MA.4.1 不定积分及其基本计算方法

一、基本概念

原函数

#原函数

设 $f(x): I \to \mathrm{R}$, 若 $\exists F(x)$ 使得 $F'(x) = f(x)(\forall x \in I)$, 则称 $F(x)$ 是 $f(x)$ 在 I 上的一个**原函数**.

全体原函数

#全体原函数

设 $F(x)$ 是 $f(x)$ 在 I 上的一个原函数, 则 $F(x) + C(C$ 为任意常数)为 $f(x)$ 在 I 上的**全体原函数**.

> ***Proof:***
> $(F(x)+c)' = F'(x) := f(x)$
> 令$G(x)$ 为$f(x)$任意原函数
> $\therefore G'(x) = f(x) = F'(x) \quad (x \in I)$
> $\xRightarrow{\exists C \in \mathbb{R}} \quad G(x) = F(x) + C$

不定积分

#不定积分

设 $f(x)$ 存在原函数, 则 $f(x)$ 的全体原函数称为 $f(x)$ 的**不定积分**, 记作

$$\int f(x)\mathrm{d}x$$

- $\int$ 一不定积分号
- $f(x)$ 一被积函数
- x 一积分变量

被积表达式

被积表达式

设 $F(x)$ 是 $f(x)$ 在 I 上的一个原函数, 则

$$\int f(x)\mathrm{d}x = F(x) + C.$$

不定积分与微分运算互逆

不定积分与微分运算互逆

即：左边不积分=右边求导数

> ***Proof:***
> $(*)$ 若f可导, 则
>
> $$\int f'(x)dx = f(x) + C$$
>
> 即:
>
> $$\int d(f(x) + C) = f(x) + C$$
>
> 若 f 存在原函数, 则
>
> $$\left(\int f(x)dx\right)' = f(x)$$
>
> 即:
>
> $$d\int f(x)dx = f(x)dx$$

不定积分的线性运算

线性运算

若 f, g 在 I 上存在原函数, 则对任意 $a \in \mathbf{R}$ 有

$$\int af(x)\mathrm{d}x = a\int f(x)\mathrm{d}x \quad \int [f(x) \pm g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$$

不定积分表

1. $\displaystyle\int k\,\mathrm{d}x = kx + C \quad \left(\int 0\,\mathrm{d}x = C\right)$;

2. $\int x^\alpha \mathrm{d}x = \frac{1}{\alpha+1}x^{\alpha+1} + C \quad (\alpha \neq -1)$;
常用形式:
- $\int x \mathrm{d}x = \frac{1}{2}x^2$
- $\int \sqrt{x} \mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} + C$
- $\int \frac{\mathrm{d}x}{\sqrt{x}} = 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C$
- $\int \frac{\mathrm{d}x}{x^2} = -x^{-1} + C = -\frac{1}{x} + C$

3. $\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + C$;
- $\int \frac{1}{ax+b} \mathrm{d}x = \frac{\ln(ax+b)}{a}$

4. $\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C \quad (0 < a \neq 1)$;
5. $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + C$;
6. $\int \sin x \, \mathrm{d}x = -\cos x + C$;
7. $\int \cos x \, \mathrm{d}x = \sin x + C$;
8. $\int \sec^2 x \, \mathrm{d}x = \tan x + C$;
- $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x|$

9. $\int \csc^2 x \, \mathrm{d}x = -\cot x + C$;
10. $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C(a \neq 0)$;

P.S. $\left(\frac{1}{a}\arctan\frac{x}{a}\right)' = \frac{1}{a} \cdot \frac{1}{a}\frac{1}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a^2+x^2}$
- $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$

11. $\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \quad (a \neq 0)$;
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

12. $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C(a > 0)$;
13. $\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$

例题

Example

例1 求 $\int\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x$

Solution

$$
\begin{aligned}
&=\int\frac{dx}{\sqrt{x}}-\int\frac{dx}{\sqrt{1-x^2}}\\
&=2\sqrt{x}-\arcsin x+c
\end{aligned}
$$

