CELEN037 2020-2021

Q1

(a)

(i)

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{x+h}{(x+h+1)} - \frac{x}{(x+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+\cancel{x}+\cancel{hx}+\cancel{h}+\cancel{x^2}-\cancel{hx}-\cancel{x}}{\cancel{h}[(x+h+1)(x+1)]}$$

$$= \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

(ii)

$$K = -\frac{1}{4x} \Rightarrow K = -\frac{1}{4(1)} \Rightarrow K = -\frac{1}{4}$$

(b)

$$\frac{dy}{dx} = \frac{(1+\cos x)\cdot\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \frac{1}{1+\cos x}$$

(c)

(i)

$$\frac{dy}{dx} = 2x + e^x$$

$$\frac{d^2y}{dx^2} = 2 + e^x$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2+1 = 3$$

(ii)

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\sin^2 x}}\cdot(\cos x)$$

$$= -\frac{\cos x}{|\cos x|} = -\frac{\cos x}{-\cos x} = 1$$

Q1

(c) (iii)

$$g'(x) = f'(x) \cdot \sec x + f(x) \cdot \sec x \tan x$$

$$g'(\pi/4) = 2 \cdot \frac{1}{\cos(\pi/4)} + 0$$

$$g'(\pi/4) = 2 \cdot \frac{2}{\sqrt{2}} = 2\sqrt{2}$$

Q2

(a) (i)

$$\frac{dy}{dx} = 1 - \sin(xy)\left[x\frac{dy}{dx} + y\right]$$

$$\frac{dy}{dx}\left[1 + x\sin(xy)\right] = 1 - y\sin(xy)$$

$$\frac{dy}{dx} = \frac{1 - y\sin(xy)}{1 + x\sin(xy)} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1-0}{1+0}$$

$$\therefore m = \left.\frac{dy}{dx}\right|_{(0,1)} = 1$$

(ii)

$$y - 1 = 1(x - 0)$$

OR $y - x - 1 = 0$

$x - y + 1 = 0$

(iii)

$$n = -\frac{1}{m} = -1$$

$$y - 1 = -1(x - 0)$$

$$x + y - 1 = 0$$

(b) (i)

$f'(x) = 0$

$\therefore 12x^2 - 2x - 2 = 0$

$(3x+1)(2x-1) = 0$

Hence: $\left(-\frac{1}{3}, \frac{38}{27}\right)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$ are stationary points.

OR: stationary points are at: $x = -\frac{1}{3}$, $x = \frac{1}{2}$

(ii)

$$\frac{d^2y}{dx^2} = 24x - 2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(-\frac{1}{3}, \frac{38}{27}\right)} = 24\left(-\frac{1}{3}\right) - 2 = -10 < 0$$

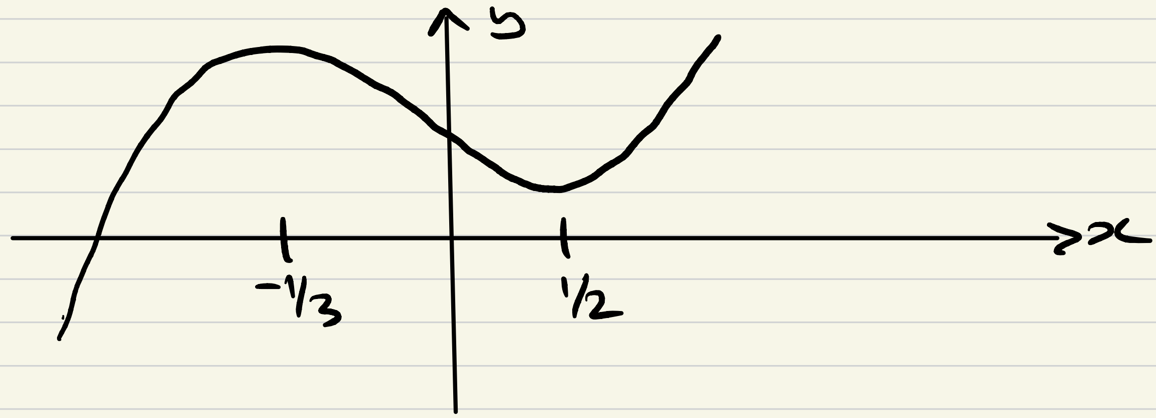
$\therefore \left(-\frac{1}{3}, \frac{38}{27}\right)$ is a maximum point

$$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{1}{2}, \frac{1}{4}\right)} = 24\left(\frac{1}{2}\right) - 2 = 10 > 0$$

