

# *Linear Algebra*



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## 2

# Vector Spaces (向量空间)

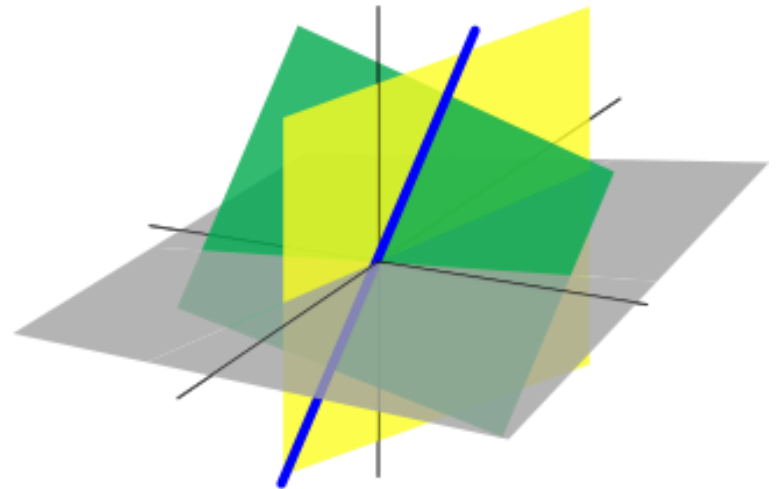
## 2.6

## LINEAR TRANSFORMATION (线性变换)

Definition & Examples

Matrix representations

Kernel (核)



# I. Linear Transformation: Definition & Examples

A **function** (函数)  $f$  from a set  $A$  to a set  $B$  is a rule that assigns to each element of  $A$  a *single* element of  $B$ . We often write

$$f : A \rightarrow B$$

$$a \mapsto f(a)$$

where  $f(a)$  is often defined by some equation, with

$\text{range}(f) = \{f(a) \mid a \in A\} \subseteq B$ . (range: 值域; domain: 定义域)

**For example,**

- $f(x) = \sin x$  is a function from  $\mathbf{R}$  to  $[-1, 1]$ .
- $f: (x, y) \mapsto (2x, 3y)$  maps  $\mathbf{R}^2$  to  $\mathbf{R}^2$ .
- $f: (x, y) \mapsto (x, y, x + y)$  maps  $\mathbf{R}^2$  to  $\mathbf{R}^3$ .
- $f: (x, y, z) \mapsto (x, z)$  maps  $\mathbf{R}^3$  to  $\mathbf{R}^2$ .

**Definition 1** A function  $f$  from a vector space  $V$  to a vector space  $W$  is called a **linear transformation** (线性变换) if

- (1)  $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$  for all vectors  $\mathbf{u}, \mathbf{v} \in V$ ;
- (2)  $f(c\mathbf{v}) = cf(\mathbf{v})$  for all vectors  $\mathbf{v} \in V$  and all  $c \in \mathbf{R}$ .

$$(1) \& (2) \Leftrightarrow f(c\mathbf{u} + d\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \in V \text{ and all } c, d \in \mathbf{R}.$$

### Examples

- $f: (x, y) \mapsto (2x, 3y)$  maps  $\mathbf{R}^2$  to  $\mathbf{R}^2$ .
- $f: (x, y) \mapsto (x, y, x + y)$  maps  $\mathbf{R}^2$  to  $\mathbf{R}^3$ .
- $f: (x, y, z) \mapsto (x, z)$  maps  $\mathbf{R}^3$  to  $\mathbf{R}^2$ .

Linear transformations  
*preserve the operations  
of vector addition and  
scalar multiplication.*

(线性变换保持加法和  
数乘运算)

### Examples

- $f(x) = x^2$  is **not** a linear transformation from  $\mathbf{R}$  to  $\mathbf{R}$ , since  

$$f(x + y) = (x + y)^2 \neq x^2 + y^2 = f(x) + f(y), \text{ except } xy=0.$$
- $f(x) = \sin x$  is **not** a linear transformation from  $\mathbf{R}$  to  $\mathbf{R}$ , since  

$$f(x + y) = \sin(x + y) \neq \sin x + \sin y = f(x) + f(y) \text{ does not always hold.}$$

## Other examples

We take as examples the spaces  $\mathbf{P}_n$ , in which the vectors are polynomials  $p(t)$  of degree  $n$ .

$$p(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n,$$

and the dimension of the vector space is  $n + 1$ .

