1. True or false. No need to justify

(1) If the row space equals the column space for the matrix A, then $A^T = A$. (False)

答案解析 反例，$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$可逆，因此其行向量和列向量均线性无关，因此

$$C(A) = C(A^T) = \mathbb{R}^2$$

显然$A \neq A^T$。

这个命题的逆命题成立，即若$A^T = A$，则$C(A) = C(A^T)$。

(2) If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent vectors, then $\mathbf{y}_1 = \mathbf{x}_1 - \mathbf{x}_2 + 2\mathbf{x}_3$, $\mathbf{y}_2 = 2\mathbf{x}_1 + \mathbf{x}_3$, $\mathbf{y}_3 = 4\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3$ are also linearly independent. (True)

答案解析 设

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

则有

$$(c_1 + 2c_2 + 4c_3)\mathbf{x}_1 + (-c_1 + c_3)\mathbf{x}_2 + (2c_1 + c_2 - 2c_3)\mathbf{x}_3 = \mathbf{0}$$

由$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$的线性无关性知

$$\begin{cases} c_1 + 2c_2 + 4c_3 = 0 \\ -c_1 + c_3 = 0 \\ 2c_1 + c_2 - 2c_3 = 0 \end{cases}$$

此线性方程组的系数矩阵

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 5 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$rank(A) = 3$故线性方程组只有零解，即

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

只有零解，故$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$线性无关。

(3) If $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s \in \mathbb{R}^n$are solutions to the linear system $A\mathbf{x} = \mathbf{0}$ and $\operatorname{rank}(A_{m\times n}) = n - s + 1$, then $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s$ are linearly dependent. (True)

答案解析 由$\operatorname{rank}(A_{m\times n}) = n - s + 1$知

$$\dim N(A) = \text{number of columns} - \operatorname{rank}(A) = n - (n - s + 1) = s - 1$$

$\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s$是$A\mathbf{x} = \mathbf{0}$的解，则这$s$个向量在矩阵$A$的零空间$N(A)$中，但$N(A)$的维数为$s-1$，故$s-1$维子空间$N(A)$中的$s(>$ 维数 $s-1)$个向量必然线性相关。

参考 2.3 节的定理：

Corollary Let V be a space of dimension $n > 0$. Then

(a) any set of n linearly independent vectors spans V(V**中任意**n**个线性无关的向量都张成**V);

(b) any n vectors that span V are linearly independent(**任何张成**V**的**n**个向量是线性无关的**);

(c) any set of less than n vectors is not a spanning set(**没有少于**n**个的线性无关向量构成的子集可以张成**V);

(d) any set of more than n vectors is linearly dependent(V**中任意含超过**n**个向量的向量组是线性相关的**);

(e) a proper subspace(**真子空间**) of V has dimension less than n.

(4) If $A\mathbf{x}=\mathbf{0}$ has infinite many solutions, then $A\mathbf{x}=\mathbf{b}(\mathbf{b}\neq\mathbf{0})$ has infinite many solutions as well.
(False)
答案解析 假设$A\in\mathbb{R}^{m\times n}$，$A\mathbf{x}=\mathbf{0}$ 有无穷多的解，只能说明$\text{rank}(A)<n$，如果$A\mathbf{x}=\mathbf{b}$在有解的情况下，也会有无穷多的解，但还有种可能是$A\mathbf{x}=\mathbf{b}$无解，例如

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix},\mathbf{b}=\begin{bmatrix}1\\1\end{bmatrix}$$

(5) If U is the reduced row echelon form of A, then A and U have the same column space (False).
答案解析
例如

$$A=\begin{bmatrix}1&1\\1&1\end{bmatrix}\to U=\begin{bmatrix}1&1\\0&0\end{bmatrix}$$

显然A和U的列空间不一样。
我们可以说的是A和U的行空间一样。

(6) If $\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\cdots,\boldsymbol{\alpha_s}\in\mathbb{R}^n$and A is an $n\times n$ matrix, then $\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\cdots,\boldsymbol{\alpha_s}$ are linearly dependent if and only if $A\boldsymbol{\alpha_1},A\boldsymbol{\alpha_2},\cdots,A\boldsymbol{\alpha_s}$ are linearly dependent. (False)
答案解析 这个命题的其中一个方向是对的，即$\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\cdots,\boldsymbol{\alpha_s}$线性相关，则一定存在不全为 0 的常数$c_1,c_2,\cdots,c_s$使得

$$c_1\boldsymbol{\alpha_1}+c_2\boldsymbol{\alpha_2}+\cdots+c_s\boldsymbol{\alpha_s}=\mathbf{0}$$

上述等式两边同时左乘上矩阵A也成立，即

$$c_1A\boldsymbol{\alpha_1}+c_2A\boldsymbol{\alpha_2}+\cdots+c_sA\boldsymbol{\alpha_s}=\mathbf{0}$$

