CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Binary Relations

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

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Definition: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

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Definition: A relation on the set A is a relation from A to itself.

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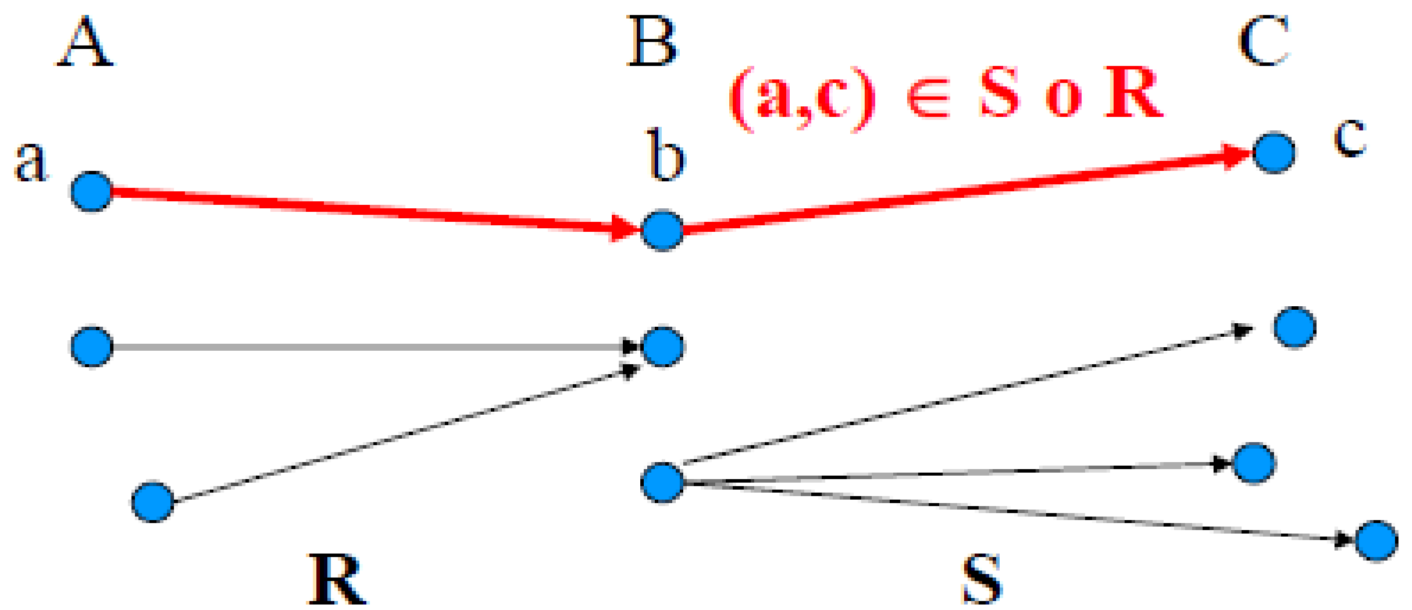
  Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.



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Theorem The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

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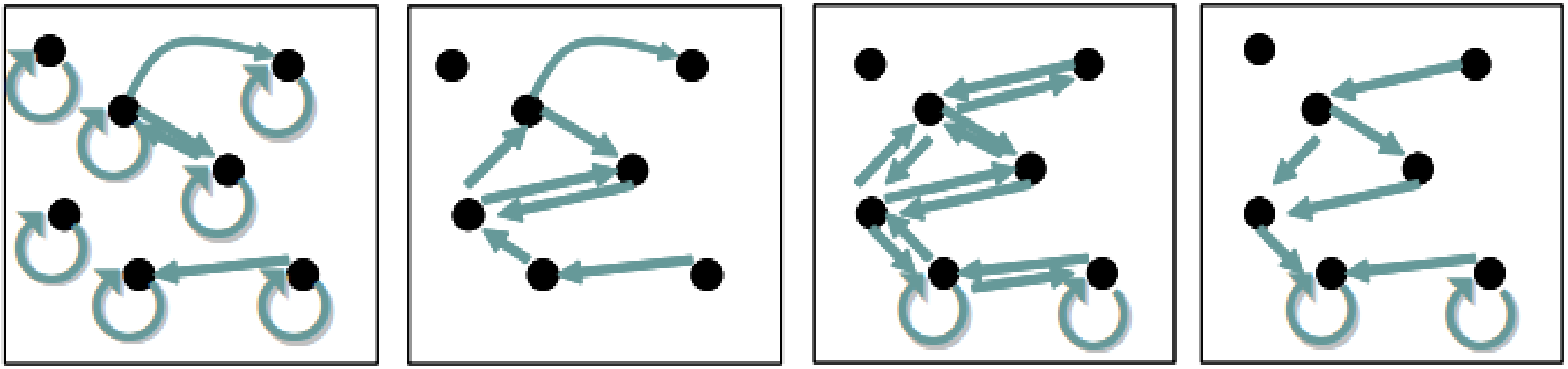

