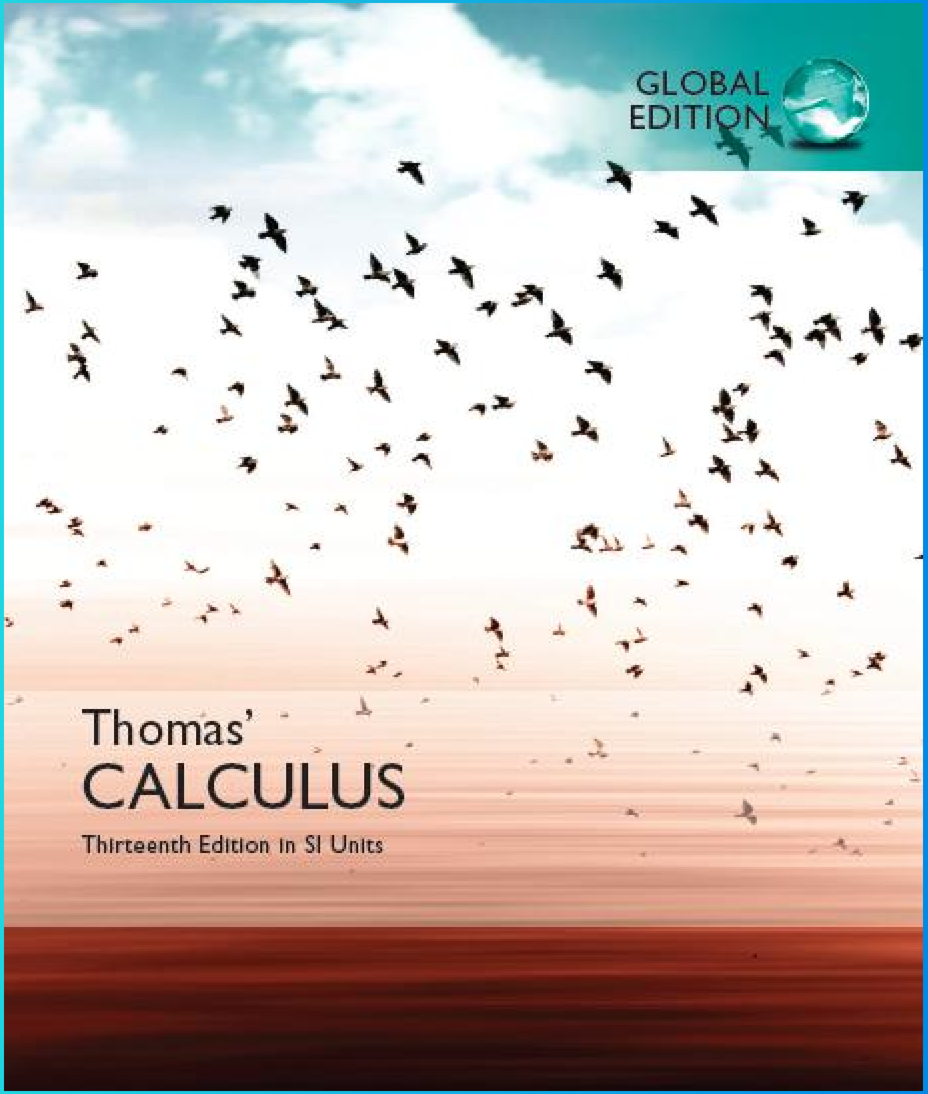

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

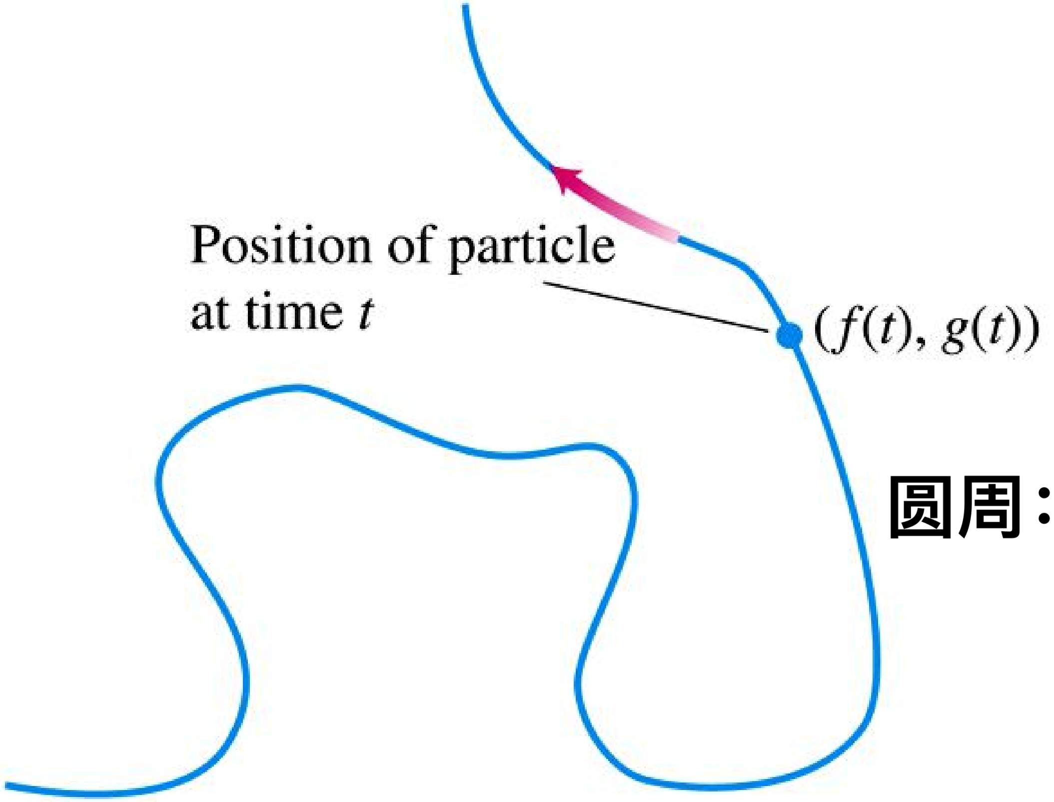
Chapter 11

Parametric Equations and Polar Coordinates

参数方程和极坐标

11.1

Parametrizations of Plane Curves
平面曲线的参数方程



圆周:$\begin{cases} x = a\cos t \\ y = a\sin t, \end{cases}$ $t \in [0, 2\pi]$

FIGURE 11.1 The curve or path traced by a particle moving in the *xy*-plane is not always the graph of a function or single equation.

DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

parameter interval. I is a closed interval, $a \leq t \leq b$,

$(f(a), g(a))$ is the **initial point**

$(f(b), g(b))$ is the **terminal point**.

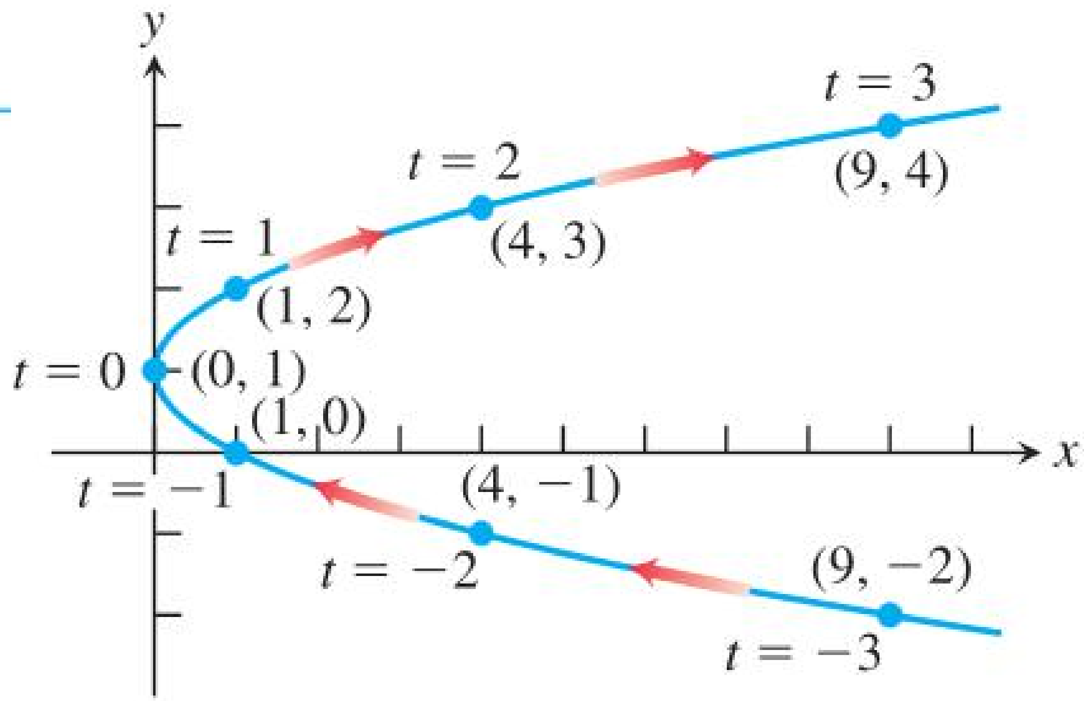
parametrized the curve.

EXAMPLE 1 Sketch the curve defined by the parametric equations

$$x = t^2, \qquad y = t + 1, \qquad -\infty < t < \infty.$$

Solution We make a brief table of values (Table 11.1),

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



EXAMPLE 2 obtaining an algebraic equation in x and y.