$c_1,c_2,\cdots,c_s$不全为 0，则$A\boldsymbol{\alpha_1},A\boldsymbol{\alpha_2},\cdots,A\boldsymbol{\alpha_s}$线性相关。
但反过来不一定成立，即$A\boldsymbol{\alpha_1},A\boldsymbol{\alpha_2},\cdots,A\boldsymbol{\alpha_s}$线性相关，$\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\cdots,\boldsymbol{\alpha_s}$不一定线性相关，比如取$A=0$。
这个结论加上条件矩阵A可逆，则是正确的，即：
If $\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\cdots,\boldsymbol{\alpha_s}\in\mathbb{R}^n$and A is an **invertible** $n\times n$ matrix, then $\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\cdots,\boldsymbol{\alpha_s}$ are linearly dependent if and only if $A\boldsymbol{\alpha_1},A\boldsymbol{\alpha_2},\cdots,A\boldsymbol{\alpha_s}$ are linearly dependent.

(7) If $A_{m\times n}$ has full row rank, then $A\mathbf{x}=\mathbf{b}$ is always consistent. (True)
答案解析 A行满秩，则$\text{rank}(A)=m$，因此

$$\dim\boldsymbol{C}(A)=m$$

则由$\boldsymbol{C}(A)\subset\mathbb{R}^m$有

$$\boldsymbol{C}(A)=\mathbb{R}^m$$

$A\mathbf{x}=\mathbf{b}$ consistent 等价于$A\mathbf{x}=\mathbf{b}$有解，也等价于$\mathbf{b}\in\boldsymbol{C}(A)$，因此对于任意的

$$\mathbf{b}\in\mathbb{R}^m=\boldsymbol{C}(A),A\mathbf{x}=\mathbf{b}\text{总是有解。}$$

(8) If P is an invertible matrix, then PA and A must have the same column space. (False)
答案解析 P可逆，P可以表示成若干个初等矩阵的乘积，则 意味着

$$A\xrightarrow{\text{若干初等行变换}}PA$$

因此PA与矩阵A有相同的行空间，但一般而言列空间不一定相同。

(9) If S and T are subspaces of a vector space V, then ,$S\cup T$ is a subspace of V. (False)
答案解析 反例 $S=\{(x,0)|x\in\mathbb{R}\}$, $T=\{(0,y)|y\in\mathbb{R}\}$
$S\cup T$为x轴并上y轴，两条过原点的直线并在一起并不是$\mathbb{R}^2$的子空间。

(10)If S and T are subspaces of a vector space V, then $S \cap T$ is a subspace of V. (True)

答案解析

(i) S is a subspace, $\mathbf{0} \in S$, T is a subspace, $\mathbf{0} \in T$, therefore $\mathbf{0} \in S \cap T$.

(ii) $\forall \mathbf{u}, \mathbf{v} \in S \cap T, \forall c \in \mathbb{R}$, then

$$\mathbf{u} + \mathbf{v} \in S, c\mathbf{u} \in S \text{ since } S \text{ is a subspace of } V,$$
$$\mathbf{u} + \mathbf{v} \in T, c\mathbf{u} \in T \text{ since } T \text{ is a subspace of } V,$$

Thus

$$\mathbf{u} + \mathbf{v} \in S \cap T, c\mathbf{u} \in S \cap T, \forall \mathbf{u}, \mathbf{v} \in S \cap T, \forall c \in \mathbb{R},$$

which implies $S \cap T$ is a subspace of V.

(11)If A is an $m \times n$ matrix, then A and A^T have the same nullity, i.e. $\dim\big(N(A)\big) = \dim\big(N(A^T)\big)$.

(False)

答案解析

$$\dim\big(N(A)\big) = n - \operatorname{rank}(A)$$
$$\dim\big(N(A^T)\big) = m - \operatorname{rank}(A)$$

若$m \neq n$，即A不是方阵时，此命题不对。

但$m = n$，即A是方阵时，有 $\dim\big(N(A)\big) = \dim\big(N(A^T)\big)$。

(12)If the rows of a matrix are linearly dependent, then the columns are also linearly dependent. (False)

答案解析 假设A是$m \times n$矩阵,

A的行向量线性无关$\Leftrightarrow$ A的行向量是A的行空间的一组基$\Leftrightarrow$ $\dim\big(C(A^T)\big) = m$

A的列向量线性无关$\Leftrightarrow$ A的列向量是A的列空间的一组基$\Leftrightarrow$ $\dim\big(C(A)\big) = n$

只有在$m = n$，即A是方阵的情况下，上述结论正确。

(13)If A is an $m \times n$ matrix and B is an $n \times m$ matrix, where $m > n$, then $AB\mathbf{x} = \mathbf{0}$ must have non-zero solutions. (True)

答案解析 首先注意矩阵B的列数为m，

$$m > n \Longrightarrow \operatorname{rank}(B) \leq n < m = B\text{的列数} \Longrightarrow B\mathbf{x} = \mathbf{0}\text{一定有非零解}$$

其次，

$$B\mathbf{x} = \mathbf{0} \Longrightarrow AB\mathbf{x} = \mathbf{0}$$

即$B\mathbf{x} = \mathbf{0}$的解一定是$AB\mathbf{x} = \mathbf{0}$的解。

2. Fill in the blanks.

(1) If $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \\ -3 & 0 & -6 & 3 \end{bmatrix}$, then $\operatorname{rank}(A) = 1$ and $A^n = 2^{n-1}\begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \\ -3 & 0 & -6 & 3 \end{bmatrix}$.

