Chapter 4: Expected Values (期望值)

- The Expected Value of a Random Variable(随机变量的期望)
- Variance and Standard Deviation(方差和标准差)
- Covariance and Correlation Coefficient (协方差和相关系数)
- Conditional Expectation (条件期望)

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Background

Example: there are two watches with daily accuracies shown as follows:

Daily Accuracy (s)	-3	-2	-1	0	1	2	3
Probability (Watch A)	0.10	0.15	0.15	0.20	0.15	0.15	0.10
Probability (Watch B)	0.05	0.05	0.10	0.60	0.10	0.05	0.05

Which watch has better quality?

Answer: Assume that the accuracy of these two watches are X and Y, then

$$E(X)=\sum_{k=1}^{7} x_k P\{X=x_k\}=0$$

$$E(Y)=\sum_{k=1}^{7} y_k P\{Y=y_k\}=0$$

Therefore, the average accuracies from the two watches are the same.

Are they of the same quality?

Thinking based on the deviation (偏差) from the average

The deviation(偏差) of r.v. X is

$$|X - E(X)|$$

If the **deviation** is small, the quality is stable

Calculation based on absolute value is not convenient

Consider the squared deviation (平方偏差)

$$[X - E(X)]^2$$

Squared deviation is still a r.v.

Consider the average of squared deviation

$$E\left(X - E(X)\right)^2$$

Variance (方差) represents the **average squared deviation** between a r.v. and its average.

Definition: For r.v. X, if the following exists

$$\mathrm{Var}(X) \triangleq D(X) \triangleq E\left(X - E(X)\right)^2$$

Then $D(X)$ is the Variance (方差) of r.v. X , $\sqrt{D(x)}$ is the Standard Deviation (标准差)

Example: Assume that the scores of shooters A and B are X and Y respectively. The frequency functions are:

x_k	8	9	10
p_k	0.15	0.40	0.45

y_k	8	9	10
p_k	0.35	0.10	0.55

Please evaluate these shooters.

Answer: First find out the **expected values:**

$$E(X) = 8 \times 0.15 + 9 \times 0.40 + 10 \times 0.45 = 9.3$$

$$E(Y) = 8 \times 0.35 + 9 \times 0.10 + 10 \times 0.55 = 9.2$$

And then find out the **variance**

$$\mathrm{Var}(X) \triangleq D(X) \triangleq E\big(X - E(X)\big)^2$$

$$D(X) = (8-9.3)^2 \times 0.15 + (9-9.3)^2 \times 0.40 + (10-9.3)^2 \times 0.45 = 0.51$$

$$D(Y) = (8-9.2)^2 \times 0.35 + (9-9.2)^2 \times 0.10 + (10-9.2)^2 \times 0.55 = 0.86$$

Based on the results, shooter A is not only better than shooter B in terms of shooting level, but also the score is more stable.

Example: there are two watches with daily accuracies shown as follows:

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Probability (Watch B)	0.05	0.05	0.10	0.60	0.10	0.05	0.05

Which watch has better quality?

Answer: Assume that the accuracies of the two watches A, B are X and Y, respectively. then

$$E(X) = \sum_{k=1}^{7} x_k P\{X = x_k\} = 0$$

$$E(Y) = \sum_{k=1}^{7} y_k P\{Y = y_k\} = 0$$

$$D(X) = E\big(X - E(X)\big)^2 = E\big(X^2\big) = 3.3$$

$$D(Y) = E\big(Y - E(Y)\big)^2 = E\big(Y^2\big) = 1.5$$

As above, the average accuracies are the same, but their variances are different. So, Watch B is better than Watch A because it is more stable.

- Explanation: Expected value —— The average value of a r.v.

  Variance —— The average deviation between a r.v. and its average.

- The calculation of variance

$$D(X)=E\big(X-E(X)\big)^2$$

is the expected value of $g(X)=\big(X-E(X)\big)^2$, then

① Assume that the frequency function of X is

$$P\{X=x_i\}=p_k, k=1,2,\cdots$$

then $$D(X)=\sum_{k=1}^{\infty}\big(x_k-E(X)\big)^2\cdot p_k$$

② Assume that the probability density of X is $f(x)$, then

$$D(X)=\int_{-\infty}^{\infty}\big(x-E(X)\big)^2 f(x)dx$$

③ $$\boxed{D(X)=E\big(X-E(X)\big)^2=E\big(X^2\big)-[E(X)]^2}$$

Example: Assume that $X \sim P(\lambda)$, find $D(X)$

Answer: Based on the calculation from the last section, we have $E(X) = \lambda$

$$
\begin{aligned}
E(X^2) &= E[X(X-1)] + E(X) \\
&= \sum_{k=0}^{\infty} k(k-1)\frac{\lambda^k}{k!}e^{-\lambda} + \lambda \\
&= \lambda^2 e^{-\lambda}\boxed{\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}} + \lambda \\
&= \lambda^2 + \lambda
\end{aligned}
$$

$$
\begin{aligned}
\therefore D(X) &= E\left(X^2\right) - [E(X)]^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

