CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Corollary 2 If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

  Corollary 3 In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

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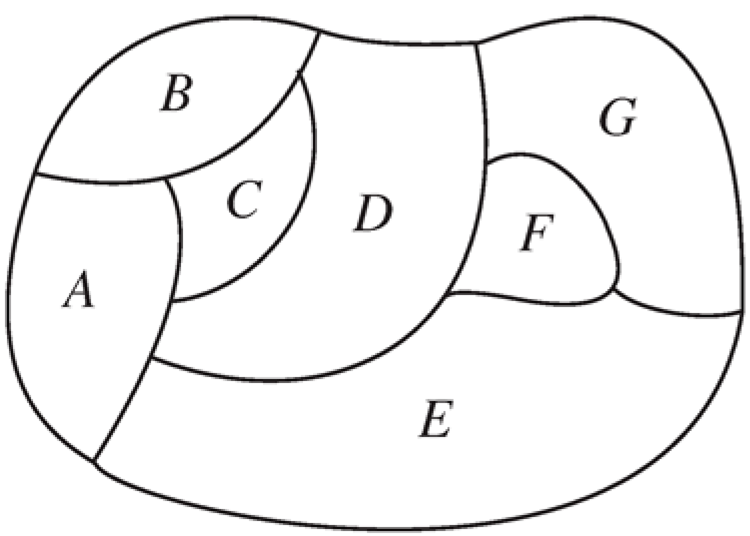

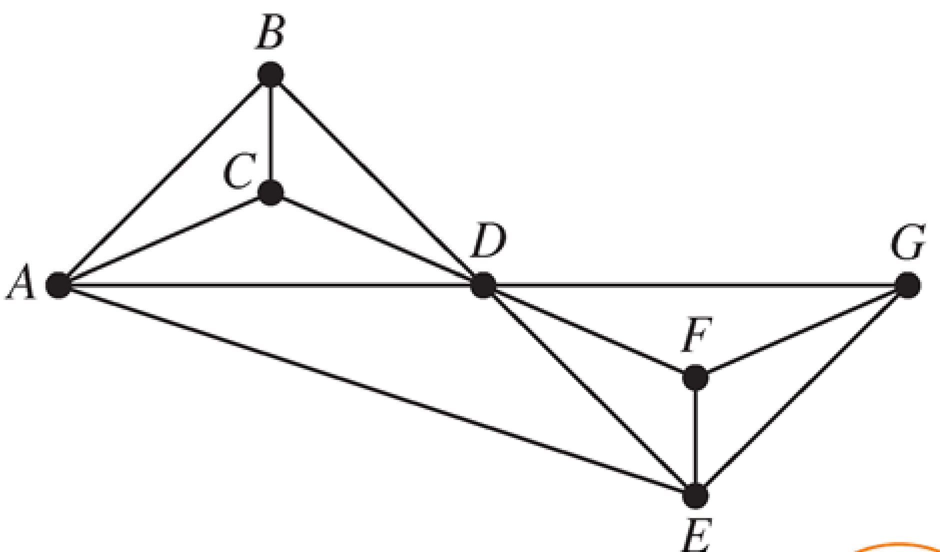

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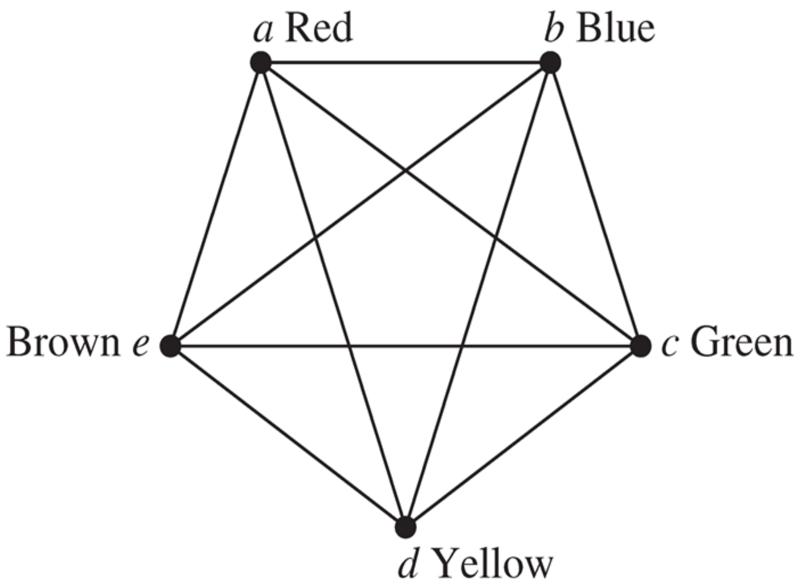

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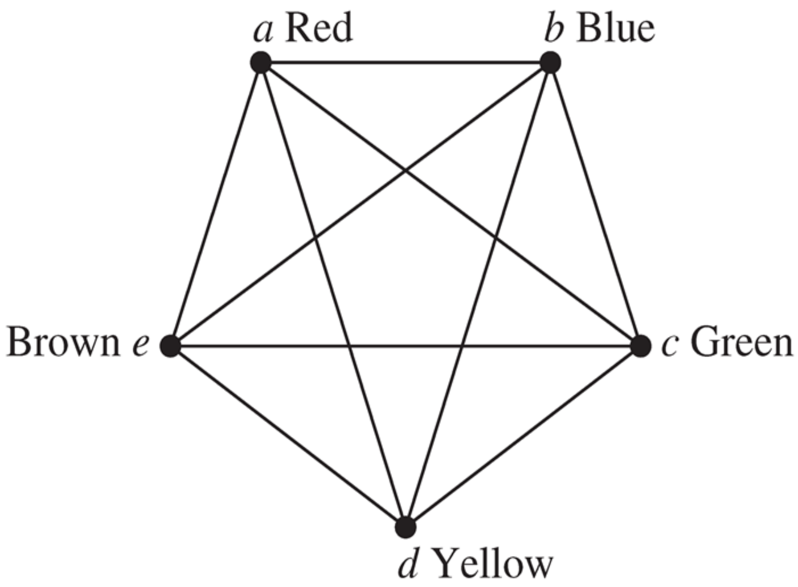

