The University of Nottingham Ningbo China

Centre for English Language Education

Semester Two 2020-2021

MID-SEMESTER EXAMINATION

FOUNDATION CALCULUS AND MATHEMATICAL TECHNIQUES

Time allowed 60 Minutes

Candidates may complete the information required on the front page of this booklet but must NOT write anything else until the start of the examination period is announced.

This paper comprises TWENTY questions. Answer all questions.

Answers must be written (with necessary steps) in this booklet.

Figures enclosed by square brackets, eg. [3], indicate marks for that question.

Only CELE approved calculator (with university logo) is allowed during this exam.

Dictionaries are not allowed with one exception.Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

Do not turn examination paper over until instructed to do so.

ADDITIONAL MATERIAL: *Useful formulae on Page 2 of this booklet.*

INFORMATION FOR INVIGILATORS:

1. *Please give a 10 minutes warning before the end of exam.*
2. *Please collect this booklet at the end of the exam.*

Student ID: ______________

Seminar Group (*e.g.* A35): __________ **Marks (out of 60):** ______

Useful formulae:

- **Differentiation: Useful results**

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(u \cdot v \cdot w) = u\,v \cdot \frac{dw}{dx} + v\,w \cdot \frac{du}{dx} + u\,w \cdot \frac{dv}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \textbf{and} \quad \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

- **Derivatives of standard functions**

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

- **Maclaurin's series**

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}\,f''(0) + \frac{x^3}{3!}\,f'''(0)$$

$$+ \ldots\ldots + \frac{x^n}{n!}\,f^{(n)}(0) + \ldots\ldots$$

- **Integration**

$$\int x^n \; dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \cos x \; dx = \sin x + C$$

$$\int \sin x \; dx = -\cos x + C$$

$$\int \sec^2 x \; dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x \; dx = -\cot x + C$$

$$\int \sec x \, \tan x \; dx = \sec x + C$$

$$\int \operatorname{cosec} x \, \cot x \; dx = -\operatorname{cosec} x + C$$

$$\int e^x \; dx = e^x + C$$

$$\int \frac{1}{x} \; dx = \ln|\,x\,| + C$$

$$\int \frac{1}{x^2+a^2} \; dx = \frac{1}{a}\,\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \; dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{|\,x\,|\sqrt{x^2-a^2}} = \frac{1}{a}\,\sec^{-1}\left|\frac{x}{a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \; dx = \ln\,|\,x + \sqrt{x^2 \pm a^2}\,| + C$$

$$\int \frac{1}{x^2-a^2} \; dx = \frac{1}{2a}\,\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int f(x)\,dx = F(x) + C$$

$$\Rightarrow \quad \int f(ax+b)\,dx = \frac{1}{a}\,F(ax+b) + C$$

- **Trigonometry**

$$\cos^2\theta + \sin^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

$$\begin{aligned} 2\sin A\cos B &= \sin(A+B) + \sin(A-B) \\ 2\cos A\sin B &= \sin(A+B) - \sin(A-B) \\ 2\cos A\cos B &= \cos(A+B) + \cos(A-B) \\ -2\sin A\sin B &= \cos(A+B) - \cos(A-B) \end{aligned}$$

1. Given $y = x^2 + 4x$, use the first principle to find $\frac{dy}{dx}$.

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 4(x+h) - x^2 - 4x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h - \cancel{x^2} - \cancel{4x}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 4h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h + 4)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h + 4$$

$$= 2x + 4$$

[2]

2. Given that $y = (x+1)^2 \cdot (2x-1)$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = (x+1)^2 \frac{d}{dx}(2x-1) + (2x-1)\frac{d}{dx}(x+1)^2$$

$$= (x+1)^2 \cdot 2 + (2x-1) \cdot 2(x+1)$$

$$= 6x^2 + 6x$$

Or

$$y = (x^2 + 2x + 1)(2x-1)$$

$$= 2x^3 + 3x^2 - 1$$

$$\frac{dy}{dx} = 6x^2 + 6x$$

[3]

3. Given that $y = e^x \cdot x^2 \cdot \tan x$, use the product rule for derivatives to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = e^x \cdot x^2 \cdot \frac{d}{dx}(\tan x) + e^x \cdot \tan x \cdot \frac{d}{dx}(x^2) + x^2 \cdot \tan x \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot x^2 \cdot \sec^2 x + e^x \cdot \tan x \cdot (2x) + x^2 \cdot \tan x \cdot e^x$$

$$= e^x \left(x^2 \cdot \sec^2 x + 2x \tan x + x^2 \tan x\right)$$

[2]

4. Given $y = \frac{\sqrt{x}}{1 - \sin x}$, use the quotient rule for derivatives to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x) \cdot \frac{1}{2\sqrt{x}} - \sqrt{x}\,(-\cos x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x}\left[\frac{1}{2x}(1-\sin x) + \cos x\right]}{(1-\sin x)^2}$$

[3]