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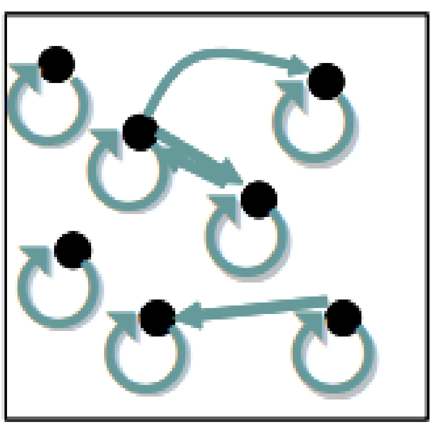
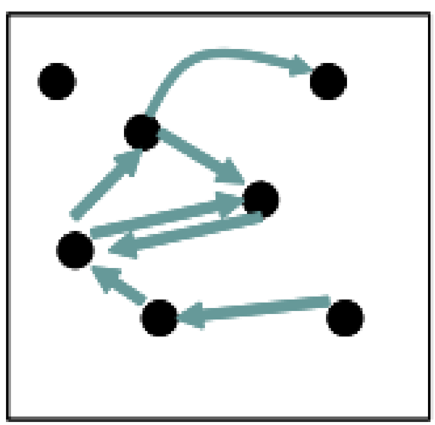


reflexive



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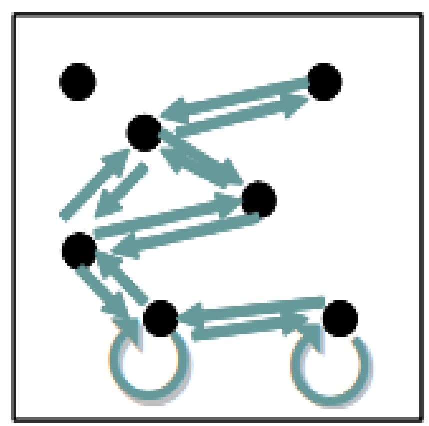
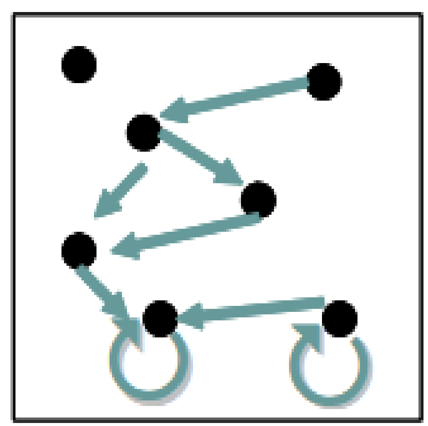


symmetric



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  The minimal set $S \supseteq R$ is called *the reflexive closure of R*.

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  ◇ contains R
  ◇ is reflexive
  ◇ is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Closures on Relations

- Relations can have different properties:
  - reflexive
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We define:

- reflexive closures
- symmetric closures
- transitive closures

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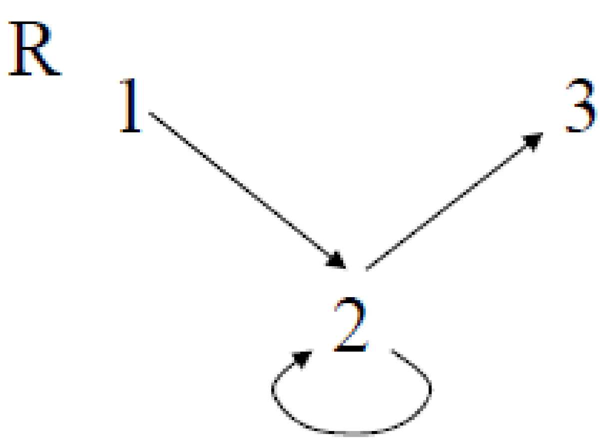

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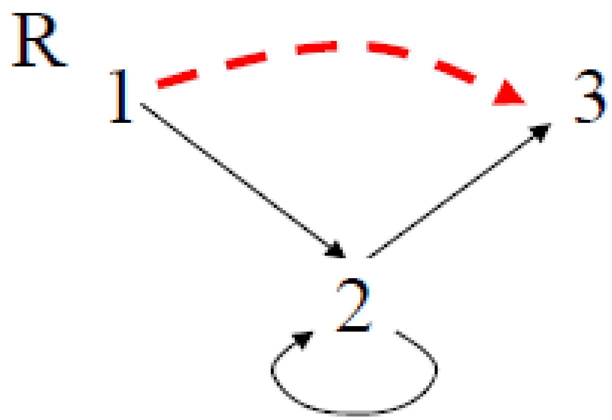

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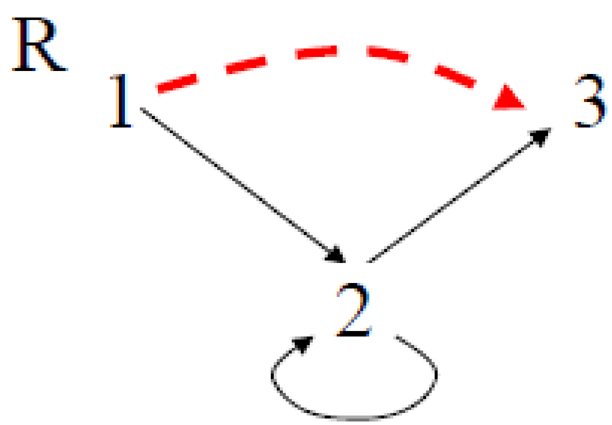

