CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Linear Congruences

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  Systems of linear congruences have been studied since ancient times.

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  About 1500 years ago, the Chinese mathematician Sun-Tsu asked: "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

Modular Inverse

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  When does an inverse of a modulo m exist?

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  How to prove the uniqueness of the inverse?

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Example. Find an inverse of 101 modulo 4620.

$$4620 = 45\cdot 101 + 75$$
$$101 = 1\cdot 75 + 26$$
$$75 = 2\cdot 26 + 23$$
$$26 = 1\cdot 23 + 3$$
$$23 = 7\cdot 3 + 2$$
$$3 = 1\cdot 2 + 1$$
$$2 = 2\cdot 1$$

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101 &= 1\cdot 75 + 26\\
75 &= 2\cdot 26 + 23\\
26 &= 1\cdot 23 + 3\\
23 &= 7\cdot 3 + 2\\
3 &= 1\cdot 2 + 1\\
2 &= 2\cdot 1
\end{aligned}$$

$$\begin{aligned}
1 &= 3 - 1\cdot 2\\
1 &= 3 - 1\cdot(23 - 7\cdot 3) = -1\cdot 23 + 8\cdot 3\\
1 &= -1\cdot 23 + 8\cdot(26 - 1\cdot 23) = 8\cdot 26 - 9\cdot 23\\
1 &= 8\cdot 26 - 9\cdot(75 - 2\cdot 26) = 26\cdot 26 - 9\cdot 75\\
1 &= 26\cdot(101 - 1\cdot 75) - 9\cdot 75\\
&= 26\cdot 101 - 35\cdot 75\\
1 &= 26\cdot 101 - 35\cdot(4620 - 45\cdot 101)\\
&= -35\cdot 4620 + 1601\cdot 101
\end{aligned}$$

Using Inverses to Solve Congruences

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  Solution: We found that -2 is an inverse of 3 modulo 7. Multiply both sides of the congruence by -2, we have $x \equiv -8 \equiv 6 \pmod{7}$.

Number of Solutions to Congruences *

- **Theorem*** Let $d = \gcd(a, m)$ and $m' = m/d$. The congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d | b$. If $d | b$, then there are exactly d solutions. If x_0 is a solution, then the other solutions are given by $x_0 + m', x_0 + 2m', \ldots, x_0 + (d-1)m'$.

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Proof.
1) "only if": If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.

2) "if": Suppose that $d|b$. Let $b = kd$. There exist integers s, t such that $d = as + mt$. Multiply both sides by k. Then $b = ask + mtk$. Let $x_0 = sk$. Then $ax_0 \equiv b \pmod{m}$.

3) "$\# = d$": $ax_0 \equiv b \pmod{m}$ $ax_1 \equiv b \pmod{m}$ imply that $m|a(x_1 - x_0)$ and $m'|a'(x_1 - x_0)$. This implies further that $x_1 = x_0 + km'$, where $k = 0, 1, \ldots, d-1$.

The Chinese Remainder Theorem

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\begin{aligned}
x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
x &\equiv 2 \pmod{7}
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- **Theorem** (*The Chinese Remainder Theorem*) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_2}$$
$$\ldots$$
$$x \equiv a_n \pmod{m_n}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

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Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$\begin{aligned} &35 \cdot 2 \equiv 1 \pmod{3} \\ &21 \equiv 1 \pmod{5} \\ &15 \equiv 1 \pmod{7} \end{aligned}$$

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$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \pmod{105}$$

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-- 程大位 《算法统要》（1593年）

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Fermat's Little Theorem

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$$\{1, 2, \ldots, p-1\} = \{x, 2x, \ldots, x(p-1) \pmod{p}\}$$

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  Theorem * There is a primitive root modulo n if and only if $n = 2, 4, p^e$ or $2p^e$, where p is an odd prime.

  $\mathcal{Q}$: proof? The number of primitive roots? $*$

Number Theory and Cryptography

- Division, Primes

- Congruence

- Greatest Common Divisor (GCD)

- Euler's Theorem / Fermart's Little Theorem

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- Greatest Common Divisor (GCD)

  Find the GCD of 286 and 503.

$$
\begin{array}{ll}
\gcd(503, 286) & 503 = 1 \cdot 286 + 217 \\
= \gcd(286, 217) & 286 = 1 \cdot 217 + 69 \\
= \gcd(217, 69) & 217 = 3 \cdot 69 + 10 \\
= \gcd(69, 10) & 69 = 6 \cdot 10 + 9 \\
= \gcd(10, 9) & 10 = 1 \cdot 9 + 1 \\
= 1 & 9 = 9 \cdot 1
\end{array}
$$

$$
\begin{array}{rcl}
1 & = & 10 - 1 \cdot 9 \\
1 & = & 7 \cdot 10 - 1 \cdot 69 \\
1 & = & 7 \cdot 217 - 22 \cdot 69 \\
1 & = & 29 \cdot 217 - 22 \cdot 286 \\
1 & = & 29 \cdot 503 - 51 \cdot 286
\end{array}
$$

- E

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  (extended) Euclidean algorithm
  find the modular inverse
  solve linear congruence $ax \equiv b \pmod{m}$ $(\gcd(a, m) = 1)$

