# Quantum-mechanical realization of a Popescu-Rohrlich box

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We consider quantum ensembles which are determined by pre- and post-selection. Unlike the case of only preselected ensembles, we show that in this case the probabilities for measurement outcomes at intermediate times satisfy causality only rarely; such ensembles can in general be used to signal between causally disconnected regions. We show that under restrictive conditions, there are certain nontrivial bipartite ensembles which do satisfy causality. These ensembles give rise to a violation of the Clauser-Horne-Shimony-Holt inequality, which exceeds the maximal quantum violation given by Tsirelson's bound $B_{\mathrm{CHSH}} \leq 2\sqrt{2}$ and obtains the Popescu-Rohrlich bound for the maximal violation, $B_{\mathrm{CHSH}} \leq 4$. This may be regarded as an *a posteriori* realization of supercorrelations, which have recently been termed Popescu-Rohrlich boxes.

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## I. INTRODUCTION

One of the most remarkable features of quantum theory is the fact that it does not violate relativistic causality or, specifically, the no-signaling condition. It seems that nothing in the formalism of quantum mechanics (QM) dictates causality. Indeed, for quantum ensembles which are both pre- and post-selected, the probability law, given by Aharonov, Bergman, and Lebowitz (ABL) [1], does not in general satisfy causality. Such ensembles can be used to signal between causally disconnected regions.

For a given pre- and post-selected ensemble, described by an initial state $|\psi_i\rangle$ and a final state $\langle\psi_f|$, the probability of measuring state $|c_n\rangle$ at the intermediate time $t_i < t < t_f$ is given by [1,2]

$$P(C=c_n|\psi_i,\psi_f) = \frac{|\langle\psi_f(t)|P_{C=c_n}|\psi_i(t)\rangle|^2}{\Sigma_i|\langle\psi_f(t)|P_{C=c_i}|\psi_i(t)\rangle|^2}, \tag{1}$$

where $P_{C=c_i}$ is a projection onto the space of eigenvalues equal to $c_i$.

Violation of causality can be shown [3], for example, by taking the initial state to be a singlet shared by Alice and Bob, $|\psi_i\rangle = (|\uparrow_z\rangle|\downarrow_z\rangle - |\downarrow_z\rangle|\uparrow_z\rangle)/\sqrt{2}$, and the final state to be $\langle\psi_f| = \langle\uparrow_x|\langle\uparrow_y|$, where the first particle belongs to Alice and the second to Bob. (We will use this convention throughout the paper unless specified otherwise.) If Bob measures the spin component of his particle along the $x$ axis, then with certainty he will get $|\downarrow_x\rangle$. However, if Alice also performs a measurement of her particle's spin along the $y$ axis, Bob's probability for obtaining $|\downarrow_x\rangle$ reduces to 0.5.

The present paper focuses on the relations between causality and nonlocality in the context of pre- and post-selected ensembles. In Sec. II we determine the generic classes of bipartite pre- and post-selected ensembles that satisfy causality. We define causality in the context of the no-signaling condition. This condition imposes specific limitations on the allowed operations of the experimenters, which we shall explicitly define. We prove that the ensembles of initial and final bipartite spin-$\frac{1}{2}$ states that satisfy the no-signaling condition belong to three generic classes.

In Sec. III we explore the amount of nonlocality in pre- and post-selected ensembles. Popescu and Rohrlich (PR) [4] have already raised a similar question: can quantum nonlocality be derived from the no-signaling condition? They discovered that it is possible to construct various causality satisfying models, which exceed the quantum mechanical bound for the Clauser-Horne-Shimony-Holt (CHSH) inequality [5], $B_{\mathrm{CHSH}} \leq 2\sqrt{2}$, derived by Tsirelson [6]. The maximal value of the CHSH inequality which satisfies causality is 4. Such models that possess supercorrelations, yet do not violate causality, have been termed PR boxes and were elaborated in [7–10]. (Note that a similar analysis, with respect to [4], has been conducted by Leonid and Tsirelson [11]). Previous research [12–15] suggested theoretical applications, using these boxes, which cannot be implemented in QM. These include reduced communication complexity, bit commitment, and simulating projective measurements that can be performed on the singlet state without communication. However, it was found that the analog of entanglement swapping [17] cannot be implemented with these boxes [16].