答案解析 注意矩阵A的 2 至 4 行都是第一行（非零向量）的倍数，因此A的秩为 1，则

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \\ -3 & 0 & -6 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}$$

$$A^n=\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\begin{bmatrix}1&0&2&-1\end{bmatrix}\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\begin{bmatrix}1&0&2&-1\end{bmatrix}\cdots\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\begin{bmatrix}1&0&2&-1\end{bmatrix}$$

$$=\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\left(\begin{bmatrix}1&0&2&-1\end{bmatrix}\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\right)\left(\begin{bmatrix}1&0&2&-1\end{bmatrix}\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\right)\cdots\left(\begin{bmatrix}1&0&2&-1\end{bmatrix}\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\right)\begin{bmatrix}1&0&2&-1\end{bmatrix}$$

$$=2^{n-1}\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}\begin{bmatrix}1&0&2&-1\end{bmatrix}=2^{n-1}A$$

(2) $\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}^8=$ ________________.

答案解析 注意矩阵$\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\cos\frac{\pi}{3}&-\sin\frac{\pi}{3}\\\sin\frac{\pi}{3}&\cos\frac{\pi}{3}\end{bmatrix}$是逆时针旋转$\frac{\pi}{3}$的旋转变换所对应的矩阵，根据映射的复合，$\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}^8$是逆时针旋转 8 次$\frac{\pi}{3}$，即逆时针旋转$\frac{8\pi}{3}$的旋转变换所对应的矩阵，因此

$$\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}^8=\begin{bmatrix}\cos\frac{8\pi}{3}&-\sin\frac{8\pi}{3}\\\sin\frac{8\pi}{3}&\cos\frac{8\pi}{3}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix}$$

(3) If A is a 5 by 4 matrix with rank(A) = 2, $x_1=[1\quad 2\quad 0\quad 1]^T$, $x_2=[2\quad 1\quad 1\quad 3]^T$ are solutions to Ax = b , $x_3=[1\quad 0\quad 1\quad 0]^T$ is a solution to Ax = 0, then the complete solutions to Ax = b are $[1\quad 2\quad 0\quad 1]^T+k_1[-1\quad 1\quad -1\quad -2]^T+k_2[1\quad 0\quad 1\quad 0]^T, k_1,k_2\in\mathbb{R}$.

答案解析 $\operatorname{rank}(A)=2$，则

$$\dim N(A)=A\text{的列数}-A\text{的秩}=4-2=2$$

故$A\mathbf{x}=\mathbf{0}$**的任意两个线性无关的解向量**都是$N(A)$的一组基。

由题意，$\mathbf{x}_1=[1\quad 2\quad 0\quad 1]^{\mathrm{T}}$，$\mathbf{x}_2=[2\quad 1\quad 1\quad 3]^{\mathrm{T}}$ 是 $A\mathbf{x}=\mathbf{b}$的两个解，因此$\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2=[-1\quad 1\quad -1\quad -2]^{\mathrm{T}}$是齐次线性方程组$A\mathbf{x}=\mathbf{0}$的解，$\mathbf{x}_3=[1\quad 0\quad 1\quad 0]^{\mathrm{T}}$也是$A\mathbf{x}=\mathbf{0}$的解，且$\mathbf{x}$与$\mathbf{x}_3$线性无关，故$A\mathbf{x}=\mathbf{0}$的一般解可以表示为

$$k_1\mathbf{x}+k_2\mathbf{x}_3=k_1[-1\quad 1\quad -1\quad -2]^{\mathrm{T}}+k_2[1\quad 0\quad 1\quad 0]^{\mathrm{T}}$$

$A\mathbf{x}=\mathbf{b}$的通解可以表示为

$$[1\quad 2\quad 0\quad 1]^T+k_1[-1\quad 1\quad -1\quad -2]^T+k_2[1\quad 0\quad 1\quad 0]^T, k_1,k_2\in\mathbb{R}$$

(4) If $\mathbf{x}_1 = [1 \quad 1 \quad 2]^T, \mathbf{x}_2 = [2 \quad -1 \quad 1]^T \in \mathbb{R}^3$ are two solutions to the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, $\text{rank}(A) = 2$, then the complete solutions to $A\mathbf{x} = \mathbf{b}$ are $[1 \quad 1 \quad 2]^T + k[-1 \quad 2 \quad 1]^T, \quad (k \in \mathbb{R})$.