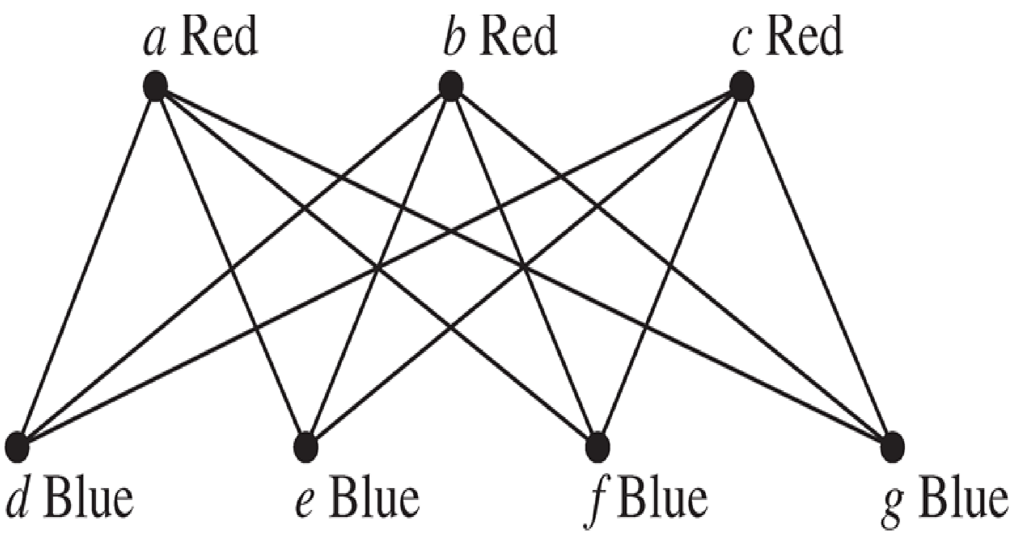

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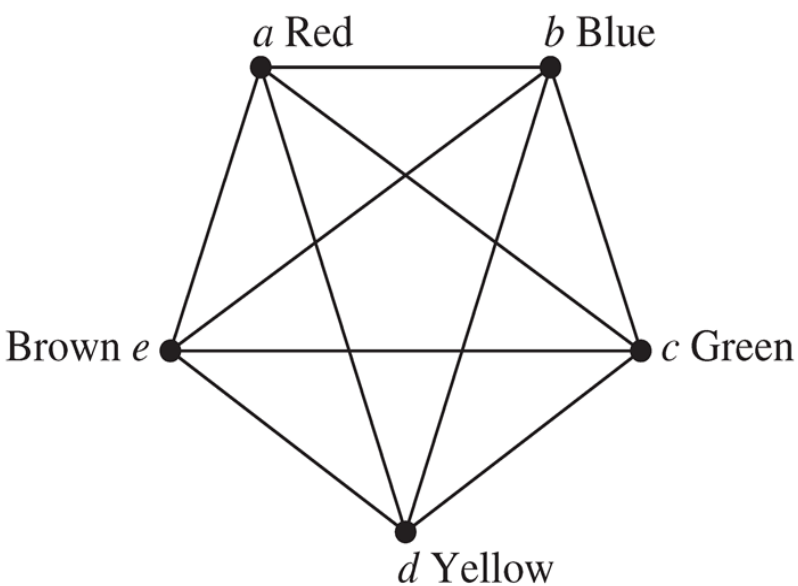