We show that for the nontrivial classes of causality satisfying initial and final states, the CHSH inequality violation exceeds Tsirelson's bound and obtains the maximal value (4). Thus these classes may be regarded as *a posteriori* realizations of PR boxes.

In Sec. IV we briefly discuss tripartite systems. Then in Sec. V we discuss the implementation of the analog of entanglement swapping [17] with pre- and post-selected ensembles. Finally, in Sec. VI we discuss the implementation of the suggested supercorrelations in an experiment.

## II. NO-SIGNALING CONDITION

We begin by defining the no-signaling condition in the context of nonlocal boxes. These boxes shall be taken here as an ensemble of pre- and post-selected states, defined by an initial state and a final state on a pair of spin-$\frac{1}{2}$ particles. No dynamics is introduced. Two causally disconnected experimenters, Alice and Bob, can each "ask" the boxes a single question—i.e., perform a von Neumann measurement on their single particle in an arbitrary direction, obtaining the probabilities for the up and down outcomes. The nosignaling condition requires that the choice of an experimenter whether to measure or not, and the direction of measurement if implemented, should not affect the other experimenter's probabilities. This condition is a rather softened condition for causality since the experimenters are neither allowed to perform positive-operator-valued measures (POVM's) nor to act on the states—that is, to apply local unitary transformations. If these restrictions are not imposed, only trivial ensembles (product initial and final states) will satisfy causality. We will elaborate on these restrictions shortly. From now on causality implies the no-signaling condition defined above. Note that, generally, "experimenters" with PR boxes ask the boxes only single bit questions, which are characterized by $a\oplus b=x\times y$, where $x$ and $y$ are the input to the boxes (the questions) and $a$ and $b$ are the outputs. This is less than is allowed in the present framework.

We now show that in bipartite systems the no-signaling condition is satisfied for the following initial-final states.

(i) Both product (trivial class): $|\psi_i\rangle=|\uparrow\rangle|\uparrow'\rangle$, $\langle\psi_f|=\langle\uparrow''|\langle\uparrow'''|$, where the tag(s) denote different bases.

(ii) Both maximally entangled:

$$|\psi_i\rangle=\frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow'\rangle+e^{i\theta_i}|\downarrow\rangle|\downarrow'\rangle),$$

$$\langle\psi_f|=\frac{1}{\sqrt{2}}(\langle\uparrow''|\langle\uparrow'''|+e^{-i\theta_f}\langle\downarrow''|\langle\downarrow'''|). \tag{2}$$

(iii) Equal states, but with their amplitudes swapped:

$$|\psi_i\rangle=\sqrt{\alpha}|\uparrow\rangle|\uparrow'\rangle+e^{i\theta}\sqrt{1-\alpha}|\downarrow\rangle|\downarrow'\rangle,$$

$$\langle\psi_f|=\sqrt{1-\alpha}\langle\uparrow|\langle\uparrow'|+e^{-i\theta}\sqrt{\alpha}\langle\downarrow|\langle\downarrow'|. \tag{3}$$

This entails that, generally, even if the amount of entanglement in the initial and final states is the same, causality may be violated.

