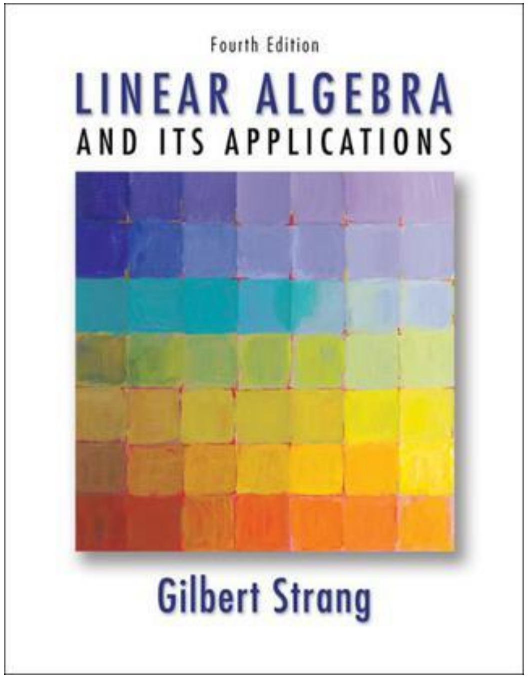
Linear Algebra



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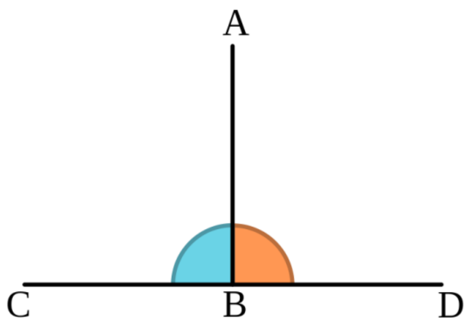
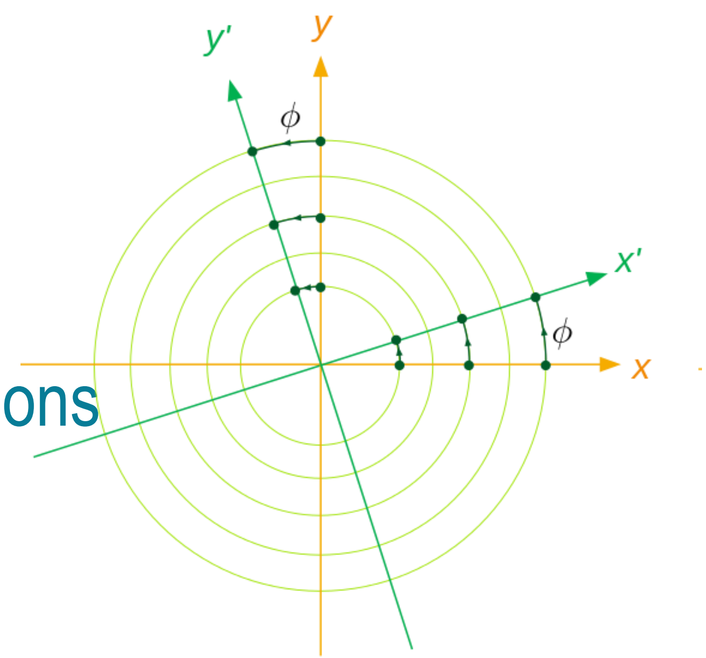
3 Orthogonality
（正交性）



