Heuristic search

- A* search
- A* properties
- Constructing heuristics
- Running A* on examples

Next: Game tree search

Readings: Chap 5.1,5.2,5.3

*Slides based on those of Sheila McIlraith

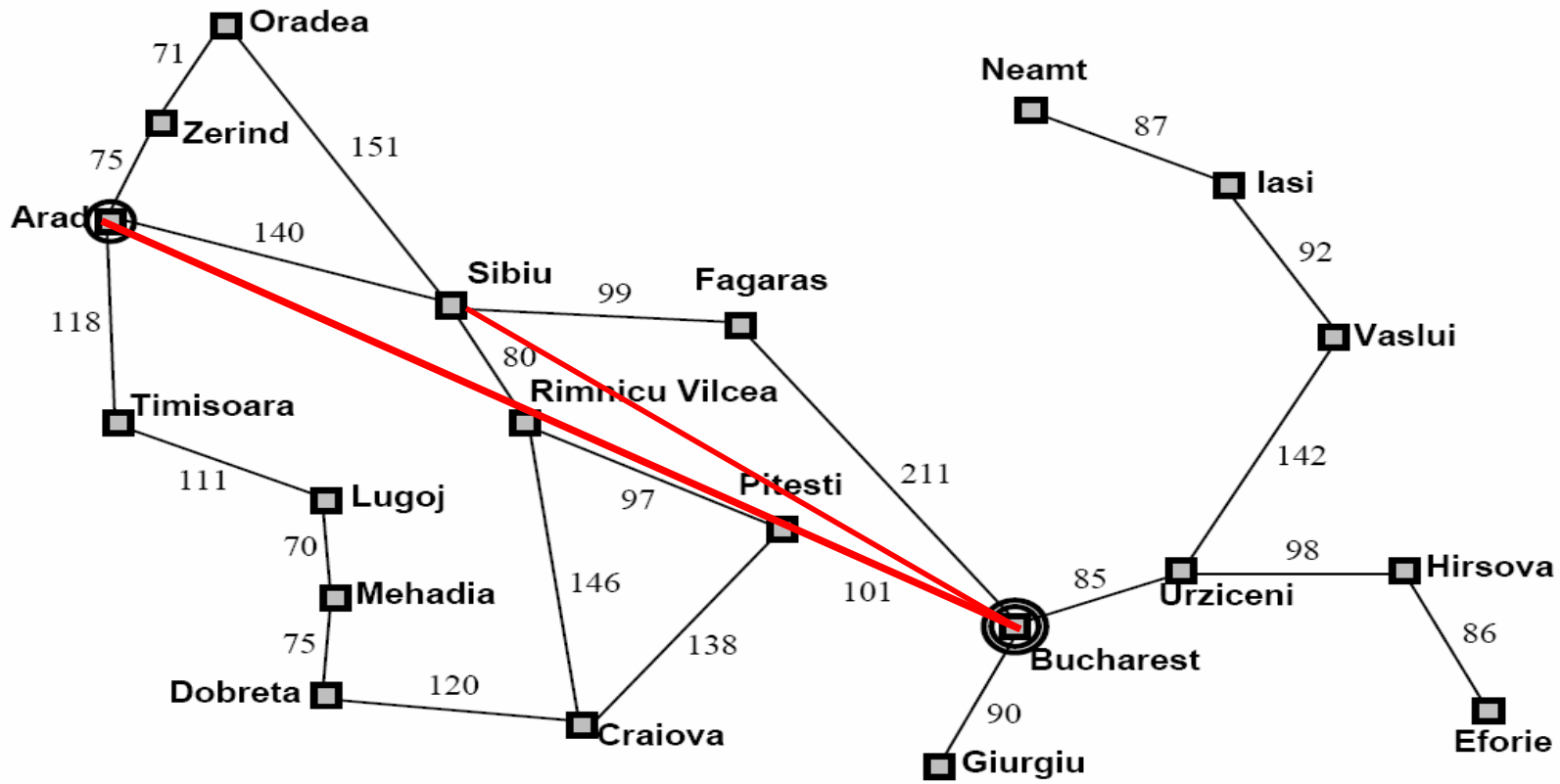
Motivation

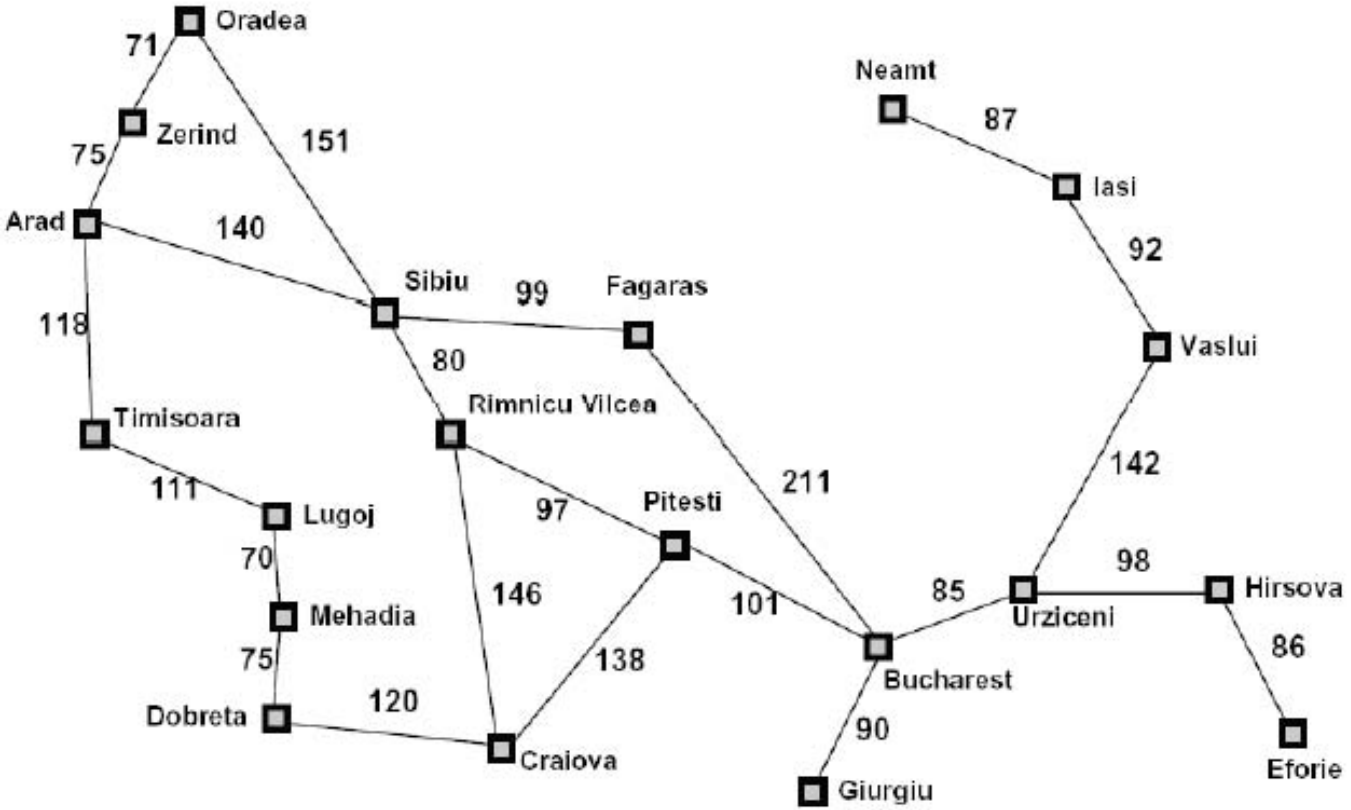
- In uninformed search, we don't try to evaluate which of the nodes on the frontier are most promising.
  - *e.g.*, in uniform cost search we always expand the cheapest path. We don't consider the cost of getting to the goal from the end of the current path.
- However, often we have some other knowledge about the merit of nodes.
  - *e.g.*, how costly it is to get to the goal from that node.

Heuristic search

- The idea is to develop a domain specific heuristic function $h(n)$, guessing the cost of getting to the goal from node n