For simplicity we choose the Schmidt decomposition basis $\{|\uparrow\rangle|\uparrow\rangle,|\downarrow\rangle|\downarrow\rangle\}$ for $|\psi_i\rangle$ and $\{\langle\tilde{\uparrow}|\langle\tilde{\uparrow}|,\langle\tilde{\downarrow}|\langle\tilde{\downarrow}|\}$ for $\langle\psi_f|$:

$$|\psi_i\rangle=\sqrt{\alpha}|\uparrow\rangle|\uparrow\rangle+e^{i\theta_i}\sqrt{1-\alpha}|\downarrow\rangle|\downarrow\rangle,$$

$$\langle\psi_f|=\sqrt{\beta}\langle\tilde{\uparrow}|\langle\tilde{\uparrow}|+e^{-i\theta_f}\sqrt{1-\beta}\langle\tilde{\downarrow}|\langle\tilde{\downarrow}|,$$

where $0\leq\alpha,\beta\leq 1$. Alice can freely choose the direction of her measurement. It can therefore be written as the projection operator $P_{A\uparrow}=V_A|\uparrow\rangle\langle\uparrow|V_A^{\dagger}$ used in Eq. (1). Correspondingly, Bob's projection operator is $P_{B\uparrow}=V_B|\uparrow\rangle\langle\uparrow|V_B^{\dagger}$. Unitary transformations $V_A$ and $V_B$ rotate the spins and are represented by

$$V_{A,B}=\begin{pmatrix}\cos(\omega_{A,B}/2) & -e^{-i\phi_{A,B}}\sin(\omega_{A,B}/2)\\ e^{i\phi_{A,B}}\sin(\omega_{A,B}/2) & \cos(\omega_{A,B}/2)\end{pmatrix}. \tag{4}$$

The no-signaling condition for Alice can be written as follows:

$$P_A(i)=P_{AB}(i,j)+P_{AB}(i,\tilde{j}), \tag{5}$$

$$P_{AB}(i,j)+P_{AB}(i,\tilde{j})=P_{AB}(i,j')+P_{AB}(i,\tilde{j}'), \tag{6}$$

where $P_{AB}(i,j)$ is the joint probability of obtaining $A=i$ and $B=j$ when both $A$ and $B$ are measured and $\langle i|\tilde{i}\rangle=\langle j|\tilde{j}\rangle=\langle i'|\tilde{i}'\rangle=\langle j'|\tilde{j}'\rangle=0$. Equation (5) implies that Alice's probability does not depend on Bob's choice of whether to measure or not, while Eq. (6) implies that Alice's probability is independent of Bob's choice of direction. An analog condition holds for Bob. We immediately see that Eq. (6) is contained in Eq. (5), since Eq. (5) does not only require that the terms be equal, but specifically determines their value. From the ABL rule (1),

$$P_{AB}(i,j)+P_{AB}(i,\tilde{j})=\frac{\hat{p}_{ij}+p_{i\tilde{j}}\hat{p}_{i\tilde{j}}}{p_{ij}\hat{p}_{ij}+p_{i\tilde{j}}\hat{p}_{i\tilde{j}}+p_{\tilde{i}j}\hat{p}_{\tilde{i}j}+p_{\tilde{i}\tilde{j}}\hat{p}_{\tilde{i}\tilde{j}}},$$

where $p_{ij}$ is the standard quantum mechanical probability for Alice obtaining $i$ and Bob $j$ when measuring $|\psi_i\rangle$, $p_{ij}=\langle\psi_i|ij\rangle\langle ij|\psi_i\rangle$. $\hat{p}_{ij}$ is the corresponding probability for $\langle\psi_f|$, $\hat{p}_{ij}=\langle\psi_f|ij\rangle\langle ij|\psi_f\rangle$. $P_A(i)$ can be expanded as

$$P_A(i)=\frac{|\langle\psi_f|ij\rangle\langle ij|\psi_i\rangle+\langle\psi_f|i\tilde{j}\rangle\langle i\tilde{j}|\psi_i\rangle|^2}{\sum_{i\in\{i,\tilde{i}\}}|\langle\psi_f|ij\rangle\langle ij|\psi_i\rangle+\langle\psi_f|i\tilde{j}\rangle\langle i\tilde{j}|\psi_i\rangle|^2}$$

