ENGG1350 Experiment Report

Viscosity by a Falling Particle

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0.1 Background Information

This section introduces the principle of the experiment, and gives the derivation of the Stokes' Law.

0.1.1 History

George G. Stokes[1] gave the analytical solution of the drag force applied on definite particle by arbitrary fluid

$$F = 6\pi\mu Rv \;^{2} \tag{1}$$

This equation is tenable only under the circumstance that the fluid is of low Reynolds number, and no turbulence occurs.

0.1.2 Derivation

The Navier-Stokes equation describe the behaviour of a certain fluid

$$\rho[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v}] = -\nabla\vec{p} + \mu\nabla^2\vec{v} \tag{2}$$

In low-Reynolds assumption, the left hand side can be neglected, we have

$$\nabla\vec{p} = \mu\nabla^2\vec{v} \tag{3}$$

Take curl in both side to eliminate the term of p, since in polar coordinate the velocity of fluid is easier to analyse.

$$\begin{aligned}\nabla\times(\nabla\times p) &= \mu[\nabla\times(\nabla^2\vec{v})]\\ &= \mu\nabla\times[\nabla(\nabla\vec{v}) - \nabla\times(\nabla\times\vec{v})]\\ &= -\mu\nabla\times[\nabla\times(\nabla\times\vec{v})]\\ &= 0\end{aligned}$$

Equality above shows that $\nabla\times[\nabla\times(\nabla\times\vec{v})] = 0$. Let

$$\nabla\times\vec{v} = \vec{\zeta} \;^{3}$$

Then we have

$$\begin{aligned}\nabla\times(\nabla\times\vec{\zeta}) &= \nabla^2\vec{\zeta} - \nabla(\nabla\vec{\zeta})\\ &= \nabla^2\vec{\zeta} = 0.\end{aligned} \tag{4}$$

[1] Stokes, G. G. (1851). "On the effect of internal friction of fluids on the motion of pendulums". Transactions of the Cambridge Philosophical Society. 9, part ii: 8–106.

[2] μ: viscous coefficient; R: radius of the sphere particle; v: flow velocity relative to the particle

[3] Moffatt, H.K. (2015), "Fluid Dynamics", in Nicholas J. Higham; et al. (eds.), The Princeton Companion to Applied Mathematics, Princeton University Press, pp. 467–476

In 2-Dimensional coordinates, $\zeta_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$. We may well assume that

$$u = -\frac{\partial \psi}{\partial y},\ v = \frac{\partial \psi}{\partial x}, \tag{5}$$

we then obtain

$$\begin{aligned} \zeta_z &= -\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \\ &= -\nabla^2 \psi. \end{aligned} \tag{6}$$

Now we are attempting to solve PDE

$$\nabla^2(\nabla^2 \psi) = 0. \tag{7}$$

We observe the fluid faraway from the sphere particle. Assume that it flows with a stable velocity u_∞. Then $v_r = u_\infty cos\theta,\ v_\theta = -u_\infty sin\theta$. We have

$$\begin{aligned} v_r &= \frac{1}{r^2 sin\theta}\frac{\partial \psi}{\partial \theta} = u_\infty cos\theta \\ v_\theta &= -\frac{1}{rsin\theta}\frac{\partial \psi}{\partial r} = -u_\infty sin\theta. \end{aligned} \tag{8}$$

From equation (8) we obtain

$$\begin{aligned} \psi &= \frac{u_\infty r^2}{2} sin^2\theta + f_1(r) \\ &= \frac{u_\infty r^2}{2} sin^2\theta + f_2(\theta). \end{aligned} \tag{9}$$

The solution shows that ψ can be written as

$$\psi = f(r) sin^2(\theta) \tag{10}$$

If we write $f(\theta) = \sum_{i \in D} C_i r^i$, then every term in the summation serve the equation above. Substitute $\psi = f(\theta)sin^2(\theta) = Cr^n sin^2\theta$ back to PDE (7), we obtain

$$\begin{aligned} \nabla^2\psi &= [\frac{\partial}{\partial r^2} + \frac{sin\theta}{r^2}\frac{\partial}{\partial \theta}(\frac{1}{sin\theta}\frac{\partial}{\partial \theta})]Cr^n sin^2\theta \\ &= C[n(n-1)-2]sin^2\theta r^{n-2} \end{aligned}$$

$$\nabla^2(\nabla^2\psi) = C[n(n-1)-2][(n-2)(n-3)-2]sin^2\theta r^{n-4} = 0. \tag{11}$$