$$x = t^2, \qquad y = t + 1,$$

Solution by eliminating the parameter t

$$x = t^2 = (y - 1)^2 = y^2 - 2y + 1.$$

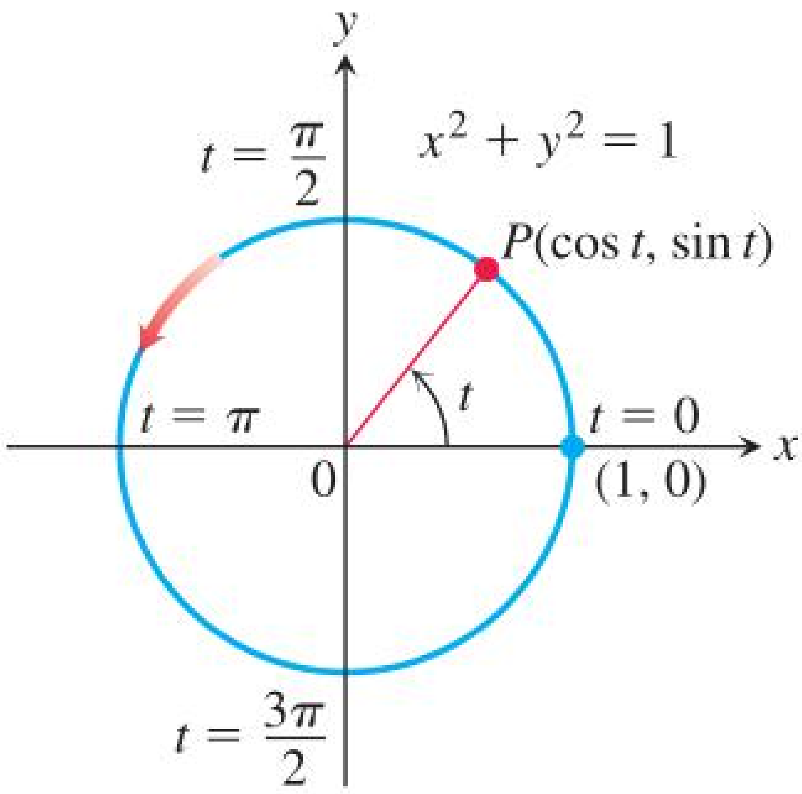
represents a parabola,

EXAMPLE 3 Graph the parametric curves

(a) $x = \cos t, \qquad y = \sin t, \qquad 0 \leq t \leq 2\pi.$

(b) $x = a\cos t, \qquad y = a\sin t, \qquad 0 \leq t \leq 2\pi.$

Solution (a) Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the unit circle



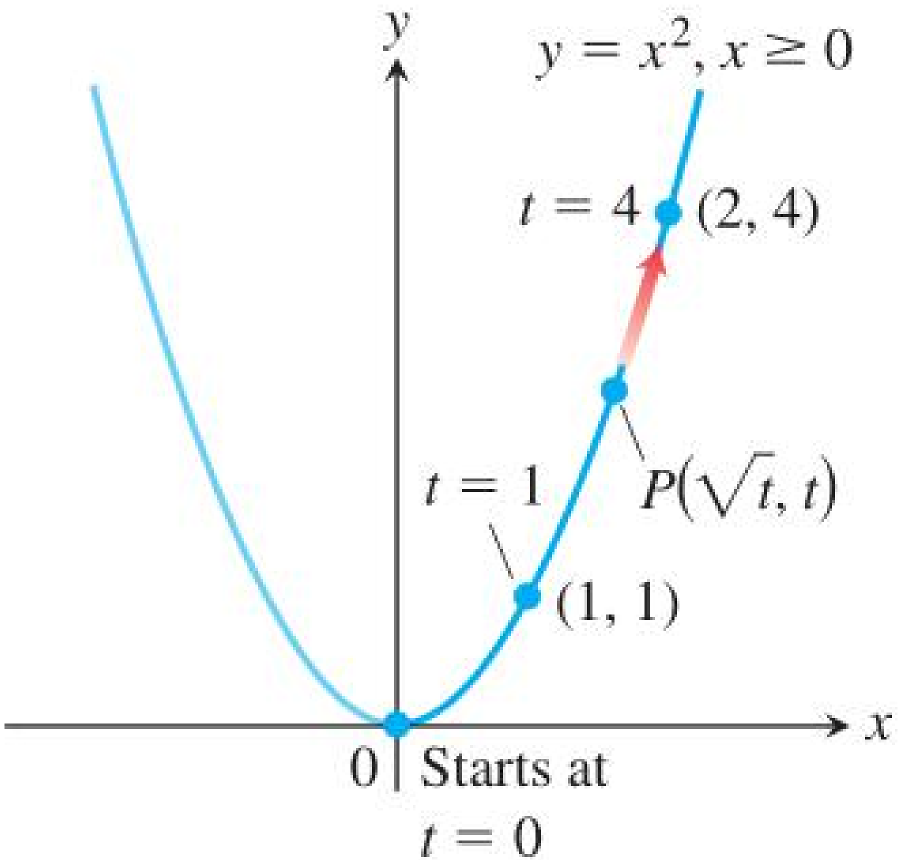
(b) the circle $x^2 + y^2 = a^2$

EXAMPLE 4 **Graph the curve**

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

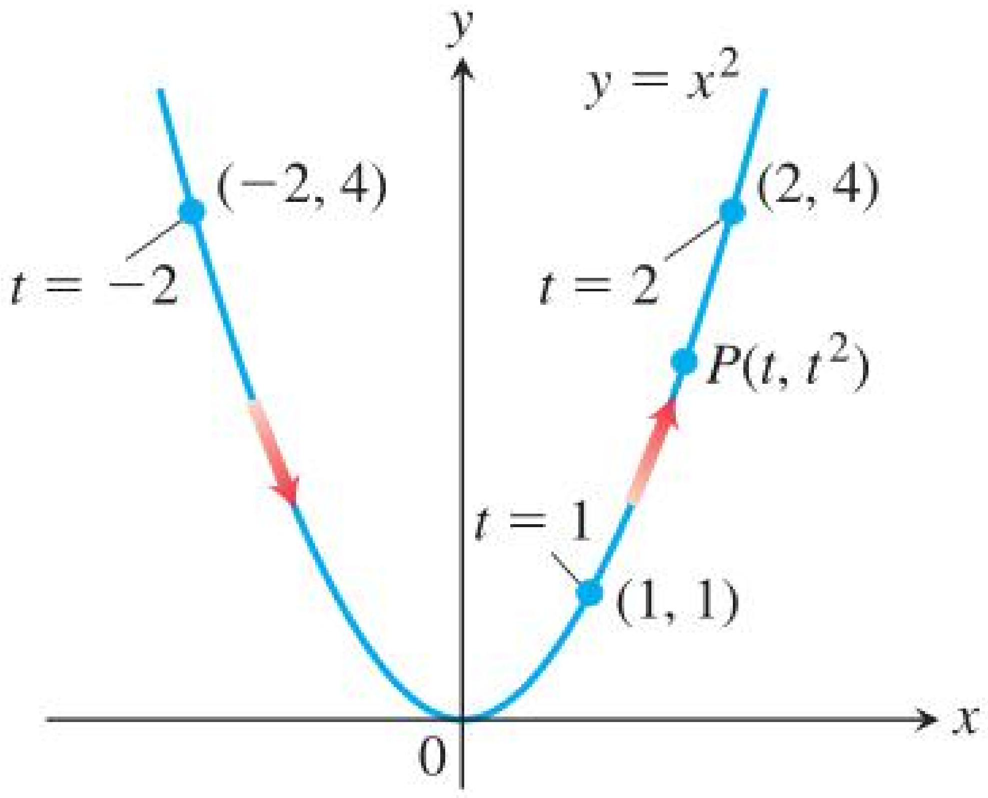
Solution

$$y = t = \left(\sqrt{t}\right)^2 = x^2.$$



EXAMPLE 5 A parametrization of the function $f(x) = x^2$ i

Solution $x = t, \qquad y = f(t) = t^2, \qquad -\infty < t < \infty.$



EXAMPLE 6

Find a parametrization for the line through the point (a, b) having slope m.

Solution $y - b = m(x - a)$. we set $t = x - a$,

$$x = a + t, \qquad y = b + mt, \qquad -\infty < t < \infty$$

EXAMPLE 7

Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}, \qquad t > 0.$$

Solution

$$x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}.$$

$$(x - y)(x + y) = 4,$$

$$x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t.$$

TABLE 11.2 Values of $x = t + (1/t)$ and $y = t - (1/t)$ for selected values of t.

t	$1/t$	x	y
0.1	10.0	10.1	−9.9
0.2	5.0	5.2	−4.8
0.4	2.5	2.9	−2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

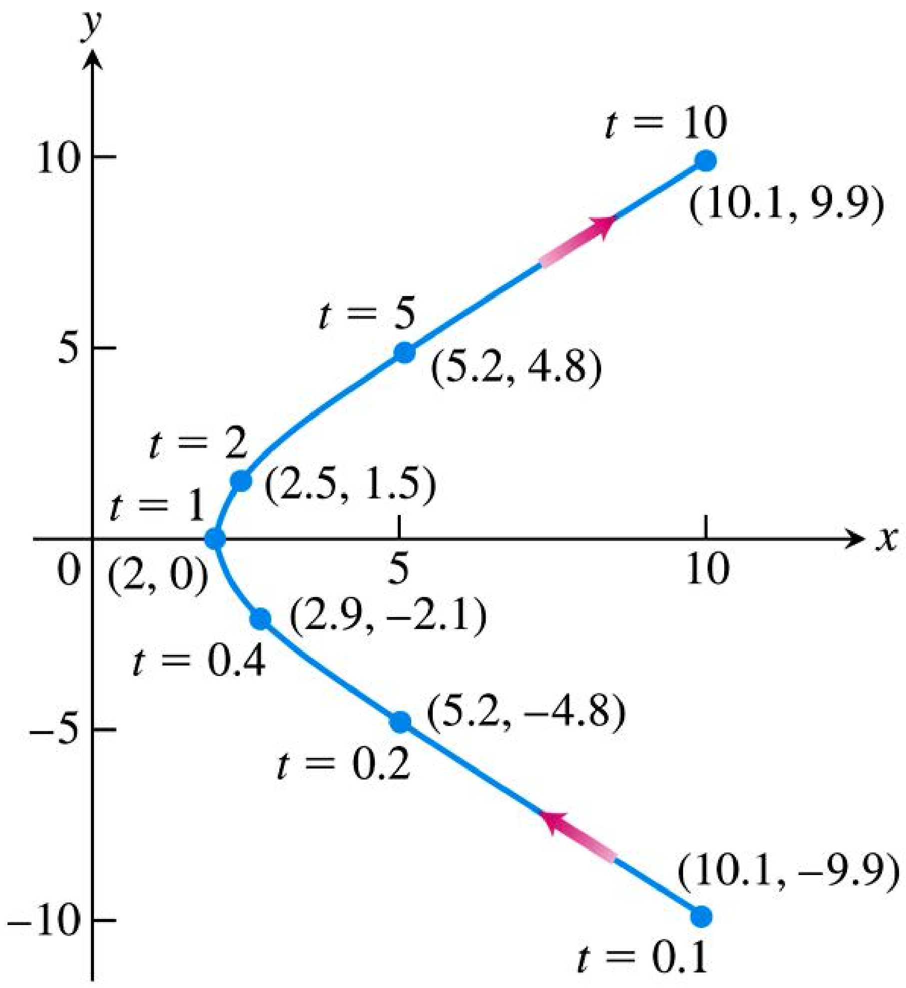


FIGURE 11.6 The curve for $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$ in Example 7. (The part shown is for $0.1 \le t \le 10$.)

$$(x - y)(x + y) = 4,$$

$$x = \sqrt{4 + t^2}, \qquad y = t, \qquad -\infty < t < \infty,$$

$$x = 2 \sec t, \qquad y = 2 \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

The cycloid curve

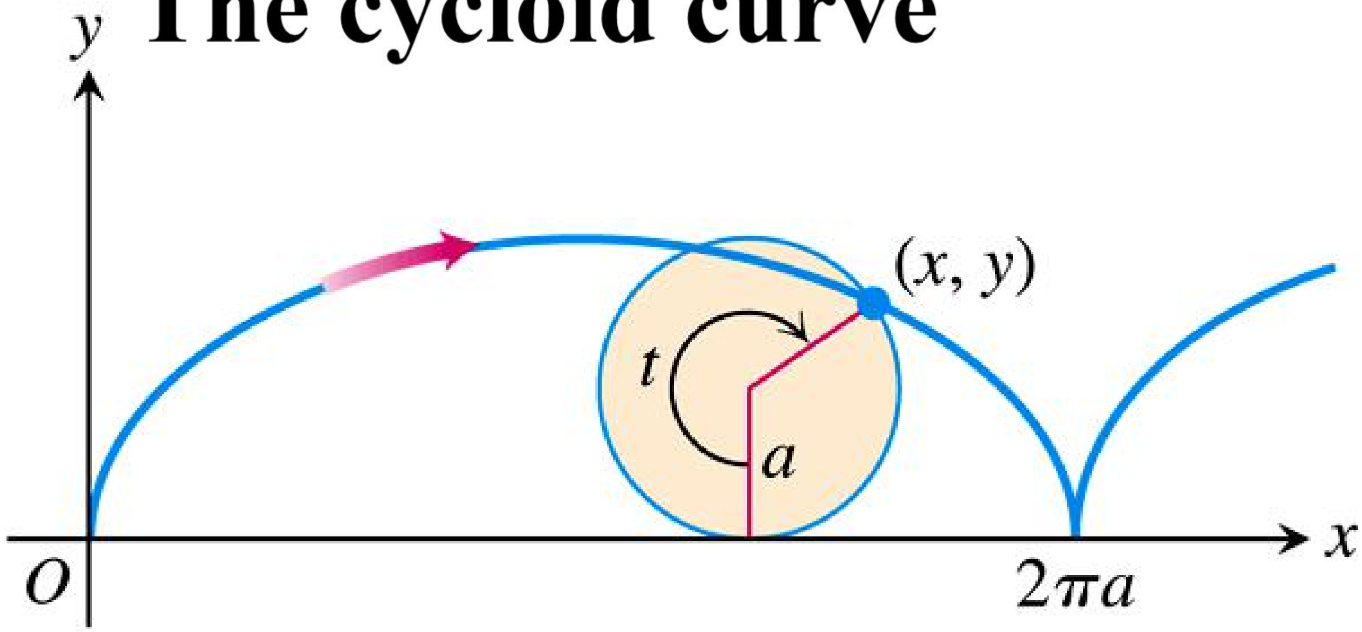


FIGURE 11.9 The cycloid curve $x = a(t - \sin t), y = a(1 - \cos t)$, for $t \geq 0$.

frequency is independent of the amplitude.