答案解析 参考第(3)题的解析，原理一样，此时注意矩阵A的列数一定为3。

(5) If $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ are two different solutions($\mathbf{x}_1 \neq \mathbf{x}_2$) to the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}(\mathbf{b} \neq \mathbf{0})$, $\text{rank}(A) = n - 1$, then the complete solutions to $A\mathbf{x} = \mathbf{b}$ are $\mathbf{x}_1 + \text{k}(\mathbf{x}_1 - \mathbf{x}_2)$, $(k \in \mathbb{R})$.

答案解析 参考第(3)题的解析，原理一样，此时注意矩阵A的列数一定为n.

(6) Let T be a linear transformation of $\mathbb{R}^2$ such that $T: (3,2)^{\mathrm{T}} \mapsto (2,0)^{\mathrm{T}}, (-4,3)^{\mathrm{T}} \mapsto (2,2)^{\mathrm{T}}$, then the matrix A so that $T(\mathbf{x}) = A\mathbf{x}$ is $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{17} \begin{bmatrix} 2 & 14 \\ -4 & 6 \end{bmatrix}$ and $T(x_1, x_2) = \frac{1}{17} \begin{bmatrix} 2x_1 + 14x\_2 \\ -4x_1 + 6x_1 \end{bmatrix}$.

答案解析 利用映射的复合，考虑映射$L_1(\mathbf{x}) = B\mathbf{x}$，$L_2(\mathbf{x}) = C\mathbf{x}$，

$$\begin{array}{c} \mathbb{R}^2 \xrightarrow{L_1} \mathbb{R}^2 \xrightarrow{L_2} \mathbb{R}^2 \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -4 \\ 3 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ \mathbf{x} \mapsto B\mathbf{x} \mapsto CB\mathbf{x} \end{array}$$

显然$B = \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}^{-1}, C = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$

则T可视作L_1和L_2的复合映射，设且有

$$T(\mathbf{x}) = L_2 L_1(\mathbf{x}) = CB\mathbf{x}$$

此处需注意矩阵乘积的顺序

则有

$$A = CB = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{17} \begin{bmatrix} 2 & 14 \\ -4 & 6 \end{bmatrix}$$

$$T(x_1, x_2) = \frac{1}{17} \begin{bmatrix} 2 & 14 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 2x_1 + 14x_2 \\ -4x_1 + 6x_1 \end{bmatrix}$$

(7) If A is a 4 by 3 matrix with $\text{rank}(A) = 2$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, then $\text{rank(AB)} = \underline{\ 2\ }$.

答案解析 首先我们有若P, Q可逆，则

$$\text{rank(A)} = \text{rank(PA)} = \text{rank(AQ)} = \text{rank(PAQ)}$$

因为P可逆，P可以表示成若干个初等矩阵的乘积，则 意味着

$$\text{A} \xrightarrow{\text{若干初等行变换}} \text{PA}$$

因此PA与矩阵A有相同的行空间，再由矩阵的秩等于行空间的维数因此$\text{rank(A)} = \text{rank(PA)}$。

Q可逆，则Q^T可逆，因此

$$\text{rank}(A^T) = \text{rank}(Q^T A^T) = \text{rank}((AQ)^T)$$

再由$\text{rank}(A^T) = \text{rank}(A)$，$\text{rank}((AQ)^T) = \text{rank}(AQ)$得出

$$\text{rank(A)} = \text{rank(AQ)}$$

$B=\begin{bmatrix}1&0&2\\0&1&2\\0&0&3\end{bmatrix}$可逆，故而$\mathrm{rank}(AB)=\mathrm{rank}(A)=2$

(8) If $\mathbf{x}_1=[1\quad 2\quad 3]^T, \mathbf{x}_2=[2\quad 1\quad 6]^T, \mathbf{x}_3=[3\quad 4\quad a]^T$ are linearly dependent, then $a=$ 9 .

答案解析 $\mathbf{x}_1=[1\quad 2\quad 3]^T, \mathbf{x}_2=[2\quad 1\quad 6]^T, \mathbf{x}_3=[3\quad 4\quad a]^T$ 线性相关，当且仅当矩阵 $A=[\mathbf{x}_1\quad \mathbf{x}_2\quad \mathbf{x}_3]$的秩小于 3。

$$A=[\mathbf{x}_1\quad \mathbf{x}_2\quad \mathbf{x}_3]=\begin{bmatrix}1&2&3\\2&1&4\\3&6&a\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&-3&-2\\0&0&a-9\end{bmatrix}$$

当$a-9=0$，即$a=9$时，$\mathrm{rank}(A)=2<3.$

(9) Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ and $T(\mathbf{x})=(x_2-x_1,x_3-x_2)$, then the kernel of T is $\{[a\quad a\quad a]^T|a\in\mathbb{R}\}$.