Example: Assume that $X \sim U(a,b)$, find $D(X)$

Answer: Based on the calculation from the last section, we have $E(X)=\frac{a+b}{2}$, the density of X is

$$f(x)=\begin{cases}\dfrac{1}{b-a}, & a<x<b\\ 0, & \text{otherwise}\end{cases}$$

$$\therefore E(X^2)=\int_a^b \frac{x^2}{b-a}dx$$

$$=\frac{b^3-a^3}{3(b-a)}=\frac{a^2+ab+b^2}{3}$$

$$\therefore D(X)=E(X^2)-[E(X)]^2$$

$$=\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2$$

$$=\frac{(b-a)^2}{12}$$

Example: Assume that the density function of X is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, x > 0 \\ 0, \qquad x \le 0 \end{cases}$$

Find $D(X)$.

Answer: Since it is the exponential distribution, then $E(X) = \theta$, so:

$$D(X) = E\left(X^2\right) - [E(X)]^2$$

$$= \int_0^\infty x^2 \frac{1}{\theta} e^{-x/\theta} dx - \theta^2$$

$$= -x^2 e^{-x/\theta} \Big|_0^\infty + 2\int_0^\infty x e^{-x/\theta} dx - \theta^2$$

$$= 2\theta^2 - \theta^2 = \theta^2$$

Basic properties of variance（方差的基本性质）

① If $X =_{a.e.} c$ (constant), then $D(X) = 0$.

② If c is a constant, then $D(cX) = c^2 D(X)$.

Proof:
$$D(cX) = E\big(cX - E(cX)\big)^2$$
$$= E\big(cX - cE(X)\big)^2$$
$$= c^2 E\big(X - E(X)\big)^2$$
$$= c^2 D(X)$$

Basic properties of variance（方差的基本性质）

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② If c is a constant, then $D(cX) = c^2 D(X)$.

③ For r.v.s X and Y:

$$D(X+Y) = D(X) + D(Y) + 2E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big]$$

Proof:
$$\begin{aligned} D(X+Y) &= E[(X+Y) - E(X+Y)]^2 \\ &= E\big[\big(X - E(X)\big) + \big(Y - E(Y)\big)\big]^2 \\ &= E[X - E(X)]^2 + E[Y - E(Y)]^2 \\ &\quad +2E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big] \\ &= D(X) + D(Y) + 2E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big] \end{aligned}$$

Basic properties of variance（方差的基本性质）

① If $X =_{a.e.} c$ (constant), then $D(X) = 0$.

② If c is a constant, then $D(cX) = c^2 D(X)$.

③ For r.v.s X and Y:

$$D(X+Y) = D(X) + D(Y) + 2E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big]$$

If X and Y are independent, then $\boxed{D(X+Y) = D(X) + D(Y)}$

Proof: $\because$ X and Y are independent

$\therefore$ $X - E(X)$ and $Y - E(Y)$ are independent

$$\therefore E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big] = E[X - E(X)] \cdot E[Y - E(Y)]$$

$$= [E(X) - E(X)] \cdot [E(Y) - E(Y)] = 0$$

If X and Y are independent, is the following right?

$$D(X - Y) = D(X) + D(Y) \quad ?$$

Example: Assume that $X \sim b(n,p)$, find $D(X)$

Answer: Since binomial distribution is from n-fold Bernoulli trials, then

$$X = X_1 + X_2 + \cdots + X_n$$

where

$$X_i = \begin{cases} 1, \text{A happened in the } i\text{th Bernoulli trial} \\ 0, \bar{\text{A}} \text{ happened in the } i\text{th Bernoulli trial} \end{cases} (i = 1,2,\dots,n)$$

And $X_1, X_2, \dots, X_n$ are independent and identically distributed (i.i.d.) with frequency functions:

$$P\{X_i = 1\} = p, P\{X_i = 0\} = 1 - p.$$

$$\therefore E(X) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$= nE(X_1) = np$$

$$D(X) = D(X_1) + D(X_2) + \cdots + D(X_n)$$

$$= nD(X_1) = n[(1-p)^2 \cdot p + (0-p)^2 \cdot (1-p)]$$

$$= np(1-p)$$

Example: Assume that $X \sim N(\mu, \sigma^2)$, find $D(X)$

Answer: Based on the last section $E(X)=\mu$, then

$$D(X)=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

let $t=\frac{x-\mu}{\sigma}$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^2 e^{-\frac{t^2}{2}}dt$$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-t)de^{-\frac{t^2}{2}}$$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\left[(-t)e^{-\frac{t^2}{2}}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}dt\right]$$