EXAMPLE 8 A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.

Solution We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. we use the angle t through which the wheel turns, in radians. the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

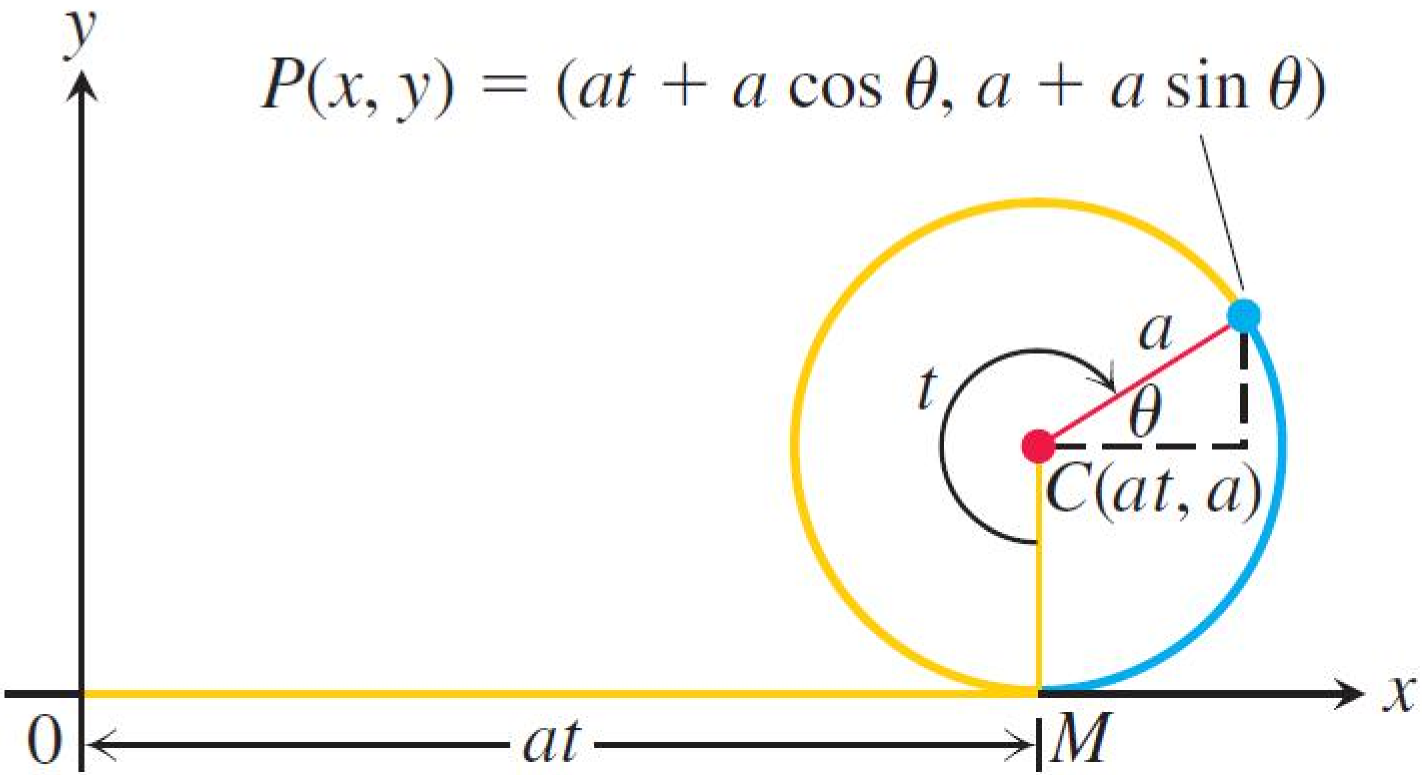


FIGURE 11.8 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

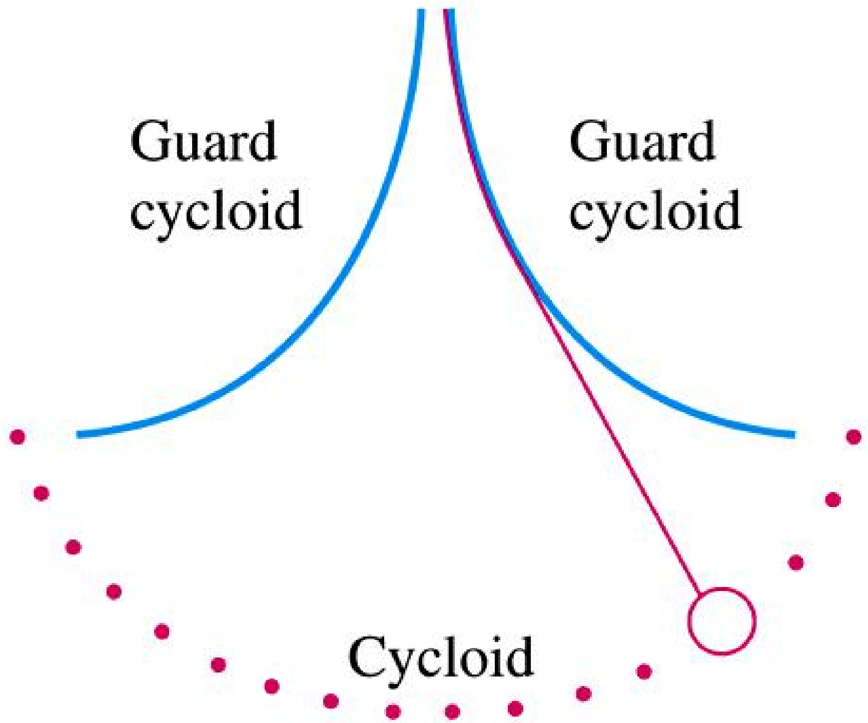
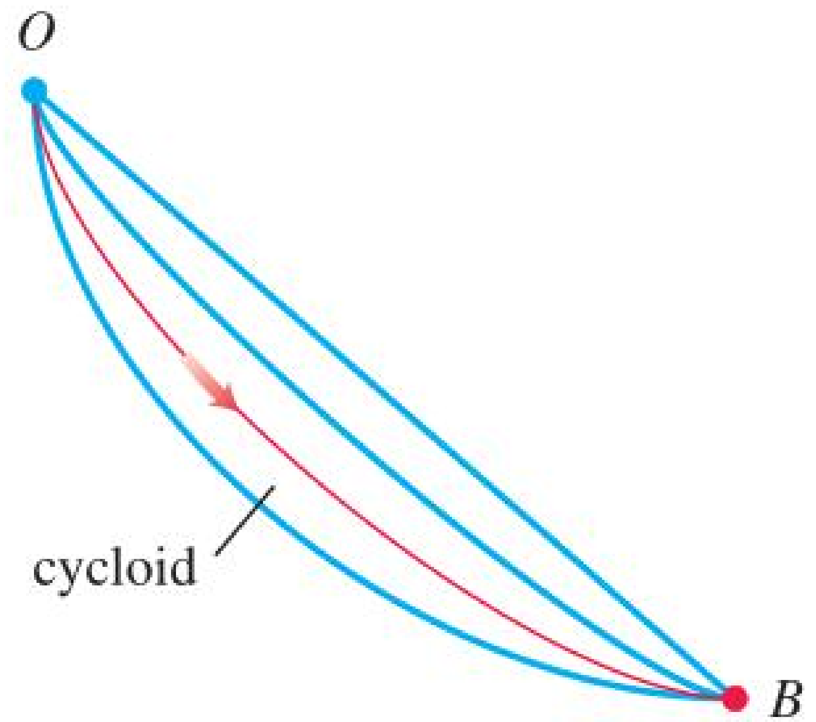
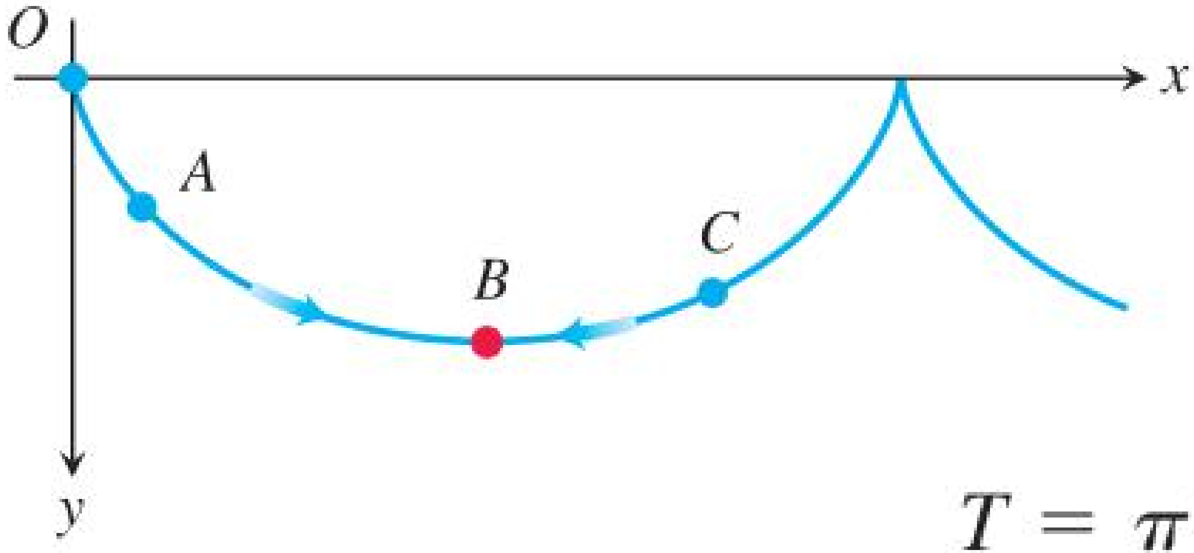


FIGURE 11.7 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\,dx.$$

minimize the value of this integral





$$T = \pi\sqrt{a/g}.$$

11.2

Calculus with Parametric Curves
参数曲线的微积分

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

by the Chain Rule: $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

EXAMPLE 1 Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $\left(\sqrt{2}, 1\right)$, where $t = \pi/4$ (Figure 11.13).