Example

例 2 求 $\int\frac{1}{\sin^2 x\cos^2 x}\,\mathrm{d}x$

Solution

$$
\begin{aligned}
&=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx\\
&=\int\left(\frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}\right)dx\\
&=\int\left(\csc^2 x+\sec^2 x\right)dx\\
&=-\cot x+\tan x+C
\end{aligned}
$$

Example

例3 求 $\int\frac{1}{x^2\left(1+x^2\right)}\mathrm{d}x$

Solution

$$
\begin{aligned}
&=\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)dx\\
&=-\frac{1}{x}-\arctan x+C
\end{aligned}
$$

二、换元积分法

第一代换法 / 凑微分法

#凑微分法

若 $\int f(u)\mathrm{d}u=F(u)+C$, 而 $\varphi(x)$ 可导, 则

$$\int f(\varphi(x))\varphi'(x)\mathrm{d}x=F(\varphi(x))+C,$$

$$\text{或} \int f(\varphi(x))\mathrm{d}\varphi(x) = F(\varphi(x)) + C.$$

- 分析

> ***Analysis***
>
> $$\begin{aligned} &[F(\varphi(x))]' = f(\varphi(x))\varphi'(x) \\ \Rightarrow &\int f(\varphi(x))\varphi'(x)dx = F(\varphi(x)) + C \end{aligned}$$

- 运算步骤

$$\begin{aligned} \int f(\varphi(x))\varphi'(x)\mathrm{d}x &= \int f(\varphi(x))\mathrm{d}\varphi(x) \\ &\xlongequal{u=\varphi(x)} \int f(u)\mathrm{d}u = F(u) + C \\ &\xlongequal{\text{回代}} F(\varphi(x)) + C. \end{aligned}$$

例题

> **Example**
>
> 例4 求下列不定积分
>
> 1. $\int \tan x dx$

> ***Solution***
>
> $$\begin{aligned} &= \int \frac{\sin x}{\cos x} dx \\ &= \int \frac{-d\cos x}{\cos x} \\ &\xlongequal{u=-\cos x} -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \end{aligned}$$

> **Example**
>
> 2. $\int (2x-1)^{10} dx$

$$= \int (2x-1)^{20} \cdot \frac{1}{2} d(2x-1)$$

$$\xlongequal{u=2x-1} \frac{1}{2} u^{10} du$$

$$= \frac{u^{11}}{22} + C$$

$$= \frac{(2x-1)^{11}}{22} + C$$

Example

3. $\int \frac{dx}{a^2+x^2}$

Solution

$$= \frac{1}{a} \int \frac{\frac{1}{a} dx}{1+\left(\frac{x}{a}\right)^2}$$

$$= \frac{1}{a} \int \frac{d\frac{x}{a}}{1+\left(\frac{x}{a}\right)^2}$$

$$= \frac{1}{a} \int \frac{du}{1+u^2}$$

$$= \frac{1}{a} \arctan \frac{x}{a} + C$$

Example

4. $\int \frac{dx}{x^2-a^2}$

Solution

$$= \int \frac{dx}{(x-a)(x+a)}$$

$$= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx$$

$$= \frac{1}{2a} \left(\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a} \right)$$

$$= \frac{1}{2a} [\ln|x-a| - \ln|x+a|]$$

$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Example

5. $\int \sec x dx$

Solution *凑微分*

$$
\begin{aligned}
&=\int \frac{1}{\cos x} dx \\
&=\int \frac{\cos x}{\cos^2 x} dx \quad \rightarrow \text{ 凑 } \sin x \\
&=\int \frac{d \sin x}{1-\sin^2 x} \quad \rightarrow \int \frac{\mathrm{d}x}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C \quad (a \neq 0) \\
&=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C \\
&=\frac{1}{2}\ln \frac{(1+\sin x)^2}{\cos^2 x}+C \\
&=\ln\left|\frac{1+\sin x}{\cos x}\right|+C \\
&=\ln|\sec x+\tan x|+C
\end{aligned}
$$

