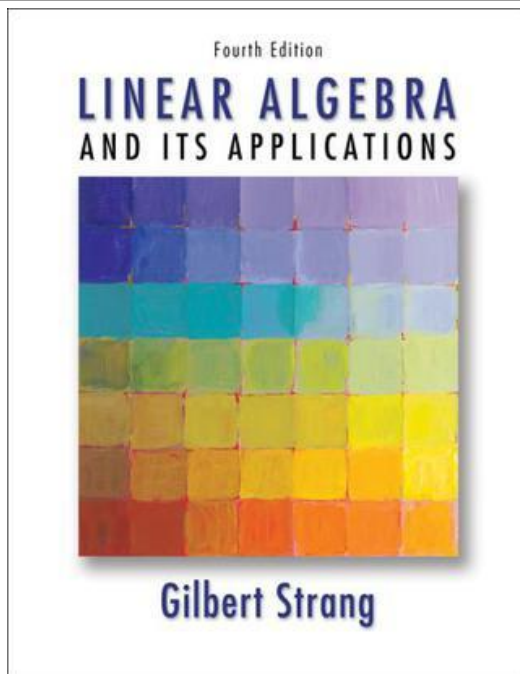


Linear Algebra



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Matrices and Gaussian Elimination

1.6

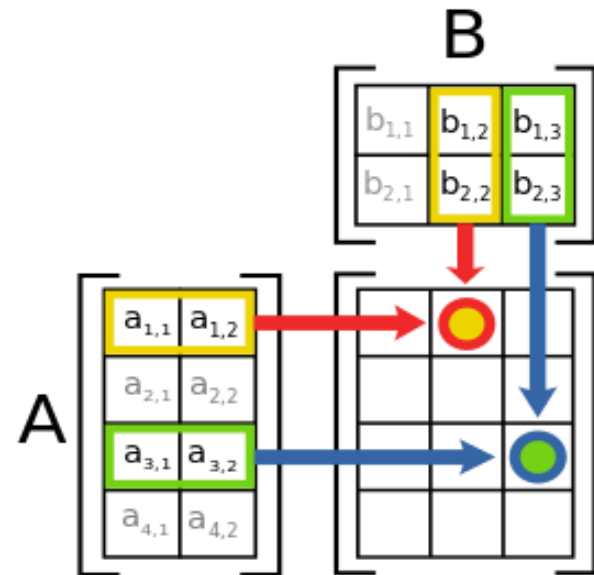
PARTITIONED MATRICES

(分块矩阵)

Introduction

Definition

Operations



- **Textbook: Problem Set 1.4, 1.6**

I. Introduction

引例：数学中的矩阵分块

如：

Let $A = \begin{bmatrix} \boxed{1} & \boxed{2} & 0 & 0 & 0 \\ \boxed{3} & \boxed{7} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{9} & \boxed{5} \\ 0 & 0 & 0 & \boxed{7} & \boxed{4} \end{bmatrix},$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}, \quad A_2 = [1],$$

$$A_3 = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}.$$

Find A^k

Is A invertible?

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}$$

II. Definition

对于行数和列数较高的矩阵 A ，经常采用**分块法**，使大矩阵的运算化成小矩阵的运算，这是矩阵运算中的一个重要技巧。

具体做法是：将矩阵 A 用若干条纵线和横线分成许多个小矩阵，每一个小矩阵称为 A 的**子块(blocks, submatrices)**，以子块为元素的形式上的矩阵称为**分块矩阵(partitioned matrices)**。

例1 已知 5 阶方阵

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

在 A 的第 2 行与第 3 行之间、第 2 列与第 3 列之间各加一条水平虚线和垂直虚线, 则 A 划分为 4 块.

此时 A 可表示为

记 $\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} = A_1, \begin{bmatrix} 1 & 0 & -1 \\ 2 & -4 & 0 \end{bmatrix} = A_2,$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = A_3, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_4,$$

如果把小矩阵 A_1, A_2, A_3, A_4 视为 4 个元素, 则 A 可视为形式上的 2 阶方阵.

常用的分块方法

1. 按行分块

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

2. 按列分块

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = [b_1 \quad b_2 \quad \cdots \quad b_n]$$

3. 按对角分块

当 n 阶矩阵 C 中非零元素都集中在主对角线附近,
如果可以分块成如下分块矩阵

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_m \end{bmatrix},$$

其中 C_i 是 r_i 阶**方阵**, $i=1,2,\dots,m$, $\sum_{i=1}^m r_i = n$,

则称 C 为**块对角矩阵(block diagonal matrix)**
或**准对角矩阵**.

