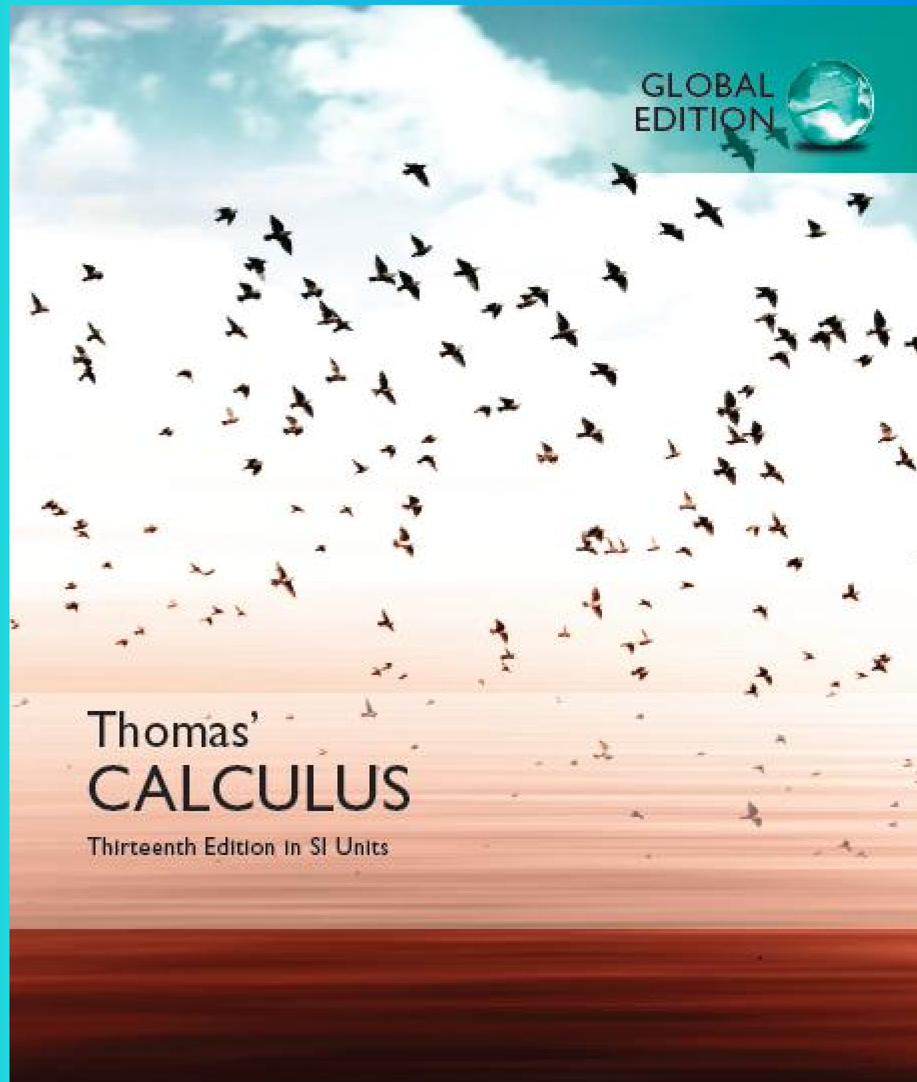

GLOBAL
EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 8

Techniques of Integration
积分技术

8.1

Using Basic Integration Formulas
用积分积分公式计算积分

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\displaystyle\int \tan x \, dx = \ln |\sec x| + C$

13. $\displaystyle\int \cot x \, dx = \ln |\sin x| + C$

14. $\displaystyle\int \sec x \, dx = \ln |\sec x + \tan x| + C$

15. $\displaystyle\int \csc x \, dx = -\ln |\csc x + \cot x| + C$

16. $\displaystyle\int \sinh x \, dx = \cosh x + C$

17. $\displaystyle\int \cosh x \, dx = \sinh x + C$

17. $\displaystyle\int \cosh x\,dx = \sinh x + C$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$

Ex. 1 **Evaluate** $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}dx.$

Solution $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}dx$

$$= \int_3^5 \frac{1}{\sqrt{x^2-3x+1}}d(x^2-3x+1).$$

$$= 2\sqrt{x^2-3x+1}\Big|_3^5 = 2(\sqrt{11}-1)$$

Ex. 2 Complete the square to evaluate $\int \frac{dx}{\sqrt{8x-x^2}}$.

Solution $\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}}$

$$= \int \frac{d(x-4)}{\sqrt{4^2-(x-4)^2}} = \sin^{-1}\left(\frac{x-4}{4}\right) + C.$$

Ex. 3 Evaluate the integral $\int \cos x \sin 2x dx$.

Solution

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$\int \cos x \sin 2x dx = \frac{1}{2}\int (\sin 3x + \sin x) dx = -\frac{\cos 3x}{6} - \frac{\cos x}{2} + C.$$

Ex. 4 Find $\displaystyle\int_0^{\pi/4} \frac{1}{1-\sin x}\,dx.$

Solution $\displaystyle\int_0^{\pi/4} \frac{1}{1-\sin x}\,dx = \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$

$$= \int_0^{\pi/4} (\sec^2 x + \tan x \sec x)\,dx = (\tan x + \sec x)\Big|_0^{\pi/4} = \sqrt{2}$$

Ex. 5 Evaluate the integral $\displaystyle\int \frac{3x^2-7x}{3x+2}\,dx.$

Solution $\displaystyle\int \frac{3x^2-7x}{3x+2}\,dx = \int \left(x-3+\frac{6}{3x+2}\right)dx$

$$= \int (x-3)\,dx + 2\int \frac{1}{3x+2}\,d(3x+2)$$

$$= \frac{(x-3)^2}{2} + 2\ln|3x+2| + C.$$

Ex. 6 Evaluate the integral $\int \frac{x+1}{x^2+x+1}dx.$

Solution $\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}d(x+\frac{1}{2})$$

$$= \frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C.$$

Ex. 7 **Evaluate the integral** $\displaystyle\int \frac{1}{(1+\sqrt{x})^3}\,dx.$

Solution $u=\sqrt{x},\, x=u^2,\quad dx = 2u\,du$

$$\int \frac{1}{(1+\sqrt{x})^3}\,dx = 2\int \frac{u}{(1+u)^3}\,du$$

$$= 2\int \frac{u+1-1}{(1+u)^3}\,du = 2\int \frac{1}{(1+u)^2}\,du - 2\int \frac{1}{(1+u)^3}\,du$$

$$= -\frac{2}{1+u} + \frac{1}{(1+u)^2} + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C$$

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(1+\sqrt{x})^3}.$$

Solution

$$u = 1 + \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx;$$

$$dx = 2\sqrt{x}\,du = 2(u-1)\,du$$

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right) du$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C = C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}.$$

EXAMPLE 8 Evaluate $\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$.

Solution

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0.$$

例9 求 $\int \frac{1}{x^2-a^2}dx.$

解 $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int (\frac{1}{x-a} - \frac{1}{x+a})dx$

$$= \frac{1}{2a}\int \frac{1}{x-a}d(x-a) - \frac{1}{2a}\int \frac{1}{x+a}d(x-a)$$

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C.$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

例10　求 $\int \frac{1}{1+e^x}dx.$

解　$$\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

例11　设 $f'(\sin^2 x) = \cos^2 x$，求 $f(x)$.

解　令 $u = \sin^2 x \quad \Rightarrow \quad \cos^2 x = 1 - u$,

$$f'(u) = 1 - u,$$

$$f(u) = \int (1 - u) du = u - \frac{1}{2} u^2 + C,$$

$$f(x) = x - \frac{1}{2} x^2 + C.$$

例12 求 $\int \frac{1}{\sqrt{1-x^2}\arcsin x}dx.$

解 $$\int \frac{1}{\sqrt{1-x^2}\arcsin x}dx$$

$$=\int \frac{1}{\arcsin x}d(\arcsin x) \quad = \ln|\arcsin x| + C.$$

例13 求 $\int \frac{\arctan^2 x}{1+x^2} dx.$

解 $\int \frac{\arctan^2 x}{1+x^2} dx$

$$= \int \arctan^2 x d(\arctan x) = \frac{\arctan^3 x}{3} + C.$$

例 求 $\int \frac{x}{\sqrt{1-x^4}} dx$.

例 求 $\int \frac{x^2}{1+x^6} dx$.

例 求 $\int \frac{1}{x(3+2\ln x)} dx$.

例 求 $\int \frac{1}{x}\sqrt{1+\ln x^3}\, dx$.

8.2

Integration by Parts

分部积分法

问题 $\int xe^{-x}dx = ?$

两个函数乘积的求导法则:

设函数$u = u(x)$和$v = v(x)$具有连续导数,

$$(uv)' = u'v + uv', \qquad uv' = (uv)' - u'v,$$

$$\int uv'dx = uv - \int u'vdx, \qquad \int udv = uv - \int vdu.$$

$$\int_a^b uv'dx = \int_a^b (uv)'dx - \int_a^b u'vdx,$$

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu,$$

分部积分公式

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du.$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$

例1 求积分$\int x\cos x dx$.

解
$$\int x\cos x dx = \int \cos x d\left(\frac{x^2}{2}\right)$$
$$= \frac{x^2}{2}\cos x + \int \frac{x^2}{2}\sin x dx$$

显然，u, v'选择不当，积分更难进行.

