H_A(t_3)]] + [H_B(t_1), [H_A(t_2), H_A(t_3)]] && \text{(B4)}\\
&\qquad + [H_A(t_3), [H_A(t_2), H_B(t_1)]] + [H_A(t_3), [H_B(t_2), H_A(t_1)]] + [H_B(t_3), [H_A(t_2), H_A(t_1)]]\Big) && \text{(B5)}\\
\Upsilon_4'(\tau) &= \frac{1}{12}\int_0^\tau dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 [[[[H_A(t_1), H_A(t_2)], H_A(t_3)]H_A(t_4)] + [H_A(t_1), [[H_A(t_2), H_A(t_3)]H_A(t_4)]] && \text{(B6)}\\
&\qquad + [H_A(t_1), [H_A(t_2), [H_A(t_3), H_A(t_4)]]] + [H_A(t_2), [H_A(t_3), [H_A(t_1), H_A(t_4)]]] && \text{(B7)}
\end{aligned}
$$

During the calculation below, we have used the commutation relations of $[\xi_k(t_i), \xi_k(t_j)]$, $[\xi_k(t_i)\xi_{k'}(t_i), \xi_{k''}(t_j)]$, etc. The derivations are provided in Appendix A.

a. term Υ_1'

$$
H_D(t) \approx \sum_j \Omega_j\sigma_j \sum_{k,l>k} D_j^{k,l}(t) \tag{B8}
$$

So we have

$$\Upsilon_1' = -i\int_0^\tau H_D(t)dt \tag{B9}$$

$$= -\frac{1}{8}i\sum_j \Omega_j\sigma_j\left(2\sum_{k,l>k}(\eta_k b_j^k)^2(\eta_l b_j^l)^2 + \sum_k(\eta_k b_j^k)^4\right)\int_0^\tau dt\,\sin(\mu t + \phi_j^{(m)}) \tag{B10}$$

$$= -i\sum_j \Omega_j\sigma_j\frac{\left(2\sum_{k,l>k}(\eta_k b_j^k)^2(\eta_l b_j^l)^2 + \sum_k(\eta_k b_j^k)^4\right)}{8\mu}\left[\cos\left(\phi_j^{(m)}\right) - \cos\left(\mu\tau + \phi_j^{(m)}\right)\right] \tag{B11}$$

$$= -i\frac{\eta^4\left[1-\cos\left(\mu\tau\right)\right]}{8\mu}\sum_j \Omega_j\sigma_j\left(2\sum_{k,l>k}(b_j^k)^2(b_j^l)^2 + \sum_k(b_j^k)^4\right) \tag{B12}$$

where $\phi_j^{(m)} = 0$, $\eta_k = \eta$.

b. term Υ_2'

Because

$$[H_A(t_1), H_C(t_2)] \tag{B13}$$

$$= \sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\sum_{k,k',l'}A_j^k(t_1)C_{j'}^{k',l'}(t_2)[\xi_k(t_1),\xi_{k'}(t_2)] \tag{B14}$$

$$= \sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\sum_{k,l'}A_j^k(t_1)C_{j'}^{k,l'}(t_2)[\xi_k(t_1),\xi_k(t_2)] \tag{B15}$$

$$\approx \sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\sum_{k,l}A_j^k(t_1)C_{j'}^{k,l}(t_2)2i\sin\omega_k(t_2-t_1) \tag{B16}$$

we have

$$-\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1}dt_2[H_A(t_1),H_C(t_2)] \tag{B17}$$

$$= -\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1}dt_2\sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\sum_{k,l}A_j^k(t_1)C_j^{k,l}(t_2)2i\sin\omega_k(t_2-t_1) \tag{B18}$$

$$= \sum_{j,j'}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}F_{j,j'}(\boldsymbol{b},\eta)\int_0^\tau dt_1\int_0^{t_1}dt_2\sin\left(\mu t_1+\phi_j^{(m)}\right)\sin(\mu t_2+\phi_{j'}^{(m)})\sin\omega_k(t_2-t_1) \tag{B19}$$

