6.1 Determinants: Definition and Basic Properties

Ch.6. Determinants

Definition of Determinants

Informal Definition

If A is an $n \times n$ matrix, then the determinant $\det(A)$ is the *signed volume* of the n-dimensional parallelepiped spanned by the columns of A.

Definition (Determinants)

The determinant of order n is a function from $n \times n$ matrices to $\mathbb{R}$, denoted by $\det(A)$ or $|A|$, such that

1. $|\vec{a}_1 \ \cdots \ \vec{a}_{i-1} \ \ s\vec{a}_i + t\vec{a}'_i \ \ \vec{a}_{i+1} \ \cdots \ \vec{a}_n| =$ $s|\vec{a}_1 \ \cdots \ \vec{a}_{i-1} \ \ \vec{a}_i \ \ \vec{a}_{i+1} \ \cdots \ \vec{a}_n| + t|\vec{a}_1 \ \cdots \ \vec{a}_{i-1} \ \ \vec{a}'_i \ \ \vec{a}_{i+1} \ \cdots \ \vec{a}_n|$ for all $s, t \in \mathbb{R}$ and all $1 \leq i \leq n$.
2. If any two columns of A are equal, then $|A| = 0$.
3. If I is the identity matrix then $|I| = 1$.

Theorem

Let the determinant of A exist. If the matrix A' is obtained from A by interchanging any two columns, then $|A'| = -|A|$.

Proof.

$$\underbrace{|\cdots \vec{a}_i + \vec{a}_j \cdots \vec{a}_i + \vec{a}_j \cdots|}_{=0} = |\cdots \vec{a}_i \cdots \vec{a}_i + \vec{a}_j \cdots| + |\cdots \vec{a}_j \cdots \vec{a}_i + \vec{a}_j \cdots|.$$

$$|\cdots \vec{a}_i \cdots \vec{a}_i + \vec{a}_j \cdots| = \underbrace{|\cdots \vec{a}_i \cdots \vec{a}_i \cdots|}_{0} + |\cdots \vec{a}_i \cdots \vec{a}_j \cdots|.$$

$$|\cdots \vec{a}_j \cdots \vec{a}_i + \vec{a}_j \cdots| = |\cdots \vec{a}_j \cdots \vec{a}_i \cdots| + \underbrace{|\cdots \vec{a}_j \cdots \vec{a}_j \cdots|}_{=0}.$$

By combining these equations,

$0 = |\cdots \vec{a}_i \cdots \vec{a}_j \cdots| + |\cdots \vec{a}_j \cdots \vec{a}_i \cdots|.$ ∎

Theorem

The determinant of order two is given by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Proof. If the determinant exist,

$$\begin{aligned}
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= |a\vec{e}_1 + c\vec{e}_2, b\vec{e}_1 + d\vec{e}_2| \\
&= a|\vec{e}_1, b\vec{e}_1 + d\vec{e}_2| + c|\vec{e}_2, b\vec{e}_1 + d\vec{e}_2| \\
&= ab|\vec{e}_1, \vec{e}_1| + ad|\vec{e}_1, \vec{e}_2| + bc|\vec{e}_2, \vec{e}_1| + cd|\vec{e}_2, \vec{e}_2| \\
&= ad|\vec{e}_1, \vec{e}_2| + bc|\vec{e}_2, \vec{e}_1| = ad|\vec{e}_1, \vec{e}_2| - bc|\vec{e}_1, \vec{e}_2| \\
&= ad - bc.
\end{aligned}$$

For the existence we show that this expression satisfies the axioms.

$$\begin{aligned}
\begin{vmatrix} a & sb + tb' \\ c & sd + td' \end{vmatrix} &= a(sd + td') - c(sb + tb') \\
&= s(ad - bc) + t(ad' - b'c) = s\begin{vmatrix} a & b \\ c & d \end{vmatrix} - t\begin{vmatrix} a & b' \\ c & d' \end{vmatrix}.
\end{aligned}$$

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = ac - ac = 0. \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1.$$

Theorem

The determinant of order three is given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ -a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{array}.$$

Proof. If the determinant exists then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}}_{=:A} + \underbrace{\begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}}_{=:B}$$

$$A = \underbrace{\begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}}_{=a_{11}a_{22}a_{33}} = a_{11}a_{22}a_{33}.$$

$$B = \underbrace{\begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}}_{=-a_{11}a_{32}a_{23}} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{vmatrix}}_{=0} = -a_{11}a_{32}a_{23}$$

The remains are similarly obtained.
The proof for the existence of determinant of order three is similar to those of the determinant of 2×2 matrices.