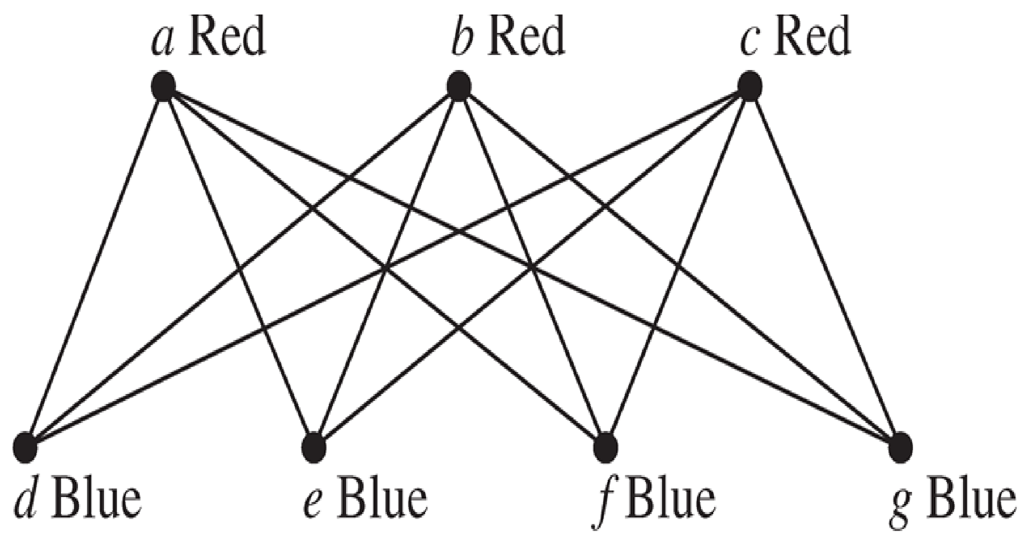

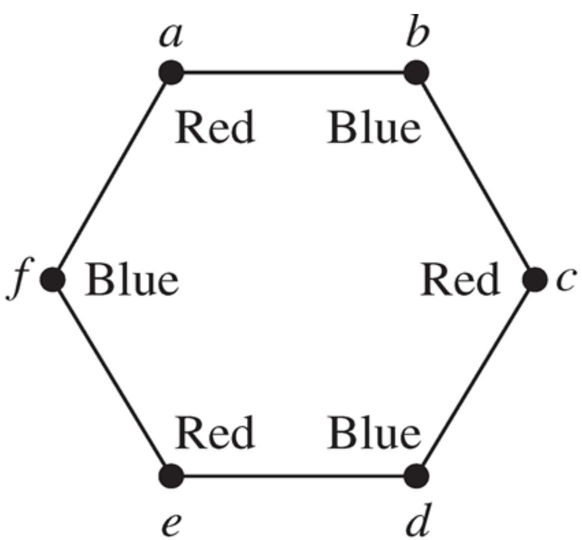

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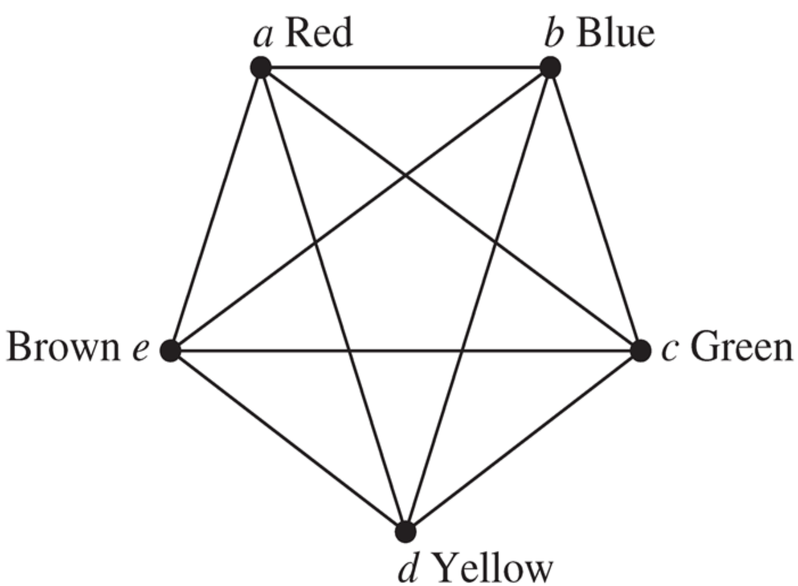
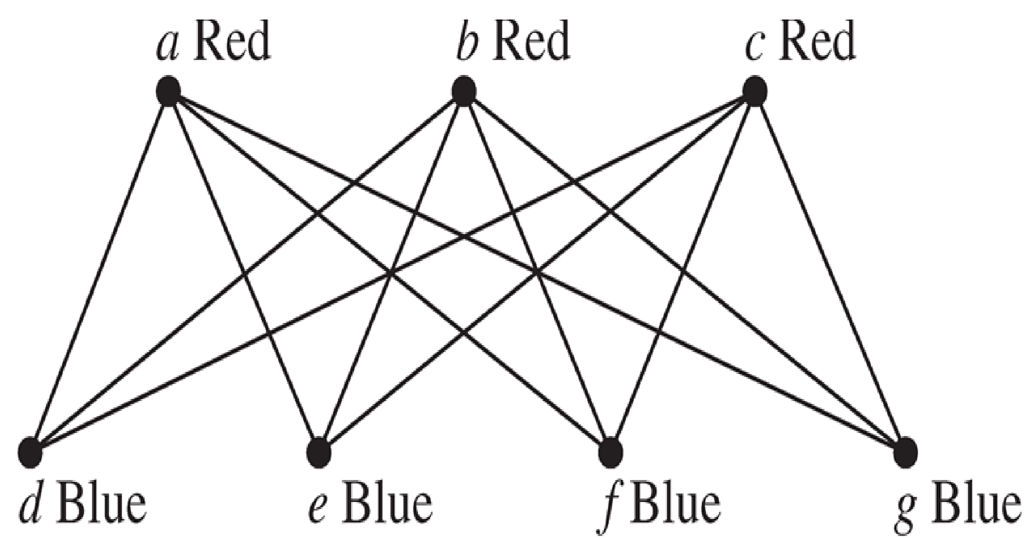
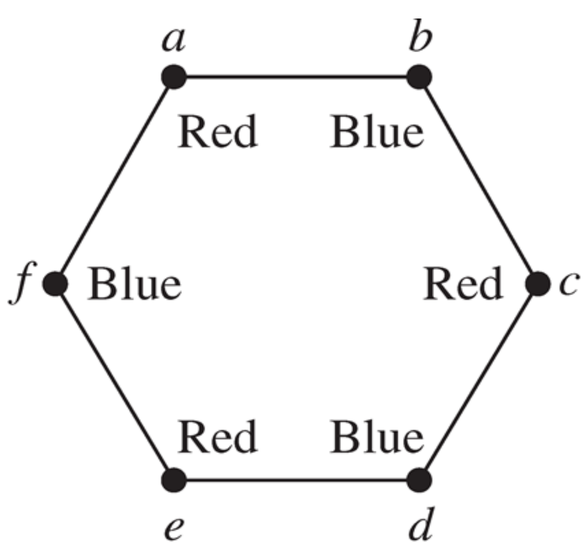
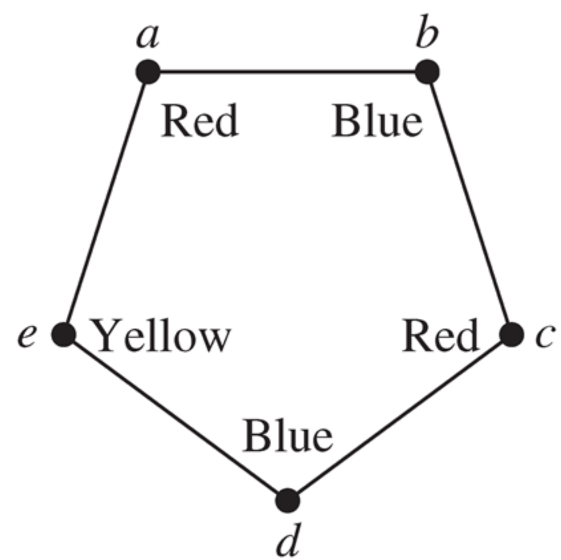


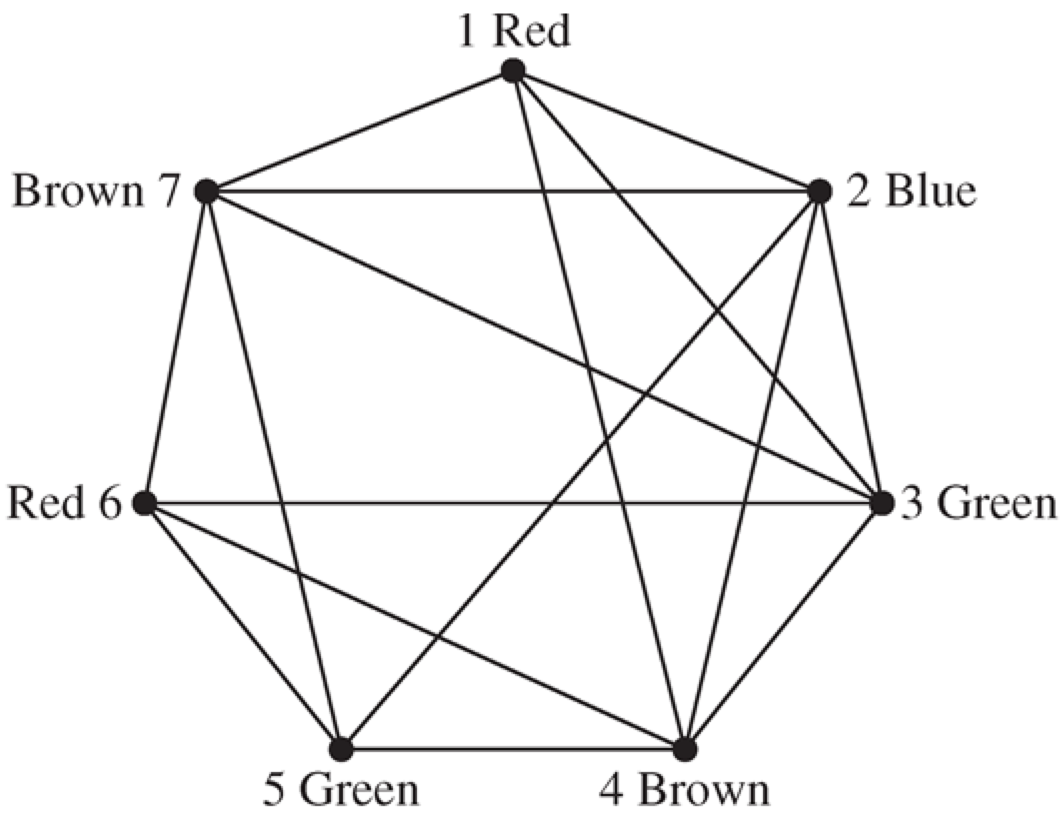






- Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

Zero Knowledge Proofs

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  In a *Zero Knowledge Proof*, Alice will prove to Bob that a statement P is true. Bob will be completely convinced that P is true, but will not learn anything as a result of this process. That is, Bob will gain zero knowledge.

Applications of ZKPs

- *Protocol design*. A *protocol* is an algorithm for interactive parties to achieve a certain goal. However, in crypto, we often want to design protocols that should achieve security even when one of the parties is "cheating". Alice can prove in *zero knowledge* that she followed the instructions.