3.2

COSINES AND PROJECTIONS ONTO LINES（余弦；向量往线上的投影）

- Cosines
- Projection onto a line
- Projections as linear transformations

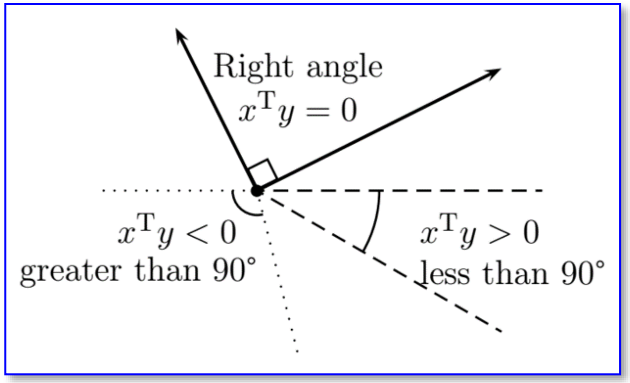


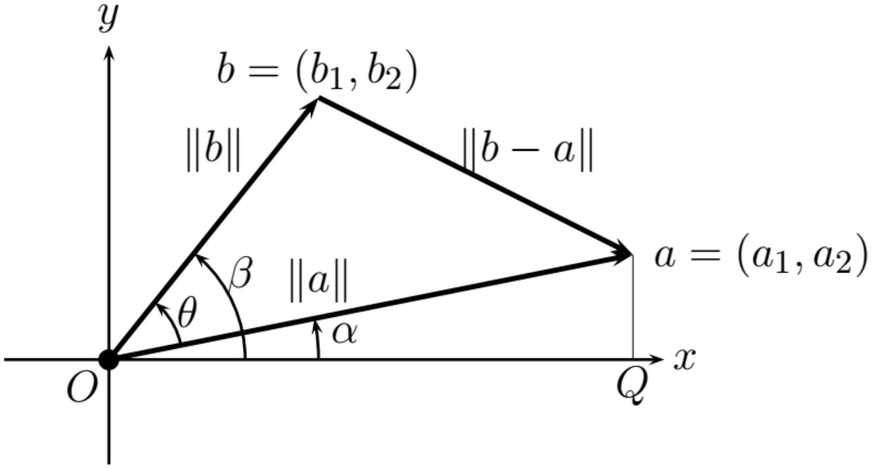
I. Cosines (余弦)

- If $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$, then $\boldsymbol{x}$, $\boldsymbol{y}$ are **orthogonal**, also called **perpendicular**.

The orthogonal case is the most important.

Now we allow inner products that are ***not zero***, and angles that are ***not right angles***.



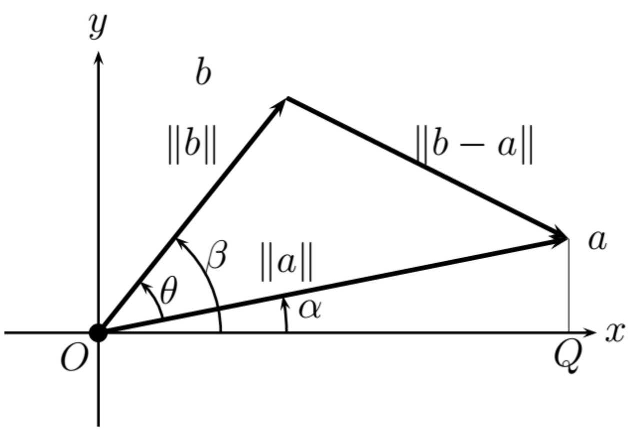


The cosine of the angle $\theta = \beta - \alpha$ using inner products.

$$\theta = \beta - \alpha$$

$$\cos\theta = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \frac{b_1}{\|\boldsymbol{b}\|}\frac{a_1}{\|\boldsymbol{a}\|} + \frac{b_2}{\|\boldsymbol{b}\|}\frac{a_2}{\|\boldsymbol{a}\|} = \frac{a_1b_1 + a_2b_2}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}.$$

- If $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} > 0$, their angle is less than 90°;
- If $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} < 0$, their angle is greater than 90°.



Law of Cosines

$$\|\boldsymbol{b}-\boldsymbol{a}\|^2 = \|\boldsymbol{b}\|^2 + \|\boldsymbol{a}\|^2 - 2\|\boldsymbol{b}\|\|\boldsymbol{a}\|\cos\theta$$

$$\Rightarrow (\boldsymbol{b}-\boldsymbol{a})^{\mathrm{T}}(\boldsymbol{b}-\boldsymbol{a}) = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} - 2\|\boldsymbol{b}\|\|\boldsymbol{a}\|\cos\theta$$

$$\Rightarrow \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - 2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} - 2\|\boldsymbol{b}\|\|\boldsymbol{a}\|\cos\theta$$

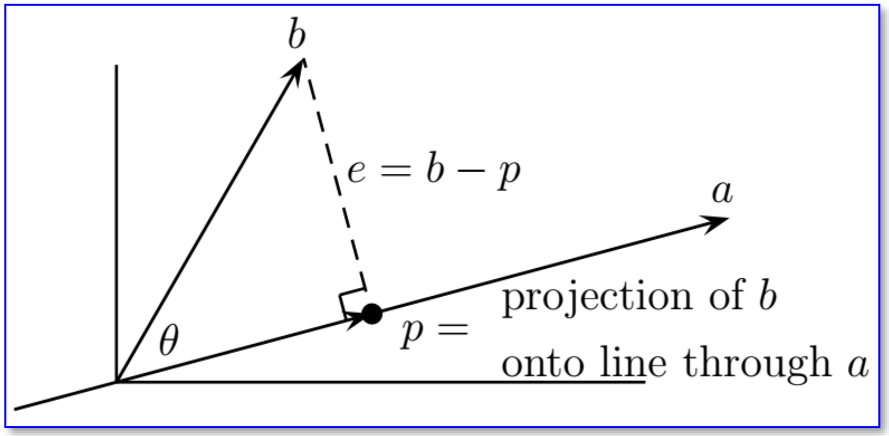
$$\Rightarrow \cos\theta = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}.$$

(the cosine of the angle between any *nonzero* vectors ***a*** and ***b***)
It holds in *n* dimensions.

