Proof of Central Limit Theorem under Not Identically Distributed Condition

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1 Introduction

Let $X_1, X_2, X_3 ... X_n$ be a series of independent random variables, each has an arbitrary distribution. With mathematical calculation I conclude that the distribution of the normalized variable

$$X_{nor} = \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \tag{1}$$

approximates a normal distribution as $n \to \infty$, where μ_i and σ_i^2 are the mean and variance of each random variable respectively.

2 Derivation

Once the distribution is decided, the mean and variance are both constants. We let

$$\sum_{i=0}^{n} \mu_i = \bar{\mu}, \ \sqrt{\sum_{i=1}^{n} \sigma_i^2} = \bar{\sigma}. \tag{2}$$

And aware that $\bar{\sigma}$ monotonously increase in the same order as n (or we say, $\bar{\sigma} = \Theta(n)$).

Consider the *p.d.f.* of the normalized variable $f_{X_{nor}}(x)$. Apply the Inverse Fourier Transform to it, we obtain

$$\begin{aligned}
\mathscr{F}^{-1}[f_{X_{nor}}(x)](f) &= \int_{-\infty}^{\infty} e^{2\pi i f x} f_{X_{nor}}(x)\ dx \\
&= \int_{-\infty}^{\infty} \sum_{N=0}^{+\infty} \frac{(2\pi i f x)^N}{N!} f_{X_{nor}}(x)\ dx \\
&= \sum_{N=0}^{+\infty} \frac{(2\pi i f)^N}{N!} \int_{-\infty}^{\infty} x^N f_{X_{nor}}(x)\ dx \\
&= \sum_{N=0}^{+\infty} \frac{(2\pi i f)^N}{N!} E(X_{nor}^N)
\end{aligned} \tag{3}$$

The expression of $E(X_{nor}^N)$ is written as

$$
\begin{aligned}
E(X_{nor}^N) &= \int \overset{\infty}{\underset{-\infty}{\cdots}} \int (\frac{\sum_{i=1}^n x_i - \sum_{i=1}^n \mu_i}{\sqrt{\sum_{i=1}^n \sigma^2}})^N f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n)\ dx_1 dx_2 \dots dx_n \\
&= \int \overset{\infty}{\underset{-\infty}{\cdots}} \int (\frac{\sum_{i=1}^n x_i - \bar{\mu}}{\bar{\sigma}})^N f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n)\ dx_1 dx_2 \dots dx_n
\end{aligned}
$$

Substitute in we obtain

$$
\begin{aligned}
&\mathscr{F}^{-1}[f_{X_{nor}}(x)](f) \\
&= \sum_{N=0}^{+\infty} \frac{(2\pi i f \frac{\sum_{i=1}^n x_i - \bar{\mu}}{\bar{\sigma}})^N}{N!} \int \overset{\infty}{\underset{-\infty}{\cdots}} \int f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n)\ dx_1 dx_2 \dots dx_n \\
&= exp(2\pi i f \frac{\sum_{i=1}^n x_i - \bar{\mu}}{\bar{\sigma}}) \int \overset{\infty}{\underset{-\infty}{\cdots}} \int f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n)\ dx_1 dx_2 \dots dx_n \\
&= e^{-\frac{2\pi i f \bar{\mu}}{\bar{\sigma}}} (\prod_{i=1}^n \int_{-\infty}^{\infty} exp(\frac{2\pi i f}{\bar{\sigma}} x_i) f_{X_i}(x_i)\ dx_i)
\end{aligned} \tag{4}
$$

Take every integral out with the dummy variable written as t.

$$
\begin{aligned}
\int_{-\infty}^{\infty} exp(\frac{2\pi i f}{\bar{\sigma}} t) f_{X_i}(t)\ dt &= \int_{-\infty}^{\infty} (1 + \frac{2\pi i f}{\bar{\sigma}} t - \frac{2\pi^2 f^2}{\bar{\sigma}^2} t^2 + \mathscr{O}(\bar{\sigma}^{-3})) f_{X_i}(t)\ dt \\
&= 1 + \frac{2\pi i f}{\bar{\sigma}} E(X_i) - \frac{2\pi^2 f^2}{\bar{\sigma}^2} E(X_i^2) + \mathscr{O}(\bar{\sigma}^{-3}) \\
&= exp\left[\ln(1 + \frac{2\pi i f}{\bar{\sigma}} E(X_i) - \frac{2\pi^2 f^2}{\bar{\sigma}^2} E(X_i^2) + \mathscr{O}(\bar{\sigma}^{-3}))\right] \\
&= exp(\frac{2\pi i f}{\bar{\sigma}} E(X_i) - \frac{2\pi^2 f^2}{\bar{\sigma}^2} (E(X_i^2) - E^2(X_i)) + \mathscr{O}(\bar{\sigma}^{-3})) \\
&\approx exp(\frac{2\pi i f}{\bar{\sigma}} E(X_i) - \frac{2\pi^2 f^2}{\bar{\sigma}^2} Var(X_i)) \\
&= exp(\frac{2\pi i f}{\bar{\sigma}} \mu_i - \frac{2\pi^2 f^2}{\bar{\sigma}^2} \sigma_i^2)
\end{aligned}
$$

Then substitute in the result for every integral in (5), we obtain

$$
\begin{aligned}
\mathscr{F}^{-1}[f_{X_{nor}}(x)](f) &= exp\left(-\frac{2\pi i f \bar{\mu}}{\bar{\sigma}}\right) exp\left(\frac{2\pi i f}{\bar{\sigma}} \sum_{i=1}^n \mu_i - \frac{2\pi^2 f^2}{\bar{\sigma}^2} \sum_{i=1}^n \sigma_i^2\right) \\
&= e^{-2\pi^2 f^2}
\end{aligned} \tag{5}
$$

Then we just need to calculate $f_{X_{nor}}(x)$ from the inverse function.

We have

$$
\begin{aligned}
f_{X_{nor}}(x) &= \int_{-\infty}^{\infty} e^{-2\pi i f x} \mathscr{F}^{-1}[f_{X_{nor}}(x)](f)\ df \\
&= \int_{-\infty}^{\infty} e^{-2\pi i f x} e^{-2\pi^2 f^2}\ df \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\end{aligned} \tag{6}
$$

This is exactly the form of the *p.d.f.* to a normal distribution. Hence, we can conclude that X_{nor} follows a normal distribution.