解 $\int x\cos x dx = \int x d\sin x$
$$= x\sin x - \int \sin x dx$$
$$= x\sin x + \cos x + C.$$

例2　求积分 $\int \ln x dx.$　　$\int x^2 \ln x dx.$

解 $\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - x + C.$

例3　求积分 $\int x^2 e^x dx.$

解 $\int x^2 e^x dx = \int x^2 d(e^x)$

$= x^2 e^x - 2\int x e^x dx = x^2 e^x - 2\int x de^x$

（再次使用分部积分法）

$= x^2 e^x - 2(xe^x - e^x) + C.$

例4 求积分$\int e^x \sin x dx$.

$$\int e^{ax} \sin bx dx.$$

解 $\boxed{\int e^x \sin x dx} = \int \sin x de^x$

$$= e^x \sin x - \int e^x d(\sin x)$$

$$= e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x de^x$$

$$= e^x \sin x - (e^x \cos x - \int e^x d\cos x)$$

$$= e^x (\sin x - \cos x) - \boxed{\int e^x \sin x dx}$$ 注意循环形式

$$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

例5　求积分的 $I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx$ 递推公式

解　$\int \sin^n x dx = -\int \sin^{n-1} x d\cos x$

$$= -\cos x \sin^{n-1} x + \int \cos x d \sin^{n-1} x$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

$$nI_n = -\left[\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)I_{n-2} = (n-1)I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2} \quad I_{10} = \int_0^{\frac{\pi}{2}} \sin^{10} x dx \quad I_{10} = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

Ex.6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from x=0 to x=4.

Solution $A = \int_0^4 xe^{-x}dx = -\int_0^4 xde^{-x}$

$$= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} - e^{-x}\Big|_0^4 = 1 - 5e^{-4} \approx 0.91$$

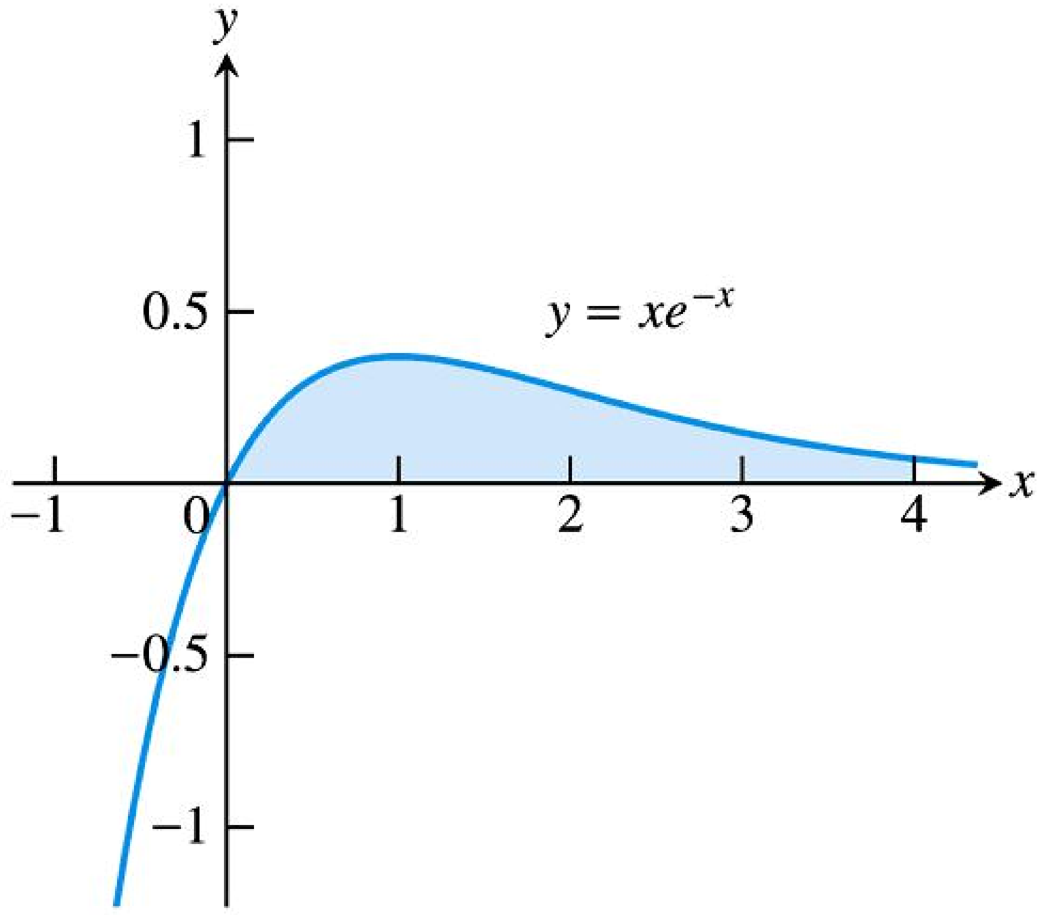


FIGURE 8.1 The region in Example 6.

例7　求积分 $\int x\arctan x\,dx$.

解 $$\int x\arctan x\,dx = \int \arctan x\,d\frac{x^2}{2}$$

$$= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\,d(\arctan x)$$

$$= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\cdot\left(1-\frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x-\arctan x) + C.$$

Tabular integration by part

$$\int f(x)g(x)dx \;= \int f(x)dg_1(x)$$

$$= f(x)g_1(x) - \int f'(x)g_1(x)dx$$

$$= f(x)g_1(x) - \int f'(x)dg_{11}(x)$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + \int f''(x)g_{11}(x)dx$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + \int f''(x)dg_{111}(x)$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + f''(x)g_{111}(x) - \int f'''(x)g_{111}(x)dx$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + f''(x)g_{111}(x) - f'''(x)g_{1111}(x)$$

$$+ \int f''''(x)g_{1111}(x)dx$$

if $f^{(k)}(x) = 0$, then $\int f(x)g(x)dx$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + f''(x)g_{111}(x) - \cdots + (-1)^k f^{(k-1)}(x)g_{(k)}(x)$$

EXAMPLE 7 Evaluate $\displaystyle\int x^2 e^x\,dx$.

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

$$\int x^2 e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C.$$

EXAMPLE 8 Find the integral $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ for $f(x) = 1$ on $[-\pi, 0)$ and $f(x) = x^3$ on $[0, \pi]$, where n is a positive integer.

Solution

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\,dx$$

$$= \frac{1}{n\pi}\sin nx\Big]_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\,dx.$$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\cos nx$
$3x^2$	$(-)$	$\frac{1}{n}\sin nx$
$6x$	$(+)$	$-\frac{1}{n^2}\cos nx$
6	$(-)$	$-\frac{1}{n^3}\sin nx$
0		$\frac{1}{n^4}\cos nx$

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos nx dx \qquad = \frac{1}{\pi}\left(\frac{3\pi^2}{n^2}(-1)^n - \frac{6((-1)^n - 1)}{n^4}\right)$$

$$= \frac{1}{\pi}\left(\frac{x^3}{n}\sin nx + \frac{3x^2}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right)_0^{\pi}$$

8.3

Trigonometric Integrals
三角函数积分法

Products of Powers of Sines and Cosines

$$\int \sin^m x \cos^n x \, dx,$$

Case 1 If m is odd, $\sin x dx = -d(\cos x)$

Case 2 If m is even and n is odd $\cos x dx = d(\sin x)$

Case 3 If both m and n are even

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Ex. 1 **Evaluate** $\int \sin^3 x \cos^2 x dx.$

Solution $\int \sin^3 x \cos^2 x dx = -\int \sin^2 x \cos^2 x d \cos x$

$$= -\int (\cos^2 x - \cos^4 x) d \cos x$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

Ex. 2 Evaluate $\int \cos^5 x dx$.

Solution $\int \cos^5 x dx = \int \cos^4 x d \sin x$

$$= \int (1 - \sin^2 x)^2 d \sin x$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) d \sin x$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.$$

Ex. 3 **Evaluate** $\int \sin^2 x \cos^4 x dx.$

Solution $\int \sin^2 x \cos^4 x dx = \frac{1}{4}\int \sin^2 2x \cos^2 x dx$

$$= \frac{1}{4}\int \sin^2 2x \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cos 2x dx$$

$$= \frac{1}{16}\int (1-\cos 4x) dx + \frac{1}{16}\int \sin^2 2x d(\sin 2x)$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

Ex. 4 Find $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos 4x}\,dx.$

Solution $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos 4x}\,dx = \int_0^{\pi/2} \sqrt{2\cos^2 2x}\,dx$

$$= \sqrt{2}\int_0^{\pi/2} |\cos 2x|\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx$$

$$= \left[\frac{\sqrt{2}\sin 2x}{2}\right]_0^{\pi/4} - \left[\frac{\sqrt{2}\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \sqrt{2}.$$

Integrals of Powers of tan *x* and sec *x*

Ex. 5 Evaluate $\int \tan^4 x dx$.

Solution $$\int \tan^4 x dx = \int \tan^2 x(\sec^2 x - 1)dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$

$$= \int \tan^2 x d \tan x - \int (\sec^2 x - 1)dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

Ex. 6 Evaluate $\int \sec^3 x\,dx$.

Solution $\int \sec^3 x\,dx = \int \sec x\,d\tan x$

$$= \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$$

Ex. 7 Evaluate $\int \tan^4 x \sec^4 x dx$.