- The operation of *differentiation* is linear

$$\frac{d}{dt}p(t) = a_1 + 2a_2t + \cdots + na_nt^{n-1}.$$

- *Integration* from 0 to  $t$  is also linear (*it takes  $\mathbf{P}_n$  to  $\mathbf{P}_{n+1}$* )

$$\int_0^t p(s)ds = a_0t + \frac{1}{2}a_1t^2 + \cdots + \frac{a_n}{n+1}t^{n+1}.$$

- *Multiplication* by a fixed polynomial like  $2 + 3t$  is linear (*it also takes  $\mathbf{P}_n$  to  $\mathbf{P}_{n+1}$* ) :

$$(2 + 3t)p(t) = 2a_0 + \cdots + 3a_nt^{n+1}.$$

## II. Transformations Represented by Matrices

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

For any vector  $\mathbf{x} \in \mathbf{R}^n$ , the product  $\mathbf{Ax}$  is a vector in  $\mathbf{R}^m$ :

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \in \mathbf{R}^m$$

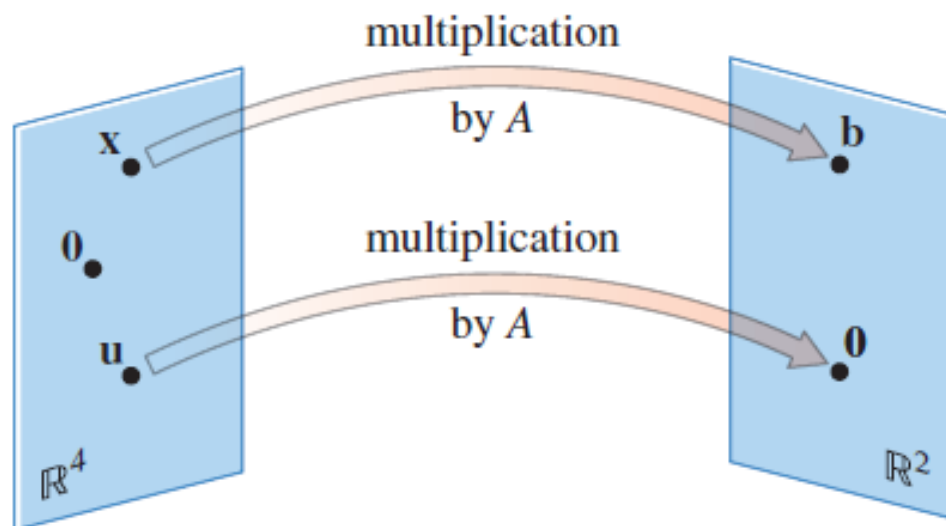
This defines a function  $f$  from  $\mathbf{R}^n$  to  $\mathbf{R}^m$ :  $f : \mathbf{x} \mapsto \mathbf{Ax}$ .

Suppose  $\mathbf{x}$  is an  $n$ -dimensional vector.

When  $\mathbf{A}$  multiplies  $\mathbf{x}$ , it *transforms* that vector into a new vector  $\mathbf{Ax}$ , which is an  $m$ -dimensional vector.

For instance,

$$\begin{array}{ccccc} \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 5 \\ 8 \end{bmatrix} & \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow \\ \mathbf{A} & \mathbf{x} & \mathbf{b} & & \mathbf{A} & \mathbf{u} & \mathbf{0} \end{array}$$



It is a *linear transformation* as, for all  $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$  and  $c \in \mathbf{R}$ ,

$$f(\mathbf{v} + \mathbf{w}) = A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = f(\mathbf{v}) + f(\mathbf{w}),$$

$$f(c\mathbf{v}) = A(c\mathbf{v}) = cA\mathbf{v} = cf(\mathbf{v}).$$

That is to say, matrix multiplication satisfies *the rule of linearity* (线性性).