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Number Theory Summary

- Division, Primes

  $a = dq + r \quad q = a \; div \; d \quad r = a \bmod d$

- Congruence

  $a \equiv b \pmod{m}$ if m divides $a - b$

- Greatest Common Divisor (GCD)

  (extended) Euclidean algorithm
  find the modular inverse
  solve linear congruence $ax \equiv b \pmod{m}$ $(\gcd(a, m) = 1)$
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- Euler's Theorem / Fermart's Little Theorem

  $x^{\phi(n)} \equiv 1 \bmod n$ if $\gcd(x, n) = 1$
  $x^{p-1} \equiv 1 \bmod p$ if $x \not\equiv 0 \bmod p$

Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
  - Pseudorandom number generators
  - Hash functions
  - Cryptography

Pseudorandom Number Generators

- *Linear congruential method*

  We choose four numbers:

  - the modulus m
  - multiplier a
  - increment c
  - seed x_0

- *Linear congruential method*

We choose four numbers:

- ◇ the modulus m
- ◇ multiplier a
- ◇ increment c
- ◇ seed x_0

We generate a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ with $0 \le x_i < m$ by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$

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- *Linear congruential method*

$$x_{n+1} = (ax_n + c) \pmod{m}$$

Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$

- x_1= 7*3+4 mod 9=25 mod 9 =7
- x_2 = 53 mod 9 = 8
- x_3 = 60 mod 9 = 6
- x_4= 46 mod 9 =1
- x_5 = 11 mod 9 =2
- x_6 = 18 mod 9 =0
- ….

Hash Functions

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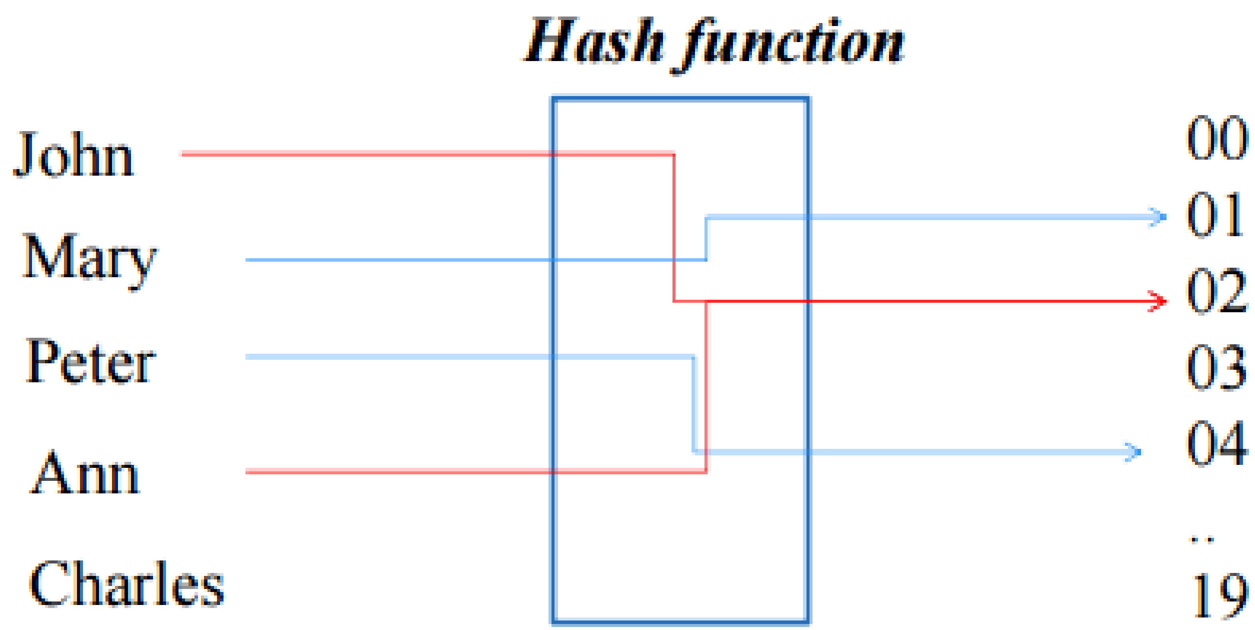
Example:



Hash Functions

- **Problem**: Given a large collection of records, how can we store and find a record quickly?

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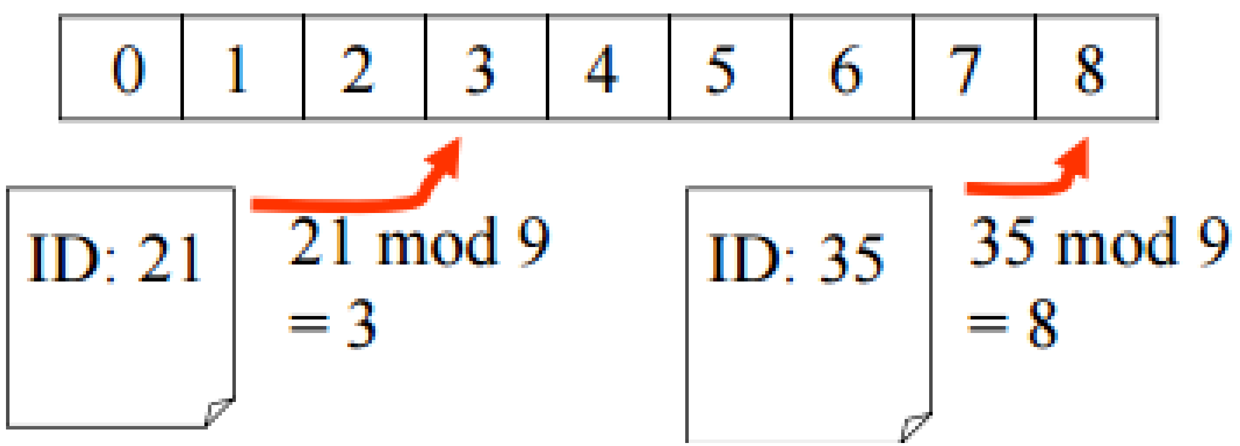
  Solution: Use a hash function, calculate the location of the record based on the record's ID.