Example

6. $\int \frac{\ln x}{x\sqrt{1+\ln x}} dx$

Solution *添项*

$$
\begin{aligned}
&\int \frac{\ln x}{x\sqrt{1+\ln x}} dx \\
&=\int \frac{\ln x \cdot d\ln x}{\sqrt{1+\ln x}} \\
&\xlongequal{u=\ln x} \int \frac{(1+u-1)du}{\sqrt{1+u}} \\
&=\int\left(\sqrt{1+u}-\frac{1}{\sqrt{1+u}}\right) d(1+u) \\
&=\frac{2}{3}(1+u)^{3/2}-2\sqrt{1+u}+C \\
&=\frac{2}{3}(1+\ln x)^{3/2}-2\sqrt{1+\ln x}+C
\end{aligned}
$$

Example

7. $\int \frac{2x-1}{x^2-4x+5} dx$

Solution *配方*

$$=\int \frac{2x-4+3}{x^2-4x+5}dx$$
$$=\int \frac{d\left(x^2-4x+5\right)}{x^2-4x+5}+\int \frac{3}{(x-2)^2+1}d(x-2)$$
$$=\ln\left|x^2-4x+5\right|+3\arctan(x-2)+C$$

Example

例 5 求 $\int \frac{x^2+1}{x^4+x^2+1}\,\mathrm{d}x.$

Solution

分式上下同时除去x^2，凑微分+配方：

$$=\int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+1+\frac{1}{x^2}}$$
$$=\int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+\sqrt{3}^2}$$
$$=\frac{1}{\sqrt{3}}\arctan\frac{x-\frac{1}{x}}{\sqrt{3}}+C$$

注意到，该式在$x=0$时无意义，继续补充

设 $f(x)=\dfrac{x^2+1}{x^4+x^2+1}$ 原函数为 $F(x)$

$$\therefore F(x)=\begin{cases}\frac{1}{\sqrt{3}}\arctan\frac{x-\frac{1}{x}}{\sqrt{3}}+C_1, & x>0\\ C, & x=0\\ \frac{1}{\sqrt{3}}\arctan\frac{x-\frac{1}{x}}{\sqrt{3}}+C_2, & x<0\end{cases}$$

$\because F(x)$ 在 $x=0$ 连续.

$\therefore F(0+0)=F(0)=F(0-0)$，此时$x-\dfrac{1}{x}$趋近正负无穷

$$\Rightarrow \frac{1}{\sqrt{3}}\cdot\left(-\frac{\pi}{2}\right)+C_1=C=\frac{1}{\sqrt{3}}\cdot\frac{\pi}{2}+C_2$$
$$\therefore C_1=C+\frac{1}{\sqrt{3}}\cdot\frac{\pi}{2}$$
$$C_2=C-\frac{1}{\sqrt{3}}\cdot\frac{\pi}{2}$$

从而原函数 $F(x)=\begin{cases}\dfrac{1}{\sqrt{3}}\arctan\dfrac{x-\frac{1}{x}}{\sqrt{3}}+\dfrac{1}{\sqrt{3}}\cdot\dfrac{\pi}{2}+C, & x\geq 0\\ \dfrac{1}{\sqrt{3}}\arctan\dfrac{x-\frac{1}{x}}{\sqrt{3}}-\dfrac{1}{\sqrt{3}}\cdot\dfrac{\pi}{2}+C, & x<0\end{cases}$

下说明 $F'(x)\big|_{x=0}=f(0)$
由于 $\lim_{x\to 0}F'(x)=\lim_{x\to 0}f(x)=f(0)=1$
又$\because$ $F(x)$ 在 $x=0$ 连续:
$\Rightarrow F'(0)=f(0)=1$ （运用 #导数极限定理 : 导函数极限→导数极限）

第二代换法/换元积分法

已知右端求左端积分为第一代换法; 已知左端, 能否求右端积分呢? 此即下面的第二换元法.