**Remark:** If  $A$  is square ( $n$  by  $n$ ):

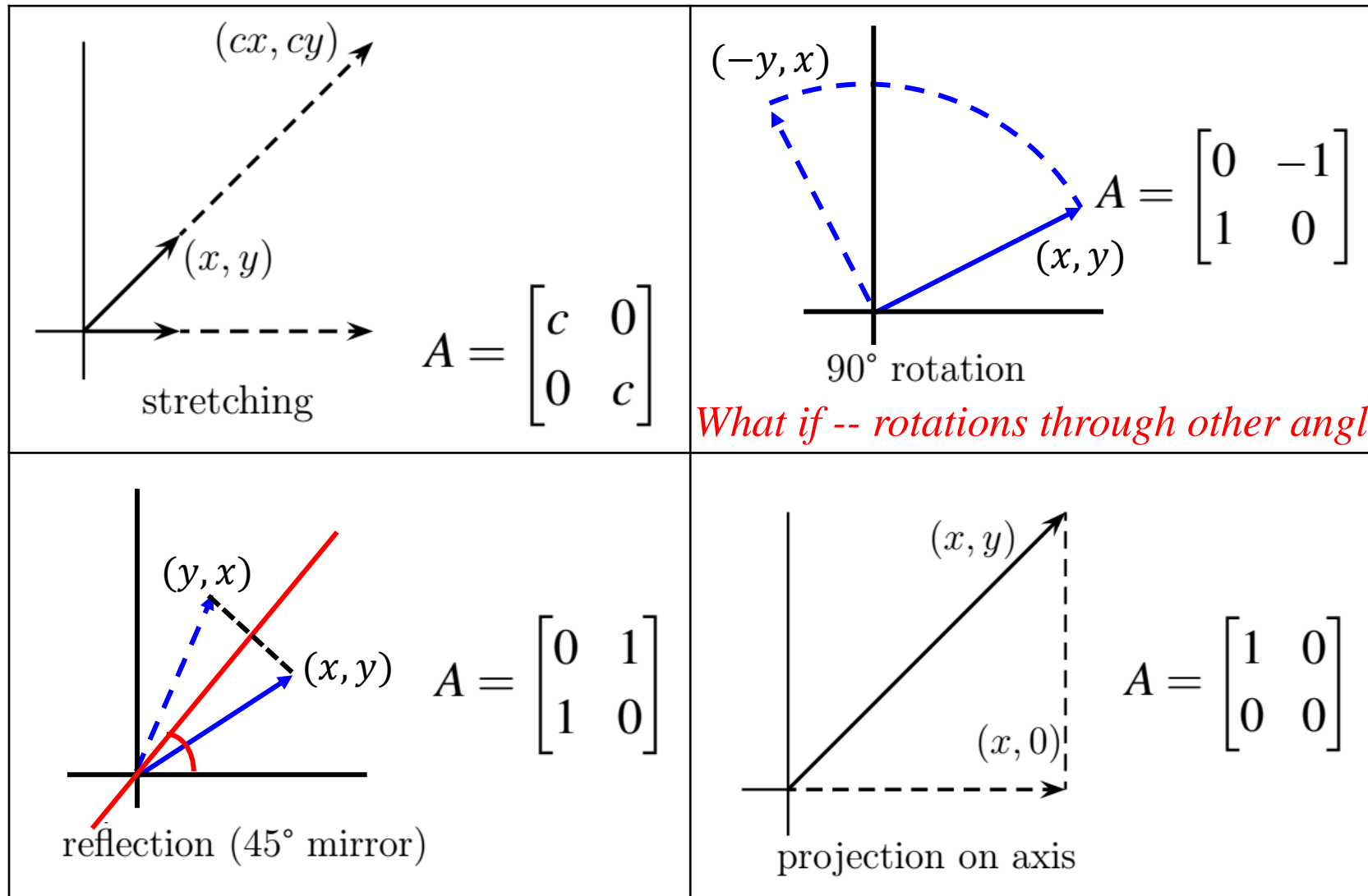
Suppose  $\mathbf{x}$  is an  $n$ -dimensional vector, then  $A\mathbf{x}$  is also an  $n$ -dimensional vector.

This happens at every point  $\mathbf{x}$  of the  $n$ -dimensional space  $\mathbf{R}^n$ .

The whole space is transformed, or “mapped into itself,” by the matrix  $A$ . (整个空间 $\mathbf{R}^n$ 在方阵 $A$ 的作用下, 变换/映射到自身:  $\mathbf{R}^n$ )



# Matrix representation - Examples in Geometry



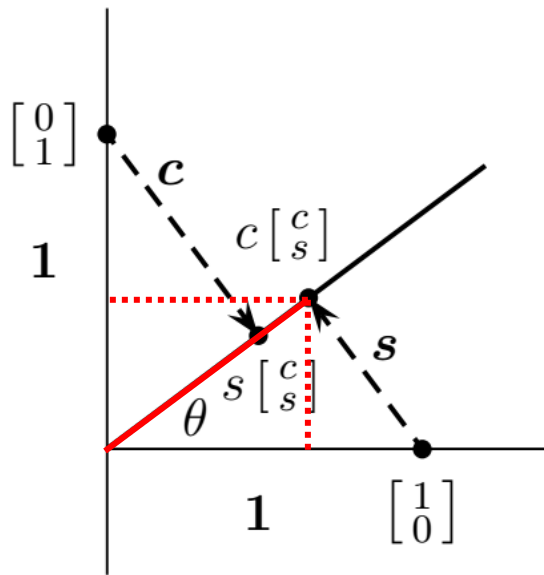
*What if -- rotations through other angles?*

*What if -- reflections in other mirrors?*

*What if -- projections onto other lines?*

**Can you find the matrix representation?**

## Projection (投影)



Projection onto the  $\theta$ -line

In general, we may find the projection to the  $\theta$ -line (the line at the angle  $\theta$  from the  $x$ -axis). Thus the linear transformation  $p$  is such that

$$p : \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow c \begin{bmatrix} c \\ s \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow s \begin{bmatrix} c \\ s \end{bmatrix} \end{cases} \quad \begin{aligned} c &= \cos \theta \\ s &= \sin \theta \end{aligned}$$

