Probability and Statistics
Tutorial 6

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Outline

Review

1. Joint Distribution Function $F_{X,Y}(x, y)$
   - (Def) $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$.
   - (Property) $F_{X,Y}(+\infty, +\infty) = 1$, $F_{X,Y}(-\infty, -\infty) = 0$.
   - (Property) $F_{X,Y}(x, y)$ is nondecreasing in x and y.
   - (Property) $F_{X,Y}(x, y)$ is right continuous in x and y.
   - (Property) $0 \leq P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$.
2. Marginal Distribution Function $F_X(x)$
   - $F_X(x) = P(X \leq x) = P(X \leq x, y < +\infty) = F_{X,Y}(x, +\infty)$.
   - $F_X(x)$ itself is a distribution function.

Review

3. Joint Distribution of Discrete Random Variables
   - Joint PMF: $P(X=i, Y=j) = p_{ij}$.
   - $\sum_{i,j} p_{ij}$=1 and $p_{ij} \geq 0$.
   - (General Case) Joint PMF $P(X_1 = i_1, X_2 = i_2, .., X_n = i_n)$.
4. Joint Distribution of Continuous Random Variables
   - (Def) Joint PDF: $f_{X,Y}(x,y)$ such that $F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x,y)dxdy$.
   - (Property) $f_{X,Y}(x,y) \geq 0$, $1 == \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy$
   - (Property) $P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)dxdy$.
   - (Property) $f_{X,Y}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$

Review

5. Marginal Distribution of Random Variables
   - (Discrete Case) Marginal PMF: $P(X = i) = \sum_j P(X = i, Y = j)$.
   - (Continuous Case) Marginal PDF: $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y) dy = \frac{\partial F(x,+\infty)}{\partial x}$.
   - Marginal PDF (PMF) is itself a PDF (PMF).

Homework

3. 三个玩家进行 10 轮独立的游戏，每个人在每轮游戏中获胜的概率都是 $\frac{1}{3}$. 计算每个人赢得游戏次数的联合分布.

Homework

Solution

$P(X_1 = i, X_2 = j, X_3 = k) = \frac{10!}{i!j!k!}(\frac{1}{3})^{10}$, *for* $i + j + k = 10$.

Homework

补充题1. 把一枚均匀硬币抛掷三次，设 X 为三次抛掷中正面出现的次数，而 Y 为正面出现次数与反面出现次数之差的绝对值，求 (X, Y) 的频率函数.

Solution

$Y \backslash X$	0	1	2	3	$P(Y)$
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\frac{3}{4}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{1}{4}$
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1.

Homework

2. 设 X 的分布为 $P(X = -1) = P(X=0) = P(X=1) = 1/3$. 令 $Y=X^2$，求(X,Y)的联合频率函数及边缘频率函数。

Homework

Solution

X \ Y	0	1	
-1	0	1/3	1/3
0	1/3	0	1/3
1	0	1/3	1/3
	1/3	2/3	

Homework

3. 设随机变量 Y 服从参数为 1 的指数分布，随机变量

$$X_k = \begin{cases} 0, & 若 Y \le k, \\ 1, & 若 Y > k, \end{cases} \quad k = 1,2$$

求二维随机变量(X_1,X_2)的联合频率函数及边缘频率函数。

Homework

Solution

解 (X_1, X_2) 的联合分布列共有如下 4 种情况:

$$P(X_1 = 0, X_2 = 0) = P(Y \leqslant 1, Y \leqslant 2) = P(Y \leqslant 1)$$
$$= 1 - e^{-1} = 0.632\ 12,$$

$$P(X_1 = 0, X_2 = 1) = P(Y \leqslant 1, Y > 2) = 0,$$

$$P(X_1 = 1, X_2 = 0) = P(Y > 1, Y \leqslant 2) = P(1 \leqslant Y \leqslant 2)$$
$$= e^{-1} - e^{-2} = 0.232\ 54,$$

$$P(X_1 = 1, X_2 = 1) = P(Y > 1, Y > 2)$$
$$= P(Y > 2) = 1 - P(Y \leqslant 2) = e^{-2} = 0.135\ 134.$$

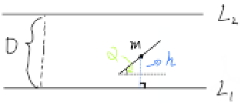
$P(X_1 = 0) = 1 - e^{-1}$, $P(X_1 = 1) = e^{-1}$.
$P(X_2 = 0) = 1 - e^{-2}$, $P(X_2 = 1) = e^{-2}$.

Homework

5. (蒲丰投针问题) 平面上画有一些平行线，它们之间的距离都是 D，一根长为 L 的针随机地投在平面上，其中 $D \geqslant L$. 证明：此针正好与一条直线相交的概率是 $2L/\pi D$. 解释为什么这个实验能够机械地估计 π 值.

Homework

Solution



m: 针的中点.

L_1: 离 m 最近的线

h: m 到 L_1 的距离

α: 与 L_1 距离最近的端点, 和水平方向(L_1方向) 所夹的 $\leq 90^\circ$ 的角.