- We require that $h(n) = 0$ for every node n whose state satisfies the goal
- There are different ways of guessing this cost in different domains. *i.e.*, heuristics are domain specific.

Example: straight line distance, *i.e.*, Euclidean distance





Straight−line distance to Bucharest	
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Greedy best-first search (Greedy BFS)

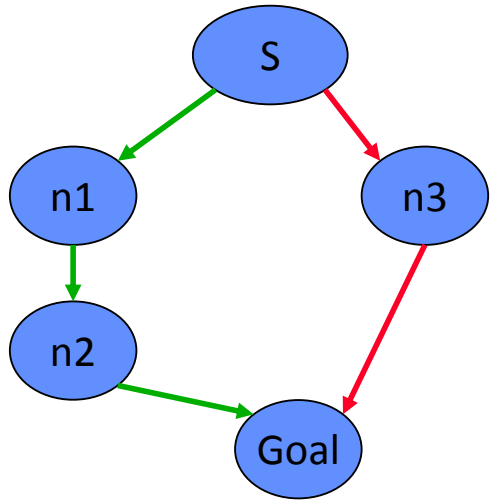
- We use $h(n)$ to order the nodes on the frontier.
- We are greedily trying to achieve a low cost solution.
- However, this method ignores the cost of getting to n, so it can be lead astray exploring nodes that cost a lot to get to but seem to be close to the goal

→ step cost = 10

→ step cost = 100

[S]
[n3,n1]
[Goal, n1]