Solution $\int \tan^4 x \sec^4 x dx$

$$= \int \tan^4 x \sec^2 x d \tan x$$

$$= \int \tan^4 x (1 + \tan^2 x) d \tan x$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$$

8.4

Trigonometric Substitutions
积分的三角代换

例1 求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0).$

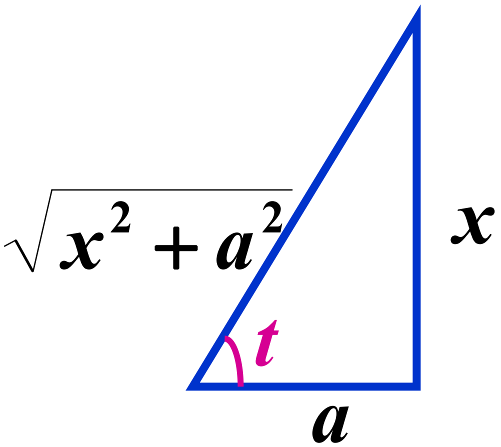
解 令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t} \cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C.$$



例2　求 $\int x^3\sqrt{4-x^2}\,dx.$

解　令 $x=2\sin t \qquad dx=2\cos t\,dt \qquad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

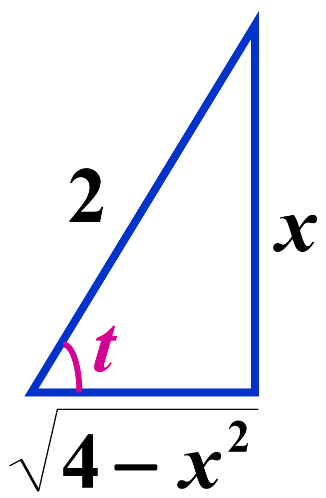
$$\int x^3\sqrt{4-x^2}\,dx=\int(2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt$$

$$=32\int\sin^3 t\cos^2 t\,dt=32\int\sin t(1-\cos^2 t)\cos^2 t\,dt$$

$$=-32\int(\cos^2 t-\cos^4 t)\,d\cos t$$

$$=-32\left(\frac{1}{3}\cos^3 t-\frac{1}{5}\cos^5 t\right)+C$$

$$=-\frac{4}{3}\left(\sqrt{4-x^2}\right)^3+\frac{1}{5}\left(\sqrt{4-x^2}\right)^5+C.$$



例3 求 $\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0).$

$x>a$

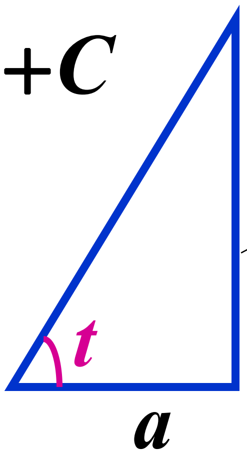
解 令$x=a\sec t$ $\quad dx=a\sec t\tan t dt \quad t\in\left(0,\frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t\cdot\tan t}{a\tan t}dt$$

$$=\int \sec t dt = \ln|\sec t+\tan t|+C$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C.$$

$$=\ln|x+\sqrt{x^2-a^2}|+C.$$



$\sqrt{x^2-a^2}$

当 $x < -a$

$$x = -a\sec t \qquad dx = -a\sec t\tan t\,dt \qquad t \in \left(0, \frac{\pi}{2}\right)$$

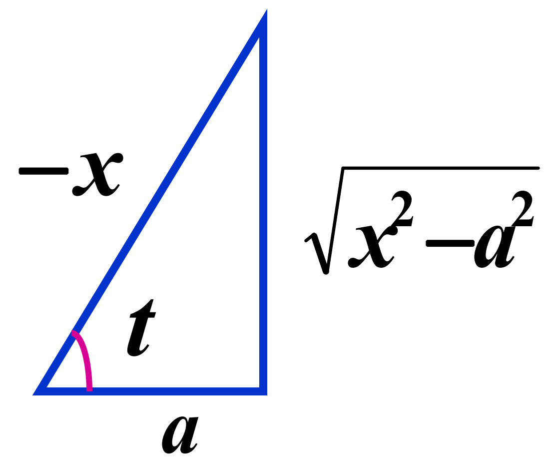
$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = -\int \frac{a\sec t \cdot \tan t}{a\tan t}dt$$

$$= -\int \sec t\,dt \quad = -\ln|\sec t + \tan t| + C$$

$$= -\ln\left|\frac{-x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C.$$

$$= -\ln|-x + \sqrt{x^2 - a^2}| + C.$$

$$= \ln|x + \sqrt{x^2 - a^2}| + C.$$



EXAMPLE 4 Evaluate $\displaystyle\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$

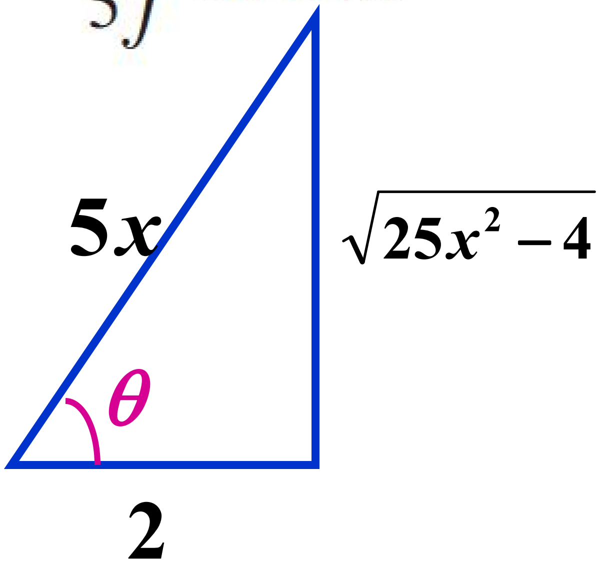
Solution We then substitute $x = \dfrac{2}{5}\sec\theta, \quad 0 < \theta < \dfrac{\pi}{2}$

$$dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta,$$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$



例8 求积分 $\int \frac{\sin x}{1+\sin x+\cos x}dx$.

解 $$\int \frac{\sin x}{1+\sin x+\cos x}dx=\int \frac{\sin x(1-\sin x-\cos x)}{1-(\sin x+\cos x)^2}dx$$

$$=-\int \frac{\sin x(1-\sin x-\cos x)}{2\sin x\cos x}dx$$

$$=-\frac{1}{2}\int (\sec x-\tan x-1)dx$$

$$=-\frac{1}{2}(\ln|\sec x+\tan x|+\ln|\cos x|-x)+C.$$

例8 求积分 $\int \frac{\sin x}{1+\sin x+\cos x} dx.$

解法2 由万能代换公式 $\tan\frac{x}{2}=u,$

$$\sin x=\frac{2u}{1+u^2}, \quad \cos x=\frac{1-u^2}{1+u^2} \quad dx=\frac{2}{1+u^2}du,$$

$$\int \frac{\sin x}{1+\sin x+\cos x} dx=\int \frac{2u}{(1+u)(1+u^2)}du$$

$$=\int \frac{2u+1+u^2-1-u^2}{(1+u)(1+u^2)}du$$

$$=\int \frac{1+u}{1+u^2}du-\int \frac{1}{1+u}du$$

$$=\arctan u+\frac{1}{2}\ln(1+u^2)-\ln|1+u|+C$$

$$\because \quad u=\tan\frac{x}{2}$$

$$=\frac{x}{2}+\ln\left|\sec\frac{x}{2}\right|-\ln\left|1+\tan\frac{x}{2}\right|+C.$$

8.5

Integration of Rational Functions by Partial Fractions

有理函数的定义：

两个多项式的商表示的函数称之.

$$\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$$

其中m、n都是非负整数；$a_0,a_1,\cdots,a_n$及

$b_0,b_1,\cdots,b_m$都是实数，并且$a_0\neq 0$，$b_0\neq 0$.

假定分子与分母之间没有公因式

(1) $n < m$，这有理函数是**真分式**；

(2) $n \geq m$，这有理函数是**假分式**；

利用多项式除法，假分式可以化成一个多项式和一个真分式之和.

例 $$\frac{x^3+x+1}{x^2+1} = x + \frac{1}{x^2+1}.$$

下面只考虑真分式的积分

$$\frac{1}{x^2-5x+6}=\frac{1}{(x-2)(x-3)}=\frac{1}{x-3}-\frac{1}{x-2}$$

$$\frac{x+3}{x^2-5x+6}=\frac{x+3}{x-3}-\frac{x+3}{x-2}=\frac{-5}{x-2}+\frac{6}{x-3}.$$

$$\frac{1}{(x^2+1)x}=\frac{1+x^2-x^2}{(x^2+1)x}=\frac{1}{x}-\frac{x}{x^2+1}$$

$$\frac{x+1}{(x^2+1)x^2}=\frac{1+x^2-x^2+x}{(x^2+1)x^2}=\frac{1}{x^2}-\frac{1}{x^2+1}+\frac{1}{x}-\frac{x}{x^2+1}$$

$$\frac{1}{(x^2+1)^2x}=\frac{1+x^2-x^2}{(x^2+1)^2x}=\frac{1}{(1+x^2)x}-\frac{x}{(x^2+1)^2}$$

$$=\frac{1}{x}-\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}$$

有理函数化为部分分式之和的一般规律：

（1）分母中若有因式 $(x-r)^k$，则拆项后有

$$\frac{A_1}{(x-r)^k}+\frac{A_2}{(x-r)^{k-1}}+\cdots+\frac{A_k}{x-r},$$

其中$A_1, A_2, \cdots, A_k$都是常数.