Then we conclude that

$$n_1 = -1,\ n_2 = 1,\ n_3 = 2,\ n_4 = 4, \tag{12}$$

that is,

$$f(r) = \frac{C_1}{r} + C_2 r + C_3 r^2 + C_4 r^4. \tag{13}$$

Boundary conditions indicate the feature of fluid in three cases: ①No penetration, ②No slip, ③No disturbance to fluid at infinity. That is

$$\begin{aligned} v_r\Big|_{r=R} &= \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta} f(r) sin^2\theta\Big|_{r=R} = 0 \\ v_\theta\Big|_{r=R} &= -\frac{1}{rsin\theta}\frac{\partial}{\partial r} f(r) sin^2\theta\Big|_{r=R} = 0 \\ v_r\Big|_{r\to\infty} &= u_\infty cos\theta \end{aligned} \tag{14}$$

Substituting equation (13) to (14), we obtain

$$C_1 = \frac{u_\infty R^3}{4},\ C_2 = -\frac{3}{4}u_\infty R,\ C_3 = \frac{u_\infty}{2},\ C_4 = 0. \tag{15}$$

Then

$$\begin{aligned}
\psi &= \frac{u_\infty r^2}{2} sin^2\theta(1 - \frac{3}{2}\frac{R}{r} + \frac{R^3}{2r^3}) \\
v_r &= u_\infty cos\theta(1 - \frac{3}{2}\frac{R}{r} + \frac{R^3}{2r^3}) \\
v_\theta &= u_\infty sin\theta(\frac{R^3}{4r^3} + \frac{3}{4}\frac{R}{r} - 1)
\end{aligned} \tag{16}$$

Back to the original equation (3)

$$\begin{aligned}
\nabla^2 \vec{v} &= [\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}](v_r\hat{e}_r + v_\theta\hat{e}_\theta) \\
&= \hat{e}_r(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{cot\theta}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}v_r - \frac{2cot\theta}{r^2}v_\theta - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}) \\
&+ \hat{e}_\theta(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} + \frac{cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{1}{r^2 sin^2\theta}v_\theta + \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta})
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\frac{1}{\mu}\frac{\partial p}{\partial r} &= \frac{3u_\infty cos\theta R}{r^3} \\
\frac{1}{\mu}\frac{\partial p}{r\partial\theta} &= \frac{3u_\infty sin\theta R}{2r^3}
\end{aligned} \tag{17}$$

Solve the equations above, we obtain

$$\begin{aligned}
p(r,\theta) &= -\frac{3u_\infty cos\theta\mu R}{2r^2} + f_1(\theta) \\
&= -\frac{3u_\infty cos\theta\mu R}{2r^2} + f_2(r)
\end{aligned} \tag{18}$$

We can simplify the equations by assuming $f_1(\theta) = f_2(r) = Const$. To find out the constant we can use boundary condition that

$$\lim_{r\to\infty} p = p_\infty \tag{19}$$

Then $Const = p_\infty$. Hence

$$p(r,\theta) = -\frac{3u_\infty cos\theta\mu R}{2r^2} + p_\infty. \tag{20}$$

For the viscous stress

$$\begin{aligned}
\tau_{rr} &= 2\mu\frac{\partial v_r}{\partial r} = 2u_\infty\mu cos\theta(\frac{3R}{2r^2} - \frac{3R^3}{2r^4}) \\
\tau_{r\theta} &= \mu[r\frac{\partial}{\partial r}(\frac{v_\theta}{r}) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}] = -\frac{3u_\infty\mu sin\theta R^3}{2r^4}
\end{aligned} \tag{21}$$