$$=\frac{p_{ij}\hat{p}_{ij}+p_{i\tilde{j}}\hat{p}_{i\tilde{j}}+a(ij,i\tilde{j})}{p_{ij}\hat{p}_{ij}+p_{i\tilde{j}}\hat{p}_{i\tilde{j}}+p_{\tilde{i}\tilde{j}}\hat{p}_{\tilde{i}\tilde{j}}+p_{\tilde{i}\tilde{j}}\hat{p}_{\tilde{i}\tilde{j}}+a(ij,i\tilde{j})+a(\tilde{i}j,\tilde{i}\tilde{j})},$$

where

$$a(ij,i\tilde{j})=\langle\psi_f|ij\rangle\langle ij|\psi_i\rangle\langle\psi_i|i\tilde{j}\rangle\langle i\tilde{j}|\psi_f\rangle+\text{c.c.}$$
$$=2\sqrt{p_{ij}\hat{p}_{ij}p_{i\tilde{j}}\hat{p}_{i\tilde{j}}}\cos(\alpha_{ij}-\hat{\alpha}_{ij}-\alpha_{i\tilde{j}}+\hat{\alpha}_{i\tilde{j}}).$$

$\alpha_{ij}$ is the argument of the complex amplitude $\langle ij|\psi_i\rangle$, while $\hat{\alpha}_{ij}$ is the argument of $\langle ij|\psi_f\rangle$. To simplify, we denote $ij$ as 1, $i\tilde{j}$ as 2, $\tilde{i}j$ as 3, and $\tilde{i}\tilde{j}$ as 4. Note that every probability for a measurement outcome performed on $|\psi_i\rangle$ is multiplied by the corresponding probability of $\langle\psi_f|$. We therefore denote $p_{ij}\hat{p}_{ij}$ by $p_1$, $\alpha_{ij}-\hat{\alpha}_{ij}$ by $\alpha_1$, $p_{i\tilde{j}}\hat{p}_{i\tilde{j}}$ by $p_2$, etc. The no-signaling condition can now be expressed as

$$(p_1+p_2)\sqrt{p_3p_4}\cos(\alpha_3-\alpha_4)=(p_3+p_4)\sqrt{p_1p_2}\cos(\alpha_1-\alpha_2),$$

$$(p_1+p_3)\sqrt{p_2p_4}\cos(\alpha_2-\alpha_4)=(p_2+p_4)\sqrt{p_1p_3}\cos(\alpha_1-\alpha_3), \tag{7}$$

for all bases $i,j$. Since the equations have a symmetric form, the general solutions for these conditions are

$$(p_1+p_2)\sqrt{p_3p_4}=(p_3+p_4)\sqrt{p_1p_2},$$

$$(p_1+p_3)\sqrt{p_2p_4}=(p_2+p_4)\sqrt{p_1p_3},$$

$$\cos(\alpha_3-\alpha_4)=\cos(\alpha_1-\alpha_2),$$

$$\cos(\alpha_2-\alpha_4)=\cos(\alpha_1-\alpha_3),$$

yielding

$$(p_3-p_2)(p_1+p_4)(p_1p_4-p_2p_3)=0,$$

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 \text{ or } \alpha_2 = \alpha_3, \quad \alpha_1 = \alpha_4. \tag{8}$$

It is now possible to identify the causality satisfying states. First, it can easily be shown that it is only for product states that $p_{ij}p_{\tilde{i}\tilde{j}}=p_{i\tilde{j}}p_{\tilde{i}j}$ and $\alpha_{ij}+\alpha_{\tilde{i}\tilde{j}}=\alpha_{i\tilde{j}}+\alpha_{\tilde{i}j}$ for all bases $i,j$. It follows that only for initial and final product states $p_1p_4=p_2p_3$ and $\alpha_1+\alpha_4=\alpha_2+\alpha_3$.