（2）分母中若有因式 $(x^2+px+q)^k$，其中 $p^2-4q<0$ 则拆项后有

$$\frac{B_1x+C_1}{(x^2+px+q)^k}+\frac{B_2x+C_2}{(x^2+px+q)^{k-1}}+\cdots+\frac{B_kx+C_k}{x^2+px+q}$$

其中 B_i,C_i 都是常数$(i=1,2,\cdots,k)$.

Ex. 1 Evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx.$

Solution $\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$

$$= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$A+B+C=1, \quad 4A+2B=4, \quad 3A-3B-C=1,$$

$$A=3/4, \quad B=1/2, \quad C=-1/4.$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx \quad = \int \left(\frac{3/4}{x-1} + \frac{1/2}{x+1} + \frac{-1/4}{x+3}\right)dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C.$$

Ex. 2 Evaluate $\int \frac{1}{x(x-1)^2}dx$.

Solution $\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$,

$$1 = A(x-1)^2 + Bx + Cx(x-1)$$

取 $x = 0, \Rightarrow A = 1$ 取 $x = 1, \Rightarrow B = 1$

取 $x = 2$, 并将 A, B 值代入 $\Rightarrow C = -1$

$$\int \frac{1}{x(x-1)^2}dx = \int \left[\frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}\right]dx$$

$$= \ln|x| - \frac{1}{x-1} - \ln|x-1| + C.$$

Ex. 3 Evaluate $\displaystyle\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$

Solution $$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \int \left(2x + \frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

Ex. 4 Evaluate $\displaystyle\int \frac{-2x+4}{(x-1)^2(1+x^2)}dx.$

Solution $\displaystyle\frac{-2x+4}{(x-1)^2(1+x^2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{1+x^2}$

$$A(x-1)(1+x^2) + B(1+x^2) + (x-1)^2(Cx+D) = -2x+4$$

let $x=1$, find $B=1$, let $x=i$, find $C=2$, $D=1$.

let $x=0$, find $A=-2$.

$$\int \frac{-2x+4}{(x-1)^2(1+x^2)}dx = \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{1+x^2}\right)dx$$

$$= -2\ln|x-1| - \frac{1}{x-1} + \ln(1+x^2) + \tan^{-1}x + C.$$

Ex. 5 Evaluate $\int \frac{1}{x(1+x^2)^2}dx.$

Solution $\frac{1}{x(1+x^2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

let $x = 0$, find $A = 1$, let $x = i$, find $E = 0$, $D = -1$.

equate the coefficient of x^4, find $B = -1$,

equate the coefficient of x^3, find $C = 0$.

$$\int \frac{1}{x(1+x^2)^2}dx = \int \left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

例6 求积分 $\int \frac{1}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}}dx.$

解 令 $t=e^{\frac{x}{6}} \Rightarrow x=6\ln t,\quad dx=\frac{6}{t}dt,$

$$\int \frac{1}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}}dx=\int \frac{1}{1+t^3+t^2+t}\cdot\frac{6}{t}dt$$

$$=6\int \frac{1}{t(1+t)(1+t^2)}dt=\int\left(\frac{6}{t}-\frac{3}{1+t}-\frac{3t+3}{1+t^2}\right)dt$$

$$= \int \left(\frac{6}{t} - \frac{3}{1+t} - \frac{3t+3}{1+t^2} \right) dt$$

$$= 6\ln t - 3\ln(1+t) - \frac{3}{2}\int \frac{d(1+t^2)}{1+t^2} - 3\int \frac{1}{1+t^2}\,dt$$

$$= 6\ln t - 3\ln(1+t) - \frac{3}{2}\ln(1+t^2) - 3\arctan t + C$$

$$= x - 3\ln(1+e^{\frac{x}{6}}) - \frac{3}{2}\ln(1+e^{\frac{x}{3}}) - 3\arctan(e^{\frac{x}{6}}) + C.$$

8.7

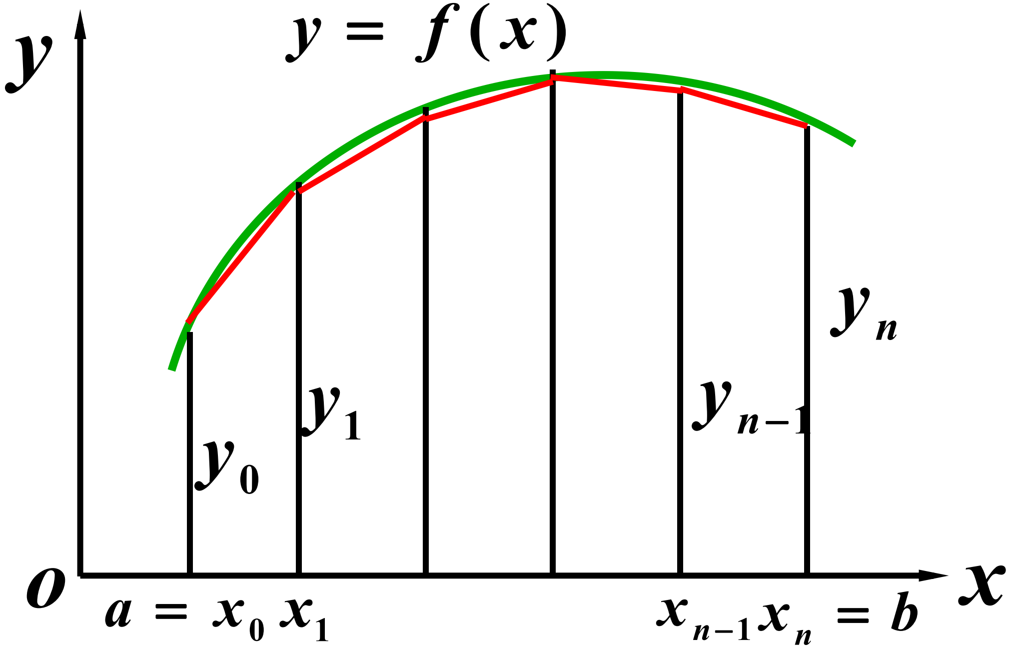
Numerical Integration

数值积分

like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1+x^4}$,

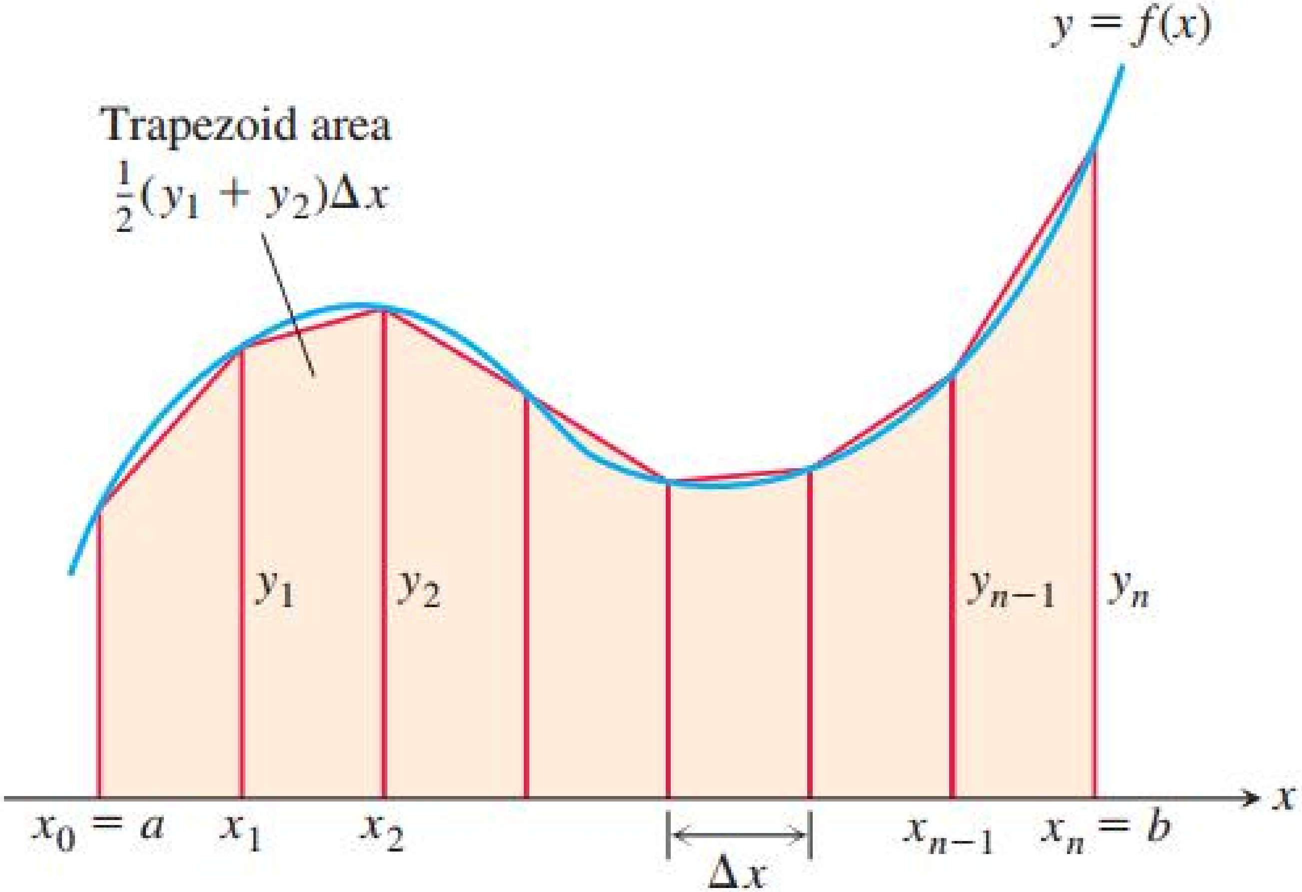
we cannot find a workable antiderivative for a function f

