MATH1851 Lecture Notes
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1 Riccati Equation

For the Riccati equation a solution $u(x)$ is given.

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \tag{1}$$

Assume that another solution has the form of $K(x) = f(u(x)) + g(x)$.Substitute in we have

$$f^{'}(u(x))u^{'}(x) + g^{'}(x) = P(x)(f(u(x)) + g(x))^2 + Q(x)(f(u(x)) + g(x)) + R(x)$$

To simplify the calculation, we assume that $f(x)$ is a linear function with the coefficient equals to 1. Then we obtain

$$\begin{aligned} g^{'}(x) &= P(x)[2u(x)g(x) + g^2(x)] + Q(x)g(x) \\ g^{'}(x) - [Q(x) + 2u(x)P(x)]g(x) &= P(x)g^2(x) \end{aligned} \tag{2}$$

This is a Bernoulli equation, the solution to which is

$$g(x)\left[e^{\int Q(x)+2u(x)P(x)dx}(-\int e^{\int Q(x)+2u(x)P(x)dx}P(x)dx + C)\right] = 1 \tag{3}$$

2 Additional Solution to Homogeneous ODE

If $y_1(x)$ is known to be a solution to the homogeneous ODE

$$y''(x) + p(x)y' + q(x)y = 0 \tag{4}$$

then we assume that another solution $y_2(x)$ has the form of $y_2(x) = uy_1(x)$, where u is also a function of x. Substitute it in we obtain

$$\begin{aligned} u''y_1 + 2u'y_1' + p(x)u'y_1 + u(y_1'' + p(x)y_1' + q(x)y_1) &= 0 \\ u''y_1 + 2u'y_1' + p(x)u'y_1 &= 0 \\ u' &= e^{-\int \frac{2y_1'}{y_1} + p(x)\ dx + C_1} \\ u &= \int e^{-\int \frac{2y_1'}{y_1} + p(x)\ dx + C_1}\ dx + C_2 \end{aligned} \tag{5}$$

3 Variation of Parameters

For an non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = f(x) \tag{6}$$

the solution $y_1(x),\ y_2(x)$ to its correspond homogeneous equation is known, so we might well assume that the solutions to the non-homogeneous one has the form of

$$Y = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{7}$$

where $v_1(x), v_2(x)$ are all functions of variable x. Substitute in we obtain

$$v_1''(x)y_1(x) + 2v_1'(x)y_1'(x) + v_2''(x)y_2(x) + 2v_2'(x)y_2'(x) + p(x)v_1'(x)y_1(x) + p(x)v_2'(x)y_2(x) + \tag{8}$$

$$v_1(x)[y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)] + v_2(x)[y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)] = f(x) \tag{9}$$

That is

$$\begin{aligned} v_1''(x)y_1(x) + 2v_1'(x)y_1'(x) + v_2''(x)y_2(x) + 2v_2'(x)y_2'(x) \\ +p(x)v_1'(x)y_1(x) + p(x)v_2'(x)y_2(x) = f(x) \end{aligned} \tag{10}$$

Let $v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0$ to simplify the calculation, we obtain

$$T(x) = v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0$$
$$\frac{dT}{dx} = v_1''(x)y_1(x) + v_1'(x)y_1'(x) + v_2''(x)y_2(x) + v_2'(x)y_2'(x) = 0$$
$$v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = f(x)$$

We obtain

$$v_1'(x) = \frac{f(x)y_2(x)}{y_1'(x)y_2(x) - y_1(x)y_2'(x)},\ v_2'(x) = \frac{f(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$
$$\boxed{Y = y_1(x)\int \frac{f(x)y_2(x)}{y_1'(x)y_2(x) - y_1(x)y_2'(x)}\ dx + y_2(x)\int \frac{f(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}\ dx}$$

Then the complete form of the solution to the original non-homogeneous ODE is

$$y(x) = C_1y_1(x) + C_2y_2(x)$$
$$+y_1(x)\int \frac{f(x)y_2(x)}{y_1'(x)y_2(x) - y_1(x)y_2'(x)}\ dx + y_2(x)\int \frac{f(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}\ dx$$

Be cautious that sometimes the $y_p(x)$ generate some terms that can be merged with some definite terms in the complementary solutions.