Example 2 The matrix

$$C = \begin{bmatrix} 0 & \cos \theta & 0 & 0 & 0 & 0 \\ \sin \theta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} C_1 & & \\ & C_2 & \\ & & C_3 \end{bmatrix},$$

where

$$C_1 = \begin{bmatrix} 0 & \cos \theta \\ \sin \theta & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}, \quad C_3 = [4].$$

For

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}.$$

Is it a block diagonal matrix?

NO

III. Operations

1. 分块矩阵的加法(Addition)

设矩阵 A, B 为同型矩阵, 将 A, B 分块为

$$A = \begin{bmatrix} \boxed{A_{11}} & \cdots & \boxed{A_{1t}} \\ \vdots & & \vdots \\ \boxed{A_{s1}} & \cdots & \boxed{A_{st}} \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{B_{11}} & \cdots & \boxed{B_{1t}} \\ \vdots & & \vdots \\ \boxed{B_{s1}} & \cdots & \boxed{B_{st}} \end{bmatrix}$$

其中 A_{ij}, B_{ij} ($i=1, 2, \dots, s; j=1, 2, \dots, t$) 为同型矩阵, 则

$$A \pm B = \begin{bmatrix} \boxed{A_{11} \pm B_{11}} & \cdots & \boxed{A_{1t} \pm B_{1t}} \\ \vdots & & \vdots \\ \boxed{A_{s1} \pm B_{s1}} & \cdots & \boxed{A_{st} \pm B_{st}} \end{bmatrix}.$$

2. 分块矩阵的数量乘法(Scalar multiplication)

设分块矩阵 $A = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{sr} \end{bmatrix}$, k 为常数, 则

$$kA = \begin{bmatrix} kA_{11} & \cdots & kA_{1r} \\ \vdots & & \vdots \\ kA_{s1} & \cdots & kA_{sr} \end{bmatrix}.$$

- If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A + B$.
- In this case, each block of $A + B$ is the (matrix) sum of the corresponding blocks of A and B .
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

3. 分块矩阵的乘法

(Multiplication of partitioned matrices)

设 A 是 $m \times n$ 矩阵, B 是 $n \times p$ 矩阵. 将 A, B 分块成

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & & \vdots \\ B_{t1} & \cdots & B_{tr} \end{bmatrix},$$

其中 A 的列的分块法和 B 的行的分块法完全相同, 则

$$AB = \begin{bmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & & \vdots \\ C_{s1} & \cdots & C_{sr} \end{bmatrix}, \quad \text{这里 } C_{ij} = \sum_{k=1}^t A_{ik} B_{kj} \\ (i = 1, 2, \dots, s; j = 1, 2, \dots, r).$$

Partitioned matrices can be multiplied by the usual **row—column rule** as if the block entries were scalars, provided that for a product AB , the **column partition of A** matches the **row partition of B** .

Example 3 Let the matrices A , B be

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

Find AB .

Solution. $A = \begin{bmatrix} I_2 & \mathbf{0}_{2 \times 3} \\ A_3 & I_3 \end{bmatrix}$, where $A_3 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 0 \end{bmatrix}$;

(B 的行与 A 的列的分块法要一致, 但 B 的列可任意分.)

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{I}_2 \\ -\mathbf{I}_3 & \mathbf{0}_{3 \times 2} \end{bmatrix}, \quad \text{where } \mathbf{B}_1 = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \mathbf{I}_2 \\ -\mathbf{I}_3 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{I}_2 \\ \mathbf{A}_3 \mathbf{B}_1 - \mathbf{I}_3 & \mathbf{A}_3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ \hline 3 & 7 & 0 & 1 & 2 \\ -1 & -2 & 0 & -1 & 1 \\ 4 & 6 & -1 & 2 & 0 \end{bmatrix}. \end{aligned}$$

(可以验算, \mathbf{AB} 直接乘与分块相乘所得的结果一致.)