To approximate $\int_a^b f(x)\,dx$,



Trapezoidal Approximations

$$\Delta x = \frac{b - a}{n}.$$



The Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x$$

$$+ \cdots + \frac{1}{2}(y_{n-1} + y_n)\Delta x$$

$$= \frac{b-a}{n}[\frac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}]$$

$$= \frac{b-a}{2n}[(y_0 + y_n) + 2y_1 + 2y_2 + \cdots + 2y_{n-1}]$$

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

EXAMPLE 1

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$.

Solution Partition $[1, 2]$ into four subintervals of equal length

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

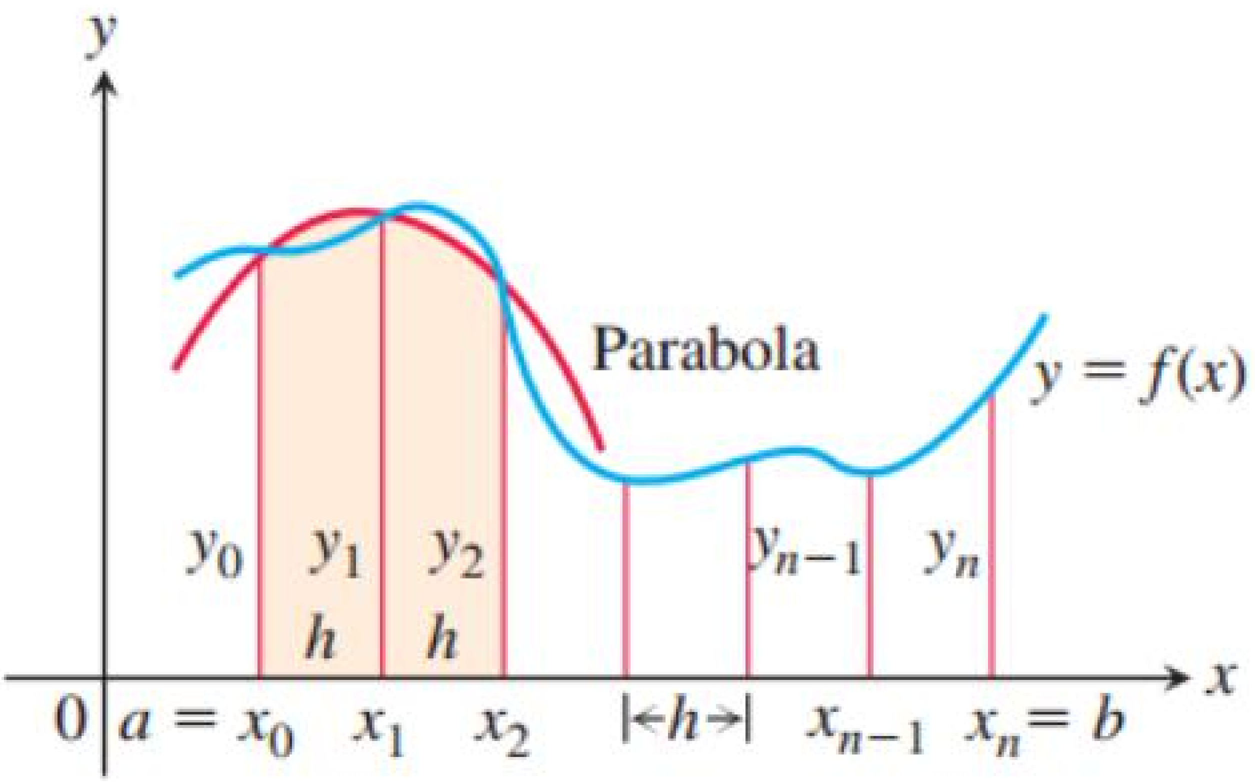
$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

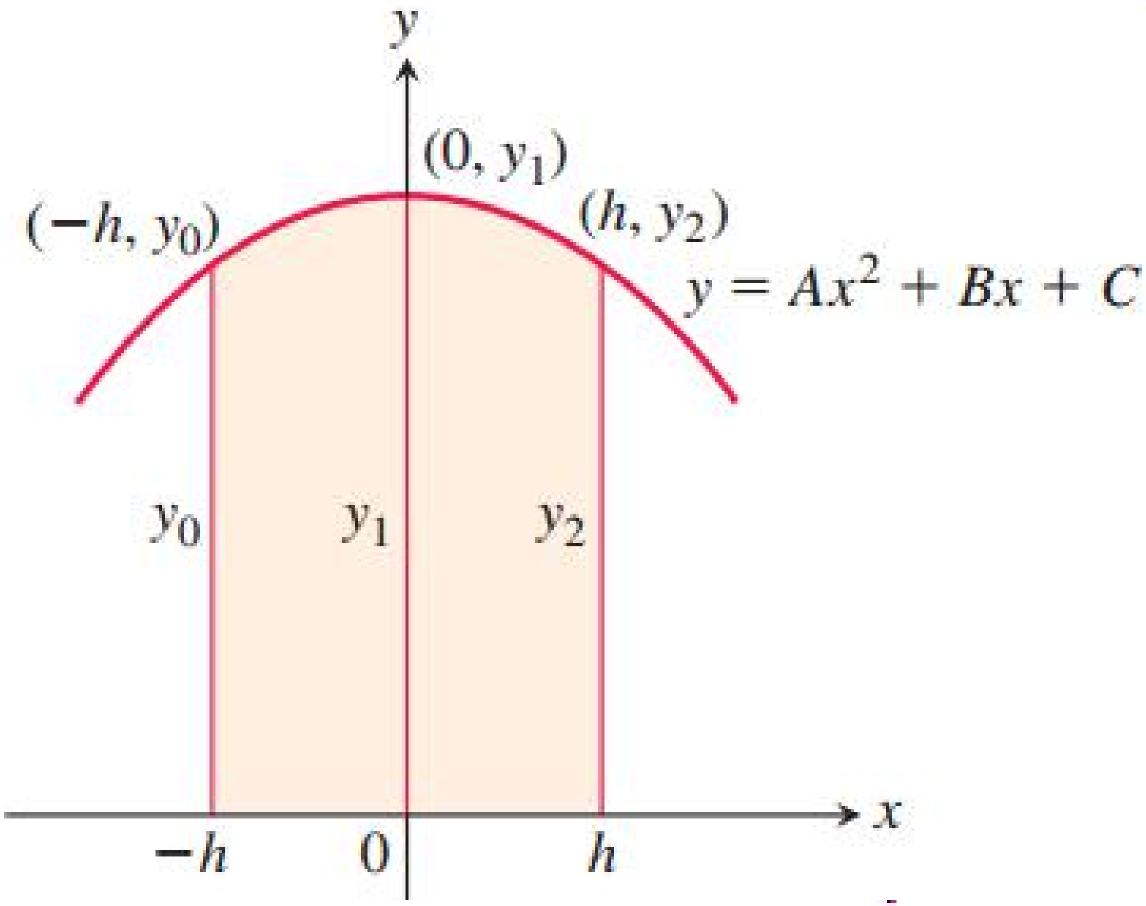
$$= \frac{75}{32} = 2.34375.$$

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

$(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%.

