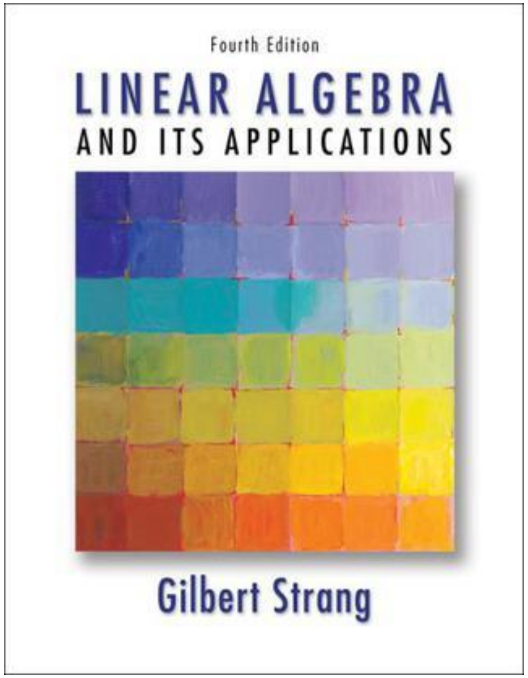
Linear Algebra



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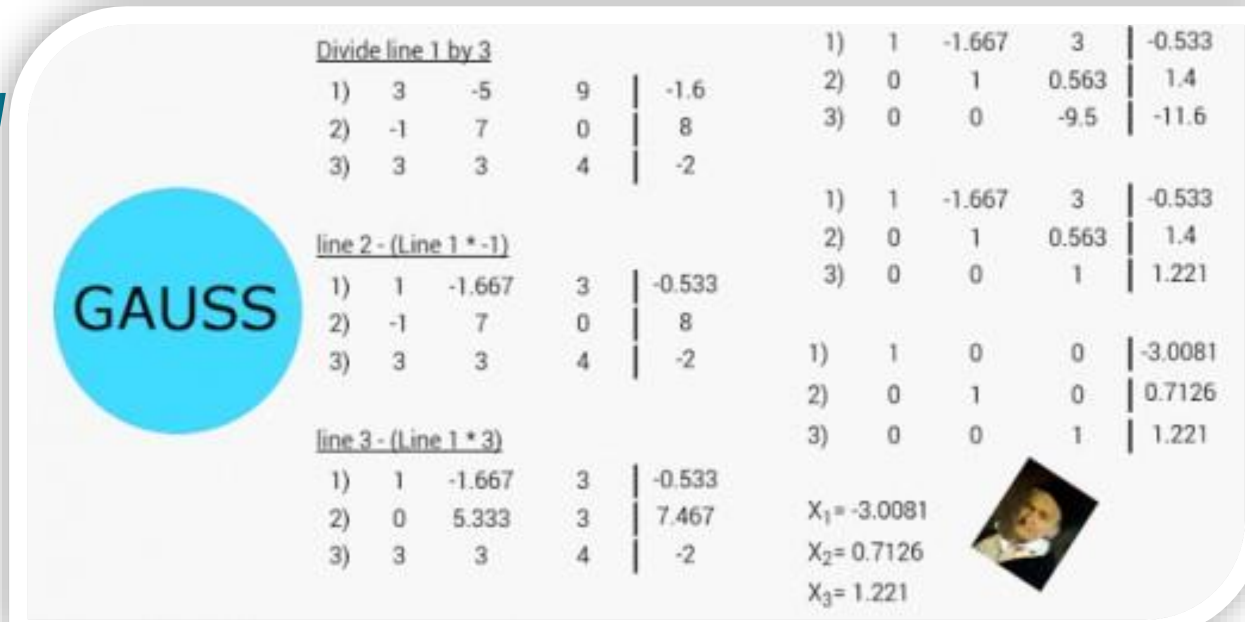
1 Matrices and Gaussian Elimination

1.1

GAUSSIAN ELIMINATION OF EQUATIONS

-INTRODUCTION

（方程组的高斯消元法）



- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

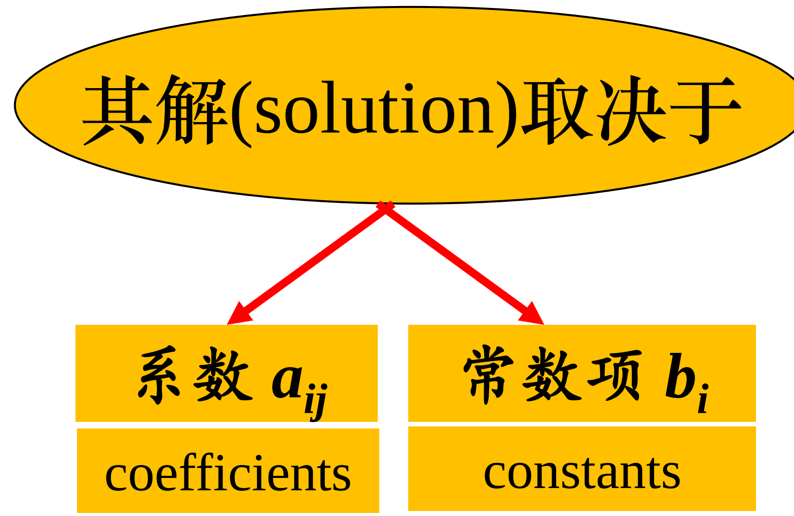
n: any positive integer

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Such systems often appear in science and engineering problems:

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases}
$$



n 个未知量(unknowns, variables),

m 个方程(equations).