In addition, $p_2-p_3=0$ only if $p_{i\tilde{j}}=p_{\tilde{i}j}$, $\hat{p}_{i\tilde{j}}=\hat{p}_{\tilde{i}j}$ or $p_{i\tilde{j}}=\hat{p}_{\tilde{i}j}$, $\hat{p}_{i\tilde{j}}=p_{\tilde{i}j}$. The first condition corresponds to the maximally entangled class (2), and the second corresponds to the swapped class (3). For both classes $p_1=p_4$. For maximally entangled states $\alpha_{ij}=\alpha_{\tilde{i}\tilde{j}}$ and $\alpha_{i\tilde{j}}=\alpha_{\tilde{i}j}$, so that $\alpha_1=\alpha_4$ and $\alpha_2=\alpha_3$. For the swapped class, $\alpha_{ij}=\hat{\alpha}_{\tilde{i}\tilde{j}}, \alpha_{\tilde{i}\tilde{j}}=\hat{\alpha}_{ij}, \alpha_{i\tilde{j}}=\hat{\alpha}_{\tilde{i}j}$, and $\alpha_{\tilde{i}j}=\hat{\alpha}_{i\tilde{j}}$, so that $\alpha_1+\alpha_4=\alpha_2+\alpha_3$. Each of the two classes, therefore, satisfies the second line in (8). In addition, both classes satisfy

$$P_A(i) = \frac{p_1+p_2+d_{12}}{p_1+p_2+p_3+p_4+d_{12}+d_{34}} = \frac{1}{2}, \tag{9}$$

where $d_{nm}=2\sqrt{p_np_m}\cos(\alpha_n-\alpha_m)$. Thus, whenever only Alice or Bob implements a measurement, each outcome probability equals $\frac{1}{2}$.

The last possibility $p_1=p_4=0$ does not yield any states since there is no state for which two of the probabilities are zero for all measurement directions. Note that in general even equal initial and final states do not satisfy causality. Finally, we remark that if the post-selected state is known to be maximally entangled, yet is otherwise unknown, then causality is trivially satisfied. We therefore recover standard QM with a Bell measurement applied to the state.

We can now discuss why Alice and Bob may only measure their states in a von Neumann manner and may not apply any unitary transformations. For otherwise, all nontrivial causality satisfying states would enable signaling. First, let us assume the ensemble contains maximally entangled initial and final states, for example, both singlet states. Then Alice can measure the spin of her particle along the $z$ direction and flip it only if it is found to be down, restricting it to the up state. However, since the state is post-selected to be a singlet, Bob's state is surely down if measured along the $z$ direction. But if Alice had not applied the conditional flip, Bob would have measured both states with equal probability. Second, for the swap class the situation is even worse. Here Alice may only implement a unitary transformation rotating the spin of her particle, without measuring. This transformation changes the pre-selected state to a different state, specifically not the swap state, which changes Bob's measurement outcome probabilities. Hence, local transformations rearrange the pre- and post-selected ensembles and therefore generally enable signaling. Consequently, extending von Neumann measurements to POVM's allows the implementation of conditional unitary transformations (for example, by using ancillas) and for this reason are excluded as well.

## III. EXCEEDING TSIRELSON'S BOUND

We proceed by deriving the bound on the CHSH inequality [5] for the nontrivial causality satisfying ensembles (2) and (3). The CHSH inequality (which holds in any classical local realistic theory) states that a certain combination of correlations is bounded, $B_{\text{CHSH}}=|C(A,B)-C(A,B')+C(A',B)+C(A',B')|\le 2$, where the observables $A$, $A'$, $B$, and $B'$ take the values of $\pm1$ and the correlation $C(A,B)$ is defined as $P_{AB}(1,1)+P_{AB}(-1,-1)-P_{AB}(1,-1)-P_{AB}(-1,1)$. In a classical theory any observable has a predefined value and the inequality is satisfied trivially. However, quantum correlations for bipartite systems violate the CHSH inequality and may reach Tsirelson's bound $B_{\text{CHSH}}\le 2\sqrt{2}$ [6].