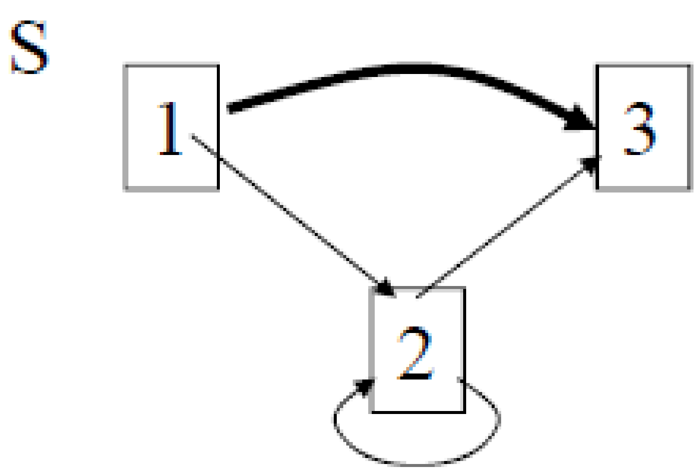

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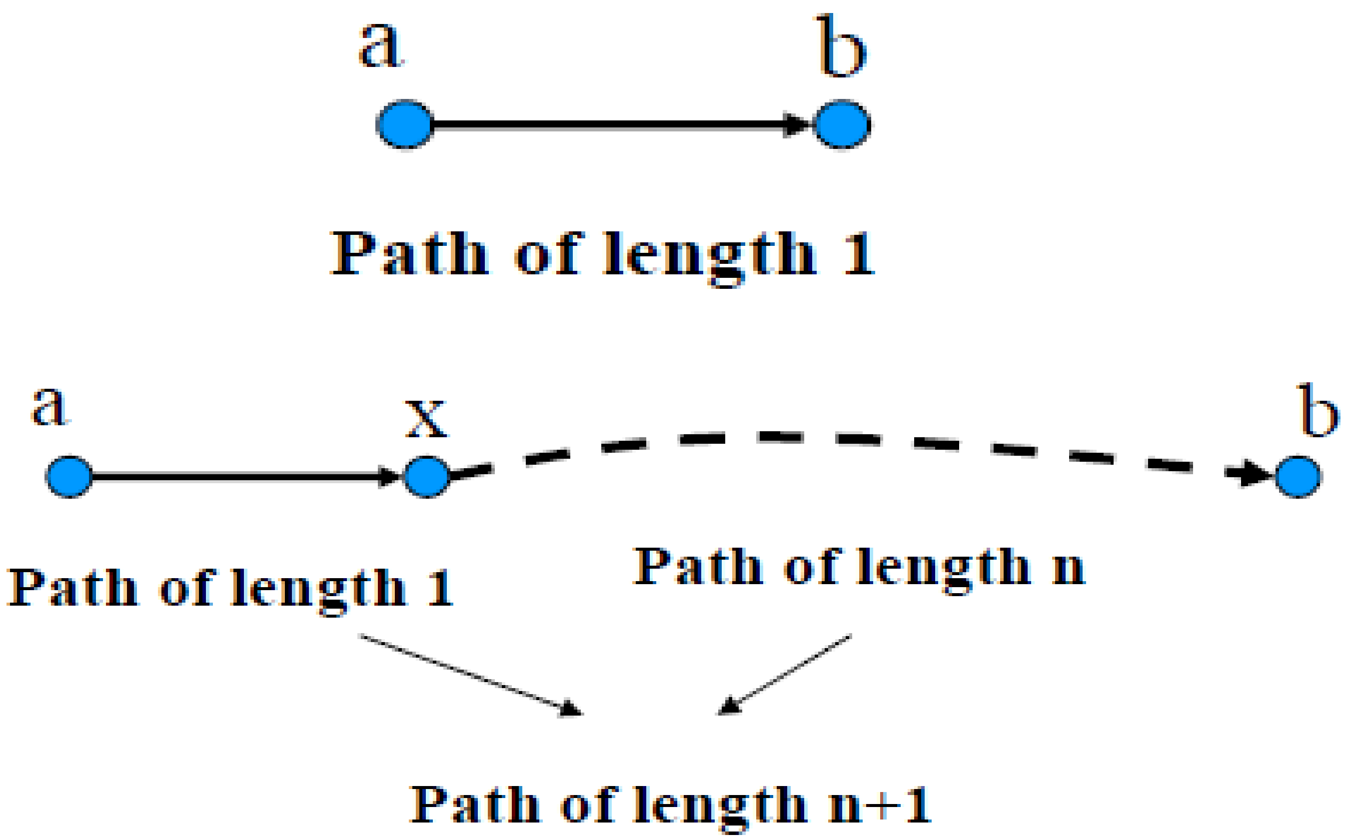