Example 4 Compute A^2 , where

$$A = \begin{bmatrix} \boxed{1} & \boxed{2} & 0 & 0 & 0 \\ \boxed{3} & \boxed{7} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{9} & \boxed{5} \\ 0 & 0 & 0 & \boxed{7} & \boxed{4} \end{bmatrix} = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} A_1^2 & & \\ & A_2^2 & \\ & & A_3^2 \end{bmatrix}$$

若 n 阶矩阵 C 和 D 分块成同型对角块矩阵, 即

$$C = \text{diag}(C_1, C_2, \dots, C_s), \quad D = \text{diag}(D_1, D_2, \dots, D_s)$$

其中 C_i 和 D_i 是同阶方阵 ($i = 1, 2, \dots, s$). 则

$$CD = \text{diag}(C_1 D_1, C_2 D_2, \dots, C_s D_s)$$

4. 分块矩阵的转置(Transpose)

Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1t} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2t} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{s1} & \mathbf{A}_{s2} & \cdots & \mathbf{A}_{st} \end{bmatrix},$$

then

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T & \cdots & \mathbf{A}_{s1}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T & \cdots & \mathbf{A}_{s2}^T \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{1t}^T & \mathbf{A}_{2t}^T & \cdots & \mathbf{A}_{st}^T \end{bmatrix}.$$

$$\text{For } \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \end{bmatrix}, \text{ we have } \mathbf{A}^T = \begin{bmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T \\ \mathbf{A}_{13}^T & \mathbf{A}_{23}^T \end{bmatrix}.$$

Example: Let $\mathbf{A} = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2]$, $\boldsymbol{\alpha}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\boldsymbol{\alpha}_2 = \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$, then $\mathbf{A}^T = ?$
 where

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 7 \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \end{bmatrix}$$

$$\text{And if } \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix}, \quad \text{then } \mathbf{B}^T = [\mathbf{b}_1^T, \mathbf{b}_2^T, \dots, \mathbf{b}_m^T].$$

5. 可逆分块矩阵的逆矩阵(Inverse)

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_m \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_1^{-1} & & & \\ & \mathbf{A}_2^{-1} & & \\ & & \ddots & \\ & & & \mathbf{A}_m^{-1} \end{bmatrix}.$$

块对角矩阵 \mathbf{A} 可逆当且仅当 \mathbf{A}_i ($i=1,2,\dots,m$)都可逆.
(A block diagonal matrix is invertible if and only if each block on the diagonal is invertible.)

$$\mathbf{B} = \begin{bmatrix} & & & \mathbf{B}_1 \\ & & & \\ & & \mathbf{B}_2 & \\ & & & \ddots \\ \mathbf{B}_m & & & \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} & & & \mathbf{B}_m^{-1} \\ & & & \\ & & \ddots & \\ & & & \mathbf{B}_2^{-1} \\ \mathbf{B}_1^{-1} & & & \end{bmatrix}.$$

Example 5 Find the inverses of the following matrices.

$$(1)A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & 7 & 4 \end{bmatrix}. \quad (2)A = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$(1)A^{-1} = \begin{bmatrix} 7 & -2 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & -7 & 9 \end{bmatrix}. \quad (2)A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1/a_n \\ 1/a_1 & 0 & 0 & \cdots & 0 \\ & 1/a_2 & & \vdots & \vdots \\ 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 1/a_{n-1} & 0 \end{bmatrix}.$$

Example 6 Let $T = \begin{bmatrix} A & \mathbf{0} \\ C & D \end{bmatrix}$, where A, D are $m \times m$ and $n \times n$ invertible matrices. Please find T^{-1} .

Solution Since $\begin{bmatrix} I_m & \mathbf{0} \\ -CA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ C & D \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix},$

and $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & D^{-1} \end{bmatrix},$

we have $T^{-1} = \left[\begin{pmatrix} I_m & \mathbf{0} \\ -CA^{-1} & I_n \end{pmatrix}^{-1} \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix} \right]^{-1}$

$$= \begin{bmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & D^{-1} \end{bmatrix} \begin{bmatrix} I_m & \mathbf{0} \\ -CA^{-1} & I_n \end{bmatrix} = \begin{bmatrix} A^{-1} & \mathbf{0} \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}.$$

Summary

分块矩阵的概念与运算 (Definition and Operations)

分块矩阵之间与一般矩阵之间的运算性质类似:

- (1) **加法(Addition)** 同型矩阵, 采用相同的分块法.
- (2) **数乘(Scalar Multiplication)** 数 k 乘矩阵, 需数 k 乘矩阵的每个子块.
- (3) **乘法(Multiplication of partitioned matrices)** 要求 A 的列分块法和 B 的行分块法一致.
- (4) **转置(Transpose)** 行变为相应的列, 每个子块都转置.
- (5) **分块矩阵的逆阵(Inverse)**.

Homework See Blackboard announcement