Analytic expressions for ANU BEC collision experiment

R. J. Lewis-Swan[1, 2]

[1] *Homer L. Dodge Department of Physics and Astronomy, The University of Oklahoma, Norman, Oklahoma 73019, USA*

[2] *Center for Quantum Research and Technology, The University of Oklahoma, Norman, Oklahoma 73019, USA*

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We compute integrated correlation functions for a Rarity-Tapster scheme with colliding BECs, based on the setup studied in the ANU experiment. The results slightly differ from those originally discussed in Phys. Rev. A 91, 052114 (2015) and can have implications for observing a violation of a CHSH-Bell inequality.

I. SETUP OF THE PROBLEM

We seek to compute the integrated pair-correlation function,

$$C_{ij} = \int_{V(\mathbf{k}_i)} d^3\mathbf{k} \int_{V(\mathbf{k}_j)} d^3\mathbf{k}'\; G^{(2)}(\mathbf{k},\mathbf{k}'), \tag{1}$$

at the output of a Rarity-Tapster protocol. This integrated quantity intrinsically depends on the nature of the two-point correlation function, which in Ref. [1] is assumed to be of the form,

$$\frac{G^{(2)}(\mathbf{k},\mathbf{k}')}{\bar{n}^2} = 1+\frac{h}{2}\left[1 \pm \cos(\Phi + \varphi(\mathbf{k},\mathbf{k}'))\right]\prod_d e^{-(\mathbf{k}+\mathbf{k}')^2/2\sigma_d^2}. \tag{2}$$

Here, σ_d is the original back-to-back (BB) correlation length and h the correlation strength. The correlation is normalized by the nominally constant halo population density $\bar{n}$. The phase-shifts of the Rarity-Tapster scheme are encoded in Φ (e.g., $\Phi = \phi_L \pm \phi_R$ depending on the details of the implementation) and $\varphi(\mathbf{k},\mathbf{k}')$ describes some phase-diffusion that occurs between scattered pairs with momenta $\mathbf{k} \neq -\mathbf{k}'$ [1].

The goal of these notes is to determine the equivalent version of Eq. (2) for the specific configuration of the ANU experiment, in particular the form of $\varphi(\mathbf{k},\mathbf{k}')$ and how it depends on the key parameters of experiment (e.g., BEC size, momentum kick, etc).

In Fig. 1 we present a comparison of the geometry of the current experiment relative to the original scheme proposed in Ref. [1]. We identify the momentum modes $\mathbf{k}_j$ for $j = 1, ..., 4$ that are involved in the Rarity-Tapster interferometer, and point out that the configuration of these modes slightly differs between the two possible protocols. In particular, modes $\mathbf{k}_2$ and $\mathbf{k}_4$ are switched [although correlations are preserved between the pairs $(\mathbf{k}_1,\mathbf{k}_2)$ and $(\mathbf{k}_3,\mathbf{k}_4)$]. This will lead to some superficial differences in the final expression for (2).

II. CALCULATING THE INTEGRATED CORRELATION FUNCTION

Without loss of generality, we will focus on a representative calculation of C_{12} for the ANU geometry, which means we must first derive $G^{(2)}(\mathbf{k},\mathbf{k}')$ for $\mathbf{k} \approx \mathbf{k}_1$ and $\mathbf{k}' \approx \mathbf{k}_2$ (see Fig. 1). Treating the Bragg pulses as instantaneous (and perfect) linear transformations, it is straightforward to adapt/rederive the result (B2) of Ref. [1]:

$$\begin{aligned} G^{(2)}(\mathbf{k},\mathbf{k}') = \bar{n}^2 + \frac{1}{4}\Big\{ & |m(\mathbf{k},\mathbf{k}',t_c) + |m(\mathbf{k}-2\mathbf{k}_L,\mathbf{k}'-2\mathbf{k}_L,t_c)|^2 \\ & - m(\mathbf{k}-2\mathbf{k}_L,\mathbf{k}'-2\mathbf{k}_L,t_c)^* m(\mathbf{k},\mathbf{k}',t_c) e^{-i\Phi + i\frac{2\hbar k_L}{m}[(\mathbf{k}+\mathbf{k}')_z - 2k_L]\Delta t_{\mathrm{free}}} \\ & - m(\mathbf{k},\mathbf{k}',t_c)^* m(\mathbf{k}-2\mathbf{k}_L,\mathbf{k}'-2\mathbf{k}_L,t_c) e^{i\Phi - i\frac{2\hbar k_L}{m}[(\mathbf{k}+\mathbf{k}')_z - 2k_L]\Delta t_{\mathrm{free}}} \Big\}, \end{aligned} \tag{3}$$

where $m(\mathbf{k},\mathbf{k}',t) = \langle \hat{a}(\mathbf{k},t)\hat{a}(\mathbf{k}',t)\rangle$ is the anomalous pair correlation, $2\mathbf{k}_L = k_0\hat{z}$ is the momentum imparted by the Bragg pulse, $k_L = |k_L|$, Δt_{free} is the free propagation time and t_c the initial collision duration.