Example: A common hash function is

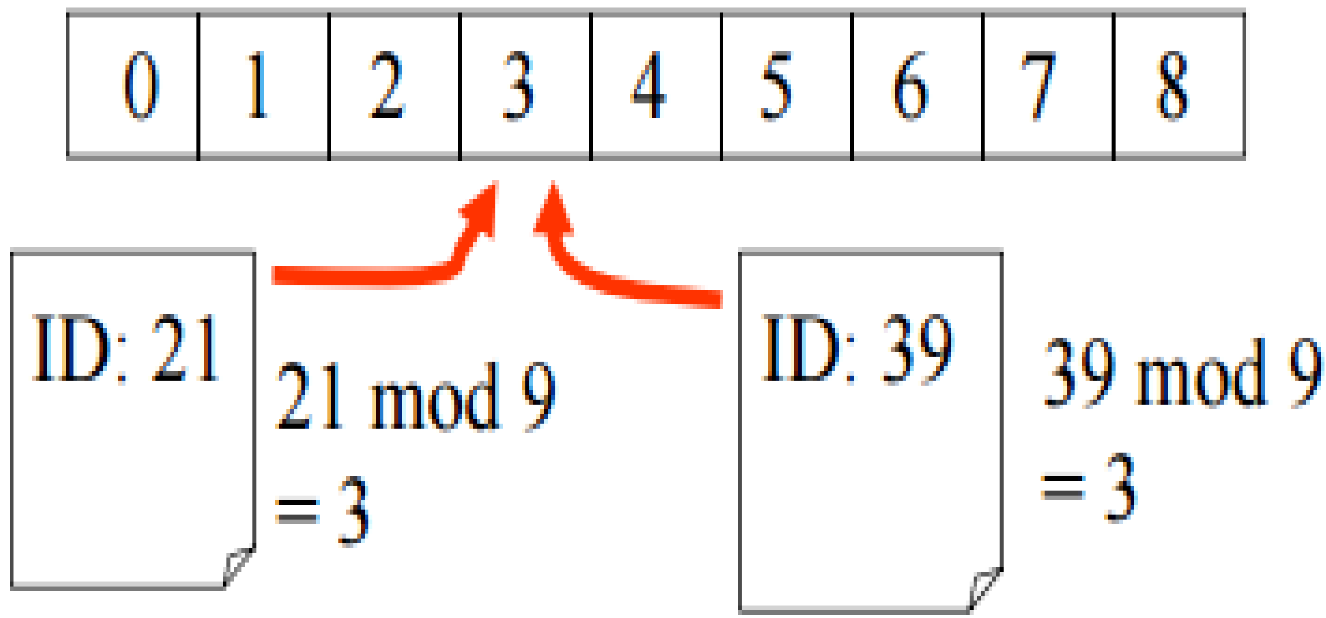
- $h(k) = k \textbf{ mod } n$,

where n is the number of available storage locations.