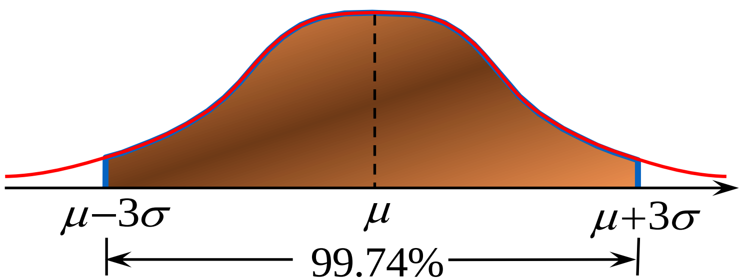
$$=\frac{\sigma^2}{\sqrt{2\pi}}\sqrt{2\pi}=\sigma^2$$

3σ Principle: Most values of the normal r.v. are in the range $(\mu - 3\sigma, \mu + 3\sigma)$

If $X \sim N(\mu, \sigma^2)$, then

$$P\{|X-\mu| < 3\sigma\} = 0.9974$$

$$P\{|X-\mu| \geq 3\sigma\} = 0.0026$$



Generally, how to find the following probability of r.v. X:

$$P\{|X-\mu| \geq \varepsilon\}$$?

where $\mu = E(X)$, $\varepsilon > 0$ are constants.

Theorem: Chebyshev's inequality （切比雪夫不等式）

If $\mu \triangleq E(X)$ and $\sigma^2 = D(X)$ both exist, then for $\forall \varepsilon > 0$, we have:

$$P\{|X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$$

Theorem: Chebyshev's inequality （切比雪夫不等式）

If $\mu \triangleq E(X)$ and $\sigma^2 = D(X)$ both exist, then for $\forall \varepsilon > 0$, we have:

$$P\{|X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$$

Proof: Only prove for continuous r.v.s, discrete r.v.s are similar. Let $f(x)$ be the density function of X, then:

$$\begin{aligned} P\{|X-\mu| \geq \varepsilon\} &= \int_{|x-\mu|\geq\varepsilon} f(x)dx \\ &\leq \int_{|x-\mu|\geq\varepsilon} \frac{(x-\mu)^2}{\varepsilon^2} f(x)dx \\ &\leq \frac{1}{\varepsilon^2} D(X) = \frac{\sigma^2}{\varepsilon^2} \end{aligned}$$

Or we can let $Y = (X-\mu)^2$, then $E(Y) = \sigma^2$, applying the Markov Inequality:

$$P\{|X-\mu| \geq \varepsilon\} = P\{Y \geq \varepsilon^2\} \leq \frac{E(Y)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}$$

Theorem: Chebyshev's inequality

If $\mu \triangleq E(X)$ and $\sigma^2 = D(X)$ both exist, then for $\forall \varepsilon > 0$, we have:

$$P\{|X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$$

Equivalently, we have:

$$P\{|X-\mu| < \varepsilon\} \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

Let $\varepsilon = 3\sigma$ and 4σ, we have:

$$P\{|X-\mu| < 3\sigma\} \geq 1 - \frac{1}{9} = 88.90\%$$

$$P\{|X-\mu| < 4\sigma\} \geq 1 - \frac{1}{16} = 93.75\%$$

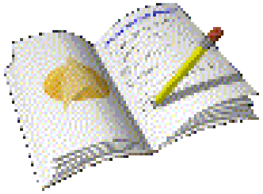
Even for ordinary r.v.s, the reliability of the 3σ rule is close to 90%.

Inferencing result from Chebyshev's inequality:

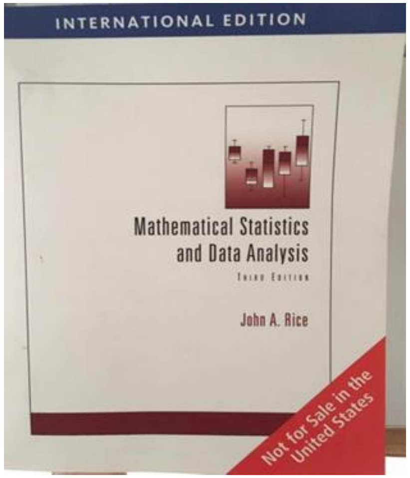
If $\sigma^2 = 0$, then $P\{X = \mu\} = 1$.

Summary
Expected values and Variances

$X\sim$	$E(X)$	$D(X)$
$X\sim p(\lambda)$	λ	λ
$X\sim b(n,p)$	np	$np(1-p)$
$X\sim U(a,b)$	$\dfrac{a+b}{2}$	$x=\dfrac{(b-a)^2}{12}$
$X\sim exp(1/\theta)$	θ	θ^2
$X\sim N(\mu,\sigma^2)$	μ	σ^2



Homework



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Supplementary Questions:

1. Suppose that X and Y are independent random variables. $E(X) = 3,\ E(Y) = 1,\ D(X) = 4,\ D(Y) = 9$. If $Z = 5X - 2Y + 15$, compute $E(Z),\ D(Z)$.

2. Suppose that $X_i (i = 1,2,3,4)$ are mutually independent to each other. $(X_i) = 2i,\ D(X_i) = 5 - i$. If $Z = 2X_1 - X_2 + 3X_3 - 0.5X_4$, compute $E(Z)$ and $D(Z)$.