In Appendix A we show that the anomalous moment can be written in the form,

$$\begin{aligned} m(\mathbf{k},\mathbf{k}',t) = \bar{n}\sqrt{h}\Big[& e^{i\theta_+(\mathbf{k},\mathbf{k}',t)} \prod_d e^{-(k_d+k'_d-k_0\hat{z})^2/4\sigma_d^2} \\ & + e^{i\theta_-(\mathbf{k},\mathbf{k}',t)} \prod_d e^{-(k_d+k'_d+k_0\hat{z})^2/4\sigma_d^2}\Big]. \end{aligned} \tag{4}$$

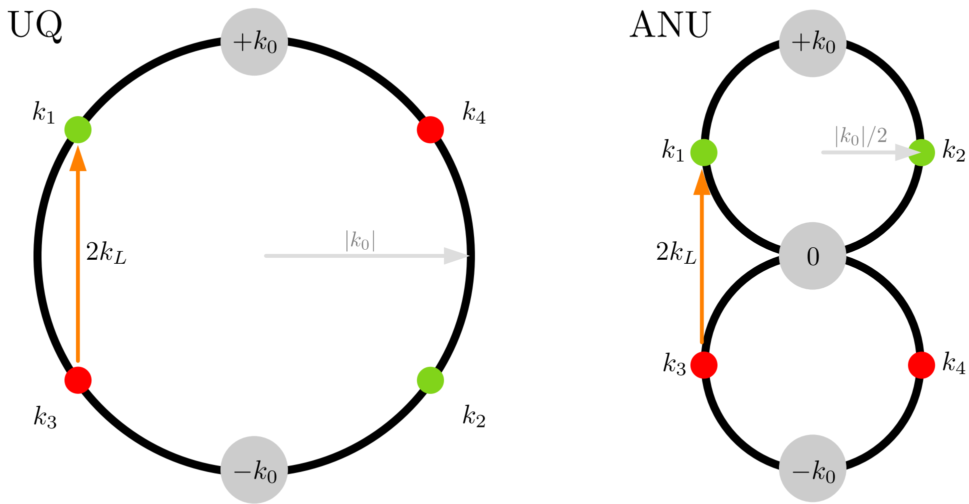

FIG. 1. Collision and Rarity-Tapster setup. (Left) The original UQ proposal involves only a single scattering halo (radius $|k_0|$ centered at $\mathbf{k} = 0$) generated by the collision of a pair of BECs with COM momenta $\pm k_0 \hat{z}$ (indicated by grey blobs). After the collision, strong atom-atom correlations are between pairs $(\mathbf{k}_1, \mathbf{k}_2)$ and $(\mathbf{k}_3, \mathbf{k}_4)$. The Rarity-Tapster scheme mixes instead the (uncorrelated) pairs $(\mathbf{k}_1, \mathbf{k}_3)$ and $(\mathbf{k}_2, \mathbf{k}_4)$. (Right) The ANU scheme involves two scattering halos (equal radius $|k_0|/2$ centered at $\mathbf{k} = \pm k_0 \hat{z}$) generated by three colliding BECs with COM momenta $0, \pm k_0 \hat{z}$. Again, we assume the correlated pairs after the collision are $(\mathbf{k}_1, \mathbf{k}_2)$ and $(\mathbf{k}_3, \mathbf{k}_4)$. However, notice that the relative location of the modes $\mathbf{k}_2$ and $\mathbf{k}_4$ have been interchanged relative to the UQ scheme.

Note that the correlation splits into two contributions corresponding to pairs in the upper and lower halos. The phase of the anomalous moment is given by,

$$\theta_\pm(\mathbf{k}, \mathbf{k}', t) \simeq -\frac{\pi}{2} - \frac{\hbar}{2m}(k^2 + k'^2)t - \frac{\sigma_{g,z}}{\sqrt{\pi} k_0} \left\{ k_0 \left[k_0 \mp (k_z + k'_z) \right] + k_0^2 - (k^2 + k'^2) \right\}, \quad (5)$$

where $\sigma_{g,z}$ is related to the rms width of the density profile of the source BEC under the assumption it may be modelled as a Gaussian (see Appendix A for detailed discussion). The phase can be loosely understood as having two contributions. The first term (modulo $\pi/2$) describes the usual phase accrued due to the free particle dispersion $\propto k^2$. The second term, $\propto \sigma_{g,z}$, is similar to that previously derived in Ref. [1]. In that case, it was clear that it described a phase accrued due to an energy mismatch between the initial and final scattered pairs. This is seemingly still the case here (consider the second quantity inside the square brackets), although there is an additional correction that is not yet understood (first quantity inside the square brackets)...