Returning to the pre- and post-selected ensembles we first consider bipartite ensembles with a maximally entangled initial and final state (2). In this class, the CHSH inequality may achieve the maximal value $B_{\text{CHSH}}=4$—e.g., for the initial and final states

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle|\uparrow_z\rangle + |\downarrow_z\rangle|\downarrow_z\rangle),$$

$$\langle\psi_f| = \frac{1}{\sqrt{2}}(\langle\uparrow_z|\langle\uparrow_x| - \langle\downarrow_z|\langle\downarrow_x|). \tag{10}$$

If Alice and Bob perform measurements along the $z$ and $x$ axes, they yield the correlations $C(Z_A,Z_B)=C(X_A,X_B)=C(Z_A,X_B)=1$ and $C(X_A,Z_B)=-1$. Therefore $B_{\text{CHSH}}=|C(Z_A,Z_B)-C(X_A,Z_B)+C(Z_A,X_B)+C(X_A,X_A)|=4$. Thus the condition $a\oplus b=x\times y$ is satisfied, where $x$ and $y$ are the inputs to the boxes, $x\in\{X_A=1,\ Z_A=0\}$ and $y\in\{X=0_B,\ Z_B=1\}$, and $a$ and $b$ are the output of the boxes, and we take spin up to be 1 and spin down to be 0.

Now let us examine the maximal bound on the CHSH inequality for the third causality satisfying class—the swapped class (3). Here the CHSH inequality may only saturate the maximal bound of 4. Interestingly, as the entanglement of the swapped states increases, the maximal achievable bound decreases. In the extremal state in which both states are maximally entangled and equal,

$$|\psi_i\rangle = |\psi_f\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle|\uparrow_z\rangle + e^{i\theta}|\downarrow_z\rangle|\downarrow_z\rangle),$$

one finds that $B_{\text{CHSH}}\le 8\sqrt{2}/3$, which is the minimal value for the swapped states. As the amplitudes differ and the entanglement is reduced, the maximal bound increases. However, when the amplitudes equal 1, corresponding to product initial and final states, the maximal bound jumps to 2. The correlation of two observables $A$ and $B$ measured by Alice and Bob is given by

$$C(A,B) = \frac{16x\cos(\omega_A)\cos(\omega_B) + 8\sqrt{x}\cos(\phi-\theta)\sin(\omega_A)\sin(\omega_B)}{x[3+\cos(2\omega_A)][3+\cos(2\omega_B)] + 2[1+2x\cos(2\phi-2\theta)]\sin^2(\omega_A)\sin^2(\omega_B) + 2\sqrt{x}\cos(\phi-\theta)\sin(2\omega_A)\sin(2\omega_B)}, \tag{11}$$

where $x=\alpha-\alpha^2$, $\phi=\phi_A+\phi_B$, and $\omega_A$, $\phi_A$, $\omega_B$, and $\phi_B$ are the measurement directions chosen by Alice and Bob, respectively, as described in Eq. (4). $C(A,B)$ equals 1 for $\omega_A=\omega_B=0$. We define the measurement directions $\omega_A$, $\omega'_A$, $\phi_A$, and $\phi'_A$ corresponding to $A$ and $A'$ and $\omega_B$, $\omega'_B$, $\phi_B$, and $\phi'_B$ similarly for $B$ and $B'$. In order to find maximal bound on the CHSH inequality, one can choose $\phi_A=\phi'_A$, $\phi_B=\phi'_B$, and $\phi_A+\phi_B=\theta$. It can then be shown that for $\omega_A=3\pi/2$, $\omega'_A=\pi$, $\omega_B=\pi+\pi d(\alpha)/4$, and $\omega'_B=\pi-\pi d(\alpha)/4$, one obtains the maximal value of the CHSH inequality for a given $\alpha$, where $d(\alpha)$ is found numerically. $d(\alpha=1/2)=1$ reduces monotonically as $\alpha$ increases (or decreases), while $d(\alpha\to 0 \text{ or } 1)\to 0$. The last term $d(\alpha\to 0 \text{ or } 1)\to 0$ corresponds to infinitesimal entanglement in which $B_{\mathrm{CHSH}}\to 4$. Already for $\alpha=0.2$, where $d(\alpha=0.2)=0.505$, the maximal bound on the CHSH inequality is $B_{\mathrm{CHSH}}\approx 3.993$.