We notice that, since $|\cos\theta| \le 1$, we have

$$\boxed{|\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}| \le \|\boldsymbol{a}\|\|\boldsymbol{b}\|.}$$ ***(Cauchy- Schwarz inequality)***

II. Projection onto a Line (往线上的投影)



Goal: find the distance from a point ***b*** to the line in the direction of the vector ***a***.

→ *find the projection* ***p***

(*The line connecting* ***b*** *to* ***p*** *is perpendicular to* ***a***)

Even though ***a*** and ***b*** are not orthogonal, the distance problem automatically brings in orthogonality.

Let $\boldsymbol{a}$ be a vector in a vector space V, and let $\mathrm{proj}_{\boldsymbol{a}}$ be the projection of the vectors of V onto the line in the direction of $\boldsymbol{a}$. Then

$$\mathrm{proj}_{\boldsymbol{a}} : \boldsymbol{b} \mapsto \boldsymbol{p} = \hat{x}\boldsymbol{a}$$

for some *scalar* $\hat{x}$, and the difference $\boldsymbol{b} - \hat{x}\boldsymbol{a}$ is perpendicular to the vector $\boldsymbol{a}$. Thus

$$0 = \boldsymbol{a}^{\mathrm{T}}(\boldsymbol{b} - \hat{x}\boldsymbol{a}) = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} - \hat{x}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a},$$

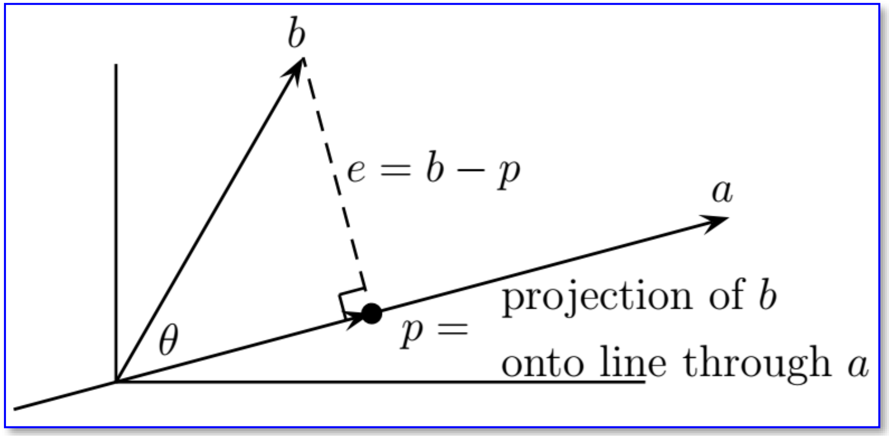
so that the scalar

$$\hat{x} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$$

We therefore have the following result.

Proposition (命题) *The projection* $\mathrm{proj}_{\boldsymbol{a}}$ *satisfies*

$$\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}.$$



Example 1 Project $\boldsymbol{b} = (1,2,3)^{\mathrm{T}}$ onto the line through $\boldsymbol{a} = (1,1,1)^{\mathrm{T}}$ to get :

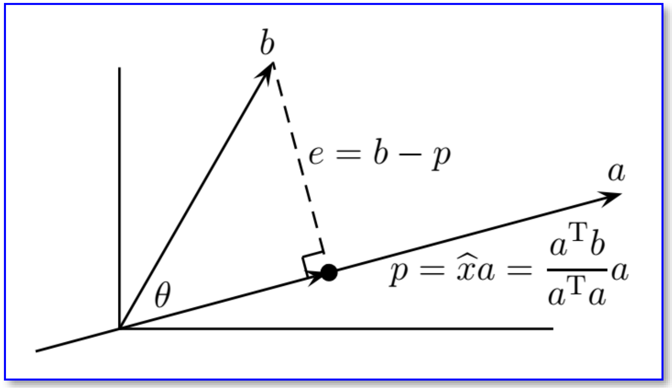
$$\hat{x} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{6}{3} = 2.$$

The projection is

$$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = (2,2,2)^{\mathrm{T}}.$$

The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ has

$$\cos\theta = \frac{\|\boldsymbol{p}\|}{\|\boldsymbol{b}\|} = \frac{\sqrt{12}}{\sqrt{14}}.$$



(*Cauchy- Schwarz inequality*)

$$\boxed{\left|\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}\right| \le \|\boldsymbol{a}\|\|\boldsymbol{b}\|.}$$

Second proof:

$$\|\boldsymbol{e}\|^2 = \|\boldsymbol{b}-\boldsymbol{p}\|^2 = \left\|\boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}\right\|^2 = \left(\boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}\right)^{\mathrm{T}}\left(\boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}\right)$$

$$= \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - 2\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} + \left(\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\right)^2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$$

$$= \frac{(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b})(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}) - (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b})^2}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} \ge 0$$

Cauchy-Schwarz inequality is equivalent to $|\cos\theta| \le 1$.