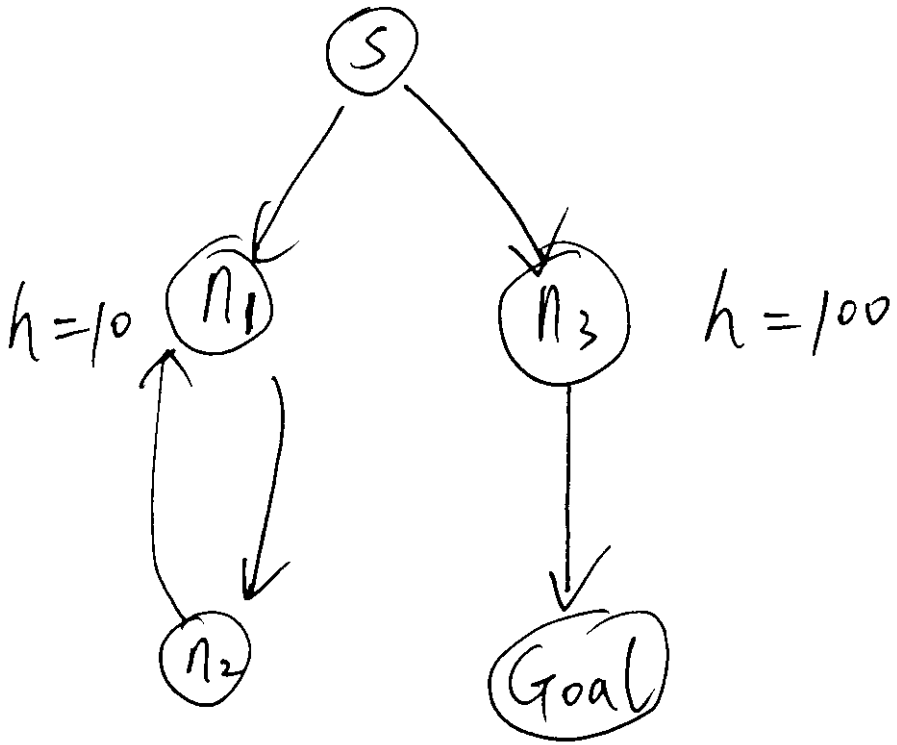
h(n1) = 70



h(n3) = 50

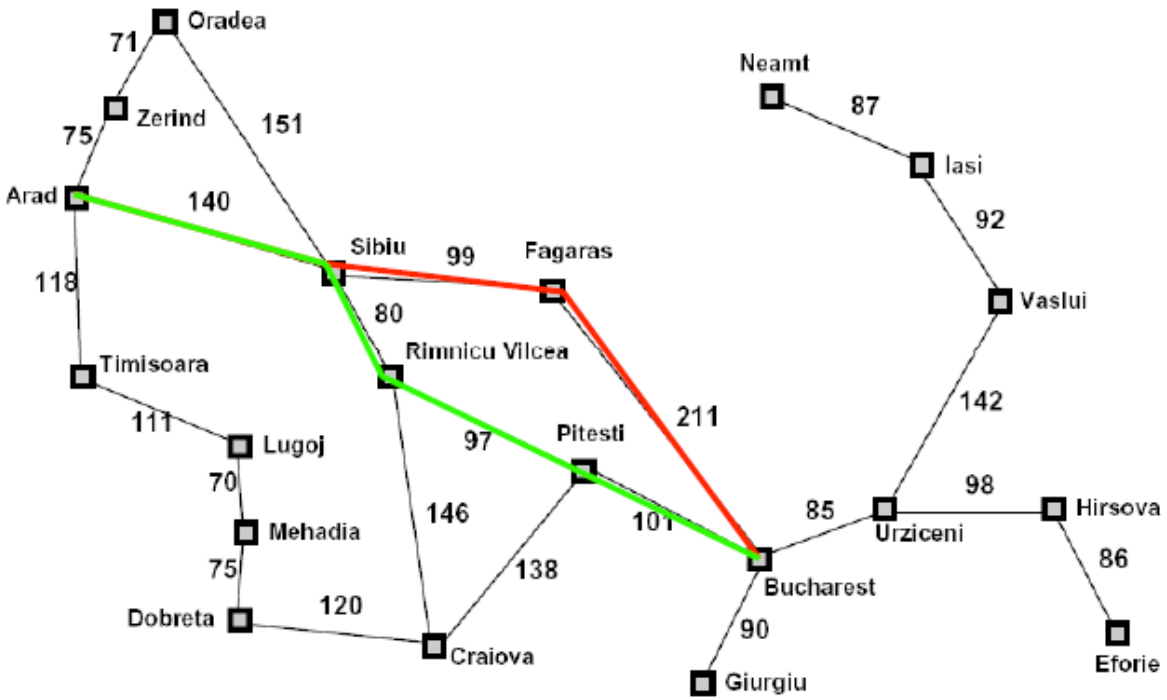
- Thus Greedy BFS is not optimal

Greedy BFS is incomplete



$h(n_2) = 10$

An example



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Arad-Sibiu-RV-Pitesli-Bucharest:

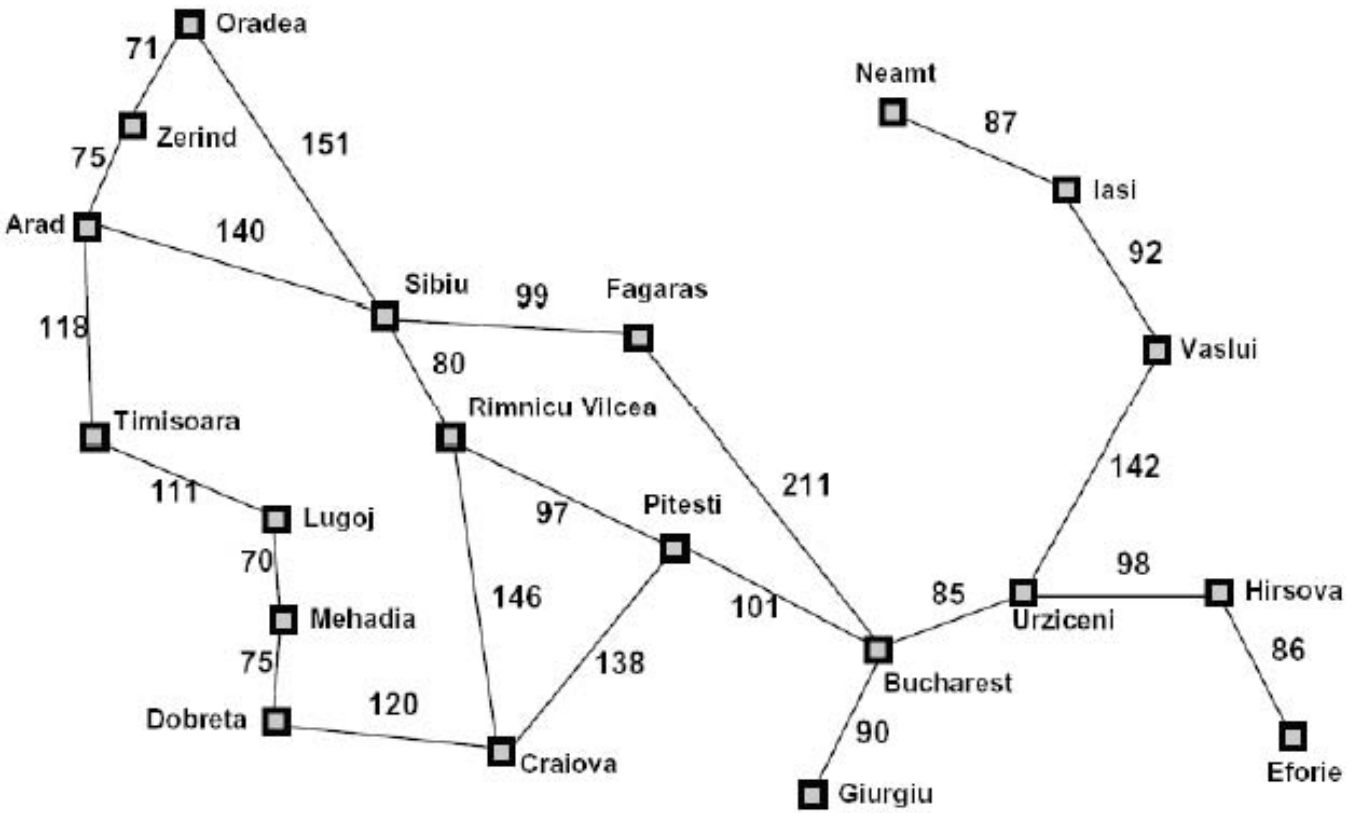
$140 + 80 + 97 + 101 = 140 + 278 = 418$

Arad-Sibiu-Fagaras-Bucharest: $140 + 99 + 211 = 140 + 310 = 450$

A* search

- Define an evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ is the cost of the path to node n
  - $h(n)$ is the heuristic estimate of the cost of getting to a goal node from n
- So $f(n)$ is an estimate of the cost of getting to the goal via node n.
- We use $f(n)$ to order the nodes on the frontier.

An example



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Arad-Sibiu-RV-Pitesli-Bucharest: 418