Simpson's Rule: Approximations Using Parabolas





$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx$$

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx = \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2),

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$C = y_1,$$

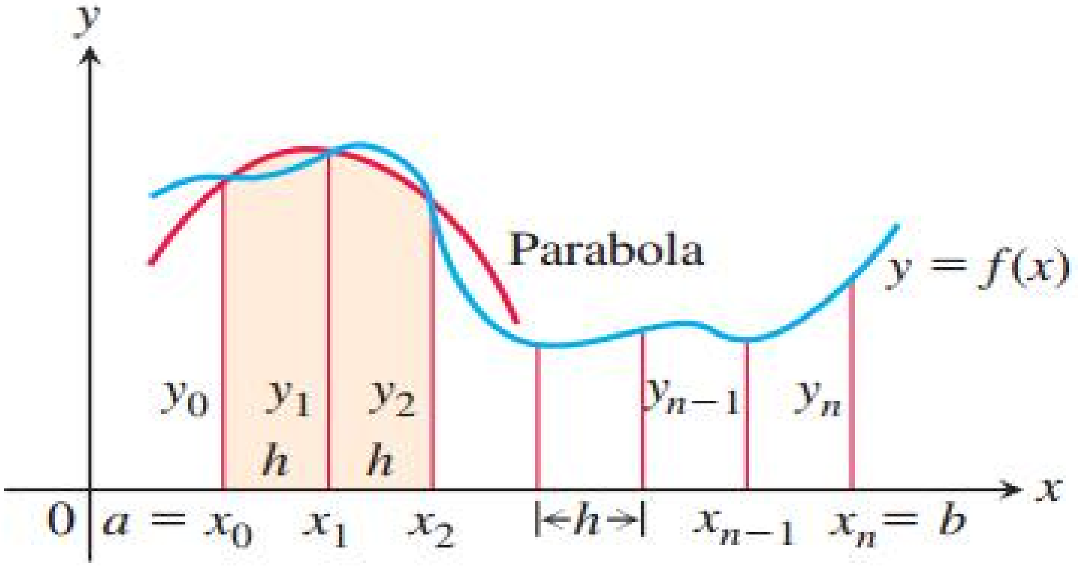
$$Ah^2 - Bh = y_0 - y_1,$$

$$Ah^2 + Bh = y_2 - y_1,$$

$$2Ah^2 = y_0 + y_2 - 2y_1.$$

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Simpson's Rule



$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots$$

$$+ \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{b-a}{3n}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

EXAMPLE 2

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right)$$

$$= 32\frac{1}{12}.$$

绝对误差$=\frac{1}{12}$。

This estimate differs from the exact value (32) by only 1/12,

Error Analysis

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

EXAMPLE 3

Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 2).

Solution

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

$f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$,

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

EXAMPLE 4

Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4}.

Solution

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

we have $M = 120$ and $b - a = 2$, so we want n to

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4} \qquad n^4 > \frac{64 \cdot 10^4}{3}.$$

$$n > 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5. \qquad n = 22.$$

EXAMPLE 5 the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

n	T_n	$\lvert\text{Error}\rvert$ less than . . .	S_n	$\lvert\text{Error}\rvert$ less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

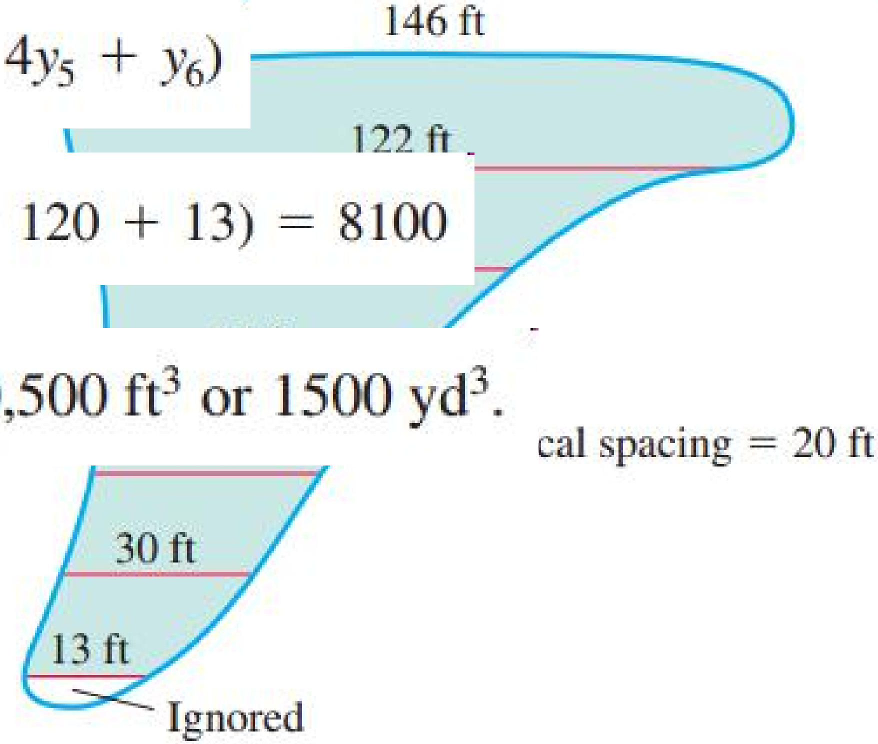
EXAMPLE 6

A town wants to drain and fill a small polluted swamp. About how many cubic yards of dirt will it take to fill the area after the swamp is drained? The swamp averages 5 ft deep.

Solution

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

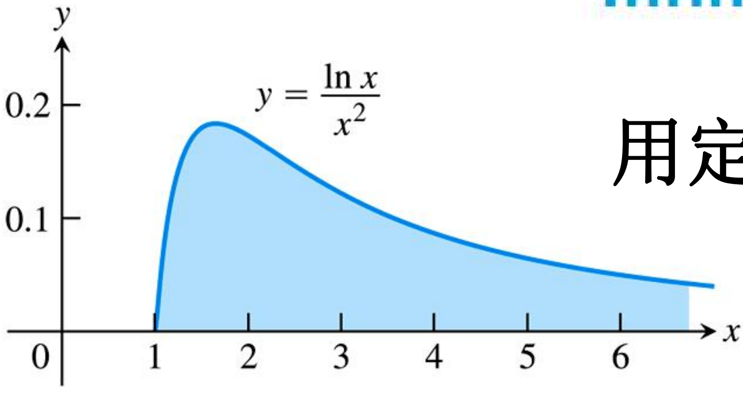


The volume is about $(8100)(5) = 40{,}500 \text{ ft}^3$ or 1500 yd^3.

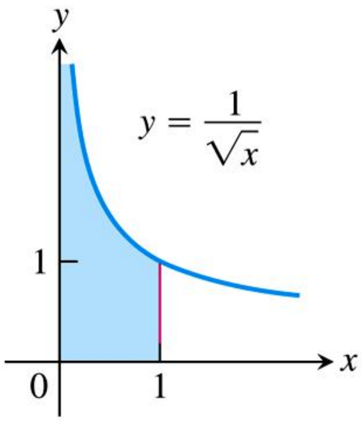
8.8

Improper Integrals
反常积分（广义积分）

Infinite Limits of Integration



(a)



(b)

FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

用定积分表示蓝色部分的面积

$$\int_1^{\infty} \frac{\ln x}{x^2} dx. \qquad \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} dx.$$

$$= \lim_{b\to\infty}\left(1 - \frac{1+\ln b}{b}\right) = 1$$

$$\int_0^1 \frac{1}{\sqrt{x}} dx.$$

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a\to 0^+} (2 - 2\sqrt{a}) = 2.$$

$$\int_0^1 \frac{1}{x} dx.$$

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{x} dx = \lim_{a\to 0^+} (-\ln a) = \infty.$$

DEFINITION

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

if the limit is finite we say that the improper integral **converges**

If the limit fails to exist, the improper integral **diverges**.

Ex. 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is the value?

Solution

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx.$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx.$$

$$= \lim_{b\to\infty} \int_1^{b} \ln x\, d\left(-\frac{1}{x}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{\ln x}{x}\Big|_1^b + \int_1^{b} \frac{1}{x^2}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b} + 1 - \frac{1}{b}\right) = 1$$

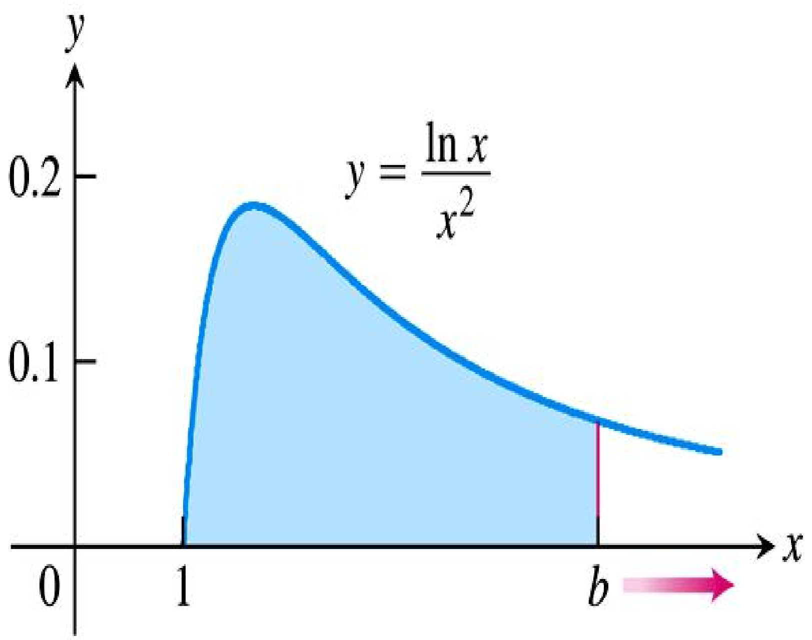
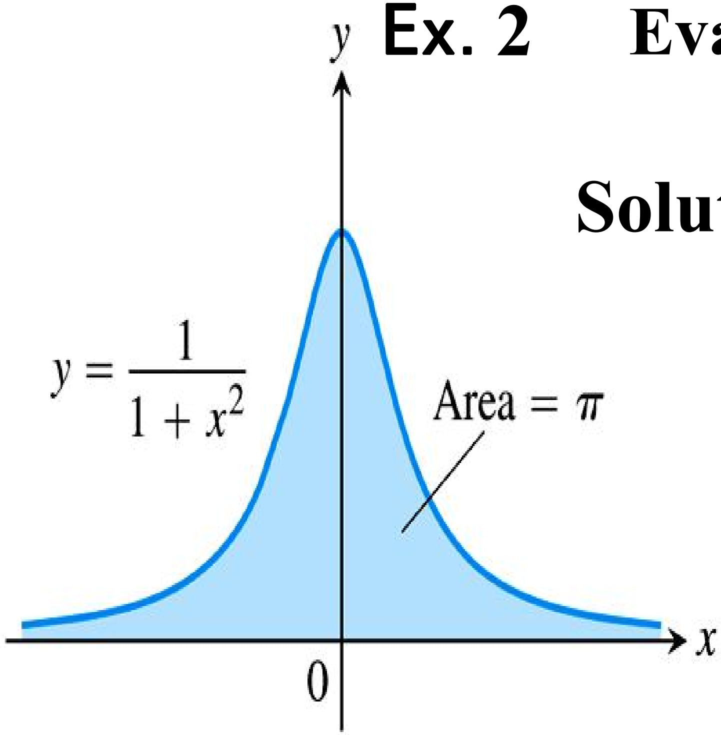


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

Ex. 2 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ $= \pi$



NOT TO SCALE

FIGURE 8.15 The area under this curve is finite (Example 2).