4 D-Operator

We use D^n to denote the derivation operation $\frac{d^n}{dx^n}$. We can easily find that

$$\begin{aligned}(D-r)y &= e^{rx}De^{-rx}\\ (D-r)^n y &= e^{rx}D^n e^{-rx}\end{aligned} \tag{11}$$

5 Convolution

Convolution in Laplace Transform gives a good way to simplify some definite transformation.

$$\boxed{\mathscr{L}(f_1(t) * f_2(t)) = \mathscr{L}(f_1(t))\mathscr{L}(f_2(t))} \tag{12}$$

where

$$f_1(x) * f_2(x) = \int_0^t f_1(\tau)f_2(t-\tau)\ \mathrm{d}\tau \tag{13}$$

5.1 Derivation

$$\begin{aligned}\mathscr{L}(f_1(t))\mathscr{L}(f_2(t)) &= \int_0^{+\infty} e^{-sw}f_1(w)\ \mathrm{d}w \int_0^{+\infty} e^{-sv}f_2(v)\ \mathrm{d}v\\ &= \int_0^{+\infty} e^{-sw}f_1(w)\ \mathrm{d}w \int_0^{+\infty} e^{-s(t-w)}f_2(t-w)\ \mathrm{d}(t-w)\\ &= \int_0^{+\infty} e^{-st}\left(\int_0^{+\infty} f_1(w)f_2(t-w)\ \mathrm{d}w\right)\mathrm{d}t\end{aligned}$$

Since functions in the transformation both have their cut-off boundary at $x = 0$, that is

$$\text{Integral Kernel} = \begin{cases} f_1(w)f_2(t-w), & 0 \le w \le t\\ 0, & \text{otherwise}\end{cases} \tag{14}$$

Then we obtain

$$\begin{aligned}\mathscr{L}(f_1(t))\mathscr{L}(f_2(t)) &= \int_0^{+\infty} e^{-st}\left(\int_0^{t} f_1(w)f_2(t-w)\ \mathrm{d}w\right)\mathrm{d}t\\ &= \mathscr{L}(f_1(t) * f_2(t))\end{aligned} \tag{15}$$

6 Taylor Series of Multivariable Function

A simple case is the functions of two variables. Assume that $u = f(x, y)$ is designed on plane D, and inside D the function has partial derivatives of n orders. A point $M(x_0, y_0)$ falls in D, and h, k is sufficiently small to guarantee that point $(x_0 + h, y_0 + k)$ falls in D as well. Define $\phi(t) = f(a + th, b + tk)$, then $\phi(t)$ has Taylor series at $t = 0$.

$$\phi(1) = \sum_{k=0}^{n} \frac{1}{k!}\phi^k(0) + \frac{\phi^{n+1}(\theta)}{(n+1)!} \tag{16}$$

where $0 < \theta < 1$. Also we have

$$\begin{aligned} \frac{d\phi}{dt} &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f \\ \frac{d^p\phi}{dt^p} &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^p f \\ &= \sum_{r=0}^{p} C_p^r h^r k^{p-r} \frac{\partial^p f}{\partial x^r \partial y^{p-r}} \end{aligned} \tag{17}$$

Substitute in we obtain

$$f(a+h, b+k) = \sum_{k=0}^{n} \frac{1}{k!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^k f(x,y)\ \Bigg|_{x=a,y=b} + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t}\right)^{n+1} f(x,y)\Bigg|_{x=a+\theta k, y=b+\theta k}$$

Or a more generalized form gives

$$\begin{aligned} f(x,y) = &\sum_{k=0}^{n} \frac{1}{k!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^k f(x,y)\Bigg|_{x=x_0, y=y_0} \\ &+ \frac{1}{(n+1)!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^{n+1} f(x,y)\Bigg|_{\substack{x=x_0+\theta(x-x_0)\\ y=y_0+\theta(y-y_0)}} \end{aligned} \tag{18}$$

We can generalize the equation and give the Taylor series for arbitrary multivariable functions.

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = &\sum_{k=0}^{n} \frac{1}{k!}\left((x_1-a_1)\frac{\partial}{\partial x_1} + (x_2-a_2)\frac{\partial}{\partial x_2} + \cdots + (x_n-a_n)\frac{\partial}{\partial x_n}\right)^k f(x_1, x_2, \ldots, x_n)\Bigg|_{x_i=a_i} \\ &+ \mathscr{O}(\theta_1, \theta_2, \ldots, \theta_n) \end{aligned}$$

7 Euler's Formula of Reflection

For $0 < 1 < p$, Euler's Formula of Reflection is defined as

$$\begin{aligned} \Gamma(p)\Gamma(1-p) &= \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} \\ &= \mathrm{B}(p, 1-p) = \int_0^1 t^{p-1}(1-t)^{-p}\ \mathrm{d}t \end{aligned} \tag{19}$$

With the transformation $t = \frac{t}{1+t}$, we obtain

$$\int_0^1 t^{p-1}(1-t)^{-p}\ \mathrm{d}t = \int_0^{+\infty} \frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t \tag{20}$$