#换元积分法

若 $\int f(\varphi(t))\varphi'(t)\mathrm{d}t=G(t)+C$, 又 $x=\varphi(t)$, 且 $\varphi'(t)\neq 0$, 则

$$\int f(x)\mathrm{d}x=G\left(\varphi^{-1}(x)\right)+C$$

- 运算步骤

$$\begin{aligned}\int f(x)\mathrm{d}x &\xlongequal{x=\varphi(t)}\int f(\varphi(t))\varphi'(t)\mathrm{d}t\\ &\xlongequal{\text{已知}} G(t)+C\\ &\xlongequal{\text{回代}} G\left(\varphi^{-1}(x)\right)+C.\end{aligned}$$

- 证明

Proof

$$\frac{d}{dx}G\left(\varphi^{-1}(x)\right)$$
$$\xlongequal{t:=\varphi^{-1}(x)} G'(t)\frac{dt}{dx}$$
$$=G'(t)\cdot\frac{1}{\frac{dx}{dt}}$$
$$=f(\varphi(t))\varphi'(t)\cdot\frac{1}{\varphi'(t)}=f(\varphi(t))$$
$$=f(x)$$

例题

三角代换去根号

含无理式 $\sqrt{a^2-x^2},\sqrt{x^2+a^2}$ 和 $\sqrt{x^2-a^2}$ 时, 可采用 $x=a\sin t, x=a\tan t$ 和 $x=a\sec t$ 等三角代换去根号.

Example

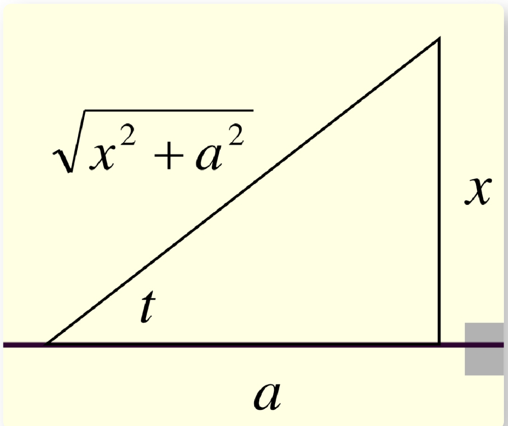
例6 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}, (a>0).$

Solution

令 $x = a\tan t, |t| < \frac{\pi}{2}$

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} &= \int \frac{a\sec^2 t\mathrm{d}t}{\sqrt{a^2\tan^2 t + a^2}} \\ &= \int \frac{\sec^2 t}{\sqrt{\sec^2 t}} dt \\ &= \int \sec t dt \\ &= \ln|\sec t + \tan t| + C \\ &= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C \\ &= \ln\left|x + \sqrt{x^2+a^2}\right| + C_1\end{aligned}$$

Tip



Example

例7 求 $\int \sqrt{a^2-x^2}\,\mathrm{d}x, (a>0).$

Solution

令 $x=a\sin t$

$$
\begin{aligned}
\text{原式} &= \int \sqrt{a^2-a^2\sin^2 t}(a\cos t dt) \\
&= a^2\int \cos^2 t dt \\
&= \frac{a^2}{2}\int \left(1+\cos^2 t\right) dt \\
&= \frac{a^2}{2}\left(t+\frac{\sin 2t}{2}\right)+c \\
&= \frac{a^2}{2}t+\frac{a^2}{2}\sin t\cos t+c \\
&= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{a^2}{2}\cdot\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}+C \\
&= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c
\end{aligned}
$$

Example

例8 求 $\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}}$.

Solution

法I.配方

$$
\begin{aligned}
\text{原式} &= \int \frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} \\
&= \arcsin(2x-1)
\end{aligned}
$$

或:

法II.灵机一动

~~注意到:~~ $\sin^2 x+\cos^x=x+(1-x)=1$

令 $x=\sin^2 t, t\in\left(0,\frac{\pi}{2}\right)$

$$
\begin{aligned}
\therefore\ \text{原式} &= \int \frac{2\sin t\cos t\mathrm{d}t}{\sqrt{\sin^2 t\cos^2 t}} \\
&= 2\int \mathrm{d}t \\
&= 2t+C \\
&= 2\arcsin\sqrt{x}+C
\end{aligned}
$$

Tip

上述例子很好地说明了原函数可不唯一（C不同）

倒数代换

分母含因子 x 时, 可用倒代换 $x=1/t$.