$\therefore \left(\frac{1}{2}, \frac{1}{4}\right)$ is a minimum point

Q2

(c)



(c) $f'(x) > 0$

$2x - \frac{3}{x} > 0$

$x^2 > \frac{3}{2}$

$\therefore \; x \in (\sqrt{3/2}, \infty)$

Or $x > \sqrt{\frac{3}{2}}$

Q3

(a) $\frac{dy}{dx} = 3\theta\cos\theta + 3\sin\theta$

$\frac{dx}{d\theta} = -(\theta - \pi)\sin\theta + \cos\theta$

$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{3\theta\cos\theta + 3\sin\theta}{-(\theta-\pi)\sin\theta + \cos\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{0+3}{\pi/2} = \frac{6}{\pi}$

(b) $x_{n+1} = x_n - \frac{x_n^4 - \cos x_n}{4x_n^3 + \sin x_n}$

n	x_n
0	0.5
1	1.33220
2	1.05284
3	0.92035
4	0.89175
5	0.89055
6	0.89055

$\therefore x_{root} = 0.89055$

(c) $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{-0.5}{2\pi r}$

$\frac{dr}{dt} = -\frac{1}{4\pi r}$

$\frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt}$

$= 2\pi \cdot \left(\frac{-1}{4\pi r}\right)$

$= -\frac{1}{2r} = -\frac{1}{2(4)}$

$= -\frac{1}{8}$

$\therefore$ circumference is reducing at a rate of $1/8$ cm/sec

Q 4

(a) (i)

$$f(x) = \cos 2x \qquad f(0) = 1$$
$$f'(x) = -2\sin 2x \qquad f'(0) = 0$$
$$f''(x) = -4\cos 2x \qquad f''(0) = -4$$
$$f'''(x) = 8\sin 2x \qquad f'''(0) = 0$$
$$f^{(4)}(x) = 16\cos 2x \qquad f^{(4)}(0) = 16$$
$$f^{(5)}(x) = -32\sin 2x \qquad f^{(5)}(0) = 0$$
$$f^{(6)}(x) = -64\cos 2x \qquad f^{(6)}(0) = -64$$

(ii)

$$\cos 2x = 1 + x(0) + \frac{x^2}{2!}(-4) + \frac{x^3}{3!} + \frac{x^4}{4!}(16) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-64) + \ldots$$

$$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots\ldots$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$

$$= \frac{1}{2}\left[1 - \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots\ldots\right)\right]$$

$$\sin^2 x = \frac{1}{2}\left(2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots\ldots\right)$$

$$\sin^2 x = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \ldots\ldots$$

Q4

(b) $f(x) = \sqrt{1-x} = (1-x)^{1/2}$ | $f(0) = 1$

$f'(x) = -\frac{1}{2}(1-x)^{-1/2}$ | $f'(0) = -\frac{1}{2}$

$f''(x) = \frac{1}{4}(1-x)^{-3/2}$ | $f''(0) = \frac{1}{4}$

$\therefore f(x) = \sqrt{1-x} \approx 1 - \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2}$

$\approx 1 - \frac{1}{2}x - \frac{1}{8}x^2$

$f(1/50) \approx \sqrt{\frac{49}{50}} \approx 1 - \frac{1}{100} - \frac{1}{20000}$

$\frac{7}{5\sqrt{2}} \approx 0.9900$

$\therefore \sqrt{2} \approx \frac{7}{5 \times 0.990} \approx 1.414$

Q5

(a)

(i)

$$I = \int \left[\frac{x^2 \cancel{(x-1)}}{\cancel{(x-1)}} + \frac{1}{\sqrt{6^2 - x^2}} \right] dx$$

$$I = \frac{x^3}{3} + \sin^{-1}\left(\frac{x}{6}\right) + c$$

(ii)

$$I = \int (e^{2x} + e^{-x}) dx$$

$$I = \frac{1}{2} e^{2x} - e^{-x} + c$$

(b) $\ln(\sin x) = t$

$$\frac{1}{\sin x} \cos x \, dx = dt$$

$$\therefore \cot x \, dx = dt$$

$$I = \int \frac{1}{10^2 - t^2} dt$$

$$= \frac{1}{2 \cdot 10} \ln \left| \frac{t - 10}{t + 10} \right| + c$$

$$\therefore I = \frac{1}{20} \ln \left| \frac{\ln(\sin x) - 10}{\ln(\sin x) + 10} \right| + c$$

