Phase due to Gravity in a Rarity-Tapster scheme

Kieran Thomas

May 2021

We wish to evaluate the phase accrued by a particle over a classical trajectory in a Rarity-Tapster interferometer, and hence determine the phase difference between the upper and lower arms of the interferometer.

Starting with the Schrödinger equation (we assume that the mean-field has dissipated)

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V(x)\psi \tag{1}$$

we look for solutions of the form $\psi = A(x,t)e^{iS(x,t)/\hbar}$. Noting that the resulting equations for A and S are separable we obtain

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\nabla S\right)^2 - \frac{i\hbar}{2m}\nabla^2 S + V(x) \tag{2}$$

In our case we wish to consider a linear potential and can hence approximate $\nabla^2 S = 0$. This leaves us with

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\nabla S\right)^2 + V(x). \tag{3}$$

The Hamilton-Jacobi equation for S is thus

$$S = \int d\tau \frac{p(\tau)^2}{2m} - V(\tau) = \int d\tau \mathcal{L} \tag{4}$$

where τ parameterises our trajectory and $\mathcal{L}$ is the Lagrangian. For our case we have $V(z) = mgz$ and therefore the phase accrued will be

$$\phi = \frac{S}{\hbar} = \frac{m}{\hbar}\int_{t_1}^{t_3} d\tau \frac{1}{2}|v(\tau)|^2 - gz(\tau), \tag{5}$$

where t_1 and t_3 are the times of our initial and final pulses. The the velocity and position can be parameterised with $\tau \in [t_1, t_3]$ as follows

$$v_z(\tau) = \begin{cases} v(0) + g\tau, \ \tau < t_c \\ v(0) + g\tau + \Delta v, \ \tau \geq t_c \end{cases} \tag{6}$$

$$z(\tau) = \begin{cases} z(0) + v(0)\tau + \frac{1}{2}g\tau^2, \ \tau < t_c \\ z(t_c) + v(t_c)(\tau - t_c) + \frac{1}{2}g(\tau - t_c)^2, \ \tau \geq t_c \end{cases} \tag{7}$$

where Δv is the difference between the two momentum modes we are trying to couple, $z(t_c) = z(0) + v(0)t_c + \frac{1}{2}gt_c^2$, and $v(t_c) = v(0) + gt_c + \Delta v$, and we've set $t_1 = 0$ for ease. Using this parameterization we can evaluate Eq. 5 analytically to give

$$\phi(v_z(0), z(0), \Delta v) = \frac{1}{2\hbar}m\left[\left(v_r^2 + v_z(0)^2 - 2gz(0)\right)t_3 + 2\left(gt_c + v_z(0)\right)(t_3 - t_c)\Delta v + (t_3 - t_c)\Delta v^2\right] \tag{8}$$

where we have taken Δv as a variable to account for the different shift in parted on the two arms of the interferometer. Now the phase difference between two paths is

$$\phi(v_{z,1}(0), z_1(0), \Delta v) - \phi(v_{z,2}(0), z_2(0), -\Delta v) = \tag{9}$$

$$-\frac{1}{2\hbar}m\left[2t_c\Delta v\left(2gt_c + \Delta v + v_{z,1}(0) + v_{z,2}(0)\right) + t_3\left(\delta v\left(\Delta v + v_{z,1}(0) + v_{z,2}(0)\right) - 2g\left(2t_c\Delta v + \delta z\right)\right)\right] \tag{10}$$

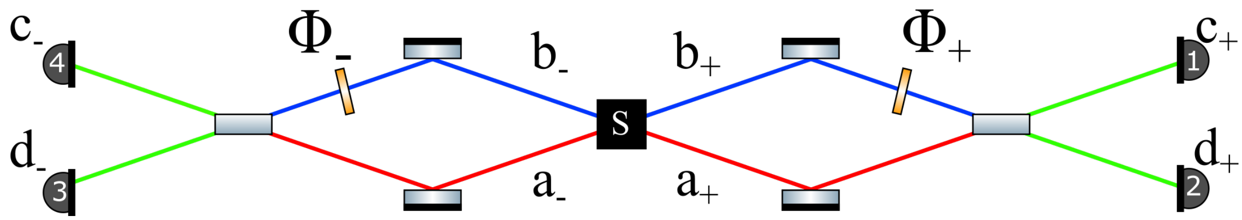

Figure 1: A diagrammatic representation of our Rarity-Tapster set up. The b-modes can be considered to be from one halo while the a-modes are from the other.