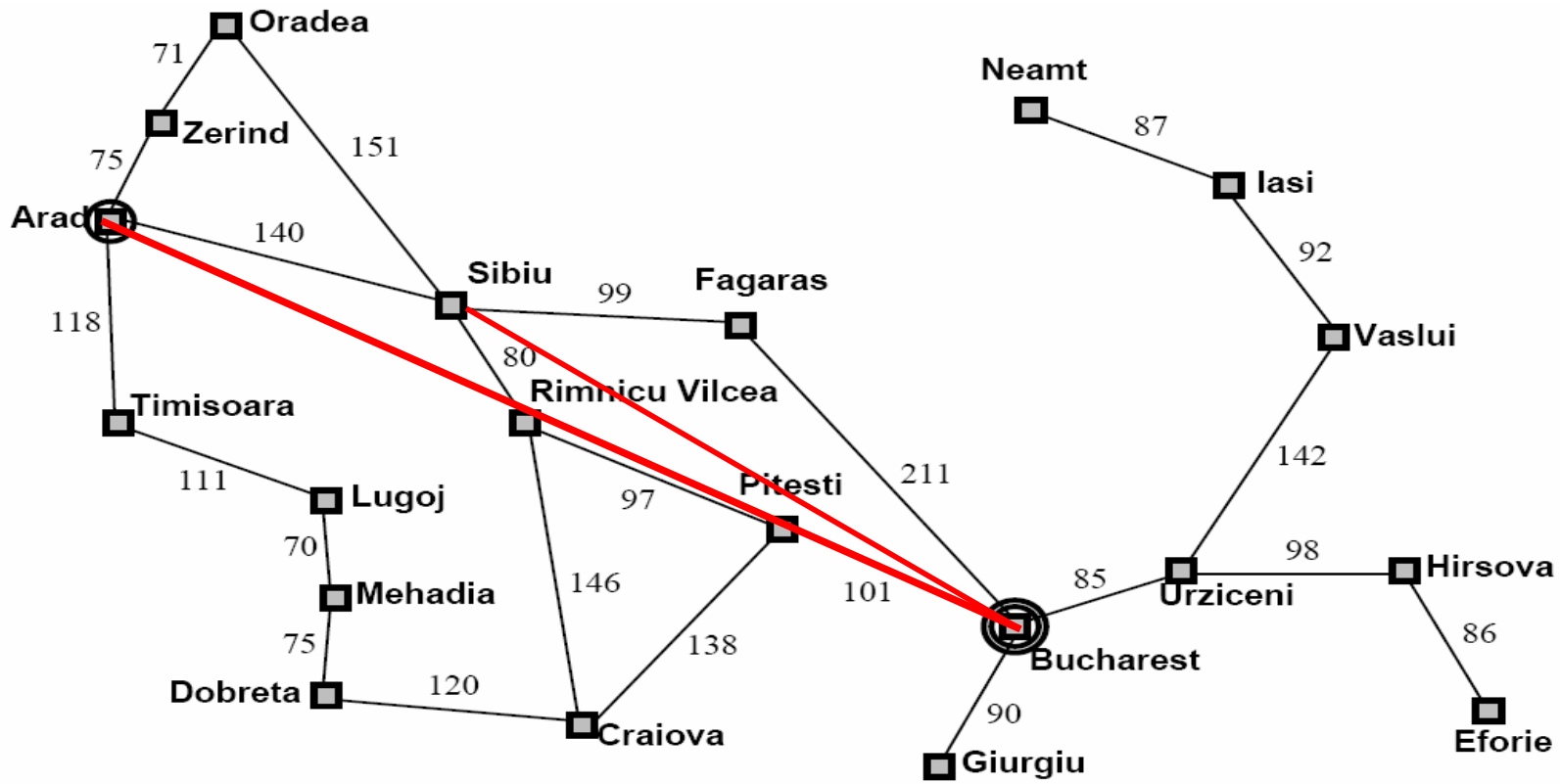
Conditions on $h(n)$: Admissible

- We always assume that $c(n1 \rightarrow n2) \geq \epsilon > 0$. The cost of any transition is greater than zero and can't be arbitrarily small.
- Let $h^*(n)$ be the cost of an optimal path from n to a goal node (∞ if there is no path).
- $h(n)$ is admissible if for all nodes n, $h(n) \leq h^*(n)$
- So an admissible heuristic underestimates the true cost to reach the goal from the current node
- Hence $h(g) = 0$ for any goal node g

Consistency (aka monotonicity)

- $h(n)$ is consistent/monotone if for any nodes n_1 and n_2, $h(n_1) \leq c(n1 \rightarrow n2) + h(n_2)$
- Note that consistency implies admissibility (proof)
  - Case 1: no path from n to the goal
  - Case 2: Let $n = n_1 \rightarrow n_2 \rightarrow \ldots \rightarrow n_k$ be an optimal path from n to a goal node. We prove by induction on i that for all i, $h(n_i) \leq h^*(n_i)$.
- Most admissible heuristics are also monotone.

An example: straight line distance



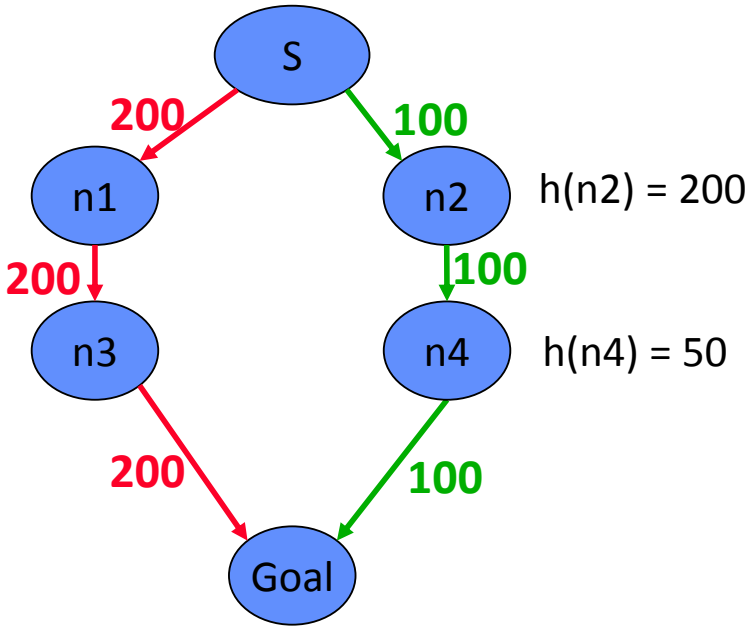
An example: admissible but nonmonotonic

The following *h* is **not consistent (i.e., not monotone)** since *h(n2)>c(n2→n4)+h(n4).*
*But it is **admissible**.*