Hash Functions

- Two records mapped to the same location

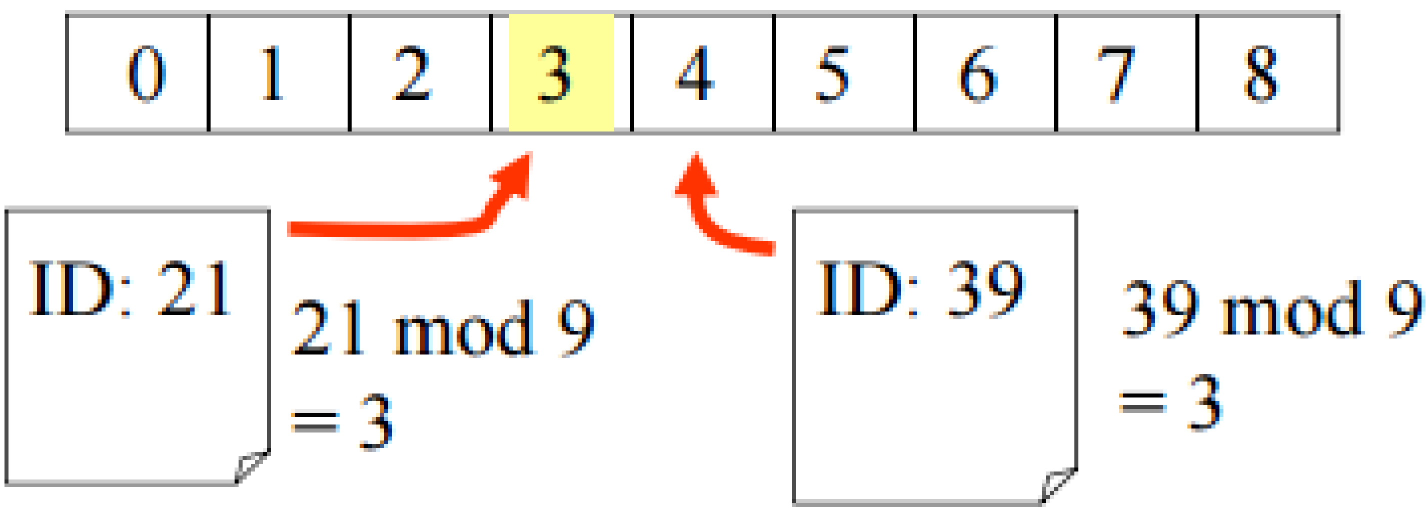


Hash Functions

- **Solution 1**: move to the next available location

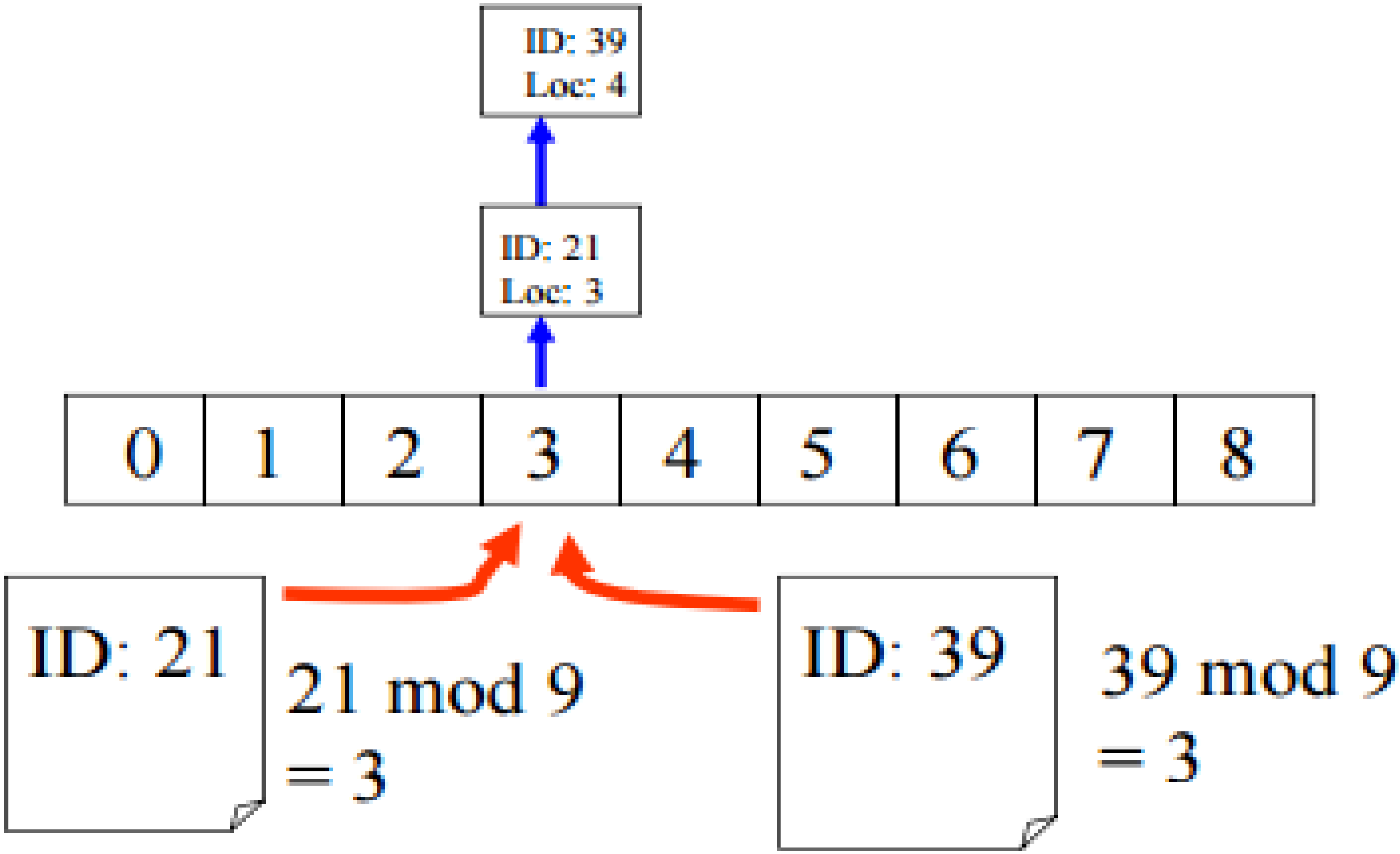
try

$$h_0(k) = k \textbf{ mod } n$$
$$h_1(k) = (k+1) \textbf{ mod } n$$
$$\ldots$$
$$h_m(k) = (k+m) \textbf{ mod } n$$



Hash Functions

- **Solution 2**: remember the exact location in a secondary structure that is searched sequentially



Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Crytocurrency, e.g., bitcoin