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Proofs that Yield Nothing But their Validity and a Methodology of Cryptographic Protocol Design

(Extended Abstract)

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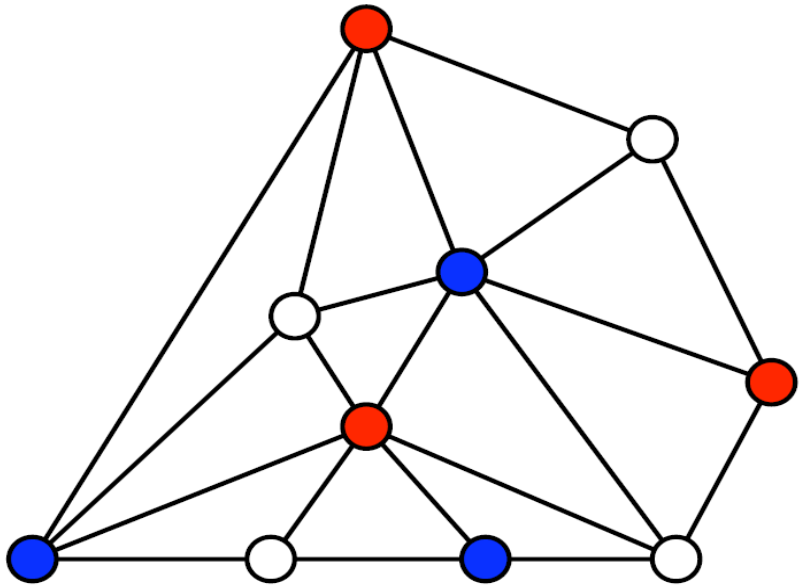
  Ideas using ZKPs:
  - Let the box contain an *instance* of a hard problem.
  - Give the authorized people the *solution* to the instance.
  - The authorized people will *prove* to the box that they know the solution in zero knowledge.

An Example



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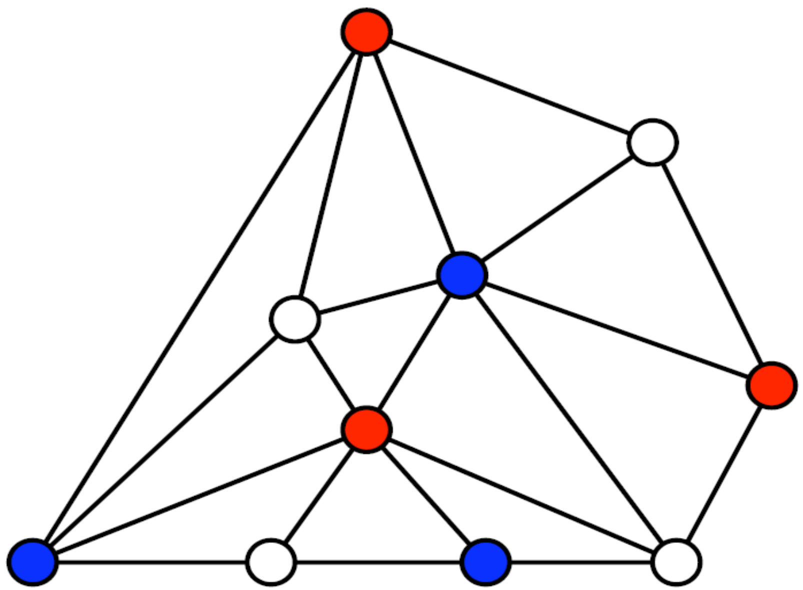

- Alice knows how to 3-color a graph: no two adjacent vertices have the same color; this is an NPC problem.
  - can impress your friends
  - useful for identification

An Example

- How can Alice convince Bob that she can 3-color the graph without
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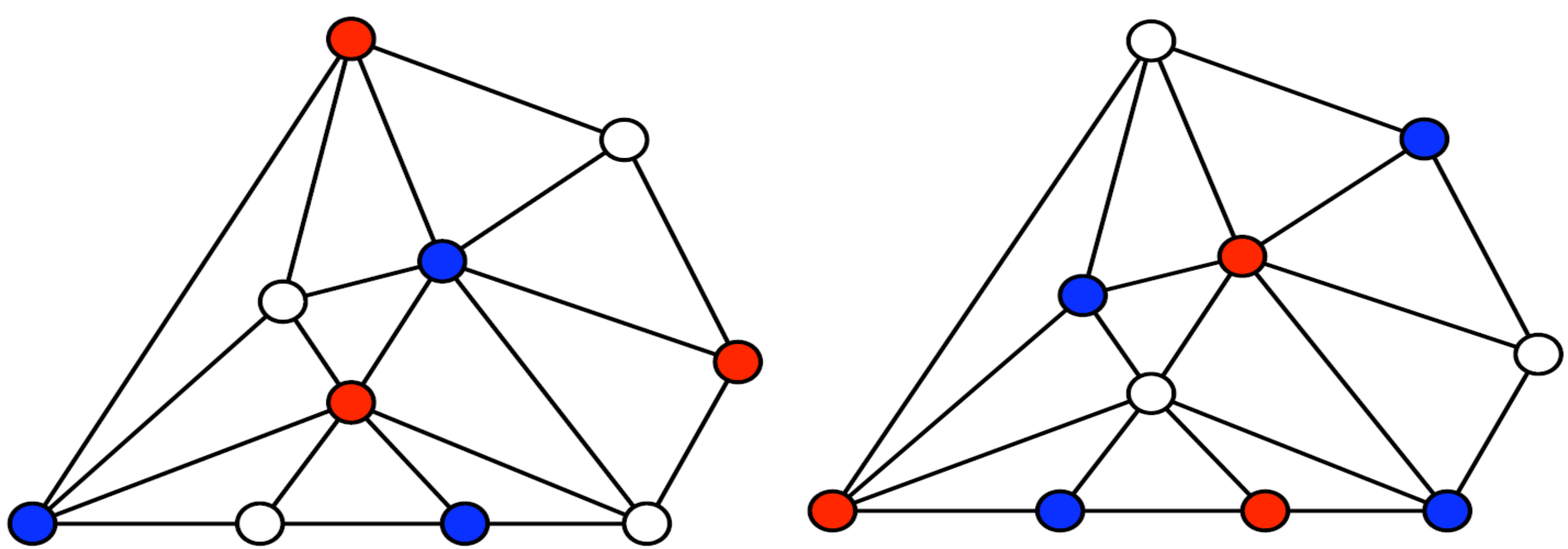
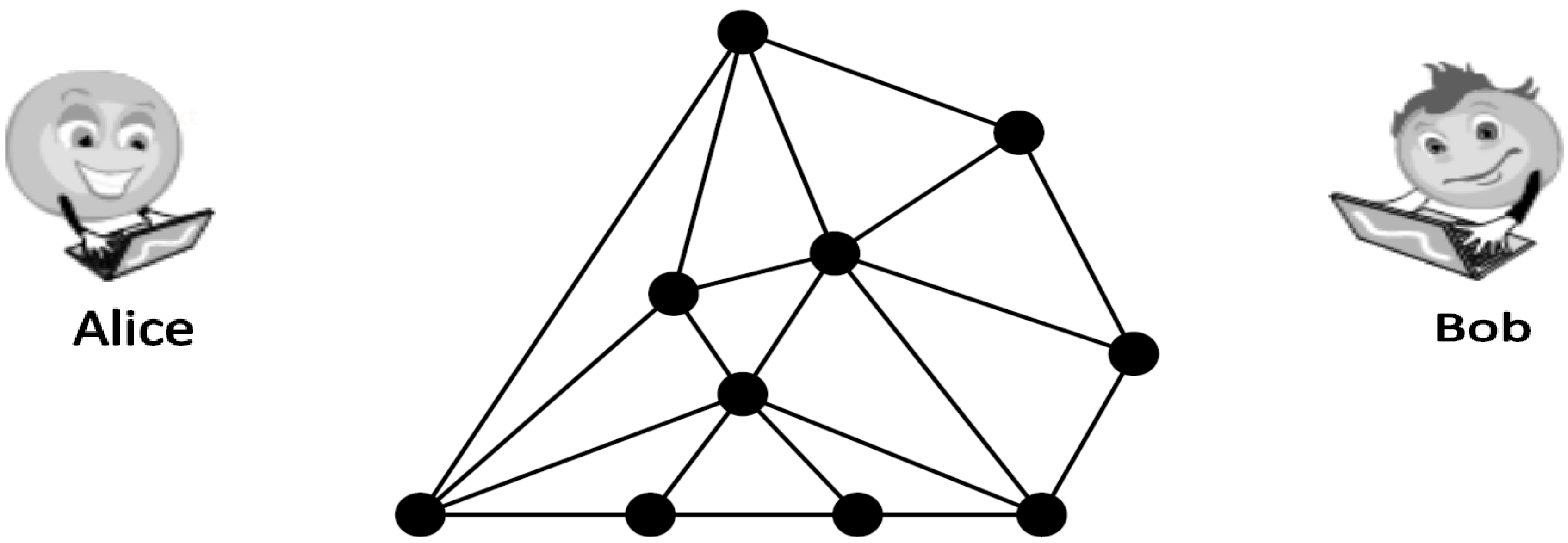
Alice may permute the vertex colors.