The point  $\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is

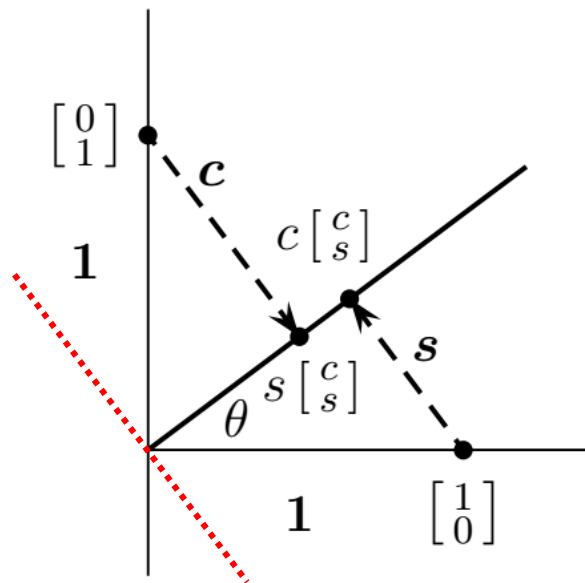
projected to:  $x \cdot p \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + y \cdot p \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$$= x \begin{bmatrix} c^2 \\ cs \end{bmatrix} + y \begin{bmatrix} cs \\ s^2 \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

**$P$**  : projection matrix

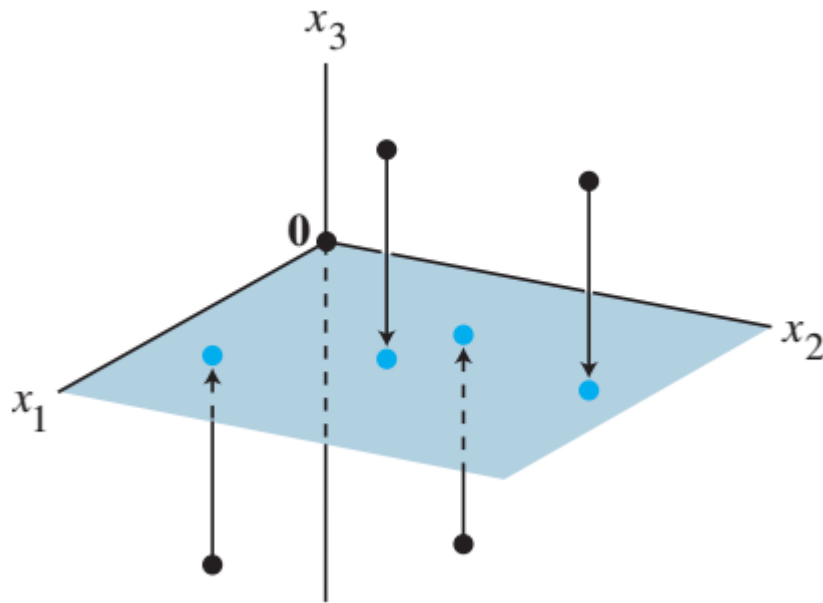
$\mathbf{P} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$  has some natural properties.



Projection onto the  $\theta$ -line

- This matrix has no inverse ( $\det(\mathbf{P})=0$ ), because the transformation has no inverse.
- Points on the perpendicular line are projected onto the origin; that line is the nullspace of  $\mathbf{P}$ .
- Points on the  $\theta$ -line are projected to themselves!
- Projecting twice is the same as projecting once, and  $\mathbf{P}^2 = \mathbf{P}$  (幂等矩阵, idempotent matrix), i.e., *a projection matrix equals its own square.* (投影矩阵等于自身的平方)

What 3 by 3 matrices represent the transformations that project every vector onto the  $x_1$ - $x_2$  plane?



$$\text{If } \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then the transformation

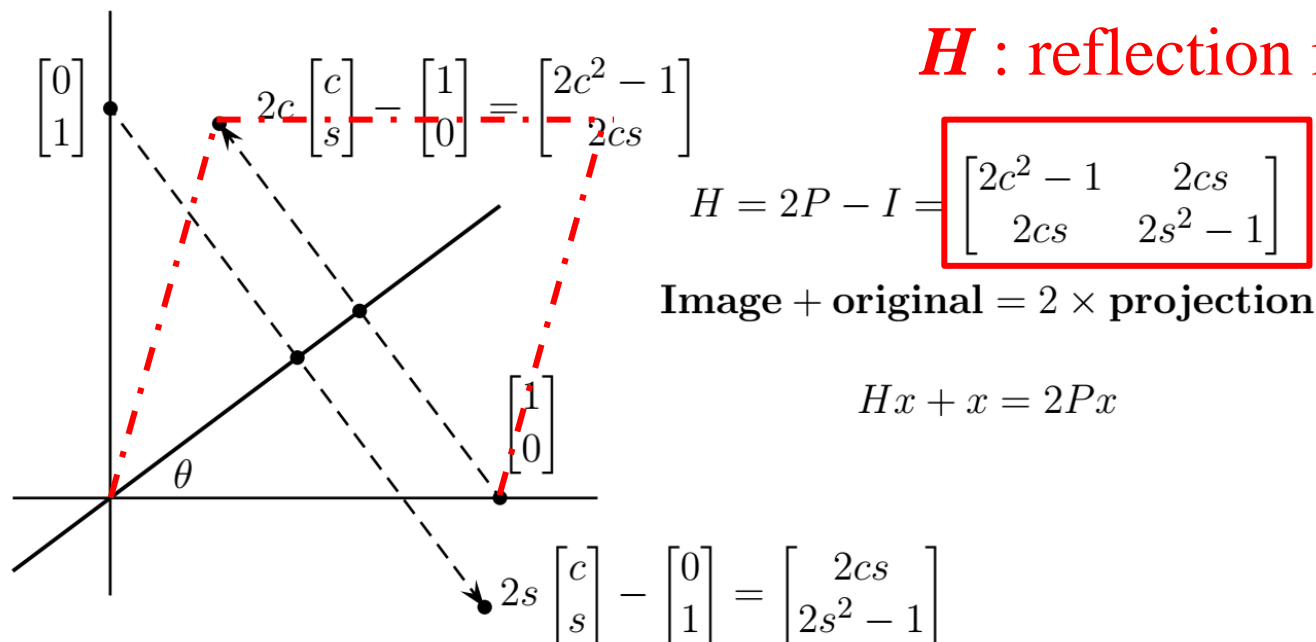
$$\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$$