答案解析 kernel of $T=\{\mathbf{x}|T(\mathbf{x})=\mathbf{0}\}$

即$T(\mathbf{x})=(x_2-x_1,x_3-x_2)=(0,0)$解得$x_2=x_1,x_3=x_1$，故

$$\text{kernel of } T=\{\mathbf{x}|T(\mathbf{x})=\mathbf{0}\}=\left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\middle| x_1=x_2=x_3\right\}=\left\{\begin{bmatrix}x_1\\x_1\\x_1\end{bmatrix}\middle| x_1\in\mathbb{R}\right\}=\left\{\begin{bmatrix}a\\a\\a\end{bmatrix}\middle| a\in\mathbb{R}\right\}$$

3. Which of the following subsets are actually subspaces? If the subset is a subspace, find its basis and dimension. If not, explain why.

(1) All skew-symmetric 3 by 3 matrices ($A^T=-A$).

答案解析 反对称矩阵可以表示为

$$\begin{bmatrix}0&-a&-b\\a&0&-c\\b&c&0\end{bmatrix}=a\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}+c\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$$

A basis: $\left\{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\right\}$

基包含三个向量，dimension = 3

(2) All symmetric 3 by 3 matrices.

答案解析 对称矩阵可以表示为

$$\begin{bmatrix}a&d&e\\d&b&f\\e&f&c\end{bmatrix}=a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+c\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$
$$+d\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+e\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+f\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

A basis: $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\}$

基包含 6 个向量，dimension = 6

(3) The set of singular 3 by 3 matrices.

答案解析 不是子空间，因为对加法不封闭，例如

$$\begin{bmatrix}1&&\\&1&\\&&0\end{bmatrix}+\begin{bmatrix}0&&\\&0&\\&&1\end{bmatrix}=\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$$

(4) The set of nonsingular 3 by 3 matrices.

答案解析 不是子空间，因为 0 矩阵不可逆。

(5) $\{(x, y, z, w) \in \mathbb{R}^4 | x + 2y - 3z - w = 0\}$.

答案解析

$$\{(x, y, z, w) \in \mathbb{R}^4 | x + 2y - 3z - w = 0\} = \{(-2y + 3z + w, y, z, w) | y, z, w \in \mathbb{R}\}$$
$$= \{y(-2,1,0,0) + z(3,0,1,0) + w(1,0,0,1) | y, z, w \in \mathbb{R}\}$$

A basis: $\{(-2,1,0,0), (3,0,1,0), (1,0,0,1)\}$

基包含三个向量，dimension = 3

4. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & a & 0 \\ 1 & 3 & 1 & a \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(1) If $a = 1$, find the dimension and a basis for the four fundamental subspaces of A.

(2) If $a = 1$, under what condition on $\mathbf{b}$ is the system $A\mathbf{x} = \mathbf{b}$ solvable? Find all the solutions when $\mathbf{b} = [1 \quad 2 \quad 3]^T$.

(3) If $a = 2$, find a matrix B so that $AB = I$.

解

$$[A, \mathbf{b}] = \begin{bmatrix} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & a & 0 & b_2 \\ 1 & 3 & 1 & a & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & a & 0 & b_2 \\ 0 & 1 & 1 & a-1 & b_3 - b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & a & 0 & b_2 \\ 0 & 0 & 1-a & a-1 & b_3 - b_1 - b_2 \end{bmatrix}$$

(1) If $a = 1$, $A \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Column space: dimension= 2, a basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right\}$

Row space: dimension = 2, a basis $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

Nullspace: dimension = 2, a basis $\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

Left nullspace: dimension = 1, a basis $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$ (solve $A^T\mathbf{x} = \mathbf{0}$)

(2) If $a = 1$, when $b_3 - b_1 - b_2 = 0$ the system $A\mathbf{x} = \mathbf{b}$ is solvable. When $\mathbf{b} = [1 \quad 2 \quad 3]^T$,

$$[A, \mathbf{b}] = \begin{bmatrix} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 0 & b_2 \\ 1 & 3 & 1 & 1 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solutions to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 + 2x_3 - x_4 \\ 2 - x_3 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x_3, x_4 \in \mathbb{R}$$

(3) If $a = 2$,

$$A \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

A的前三列线性无关，则A有右逆，由于$A = [A_1\ X]$，其中$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

$$A_1^{-1} = \begin{bmatrix} 5 & 2 & -4 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