Cryptography

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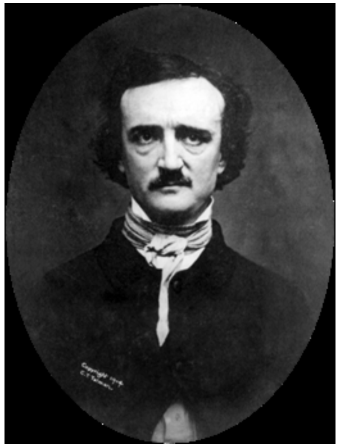
Cryptography = kryptos + graphos
(secret) (writing)

The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



Cryptography

- History of almost 4000 years (from 1900 B.C.)

$$\text{Cryptography} = \underset{(\text{secret})}{\text{kryptos}} + \underset{(\text{writing})}{\text{graphos}}$$

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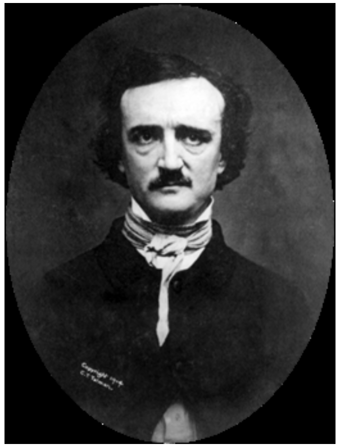


"Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve." – 1941

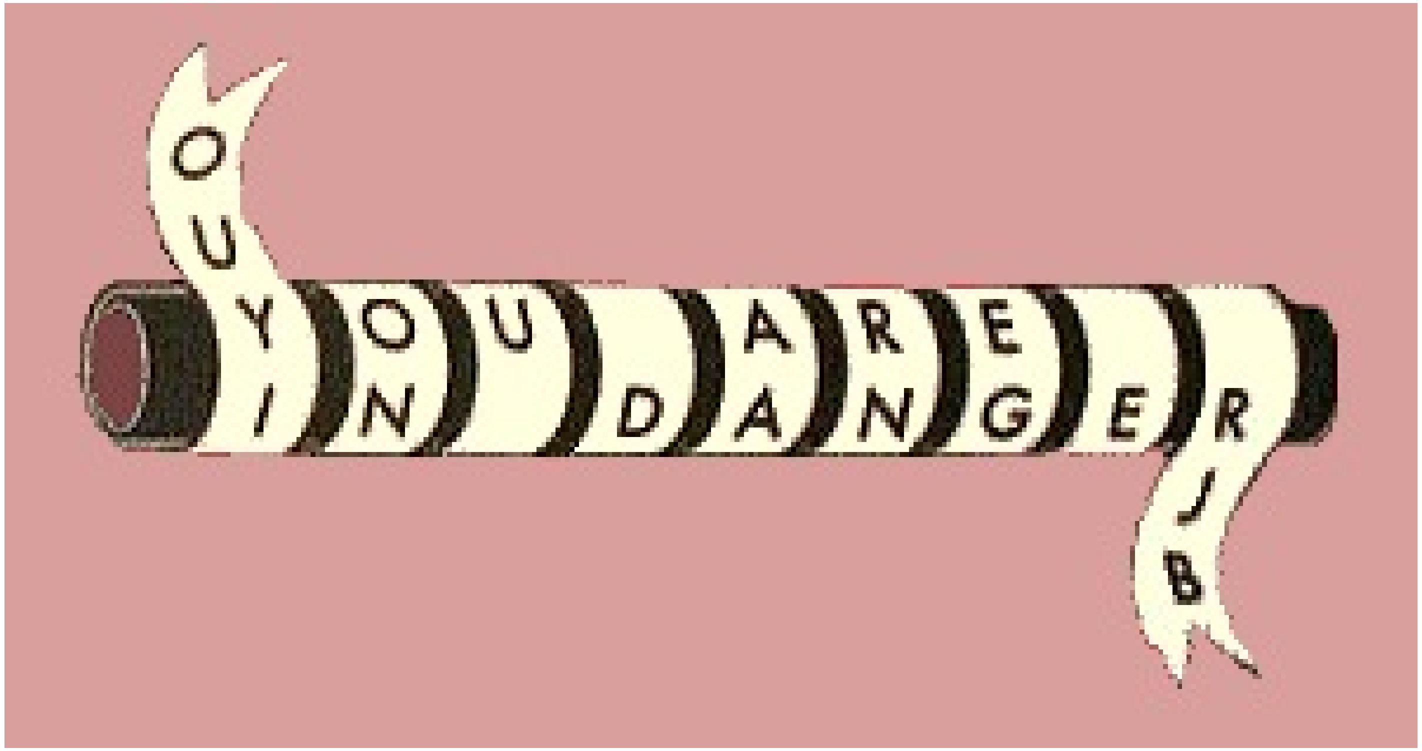
Cryptography

- One-sentence definition:

  "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*." – Ronald L. Rivest

Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



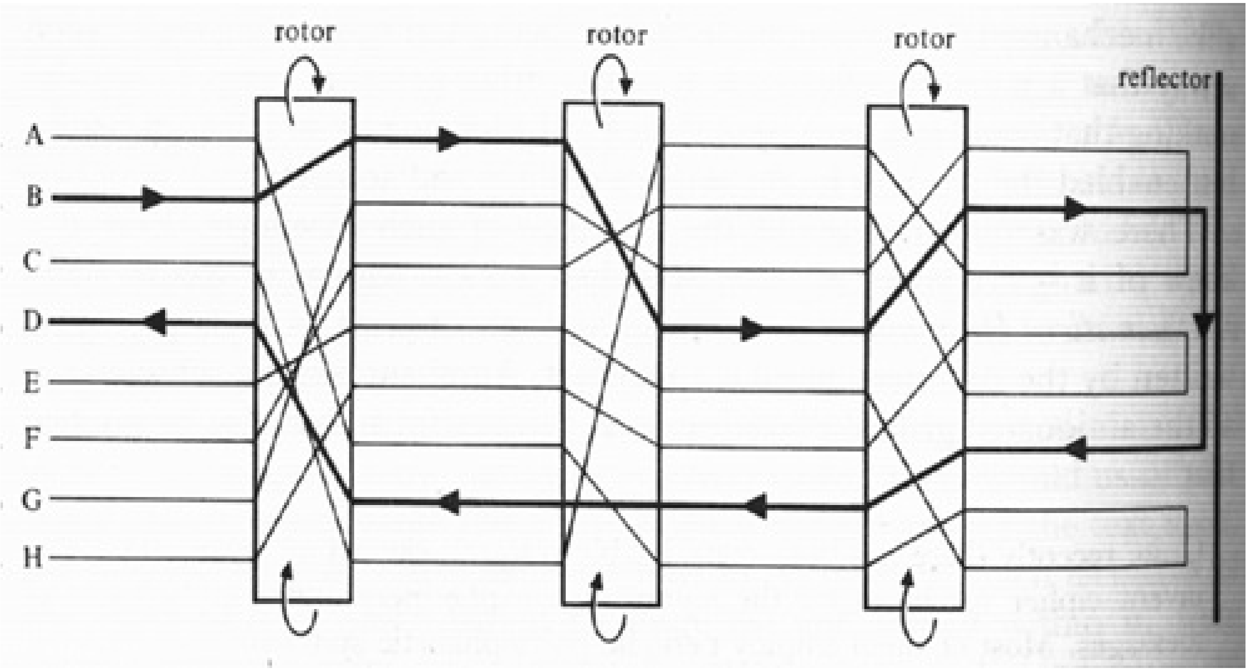
Some Examples

- The Greeks also invented a cipher which changed letters to numbers. A form of this code was still being used during *World War I*.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Some Examples

- Enigma, Germany coding machine in *World War II.*



Cryptography History

- History (until 1970's)

  "*Symmetric*" cryptography

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$\mathcal{Q}$: What if Bob could send Alice a "special key" useful only for encryption but no help for decryption?

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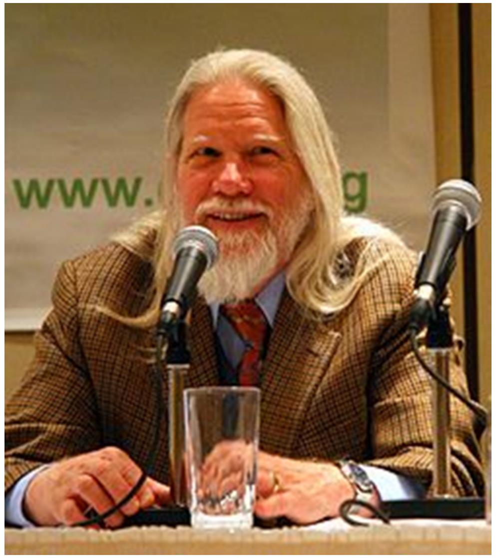
- History (from 1976)
  - ◇ W. Diffie, M. Hellman, “New direction in cryptography”, *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

“We stand today on the brink of a revolution in cryptography.”



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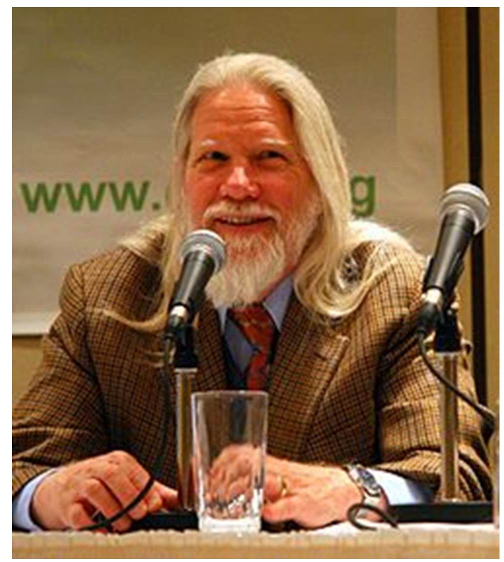
2015 **Turing Award**



Bailey W. Diffie

Martin E. Hellman

2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

Public Key Cryptography

- Alice wants to send a message to Bob

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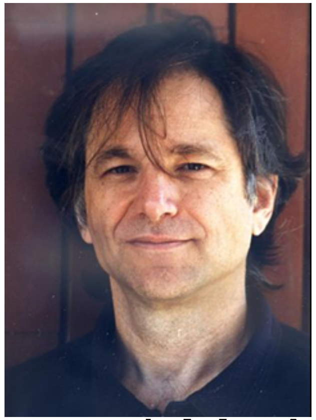
Adi Shamir


Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, "A method for obtaining digital signatures and public-key cryptosystems", *Communications of the ACM*, vol. 21-2, pages 120-126, 1978.

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$\mathcal{Q}$: How to prove this?