Substitution of Eq. (4) into Eq. (3) yields the new result,

$$\frac{G^{(2)}(\mathbf{k}, \mathbf{k}')}{\bar{n}^2} - 1 = \frac{h}{2} \left[1 \pm \cos(\Phi + \varphi(\mathbf{k}, \mathbf{k}')) \right] \times \prod_d e^{-(\mathbf{k}+\mathbf{k}'-k_0)^2/2\sigma_d^2}, \quad (6)$$

with

$$\begin{aligned} \varphi(\mathbf{k}, \mathbf{k}') &\equiv \theta_-(\mathbf{k} - 2\mathbf{k}_L, \mathbf{k}' - 2\mathbf{k}_L, t_c) - \theta_+(\mathbf{k}, \mathbf{k}', t_c) \\ &\quad + \frac{2\hbar k_L}{m} [(\mathbf{k} + \mathbf{k}')_z - 2k_L] \Delta t_{\text{free}}, \\ &= -4k_L \left[(\mathbf{k} + \mathbf{k}')_z - 2k_L \right] \\ &\quad \times \left[\frac{\hbar}{2m} (\Delta t_{\text{free}} - t_c) + \frac{2\sigma_{g,z}}{k_0 \sqrt{\pi}} \right]. \end{aligned} \quad (7)$$

where we have used that $2k_L = k_0$. Within the square brackets we identify two contributions: i) a phase difference that accumulates due to a mismatch in the momentum states during the collision/free propagation (a simple consequence of $|\mathbf{k}| \neq |\mathbf{k} - 2\mathbf{k}_L|$), and ii) the forementioned "energy mismatch" that is different for the scattered pairs we mix with the Bragg pulses. The form of Eq. (7) is similar to that derived previously in Ref. [1], but has two important differences.

First, the dependence on the momentum becomes of the form $\mathbf{k} + \mathbf{k}'$, whereas in Ref. [1] it was $\mathbf{k} - \mathbf{k}'$. This difference is a simple consequence of the different configuration of how the modes are mixed (e.g., $\mathbf{k}_2$ and $\mathbf{k}_4$ are swapped in Fig. 1). Even though this leads to some differences in the final expressions we calculate, we find it is not quantitatively important.

The second difference is that the contribution of the term $\propto \sigma_{g,z}$ is increased fourfold compared to Ref. [1]. This effect does generate a quantitative difference and indicates that the effects of dephasing may be worse than previously estimated. Half of this increase could be understood in a straightforward manner by recognizing that the protocol of Ref. [1] involves only a single pair of BECs moving apart with momenta $+/-k_0$, and so we should interchange $k_0 \to 2k_0$ in our expression. However, this does not reconcile a remaining factor of two and a physical interpretation remains unclear at the moment...

Given there has been some confusion about the fundamental scaling and parameter dependence of the phase term $\varphi(\mathbf{k},\mathbf{k}')$ (e.g., the variation with $\sigma_{g,z}$), it is useful to rewrite Eq. (7) in terms of a separation timescale[1] $t_{\rm sep} = 2\sigma_{g,z}m/(\hbar k_0)$,

$$\varphi(\mathbf{k},\mathbf{k}') = -\frac{2\hbar k_L\left[(\mathbf{k}+\mathbf{k}')_z - 2k_L\right]}{m} \times\left[(\Delta t_{\rm free} - t_c) + \frac{2t_{\rm sep}}{\sqrt{\pi}}\right]. \quad (8)$$

For comparison, the result of Ref. [1] [see Eq. (B7)] would instead be written as,

$$\varphi(\mathbf{k},\mathbf{k}') = -\frac{2\hbar k_L\left[(\mathbf{k}-\mathbf{k}')_z - 2k_L\right]}{m} \times\left[(\Delta t_{\rm free} - t_c) + \frac{t'_{\rm sep}}{\sqrt{\pi}}\right], \quad (9)$$

with $t'_{\rm sep} = \sigma_{g,z}m/(\hbar k_0)$.

Finally, we are ready to compute the integrated correlation function C_{ij} from the definitions of Eqs. (1) and (6). We find,

$$C_{ij} = \bar{n}^2V + \frac{\bar{n}^2h}{2}\left[1\pm\cos(\Phi)\right]\alpha_x\alpha_y\beta_z\prod_d\left(2\sigma_d^2\right), \quad (10)$$

with

$$\alpha_d = e^{-2\lambda_d^2} - 1 + \sqrt{2\pi}\lambda_d\mathrm{erf}(\sqrt{2}\lambda_d), \quad (11)$$