Integral kernel $\frac{t^{p-1}}{(1+y)^p}$ has a p-order singular point $y = -1$.Then setting up a circular contour integral, we obtain

$$\int_\epsilon^R \frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t + \oint_{C_\epsilon} \frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t + \int_R^\epsilon (te^{2\pi i})^{p-1}\frac{1}{(1+y)^p}\ \mathrm{d}t + \oint_{C_R} \frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t = 2\pi i\ \mathrm{Res}\left(\frac{t^{p-1}}{(1+y)^p}, -1\right)$$

where

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t=\lim_{R\to\infty}\oint_{C_R}\frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t=0$$

$$\mathrm{Res}\left(\frac{t^{p-1}}{(1+y)^p},-1\right)=e^{-i\pi}$$

Substitute in we obtain

$$(1-e^{2(p-1)\pi i})\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int_\epsilon^R\frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t=2\pi ie^{-i\pi}$$

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int_\epsilon^R\frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t=\int_0^{+\infty}\frac{t^{p-1}}{(1+y)^p}\ \mathrm{d}t=\frac{2\pi ie^{-i\pi}}{1-e^{2(p-1)\pi i}}=\frac{\pi}{\sin p\pi}$$

Hence,

$$\boxed{\Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin p\pi}}\tag{21}$$

8 Duplication Formula

For $p>0$,

$$\Gamma(2p)=\frac{2^{2p-1}}{\sqrt{\pi}}\Gamma(p)\Gamma(p+\frac{1}{2})\tag{22}$$

8.1 Derivation

$$\begin{aligned}\frac{2^{2p-1}}{\sqrt{\pi}}\Gamma(p)\Gamma(p+\frac{1}{2})&=\frac{2^{2p-1}}{\sqrt{\pi}}\Gamma(p)\frac{\Gamma(p)\Gamma(\frac{1}{2})}{\mathrm{B}(p,\frac{1}{2})}\\&=2^{2p-1}\frac{\Gamma(p)\Gamma(p)}{\mathrm{B}(p,\frac{1}{2})}\end{aligned}\tag{23}$$

As

$$\begin{aligned}\mathrm{B}(p,p)&=\int_0^1 t^{p-1}(1-t)^{p-1}\ \mathrm{d}t\\&=\int_0^1(\frac{1+t}{2})^{p-1}(\frac{1-t}{2})^{p-1}\ \mathrm{d}(\frac{1+t}{2})\\&=\int_{-1}^1 4^{1-p}(1-t^2)^{p-1}\ \mathrm{d}t\\&=2^{2-2p}\int_0^1(1-t^2)^{p-1}\ \mathrm{d}t\\&=2^{1-2p}\int_0^1(1-t)^{p-1}t^{-\frac{1}{2}}\ \mathrm{d}t=2^{1-2p}\ \mathrm{B}(p,\frac{1}{2})\end{aligned}\tag{24}$$

Substitute (25) into (24) we obtain

$$2^{2p-1}\frac{\Gamma(p)\Gamma(p)}{\mathrm{B}(p,\frac{1}{2})}=\frac{\Gamma(p)\Gamma(p)}{\mathrm{B}(p,p)}=\Gamma(2p)\quad\text{Q.E.D.}$$

9 Spherical Harmonics (some pure mathematics)

We are interested in a general solution to Laplace Equation in spherical coordinate system. Denote the solution as $f(r,\theta,\phi)$

$$\nabla^2 f=0\tag{25}$$

9.1 Variable Separation in Spherical Coordinate System

The following gives the mapping from spherical coordinate system to Cartesian coordinate system.

$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\cos\theta\sin\phi \\ z = r\sin\theta \end{cases} \tag{26}$$

Hence the corresponding differential operator is

$$\nabla^2 f = \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 sin\theta} \frac{\partial}{\partial \theta} (sin\theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right) f \tag{27}$$

Let $f = R(r)Y(\theta, \phi)$. Substitute in we obtain

$$\nabla^2 R(r)Y(\theta,\phi) = \frac{Y}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) + \frac{R}{r^2 sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial Y}{\partial \theta} \right) + \frac{R}{r^2 sin^2\theta} \frac{\partial^2 Y}{\partial \phi^2} = 0 \tag{28}$$

That gives

$$\frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) = -\frac{1}{Y sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial Y}{\partial \theta} \right) - \frac{1}{Y sin^2\theta} \frac{\partial^2 Y}{\partial \phi^2} \tag{29}$$

The LHS only depends on variable r, while the RHS depends on θ and ϕ. Hence, the both side all equals on a CONSTANT, denoted as $l(l+1)$.

$$\frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) = l(l+1), \quad \frac{1}{Y sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{Y sin^2\theta} \frac{\partial^2 Y}{\partial \phi^2} = -l(l+1) \tag{30}$$