## IV. TRIPARTITE SYSTEMS

We now briefly discuss the no-signaling condition for initial and final tripartite states. Here the no-signaling condition becomes even worse—even with maximally entangled initial and final states such as the Greenberger-Horne-Zeilinger (GHZ) states (shared by Alice, Bob, and Clare), signaling is possible. Take, for example,

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|\uparrow_y\rangle|\uparrow_y\rangle|\uparrow_y\rangle + |\downarrow_y\rangle|\downarrow_y\rangle|\downarrow_y\rangle),$$

$$\langle\psi_f| = \frac{1}{\sqrt{2}}(\langle\uparrow_x|\langle\uparrow_x|\langle\uparrow_x| - \langle\downarrow_x|\langle\downarrow_x|\langle\downarrow_x|). \tag{12}$$

If Alice chooses to measure her spin along the $z$ direction, then $P_A(\downarrow)=0$, while if Bob implements a measurement along the $x$ direction, then $P_A(\uparrow)=0.5$. We estimate that only the trivial initial and final product states generally satisfy causality.

## V. ENTANGLEMENT SWAPPING

Recently, Short, Popescu, and Gisin [16] showed that it is impossible to implement the analog of entanglement swapping [17] with PR boxes. Assume that Alice and Bob share a PR box and Bob and Clare share a PR box too. It was found that Bob cannot swap the correlations; that is, he cannot create any nonlocal correlations between Alice and Clare.

In our case, if we limit Bob to perform only single-particle measurements, analogous with [16], then the same conclusion is reaffirmed: no nonlocal correlations can be created between Alice and Clare. However, if we allow Bob to perform any operations on his particles, then he can create a PR box between Alice and Clare, at least if the PR boxes are constructed from maximally entangled initial and final states as in Eqs. (10).

The method to achieve this is similar to entanglement swapping in standard QM [17], as it is also based on Bell measurements. Alice and Bob and Clare and Bob share an ensemble of pre- and post-selected states in the form of Eqs. (10). In order to perform swapping, Bob performs Bell measurements on his particles and the Hadamard operation

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

on the particle that he shares with Clare. If Bob obtains $|\phi^+\rangle=(|\uparrow_z\rangle|\uparrow_z\rangle+|\downarrow_z\rangle|\downarrow_z\rangle)/\sqrt{2}$, then Alice and Clare share an ensemble of pre- and post-selected states in the form of Eqs. (10), which is a PR box. If Bob obtains other outcomes in his Bell measurements, then we still get pre- and post-selected ensembles that possess maximal correlations, but with other variables. Thus Alice and Clare obtain a PR box only after Bob transmits the classical information of the measurement outcome.

It should be mentioned, though, that if we let Bob perform any bipartite measurements on his particles, in general he will also create signaling between Alice and Clare. After a measurement in a basis of states that are entangled, but not maximally entangled, the states shared by Alice and Clare do not belong to any of the classes that satisfy the no-signaling condition, which were found in Sec. II. Therefore, the moment the information of Bob's observed outcome reaches Alice and Clare, they can superluminally signal each other.

