APPENDIX

(Dated: November 18, 2019)

COMBINATORIAL ALGORITHM FOR EVALUATING CORRELATION FUNCTIONS

Correlations are central to the understanding of many-body quantum systems which admit no local theory [1], as well as to generalising the concept of classical coherence to nonclassical fields [2]. A theoretical tool for characterising correlations in quantum fields was introduced by Glauber [2] who defined the normalised n-th order correlation function as

$$g^{(n)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \frac{\langle : \hat{n}(\mathbf{x}_1)\, \hat{n}(\mathbf{x}_2) \ldots \hat{n}(\mathbf{x}_n) : \rangle}{\langle \hat{n}(\mathbf{x}_1) \rangle \langle \hat{n}(\mathbf{x}_2) \rangle \ldots \langle \hat{n}(\mathbf{x}_n) \rangle}, \tag{1}$$

where $\hat{n}(\mathbf{x})$ is the number operator for variable $\mathbf{x}$ (e.g. position, momentum, time), and the :: symbol denotes normal ordering of the operator product (annihilation operators are placed to the right of creation operators). Glauber's correlation function characterises the likelihood of observing the coincidence of n-fold joint detection event $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ in a single experiment, therefore for a many-body system a large value of $g^{(n)}$ indicates the presence of an n-particle interaction. As a special case, if the particle numbers at all $\mathbf{x}_i$ are uncorrelated, such that their fluctuations are independent of each other, the expectation value in the numerator of Eq. (1) factorises to yield $g^{(n)} = 1$. Here I describe the algorithm – based on counting all n-th order coincident events – for determining the correlation function from experimental data. For simplicity, I present the case for second order correlation function used throughout this thesis, from which algorithms for higher order correlation functions follow by extension.

Let an *experiment* $\mathcal{E} = \{e_i\}$ (referred to as a shot) consist of a set of *events* e observed from the particular realisation (e.g. atoms/photons are detected somewhere at sometime: $e = (\mathbf{r}, t)$). The phenomenon is investigated by repeating the experiment many times, giving a set of experiments $\{\mathcal{E}_i\}$ which is the data set. We assume that any two events occurring in the same experiment are distinguishable, which is reasonable since in an experiment individual particles are resolved as separate "clicks" on a detector.

Operationally, finite sized bins of size ϵ are used in counting the number of particles $n(\mathbf{x})$ for the continuous variable $\mathbf{x}$, such that for the i-th shot $n_i(\mathbf{x}) = \#\{\mathbf{r} : |\mathbf{r} - \mathbf{x}| < \epsilon, \mathbf{r} \in \mathcal{E}_i\}$, where $\#A$ denotes the number of elements of a set A. By definition the second order correlation function is given by

$$g^{(2)}(\mathbf{x}, \mathbf{x}') = \frac{\langle : \hat{n}(\mathbf{x})\, \hat{n}(\mathbf{x}') : \rangle}{\langle \hat{n}(\mathbf{x}) \rangle \langle \hat{n}(\mathbf{x}') \rangle} \tag{2}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^{n} \left[\#\{\mathbf{r} : |\mathbf{r} - \mathbf{x}| < \epsilon, \mathbf{r} \in \mathcal{E}_i\} \#\{\mathbf{r}' : |\mathbf{r}' - \mathbf{x}'| < \epsilon, \mathbf{r}' \in \mathcal{E}_i\}\right]_{\text{unique}}}{\left(\frac{1}{n} \sum_{i=1}^{n} \#\{\mathbf{r} : |\mathbf{r} - \mathbf{x}| < \epsilon, \mathbf{r} \in \mathcal{E}_i\}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \#\{\mathbf{r}' : |\mathbf{r}' - \mathbf{x}'| < \epsilon, \mathbf{r}' \in \mathcal{E}_i\}\right)}, \tag{3}$$

where the subscripted expression in the numerator indicates that no single event should belong to multiple terms in the product. The reason we must avoid such self-counting is naturally tied to the normal ordering of number product in Eq. (1) to properly describe correlation measurements realised in experiments [3], since physically a particle is annihilated with its detection. This is a crucial point that broadly means that a particle cannot be correlated with itself, which is evident when the numerator of Eq. (3), called the unnormalised correlation function and denoted by a capitalised symbol, is written as

$$\begin{aligned} G^{(2)}(\mathbf{x}, \mathbf{x}') &= \langle : \hat{n}(\mathbf{x})\, \hat{n}(\mathbf{x}') : \rangle \\ &= \frac{1}{n} \sum_{i=1}^{n} \#\{(\mathbf{r}, \mathbf{r}') : \mathbf{r} \approx \mathbf{x}, \mathbf{r}' \approx \mathbf{x}',\, \mathbf{r}, \mathbf{r}' \in \mathcal{E}_i, \mathbf{r} \neq \mathbf{r}'\}, \end{aligned} \tag{4}$$