A的其中一个右逆是

$$B = \begin{bmatrix} A_1^{-1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -4 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Let $\alpha = (4,3,3,1)^{\mathrm{T}}, \alpha_1 = (1,2,3,4)^{\mathrm{T}}, \alpha_2 = (0,1,2,3)^{\mathrm{T}}, \alpha_3 = (0,0,1,2)^{\mathrm{T}}, \alpha_4 = (0,0,0,1)^{\mathrm{T}}$.
   (1) Can α be represented by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$? If so, write the linear combination.
   (2) Can α_4 be represented by $\alpha_1, \alpha_2, \alpha_3$? If so, write the linear combination.
   答案解析 令

$$A = [\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \boldsymbol{\alpha}_3 \quad \boldsymbol{\alpha}_4 \quad \boldsymbol{\alpha}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 3 \\ 4 & 3 & 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 2 & 1 & 0 & -9 \\ 0 & 3 & 2 & 1 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

   (1) $\boldsymbol{\alpha} = 4\boldsymbol{\alpha}_1 - 5\,\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 - 2\,\boldsymbol{\alpha}_4$
   (2) $\boldsymbol{\alpha}_4$ cannot be a linear combination of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$.

6. Let $\alpha_1 = (1,0,2,1)^{\mathrm{T}}, \alpha_2 = (1,2,0,1)^{\mathrm{T}}, \alpha_3 = (2,1,3,0)^{\mathrm{T}}, \alpha_4 = (2,5,-1,4)^{\mathrm{T}}, \alpha_5 = (1,-1,3,-1)^{\mathrm{T}}$, and V=Span$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Choose a basis of V from the set of vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, and use the basis to represent the other vectors.

   答案解析 令$A = [\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \boldsymbol{\alpha}_3 \quad \boldsymbol{\alpha}_4 \quad \boldsymbol{\alpha}] = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & 1 & 5 & -1 \\ 2 & 0 & 3 & -1 & 3 \\ 1 & 1 & 0 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   A basis for V from the set of vectors $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \alpha_4, \alpha_5\}$: $\{\alpha_1, \alpha_2, \alpha_3\}$

$$\alpha_4 = \boldsymbol{\alpha}_1 + 3\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3$$
$$\alpha_5 = -\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3$$

7. Let

$$\mathrm{A} = \begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & -3 \end{bmatrix}$$

   (1) Find all the solutions to $\mathrm{A}\mathbf{x} = \mathbf{0}$.
   (2) Find all the matrices B satisfying $AB = I$.
   答案解析 (1)

$$A=\begin{bmatrix}1&-2&3&-4\\0&1&-1&1\\1&2&0&-3\end{bmatrix}\to\begin{bmatrix}1&-2&3&-4\\0&1&-1&1\\0&4&-3&-1\end{bmatrix}\to\begin{bmatrix}1&-2&3&-4\\0&1&-1&1\\0&0&1&-5\end{bmatrix}$$

The solutions to $A\mathbf{x}=\mathbf{0}$：

$$k[-1\quad 2\quad 3\quad 1]^T, k\in\mathbb{R}$$

(2)要求矩阵A的所有右逆，先求一个右逆，利用A的前三列线性无关，令

$$A_1=\begin{bmatrix}1&-2&3\\0&1&-1\\1&2&0\end{bmatrix}$$

利用 Gauss-Jordan 法求出

$$A_1^{-1}=\begin{bmatrix}2&6&-1\\-1&-3&1\\-1&-4&1\end{bmatrix}$$

令$B_p=\begin{bmatrix}A_1^{-1}\\\mathbf{0}\end{bmatrix}=\begin{bmatrix}2&6&-1\\-1&-3&1\\-1&-4&1\\0&0&0\end{bmatrix}$，则$AB_p=I$。

注意若$AB_1=I, AB_2=I$，则$A(B_1-B_2)=0$，因此B_1-B_2的列向量均为$A\mathbf{x}=\mathbf{0}$的解，因此所有A的右逆有如下形式：

$$B_p+[\alpha_1\quad\alpha_2\quad\alpha_3]$$

其中$\alpha_1,\alpha_2,\alpha_3$为$A\mathbf{x}=\mathbf{0}$的解，故

$$B=\begin{bmatrix}2&6&-1\\-1&-3&1\\-1&-4&1\\0&0&0\end{bmatrix}+\begin{bmatrix}-x&-y&-z\\2x&2y&2z\\3x&3y&3z\\x&y&z\end{bmatrix}=\begin{bmatrix}2-x&6-y&-1-z\\-1+2x&-3+2y&1+2z\\-1+3x&-4+3y&1+3z\\x&y&z\end{bmatrix}$$

8. If $A_{m\times n}B_{n\times s}=0$,then

$$\mathrm{rank}(A)+\mathrm{rank}(B)\le n$$

证明: $A_{m\times n}B_{n\times s}=0\Longrightarrow A[\alpha_1\quad\alpha_2\quad\cdots\quad\alpha_s]=\mathbf{0}\Longrightarrow A\alpha_1=\mathbf{0},A\alpha_2=\mathbf{0},\cdots,A\alpha_s=\mathbf{0}.$

即B的列向量在A的零空间$\boldsymbol{N}(A)$中，因此B的列空间是A的零空间$\boldsymbol{N}(A)$的子空间，故而

$$\mathrm{rank}(B)=\dim\boldsymbol{C}(B)\le\dim\boldsymbol{N}(A)=n-\mathrm{rank}(A)$$
$$\mathrm{rank}(A)+\mathrm{rank}(B)\le n$$

9. Let B be a square matrix of order n and C be a $n\times s$ matrix with $\mathrm{rank}(C)=n$. Show that if $BC=0$ then $B=0$.