$$= \sum_{j,j'>j}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\left(F_{j,j'}(\boldsymbol{b},\eta)+F_{j',j}(\boldsymbol{b},\eta)\right)\int_0^\tau dt_1\int_0^{t_1}dt_2\sin\left(\mu t_1+\phi^{(m)}\right)\sin(\mu t_2+\phi^{(m)})\sin\omega_k(t_2-t_1) \tag{B20}$$

where $F_{j,j'}(\boldsymbol{b},\eta) = \sum_{k,l}\frac{\eta^4 i}{6}b_j^k\left(3b_{j'}^k(b_{j'}^l)^2 + (b_{j'}^k)^3\times(k==l)\right)$. Similarly

$$-\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1}dt_2[H_C(t_1),H_A(t_2)] \tag{B21}$$

$$=\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1}dt_2[H_A(t_2),H_C(t_1)] \tag{B22}$$

$$= \sum_{j,j'>j}\Omega_j\Omega_{j'}\sigma_j\sigma_{j'}\left(F_{j,j'}(\boldsymbol{b},\eta)+F_{j',j}(\boldsymbol{b},\eta)\right)\int_0^\tau dt_1\int_0^{t_1}dt_2\sin\left(\mu t_2+\phi^{(m)}\right)\sin(\mu t_1+\phi^{(m)})\sin\omega_k(t_1-t_2) \tag{B23}$$

$$=\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1}dt_2[H_A(t_1),H_C(t_2)] \tag{B24}$$

So we have

$$-\frac{1}{2}\int_0^{\tau} dt_1 \int_0^{t_1} dt_2 \left([H_A(t_1), H_C(t_2)] + [H_C(t_1), H_A(t_2)]\right) \approx 0 \tag{B25}$$

where we have discarded the non-phonon-conserving terms. The remaining term

$$[H_B(t_1), H_B(t_2)] \tag{B26}$$
$$=\sum_{j,j'} \Omega_j \Omega_{j'} \sum_{k,k',l,l'} B_j^{k,l}(t_1) B_{j'}^{k',l'}(t_2) \sigma_j \sigma_{j'} [\xi_k(t_1)\xi_l(t_1), \xi_{k'}(t_2)\xi_{l'}(t_2)] \tag{B27}$$
$$=\eta^4 \frac{1}{4}\cos(\mu t_1)\cos(\mu t_2) \sum_{j,j'} \Omega_j \Omega_{j'} \sigma_j \sigma_{j'} \left(\sum_{k,l>k} (b_j^k b_j^l)(b_{j'}^k b_{j'}^l) \times 4 \times 2i \sin 2\omega(t_2 - t_1) + \sum_k (b_j^k)^2 (b_{j'}^k)^2 \times 1 \times 4i \sin 2\omega(t_2 - t_1)\right) \tag{B28}$$
$$=\eta^4 i \cos(\mu t_1)\cos(\mu t_2) \sin 2\omega(t_2 - t_1) \sum_{j,j'} \Omega_j \Omega_{j'} \sigma_j \sigma_{j'} \left(2\sum_{k,l>k} (b_j^k b_j^l)(b_{j'}^k b_{j'}^l) + \sum_k (b_j^k)^2 (b_{j'}^k)^2\right) \tag{B29}$$

where we have discarded the non-phonon-conserving terms. When $\omega_k \approx \omega, \eta_k \approx \eta$, we have

$$\Upsilon_2' \approx -\frac{1}{2}\int_0^{\tau} dt_1 \int_0^{t_1} dt_2 [H_B(t_1), H_B(t_2)] \tag{B30}$$
$$\approx -\frac{\eta^4 i}{2}\int_0^{\tau} dt_1 \int_0^{t_1} dt_2 \cos(\mu t_1)\cos(\mu t_2) \sin 2\omega(t_2 - t_1) \sum_{j,j'} \Omega_j \Omega_{j'} \sigma_j \sigma_{j'} \left(2\sum_{k,l>k} (b_j^k b_j^l)(b_{j'}^k b_{j'}^l) + \sum_k (b_j^k)^2 (b_{j'}^k)^2\right) \tag{B31}$$