projects points in  $\mathbf{R}^3$  onto the  $x_1$ - $x_2$  plane because

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

## Reflection (反射)

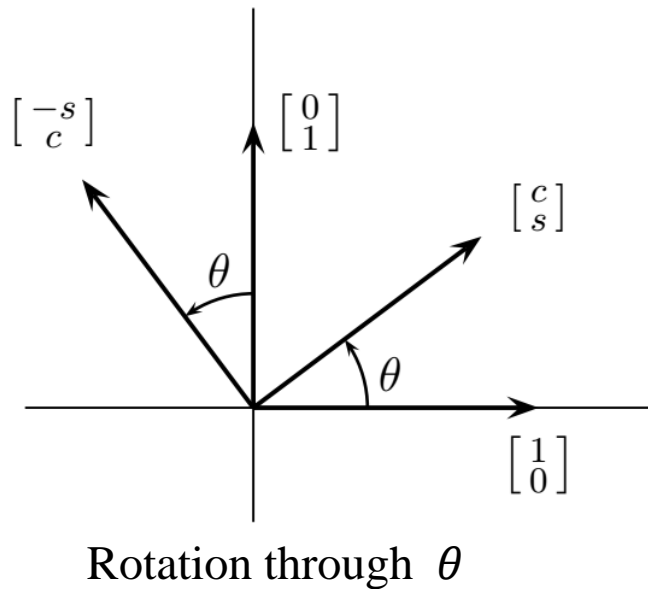
In general, we may find reflection along  $\theta$ -line in a similar way:



$H$  has some remarkable properties:

- $H^2 = I$ . Two reflections bring back the original.  
( $H^2 = (2P - I)^2 = 4P^2 - 4P + I = I$ , since  $P^2 = P$ .)
- $H^{-1} = H$ . A reflection is its own inverse.

## Rotation (旋转)



$$\mathbf{Q}_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} : \text{rotation matrix}$$

is a perfect example showing the correspondence between transformations and matrices:

- $\mathbf{Q}_\theta \mathbf{Q}_{-\theta} = \mathbf{I}$   
The inverse of  $\mathbf{Q}_\theta$  equals  $\mathbf{Q}_{-\theta}$  (rotation backward through  $\theta$ )
- $\mathbf{Q}_\theta^2 = \mathbf{Q}_{2\theta}$   
The square of  $\mathbf{Q}_\theta$  equals  $\mathbf{Q}_{2\theta}$  (rotation through a double angle)
- $\mathbf{Q}_\theta \mathbf{Q}_\phi = \mathbf{Q}_{\theta+\phi}$   
The product of  $\mathbf{Q}_\theta$  and  $\mathbf{Q}_\phi$  equals  $\mathbf{Q}_{\theta+\phi}$  (rotation through  $\phi$  then  $\theta$ )

# Matrix Representations of Linear Transformations

(线性变换的矩阵表示: *general case*)

**Example 1** Let  $f$  be a linear transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^3$  such that

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix},$$

Determine  $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ .

**Solution**

$$\begin{aligned} f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= f\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= xf\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yf\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ 2x - y \\ 3x + 2y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

**Theorem 1** Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear transformation. Then there exists a unique matrix  $A$  such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact,  $A$  is the  $m \times n$  matrix whose  $j$ -th column is the vector  $T(\mathbf{e}_j)$ , where  $\mathbf{e}_j$  is the  $j$ -th column of the identity matrix in  $\mathbf{R}^n$ :

$$A = [T(\mathbf{e}_1) \quad \dots \quad T(\mathbf{e}_n)].$$

**Proof** Write  $\mathbf{x} = I_n \mathbf{x} = [\mathbf{e}_1 \quad \dots \quad \mathbf{e}_n] \mathbf{x} = x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n$ , and use the linearity of  $T$  to compute

$$T(\mathbf{x}) = T(x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n) = x_1 T(\mathbf{e}_1) + \dots + x_n T(\mathbf{e}_n)$$

$$= \boxed{[T(\mathbf{e}_1) \quad \dots \quad T(\mathbf{e}_n)]} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}.$$

The matrix  $A$  is called the **standard matrix for the linear transformation**  $T$ .