Example.

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{array}{l} 45 + 84 + 96 \\ \qquad -48 - 72 - 105 \end{array} = 0.$$

Determinant of order n

Definition

An arrangement of members of a finite set into a sequence is called a *permutation*.

Example. All permutations of $\{1, 2, 3\}$ are
$(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$.

Lemma

The number of permutations of n elements is $n!$.

Definition

An *inversion* of a permutation is a pair (i, j) of positions where the entries of a permutation are in the opposite order; if $(p_1, p_2, \cdots, p_n)$ is a permutation then i < j and $p_i > p_j$.

Example. The permutation $(3, 2, 4, 5, 1)$ has the five inversions $(3, 2)$, $(3, 1)$, $(2, 1)$, $(4, 1)$, $(5, 1)$.

Definition

A permutation is called *even* if it has an even number of inversions and *odd* otherwise.

Theorem

Let $A = [a_{ij}]$ be an $n \times n$. Then,

$$\det(A) = \sum_{P=(p_1,p_2,\cdots,p_n)} \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n},$$

where the sum ranges over all permutations P of $\{1, 2, \cdots, n\}$ and

$$\epsilon(P) = \begin{cases} 1 & \text{if } P \text{ is even,} \\ -1 & \text{if } P \text{ is odd.} \end{cases}$$

Note This formula is inefficient for calculating, since the number of terms is $n!$. However, it is theoretically useful.

Theorem

If $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Proof. Assume that A is an upper triangular matrix. Consider

$$\det(A) = \sum_{P=(p_1,p_2,\cdots,p_n)} \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

In the 1st column, all entries except a_{11} are zero. In the determinant formula we can set $p_1 = 1$ and see that all $p_2, p_3, \cdots, p_n \neq 1$.
In the 2nd column, all entries except a_{12} and a_{22} are zero. In the determinant formula we can set $p_2 = 2$, since p_2 cannot take 1. And all $p_3, p_4 \cdots, p_n \neq 2$. By continuing this argument we can obtain $p_i = i$ for all $1 \leq i \leq n$. ∎

Theorem

For every square matrix A,
$$\det(A^T) = \det(A).$$

Proof. Consider $\det(A) = \sum_{P=(p_1,p_2,\cdots,p_n)} \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}$.

We rearrange $a_{p_1 1} a_{p_2 2} \cdots a_{p_n n} = a_{1 q_1} a_{2 q_2} \cdots a_{n q_n}$.
Observe that if we consider p_i as an one-to-one function from $\{1, 2, \cdots, n\}$ onto itself by $i \mapsto p_i$ then q_i is the inverse of p_i, that is, $p_i = j \Leftrightarrow q_j = i$. Thus,

$$\begin{aligned}
\det(A) &= \sum_{Q=(q_1,q_2,\cdots,q_n)} \epsilon(Q) a_{1 q_1} a_{2 q_2} \cdots a_{n q_n} \\
&= \sum_{Q=(q_1,q_2,\cdots,q_n)} \epsilon(Q) a^t_{q_1 1} a^t_{q_2 2} \cdots a^t_{q_n n} \\
&= \det(A^T),
\end{aligned}$$

where $A^T = [a^t_{ij}]$.

Theorem

If A and B be square matrices of the same size then

$$\det(AB) = \det(A)\det(B).$$

Proof.

$$
\begin{aligned}
AB &= \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \\
&= \left[\sum_{i_1=1}^{n} b_{i_1 1}\vec{a}_{i_1} \quad \sum_{i_2=1}^{n} b_{i_2 2}\vec{a}_{i_2} \quad \cdots \quad \sum_{i_n=1}^{n} b_{i_n n}\vec{a}_{i_n} \right]
\end{aligned}
$$

$$
\det(AB) = \left| \sum_{i_1=1}^{n} b_{i_1 1}\vec{a}_{i_1} \quad \sum_{i_2=1}^{n} b_{i_2 2}\vec{a}_{i_2} \quad \cdots \quad \sum_{i_n=1}^{n} b_{i_n n}\vec{a}_{i_n} \right|
$$