RSA Public Key Cryptosystem: Example

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

RSA Public Key Cryptosystem: Parameters

Parameters: $p \quad q \quad n \quad \phi(n) \quad e \quad d$

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \mathrm{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Applications of RSA

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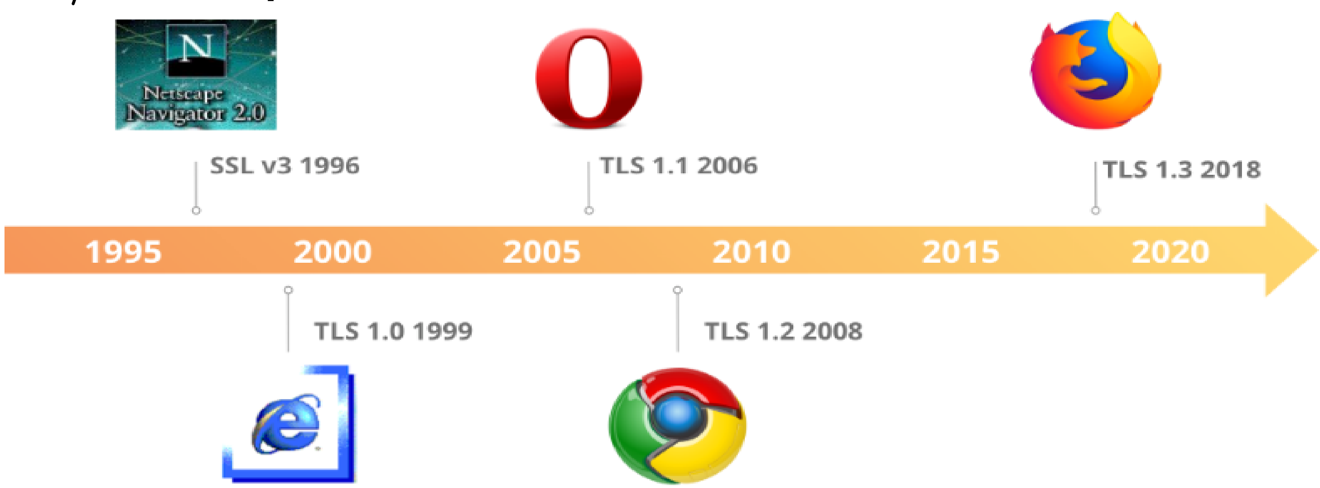

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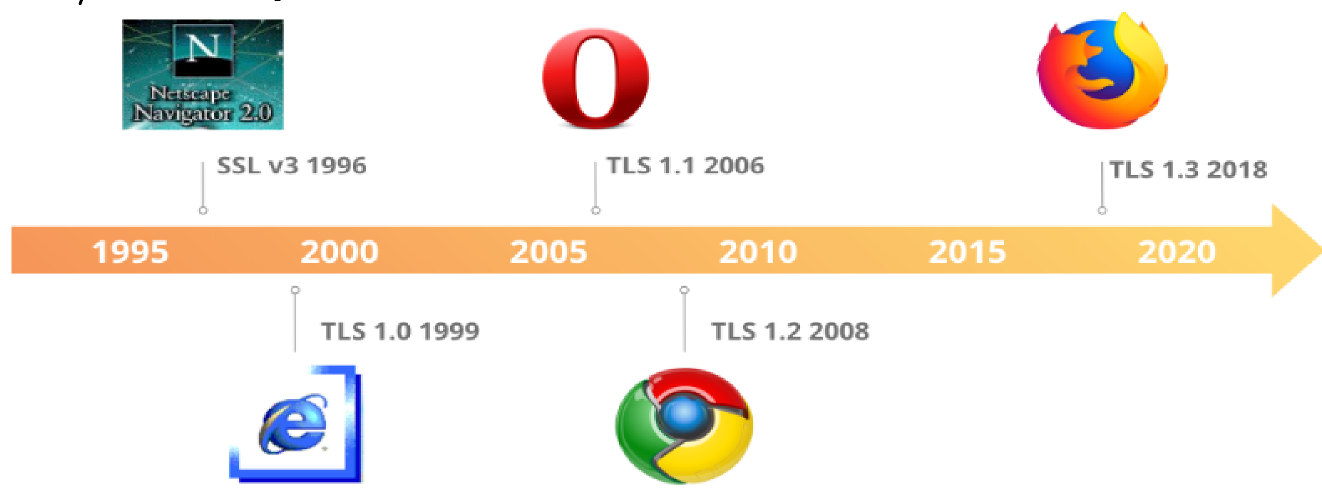