c. term Υ_3'

$$[H_B(t_3), [H_A(t_2), H_A(t_1)]] \tag{B32}$$
$$\approx \sum_{j,j'',j'''} \Omega_j \Omega_{j'} \Omega_{j''} \sum_{k,k'} 2i \sin \omega_k (t_1 - t_2) \left(A_j^k(t_1) A_{j'}^k(t_2) + A_{j''}^k(t_1) A_j^k(t_2)\right) B_j^{k',l'}(t_3) \xi_{k'}(t_3) \xi_{l'}(t_3) [\sigma_{j''}, \sigma_j \sigma_{j'}] \tag{B33}$$
$$=0 \tag{B34}$$

Moreover, according to Eq. (A11), we have

$$[H_A(t_1), H_B(t_2)] = -[H_B(t_2), H_A(t_1)] \tag{B35}$$
$$=\sum_{j,j'} \Omega_j \Omega_{j'} \sum_{k,k',l'} A_j^k(t_1) B_{j'}^{k',l}(t_2) \sigma_j \sigma_{j'} [\xi_k(t_1), \xi_{k'}(t_2)\xi_{l'}(t_2)] \tag{B36}$$
$$\approx \sin \omega(t_2 - t_1) \sin \mu t_1 \cos \mu t_2 \sum_{j,j'} \Omega_j \Omega_{j'} \sigma_j \sigma_{j'} \sum_k M_{k,j,j''}(A, B) \xi_k(t_2) \tag{B37}$$

where

$$M_{k,j,j''}(A, B) = -i\eta^3 b_j^k \left(2 b_{j'}^k b_{j'}^k + \sum_{k' \neq k} b_{j'}^{k'} b_{j'}^{k'}\right) \tag{B38}$$

Here, we have approximated all phonon frequencies as the average of them $\omega_k \approx \omega$. We then have

$$[H_A(t_3), [H_A(t_2), H_B(t_1)]] \tag{B39}$$

$$\approx \left[\sum_{j,k} \Omega_j A_j^k(t_3)\xi_k(t_3)\sigma_j, \sin\omega(t_1 - t_2)\sin\mu t_2 \cos\mu t_1 \sum_{j,j'} \Omega_j\Omega_{j'}\sigma_j\sigma_{j'} \sum_k M_{k,j,j''}(A,B)\xi_k(t_1)\right] \tag{B40}$$

$$= \sin\omega(t_1 - t_2)\sin\mu t_2 \cos\mu t_1 \sum_{j,j',j''} \Omega_j\Omega_{j'}\Omega_{j''}\sigma_j\sigma_{j'}\sigma_{j''} \sum_k \left[A_{j''}^k(t_3)\xi_k(t_3), M_{k,j,j''}(A,B)\xi_k(t_1)\right] \tag{B41}$$

$$\approx \sin\omega(t_1 - t_2)\sin\omega(t_1 - t_3)\sin\mu t_2 \cos\mu t_1 \sin\mu t_3 \sum_{j,j',j''} \Omega_j\Omega_{j'}\Omega_{j''}\sigma_j\sigma_{j'}\sigma_{j''} \sum_k M_{k,j,j',j''}(A,A,B) \tag{B42}$$

$$= \sin\omega(t_1 - t_2)\sin\omega(t_1 - t_3)\sin\mu t_2 \cos\mu t_1 \sin\mu t_3 M(A,A,B) \tag{B43}$$

where

$$M_{k,j,j',j''}(A,A,B) = \eta b_{j''}^k M_{k,j,j'}(A,B)$$

and

$$M(A,A,B) \equiv \sum_{j,j',j''} \Omega_j\Omega_{j'}\Omega_{j''}\sigma_j\sigma_{j'}\sigma_{j''} \sum_k M_{k,j,j',j''}(A,A,B).$$