→ step cost = 200
→ step cost = 100

h(n1) =50

h(n3) =50



g(n)+h(n)=f(n)

{S} → {n1 [200+50=**250**], n2 [200+100=**300**]}
→ {n2 [100+200=**300**], n3 [400+50=**450**]}
→ {n4 [200+50=**250**], n3 [400+50=**450**]}
→ {goal [300+0=**300**], n3 [400+50=**450**]}

We **do find** the optimal path as the heuristic is still admissible. **But** we are mislead into ignoring n2 until after we expand n1.

Time and space complexities

- When $h(n) = 0$, for all n, h is monotone. A* becomes uniform-cost
- Hence the same bounds as uniform-cost apply. (These are worst case bounds). Still exponential unless we have a very good h!

Admissibility implies optimality

- Suppose that an optimal solution has cost C^*
- Any optimal solution will be expanded before any path with cost $> C^*$ (to be proved later)
- Note that in general, paths are not expanded in the order of their costs (see the example on Pg. 14)
- So the paths expanded before an optimal solution must have cost $\leq C^*$
- There are finitely many paths with cost $\leq C^*$
- Eventually we must examine an optimal solution, and a sub-optimal solution will not be examined before an optimal solution

Any optimal path will be expanded before any path of cost $> C^*$

Proof:

- Let p^* be an optimal solution
- Assume that p is a path s.t. $c(p) > c(p^*)$ and p is expanded before p^*
- Then there must be a node n on p^* which is still in the frontier
- So $c(p) \leq f(p) \leq f(n) = g(n) + h(n)$
  $\leq g(n) + h^*(n) = c(p^*)$, contradicting $c(p) > c(p^*)$

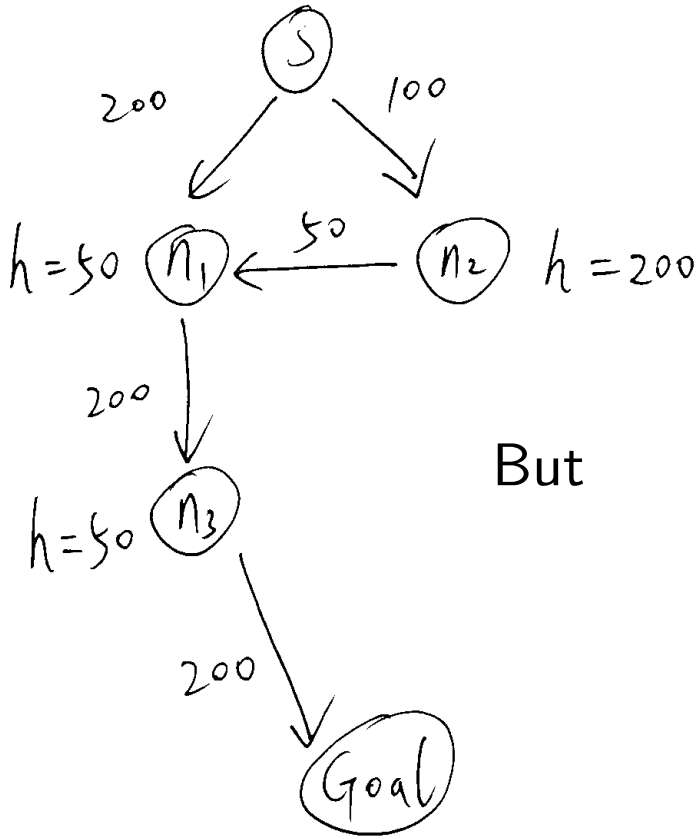
What about cycle checking?

- We will show that monotonicity guarantees we have found an optimal path to a node the first time we visit it
- Thus with monotonicity, cycle checking preserves optimality
- However, with only admissibility, cycle checking might not preserve optimality.
- To fix this: for previously visited nodes, must remember cost of previous path. If new path is cheaper must explore again.