(寻找矩阵 $A$ 的关键, 是知道线性变换 $T$ 对于标准基各列的作用)



**Example 2** Define the linear transformation  $L: \mathbf{R}^3 \rightarrow \mathbf{R}^2$  by

$$L(\mathbf{x}) = (x_1 + x_2, x_2 + x_3)^T.$$

for each  $\mathbf{x} = (x_1, x_2, x_3)^T$  in  $\mathbf{R}^3$ .

We wish to find a matrix  $\mathbf{A}$  such that  $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$  for each  $\mathbf{x} \in \mathbf{R}^3$ .

**Solution:** To do this, we calculate

$$L(\mathbf{e}_1) = L((1,0,0)^T) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L(\mathbf{e}_2) = L((0,1,0)^T) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$L(\mathbf{e}_3) = L((0,0,1)^T) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We choose these vectors to be the columns of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

To check the result, we compute  $\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}.$

## *What if – general case:* 非自然基

To transform a space to itself, one basis is enough.

A transformation from one space to another requires a basis for each. They can be bases other than the standard bases.

**Theorem 2** Suppose the vectors  $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  are a basis for the space  $V$ , and vectors  $F = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$  are a basis for  $W$ .

*Each linear transformation  $T$  from  $V$  to  $W$  is represented by a matrix  $\mathbf{A}$ . The  $j$ th column of  $\mathbf{A}$  is found by applying  $T$  to the  $j$ th basis vector  $\mathbf{v}_j$ , and writing  $T(\mathbf{v}_j)$  as a combination of the  $\mathbf{w}$ 's:*

**Column  $j$  of  $\mathbf{A}$**  is the coordinate vector of  $T(\mathbf{v}_j)$  with respect to  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ , which means

$$T(\mathbf{v}_j) = \mathbf{A}\mathbf{v}_j = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \dots + a_{mj}\mathbf{w}_m.$$

$E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  are a basis for  $V$ ,

$F = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$  are a basis for  $W$ .

$$T(\mathbf{v}_j) = \mathbf{A}\mathbf{v}_j = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \dots + a_{mj}\mathbf{w}_m$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$\downarrow$   
 $[T(\mathbf{v}_1)]_F$

$\downarrow$   
 $[T(\mathbf{v}_2)]_F$

$\downarrow$   
 $[T(\mathbf{v}_n)]_F$

where

$$T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \dots + a_{m1}\mathbf{w}_m$$

$$T(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \dots + a_{m2}\mathbf{w}_m$$

$\vdots$

$$T(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \dots + a_{mn}\mathbf{w}_m$$

**Example 3** Define the linear transformation  $L: \mathbf{R}^3 \rightarrow \mathbf{R}^2$  by

$$L(\mathbf{x}) = x_1 \mathbf{b}_1 + (x_2 + x_3) \mathbf{b}_2$$

for each  $\mathbf{x} = (x_1, x_2, x_3)^T$  in  $\mathbf{R}^3$ , where  $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

Find the matrix  $\mathbf{A}$  representing  $L$  with respect to the ordered bases  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  and  $\{\mathbf{b}_1, \mathbf{b}_2\}$ .

**Solution:**

$$L(\mathbf{e}_1) = 1\mathbf{b}_1 + 0\mathbf{b}_2$$

$$L(\mathbf{e}_2) = 0\mathbf{b}_1 + 1\mathbf{b}_2$$

$$L(\mathbf{e}_3) = 0\mathbf{b}_1 + 1\mathbf{b}_2$$

The  $j$ th column of  $\mathbf{A}$  is determined by the coordinates of  $L(\mathbf{e}_j)$  with respect to  $\{\mathbf{b}_1, \mathbf{b}_2\}$  for  $i = 1, 2, 3$ . Thus

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

**Example 4** Let  $L$  be a linear transformation mapping  $\mathbf{R}^2$  into itself defined by

$$L(\alpha \mathbf{b}_1 + \beta \mathbf{b}_2) = (\alpha + \beta) \mathbf{b}_1 + 2\beta \mathbf{b}_2.$$

where  $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

Find the matrix  $\mathbf{A}$  representing  $L$  with respect to  $\{\mathbf{b}_1, \mathbf{b}_2\}$ .

**Solution:**

$$L(\mathbf{b}_1) = 1\mathbf{b}_1 + 0\mathbf{b}_2$$

$$L(\mathbf{b}_2) = 1\mathbf{b}_1 + 2\mathbf{b}_2$$

Thus

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

**Example 5** Let  $L: \mathbf{R}^2 \rightarrow \mathbf{R}^3$  be the linear transformation defined by

$$L(\mathbf{x}) = (x_2, x_1 + x_2, x_1 - x_2)^T.$$

Find the matrix representation of  $L$  with respect to the ordered bases  $\{\mathbf{u}_1, \mathbf{u}_2\}$  and  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ , where

$$\mathbf{u}_1 = (1, 2)^T, \mathbf{u}_2 = (3, 1)^T$$

and

$$\mathbf{b}_1 = (1, 0, 0)^T, \mathbf{b}_2 = (1, 1, 0)^T, \mathbf{b}_3 = (1, 1, 1)^T.$$

**Solution 1**

$$L(\mathbf{u}_1) = (2, 3, -1)^T, \quad L(\mathbf{u}_2) = (1, 4, 2)^T.$$

We need to write them as combinations of  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ :

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{21} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{31} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = a_{12} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{22} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{32} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus  $\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}.$

**Theorem 3 (An equivalent way to find the matrix  $A$ )**

Let  $E = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  and  $F = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$  be ordered bases for  $\mathbf{R}^n$  and  $\mathbf{R}^m$ , respectively.