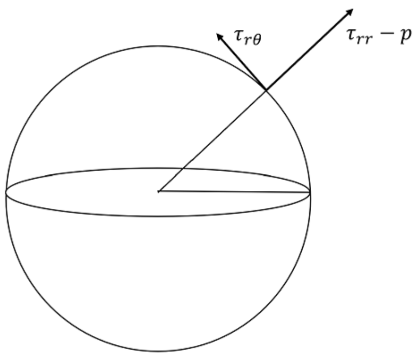


Figure 1.2.1 Diagram of Drag Force

The contribution of every area element is

$$dF = [\tau_{r\theta}\Big|_{r=R} sin\theta - (\tau_{rr}\Big|_{r=R} - p)cos\theta]\ 2\pi R^2 sin\theta d\theta$$

Then we have

$$\begin{aligned} F &= \int_0^{\pi} (-\frac{3u_\infty \mu}{2R} + p_\infty cos\theta) 2\pi R^2 sin\theta\ d\theta \\ &= 6\mu R\pi u_\infty \end{aligned} \tag{22}$$

Eventually, we verify the analytical solution to the drag force.

0.2 Experiment

In this section we mainly focus on the data analysis of the experiment and set up some discussions.

0.2.1 Force Analysis

The forces applying on the moving particle include ①Buoyant force ② Gravity ③ Drag force. Using Newton's law we obtain

$$\rho_s V_s g - \rho_f V_s g - 6\pi\mu R v = \rho_s V_s \frac{dv}{dt} \tag{23}$$

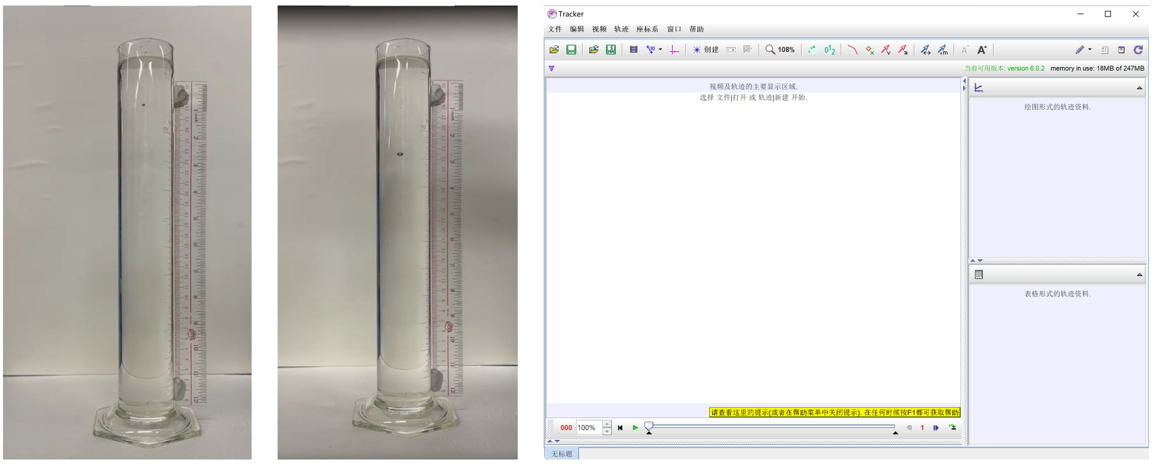
As the particle reaches stationary, we have $\frac{dv}{dt} = 0$, then we obtain

$$\begin{aligned} \mu &= \frac{(\rho_s - \rho_f) g D^2}{18v} \\ &= \frac{(\frac{6M}{\pi D^3} - \rho_f) g D^2}{18v} \end{aligned} \tag{24}$$

With the formula above we can measure the viscous coefficient μ.

0.2.2 Experiment Instruments

1. A measuring cylinder containing a colorless viscous fluid. A 30-cm ruler glued vertically on the outside of the cylinder.
2. Two metal spherical particles of different sizes.
3. Smart phone; phone stand tripod.
4. Fine tip tweezers; calipers; magnet; electronic weighing machine.
5. Tracker[4]



(a) Small Particle (b) Large Particle (c) Tracker Software

[4] a free video analysis and modeling tool,https://physlets.org/tracker/

0.2.3 Result and Analysis

Data before Experiment

Smaller particle	$D_1 = 1.98 \times 10^{-3} m/s$	$M_1 = 0.033 \times 10^{-3} kg$
Larger particle	$D_2 = 3.90 \times 10^{-3} m/s$	$M_2 = 0.263 \times 10^{-3} kg$