应用：营养医学

全国18岁及以上成人超重率为30.1%，肥胖率为11.9%；6-17岁儿童青少年超重率为9.6%，肥胖率为6.4%。【国家卫生计生委在线发布（2015年7月3日）http://www.nhfpc.gov.cn/】

应用：营养医学

每100g食物营养成分表（数据来源：人民网）

	大米	面粉	萝卜	白菜	丝瓜	猪肉	鸭蛋	鲫鱼	需求
蛋白质(g)	7.5	12	2.0	1.1	1.5	17	13	13	b_1
脂肪(g)	0.5	0.8	0.4	0.1	0.1	29	15	1.1	b_2
碳水化合物(g)	79	70	5	2	5	1.1	0.5	0.1	b_3
无机盐类(g)	0.4	1.5	1.4	0.8	0.5	0.9	1.8	0.8	b_4
钙(mg)	10	22	19	86	28	11	71	54	b_5
磷(mg)	100	180	23	27	45	170	210	20	b_6
铁(mg)	1.0	7.6	1.9	1.2	0.8	0.4	3.2	2.5	b_7
向量 vectors	α_1	α_2			$\cdots$			α_k	β
未知量 unknowns	x_1	x_2			$\cdots$			x_k	

系数矩阵 coefficient matrix

增广矩阵 augmented matrix

线性方程组
system of linear equations

$$\boxed{x_1\alpha_1 + x_2\alpha_2 + \cdots + x_k\alpha_k = \beta}$$

量体裁衣

Matrix Notation

n 个未知量m个方程的线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.

方程组的系数(及常数项)排成的数表(即矩阵, matrix)

$$[\boldsymbol{A}, \boldsymbol{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

$\boldsymbol{A}$ 称为方程组的**系数矩阵** (coefficient matrix), $[\boldsymbol{A}, \boldsymbol{b}]$ 称为**增广矩阵** (augmented matrix).

对线性方程组的研究可转化为对这张表的研究

- Given the system:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

coefficient matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

augmented matrix

3 by 4 matrix

3行4列矩阵

The size of a matrix tells how many rows and columns it has.

If *m* and *n* are positive integers, an $\boldsymbol{m \times n}$ (read "*m* by *n*") **matrix** is a rectangular array of numbers with *m* rows and *n* columns.

- A **solution (解)** of the system is a list $(k_1, k_2, \ldots, k_n)$ of numbers that makes each equation a true statement when the values $k_1, \ldots, k_n$ are substituted for $x_1, \ldots, x_n$, respectively.

- The set of all possible solutions is called the **solution set (解集)** of the linear system.

- Two linear systems are called **equivalent (等价, 同解)** if they have the same solution set.

- The basic strategy for solving a linear system is to *replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve*.

消元法的基本思想：通过消元变形把方程组化成容易求解的同解方程组.

Example 1 Solve the given system of equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

Solution: The elimination procedure is shown here *with* and *without* matrix notation, and the results are placed side by side for comparison.

（1）消元法的基本思想: 通过消元变形把方程组化成容易求解的同解方程组.

（2）用消元法解线性方程组的消元步骤可以在增广矩阵上实现

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Forward elimination

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

triangular form

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

triangular form

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Back substitution

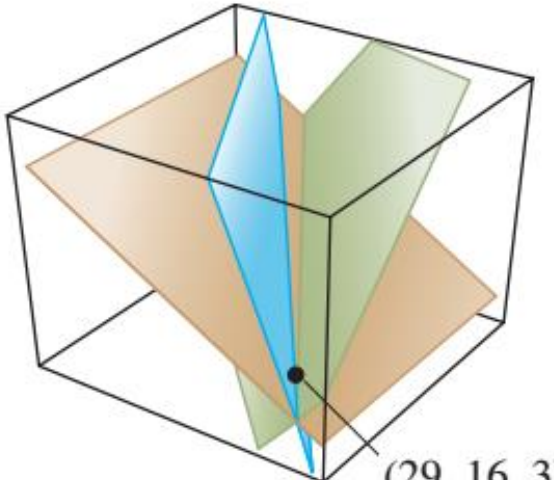
$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 1 Solve the given system of equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

https://www.geogebra.org/m/nngryjuc



Each of the original equations determines a plane in three-dimensional space. The point (29, 16, 3) lies in all three planes.

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

线性方程组的初等变换
(operations on equations in a linear system)

增广矩阵的初等行变换
(operations on the rows of the augmented matrix)

Elementary operations on linear systems (线性方程组的初等变换)

① Interchange two equations
(交换两个方程的位置);

② Multiply an equation by a nonzero number
(用一个非零的数乘某一个方程);

③ Add a multiple of an equation to another
(将一个方程的倍数加到另一个方程上).

Remark : The linear system is changed to a new **equivalent** system by a series of elementary operations.
(线性方程组经初等变换后, 得到的线性方程组与原线性方程组同解).

Elementary Row Operations (ERO：矩阵的初等行变换)

- Elementary row operations include the following:
  1. ***(Interchange*, 对换)** Interchange two rows.
  2. ***(Scaling*, 倍乘)** Multiply all entries in a row by a nonzero constant.
  3. ***(Replacement*, 倍加)** Replace one row by the sum of itself and a multiple of another row.

- Two matrices are called **row equivalent (行等价)** if there is a sequence of *elementary row operations* that transforms one matrix into the other.
- It is important to note that row operations are *reversible*(可逆).
- If the augmented matrices of two linear systems <u>are row equivalent</u>, then the two systems have the same solution set.

- A system of linear equations is said to be **consistent (相容)** if it has either *one solution* or *infinitely many solutions*.
- A system is **inconsistent (不相容)** if it has *no solution*.

- **Two fundamental questions** about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution ***exist***?
  2. If a solution exists, is it the *only* one; that is, is the solution ***unique***?

Existence(存在性) and Uniqueness(唯一性) of solutions