If  $L: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is a linear transformation and  $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_n]$  is the matrix representing  $L$  with respect to  $E$  and  $F$ , then

$$\mathbf{a}_j = \mathbf{B}^{-1}L(\mathbf{u}_j) \text{ for } j = 1, \dots, n$$

where  $\mathbf{B} = [\mathbf{b}_1, \dots, \mathbf{b}_m]$ .

**Proof.** If  $\mathbf{A}$  is representing  $L$  with respect to  $E$  and  $F$ , then, for  $j = 1, \dots, n$ ,

$$\begin{aligned} L(\mathbf{u}_j) &= a_{1j}\mathbf{b}_1 + a_{2j}\mathbf{b}_2 + \cdots + a_{mj}\mathbf{b}_m = [\mathbf{b}_1, \dots, \mathbf{b}_m] \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \\ &= \mathbf{B}\mathbf{a}_j \end{aligned}$$

The matrix  $\mathbf{B}$  is nonsingular since its column vectors form a basis for  $\mathbf{R}^m$ . Hence

$$\mathbf{a}_j = \mathbf{B}^{-1}L(\mathbf{u}_j) \text{ for } j = 1, \dots, n.$$

The way to find the matrix representation of the transformation is :

*by computing the reduced row echelon form of an augmented matrix.*

**Remark.** If  $\mathbf{A}$  is the matrix representing the linear transformation  $L: \mathbf{R}^n \rightarrow \mathbf{R}^m$  with respect to the bases  $E = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  and  $F = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ , then the reduced row echelon form of

$$[\mathbf{b}_1, \dots, \mathbf{b}_m \mid L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)]$$

is

$$[\mathbf{I} \mid \mathbf{A}] .$$



**Example 5 (continued)**

Let  $L: \mathbf{R}^2 \rightarrow \mathbf{R}^3$  be the linear transformation defined by

$$L(\mathbf{x}) = (x_2, x_1 + x_2, x_1 - x_2)^T.$$

Find the matrix representations of  $L$  with respect to the ordered bases  $\{\mathbf{u}_1, \mathbf{u}_2\}$  and  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ , where  $\mathbf{u}_1 = (1, 2)^T$ ,  $\mathbf{u}_2 = (3, 1)^T$  and  $\mathbf{b}_1 = (1, 0, 0)^T$ ,  $\mathbf{b}_2 = (1, 1, 0)^T$ ,  $\mathbf{b}_3 = (1, 1, 1)^T$ .

**Solution 2** We must compute  $L(\mathbf{u}_1)$  and  $L(\mathbf{u}_2)$  and then transform the augmented matrix  $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \mid L(\mathbf{u}_1), L(\mathbf{u}_2)]$  to reduced row echelon form:

$$L(\mathbf{u}_1) = (2, 3, -1)^T, \quad L(\mathbf{u}_2) = (1, 4, 2)^T.$$

$$\left[ \begin{array}{ccc|cc} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array} \right].$$

The matrix representing  $L$  with respect to the given ordered bases is

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}.$$

Next we find matrices that represent **differentiation**.

**Basis for  $P_3: E = \{p_1, p_2, p_3, p_4\}$ .**  $p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3$ .

The derivatives of those four basis vectors

**Action of  $\frac{d}{dt}$ :**  $\frac{d}{dt}p_1 = 0, \frac{d}{dt}p_2 = p_1, \frac{d}{dt}p_3 = 2p_2, \frac{d}{dt}p_4 = 3p_3$ .

Coordinate vectors:  $[p_1]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [p_2]_E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [p_3]_E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, [p_4]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

**Differentiation matrix**

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$   
 $Ap_1 \quad Ap_2 \quad Ap_3 \quad Ap_4$

$$p_1 = 1, \quad p_2 = t, \quad p_3 = t^2, \quad p_4 = t^3$$

For the matrix  $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,

$$N(\mathbf{A}) = \text{Span}\{p_1\}, \quad \text{nullity}(\mathbf{A}) = 1;$$

$$C(\mathbf{A}) = \text{Span}\{p_2, p_3, p_4\}, \quad \text{rank}(\mathbf{A}) = 3.$$

For any vector in the vector space  $\mathbf{P}_3$ , the derivative is decided by linearity.

For example, for  $p = 2 + t - t^2 - t^3$ ,

$$\frac{dp}{dt} = \mathbf{A}p \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \end{bmatrix},$$

so the derivative for  $p$  is  $1 - 2t - 3t^2$ .