Connectivity Relation

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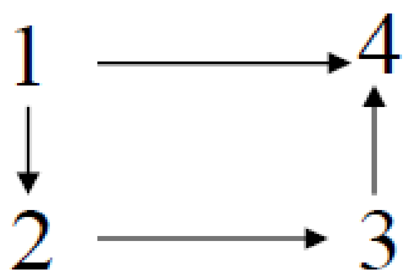

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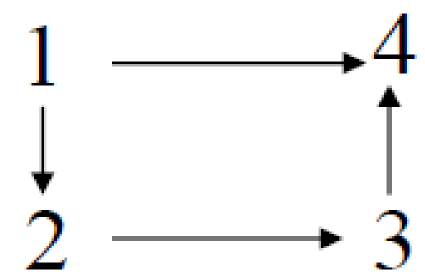

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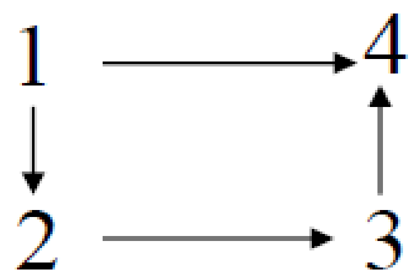

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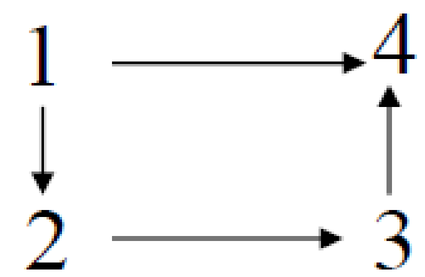

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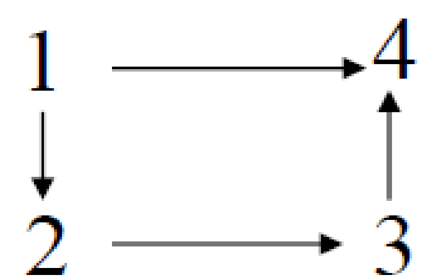

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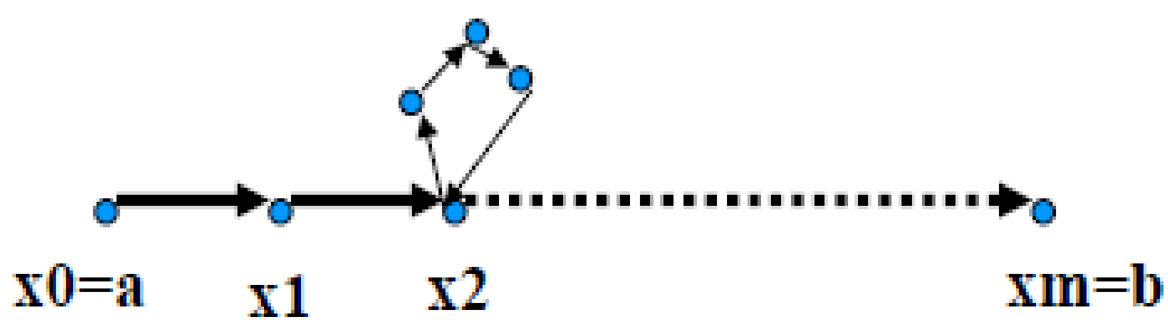

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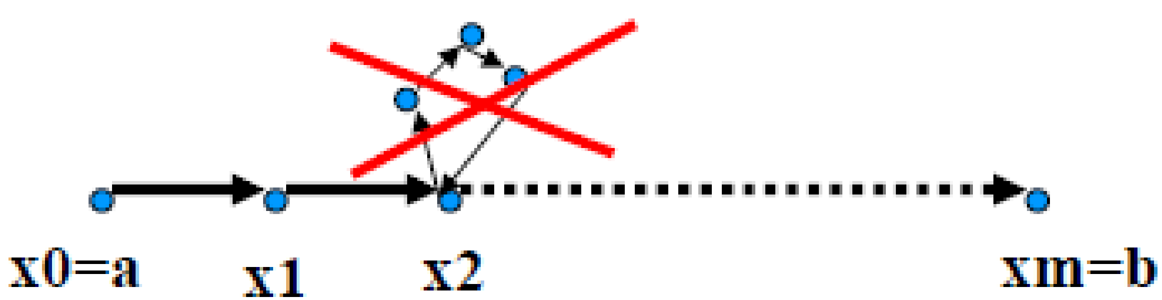

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Proof

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2. $R^* \subseteq S$ whenever S is a transitive relation containing R

1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

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We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

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Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R^*} = ?$

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for $i := 2$ to n
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return B
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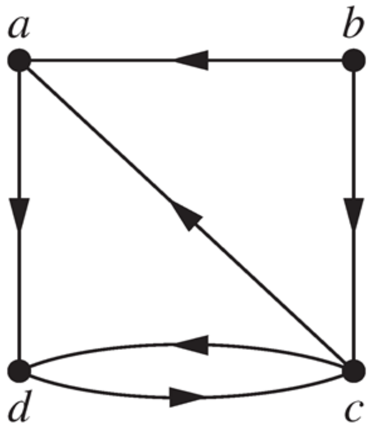
Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