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}. = \frac{1}{\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2}.$$

The tangent line is $y - 1 = \sqrt{2}\left(x - \sqrt{2}\right)$

$$y = \sqrt{2}\,x - 1.$$

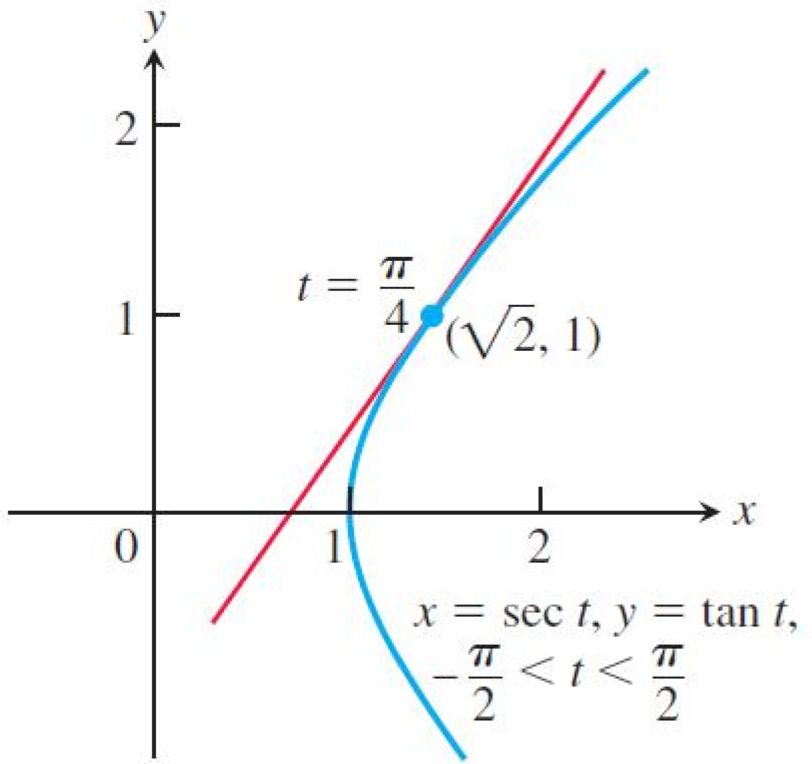


FIGURE 11.13 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

EXAMPLE 2

Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$.

Solution

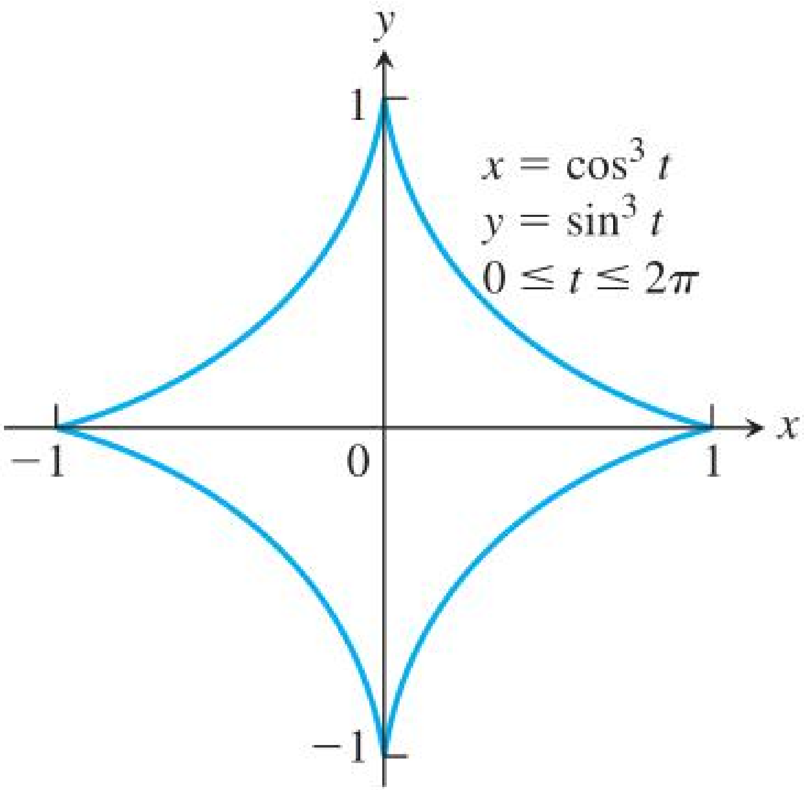
$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3}$$

EXAMPLE 3 Find the area enclosed by the astroid (Figure 11.14)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$



Solution By symmetry,

$$A = 4\int_0^1 y\,dx = 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t \sin t\,dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1 - \cos 2t}{2}\right)^2 \left(\frac{1 + \cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\,dt$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt$$

$$= \frac{3\pi}{8}.$$

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \qquad \text{and} \qquad y = g(t), \qquad a \leq t \leq b.$$

f and g are **continuously differentiable**

$f'(t)$ and $g'(t)$ are not simultaneously zero,

$$L = \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE 5 Find the length of the astroid

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3|\cos t \sin t| = 3\cos t \sin t.$$

Because of the curve's symmetry

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t \, dt = \frac{3}{2}.$$

The length of the astroid is four times this: $4(3/2) = 6$.

EXAMPLE 6 Find the perimeter of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution
$$x = a \sin t$$
$$y = b \cos t,\ a > b \text{ and } 0 \le t \le 2\pi.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\cos^2 t + b^2 \sin^2 t = a^2 - (a^2 - b^2)\sin^2 t$$
$$= a^2\left[1 - e^2\sin^2 t\right]$$

$e = 1 - \dfrac{b^2}{a^2},$

$$P = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2 t}\,dt.$$

elliptic integral

$|e \sin t| \le e < 1$

$$\sqrt{1 - e^2\sin^2 t} = 1 - \frac{1}{2}e^2\sin^2 t - \frac{1}{2\cdot 4}e^4\sin^4 t - \cdots,$$

$$P = 2\pi a\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5} - \cdots\right].$$

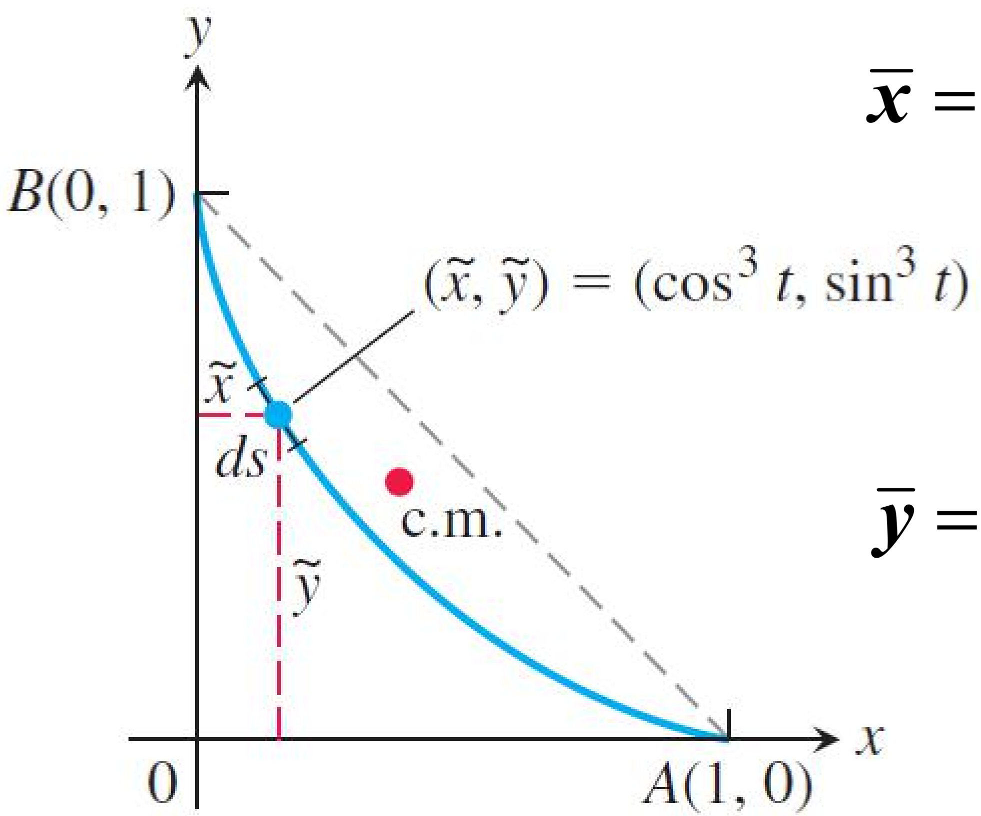
The Arc Length Differential

curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$,

$$s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2}\, dz.$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$ds = \sqrt{1 + f'(x)^2}\, dx \qquad ds = \sqrt{dx^2 + dy^2}.$$



$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} = \frac{\int \tilde{x}\,\delta\,ds}{\int \delta\,ds}$$

$$\bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} = \frac{\int \tilde{y}\,\delta\,ds}{\int \delta\,ds}$$

FIGURE 11.17 The centroid (c.m.) of the astroid arc in Example 7.

EXAMPLE 7 Find the centroid of the first-quadrant arc of the astroid

Solution take the curve's density to be $\delta = 1$

mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$.

$$dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3\cos t \sin t\, dt.$$

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3\cos t \sin t\, dt = \frac{3}{2}.$$

$$M_x = \int \tilde{y}\ dm = \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t\, dt = 3 \cdot \frac{\sin^5 t}{5}\bigg]_0^{\pi/2} = \frac{3}{5}.$$

$$\bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}.$$ The centroid is the point $(2/5, 2/5)$.

Areas of Surfaces of Revolution

$S = \int 2\pi y \, ds$ for revolution about the x-axis,

$x = f(t)$ and $y = g(t), a \le t \le b,$

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

EXAMPLE 9

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \leq t \leq 2\pi.$$

find the area of the surface swept out by revolving the circle about the x-axis

Solution

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt = 4\pi^2.$$

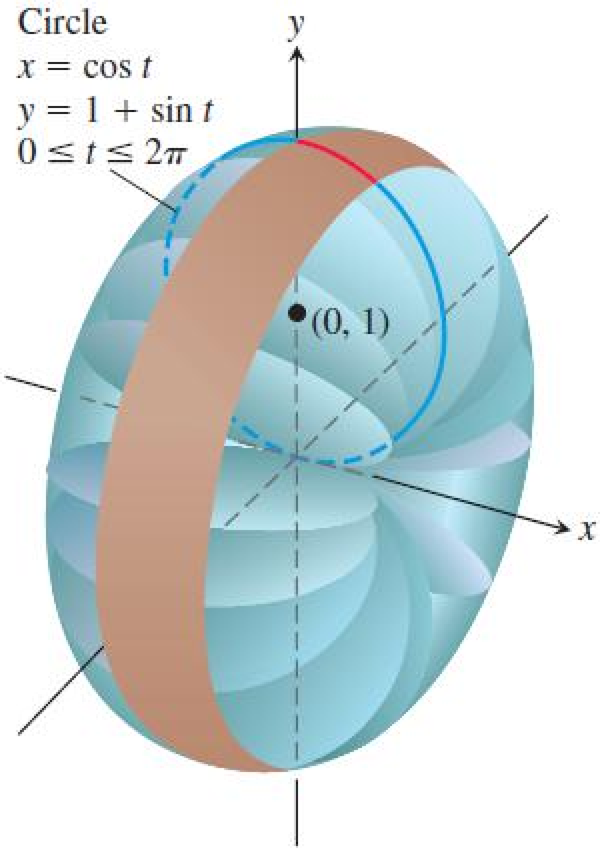
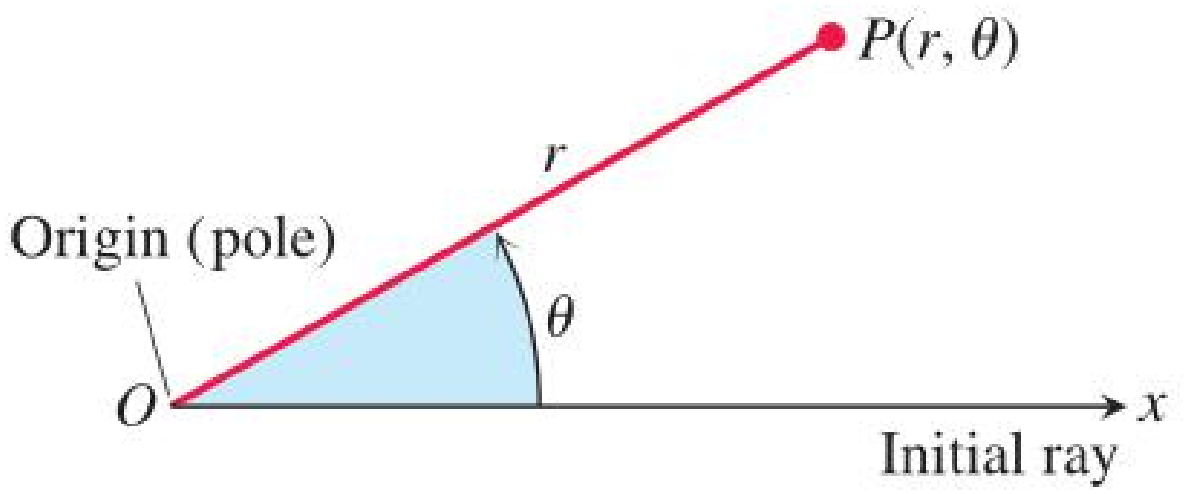