Key exchange/agreement and authentication

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RSA	Yes	Yes	Yes	Yes	Yes	No
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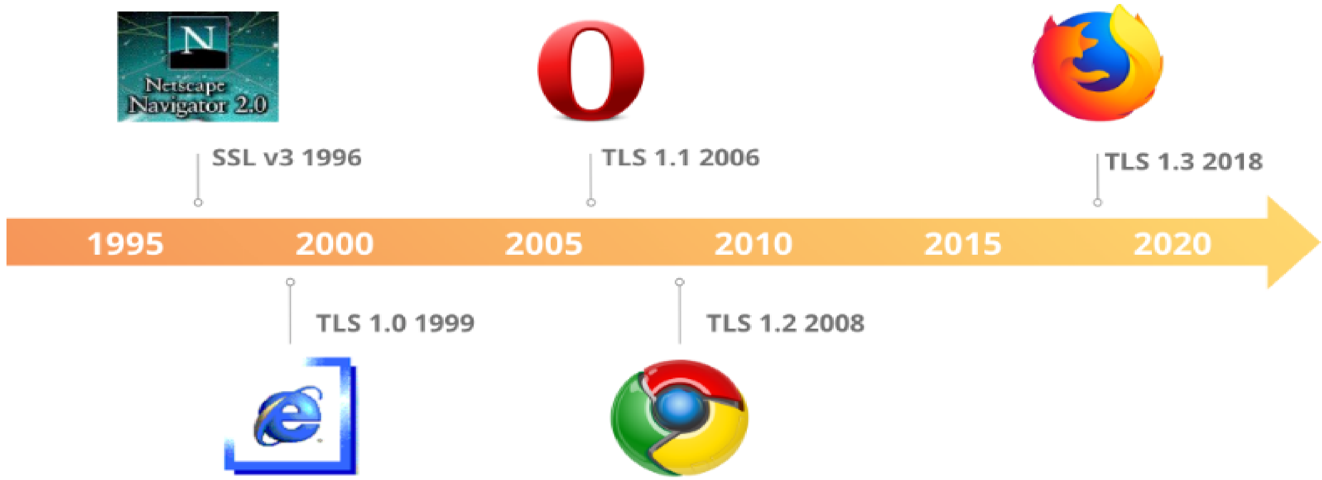

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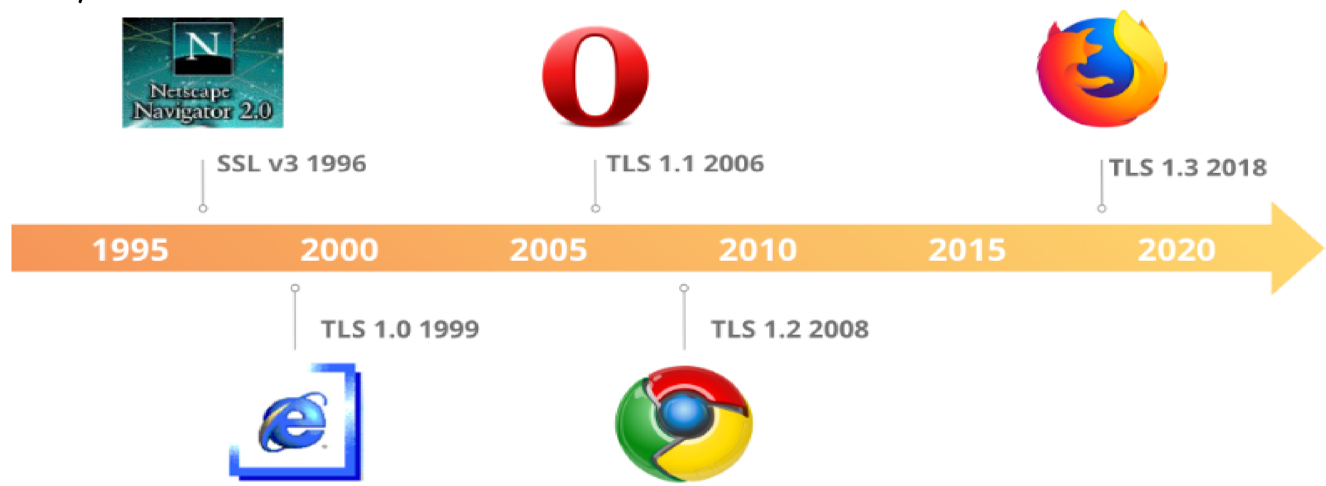

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Using RSA for Digital Signature

$S = M^d \bmod n$ (RSA **signature**)

$M = S^e \bmod n$ (RSA **verification**)

Why?

The Discrete Logrithm

- **The discrete logarithm** of an integer y to the base b is an integer x, such that

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This is very hard!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of $\mathbb{Z}_p$. The private key x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The public key for *El Gamal encryption* is (p, g, y).

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  El Gamal Encryption: Pick a random integer k from $\mathbb{Z}_{p-1}$,

  $$a = g^k \bmod p$$
  $$b = My^k \bmod p$$

  The ciphertext C consists of the pair (a, b).

  El Gamal Decryption:

  $$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$a = g^k \bmod p$$
$$b = k^{-1}(M - xa) \bmod (p-1)$$

(El Gamal **signature)**

$$y^a a^b \equiv g^M \pmod{p}$$

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$\mathcal{Q}$: How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence $y = 2^{765} \bmod 2579 = 949$.

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Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p)\\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579)\\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

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Attack 1: Use $M = by^{-k}$. However, k is randomly picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is secret.

Diffie-Hellman Key Exchange Protocol

User A		User B
Generate random $X_A < p$	$\xrightarrow{Y_A}$	Generate random $X_B < p$
calculate $Y_A = \alpha^{X_A} \bmod p$	$\xleftarrow{Y_B}$	Calculate $Y_B = \alpha^{X_B} \bmod p$
Calculate $k = (Y_B)^{X_A} \bmod p$		Calculate $k = (Y_A)^{X_B} \bmod p$

Next Lecture

- induction ...