(c)

(i)

$$I = \int \sin^9 x \cos^5 x \, dx$$

$$= \int \sin^9 x (1 - \sin^2 x)^2 \cdot \cos x \, dx$$

let $\sin x = t$

then $\cos x \, dx = dt$

$$\therefore I = \int t^9 (1 - t^2)^2 \cdot dt$$

$$= \int t^9 (1 - 2t^2 + t^4) dt$$

$$\therefore I = \int (t^9 - 2t^{11} + t^{13}) dt$$

$$I = \frac{t^{10}}{10} - \frac{t^{12}}{6} + \frac{t^{14}}{14} + c$$

$$I = \frac{1}{10} \sin^{10} x - \frac{1}{6} \sin^{12} x + \frac{1}{14} \sin^{14} x + c$$

Q5

(c) (ii)

$$I = \frac{1}{2}\int \left(\sin 14x - \sin 4x\right) dx$$

$$= \frac{1}{2}\left(-\frac{\cos 14x}{14} + \frac{\cos 4x}{4}\right) + C$$

$$= -\frac{\cos 14x}{28} + \frac{\cos 4x}{8} + C$$

$$\therefore I = \frac{\cos 4x}{8} - \frac{\cos 14x}{28} + C$$

Q6

(a) Let $\tan\left(\frac{x}{2}\right) = t$

$\therefore \quad dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

Hence: $I = \int \frac{1}{2\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$

$$I = 2\int \frac{1}{4t - 1 + t^2}\,dt = 2\int \frac{1}{(t+2)^2 - \sqrt{5}^2}\,dt$$

$$= \frac{\cancel{2}}{\cancel{2}\cdot\sqrt{5}} \ln\left|\frac{t+2-\sqrt{5}}{t+2+\sqrt{5}}\right| + C$$

$$\therefore\ I = \frac{1}{\sqrt{5}} \ln\left|\frac{\tan(x/2)+2-\sqrt{5}}{\tan(x/2)+2+\sqrt{5}}\right| + C$$

(b) (i)

$$I = \int \frac{\cancel{e^x}(e^x - e^{-x})}{\cancel{e^x}(e^x+e^{-x})}\,dx$$

$\therefore\ I = \ln(e^x + e^{-x}) + C$

Any equivalent form is accepted.

Q6

(b) (ii)

$$I = \int \frac{\frac{1}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x} - \frac{2}{\cos^2 x}}\,dx = \int \frac{\sec^2 x}{1 - 2\sec^2 x}\,dx$$

$$\therefore I = \int \frac{\sec^2 x}{1 - 2 - 2\tan^2 x}\,dx = \int \frac{\sec^2 x}{-1 - 2\tan^2 x}\,dx$$

let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore I = -\int \frac{1}{2t^2 + 1}\,dt$$

Hence $$I = -\int \frac{1}{(\sqrt{2}t)^2 + 1}\,dt$$

or $$I = -\frac{1}{2}\int \frac{1}{t^2 + (1/\sqrt{2})^2}\,dt$$

$$\therefore I = -\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t) + C$$

hence $$I = -\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$$

Q6

(c)

(i)

$$\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$x = A(x+2) + B(x+1)$

let $x = -1$: $A = -1$

let $x = -2$: $-B = -2$ $\therefore$ $B = 2$

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2}$$

$$= \frac{2}{(x+2)} - \frac{1}{(x+1)}$$

(ii)

$$\int f(x)\,dx = \int \left[\frac{2}{(x+2)} - \frac{1}{(x+1)}\right] dx$$

$$= \ln\left|\frac{(x+2)^2}{(x+1)}\right| + c$$

(iii)

$$\int_0^1 f(x)\,dx = \left[\ln\left|\frac{(x+2)^2}{(x+1)}\right|\right]_0^1$$

$$= \left\{\ln\left(\frac{3^2}{2}\right) - \ln\left(\frac{2^2}{1}\right)\right\} = \ln\left(\frac{9}{2}\cdot\frac{1}{2^2}\right)$$

$$\therefore \int_0^1 f(x)\,dx = \ln\left(9/8\right)$$

■

Q7

(a)

(i)