and

$$\beta_d = e^{-\frac{\Delta k_d^2}{2\sigma_d^2}}\cos(4Ak_L\Delta k_d) - 1 + \sqrt{\frac{\pi}{2}}\frac{e^{-8A^2k_L^2\sigma_d^2}}{2\sigma_d}\left[(\Delta k_d - 4iAk_L\sigma_d^2)\mathrm{erf}\left(\frac{\Delta k - 4iAk_L\sigma_d^2}{\sqrt{2}\sigma_d}\right) + (\Delta k_d + 4iAk_L\sigma_d^2)\mathrm{erf}\left(\frac{\Delta k + 4iAk_L\sigma_d^2}{\sqrt{2}\sigma_d}\right)\right] + 4\sqrt{2}Ak_L\sigma_d D_{\rm F}\left(2\sqrt{2}Ak_L\sigma_d\right), \quad (12)$$

for $A = \hbar/(2m)[\Delta t_{\rm free} - t_c + 2t_{\rm sep}/\sqrt{\pi}]$ and $k_L = |\mathbf{k}_L| = k_0/2$. As was foreshadowed, the deviation of the functional form of β_d relative to the prior notes is a result of the different configuration of the targetted momentum modes (see Fig. 1).

For future clarity, Eq. (12) can also be expressed in a streamlined form,

$$\beta_d = e^{-2\lambda_d^2}\cos(4\mathcal{A}_d\lambda_d) - 1 + 2\sqrt{2}\mathcal{A}_d D_{\rm F}\left(\sqrt{2}\mathcal{A}_d\right) + \sqrt{\frac{\pi}{2}}e^{-2\mathcal{A}_d^2}\left[(\lambda_d - i\mathcal{A}_d)\mathrm{erf}\left(\sqrt{2}\lambda_d - \sqrt{2}i\mathcal{A}_d\right) + (\lambda_d + i\mathcal{A}_d)\mathrm{erf}\left(\sqrt{2}\lambda_d + \sqrt{2}i\mathcal{A}_d\right)\right], \quad (13)$$

where we define $\mathcal{A}_d = 2Ak_L\sigma_d$.

[1] The form of $t_{\rm sep}$ follows from the assumption of a Gaussian density profile for the source BEC. See Appendix A).

III. NUMERICAL EXAMPLES

The results of the previous sections enable us to write a succinct expression for the correlation coefficient,

$$E(\Phi) = \left.\frac{C_{14}+C_{23}-C_{12}-C_{34}}{C_{14}+C_{23}+C_{12}+C_{34}}\right|_\Phi, \quad (14)$$

$$= \frac{h\alpha_x\alpha_y\beta_z}{16\prod_d\lambda_d^2 + h\alpha_x\alpha_y\beta_z}\cos(\Phi), \quad (15)$$

with α_d and β_d as given above. This form is unchanged from Ref. [1] as all differences are encoded in β_d.

In Fig. 2 we plot the predicted behaviour of $|E(0)|$ as a function of the side length of the integration volume, which is assumed to be identical in all directions, $\Delta k_d = \Delta k$ (following the convention of the current experiment). We compare: i) the result obtained using Eqs. (11) and (12), and ii) original expressions of Ref. [1] (see Appendix B), iii) the case with no dephasing contribution [i.e., setting $\varphi(\mathbf{k},\mathbf{k}') = 0$ in Eq. (6)]. The parameters used for Fig. 2 can be found in Table I.

The key result is that dephasing is predicted to be enhanced in the ANU scheme, relative to the original UQ

TABLE I. Parameters for current ANU experiment.

Parameters	**Weak trap**	**Tight trap**
$(\omega_x, \omega_y, \omega_z)/(2\pi)$	$(20, 20, 20)$ Hz	$(50, 400, 400)$ Hz
N	2×10^5	2×10^5
k_0	4.1×10^6 m^{-1}	4.1×10^6 m^{-1}
(R_x, R_y, R_z)	$(48, 48, 48)$ μm	$(76, 10, 10)$ μm
$\sigma_{g,z}$	$0.58R_z$	$0.58R_z$
t_c	1 ms	130 μs
$\Delta t_{\rm free}$	1 ms	96 μs
$t_{\rm sep}$	1 ms	170 μs
h	9	2
$(\sigma_x, \sigma_y, \sigma_z)$	$(0.14, 0.13, 0.16)$ μm^{-1}	$(0.38, 5.0, 5.0)$ μm^{-1}
Δk	0.063 μm^{-1}	0.63 μm^{-1}

configuration with equivalent parameters. In particular, for the tight trap we have that $|E(0)|$ is effectively zero. A few broad comments should be made at this point, to put these results into the correct context. Firstly, this calculation should only be taken as a qualitative prediction. This is primarily due to the fact that the underpinning perturbative model that is used to calculate $\varphi(\mathbf{k}, \mathbf{k}')$ (see Appendix A) assumes a Gaussian density profile for the source BECs. This is quantitatively incorrect for the BECs in the real experiment. Related to this, the parameter $\sigma_{g,z}$ (the rms width of the source BEC and that controls A) is not rigorously defined in terms of experimental parameters. Currently we are plugging in an estimated value for $\sigma_{g,z}$ based on the calculated Thomas-Fermi radius for the source BECs, but the scaling factor between these two has not been carefully optimized [1, 2]. Other relevant details we have ignored include the finite efficiency of the Bragg pulses and their non-zero duration.