FIGURE 11.18 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

11.3

Polar Coordinates
极坐标



$$P(r, \theta)$$

Directed distance from O to P

Directed angle from initial ray to OP

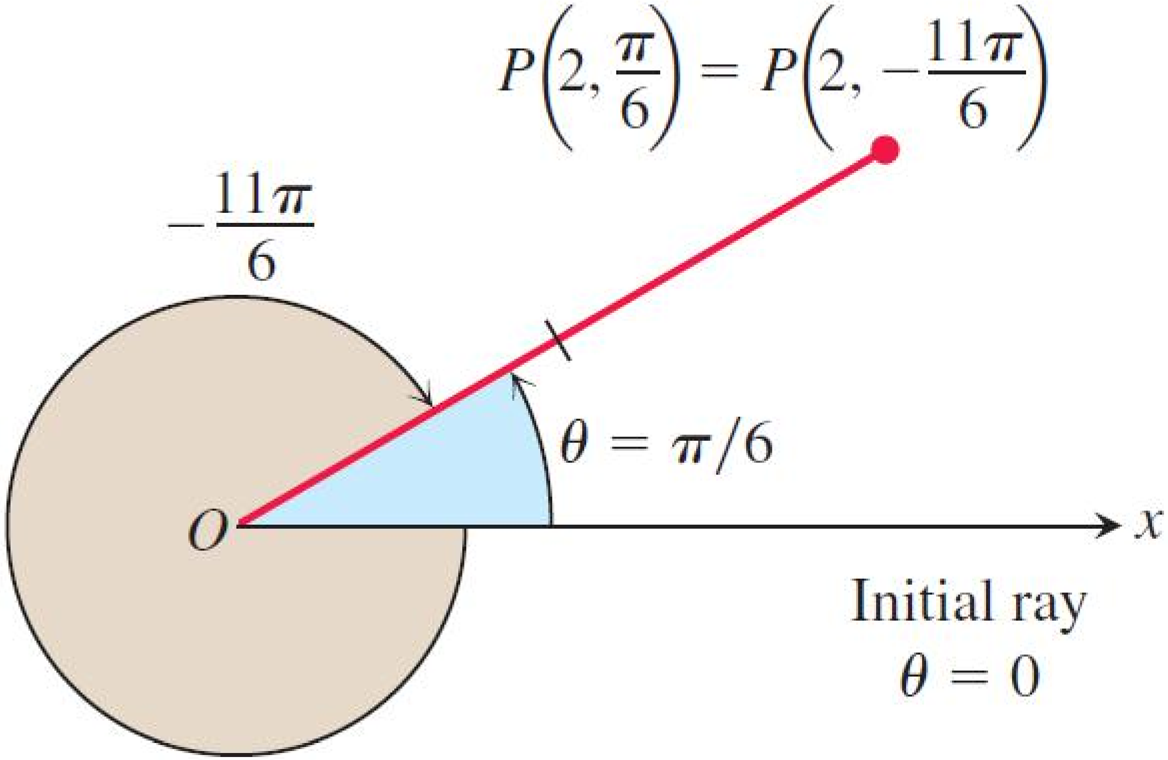


FIGURE 11.20 Polar coordinates are not unique.

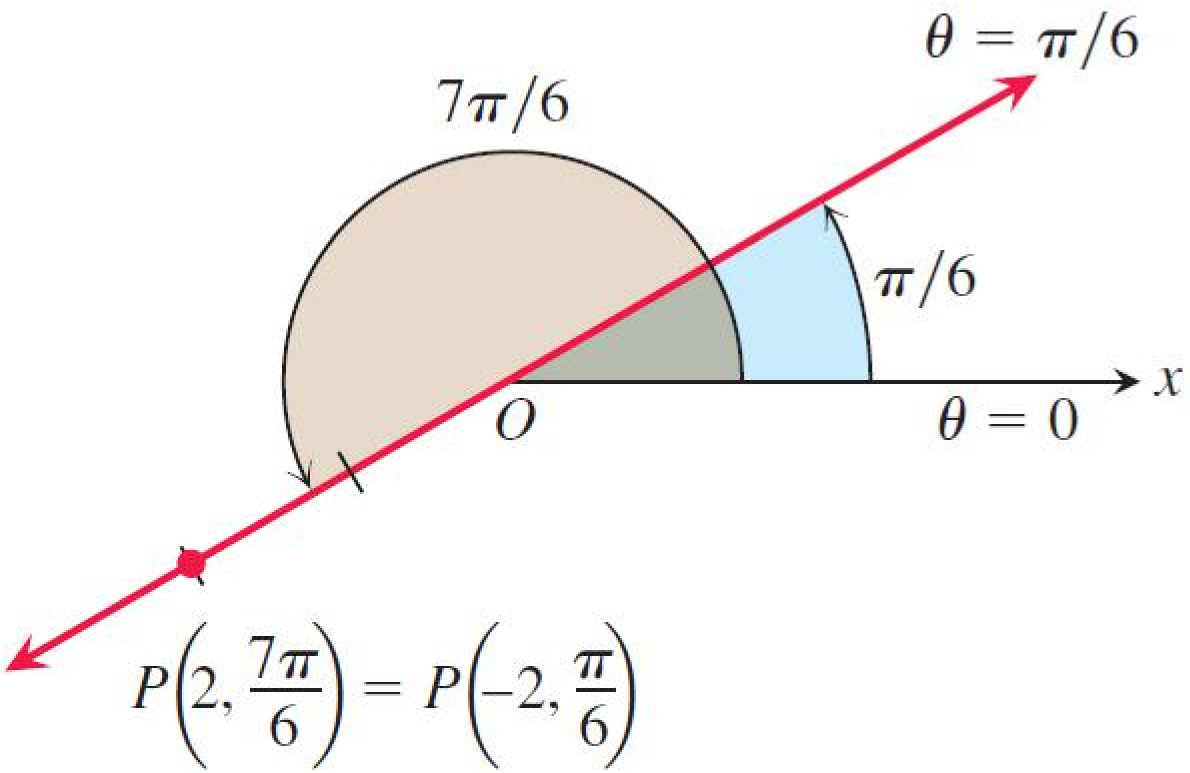


FIGURE 11.21 Polar coordinates can have negative r-values.

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots .$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

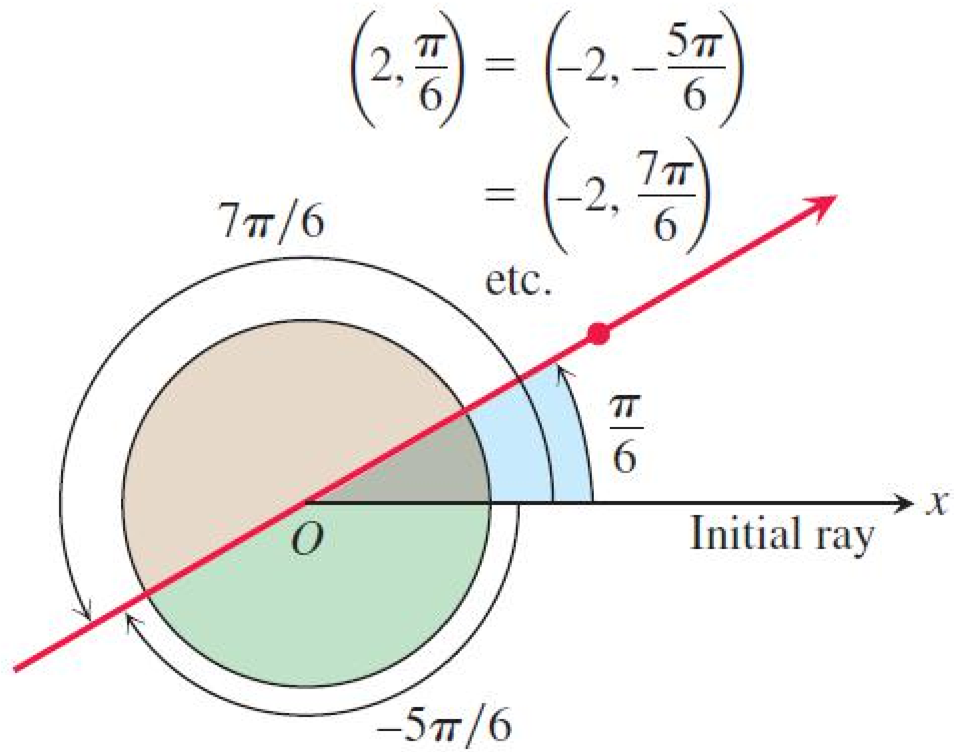
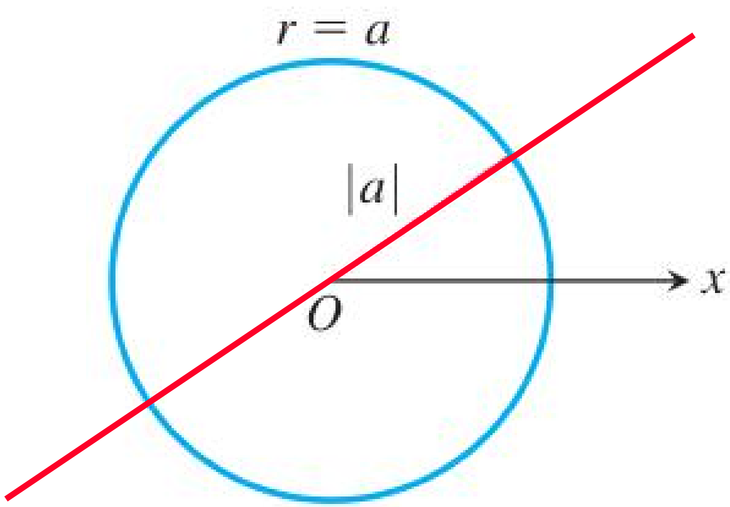


FIGURE 11.22 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

Polar Equations and Graphs

$\theta = \theta_0$ and let r vary between $-\infty$ and ∞,



$r = a \neq 0$, θ varies over any interval of length 2π,

EXAMPLE 2 A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 11.22.