5. Given $f(1) = 2$, $f'(1) = 0$, $g(1) = 3$, and $g'(1) = 4$, find $h'(1)$ if $h(x) = \dfrac{f(x)}{g(x)}$.

$$h'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

$$h'(1) = \frac{g(1)\cdot f'(1) - f(1)\cdot g'(1)}{[g(1)]^2}$$

$$= \frac{3\times 0 - 2\times 4}{3^2}$$

$$= -\frac{8}{9}$$

[2]

6. Given $y = \sin\left(\tan\left(e^x\right)\right)$, use the chain rule to find $\dfrac{dy}{dx}$.

$$v = e^x \rightarrow \frac{dv}{dx} = e^x \; ; \quad u = \tan v \rightarrow \frac{du}{dv} = \sec^2 v$$

$$y = \sin(u) \longrightarrow \frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} = \cos u\cdot\sec^2 v\cdot e^x$$

$$= \cos(\tan(e^x))\cdot\sec^2(e^x)\cdot e^x$$

Or

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\tan(e^x))\right] = \cos(\tan(e^x))\cdot\frac{d}{dx}\left[\tan(e^x)\right]$$

$$= \cos(\tan(e^x))\cdot\sec^2(e^x)\cdot\frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = \cos(\tan(e^x))\cdot\sec^2(e^x)\cdot e^x$$

[4]

7. Given $x^3y+y^3x=2$, use implicit differentiation to find the gradient at the point $(1,1)$.

$$3x^2y + x^3\frac{dy}{dx} + y^3 + x3y^2\frac{dy}{dx} = 0$$

$$x^3\frac{dy}{dx} + 3xy^2\frac{dy}{dx} = -3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{-(3x^2y+y^3)}{(x^3+3xy^2)}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-(3+1)}{(1+3)}$$

$$= -1$$

[3]

8. Given $y = \dfrac{\sqrt[3]{(x-1)^2}\cdot(x-3)^5}{e^{3x}}$, use the method of logarithmic differentiation to find $\dfrac{dy}{dx}$.

$$\ln y = \ln(x-1)^{\frac{2}{3}} + \ln(x-3)^5 - \ln(e^{3x})$$

$$= \frac{2}{3}\ln(x-1) + 5\ln(x-3) - 3x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3(x-1)} + \frac{5}{(x-3)} - 3$$

$$\frac{dy}{dx} = y\left[\frac{2}{3(x-1)} + \frac{5}{(x-3)} - 3\right]$$

$$= \frac{\sqrt[3]{(x-1)^2}\cdot(x-3)^5}{e^{3x}}\left[\frac{2}{3(x-1)} + \frac{5}{(x-3)} - 3\right]$$

[4]

9. Given equation of the curve $x^2 - y^2 = 7$, find the equation of the normal line to the curve at the point $(4, -3)$.

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(4,-3)} = -\frac{4}{3} \; ; \quad n = \frac{-1}{(-4/3)}$$

$$n = 3/4$$

$$(y+3) = \frac{3}{4}(x-4)$$

$$4y - 3x + 24 = 0$$

[3]

10. Given parametric equations of the curve: $x = e^{-t} \cdot \cos 2t, \quad y = e^{-2t} \cdot \sin 2t$, find the slope of the curve at the point $t = 0$.

$$\frac{dy}{dt} = -2e^{-2t}\sin 2t + 2e^{-2t}\cos 2t$$

$$= 2e^{-2t}(\cos 2t - \sin 2t)$$

$$\frac{dx}{dt} = -e^{-t}\cos 2t + (-2e^{-t}\sin 2t)$$

$$= -e^{-t}(\cos 2t + 2\sin 2t)$$

$$\frac{dy}{dx} = \frac{2e^{-2t}(\cos 2t - \sin 2t)}{-e^{-t}(\cos 2t + 2\sin 2t)}$$

$$= \frac{-2e^{-2t}(\cos 2t - \sin 2t)}{e^{-2t}(\cos 2t + 2\sin 2t)}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{-2(1-0)}{1(1+0)} = -2$$

[3]

11. Given $f(x) = x^4 - 8x^2$, find and classify the stationary points of f.

$$f'(x) = 4x^3 - 16x = 0$$

$$4x(x^2 - 4) = 0$$

The stationary points are at:
$(-2, -16)$, $(0,0)$, and $(2, -16)$.