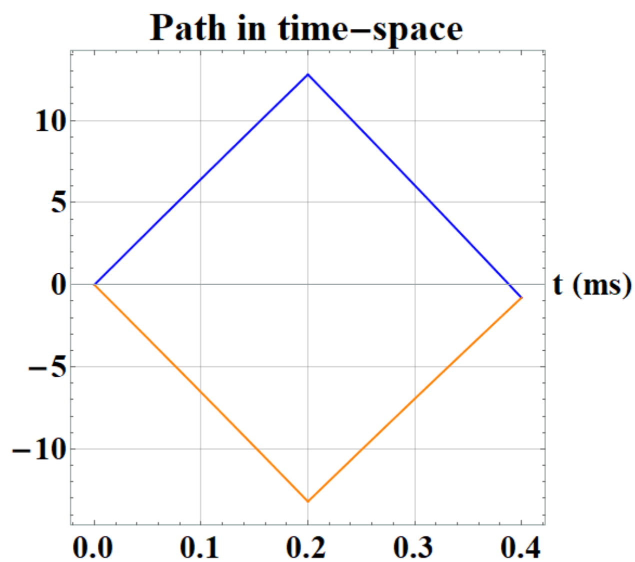

Figure 2: Path of atoms in a single arm of a Rarity-Tapster interferometer for $\Delta v = 0.13$ m/s and $t_3 = 2t_c = 0.4$ ms.

where $\delta v = v_{z,2}(0) - v_{z,1}(0)$, and $\delta z = z_2(0) - z_1(0)$. Note his does assume that the pairs are all generated at the same time, however I believe pairs being generated at different times is in effect equivalent to having a different initial velocity and position. If we let $z_1(0) = z_2(0)$ and $v_{z,1}(0) + \Delta v = v_{z,2}(0)$, Eq. 10 simplifies to

$$\phi(v_{z,1}(0), z_1(0), \Delta v) - \phi(v_{z,2}(0), z_2(0), -\Delta v) = \frac{1}{2\hbar} m\Delta v \left[4gt_c(t_3 - t_c) + (t_3 - 2t_c)(2v_{z,1}(0) + \Delta v)\right] \quad (11)$$

which can be simplified again if we set $t_3 = 2t_c$

$$\phi(v_{z,1}(0), z_1(0), \Delta v) - \phi(v_{z,2}(0), z_2(0), -\Delta v) = \frac{2gmt_c^2 \Delta v}{\hbar} = 4gk_0 t_c^2 = gk_0 t_3^2 \quad (12)$$

where $k_0 = \frac{m\Delta v}{2\hbar}$ is the wave-vector of the Bragg lattice. We can see that this is the standard expression for gravitational phase difference used in atom interferometery.

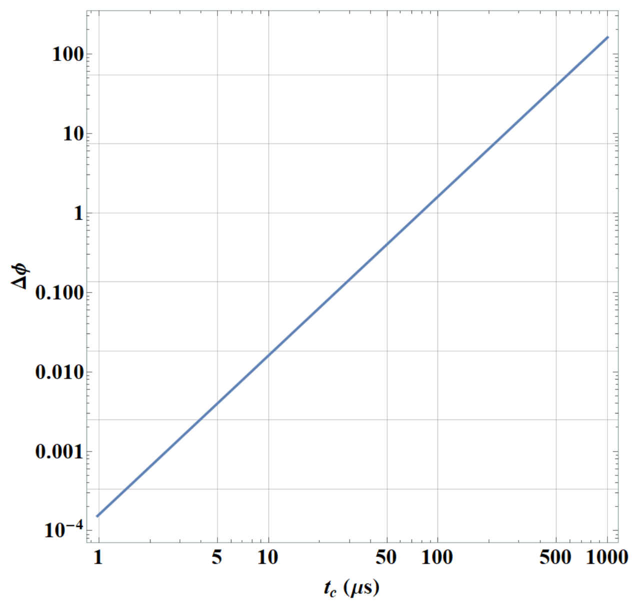

Figure 3: The gravitational phase difference $\Delta\phi$ against collision time t_c for the two paths of the interferometer for particles created with coupled momenta and at the same position and time.