An example: cycle checking destroys optimality

h is admissible but not monotone, since
$h(n_2) > c(n_2 \to n_1) + h(n_1)$

$$
\begin{aligned}
\{S\} &\to \{n_1[200+50=250], \\
&\quad\ n_2[200+100=300]\} \\
&\to \{n_2[200+100=300], \\
&\quad\ n_3[400+50=450]\} \\
&\to \{n_3[400+50=450]\} \\
&\to \{goal[600+0=600]\}
\end{aligned}
$$



But

the optimal path is $S \to n_2 \to n_1 \to n_3 \to Goal$

Consequences of consistency

Proposition 1. The f-values of nodes along a path must be non-decreasing

Note that with only admissibility, this does not hold
(see the example on Pg. 14)

Consequences of consistency

Proposition 2. If n_2 is expanded after n_1, then $f(n_1) \leq f(n_2)$

Proof. There are two cases:

1. When n_1 is expanded, n_2 is on the frontier.
2. When n_1 is expanded, some ancestor n of n_2 is on the frontier (use Proposition 1).

Note that with only admissibility, this does not hold
(see the example on Pg. 14)

Consequences of consistency

Proposition 3. When n is expanded every path with lower f-value has already been expanded.

Proof. Suppose that $p = n_1 \to n_2 \ldots \to n_k$ has not expanded but $f(n_k) < f(n)$

- Suppose that n_i is the last expanded node on p
- Then n_{i+1} must be on the frontier when n is expanded
- So $f(n) \leq f(n_{i+1})$
- By Proposition 1, $f(n_{i+1}) \leq f(n_k)$,
- Hence $f(n) \leq f(n_k)$, a contradiction

Note that with only admissibility, this does not hold
(see the example on Pg. 14)

Consequences of consistency

Proposition 4. The first time A* expands a state, it has found the minimum cost path to that state.

Proof.

- Let $p = n_1 \to n_2 \ldots \to n_k = n$ be the first path to n found.
- Let $p' = m_1 \to m_2 \ldots \to m_j = n$ be a path to n found later
- By Proposition 2, $g(p) + h(n) \leq g(p') + h(n)$
- Hence $g(p) \leq g(p')$.

IDA*

- A* has the same potential space problems as BFS or UCS
- IDA* - Iterative Deepening A* is similar to Iterative Deepening Search and similarly addresses space issues.
- Like iterative deepening, but now the cutoff is the f-value (g+h) rather than the depth
- At each iteration, the cutoff value is the smallest f-value of any node that exceeded the cutoff on the previous iteration

A* search: Summary

- Define an evaluation function $f(n) = g(n) + h(n)$
- We use $f(n)$ to order the nodes on the frontier.
- $h(n)$ is admissible if for all nodes n, $h(n) \leq h^*(n)$
- $h(n)$ is consistent/monotone if for any nodes n_1 and n_2, $h(n_1) \leq c(n1 \rightarrow n2) + h(n_2)$
- Consistency implies admissibility
- Admissibility implies optimality
- Exponential time and space complexity

Consequences of consistency: Summary

1. The f-values of nodes along a path must be non-decreasing
2. If n_2 is expanded after n_1, then $f(n_1) \leq f(n_2)$
3. When n is expanded every path with lower f-value has already been expanded.
4. The first time A* expands a state, it has found the minimum cost path to that state.

Thus with monotonicity, cycle checking preserves optimality

Building heuristics: Relaxed problem

- One useful technique is to consider an easier problem, and let $h(n)$ be the cost of reaching the goal in the easier problem.
- 8-puzzle moves: can move a tile from square A to B if
  - A is adjacent (left, right, above, below) to B
  - and B is blank
- Can relax some of these conditions
  1. can move from A to B if A is adjacent to B (ignore whether B is blank)
  2. can move from A to B if B is blank (ignore adjacency)
  3. can move from A to B (ignore both conditions)

Building heuristics: Relaxed problem

- #3 leads to the misplaced tiles heuristic
  - $h(n) =$ number of misplaced tiles
  - admissible: for each misplaced tile, we need at least one action to move it to its goal position; such actions are different for any two different misplaced tiles
  - monotone: any action can remove at most one misplaced title, hence for any nodes n_1 and n_2, $h(n_1) - h(n_2) \leq c(n1 \rightarrow n2)$
- #1 leads to the Manhattan distance heuristic
  - $h(n) =$ sum of the Manhattan distances
  - admissible: for each misplaced tile, we need at least d actions to move it to its goal position, where d is the Manhattan distance between the initial and the goal positions; the sets of such actions are disjoint for any two different misplaced tiles
  - monotone: any action can decrease $h(n)$ by at most 1

Building heuristics: Relaxed problem

Theorem. The optimal cost to nodes in the relaxed problem is an admissible heuristic for the original problem!

Proof:

- Let P be the original problem, and P' the relaxed problem
- Then $Sol(P) \subseteq Sol(P')$
- So $mincost(Sol(P')) \leq mincost(Sol(P))$
- Thus $h(n) \leq h^*(n)$

Comparing two heuristics

Definition. We say that h_2 dominates h_1 (or is more informed than h_1), if both are admissible and for every node n other than the goal nodes, we have $h_1(n) \leq h_2(n)$.

Theorem. If h_2 dominates h_1, then every node that is expanded by A* using h_2 is also expanded by A* using h_1.