$h \sim \text{Uniform}\left(0, \frac{D}{2}\right)$

$\alpha \sim \text{Uniform}\left(0, \frac{\pi}{2}\right)$

$$\mathbb{P}(\text{相交}) = \mathbb{P}\left(\frac{L}{2}\sin\alpha \geq h\right)$$

$$= \frac{1}{\frac{D}{2}\cdot\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\sin\alpha} dh\, d\alpha$$

$$= \frac{2L}{D\pi}\int_0^{\frac{\pi}{2}} \sin\alpha\, d\alpha = \frac{2L}{\pi D}.$$

$N = \#$ of experiments

$N_1 = \#$ of success

When N Large,

$$\frac{N_1}{N} \approx \mathbb{P}(\text{相交}) = \frac{2L}{\pi D}.$$

Then, $\pi \approx \dfrac{2LN}{DN_1}$.

Homework

6. 从椭圆内部随机地选择一个点，椭圆方程为：

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

计算该点坐标 x 和 y 的边际密度.

Homework

Solution

Solution. $f_{X,Y}(x,y) = \frac{1}{\pi ab} 1_{\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}}$.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

$$= \begin{cases} \frac{1}{\pi ab}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy, & x \in [-a,a] \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{2}{a\pi}\sqrt{1-\frac{x^2}{a^2}}, & x \in [-a,a] \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{2}{b\pi}\sqrt{1-\frac{y^2}{b^2}}, & y \in [-b,b] \\ 0, & \text{otherwise} \end{cases}$$

Homework

7. 计算相应于如下 cdf 的联合密度和边际密度

$$F(x,y)=(1-\mathrm{e}^{-\alpha x})(1-\mathrm{e}^{-\beta y}),\quad x\geqslant 0,\quad y\geqslant 0,\quad \alpha>0,\quad \beta>0$$

Homework

Solution

$f_{X,Y}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \alpha\beta e^{-(\alpha x+\beta y)} 1_{x\geq 0, y\geq 0}.$
$f_X(x) = \frac{\partial F(x,+\infty)}{\partial x} = \alpha e^{-(\alpha x)} 1_{x\geq 0}.$
$f_Y(y) = \frac{\partial F(+\infty,y)}{\partial y} = \beta e^{-(\beta x)} 1_{y\geq 0}.$

Homework

8. 若 X 和 Y 具有联合密度

$$f(x,y)=\frac{6}{7}(x+y)^2, \quad 0\leqslant x\leqslant 1, \quad 0\leqslant y\leqslant 1$$

a. 利用合适区域上的积分，计算 (i) $P(X>Y)$，(ii) $P(X+Y\leqslant 1)$，(iii) $P\left(X\leqslant \frac{1}{2}\right)$.

b. 计算 x 和 y 的边际密度.

c. 计算这两个变量的条件密度.

Homework

Solution

a. $P(X > Y) = \int_0^1 \int_y^1 \frac{6}{7}(x+y)^2 dxdy = \frac{1}{2}$.

$P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} \frac{6}{7}(x+y)^2 dxdy = \frac{3}{14}$

$P(X \leq \frac{1}{2}) = \int_0^1 \int_0^{\frac{1}{2}} \frac{6}{7}(x+y)^2 dxdy = \frac{2}{7}$

b. For $0 \leq x \leq 1$, $f_X(x) = \int_0^1 \frac{6}{7}(x+y)^2 dy = \frac{6}{7}x^2 + \frac{6}{7}x + \frac{2}{7}$; otherwise, $f_X(x) = 0$.

For $0 \leq y \leq 1$,
$f_Y(y) = \int_0^1 \frac{6}{7}(x+y)^2 dx = \frac{6}{7}(x+y)^2 dy = \frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}$; otherwise, $f_Y(y) = 0$.

c. For $0 \leq x \leq 1$, $0 \leq y \leq 1$,

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{3(x+y)^2}{3y^2+3y+1}$.

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{3(x+y)^2}{3x^2+3x+1}$.

Homework

1. 设二维连续随机变量(X,Y)的联合分布函数为

$$F(x,y)=\begin{cases} k(1-e^{-x})(1-e^{-y}), & x>0, y>0, \\ 0, & \text{其他}, \end{cases}$$

求边缘密度函数及 $P(1<X<3,\ 1<Y<2)$。

Homework

Solution

Since $F(+\infty, +\infty) = 1$, *then* $k = 1$.
$f_X(x) = \frac{\partial F(x,+\infty)}{\partial x} = e^{-x}1_{x>0}$.
$f_Y(y) = \frac{\partial F(+\infty,y)}{\partial y} = e^{-y}1_{y>0}$.
$P(1 < X < 3, 1 < Y < 2) = \int_1^3 \int_1^2 e^{-(x+y)} dydx = (e^{-1} - e^{-3})(e^{-1} - e^{-2})$.