With this in mind, the key conclusion that can be made from Fig. 2 is that it is likely the ANU experiment is not working in a limit where dephasing can be readily ignored, e.g., the current integration volume is large relative to the momentum scales over which the dephasing contribution matters [effectively set by $1/(Ak_L)$].

Figure 3 shows a calculation for $|E(0)|$ where we choose the integration volume to be scaled with the BB correlation length (i.e., we fix $\lambda_d = \lambda$ rather than $\Delta k_d = \Delta k$) and the free propagation duration $\Delta t_{\rm free}$ is allowed to vary. The purpose of this figure is to illustrate that there exists a "magic time" where the dephasing contribution vanishes, $\Delta t_{\rm free} = t_c - 4m\sigma_{g,z}/(\hbar k_0\sqrt{\pi})$ (or equivalently $t_c - 2t_{\rm sep}/\sqrt{\pi}$). Clearly, for certain choices of t_c the optimal time can be negative, and in fact this is almost the case for the weak trap scenario under current conditions [see Fig. 3]. However, this can be remedied by simply waiting longer to implement the first π-pulse.

[1] R. J. Lewis-Swan and K. V. Kheruntsyan, Phys. Rev. A **91**, 052114 (2015).

[2] J. Chwedeńczuk, P. Ziń, M. Trippenbach, A. Perrin, V. Leung, D. Boiron, and C. I. Westbrook, Phys. Rev. A **78**, 053605 (2008).

Appendix A: Computing the anomalous moment

We will solve for the anomalous moment $m(\mathbf{k}, \mathbf{k}', t)$ using a perturbative treatment adapted from [2]. We start from the Heisenberg equation of motion for the Bogoliubov operator $\hat{\delta}(\mathbf{r}, t)$ that describes the bosonic field in the scattering halo is,

$$i\hbar \frac{d\hat{\delta}(\mathbf{r}, t)}{dt} = -\frac{\hbar^2}{2m} + g(\mathbf{r}, t)\hat{\delta}^\dagger(\mathbf{r}, t), \tag{A1}$$

where $g(\mathbf{r}, t)$ is a spatially and time-varying function that characterizes the colliding BECs that act as a source for the scattered atoms. For the configuration of the ANU experiment we may write $g(\mathbf{r}, t) = g_+(\mathbf{r}, t) + g_-(\mathbf{r}, t)$ where

$$g_\pm(\mathbf{r}, t) = 2U_0\psi_0(\mathbf{r}, t)\psi_\pm(\mathbf{r}, t) \tag{A2}$$

and

$$\psi_0(\mathbf{r}, t) = \sqrt{\frac{\rho_0}{3}} e^{-\frac{x^2}{2\sigma_{g,x}^2} - \frac{y^2}{2\sigma_{g,y}^2} - \frac{z^2}{2\sigma_{g,z}^2}},$$
$$\psi_\pm(\mathbf{r}, t) = \sqrt{\frac{\rho_0}{3}} e^{\mp i k_0 z - i\frac{\hbar k_0^2}{2m}t} e^{-\frac{x^2}{2\sigma_{g,x}^2} - \frac{y^2}{2\sigma_{g,y}^2} - \frac{(z \mp v_0 t)^2}{2\sigma_{g,z}^2}} \tag{A3}$$

describe the BECs with COM momenta $\mathbf{k} = 0, \pm k_0\hat{z}$, respectively, and $v_0 = \hbar k_0/m$. In this treatment, we have assumed the BECs have a Gaussian profile and this is an important simplification so that we can tractably treat the time-dependent overlap and thus separation of the source BECs (e.g., $g(\mathbf{r}, t) \to 0$ as $t \gg \sigma_{g,z}/v_0$). Moreover, our definition of $g(\mathbf{r}, t)$ explicitly ignores collisions between the BECs with $\mathbf{k} = \pm k_0\hat{z}$, as these will not be relevant if we restrict our analysis to atoms scattered into the two primary halos.