For two qubit gates, it can be simplified as

Therefore, we have

$$[H_A(t_1), [H_A(t_2), H_B(t_3)]] - [H_A(t_1), [H_A(t_3), H_B(t_2)]] + [H_A(t_3), [H_A(t_2), H_B(t_1)]] - [H_A(t_3), [H_A(t_1), H_B(t_2)]]$$

$$= [\sin\omega(t_3 - t_2)\sin\omega(t_3 - t_1)\sin\mu t_2 \cos\mu t_3 \sin\mu t_1 \tag{B44}$$

$$- \sin\omega(t_2 - t_3)\sin\omega(t_2 - t_1)\sin\mu t_3 \cos\omega t_2 \sin\mu t_1 \tag{B45}$$

$$+ \sin\omega(t_1 - t_2)\sin\omega(t_1 - t_3)\sin\mu t_2 \cos\mu t_1 \sin\mu t_3 \tag{B46}$$

$$- \sin\omega(t_2 - t_1)\sin\omega(t_2 - t_3)\sin\mu t_1 \cos\mu t_2 \sin\mu t_3] M(A,A,B) \tag{B47}$$

d. term Υ_4'

$$[[[[H_A(t_1), H_A(t_2)], H_A(t_3)]H_A(t_4)] \tag{B48}$$

$$= [[[\sum_{j,j'} \Omega_j\Omega_{j'}\sigma_j\sigma_{j'} \sum_k 2\sin\omega_k(t_2 - t_1), H_A(t_3)]H_A(t_4)] = 0 \tag{B49}$$

other terms are similar, so

$$\Upsilon_4' = 0$$

Appendix C: Approximating the dissipation dynamics

The evolution can be expressed as

$$\dot{\rho} = -i[H,\rho] + \mathbb{L}(\rho) \tag{C1}$$
$$\tag{C2}$$

with

$$H = \sum_{k=1}^{K} H_k = \sum_{j=0,1}\sum_{k=1}^{K} H_{j,k}, \tag{C3}$$
$$H_{j,k} = \eta \sin(\mu t)\, b_j^k \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right)\sigma_j^z, \tag{C4}$$
$$\tag{C5}$$

and

$$\mathbb{L}(\rho) = \sum_{k=1}^{K} \mathbb{L}_k(\rho), \tag{C6}$$
$$\mathbb{L}_k(\rho) = \cdots . \tag{C7}$$

k represents the kth phonon mode. It can be verified that $[H_{j,k}, H_{j',k'}] = 0$

We denote $\tilde{\rho}(t)$ as the vectorized density matrix ρ. The operators can be reexpressed in the matrix form:

$$-i[H_{j,k},\cdot] \to \mathcal{H}_{j,k} \tag{C8}$$
$$\mathbb{L}_k(\cdot) \to \mathcal{L}_{j,k} \tag{C9}$$

The evolution can be rewritten as

$$\dot{\tilde{\rho}}(t) = (\mathcal{H}(t) + \mathcal{L})\tilde{\rho}(t) = \left(\sum_k (\mathcal{H}_k(t) + \mathcal{L}_k)\right)\tilde{\rho}(t) = \left(\sum_{j,k} \mathcal{H}_{j,k}(t) + \sum_k \mathcal{L}_k\right)\tilde{\rho}(t) \tag{C10}$$

Magnus expansion to the second order gives

$$\tilde{\rho}(\tau) = \mathcal{T}\int_0^\tau e^{(\mathcal{H}(t)+\mathcal{L})dt}\tilde{\rho}_0 \tag{C11}$$
$$= \exp\left(\int_0^\tau (\mathcal{H}(t) + \mathcal{L})dt + \frac{1}{2}\int_0^\tau dt_1 \int_0^{t_1} dt_2[\mathcal{H}(t_1) + \mathcal{L}, \mathcal{H}(t_2) + \mathcal{L}]dt\right)\tilde{\rho}_0 + O(\varepsilon^3) \tag{C12}$$
$$= \exp\left(\int_0^\tau \mathcal{H}(t)dt + \mathcal{L}\tau + \frac{1}{2}\int_0^\tau\int_0^{t_1}[\mathcal{L}, \mathcal{H}(t_2) - \mathcal{H}(t_1)] + \frac{1}{2}\int_0^\tau dt_1\int_0^{t_1} dt_2[\mathcal{H}(t_1), \mathcal{H}(t_2)]dt\right)\tilde{\rho}_0 + O(\varepsilon^3) \tag{C13}$$