$$f''(x) = 12x^2 - 16$$

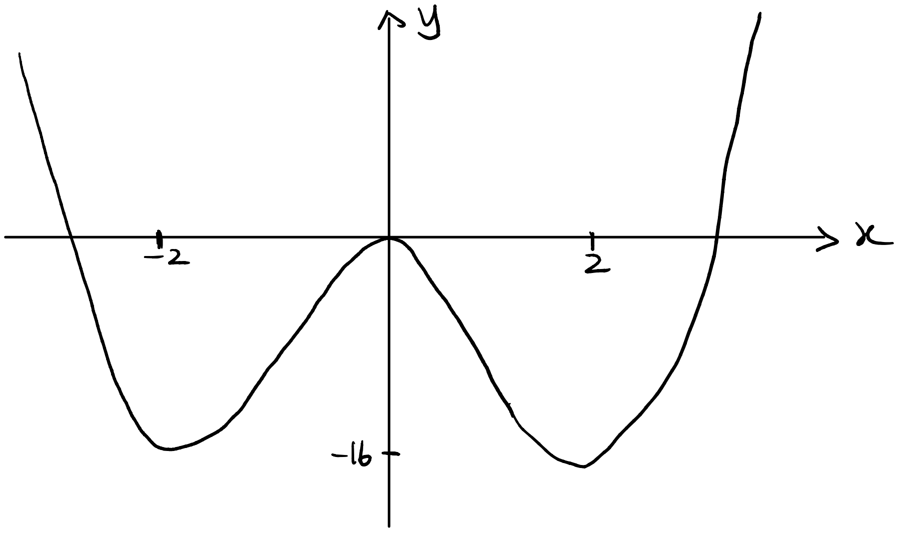
$f''(x)\big|_{x=-2} = 32 > 0$, $\therefore$ $(-2,-16)$ is a minimum point

$f''(x)\big|_{x=0} = -16 < 0$, $\therefore$ $(0,0)$ is a maximum point

$f''(x)\big|_{x=2} = 32 > 0$, $\therefore$ $(2,-16)$ is a minimum point

[3]

12. For the function f defined in Q.11, sketch the graph of $y = f(x)$.



[1]

13. The Newton-Raphson iteration formula is given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. (13.1)

(a) Given $f(x) = x^3 + 3x + 1$, obtain an expression for x_{n+1} for the given function f.

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$$

(b) Starting with $x_0 = 1$, apply equation (13.1) to determine the root of $f(x) = 0$, correct to 5 d.p.

n	x_n
0	1
1	0.16667
2	−0.32132
3	−0.32219
4	−0.32219

$x_{root} = -0.32219$

[4]

14. Given $f(x) = \sin x$, obtain the Maclaurin's series of $f(x)$ up to the terms in x^5.

$f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x$ $\quad f^{(5)}(0) = 1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

[4]

15. Evaluate $\int \left(x^3 - 3\sqrt{x^3} + \frac{1}{x^2} - \sec^2 x\right) dx$.

$$\int x^3\,dx - 3\int x^{3/2}\,dx + \int x^{-2}\,dx - \int \sec^2 x\,dx$$

$$\frac{x^4}{4} - \frac{3}{(5/2)} x^{5/2} + \frac{x^{-1}}{(-1)} - \tan x + C$$

$$\frac{x^4}{4} - \frac{6}{5} x^{5/2} - \frac{1}{x} - \tan x + C$$

[3]

16. Use the substitution $\frac{1}{x} = t$, to evaluate the integral $\int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$.

$$\frac{1}{x} = t \Rightarrow -\frac{1}{x^2}\,dx = dt$$

$$\int \sin(t)\cdot(-dt) = -\int \sin(t)\,dt$$

$$= -(-\cos(t)) + C$$

$$= \cos(t) + C$$

$$= \cos\left(\frac{1}{x}\right) + C$$

[3]

17. Use appropriate substitution to evaluate the integral $\int \frac{x^3}{1+x^8}\,dx$.

$t = x^4 \Rightarrow dt = 4x^3\,dx$

$\frac{1}{4}dt = dx$

$I = \int \frac{1}{1+t^2}\left(\frac{1}{4}dt\right) = \frac{1}{4}\int \frac{1}{1+t^2}\,dt$

$= \frac{1}{4}\tan^{-1}(t) + C$

$= \frac{1}{4}\tan^{-1}(x^4) + C$

[3]

18. Evaluate the integral $\int \sin^6 x\ \cos^3 x\,dx$ by using appropriate substitution.

$\cos^2 x = 1 - \sin^2 x$

$I = \int \sin^6 x \cos^2 x \cos x\,dx$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

$I = \int t^6(1-t^2)\,dt = \int (t^6 - t^8)\,dt$

$= \frac{t^7}{7} - \frac{t^9}{9} + C$

$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C$

[3]

19. Evaluate the integral $\int \cos 5x \ \sin 7x \ dx$.

$$\int \cos 5x \sin 7x \, dx = \frac{1}{2}\int \left(\sin(12x) - \sin(-2x)\right) dx$$

$$= \frac{1}{2}\int \left(\sin(12x) + \sin(2x)\right) dx$$

$$= \frac{1}{2}\left[-\frac{\cos(12x)}{12} - \frac{\cos(2x)}{2}\right] + C$$

$$I = -\frac{\cos(12x)}{24} - \frac{\cos(2x)}{4} + C$$

[3]

20. Evaluate the integral $\int \frac{1}{\sqrt{10x - x^2}} dx$ by completing the square for the term inside the square root.

Completing the square in the denominator:

$$-(x^2 - 10x) = -(x^2 - 10x + 5^2) + 5^2$$

$$= 5^2 - (x-5)^2$$

$$I = \int \frac{1}{\sqrt{5^2 - (x-5)^2}} dx$$

$$= \sin^{-1}\left(\frac{x-5}{5}\right) + C$$

[4]