Example

例9 求 $\int \frac{\mathrm{d}x}{x\sqrt{x^2+x+1}}$.

Solution

$$\begin{aligned}&\xlongequal[t>0]{x=1/t}\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+\frac{1}{t}+1}}\\&=-\int \frac{dt}{\sqrt{1+t+t^2}}\\&=-\int \frac{d\left(t+\frac{1}{2}\right)}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}}\\&=-\ln\left|t+\frac{1}{2}+\sqrt{1+t+t^2}\right|+C\\&=-\ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x}\right|+C\end{aligned}$$

三、分部积分法

#分部积分法

若 $u'(x),v'(x)$ 连续, 则

$$\int u(x)v'(x)\mathrm{d}x=u(x)v(x)-\int v(x)u'(x)\mathrm{d}x.$$

- 记忆
  - $\int u(x)\mathrm{d}v(x)=u(x)v(x)-\int v(x)\mathrm{d}u(x).$
  - 前后相乘 - ∫ 交换位置
- 凑微分函数依次是: 指数函数, 三角函数, 幂函数,对数函数, 反三角函数.
- 证明

Proof

$$\begin{aligned}&(u(x)v(x))'=u'(x)v(x)+u(x)v'(x)\\&\text{有}\int\left[u'(x)v(x)+u(x)v'(x)\right]=u(x)v(x)+c\\&\text{即:}\int u(x)dv(x)=u(x)v(x)-\int v(x)du(x)\end{aligned}$$

例题

Example

例 10 求 $\int \ln x \, \mathrm{d}x$.

Solution

$$
\begin{aligned}
&= x\ln x - \int x d\ln x \\
&= x\ln x - \int x \cdot \frac{1}{x} dx \\
&= x\ln x - x + C
\end{aligned}
$$

Example

例11 求 $I = \int \mathrm{e}^x \sin x \, \mathrm{d}x$.

Solution

$$
\begin{aligned}
I &= \int \mathrm{e}^x \sin x \, \mathrm{d}x && e^x, \mathrm{d}x\text{合并} \\
&= \int \sin x \mathrm{d}e^x && \text{分部积分} \\
&= e^x \sin x - \int e^x \mathrm{d} \sin x && e^x, \mathrm{d}x\text{合并} \\
&= e^x \sin x - \int \cos x \mathrm{d}e^x && \text{分部积分} \\
&= e^x \sin x - e^x \cos x + \int e^x \mathrm{d} \cos x \\
&= e^x(\sin x - \cos x) - \underbrace{\int \sin x \mathrm{d}e^x}_{I} \\
\therefore I &= e^x(\sin x - \cos x) - I \\
\therefore 2I &= e^x(\sin x - \cos x) + C \\
I &= \frac{1}{2}e^x(\sin x - \cos x) + C
\end{aligned}
$$

Example

例12 求 $I_n = \int \frac{\mathrm{d}x}{(x^2+a^2)^n}, (a>0, n \in \mathbb{N})$.

Solution

$$
\begin{aligned}
I_n &= \frac{\alpha}{(x^2+a^2)^n} - \left(-2\int \frac{(x^2+a^2-a^2}{(x^2+a^2)^{n+1}} dx\right) \\
&= \frac{x}{(x^2+a^2)} + 2nI_n - 2na^2 I_{n+1} \\
\Rightarrow I_n &= \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n
\end{aligned}
$$

其中: $I_1 = \frac{1}{a}\arctan\frac{x}{a}$

Example

例13 求 $\int x\mathrm{e}^x(1+\mathrm{e}^x)^{\frac{-3}{2}}\,\mathrm{d}x.$

Solution

$$
\begin{aligned}
I &= \int x(1+e^x)^{-\frac{3}{2}} d\left(e^x+1\right) \\
&= -2\int xd(1+e^x)^{-\frac{1}{2}} \\
&= -\frac{2x}{\sqrt{1+e^x}} + 2\int \frac{1}{\sqrt{1+e^x}} dx \\
&\text{其中} \int \frac{1}{\sqrt{1+e^x}} dx
\end{aligned}
$$