答案解析 先证第 8 题，再由$BC=0$和$\mathrm{rank}(C)=n$得出

$$\mathrm{rank}(B)\le n-\mathrm{rank}(C)=n-n=0$$

即矩阵B的秩为 0，因此$B=0$。

10. Suppose $A_{s\times n}B_{n\times r}=A_{s\times n}C_{n\times r}$, show that if $\mathrm{rank}(A)=n$ then $B=C$.

答案解析 先证第 8 题，再由$A_{s\times n}B_{n\times r}=A_{s\times n}C_{n\times r}\Longrightarrow A(B-C)=0$和$\mathrm{rank}(A)=n$ 得出

$$\mathrm{rank}(B-C)\le n-\mathrm{rank}(A)=n-n=0$$

因此$B=C$。

11. Let B be a square matrix of order n and C be a $n\times s$ matrix with $\mathrm{rank}(C)=n$. Show that if $BC=C$ then $B=I$.

答案解析 先证第 8 题，再由$BC=C \Longrightarrow (B-I)C=0$和$\operatorname{rank}(C)=n$ 得出

$$\operatorname{rank}(B-I) \le n-\operatorname{rank}(C)=n-n=0$$

因此$B=I$。

12. Suppose $\operatorname{rank}(A_{n\times n})=r$, show that we can find a square matrix $B_{n\times n}$ with rank $n-r$ so that $BA=0$.

参考答案： 由$\operatorname{rank}(A_{n\times n})=r$知$A^T\mathbf{x}=\mathbf{0}$的解空间，即$\boldsymbol{N}(A^T)$，的维数为$n-r$。设$\boldsymbol{N}(A^T)$的一组基为

$$\alpha_1,\alpha_2,\cdots,\alpha_{n-r}$$

则$\alpha_1,\alpha_2,\cdots,\alpha_{n-r}$线性无关，令

$$B^T=[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-r} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}]$$

显然$\operatorname{rank}(B)=\operatorname{rank}(B^T)=n-r$

且有

$$A^TB^T=A^T[\alpha_1 \quad \alpha_2\cdots \quad \alpha_{n-r} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\,]=[A^T\alpha_1 \quad A^T\alpha_2\cdots \quad A^T\alpha_{n-r} \quad A^T\mathbf{0} \quad \cdots \quad A^T\mathbf{0}\,]=0$$

故而$A^TB^T=(BA)^T=0$, 因此

$$BA=0$$

13. (1) Show that $\operatorname{rank}(AB)\le \min\{\operatorname{rank}(A),\operatorname{rank}(B)\}$, where A is an $m\times n$ matrix, and B is an $n\times s$ matrix.

(2) Let A be an $m\times n$ matrix, and $m<n$. Prove that the homogenous system of linear equations $(A^TA)x=0$ has <u>nonzero</u> solutions.

参考答案： (1) 令$B=[\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \cdots \quad \boldsymbol{\beta}_s]$

$$AB=A[\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \cdots \quad \boldsymbol{\beta}_s]=[A\boldsymbol{\beta}_1 \quad A\boldsymbol{\beta}_2 \quad \cdots \quad A\boldsymbol{\beta}_s]$$

说明AB的列向量$A\boldsymbol{\beta}_1,A\boldsymbol{\beta}_2,\cdots,A\boldsymbol{\beta}_s$是$A$的列向量的线性组合，故而

$$A\boldsymbol{\beta}_1\in\boldsymbol{C}(A),A\boldsymbol{\beta}_2\in\boldsymbol{C}(A),,\cdots,A\boldsymbol{\beta}_s\in\boldsymbol{C}(A)$$
$$\Longrightarrow \boldsymbol{C}(AB)\subset\boldsymbol{C}(A)$$

因此

$$\operatorname{rank}(AB)=\dim\big(\boldsymbol{C}(AB)\big)\le\dim\boldsymbol{C}(A)=\operatorname{rank}(A)$$

类似的，令$A=\begin{bmatrix}\boldsymbol{\alpha}_1\\\boldsymbol{\alpha}_2\\\vdots\\\boldsymbol{\alpha}_m\end{bmatrix}$，则

$$AB=\begin{bmatrix}\boldsymbol{\alpha}_1\\\boldsymbol{\alpha}_2\\\vdots\\\boldsymbol{\alpha}_m\end{bmatrix}B=\begin{bmatrix}\boldsymbol{\alpha}_1B\\\boldsymbol{\alpha}_2B\\\vdots\\\boldsymbol{\alpha}_mB\end{bmatrix}$$