Solution $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \left(-\tan^{-1} a\right) = \pi/2$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b\right) = \pi/2$$

Ex. 3 Investigate the convergence of $\int_1^{+\infty} \frac{1}{x^p} dx$.

Solution if $p = 1$,

$$\int_1^{+\infty} \frac{1}{x^p} dx = \int_1^{+\infty} \frac{1}{x} dx = [\ln x]_1^{+\infty} = \lim_{x\to\infty} \ln x = +\infty,$$

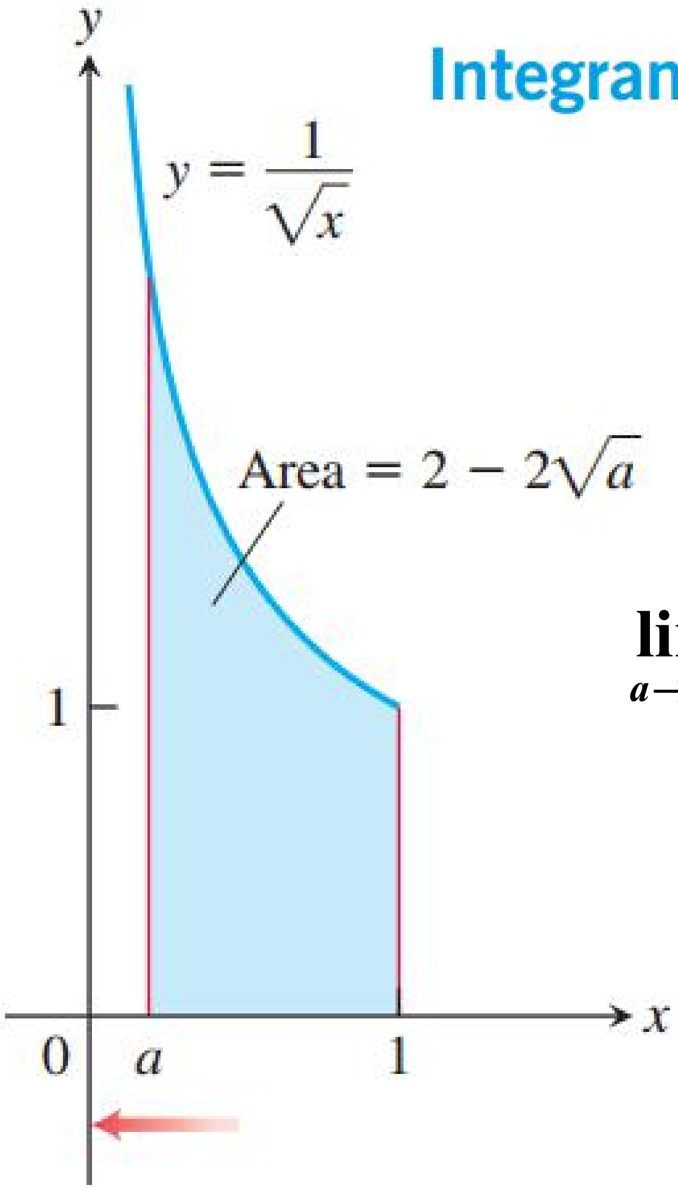
if $p \neq 1$,

$$\int_1^{+\infty} \frac{1}{x^p} dx = \left[\frac{x^{1-p}}{1-p}\right]_1^{+\infty} = \begin{cases} +\infty, & p < 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$$

Therefore, the integral converges to $\frac{1}{p-1}$ if $p > 1$

and it diverges if $p \leq 1$.

Integrands with Vertical Asymptotes



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}(2-2\sqrt{a}) = 2.$$

DEFINITION

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is $\lim_{x \to c} f(x) = \pm\infty$ where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

converges
diverges.

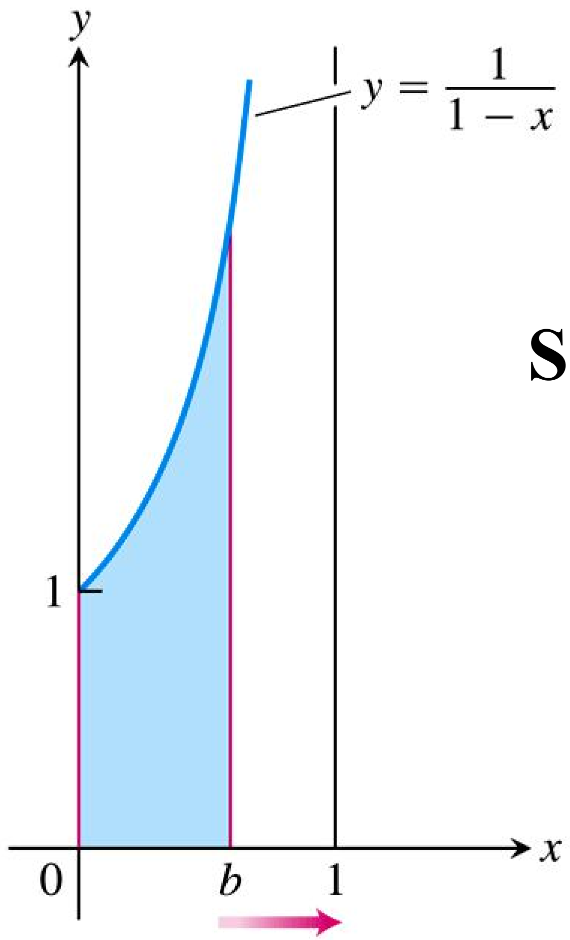


FIGURE 8.17 The area beneath the curve and above the x-axis for [0, 1) is not a real number (Example 4).

Ex. 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution

$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx$$

$$= \lim_{b\to 1^-}\left(-\ln(1-x)\Big|_0^b\right)$$

$$= \lim_{b\to 1^-}\left(-\ln(1-b)\right)$$

$$= +\infty,$$

So the integral diverges.

Ex. 5 Evaluate $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx.$

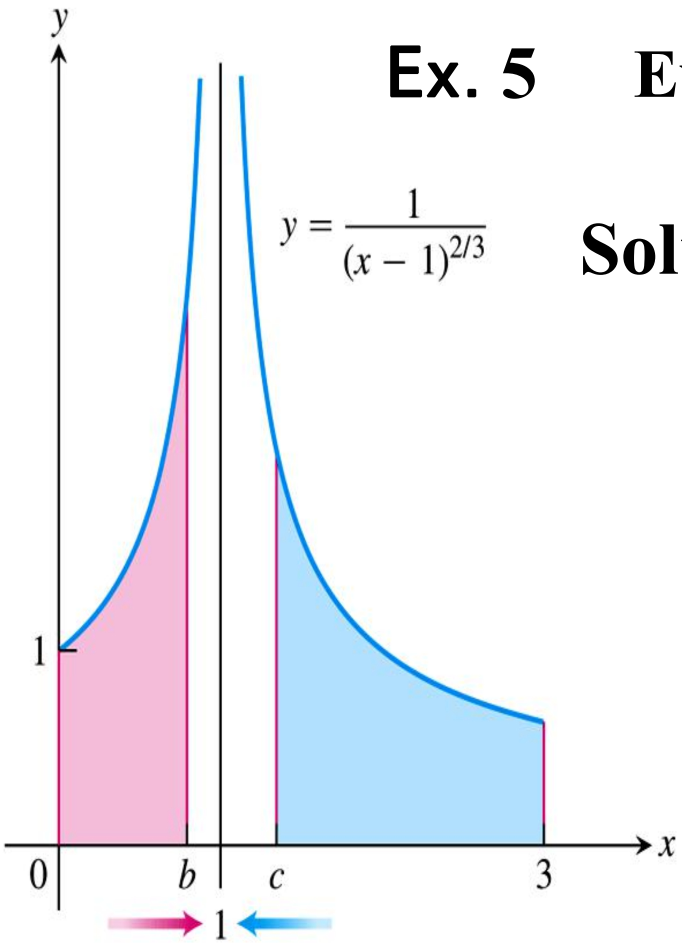


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

Solution $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx =$

$$\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{b\to 1^-} 3\sqrt[3]{x-1}\,\Big|_0^b = \lim_{b\to 1^-} 3(\sqrt[3]{b-1}+1) = 3$$

$$\int_1^3 \frac{1}{(x-1)^{2/3}}\,dx = 3\sqrt[3]{x-1}\,\Big|_1^3 = 3\sqrt[3]{2}$$