**Example: Integration****Integration:**  $V(= P_3) \rightarrow W(= P_4)$ **Basis for  $P_3$ :**  $\mathbf{x}_1 = 1, \mathbf{x}_2 = t, \mathbf{x}_3 = t^2, \mathbf{x}_4 = t^3$ **Basis for  $P_4$ :**  $\mathbf{y}_1 = 1, \mathbf{y}_2 = t, \mathbf{y}_3 = t^2, \mathbf{y}_4 = t^3, \mathbf{y}_5 = t^4$ 

$$\int_0^t 1 ds = t \text{ or } A\mathbf{x}_1 = \mathbf{y}_2, A\mathbf{x}_2 = \frac{1}{2}\mathbf{y}_3, \int_0^t s^3 ds = \frac{t^4}{4} \text{ or } A\mathbf{x}_4 = \frac{1}{4}\mathbf{y}_5$$

$$A_{\text{int}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

$$A\mathbf{x}_3 = \frac{1}{3}\mathbf{y}_4$$

**Theorem 4** Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are linear transformations from  $V$  to  $W$  and from  $U$  to  $V$ . Their product  $\mathbf{AB}$  starts with a vector  $\mathbf{u}$  in  $U$ , goes to  $\mathbf{Bu}$  in  $V$ , and finishes with  $\mathbf{ABu}$  in  $W$ . This “composition”  $\mathbf{AB}$  is again a linear transformation (from  $U$  to  $W$ ). Its matrix is the product of the individual matrices representing  $\mathbf{A}$  and  $\mathbf{B}$ .

$$\begin{array}{ccccc} U & \xrightarrow{\mathbf{B}} & V & \xrightarrow{\mathbf{A}} & W \\ \mathbf{u} & \mapsto & \mathbf{Bu} & \mapsto & \mathbf{ABu} \end{array}$$

**Example 6** Let  $T$  be a linear transformation of  $\mathbf{R}^2$  such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Find the matrix for  $T$ .

**Solution** The matrix  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$  is such that  $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

The matrix  $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$  is such that  $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ .

Thus the matrix

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 7 & 1 \\ 8 & 2 \end{bmatrix}$$

is such that  $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ , and  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3} \begin{bmatrix} 7 & 1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ .

### III. Ranges (值域)

We notice that, if  $f$  is a linear transformation, then

$$f(c\mathbf{u} + d\mathbf{v}) = f(c\mathbf{u}) + f(d\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v}).$$

**Theorem 5** Let  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear transformation. Then the range of  $f$  is a **subspace** of  $\mathbf{R}^m$ .

**Proof** Let  $S = \text{range}(f)$ . Then  $f(\mathbf{0}) = \mathbf{0} \in S$ ,

because  $f(\mathbf{0}) = f(\mathbf{0} - \mathbf{0}) = f(\mathbf{0}) - f(\mathbf{0}) = \mathbf{0}$ .

Let  $\mathbf{x}, \mathbf{y} \in S$ , i.e.,  $\mathbf{x} = f(\mathbf{u})$  and  $\mathbf{y} = f(\mathbf{v})$  for some  $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ .

Thus  $\mathbf{x} + \mathbf{y} = f(\mathbf{u}) + f(\mathbf{v}) = f(\mathbf{u} + \mathbf{v}) \in S$ .

Let  $a \in \mathbf{R}$  and  $\mathbf{x} \in S$ , i.e.,  $\mathbf{x} = f(\mathbf{u})$  for some  $\mathbf{u} \in \mathbf{R}^n$ .

Thus  $a\mathbf{x} = af(\mathbf{u}) = f(a\mathbf{u}) \in S$ .

Therefore,  $S$  is a subspace of  $\mathbf{R}^m$ .

**Note:** *A linear transformation is determined by the effect it has on a basis.*

## Rank-nullity theorem

Let  $f$  be a linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$ .

**Definition 2** The **kernel (核)**  $\ker(f)$  is the set  $\{\mathbf{u} \in \mathbf{R}^n \mid f(\mathbf{u}) = \mathbf{0}\}$ .

**Example 7** Let  $f$  be a linear transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  defined by

$$f(x, y) = (x + y, x + y).$$

Then the range

$$\text{range}(f) = \{(a, a) \mid a \in \mathbf{R}\},$$

which is a line. The kernel is also a line:

$$\ker(f) = \{(a, -a) \mid a \in \mathbf{R}\}.$$

### Theorem 6 (Rank-nullity theorem)

*Let  $f$  be a linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$ .*

*Then the range of  $f$  is a subspace of  $\mathbf{R}^m$ , and the kernel of  $f$  is a subspace of  $\mathbf{R}^n$ . Moreover,*

$$\dim(\ker(f)) + \dim(\text{range}(f)) = n.$$



**Key words:**

*Linear transformation: definition and examples;  
matrix of linear transformation;  
Range, kernel*

**Homework**

**See Blackboard**