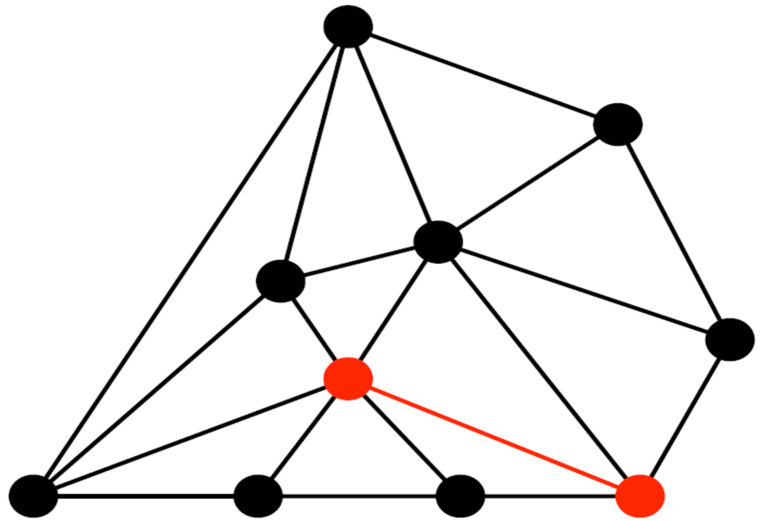
An Example

- Alice then encrypts all vertex colors (one key per vertex), and sends the graph to Bob.



An Example

- Bob picks an edge at random.



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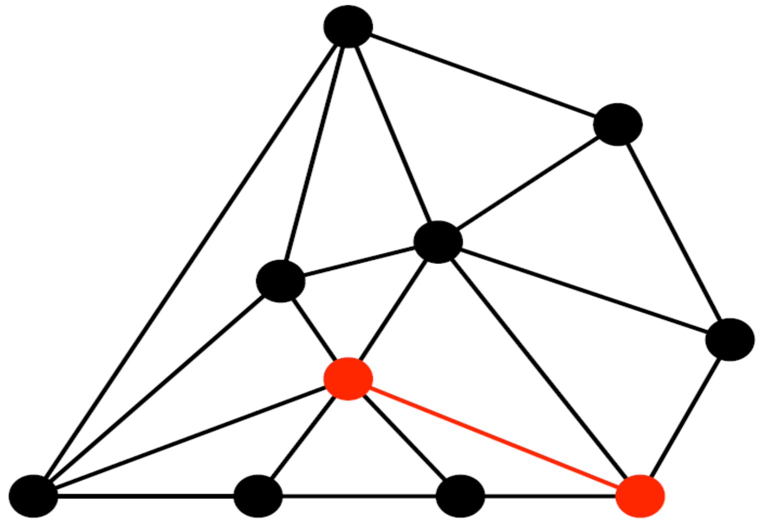
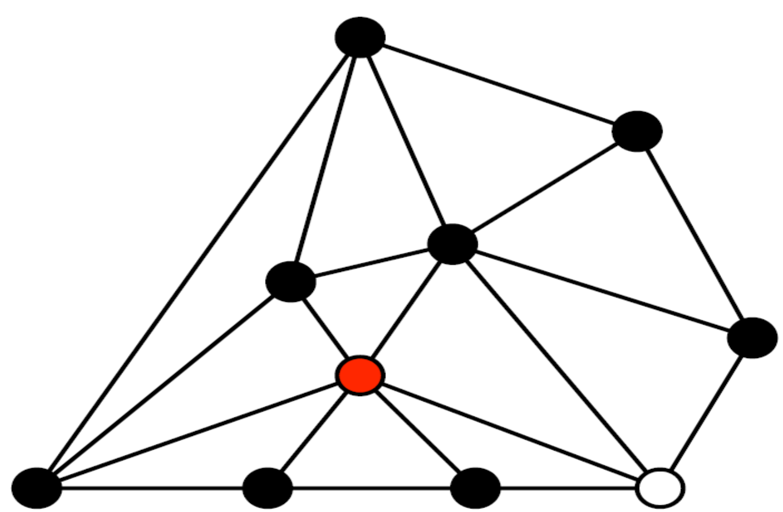


Alice reveals colors of those two keys.



An Example

- Repeat as much as needed:
  - Alice permutes graph coloring
  - Alice encrypts all vertices with distinct keys
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After k repetitions, the probability she fools Bob is $(1 - \frac{1}{|E|})^k$.

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- What does Bob see?
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  Because Bob could have generated those keys and colors by himself, he learns nothing from the graph coloring.