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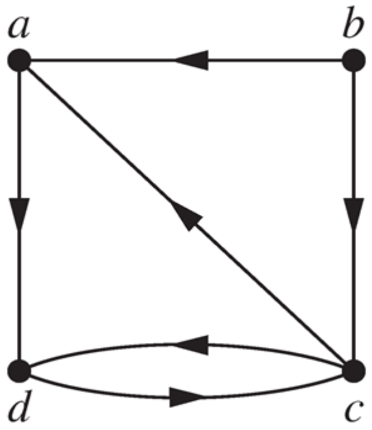

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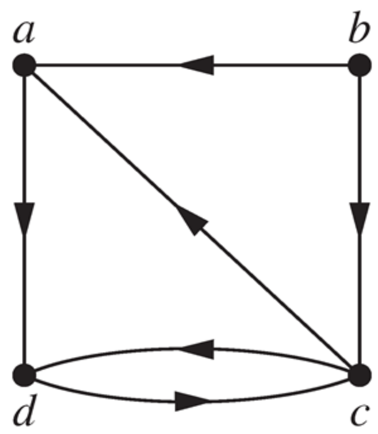

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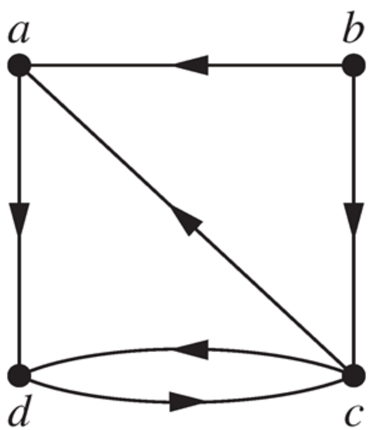

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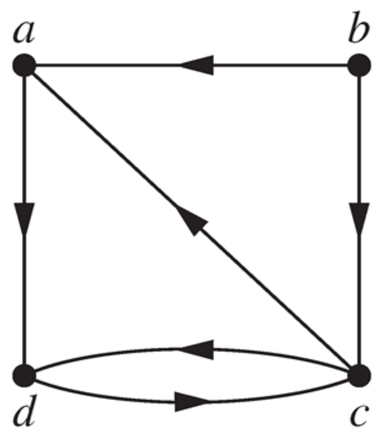

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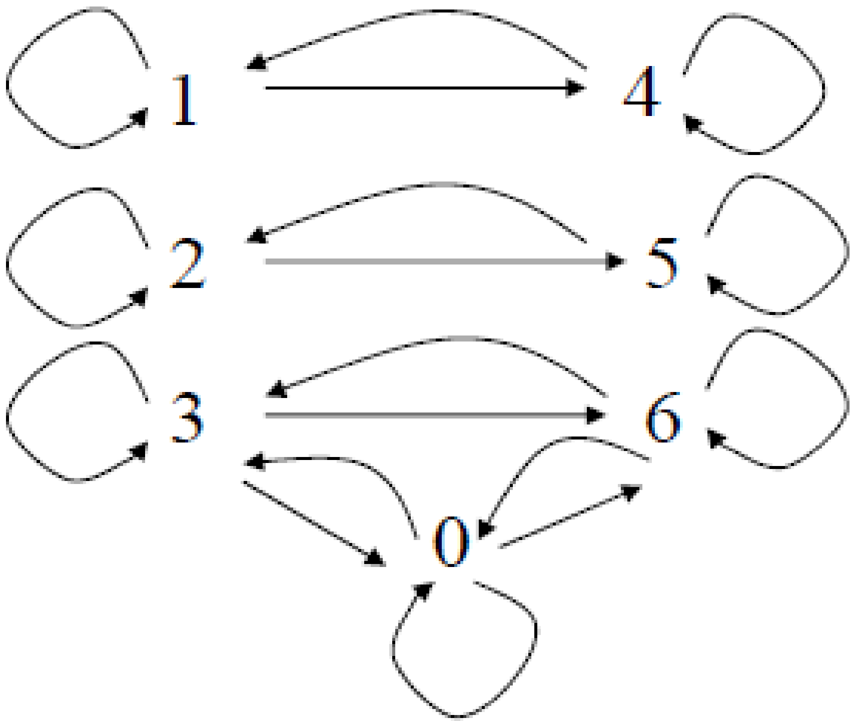