EXAMPLE 3 Graph the sets of points whose polar coordinates satisfy

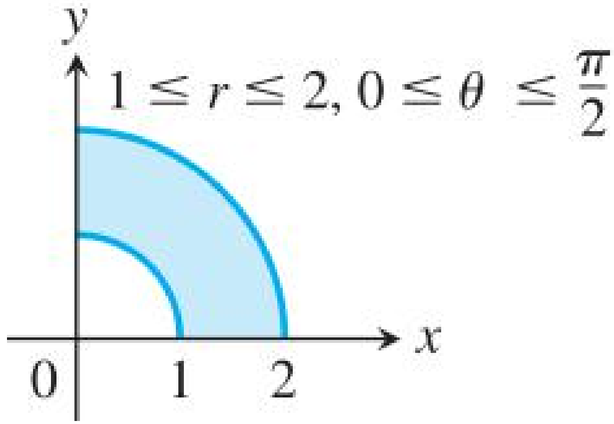
(a) $1 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2 \quad \text{and} \quad \theta = \dfrac{\pi}{4}$

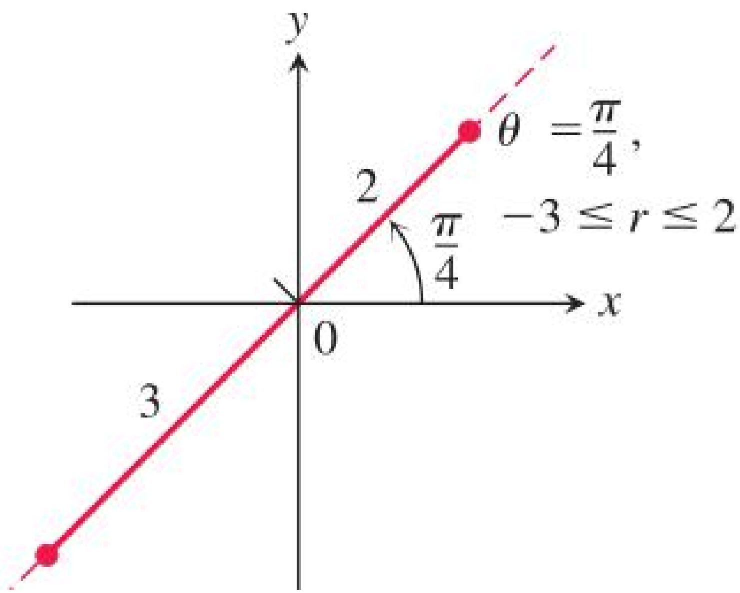
(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

Solution

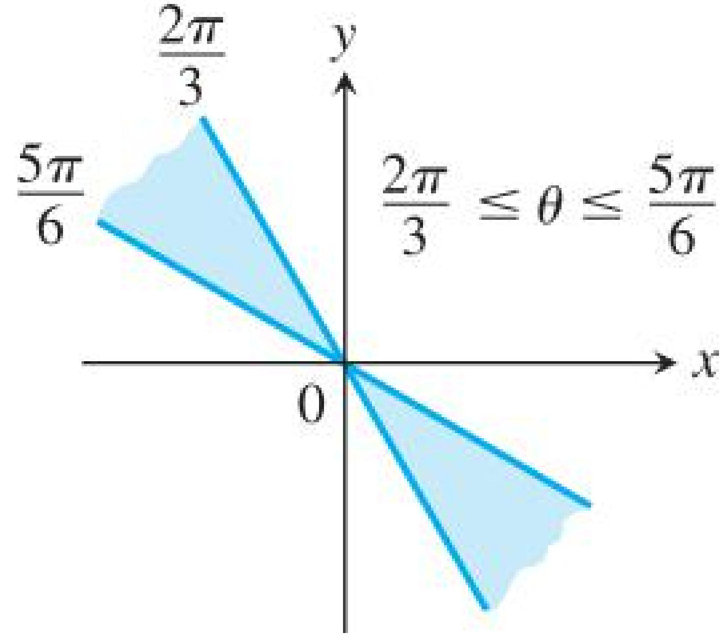
(a)



(b)



(c)

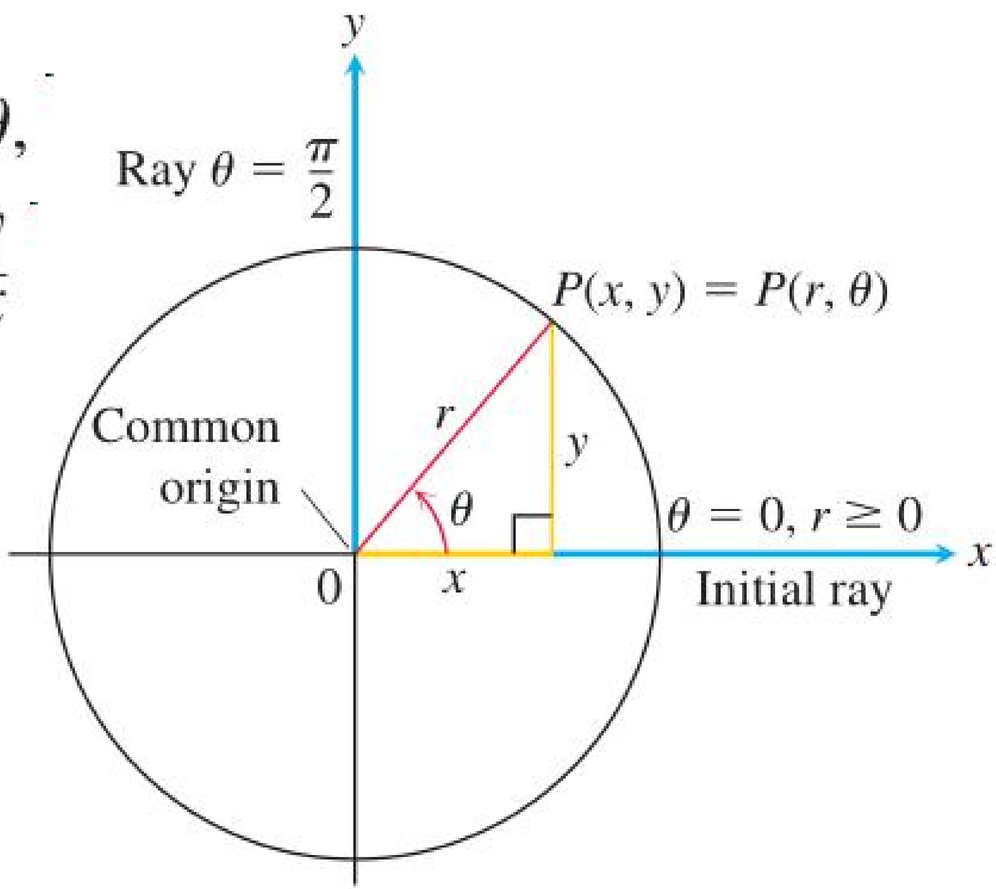


Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis.

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$



there is a unique $\theta \in [0, 2\pi)$

$$\mathbf{0 < r < \infty}$$

EXAMPLE 4 Here are some plane curves expressed in terms of both

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution

$$x^2 + y^2 - 6y = 0$$

$$r^2 - 6r \sin\theta = 0$$

$$r = 0 \quad \text{or} \quad r - 6 \sin\theta = 0$$

$$r = 6 \sin\theta$$

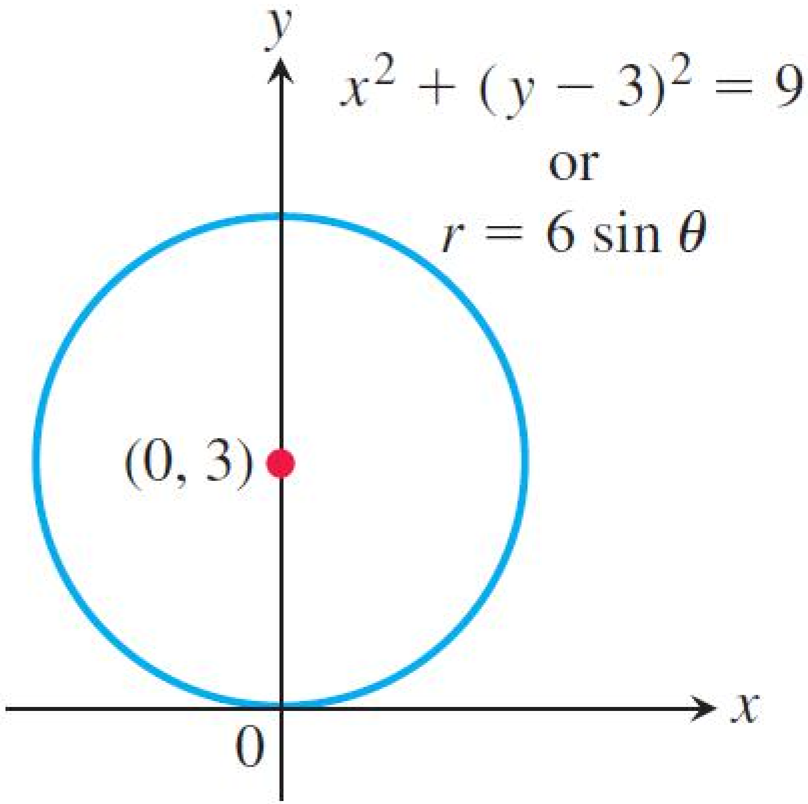


FIGURE 11.26 The circle in Example 5.

EXAMPLE 6

Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$ **(b)** $r^2 = 4r\cos\theta$ **(c)** $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

(a) $x = -4$ **(b)** $x^2 + y^2 = 4x$

(c) $r(2\cos\theta - \sin\theta) = 4$ $\quad 2x - y = 4$

11.4

Graphing Polar Coordinate Equations

极坐标方程的做图

Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is still given by dy/dx,

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\dfrac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $dx/d\theta \neq 0$ at (r, θ).

例. 求螺线 $r=\theta$ 在 $\theta=\pi$ 处的斜率.

解. $x=r\cos\theta,\ y=r\sin\theta,$

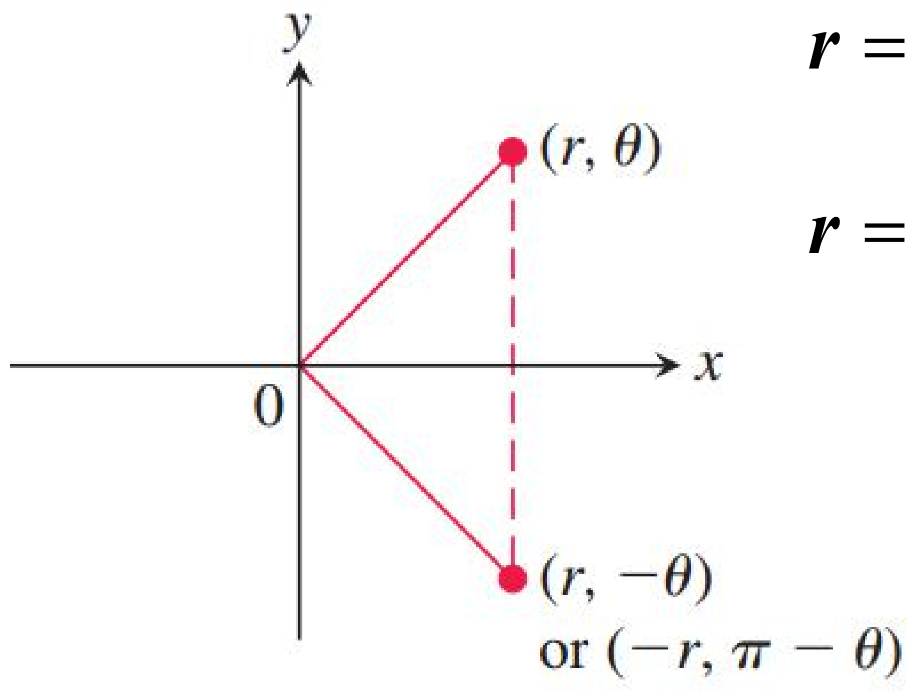
$x=\theta\cos\theta,\ y=\theta\sin\theta,$

$$\frac{dy}{dx}=\frac{\sin\theta+\theta\cos\theta}{\cos\theta-\theta\sin\theta},$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi}=\frac{-\pi}{-1}=\pi.$$

Symmetry

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.27a).