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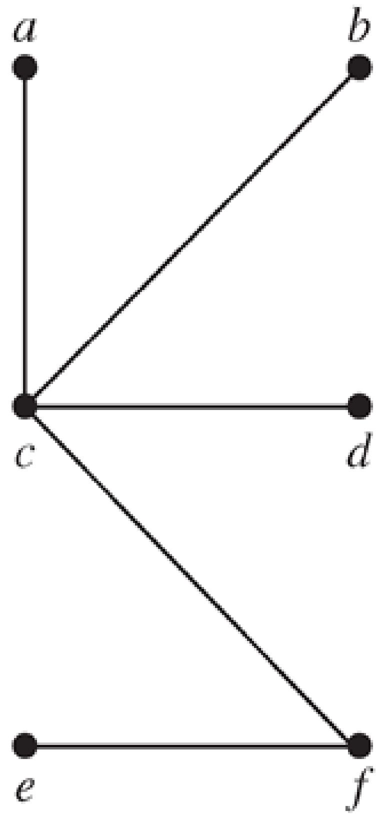

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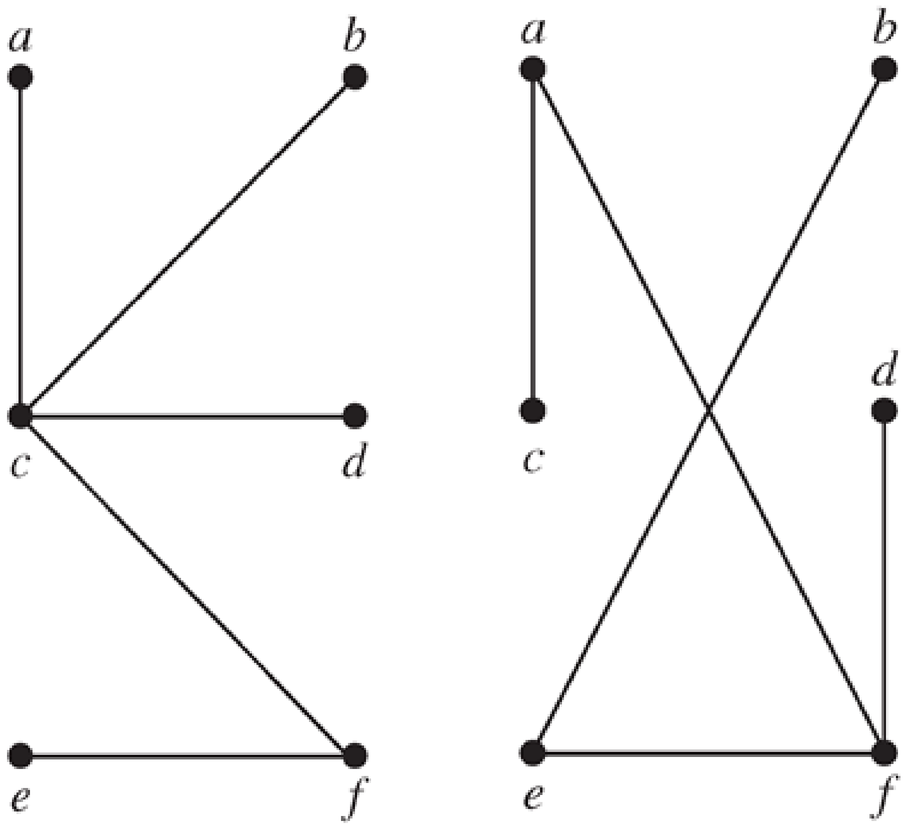

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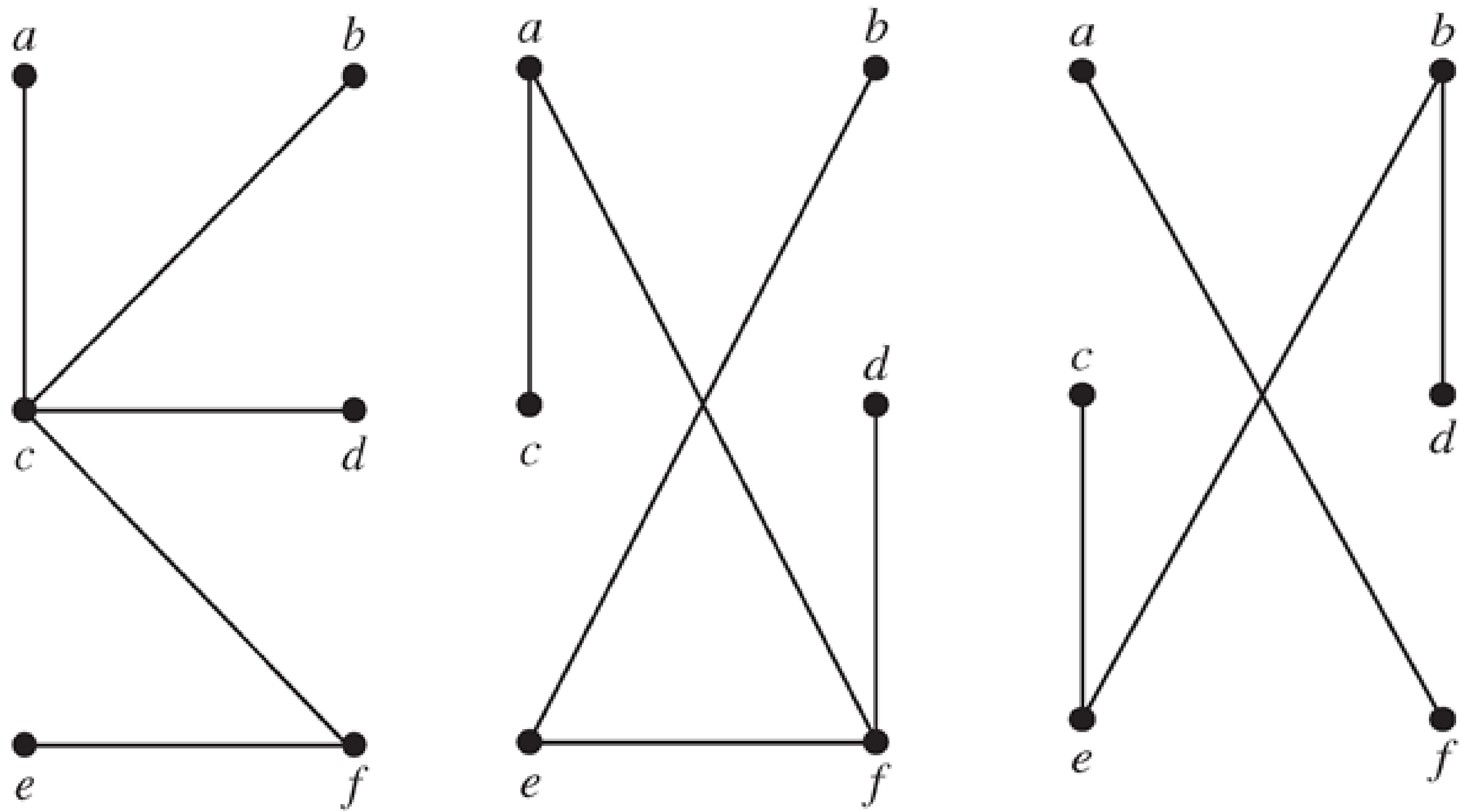