When solving the spherical luminosity, we are interested in the angular terms(variation of reference frame of radius terms is easy to implement). Then separate Y into

$$Y = \Theta(\theta)\Phi(\phi) \tag{31}$$

Substitute Y into (31) we obtain

$$\frac{sin\theta}{\Theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial \Theta}{\partial \theta} \right) + l(l+1) sin\theta = -\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} \tag{32}$$

Likewise, we use a new constant to connect both sides.

$$\frac{sin\theta}{\Theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial \Theta}{\partial \theta} \right) + l(l+1) sin^2\theta = \lambda, \quad \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} = -\lambda \tag{33}$$

$$\frac{1}{sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial \Theta}{\partial \theta} \right) + \left[l(l+1) - \frac{\lambda}{sin^2\theta} \right] \Theta = 0, \quad \frac{\partial^2 \Phi}{\partial \phi} + \lambda \Phi = 0 \tag{34}$$

For the second PDE, an implicit condition is that $\Phi(\phi) = \Phi(\phi + 2\pi)$, the solution can be solved by characteristic equation by assuming $\Phi = e^{im\phi}$

$$-m^2 + \lambda = 0, \quad e^{im\phi} = e^{im(\phi+2\pi)} \tag{35}$$

Hence,

$$\Phi(\phi) = e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \ldots \tag{36}$$

As for the first PDE, we execute a transformation of the differential operator

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial cos\theta} \frac{\partial cos\theta}{\partial \theta} = -sin\theta \frac{\partial}{\partial cos\theta} \tag{37}$$

Then we can transfer (36) into

$$\frac{\partial}{\partial cos\theta} \left(sin^2\theta \frac{\partial \Theta}{\partial cos\theta} \right) + \left[l(l+1) - \frac{\lambda}{sin^2\theta} \right] \Theta = 0 \tag{38}$$

Applying the variable substitution $x = cos\theta$, we obtain:

$$\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial\Theta}{\partial x}\right)+\left[l(l+1)-\frac{\lambda}{1-x^2}\right]\Theta=0 \tag{39}$$

$$(1-x^2)\frac{\partial^2\Theta}{\partial x^2}-2x\frac{\partial\Theta}{\partial x}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\Theta=0 \tag{40}$$

With the aforementioned conclusion, $\lambda = m^2$, where $m = 0, \pm1, \pm2, \ldots$ The equation has an periodic solution if and only if $m^2 = l(l+1), \quad l = 0, \pm1, \pm2 \ldots$ Assume that $\Theta(x)$ has series solution

$$\Theta(x)=\sum_{k=0}^{\infty}c_kx^k \tag{41}$$

Substitute in and let $m = 0$(corresponds to Legendre Equation of order l) , we obtain

$$(1-x^2)\sum_{k=2}^{+\infty}k(k-1)c_kx^{k-2}-2x\sum_{k=1}^{+\infty}kc_kx^{k-1}+l(l+1)\sum_{k=0}^{+\infty}c_kx^k=0$$

$$\sum_{k=0}^{+\infty}(k+2)(k+1)c_{k+2}x^k-(k-l)(k+l+1)c_kx^k=0 \quad \textbf{for all x} \tag{42}$$

This gives a recurrence that

$$c_{k+2}=\frac{(k-l)(k+l+1)}{(k+2)(k+1)}c_k \tag{43}$$

The base cases for odd terms and even terms are, respectively

$$c_0=1, \quad c_1=1 \tag{44}$$

Hence we can give a general form of the coefficient to odd and even terms respectively.

$$c_k=\begin{cases}\frac{1}{k!}\frac{(l+k-1)!!}{(l-1)!!}\frac{l!!}{(l-k)!!}(-1)^{\frac{k}{2}}c_0 & \text{k is even}\\ \frac{1}{k!}\frac{(l+k-1)!!}{(l-1)!!}\frac{l!!}{(l-k)!!}(-1)^{\frac{k-1}{2}}c_1 & \text{k is odd}\end{cases} \tag{45}$$

Then the solution to the Legendre Equation will be

$$P_l(x)=\sum_{k=0}^{+\infty}c_kx^k \tag{46}$$

Or we have **Rodrigues Formula** to simply generate the solution

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \tag{47}$$

10 Lipschitz Continuous Condition

10.1 Content

Given that two arbitrary points on the curve of function f, denoted as x_1, x_2. The slope of the line passing through both is bounded, i.e.

$$||f(x_1)-f(x_2)|| \leq K||x_1-x_2|| \tag{48}$$

where K is a constant. Lipschitz continuous condition gives a stronger continuity with higher smoothness.