whereas the denominator is given by

$$\langle \hat{n}(\mathbf{x}) \rangle \langle \hat{n}(\mathbf{x}') \rangle = \frac{1}{n^2} \sum_{i,j=1}^{n} \#\{(\mathbf{r}, \mathbf{r}') : \mathbf{r} \approx \mathbf{x}, \mathbf{r}' \approx \mathbf{x}', \mathbf{r} \in \mathcal{E}_i, \mathbf{r}' \in \mathcal{E}_j\}, \tag{5}$$

where we have simplified the notation $|a - b| < \epsilon$ for $a \approx b$. In words, the second order correlation function formally defined in Eq. (2) has an equivalent definition in a combinatorial form, given by the ratio of the average number of appropriate coincident pairs, not self-counting, observed in an individual experiment to that observed indiscriminately across all experiments in the data set. A simple extension reveals the modus operandi of the algorithm used in this thesis: evaluate the n-th order correlation function by counting n-fold coincident events occurring within the same shot, normalised by the frequency when all shots are collated.

For completeness, I give an explicit justification of the algorithm used for the back-to-back (BB) correlation function used extensively throughout this thesis. The BB correlation function is then characterised by a single variable $\Delta = \hat{\mathbf{x}} + \hat{\mathbf{x}}'$ quantifying the mismatch in the BB condition, since the absolute momenta $\mathbf{x}$ of one of the particles is integrated over all space V, given explicitly by

$$g^{(2)}_{\mathrm{BB}}(\Delta) = \frac{\frac{1}{V}\int_V G^{(2)}(\mathbf{x}, -\mathbf{x}+\Delta)\mathrm{d}\mathbf{x}}{\frac{1}{V}\int_V \langle \hat{n}(\mathbf{x})\rangle \langle \hat{n}(-\mathbf{x}+\Delta)\rangle \, \mathrm{d}\mathbf{x}}. \tag{6}$$

The numerator of Eq. (6) simplifies after substituting Eq. (4), given explicitly as follows

$$\begin{aligned} G^{(2)}_{\mathrm{BB}}(\Delta) &= \frac{1}{V}\int_V \left(\frac{1}{n}\sum_{i=1}^{n} \#\{(\mathbf{r},\mathbf{r}') : \mathbf{r}\approx\mathbf{x}, \mathbf{r}'\approx -\mathbf{x}+\Delta,\, \mathbf{r},\mathbf{r}'\in\mathcal{E}_i, \mathbf{r}\neq\mathbf{r}'\}\right)\mathrm{d}\mathbf{x} &(7)\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{V}\int_V \#\{(\mathbf{r},\mathbf{r}') : \mathbf{r}\approx\mathbf{x}, \mathbf{r}'\approx -\mathbf{x}+\Delta,\, \mathbf{r},\mathbf{r}'\in\mathcal{E}_i, \mathbf{r}\neq\mathbf{r}'\}\mathrm{d}\mathbf{x}\right) &(8)\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{V}\int_V \#\{(\mathbf{r},\mathbf{r}') : \mathbf{r}\approx\mathbf{x}, \mathbf{r}+\mathbf{r}'\approx \Delta,\, \mathbf{r},\mathbf{r}'\in\mathcal{E}_i, \mathbf{r}\neq\mathbf{r}'\}\mathrm{d}\mathbf{x}\right) &(9)\\ &= \frac{1}{n}\sum_{i=1}^{n} \#\{(\mathbf{r},\mathbf{r}') : \mathbf{r}+\mathbf{r}'\approx \Delta,\, \mathbf{r},\mathbf{r}'\in\mathcal{E}_i, \mathbf{r}\neq\mathbf{r}'\}. &(10)\end{aligned}$$

The denominator of Eq. (6) similarly simplifies to

$$\frac{1}{n^2}\sum_{i,j=1}^{n} \#\{(\mathbf{r},\mathbf{r}') : \mathbf{r}+\mathbf{r}'\approx \Delta,\, \mathbf{r}\in\mathcal{E}_i, \mathbf{r}'\in\mathcal{E}_j\}. \tag{11}$$

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[3] Reid, M. D. & Walls, D. F. Violations of classical inequalities in quantum optics. *Phys. Rev. A* **34**, 1260–1276 (1986). URL `https://link.aps.org/doi/10.1103/PhysRevA.34.1260`.