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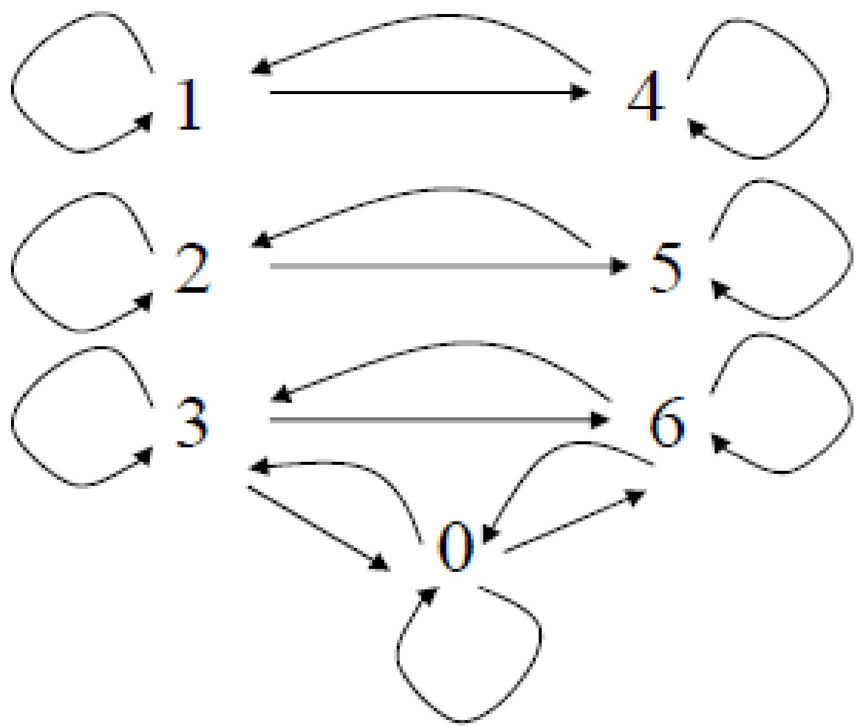

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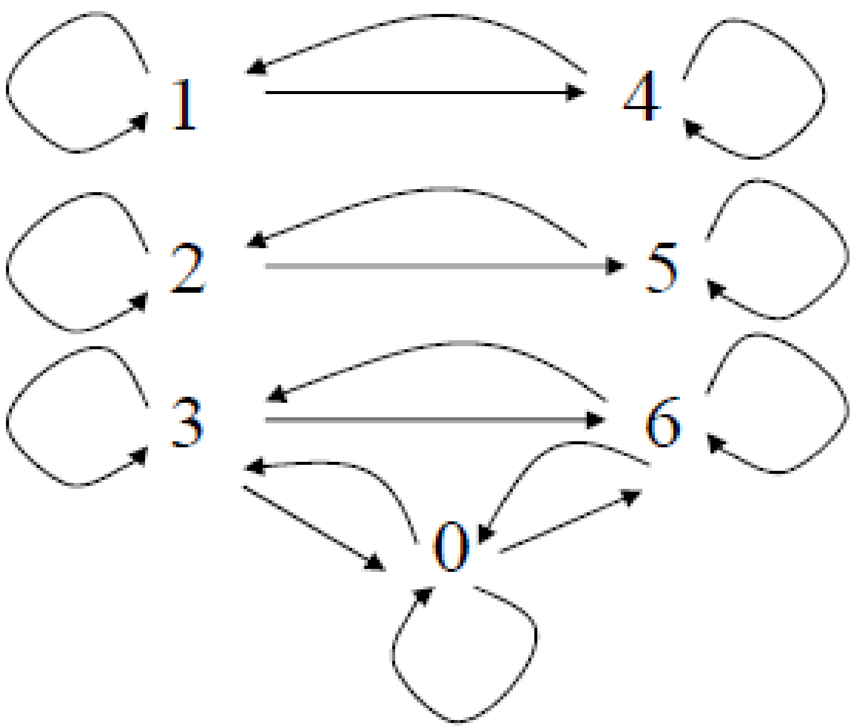

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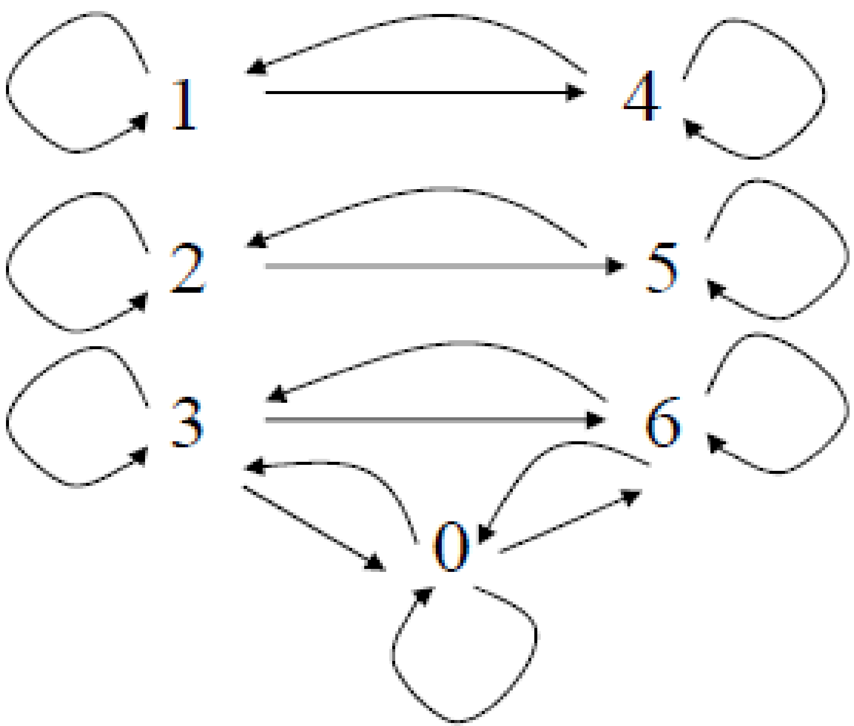