故原式 $= -\frac{2x}{\sqrt{1+e^x}} + 2\int \frac{1}{\sqrt{1+e^2}} dx$

$= -\frac{2x}{\sqrt{1+e^{-2}}} + \ln\left|\frac{\sqrt{10x-1}-1}{\sqrt{1+c^2}+1}\right| + c$

Intro

$$\int \frac{x}{(x^2+1)^{3/2}}\, dx$$

Wolframalpha

Soution

Take the integral:

$\int \frac{x}{(x^2+1)^{3/2}}\, dx$

For the integrand $\frac{x}{(x^2+1)^{3/2}}$, substitute $u = x^2 + 1$ and $du = 2x\, dx$:

$= \frac{1}{2} \int \frac{1}{u^{3/2}}\, du$

The integral of $\frac{1}{u^{3/2}}$ is $-\frac{2}{\sqrt{u}}$:

$= -\frac{1}{\sqrt{u}} + \text{constant}$

Substitute back for $u = x^2 + 1$:

$$\boxed{\text{Answer: } = -\frac{1}{\sqrt{x^2+1}} + \text{constant}}$$

Example

⇒习题P154.3.(6) $\int \frac{x \log(x)}{(x^2+1)^{3/2}}\, \mathrm{d}x$

Wolframalpha

Solution

For the integrand $\frac{x\log(x)}{(x^2+1)^{3/2}}$, integrate by parts, $\int f\,dg = fg - \int g\,df$, where

$f = \log(x), dg = \frac{x}{(x^2+1)^3 2}\,dx, df = \frac{1}{x}\,dx,$

$g = \frac{1}{\sqrt{x^2+1}}$:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} + \int \frac{1}{x\sqrt{x^2+1}}\,dx$$

For the integrand $\frac{1}{x\sqrt{x^2+1}}$, substitute $x = \tan(u)$and $dx = \sec^2(u)\,du$.

Then $\sqrt{x^2+1}= \sqrt{\tan^2(u)+1}= \sec(u)$and $u = \tan^{-1}(x)$:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} + \int \csc(u)\,du$$

Multiply numerator and denominator of $\csc(u)$by $\cot(u)+\csc(u)$:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} + \int -\frac{-\cot(u)\csc(u)-\csc^2(u)}{\cot(u)+\csc(u)}\,du$$

For the integrand $-\frac{-\cot(u)\csc(u)-\csc^2(u)}{\cot(u)+\csc(u)}$, substitute $s = \cot(u)+\csc(u)$and

$ds = \left(-\csc^2(u) - \cot(u)\csc(u)\right)du$:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} + \int -\frac{1}{s}\,ds$$

Factor out constants:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} - \int \frac{1}{s}\,ds$$

The integral of $\frac{1}{s}$is $\log(s)$:

$$= -\log(s) - \frac{\log(x)}{\sqrt{x^2+1}} + \text{constant}$$

Substitute back for $s = \cot(u)+\csc(u)$:

$$= -\log(\cot(u)+\csc(u)) - \frac{\log(x)}{\sqrt{x^2+1}} + \text{constant}$$

Substitute back for $u = \tan^{-1}(x)$:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} - \log\left(\cot\left(\tan^{-1}(x)\right) + \csc\left(\tan^{-1}(x)\right)\right) + \text{constant}$$

Simplify using $\cot\left(\tan^{-1}(z)\right) = \frac{1}{z}$and $\csc\left(\tan^{-1}(z)\right) = \frac{\sqrt{z^2+1}}{z}$:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} - \log\left(\frac{\sqrt{x^2+1}+1}{x}\right) + \text{constant}$$

An alternative form of the integral is:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} - \operatorname{csch}^{-1}(x) + \text{constant}$$

Which is equivalent for restricted $x$values to:

Answer:

$$= -\frac{\log(x)}{\sqrt{x^2+1}} - \log\left(\sqrt{x^2+1}+1\right) + \log(x) + \text{constant}$$