It is straightforward to rewrite Eq. (A1) in Fourier space in terms of the bosonic operators $\hat{a}(\mathbf{k}, t) = \int d^3\mathbf{r}/(2\pi)^{3/2} e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\delta}(\mathbf{r}, t)$, and then move to a rotating frame $\hat{b}(\mathbf{k}, t) = \hat{a}(\mathbf{k}, t)e^{i\hbar k^2 t/(2m)}$ to arrive at the equivalent equation of motion,

$$\frac{d\hat{b}(\mathbf{k}, t)}{dt} = -\frac{i}{\hbar} \int \frac{d^3\mathbf{q}}{(2\pi)^{3/2}}\, \tilde{g}(\mathbf{q}+\mathbf{k}, t)\hat{b}^\dagger(\mathbf{q}, t)e^{i\hbar(k^2+q^2)t/(2m)}, \tag{A4}$$

where $k \equiv |\mathbf{k}|$. Here, we have defined $\tilde{g}(\mathbf{k}, t) = \tilde{g}_+(\mathbf{k}, t) +$

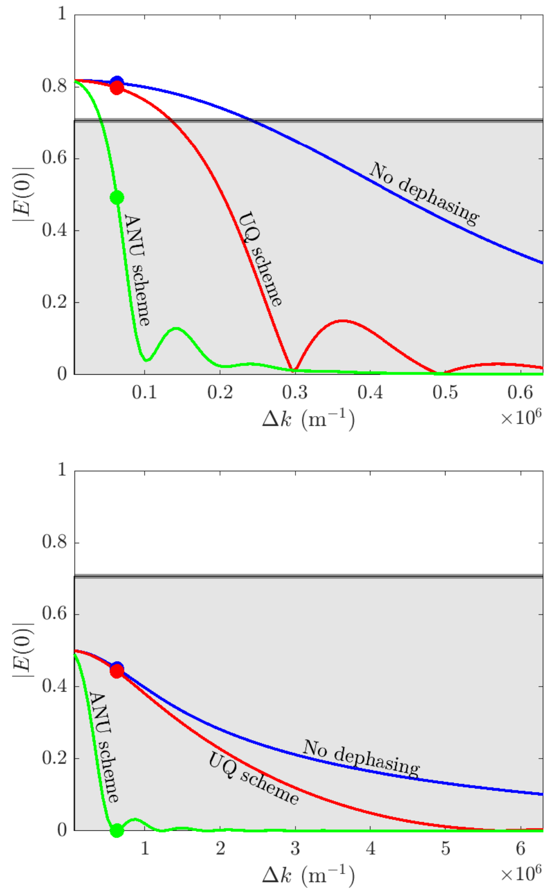

FIG. 2. Numerical prediction for amplitude of correlation coefficient in (top) weak and (bottom) tight trap configurations. Parameters for each case are given in Table I. We vary the side length of the integration volume, which is assumed to be identical in all directions, $\Delta k_d = \Delta k$. Different colored lines indicate calculated $|E(0)|$ [Eq. (15)] with: i) no dephasing contribution, and dephasing according to ii) UQ and iii) ANU schemes (see text and Appendix B). Grey shading region indicates $|E(0)| \leq 1/\sqrt{2}$, which is insufficient to achieve a violation of the CHSH-Bell inequality.

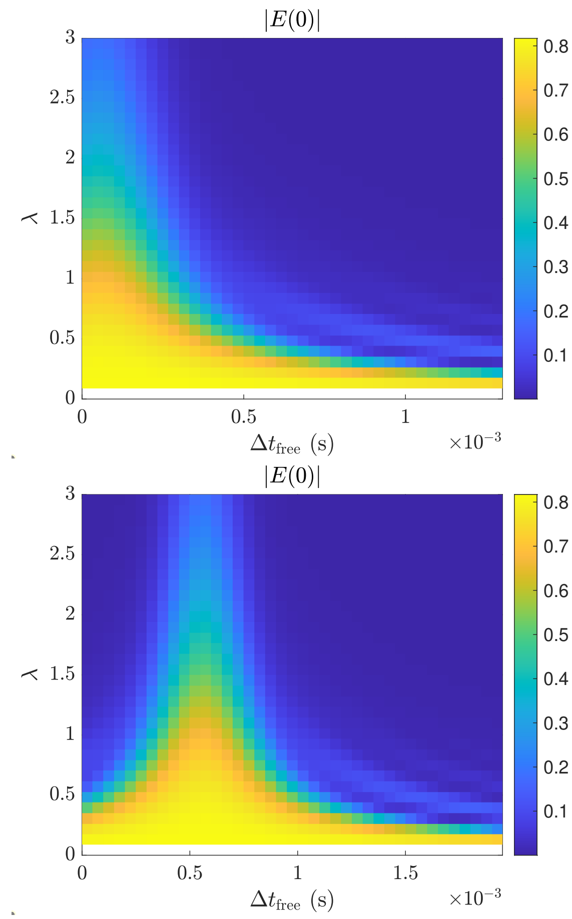

FIG. 3. Numerical prediction for amplitude of correlation coefficient in the weak trap configurations. Parameters for each case are given in Table I. We vary the side length of the integration volume in an unbiased way, $\lambda_d = \lambda$. For the upper panel we choose $t_c = 1$ ms, while for the lower we increase this to $t_c = 1.5$ ms. All calculations are from Eq. (15).