where $\varepsilon \equiv \tau\|\mathcal{H} + \mathcal{L}\|$. It can be verified that $[\mathcal{H}(t_1), \mathcal{H}(t_2)]$ commutes with other terms. Moreover, $\exp\left(\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1} dt_2[\mathcal{H}(t_1), \mathcal{H}(t_2)]dt\right)\tilde{\rho}_0 = \tilde{\rho}_{\text{ideal}}$ corresponds to the ideal ion-ion coupling terms with single-ion rotation terms excluded. We therefore have

$$= \exp\left(\int_0^\tau \mathcal{H}(t)dt + \mathcal{L}\tau + \frac{1}{2}\int_0^\tau\int_0^{t_1}[\mathcal{L}, \mathcal{H}(t_2) - \mathcal{H}(t_1)]\right)\exp\left(\frac{1}{2}\int_0^\tau dt_1\int_0^{t_1} dt_2[\mathcal{H}(t_1), \mathcal{H}(t_2)]dt\right)\tilde{\rho}_0 \tag{C14}$$
$$= \exp\left(\sum_k \int_0^\tau \mathcal{H}_k(t)dt + \mathcal{L}_k\tau + \frac{1}{2}\int_0^\tau\int_0^{t_1}[\mathcal{L}_k, \mathcal{H}_k(t_2) - \mathcal{H}_k(t_1)]\right)\tilde{\rho}_{\text{ideal}} \tag{C15}$$
$$\equiv \exp\left(\sum_k \mathcal{E}_k\right)\tilde{\rho}_{\text{ideal}} \tag{C16}$$

It can be verified that $[\mathcal{E}_k, \mathcal{E}_{k'}] = 0$, and $\exp(\mathcal{E}_k) = \exp\left(\hat{\mathcal{E}}_k\right) + O\left(\varepsilon_k^3\right) = \mathcal{T} \int_0^\tau (\mathcal{H}_k(t) + \mathcal{L}_k)\, dt + O\left(\varepsilon_k^3\right)$, where $\exp\left(\hat{\mathcal{E}}_k\right)$ corresponds to the evolution $\dot{\rho} = -i[H_k(t), \rho] + \mathbb{L}_k(\rho)$, and $\varepsilon_k \equiv \|\mathcal{H}_k + \mathcal{L}_k\|$. So the final state can be approximated as

$$\tilde{\rho}(\tau) = \prod_k \exp\left(\hat{\mathcal{E}}_k\right) \tilde{\rho}_{\text{ideal}} + O(??). \tag{C17}$$

Transforming back to the density matrix representation, $\rho(\tau)$ can be approximated as follow.

Step.1 obtain the optimized pulse sequence, $H(t)$, for generating Bell's state. Set $\rho_0 \otimes |\text{vac}\rangle\langle\text{vac}| \to \rho$, with ρ_0 the quantum state of ions, and $|\text{vac}\rangle\langle\text{vac}|$ the initial state of phonon.

Step.2.1 evolve ρ according to

$$\dot{\rho} = -i[H_k(t), \rho] + \mathbb{L}_k(\rho)$$

from $t = 0$ to $t = \tau$. Here, $k = 1$.

Step.2.2 $\text{tr}_{\text{phonon}}[\rho] \otimes |\text{vac}\rangle\langle\text{vac}| \to \rho$. In other words, trace out the first phonon mode, and then tensor product the second phonon mode.

Step.3 repeat step 2 from $k = 2$ to $k = K$. The target state is $\text{tr}_{\text{phonon}}[\rho]$