Therefore, $\boxed{\left|\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}\right| \le \|\boldsymbol{a}\|\|\boldsymbol{b}\|.}$

Equality holds if and only if ***b*** *is a multiple of* ***a***.

Triangle inequality: $\boxed{\|\boldsymbol{a}+\boldsymbol{b}\| \le \|\boldsymbol{a}\| + \|\boldsymbol{b}\|.}$

III. Projection as a Linear Transformation

(Projection Matrix of Rank 1: 秩为1的投影矩阵)

$$\mathrm{proj}_{\boldsymbol{a}} : \boldsymbol{b} \mapsto \boldsymbol{p} = \hat{x}\boldsymbol{a}\ .$$

Rewrite the projection $\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b})$:

$$\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \boxed{\frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}}\boldsymbol{b}.$$

Projection onto a line is carried out by a ***projection matrix*** $\boldsymbol{P}$:

$$\boldsymbol{P} = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$$

(a column times a row—a square matrix—divided by the number $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.)

$\boldsymbol{P}$ is a matrix of rank 1, and as a linear transformation, it transforms a vector $\boldsymbol{b}$ to its projection $\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \boldsymbol{P}\boldsymbol{b}$.

Theorem. *Let $\boldsymbol{a}$ be a nonzero vector of a vector space V, and let T be a linear transformation which transforms vector $\boldsymbol{b}$ to its projection onto the line in the direction of $\boldsymbol{a}$. Then the matrix of T is*

$$\boldsymbol{P} = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\|\boldsymbol{a}\|^2}.$$

Example 2 Let $\boldsymbol{a}=(1,1,1)^{\mathrm{T}}$.

Then the matrix that projects onto the line through $\boldsymbol{a}$ is

$$P=\frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}=\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{bmatrix}.$$

This matrix has two properties (typical of projections):

1. $\boldsymbol{P}$ is a symmetric matrix: $\boldsymbol{P}^{\mathrm{T}}=\boldsymbol{P}$.

2. Its square is itself: $\boldsymbol{P}^2=\boldsymbol{P}$.

The column space consists of the line through $\boldsymbol{a}=(1,1,1)^{\mathrm{T}}$.

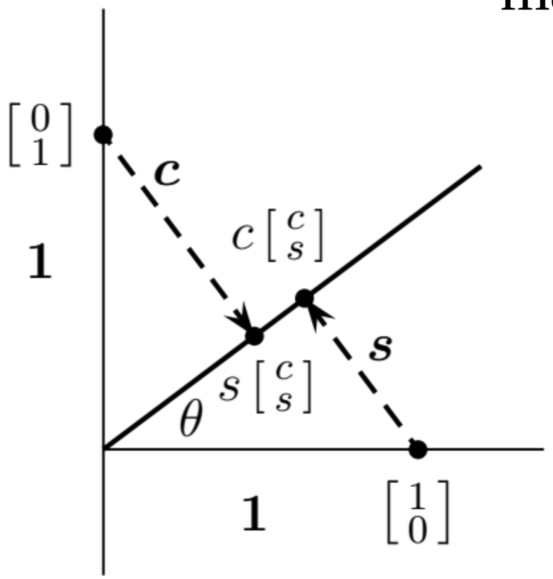
The nullspace consists of the plane perpendicular to $\boldsymbol{a}$.

The rank is $r=1$.

Remark: The nullspace should be orthogonal to the *row space*.

But because $\boldsymbol{P}$ is symmetric, its row and column spaces are the same.

Example 3 Project onto the "θ-direction" in the x-y plane. ($\mathbf{R}^2$)



Projection onto the θ-line

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

(c=cosθ, s=sinθ)

The line goes through $\boldsymbol{a} = (\cos\theta, \sin\theta)^{\mathrm{T}}$ and the matrix is symmetric with $\boldsymbol{P}^2 = \boldsymbol{P}$.

$$\boldsymbol{P} = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{\begin{bmatrix} c \\ s \end{bmatrix}\begin{bmatrix} c & s \end{bmatrix}}{\begin{bmatrix} c & s \end{bmatrix}\begin{bmatrix} c \\ s \end{bmatrix}} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$

Note:

$\boldsymbol{P}$ in any number of dimensions: $\boldsymbol{P} = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$.

We emphasize that it produces the projection $\boldsymbol{p}$:

To project $\boldsymbol{b}$ *onto* $\boldsymbol{a}$*, multiply by the projection matrix* $\boldsymbol{P}$: $\boldsymbol{p} = \boldsymbol{P}\boldsymbol{b}$.

Key words:

Cosine of the angle

Projection onto a Line

Projection as Linear Transformation: Projection matrix

Homework

See Blackboard