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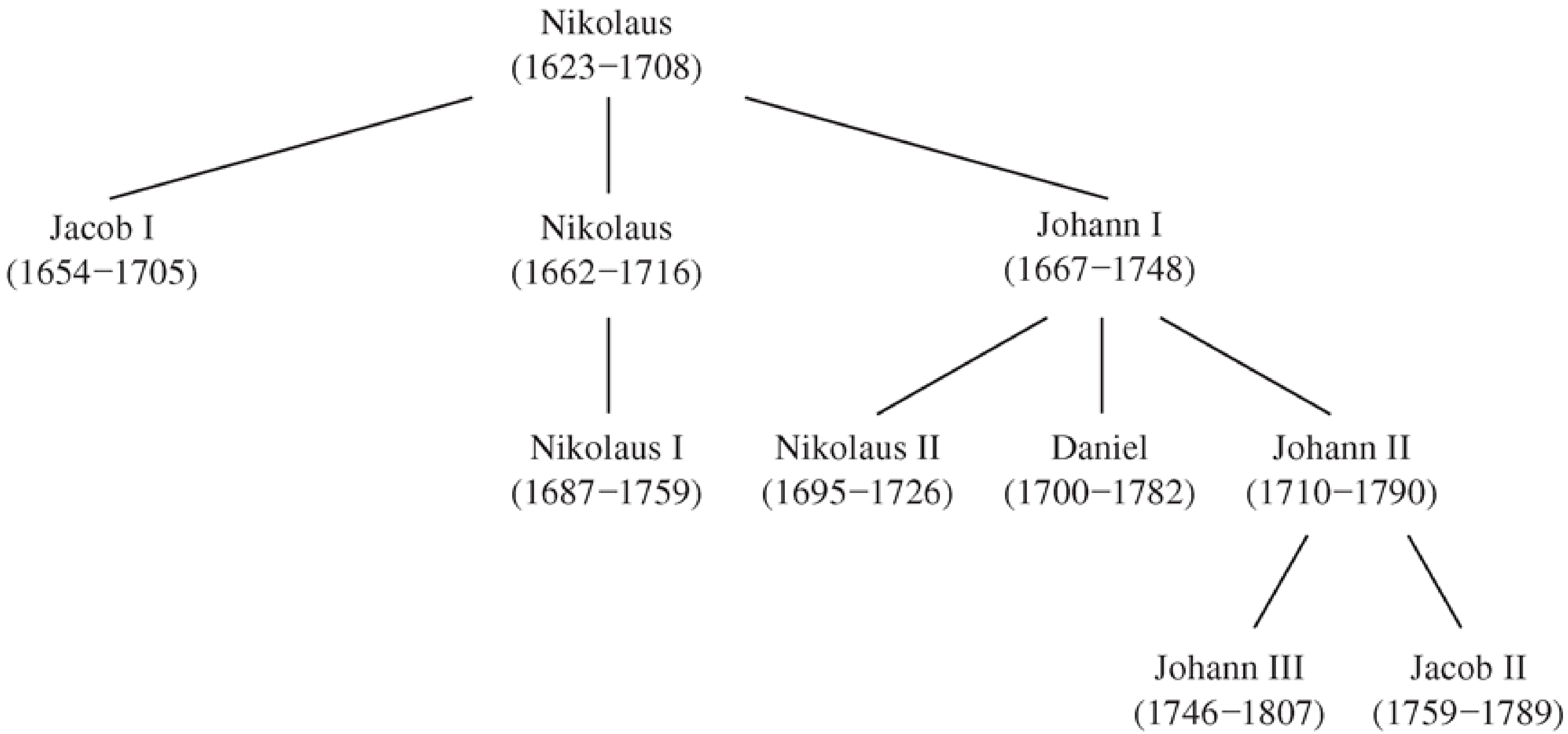

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Two properties of tree: connected, no circuit

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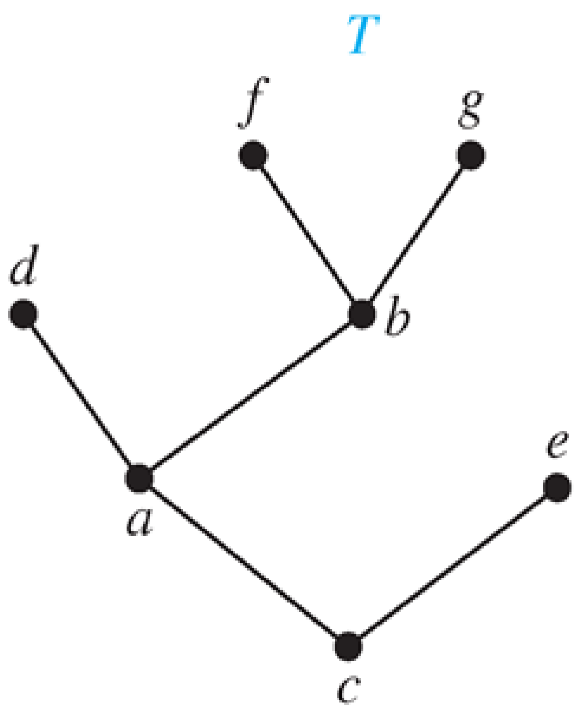

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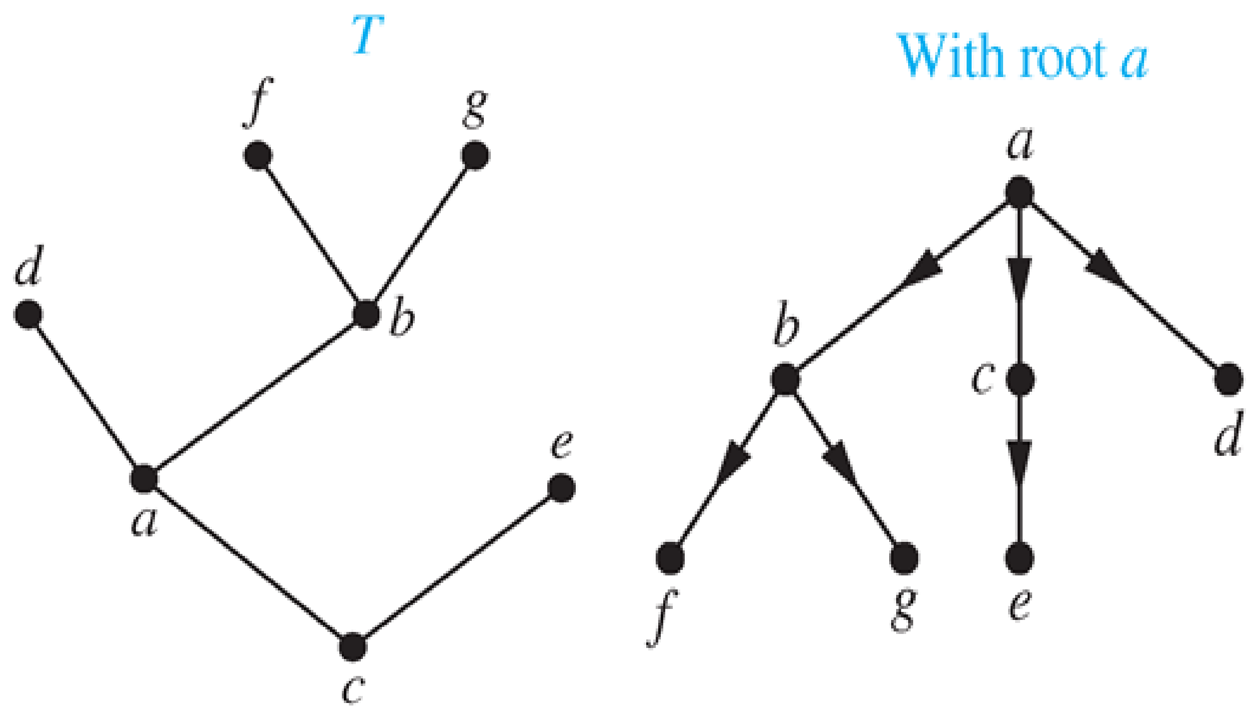