Depth	IDS	A*(Misplaced) **h1**	A*(Manhattan) **h2**
10	47,127	93	39
14	3,473,941	539	113
24	---	39,135	1,641

Running A* with cycle checking on the 8-puzzle problem

采用Manhattan启发式函数，用带环检测的A*搜索初始状态和目标状态如下图所示的8数码问题，画出搜索图，图中标明所有节点的f, g, h值。

初始：

2	8	3
1	6	4
7		5

目标：

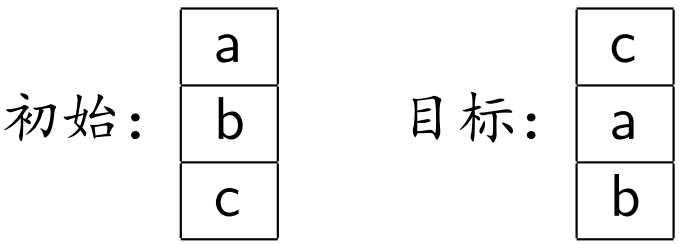
1	2	3
8		4
7	6	5

积木世界规划

现有积木若干，积木可以放在桌子上，也可以放在另一块积木上面。有两种操作：

1. $move(x, y)$：把积木x放到积木y上面。前提是积木x和y上面都没有其他积木。
2. $moveToTable(x)$：把积木x放到桌子上，前提是积木x上面无其他积木，且积木x不在桌子上。

设计本问题的一个启发式函数$h(n)$，满足$h(n) \leq h^*(n)$，然后用A*搜索初始状态和目标状态如下图所示的规划问题：



积木世界规划

- 我们说一块积木在其目标位置如果以这块积木为顶的子塔出现在目标状态。
- 令$h(n)$为不在目标位置的积木个数。
- admissible: for each misplaced block, we need at least one action to move it to its goal position; such actions are different for any two different misplaced blocks
- monotone: any action can remove at most one misplaced block

积木世界规划

Can you design a better admissible heuristic function?

- We say x is a good tower if x is in its goal position
- If it is at all possible to create a good tower, do so;
- else, move a block to the table, but be sure it doesn't come from a good tower.

滑动积木块游戏

- 一个盒子中有七个格子，里面放了黑色，白色两种木块;
- 三个黑色在左边，三个白色在右边，最右边一个格子空着;
- 一个木块移入相邻空格，耗散值为1;
- 一个木块相邻一个或两个其他木块跳入空格，耗散值为跳过的木块数;
- 游戏中将所有白色木块跳到黑色木块左边为成功;

滑动积木块游戏

- 令$h(n)$为每个白色木块前的黑色木块数目和
- EX=>XE, EXY=>YXE, EXYZ=>ZXYE
- 每个代价为1的动作使$h(n)$至多下降1
- 每个代价为2的动作使$h(n)$至多下降2
- 因此$h(n)$是单调的

The missionaries and cannibals problem

- N missionaries and N cannibals are at the left bank of a river
- There is a boat that can hold K people
- Find a way to get everyone to the right bank
- So that at any time, at any place (on either bank, or in the boat), #missionaries $\geq$ #cannibals or #missionaries $=0$

Formulation of the MC problem

- States (M, C, B) where M – #missionaries, C – #cannibals at the left bank, $B = 1$ indicates the boat is at the left bank
- Actions (m, c) where m – #missionaries, c – #cannibals on the boat
- Precondition: #missionaries and #cannibals satisfy the constraint
- Effects: $(M, C, 1) \overset{(m,c)}{\Rightarrow} (M - m, C - c, 0)$ and $(M, C, 0) \overset{(m,c)}{\Rightarrow} (M + m, C + c, 1)$

Heuristics for $K \leq 3$

- Is $h_1(n) = M + C$ admissible? No, in case of $(1, 1, 1)$, $h_1(n) = 2$, but $h^*(n) = 1$
- Let $h(n) = M + C - 2B$
- monotone:
  - $(M, C, 1) \overset{(m,c)}{\Rightarrow} (M - m, C - c, 0)$:
    $h(n_1) - h(n_2) = m + c - 2 \leq K - 2 \leq 1$
  - $(M, C, 0) \overset{(m,c)}{\Rightarrow} (M + m, C + c, 1)$:
    $h(n_1) - h(n_2) = 2 - (m + c) \leq 1$, since $m + c \geq 1$

Directly proving admissibility

When $B = 1$, in the best situation

- In the last step, we can get 3 people to the right bank
- Preceding that, we can get 3 people to the right bank, and then a person gets the boat back to the left bank
- Thus we need $\geq 2 \cdot \lceil \frac{M+C-3}{2} \rceil + 1 \geq M + C - 2$ actions

When $B = 0$,

- We need a person to get the boat back to the left bank
- From above, now to get $M + C + 1$ people to the right bank, we need $\geq M + C + 1 - 2$ actions
- Thus in total, we need $\geq M + C$ actions

Exercise

Running breadth-first with cycle-checking for $M = 3$ and $K = 2$

Exercise

Running A* with cycle checking for $M = 5$ and $K = 3$