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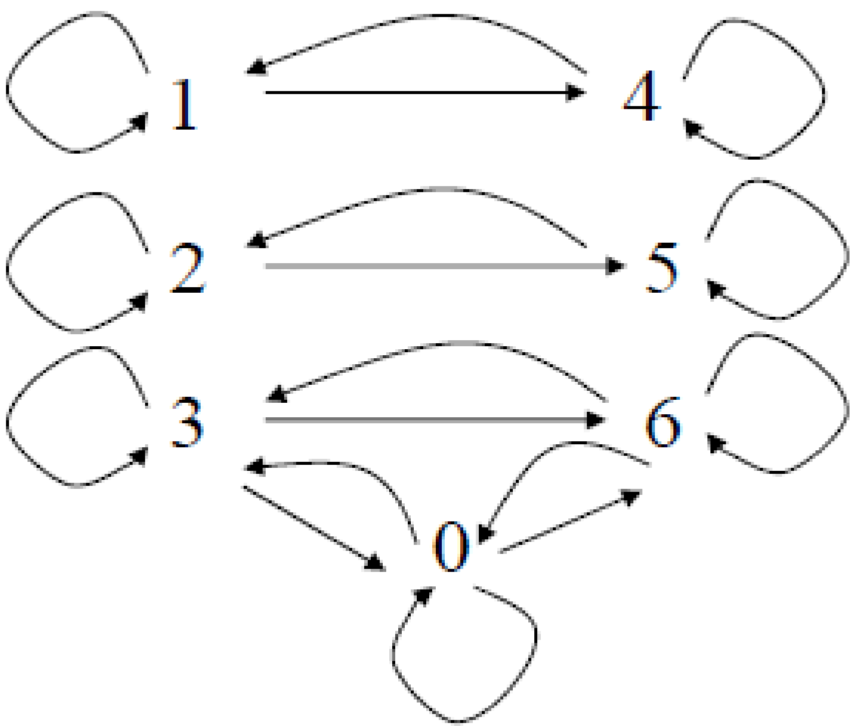

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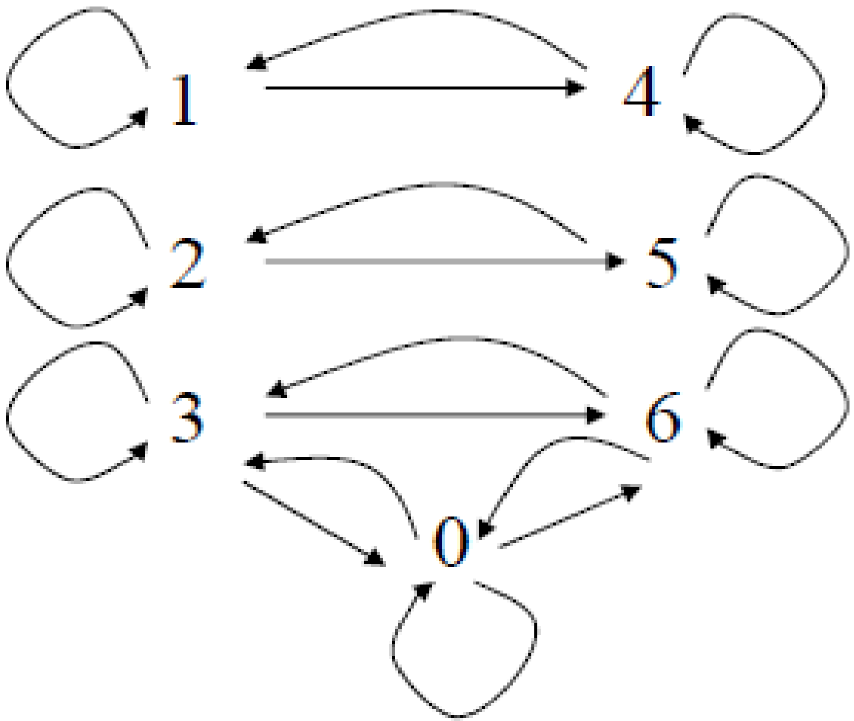