Table 1: Diameter & Mass

Smaller particle	$\rho_{s_1} = \frac{6M_1}{\pi D_1^3} = 8119.321 kg/m^3$
Larger particle	$\rho_{s_2} = \frac{6M_2}{\pi D_2^3} = 8467.658 kg/m^3$

Table 2: Density

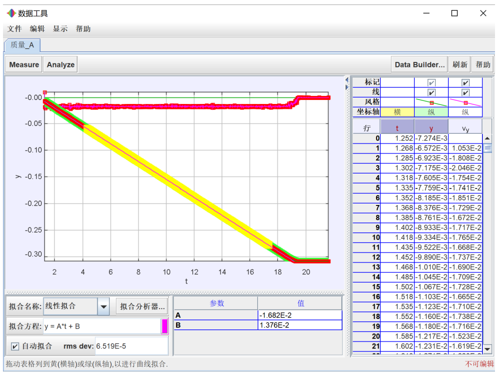
Data in Experiment

Particle	Terminal Velocity (m/s)			Average terminal velocity $\bar{V}$ (m/s)
	First trail	Second trial	Third trial	
Large	0.06082	0.06068	0.06077	0.06076
Small	0.01682	0.01677	0.01679	0.01679

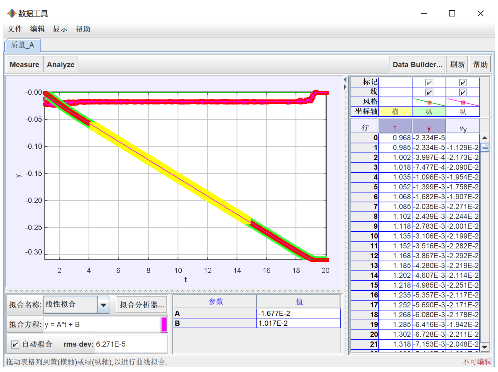
Table 3: Velocity

Particle	Dynamic viscosity $\mu = \frac{(\rho_s - \rho_f) g D^2}{18V}$	Reynolds number $Re = \frac{\rho_f D \bar{V}}{\mu}$
Large	0.9840	0.3005
Small	0.8734	0.04751