$\tilde{g}_-(\mathbf{k}, t)$ is the Fourier transform of $g(\mathbf{r}, t)$,

$$\begin{aligned}\tilde{g}_\pm(\mathbf{k}, t) &= \int \frac{d^3\mathbf{r}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} g_\pm(\mathbf{r}, t), \\ &= \frac{U_0 \rho \sigma_{g,x}\sigma_{g,y}\sigma_{g,z}}{3\sqrt{2}} e^{-ia_\pm t - bt^2} \\ &\quad \times e^{-\frac{k_x^2\sigma_{g,x}^2}{4} - \frac{k_y^2\sigma_{g,y}^2}{4} - \frac{\sigma_{g,z}^2}{4}(k_0 \mp k_z)^2}, \end{aligned} \tag{A5}$$

for

$$\begin{aligned} a_\pm &= \frac{\hbar k_0}{2m}[2k_0 \mp (k_z + k_z')], \\ b &= \frac{\hbar^2 k_0^2}{4m^2\sigma_{g,z}^2}. \end{aligned} \tag{A6}$$

It is useful to note that the latter parameter is connected to the intrinsic timescale of separation for the colliding BECs, $t_{\rm sep}$. In particular, we have that the momentum kick k_0 of the BECs corresponds to a velocity $v_0 = \hbar k_0/m$. Then, within the approximation of a Gaussian density profile, the time-scale for each pair of colliding BECs (e.g., one with COM momentum $\mathbf{k} = 0$ and the other with $\mathbf{k} = \pm k_0 \hat{z}$) to separate is $t_{\rm sep} = 2\sigma_{g,z}/v_0 = b^{-1/2}$.

Equation (A4) can be formally integrated to yield the formal solution,

$$\begin{aligned}\hat{b}(\mathbf{k}, t) = -\frac{i}{\hbar}\int_0^t d\tau \int \frac{d^3\mathbf{q}}{(2\pi)^{3/2}} \Big[\tilde{g}(\mathbf{q} + \mathbf{k}, \tau) \\ \times \hat{b}^\dagger(\mathbf{q}, \tau) e^{i\hbar(k^2+q^2)\tau/(2m)} \Big]. \end{aligned} \tag{A7}$$

This form is not useful by itself, but we can invoke a perturbative approach, $\hat{b}(\mathbf{k}, t) = \sum_j \hat{b}^{(j)}(\mathbf{k}, t)$, and calculate relevant expectation values [2]. Using that the zeroth-order term is $\hat{b}^{(0)}(\mathbf{k}, t) = \hat{b}(\mathbf{k}, 0)$, we can compute the anomalous moment to lowest non-trivial order,

$$M(\mathbf{k},\mathbf{k}',t) = \langle \hat{b}(\mathbf{k},t)\hat{b}(\mathbf{k}',t)\rangle,$$
$$\simeq -\frac{i}{\hbar}\int_0^t d\tau \int \frac{d^3\mathbf{q}}{(2\pi)^{3/2}}\ \tilde{g}(\mathbf{q}+\mathbf{k},\tau)e^{i\frac{\hbar(k^2+q^2)\tau}{2m}}.$$

Plugging in Eq. (A5) and evaluating the integrals we find $M(\mathbf{k},\mathbf{k}',t) = M_+(\mathbf{k},\mathbf{k}',t) + M_-(\mathbf{k},\mathbf{k}',t)$

$$M_\pm(\mathbf{k},\mathbf{k}',t) = -i\frac{A_\pm(\mathbf{k}+\mathbf{k}')}{\sqrt{b}}\left[D_\mathrm{F}\left(\frac{a_\pm - c}{2\sqrt{b}}\right) + e^{-i(a_\pm - c)t - bt^2}D_\mathrm{F}\left(\frac{c-a_\pm+2bt}{2\sqrt{b}}\right)\right], \quad \text{(A8)}$$

where D_F is the Dawson F function and we have defined

$$A_\pm(\mathbf{k}) = \frac{U_0\rho\sigma_{g,x}\sigma_{g,y}\sigma_{g,z}}{3\sqrt{2}}e^{-\frac{k_x^2\sigma_{g,x}^2}{4}-\frac{k_y^2\sigma_{g,y}^2}{4}-\frac{\sigma_{g,z}^2}{4}(k_0\mp k_z)^2}$$
$$c = \frac{\hbar}{2m}(k^2+k'^2). \quad \text{(A9)}$$

The anomalous moment in the original frame is then obtained as $m(\mathbf{k},\mathbf{k}',t) = M(\mathbf{k},\mathbf{k}',t)e^{-i\frac{\hbar}{2m}(k^2+k'^2)t}$.