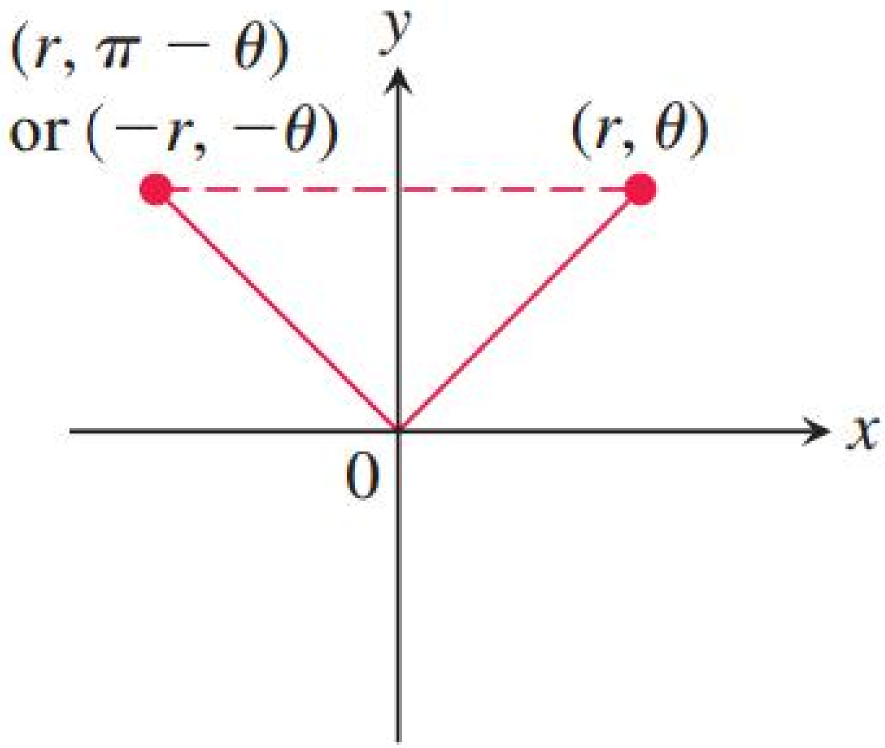


(a) About the x-axis

$$r = \cos\theta$$

$$r = \cos 2\theta$$

2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.27b).

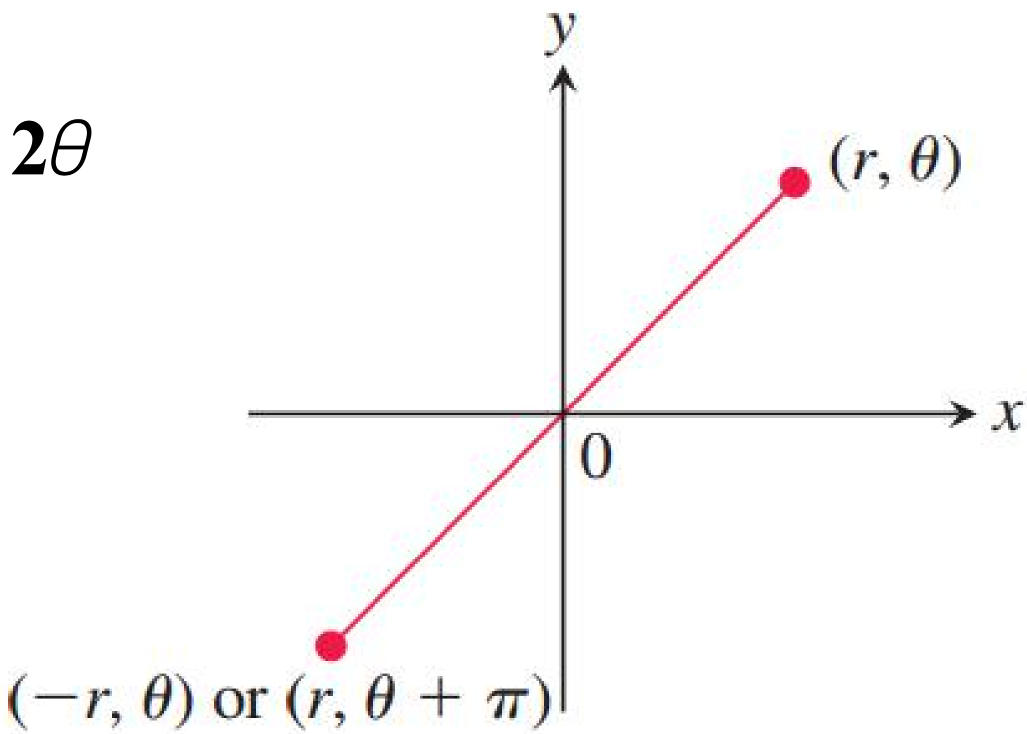


(b) About the y-axis

$$r = \cos 2\theta$$

3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.27c).

$$r = \cos 2\theta$$



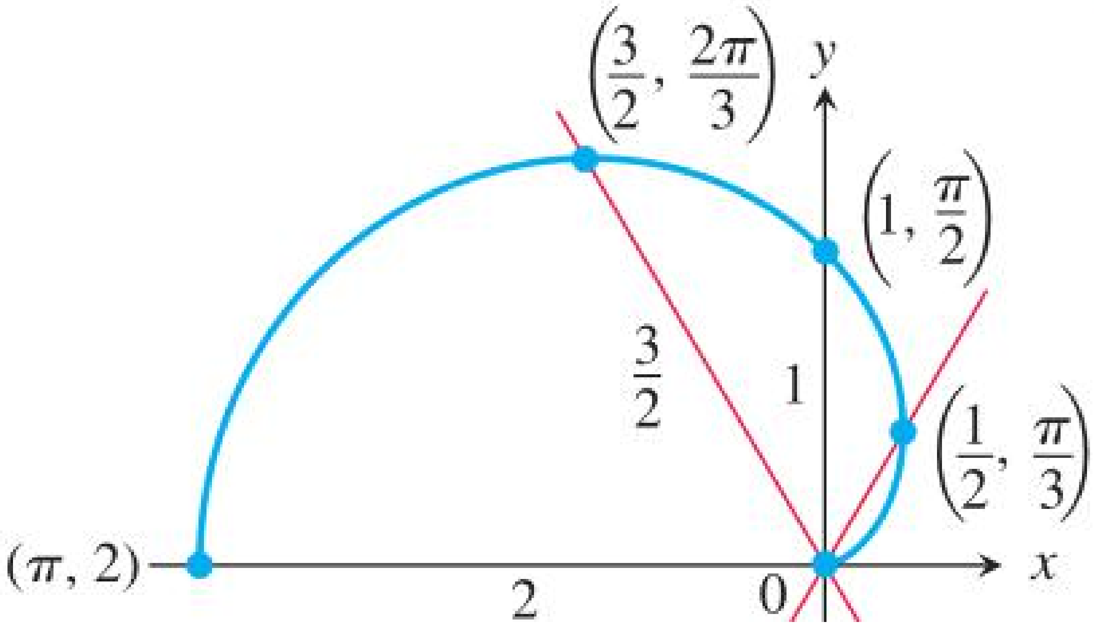
(c) About the origin

EXAMPLE 1 Graph the curve $r = 1 - \cos\theta$ in the Cartesian xy-plane.

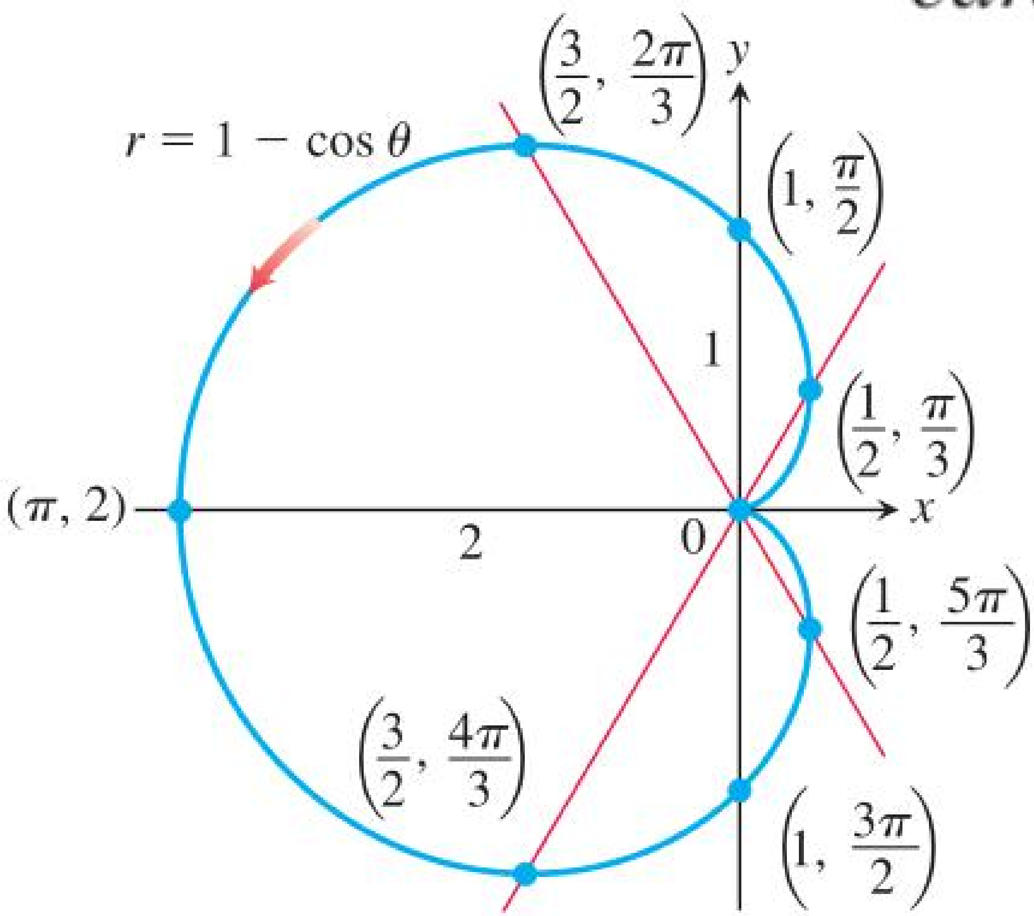
Solution The curve is symmetric about the x-axis

(r, θ) on the graph $\Rightarrow (r, -\theta)$ on the graph.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2



cardioid



$$r = 1 + \cos\theta$$

EXAMPLE 2 Graph the curve $r^2 = 4\cos\theta$ in the Cartesian xy-plane.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$,

θ from $-\pi/2$ to $\pi/2$.