$$
= \sum_{i_1=1}^{n} b_{i_1 1} \left| \vec{a}_{i_1} \quad \sum_{i_2=1}^{n} b_{i_2 2}\vec{a}_{i_2} \quad \cdots \quad \sum_{i_n=1}^{n} b_{i_n n}\vec{a}_{i_n} \right|
$$

$$= \sum_{i_1=1}^{n} b_{i_1 1} \sum_{i_2=1}^{n} b_{i_2 2} \cdots \sum_{i_n=1}^{n} b_{i_n n} \left| \vec{a}_{i_1} \quad \vec{a}_{i_2} \quad \cdots \quad \vec{a}_{i_n} \right|$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} b_{i_1 1} b_{i_2 2} \cdots b_{i_n n} \left| \vec{a}_{i_1} \quad \vec{a}_{i_2} \quad \cdots \quad \vec{a}_{i_n} \right|$$

Since the determinant of the matrices with same columns is zero, we remove such matrices.

$$= \sum_{P=(i_1,i_2,\cdots,i_n)} b_{i_1 1} b_{i_2 2} \cdots b_{i_n n} \left| \vec{a}_{i_1} \quad \vec{a}_{i_2} \quad \cdots \quad \vec{a}_{i_n} \right|$$

$$= \sum_{P=(i_1,i_2,\cdots,i_n)} b_{i_1 1} b_{i_2 2} \cdots b_{i_n n} \epsilon(P) \left| \vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n \right|$$

$$= \det(A) \sum_{P=(i_1,i_2,\cdots,i_n)} b_{i_1 1} b_{i_2 2} \cdots b_{i_n n} \epsilon(P)$$

$$= \det(A) \det(B).$$

Corollary

$\det(A^{-1}) = 1/\det(A)$.

Calculation of Determinants

Lemma

Let A be an $n \times n$ matrix.

1. $|\cdots \ \vec{0} \ \cdots| = 0$.
2. $|\cdots \ c\vec{a}_i \ \cdots| = c|\cdots \ \vec{a}_i \ \cdots|$.
3. $|\cdots \ \vec{a}_i \ \cdots \ \vec{a}_j \ \cdots| = -|\cdots \ \vec{a}_j \ \cdots \ \vec{a}_i \ \cdots|$.
4. $|\cdots \ \vec{a}_i \ \cdots \ \vec{a}_j \ \cdots = |\cdots \ \vec{a}_i - c\vec{a}_j \ \cdots \ \vec{a}_j \ \cdots| \quad$ for $i \neq j$.

1. $|\cdots \ \vec{0} \ \cdots| = |\cdots \ (1-1)\vec{0} \ \cdots| = |\cdots \ \vec{0} \ \cdots| - |\cdots \ \vec{0} \ \cdots| = 0.$
2. By the axiom 1 of determinants we have it.
3. It is proved already.
4. $|\cdots \ \vec{a}_i - c\vec{a}_j \ \cdots \ \vec{a}_j \ \cdots| =$
   $|\cdots \ \vec{a}_i \ \cdots \ \vec{a}_j \ \cdots| - c\underbrace{|\cdots \ \vec{a}_j \ \cdots \ \vec{a}_j \ \cdots|}_{=0} = |\cdots \ \vec{a}_i \ \cdots \ \vec{a}_j \ \cdots|.$

Remark

By $\det(A^T) = \det(A)$, every property described in the above lemma for the columns of determinants holds for the rows.

Example.

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{array}{c} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ 3 & -6 & -12 \end{vmatrix} \\ \quad C_2 - 4C_1 \; C_3 - 7C_1 \end{array} = (-1)(-2) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} = 2 \begin{vmatrix} 1 & 4 & 7 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 9 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$= 9 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 9 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 9.$$

Theorem

A matrix A is not invertible if and only if $\det(A) = 0$.

Proof. ($\Rightarrow$) If A is not invertible then by the Gaussian elimination we can reduce A to an echelon matrix U with a zero row. If we apply the same elementary row operations to $\det(A)$ then by the above lemma, $\det(A) = c\det(U)$ for some $c \in \mathbb{R}$. Since $\det(U) = 0$, we obtain $\det(A) = 0$.
($\Leftarrow$) We prove the contrapositive. Suppose that A is invertible. By the Gauss–Jordan elimination we can reduce A to an identity matix I. By the same elementary row operations we have $\det(A) = c\det(I) = c$ for a nonzero $c \in \mathbb{R}$. Thus $\det(A) \neq 0$.