## VI. PHYSICAL REALIZATION

We proceed now by suggesting a physical realization of bipartite supercorrelations using the maximally entangled initial and final states (10). The scheme to demonstrate such correlations in an experiment includes three steps. First the desired Einstein-Podolsky-Rosen (EPR) state is prepared. The next step is a simulation of the supercorrelations by implementation of local measurements for each site in the $x$ or $z$ direction. Finally, a measurement is made to verify that the correct EPR state has been obtained. This happens in a quarter of the case, in which the intermediate correlations yield $B_{\mathrm{CHSH}}=4$. The most practical implementations of such experiments can be realized with photons. Preparation and verification of EPR states with photons have been conducted in teleportation experiments [18]. The intermediate measurements are implemented with polarization filters in the desired orientations. However, in such experimental setups, there will be no clicks in the detectors in the intermediate measurements and their success or failure is given *a posteriori*. A more illustrative yet difficult to implement experiment can be conducted with ion traps. Preparation and verification of EPR states with ion traps have recently been conducted [19]. It should be mentioned that the obtained maximal correlations require communication in the post-selection procedures.

## VII. DISCUSSION

Clearly, quantum mechanics satisfies causality. However, variants of the theory, such as nonlinear dynamical theories [20], generally violate causality. The existence of a final state and its relation to causality in the context of a universal wave function have been discussed in [21–23]. In the present paper we showed that though an additional boundary condition generally leads to causality violation, one can define natural constraints of single-particle measurements, for which there are nontrivial pre- and post-selected states that satisfy causality. These pre- and post-selected states give rise to a violation of the CHSH inequality, which exceeds the regular quantum mechanical bound and reaches the maximal value of 4. Cabello [24] has proposed to reach this bound using post-selection on GHZ states. Our method provides an *a posteriori* PR box which can be implemented with today's technology.

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[1] Y. Aharonov, P. G. Bergman, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).
[2] Y. Aharonov and L. Vaidman, J. Phys. A **24**, 2315 (1991).
[3] Y. Aharonov and L. Vaidman, Lect. Notes Phys. **72**, 369 (2002); e-print quant-ph/0105101.
[4] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994); e-print quant-ph/9709026.
[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[6] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).
[7] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Phys. Rev. A **71**, 022101 (2005).
[8] J. Barrett and S. Pironio, Phys. Rev. Lett. **95**, 140401 (2005).
[9] M. Piani, M. Horodecki, and R. Horodecki, Phys. Rev. A **74**, 012305 (2006).
[10] H. Buhrman and S. Massar, Phys. Rev. A **72**, 052103 (2005).
[11] K. Leonid and B. Tsirelson, in *Symposium on the Foundations of Modern Physics*, edited by P. Lahti *et al.* (World Scientific, Singapore, 1985).
[12] T. Short, N. Gisin, and S. Popescu, Quantum Inf. Process. **5**, 131–138 (2006).
[13] W. van Dam, e-print quant-ph/0501159.
[14] N. J. Cerf, N. Gisin, S. Massar, and S. Popescu, Phys. Rev. Lett. **94**, 220403 (2005).
[15] H. Buhrman, M. Christandl, F. Unger, S. Wehner, and A. Winter, Proc. R. Soc. London, Ser. A **462**, 1919 (2006).
[16] A. J. Short, S. Popescu, and N. Gisin, Phys. Rev. A **73**, 012101 (2006).
[17] M. Zukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. **71**, 4287 (1993).
[18] D. Bouwmeester, J. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997).
[19] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and C. Monroe, Nature (London) **404**, 256 (2000).
[20] C. Simon, V. Bužek, and N. Gisin, Phys. Rev. Lett. **87**, 170405 (2001).
[21] Y. Aharonov and E. Gruss, e-print quant-ph/0507269.
[22] M. Gell-Mann and J. B. Hartle, in *Proceedings of the NATO Workshop on the Physical Origins of Time Asymmetry, Mazagón, Spain, 1991*, edited by J. Halliwell, J. Pérez-Mercader, and W. Zurek (Cambridge University Press, Cambridge, England, 1994).
[23] A. Kent, Phys. Rev. D **59**, 043505 (1998).
[24] A. Cabello, Phys. Rev. Lett. **88**, 060403 (2002).