The curve is symmetric about the x-axis

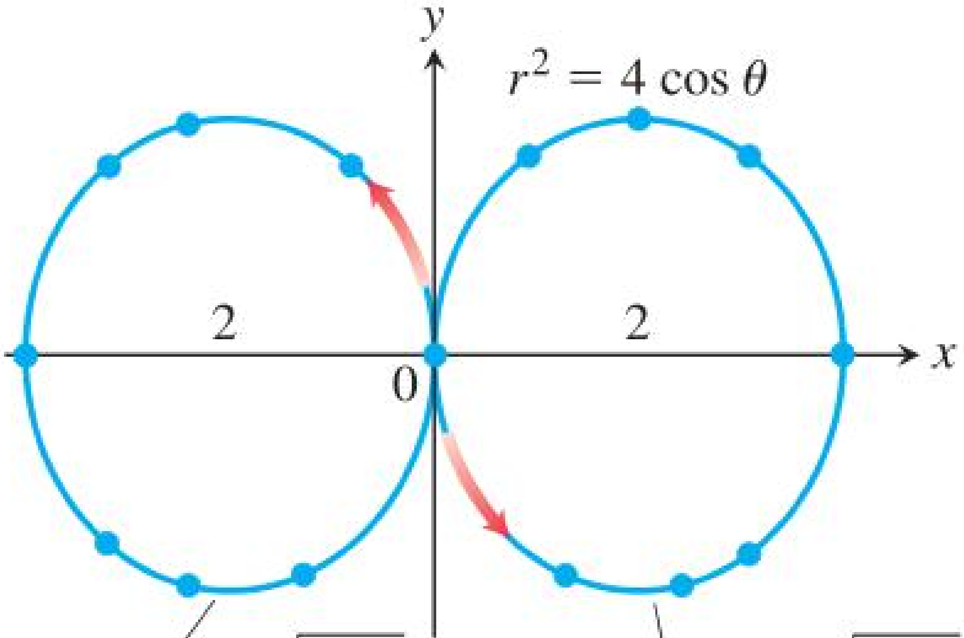
$$(r, \theta) \text{ on the graph } \Rightarrow (r, -\theta) \text{ on the graph.}$$

The curve is also symmetric about the origin

$$(r, \theta) \text{ on the graph } \Rightarrow (-r, \theta) \text{ on the graph.}$$

$$r = \pm 2\sqrt{\cos\theta}.$$

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0



Loop for $r = -2\sqrt{\cos\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Loop for $r = 2\sqrt{\cos\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

11.5

Areas and Lengths in Polar Coordinates
极坐标下计算面积和长度

Area in the Plane

The region is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS.

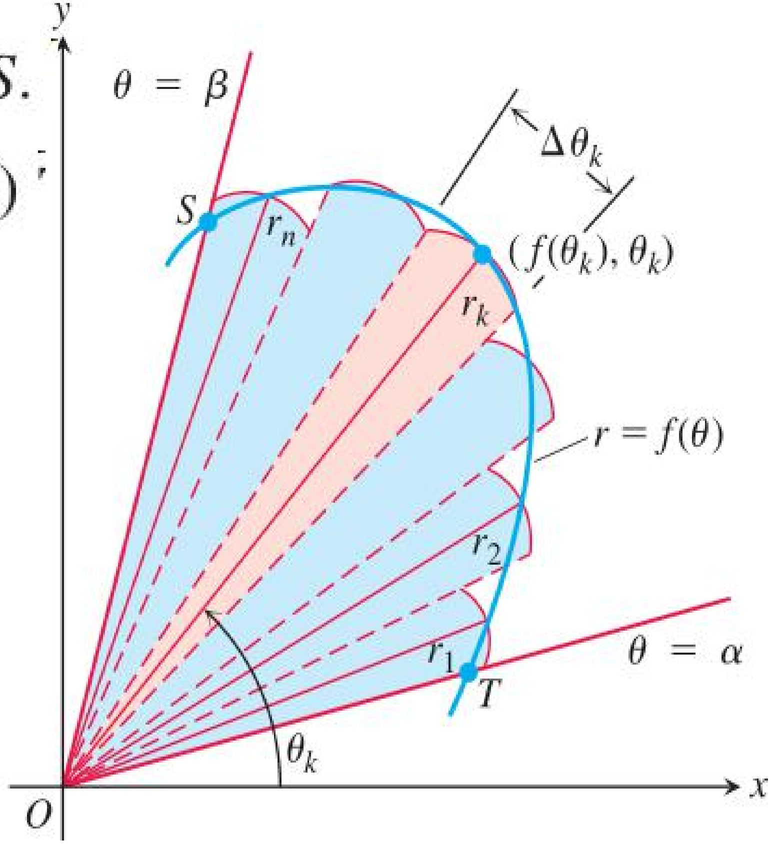
The typical sector has radius $r_k = f(\theta_k)$

$$A_k = \frac{1}{2} r_k{}^2 \, \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$



Area of the Fan-Shaped Region Between the Origin and the Curve

$$r = f(\theta), \quad \alpha \leq \theta \leq \beta$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

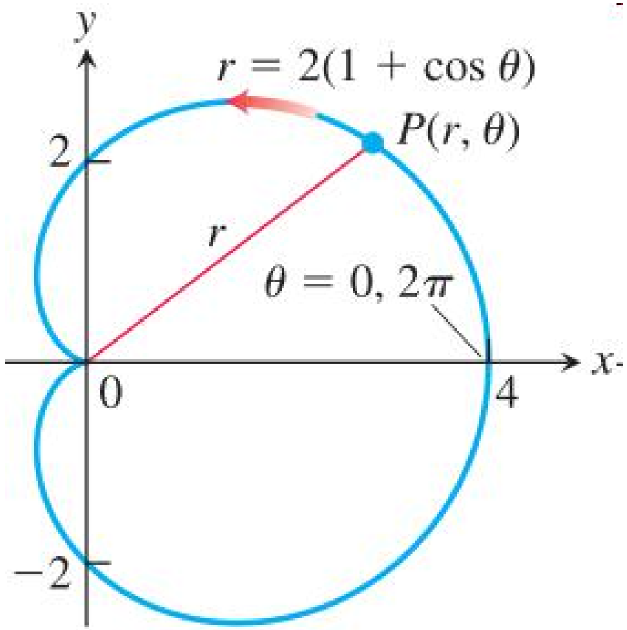
the **area differential** $dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta) \right)^2 d\theta.$

EXAMPLE 1

Find the area of the region in the xy-plane enclosed by the cardioid

$r = 2(1 + \cos \theta).$

Solution

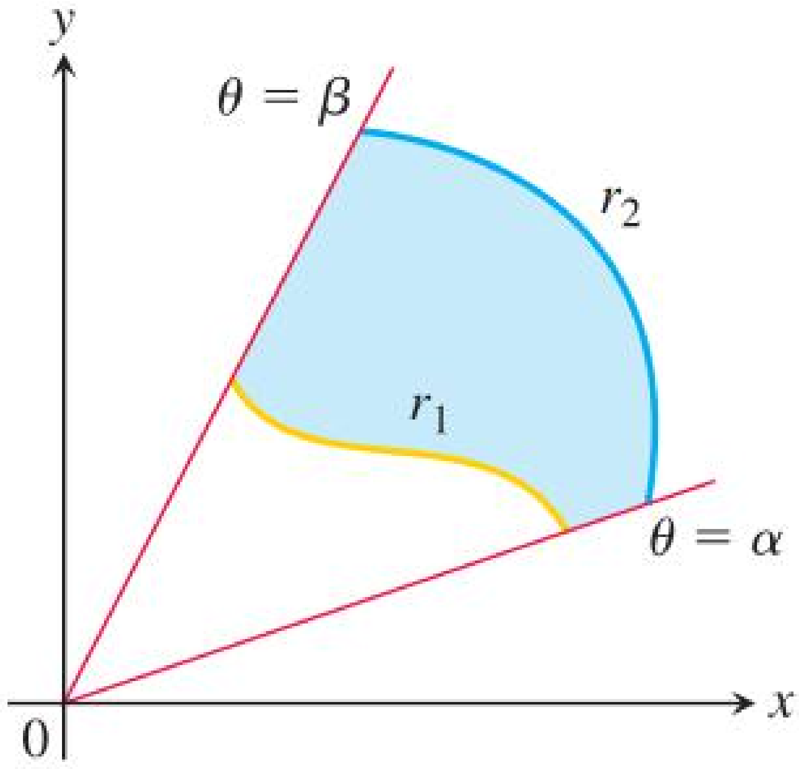


$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta = 6\pi.$$

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^{\,2}\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^{\,2}\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$



EXAMPLE 2

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

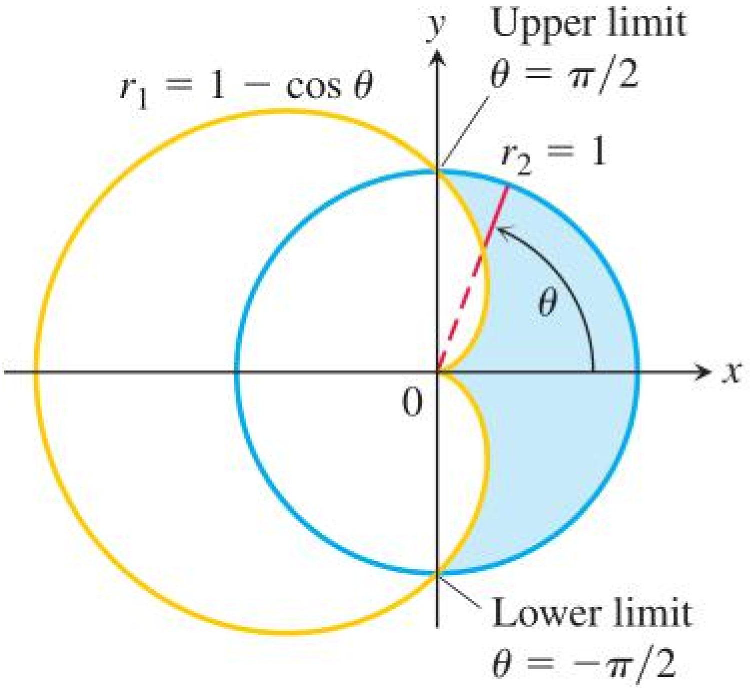
Solution

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$

$$= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= 2 - \frac{\pi}{4}.$$



Length of a Polar Curve

$r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$

$P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β,

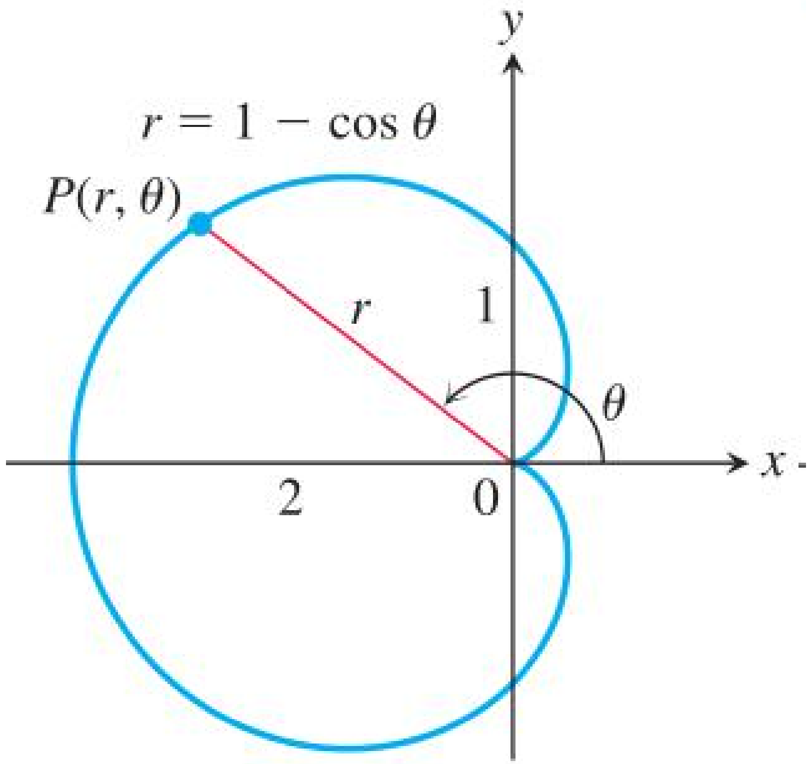
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE 3 Find the length of the cardioid $r = 1 - \cos\theta$.

Solution

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_{0}^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta = 8.$$



y
$r = 1 - \cos\theta$
$P(r, \theta)$
r
1
θ
x
2
0

11.6

Conic Sections
圆锥曲线

11.7

Conics in Polar Coordinates