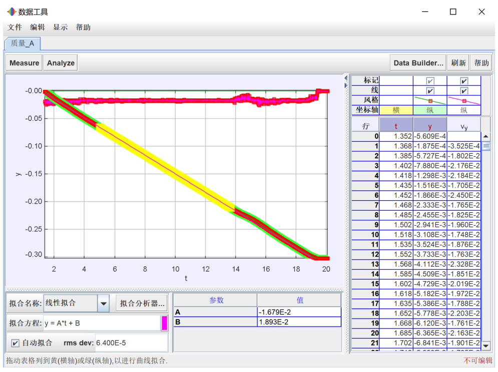
Table 4: Outcome



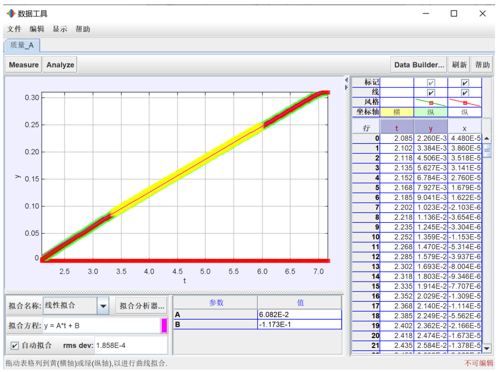
(d) Small Trial 1



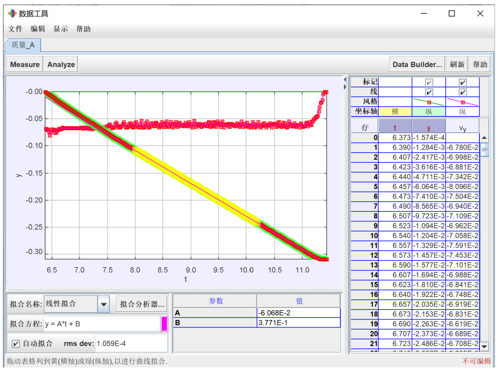
(e) Small Trial 2



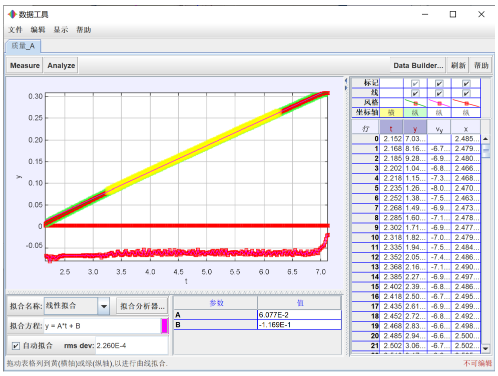
(f) Small Trial 3



(g) Large Trial 1 (Coordinate reversed)



(h) Large Trial 2



(i) Large Trial 3 (Coordinate reversed)

0.3 Analysis and Discussion

① To analyse the two measured values, we define ***relative difference*** as

$$\begin{aligned}\eta &= \frac{|\mu_1 - \mu_2|}{\frac{\mu_1+\mu_2}{2}} \\ &= 0.119\end{aligned} \tag{25}$$

The two measured values are slightly different.The difference may be caused by **insufficiency of instrumental accuracy, error of experiment operation, inaccuracy of object tracking in *Tracker* software, some turbulence occurred in the falling process, etc.**

② For the small particle the prerequisite that $Re << 1$ is satisfied. Nevertheless, for the large particle we can not obtain $Re << 1$. Thus the LHS in equation (2) can not be simply neglected. The existence of extra viscosity brought by Reynolds number provides the particle with an extra drag force, which contributes to the error of the measured value in this experiment.

③ According to the table of dynamic viscosity [5], the density and average dynamic viscosity of the fluid we measured in this experiment best corresponds **Glycerine**.

Fluid	Absolute Viscosity		
	(N s/m^2, Pa s)	*(centipoise, cP)*	*(10^{-4} lb/s ft)*
Acetic acid	0.001155	1.155	7.76
Acetone	0.000316	0.316	2.12
Alcohol, ethyl (ethanol)	0.001095	1.095	7.36
Alcohol, methyl (methanol)	0.00056	0.56	3.76
Alcohol, propyl	0.00192	1.92	12.9
Benzene	0.000601	0.601	4.04
Blood	0.003 - 0.004		
Bromine	0.00095	0.95	6.38
Carbon Disulfide	0.00036	0.36	2.42
Carbon Tetrachloride	0.00091	0.91	6.11
Castor Oil	0.650	650	
Chloroform	0.00053	0.53	3.56
Decane	0.000859	0.859	5.77
Dodecane	0.00134	1.374	9.23
Ether	0.000223	0.223	1.50
Ethylene Glycol	0.0162	16.2	109
Trichlorofluoromethane refrigerant R-11	0.00042	0.42	2.82
Glycerine	0.950	950	6380
Heptane	0.000376	0.376	2.53
Hexane	0.000297	0.297	2.00
Kerosene	0.00164	1.64	11.0
Linseed Oil	0.0331	33.1	222
Mercury	0.0015	1.53	10.3
Milk	0.003		
Octane	0.00051	0.51	3.43
Phenol	0.0080	8.0	54
Propane	0.00011	0.11	0.74
Propylene	0.00009	0.09	0.60
Propylene glycol	0.042	42	
Toluene	0.000550	0.550	3.70
Turpentine	0.001375	1.375	9.24
Water, Fresh	0.00089	0.89	6.0

Figure 1: Fluid Feature Table

[5] https://www.engineeringtoolbox.com/absolute-viscosity-liquids-d1259.html

References

[1] Nptelhrd. (2015, June 22). Mod-01 Lec-18 Stokes Drag on a Sphere (Contd.) and Introduction to Lubrication Theory. Youtube. https://www.youtube.com/watch?v=v8juW2d2tYc&t=989s

[2] Nptelhrd. (2015, June 22). Mod-01 Lec-17 Stokes Drag on a Sphere. Youtube. https://www.youtube.com/watch?v=v8juW2d2tYc&t=989s

[3] 吴耀祖. (1981). 低雷诺数流体力学介绍. 力学与实践, 34–40+54.