$\tan x = t$
$\sec^2 x\,dx = dt$

x	0	$\pi/4$
t	0	1

$$I = \int_0^{\pi/4} \tan^7 x \sec^2 x\,dx = \int_0^1 t^7\,dt$$

$$I = \frac{1}{8}\left[t^8\right]_0^1 = \frac{1}{8}\left[1-0\right]$$

$$\therefore \underline{I = 1/8}$$

(ii)

$$I = \int_2^K \frac{1}{|x|\sqrt{x^2-2^2}}\,dx = \frac{1}{2}\left[\sec^{-1}\left|\frac{x}{2}\right|\right]_2^K$$

$$\therefore \frac{1}{2}\left(\sec^{-1}(K/2) - 0\right) = \frac{\pi}{6}$$

$$\sec^{-1}(K/2) = \pi/3 \Rightarrow \cos^{-1}\left(\frac{2}{K}\right) = \frac{3}{\pi}$$

$$\frac{2}{K} = \frac{1}{2} \Rightarrow K = 4$$

(b) $p\int_0^{\pi} x\cos x\,dx = 1$

$$p\left(\left[x\sin x\right]_0^{\pi} - \int_0^{\pi}\sin x\,dx\right) = 1$$

$$p\left(0 + \left[\cos x\right]_0^{\pi}\right) = 1 \Rightarrow p(-1-1) = 1$$

$$\therefore p(-2) = 1 \Rightarrow \underline{p = -1/2}$$

Turn over

Q7

(c)

(i)

$$A_R = \int_{-1}^{0} e^{x/2}\,dx = 2\left[e^{x/2}\right]_{-1}^{0}$$

$$A_R = 2\left\{1 - e^{-1/2}\right\} \approx 0.78694$$

(ii)

$$V_R = \pi \int_{-1}^{0} \left(e^{x/2}\right)^2 dx = \pi \int_{-1}^{0} e^{x}\,dx$$

$$V_R = \pi\left[e^{x}\right]_{-1}^{0} = \pi\left\{1 - e^{-1}\right\}$$

$$\therefore \quad V_R \approx 1.98587$$

Q8

(a)

(i)

$$\int (1-\cos y)\,dy = \int (1+\sin x)\,dx$$

$$y - \sin y = x - \cos x + c$$

(ii)

$$\int \left(\frac{1+y^2}{y}\right) dy = \int \tan^2 x\,dx$$

$$\int \left(\frac{1}{y} + y\right) dy = \int (\sec^2 x - 1)\,dx$$

$$\ln|y| + \frac{y^2}{2} = \tan x - x + c$$ general solution

when $x_0 = 0$; $y_0 = 1$

$$0 + \frac{1}{2} = 0 - 0 + c \Rightarrow c = 1/2$$

$$\ln|y| + \frac{y^2}{2} = \tan x - x + \frac{1}{2}$$ particular solution

(b)

$$y = \frac{\sin x + a}{x} - \cos x$$

$$\frac{dy}{dx} = \frac{x(\cos x) - (\sin x + a)}{x^2} + \sin x$$

$$LHS = x \cdot \left[\frac{x(\cos x) - (\sin x + a)}{x^2} + \sin x\right]$$

$$+ \frac{\sin x + a}{x} - \cos x$$

$$= \cancel{\cos x} - \cancel{\frac{(\sin x + a)}{x}} + x\sin x + \cancel{\frac{(\sin x + a)}{x}} \cancel{- \cos x}$$

$$= x \sin x = RHS$$

∎

Q 8

(c)

$$\frac{dP}{dt} = kP$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

at $t = 0$ $P = P_0$

(i)

$$\ln|P_0| = k(0) + C \Rightarrow C = \ln|P_0|$$

$$\therefore \ln|P| = kt + \ln|P_0|$$

$$\ln\left|\frac{P}{P_0}\right| = kt \Rightarrow \frac{P}{P_0} = e^{kt}$$

$$\therefore P = P_0 e^{kt}$$

$$P(t) = P_0 e^{kt}$$

t	20	50
P	$2P_0$	?

$$2P_0 = P_0 e^{k20}$$

$$\therefore k = \frac{\ln(2)}{20}$$

(ii)

$$P(50) = P_0 e^{\frac{\ln(2)}{20} \cdot 50}$$

$$= P_0 e^{[\ln(2)]^{5/2}}$$

$$\therefore P(50) = (2)^{5/2} P_0 \approx \underline{5.66 P_0}$$

Thus the population after 50 years is approximately 6 times P_0.