说明AB的行向量是B的行向量的线性组合，故而

$$\operatorname{rank}(AB)=\dim\big(\boldsymbol{C}((AB)^T)\big)\le\dim\boldsymbol{C}(B^T)=\operatorname{rank}(B)$$

(2) 由第一问$\operatorname{rank}(A^TA)\le\min\{\operatorname{rank}(A),\operatorname{rank}(B)\}\le m<n$，故$A^TA$不是满秩矩阵，$(A^TA)\mathbf{x}=\mathbf{0}$一定有非零解。

14. Prove that

(1) $\operatorname{rank}(A+B)\le\operatorname{rank}(A)+\operatorname{rank}(B)$, where A and B are matrices of the same size.

(2) Suppose that $A=aa^T+bb^T$, where a,b are n-dimensional vectors. Prove that: (a) $\operatorname{rank}(A)\le 2$; and (b) $\operatorname{rank}(A)\le 1$ when a,b are linearly dependent.

参考答案：(1) 设A,B均是$m\times n$矩阵，$\operatorname{rank}(A)=p,\operatorname{rank}(B)=q$，将$A,B$按列分快为

$$A=(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_n),B=(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_n)$$

于是

$$A+B=(\boldsymbol{\alpha}_1+\boldsymbol{\beta}_1,\boldsymbol{\alpha}_2+\boldsymbol{\beta}_2,\cdots,\boldsymbol{\alpha}_n+\boldsymbol{\beta}_n)$$

不妨设$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p$为$A$的列空间的一组基，$\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_q$为$B$的列空间的一组基, 则显然$A+B$的列向量可由$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_q$线性表出，令矩阵$D=\left(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_p,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_q\right)$，其中矩阵$D$包含$p+q$列，我们有

$$\boldsymbol{C}(A+B)\subset\boldsymbol{C}(D)$$

故

$$\operatorname{rank}(A+B)=\dim\left(\boldsymbol{C}(A+B)\right)\le\dim\left(\boldsymbol{C}(D)\right)\le p+q=\operatorname{rank}(A)+\operatorname{rank}(B)$$

(2) 由第(1)问，

$$\operatorname{rank}(A)=\operatorname{rank}(aa^T+bb^T)\le\operatorname{rank}(aa^T)+\operatorname{rank}(bb^T)\le 1+1=2$$

当a,b线性相关时，若a,b皆为 0 向量，则$A=0$，$\operatorname{rank}(A)=0$.

若a,b不都为 0 向量时，不妨设$a\neq 0$, 由a,b线性相关，得出存在常数k，使得$b=ka$，此时

$$A=aa^T+bb^T=(1+k^2)aa^T$$
$$\operatorname{rank}(A)=\operatorname{rank}(aa^T)=1$$

15. Suppose that A is a full column rank matrix, and AB = C. Show that Bx = 0 has the same solution set as Cx = 0. (i.e., Bx = 0 ⟺ Cx = 0)

参考答案 显然对于任意矩阵A,B，只要乘积AB有意义，一定有

$$B\mathrm{x}=0\Longrightarrow AB\mathrm{x}=0$$

其次，由于矩阵A列满秩，则矩阵A存在左逆D使得$DA=I$，因此

$$AB\mathrm{x}=0\Longrightarrow DAB\mathrm{x}=D0=0\Longrightarrow B\mathrm{x}=0$$

16. (1) If $A_{m\times n}B_{n\times s}=0$, prove that $\operatorname{rank}(A)+\operatorname{rank}(B)\le n$.

(2) Suppose that $A^2=A+2I$ holds for an $n\times n$ matrix A. Show that: $\operatorname{rank}(A+I)+\operatorname{rank}(A-2I)=n$.

参考答案 (1)的证明见第 8 题。

（2）$A^2=A+2I\Longrightarrow A^2-A-2I=0\Longrightarrow(A+I)(A-2I)=0$

由第(1)问

$$\operatorname{rank}(A+I)+\operatorname{rank}(A-2I)\le n$$

再由$\operatorname{rank}(A-2I)=\operatorname{rank}(2I-A)$以及不等式$\operatorname{rank}(A+B)\le\operatorname{rank}(A)+\operatorname{rank}(B)$得

$$\begin{aligned}&\operatorname{rank}(A+I)+\operatorname{rank}(A-2I)\\&=\operatorname{rank}(A+I)+\operatorname{rank}(2I-A)\\&\ge\operatorname{rank}(A+I+2I-A)=\operatorname{rank}(3I)=n\end{aligned}$$

故

$$\operatorname{rank}(A+I)+\operatorname{rank}(A-2I)=n$$