Homework

2. 设二维连续随机变量(X,Y)的概率密度为

$$f(x,y)=\begin{cases} x+y, & 0<x,y<1, \\ 0, & 其他, \end{cases}$$

(1) 求边缘密度函数;　　(2)求 $P(X>Y)$;

(3)求 $P(X<0.5)$

Homework

Solution

(1) $f_X(x) = (x + \frac{1}{2})1_{0<x<1}$. $f_Y(y) = (y + \frac{1}{2})1_{0<y<1}$.
(2) $P(X > Y) = \int_0^1 \int_y^1 (x + y)dxdy = \frac{1}{2}$.
(3) $P(X < 0.5) = \int_0^{0.5} (x + \frac{1}{2})dx = \frac{3}{8}$.

Supplement Exercises

Exercise 1

15. 从(0,1)中随机地取两个数,求其积不小于 3/16,且其和不大于 1 的概率.

Solution

解 设取出的两个数分别为 X 和 Y,则 (X,Y) 的联合密度函数为

$$p(x,y)=\begin{cases}1, & 0<x<1,0<y<1,\\ 0, & \text{其他}.\end{cases}$$

因为 $p(x,y)$ 的非零区域与 $\{xy \geqslant 3/16, x+y \leqslant 1\}$ 的交集为图 3.6 阴影部分.

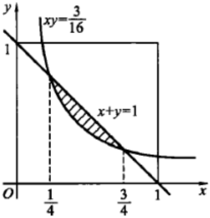


图 3.6

所以

$$P\{XY \geqslant 3/16, X+Y \leqslant 1\} = \int_{1/4}^{3/4}\int_{\frac{3}{16x}}^{1-x} \mathrm{d}y\mathrm{d}x = \int_{1/4}^{3/4}\left(1-x-\frac{3}{16x}\right)\mathrm{d}x$$

$$=\left(x-\frac{1}{2}x^2-\frac{3}{16}\ln x\right)\Bigg|_{1/4}^{3/4} = \frac{1}{4}-\frac{3}{16}\ln 3 = 0.044\,0.$$

Supplement Exercises

Exercise 2

4. 设随机变量 $X_i, i = 1,2$ 的分布列如下，且满足 $P(X_1X_2 = 0) = 1$，试求$P(X_1 = X_2)$.

X_i	-1	0	1
P	0.25	0.5	0.25

Supplement Exercises

Solution

解　记(X_1,X_2)的联合分布列为

X_1 \ X_2	-1	0	1
-1	p_{11}	p_{12}	p_{13}
0	p_{21}	p_{22}	p_{23}
1	p_{31}	p_{32}	p_{33}

由 $P(X_1X_2=0)=1$ 知：$p_{12}+p_{21}+p_{22}+p_{23}+p_{32}=1$，所以 $p_{11}=p_{13}=p_{31}=p_{33}=0$. 即

X_1 \ X_2	-1	0	1
-1	0	p_{12}	0
0	p_{21}	p_{22}	p_{23}
1	0	p_{32}	0

又因为

$$
\begin{aligned}
0.25 &= P(X_1=-1)\\
&= P(X_1=-1,X_2=-1)+P(X_1=-1,X_2=0)+P(X_1=-1,X_2=1)\\
&= p_{11}+p_{12}+p_{13}=p_{12},
\end{aligned}
$$

Solution

同理由 $P(X_1=1)=P(X_2=-1)=P(X_2=1)=0.25$ 可知 $p_{32}=p_{21}=p_{23}=0.25$，即

X_1 \ X_2	-1	0	1
-1	0	0.25	0
0	0.25	p_{22}	0.25
1	0	0.25	0

又由分布列的正则性得 $p_{22}=0$，因此

$$P(X_1=X_2)=p_{11}+p_{22}+p_{33}=0.$$

Exercise 3

7. 设二维随机变量(X,Y)的联合密度函数为

$$p(x,y)=\begin{cases}4xy, & 0<x<1, 0<y<1,\\ 0, & \text{其他}.\end{cases}$$

试求

(1) $P(0<X<0.5, 0.25<Y<1)$;

(2) $P(X=Y)$;

(3) $P(X<Y)$;

(4) (X,Y)的联合分布函数.

Supplement Exercises

Solution

(4) (X,Y) 的联合分布函数 $F(x,y)$ 要分如下 5 个区域表示：

$$F(x,y)=\begin{cases}\int_{-\infty}^{x}\int_{-\infty}^{y}0\mathrm{d}x\mathrm{d}y\\ 4\int_{0}^{x}\int_{0}^{y}t_1t_2\mathrm{d}t_2\mathrm{d}t_1\\ 4\int_{0}^{x}\int_{0}^{1}t_1t_2\mathrm{d}t_2\mathrm{d}t_1\\ 4\int_{0}^{1}\int_{0}^{y}t_1t_2\mathrm{d}t_2\mathrm{d}t_1\\ 4\int_{0}^{1}\int_{0}^{1}t_1t_2\mathrm{d}t_2\mathrm{d}t_1\end{cases}=\begin{cases}0, & x<0,\text{或 } y<0,\\ x^2y^2, & 0\leqslant x<1,0\leqslant y<1,\\ x^2, & 0\leqslant x<1,1\leqslant y,\\ y^2, & 1\leqslant x,0\leqslant y<1,\\ 1, & x\geqslant 1,y\geqslant 1.\end{cases}$$

Thank you!