That $M(\mathbf{k},\mathbf{k}')$ breaks into two components, $M_\pm(\mathbf{k},\mathbf{k}')$, is a reflection of the generation of back-to-back correlated pairs in two *independent* halos in the ANU experiment. In particular, inspecting Eq. (A9), we have that $M(\mathbf{k},\mathbf{k}') \simeq M_+(\mathbf{k},\mathbf{k}')$ for $\mathbf{k}+\mathbf{k}' \simeq k_0\hat{z}$ and $M(\mathbf{k},\mathbf{k}') \simeq M_-(\mathbf{k},\mathbf{k}')$ for $\mathbf{k}+\mathbf{k}' \simeq -k_0\hat{z}$.

To connect to the form proposed in Eq. (4), we would like to re-express the complex anomalous moment in terms of an amplitude and phase, e.g., $M_\pm \equiv |M_\pm|e^{i\vartheta_\pm}$. The phase is determined as

$$\vartheta_\pm(\mathbf{k},\mathbf{k}',t) = -\frac{\pi}{2} + \arg\left[D_\mathrm{F}\left(\frac{a_\pm - c}{2\sqrt{b}}\right) + e^{-i(a_\pm - c)t - bt^2}D_\mathrm{F}\left(\frac{c-a_\pm+2bt}{2\sqrt{b}}\right)\right]. \quad \text{(A10)}$$

This expression is too complicated to be useful [i.e., the subsequent integral in Eq. (1) will become intractable]. Nevertheless, we can obtain useful insight by making a series of approximations.

First, let us assume that $(a_\pm - c)/(2\sqrt{b}) \ll \sqrt{b}t, 1$ and thus we can treat it as a small parameter and Taylor expand (A8) to lowest order. This assumption essentially requires that $|\mathbf{k}+\mathbf{k}'| - k_0 \ll 1/\sigma_{g,z}$, or equivalently that we restrict the dimensions of the integration volume in Eq. (1) for C_{ij} to be much smaller than the BB correlation width [we identify that $\sigma_z \sim 1/\sigma_{g,z}$ from Eq. (A9)]. Clearly we will not always satisfy this assumption, but nevertheless the results we generate remain useful. Second, we assume that $\sqrt{b}t \gg 1$. As was previously pointed out, this corresponds to $t/t_\mathrm{sep} \gg 1$ and corresponds to spatial separation of the colliding BECs. This assumption is reasonable but sometimes dubious: We typically use $t_c \simeq t_\mathrm{sep}$! However, the assumption allows us to gain useful insight.

Together, these assumptions give:

$$\vartheta_\pm(\mathbf{k},\mathbf{k}') \simeq -\frac{\pi}{2} - \frac{a_\pm - c}{\sqrt{\pi b}}, \quad \text{(A11)}$$

or for the anomalous moment in the original frame, $m(\mathbf{k},\mathbf{k}',t) = M(\mathbf{k},\mathbf{k}',t)e^{-i\frac{\hbar}{2m}(k^2+k'^2)t}$,

$$\theta_\pm(\mathbf{k},\mathbf{k}',t) \simeq -\frac{\pi}{2} - \frac{\hbar}{2m}(k^2+k'^2)t - \frac{\sigma_{g,z}}{\sqrt{\pi}k_0}\left\{k_0\left[2k_0 \mp (k_z+k_z')\right] - (k^2+k'^2)\right\} \quad \text{(A12)}$$

Appendix B: Previous expressions for integrated correlation function

For completeness, the integrated correlation function for the scheme proposed in Ref. [1] (UQ scheme in Fig. 1) is given by:

$$C_{ij} = \bar{n}^2V + \frac{\bar{n}^2h}{2}\left[1\pm\cos(\Phi)\right]\alpha_x\alpha_y\beta_z\prod_d\left(2\sigma_d^2\right), \quad \text{(B1)}$$

This form assumes the Bragg pulses act along $\hat{z}$ and

$$\beta_d = i\sqrt{\frac{\pi}{2}}\frac{e^{-8A^2|\mathbf{k}_L|^2\sigma_d^2}}{8\sigma_d A|\mathbf{k}_L|} \times\left[e^{-4iA|\mathbf{k}_L|\Delta k}\mathrm{erf}\left(\frac{\Delta k - 4iA|\mathbf{k}_L|\sigma_d^2}{\sqrt{2}\sigma_d}\right) - e^{4iA|\mathbf{k}_L|\Delta k}\mathrm{erf}\left(\frac{\Delta k + 4iA|\mathbf{k}_L|\sigma_d^2}{\sqrt{2}\sigma_d}\right) + 2\cos(4A|\mathbf{k}_L|\Delta k)\mathrm{erf}(i2\sqrt{2}A|\mathbf{k}_L|\sigma_d)\right], \quad \text{(B2)}$$

with

$$A = \frac{\hbar}{2m}\left(\Delta t_\mathrm{free} - t_2\right) + \frac{\sigma_{g,z}}{2|\mathbf{k}_0|\sqrt{\pi}}. \quad \text{(B3)}$$

The expression for α_d is identical for both the UQ and ANU schemes, e.g., given by Eq. (11).