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2}$$

Ex. **Investigate the convergence of** $\int_0^1 \frac{1}{x^q}dx \quad \int_a^b \frac{1}{(x-a)^q}dx$

Solution

$$(1)\ \ q=1,\quad \int_0^1 \frac{1}{x^q}dx = \int_0^1 \frac{1}{x}dx = [\ln x]_0^1 = +\infty,$$

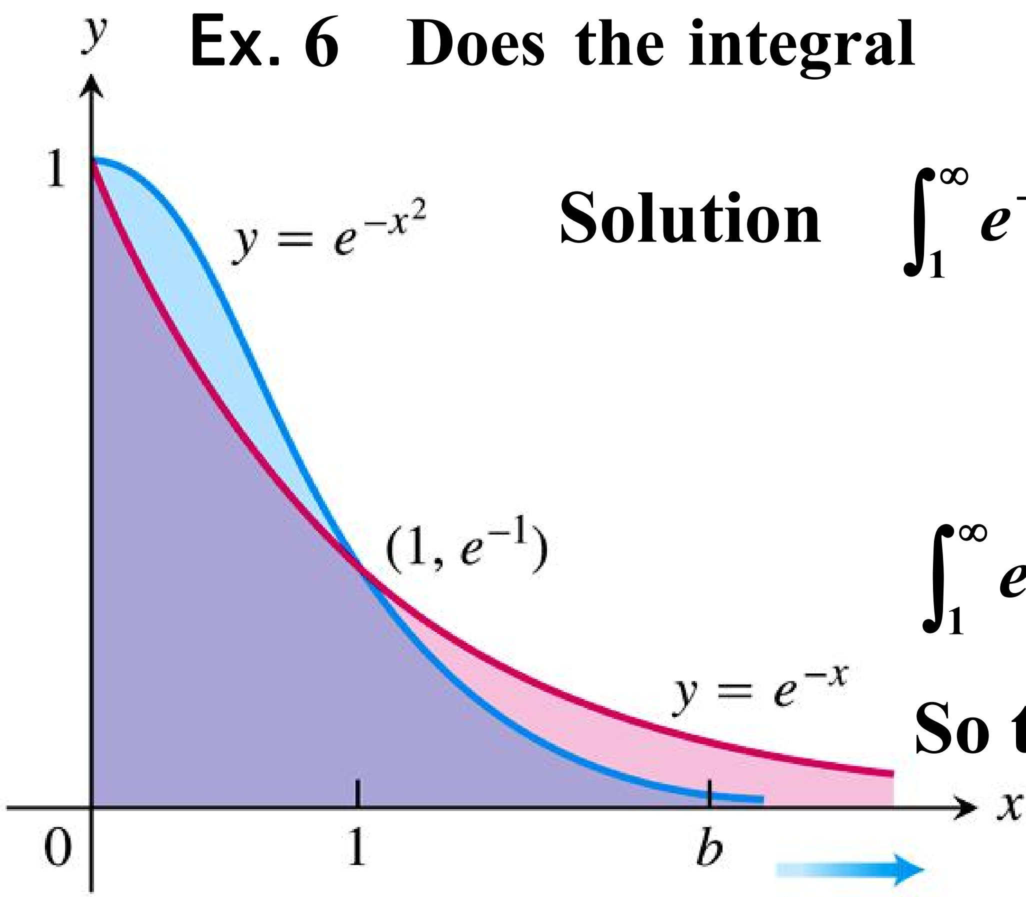
$$(2)\ \ q\neq 1,\quad \int_0^1 \frac{1}{x^q}dx = \left[\frac{x^{1-q}}{1-q}\right]_0^1 = \begin{cases} +\infty, & q>1 \\ \dfrac{1}{1-q}, & q<1 \end{cases}$$

因此当$q<1$时反常积分收敛，其值为

$\dfrac{1}{1-q}$；当$q\geq 1$时反常积分发散.

Tests for Convergence and Divergence

Ex. 6 **Does the integral** $\int_1^\infty e^{-x^2}\,dx$ **converge?**



Solution $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx$

$= \lim_{b\to\infty}(e^{-1} - e^{-b}) = e^{-1}$

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx$$

So the integral converges.

FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6).

比较检验法

THEOREM 2—Direct Comparison Test Let f and g be continuous on $\boxed{[a,b)}$ with $0 \le f(x) \le g(x)$ for all $\boxed{[a,b)}$ Then

1. $\boxed{\int_a^b f(x)dx}$ converges if $\boxed{\int_a^b g(x)dx}$ converges.

2. $\boxed{\int_a^b g(x)dx}$ diverges if $\boxed{\int_a^b f(x)dx}$ diverges.

$$\int_a^b f(x)dx \le \int_a^b g(x)dx \qquad \lim_{b\to\infty}\int_a^b f(x)dx \le \lim_{b\to\infty}\int_a^b g(x)dx$$

Ex. Test the convergence for $\int_1^{\infty} \frac{1}{x\sqrt{2x+1}}\,dx$

Solution $\int_1^{\infty} \frac{1}{x\sqrt{2x+1}}\,dx = \int_{\sqrt{3}}^{\infty} \frac{2}{u^2-1}\,du$ $\quad \sqrt{2x+1} = u,$

$$= \int_{\sqrt{3}}^{\infty} \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \ln\left|\frac{u-1}{u+1}\right|\Bigg|_{\sqrt{3}}^{+\infty} = \ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right|$$

$$\frac{1}{x\sqrt{2x+1}} < \frac{1}{x\sqrt{2x}} = \frac{1}{\sqrt{2}}\frac{1}{x^{3/2}}$$

converges

Ex. Test the convergence for $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$

Solution $\frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$

converges

$$\int_1^{\infty} \frac{1}{x\sqrt{2x-1}}\,dx?$$

比较检验法的极限形式

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $\boxed{[a,b)}$ and if

$$\boxed{\lim_{x\to b^-}\frac{f(x)}{g(x)}=L} \qquad 0<L<\infty,$$

then

$$\boxed{\int_a^b f(x)dx} \quad \text{and} \quad \boxed{\int_a^b g(x)dx}$$

both converge or both diverge.

$$\frac{L}{2}<\frac{f(x)}{g(x)}\le\frac{3L}{2}(x>M>a) \qquad \frac{L}{2}g(x)<f(x)\le\frac{3L}{2}g(x)(x>M>a)$$

Ex. 7 Test the convergence for the next integrals

$$(a)\int_2^{\infty}\frac{\cos\frac{1}{x}}{\sqrt{x(x-1)(x+1)}}\,dx$$

converges

$$(b)\int_1^{\infty}\frac{1}{\sqrt{x^2-0.1}}\,dx$$

diverges

$$(c)\int_0^{\pi/2}\frac{\cos x}{\sqrt{x}}\,dx$$

converges

Ex. 8 Shou that $\int_1^{\infty} \frac{1}{x\sqrt{1+x^2}}dx$ converges by comparison test.

Solution $\lim_{x\to\infty} \frac{1}{x\sqrt{1+x^2}} / \frac{1}{x^2} = 1$

$\int_1^{\infty} \frac{1}{x\sqrt{1+x^2}}dx$ converges because $\int_1^{\infty} \frac{1}{x^2}dx$ converges.

Ex. 9 Investigate the converges of $\int_1^{\infty} \frac{1-e^{-x}}{x}dx$

Solution $\lim_{x\to\infty} \frac{1-e^{-x}}{x} / \frac{1}{x} = 1$

$\int_1^{\infty} \frac{1-e^{-x}}{x}dx$ diverges because $\int_1^{\infty} \frac{1}{x}dx$ diverges.

例 判别下列反常积分的敛 散性：

$$(1)\int_0^1 \frac{e^x dx}{\sqrt{1-x}}, \qquad (2)\int_0^1 \frac{\ln x dx}{\sqrt{x}}.$$

解（1） $\because$ 被积函数在点 $x=1$ 的左邻域内无界.

$$\lim_{x\to 1-0} \frac{\dfrac{e^x}{\sqrt{1-x}}}{\dfrac{1}{\sqrt{1-x}}} = e,$$ 所给反常积分（1）收敛.

解（2） $$\int_0^1 \frac{\ln x dx}{\sqrt{x}} = \int_0^1 \ln x d2\sqrt{x}$$

$$= 2\sqrt{x}\ln x\Big|_0^1 - 2\int_0^1 \frac{1}{\sqrt{x}} dx = -4$$

所给反常积分（2）收敛.

Testing for Convergence

$$\int_{-1}^{1} \ln |x| \, dx$$

$$\int_{0}^{1} \frac{dt}{t - \sin t}$$

$$\int_{1}^{\infty} \frac{\sqrt{x+1}}{x^2} \, dx$$

$$\int_{0}^{\pi} \frac{dt}{\sqrt{t} + \sin t}$$

$$\int_{\pi}^{\infty} \frac{1 + \sin x}{x^2} \, dx$$

$$\int_{e^e}^{\infty} \ln(\ln x) \, dx$$

For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx$$

converge? Evaluate the corresponding integral(s).

$$a = \frac{1}{2} \qquad -\frac{1}{4} \ln 2$$