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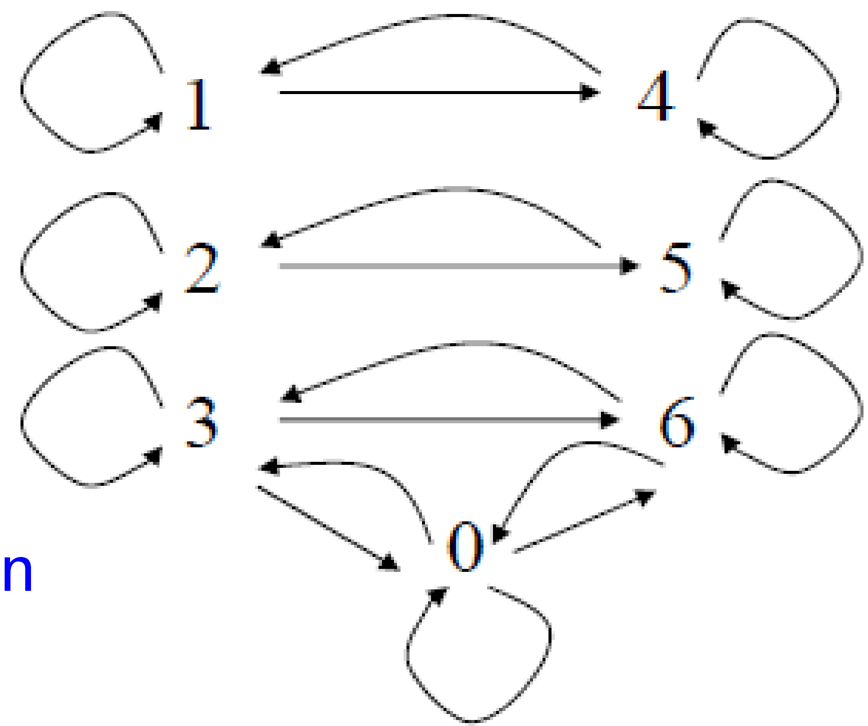

R is an equivalence relation

Examples of Equivalence Relations

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  "Strings a and b have the same length."

  "Integers a and b have the same absolute value."

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  "The relation $\geq$ between real numbers."

  "has a common factor greater than 1 between natural numbers."

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  $[a]$ = the set of all strings of the same length as a

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  $[a]$ = the set $\{a, -a\}$

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

  $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

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Partition of a Set S

- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

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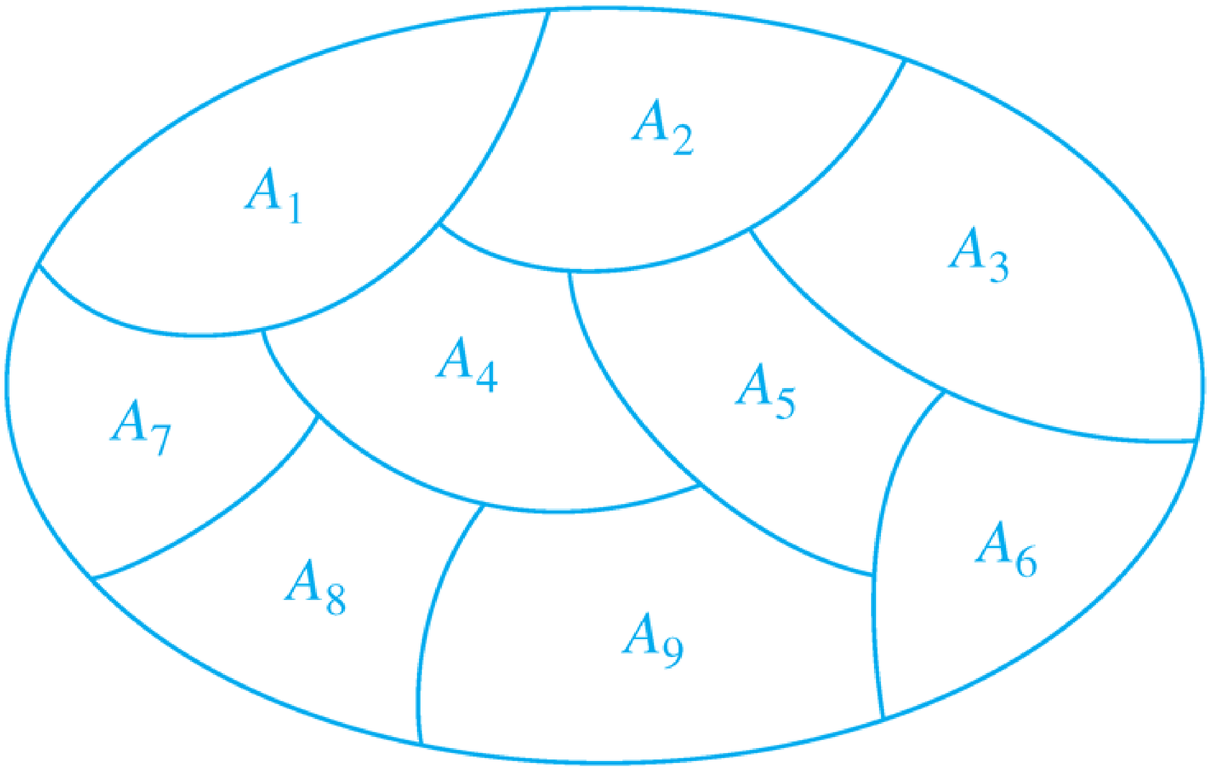

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Is A_1, A_2, A_3 a partition of S?

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Theorem Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Next Lecture

- relation, graph ...