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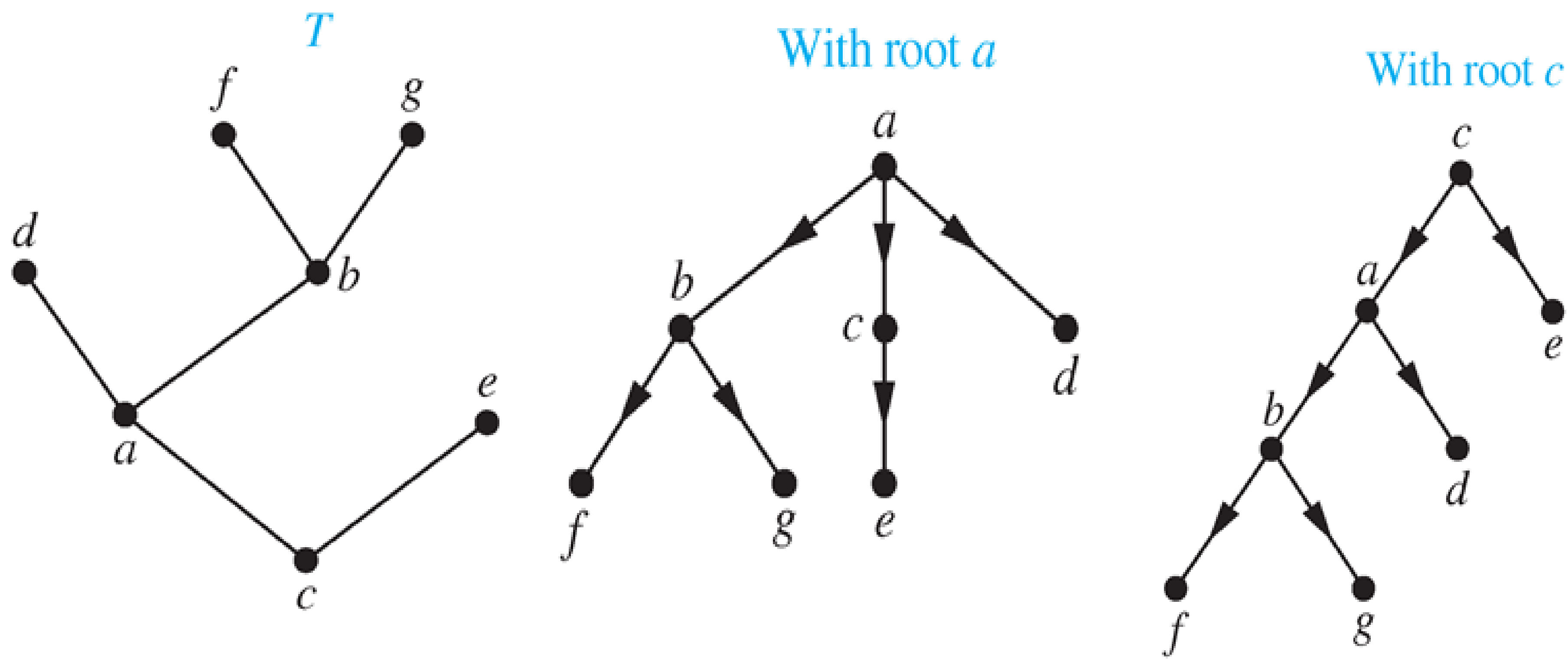

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  subtree with a as its root: consists of a and its descendants and all edges incident to these descendants

m-Ary Trees

- **Definition** A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. In particular, an m-ary tree with $m = 2$ is called a *binary tree*.

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  left subtree, *right subtree*

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  (i) n vertices, $i = (n-1)/m$, $\ell = [(m-1)n+1)]/m$
  (ii) i internal vertices, $n = mi + 1$, $\ell = (m-1)i + 1$
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  using $n = mi + 1$ and $n = i + \ell$

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  Definition A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h-1$. (differ no greater than 1)

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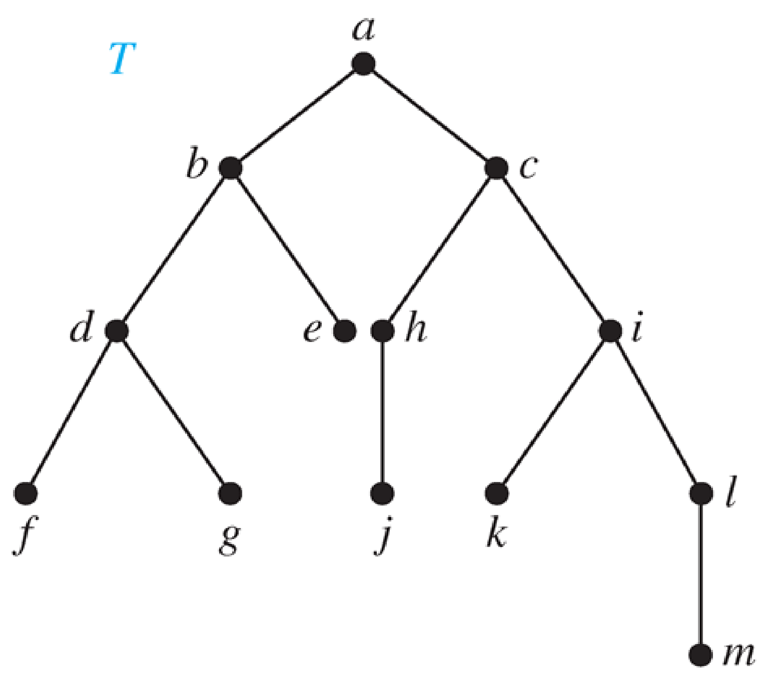

Tree Traversal

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  The three most commonly used traversals are *preorder traversal*, *inorder traversal*, *postorder traversal*.

Preorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *preorder traversal* begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.

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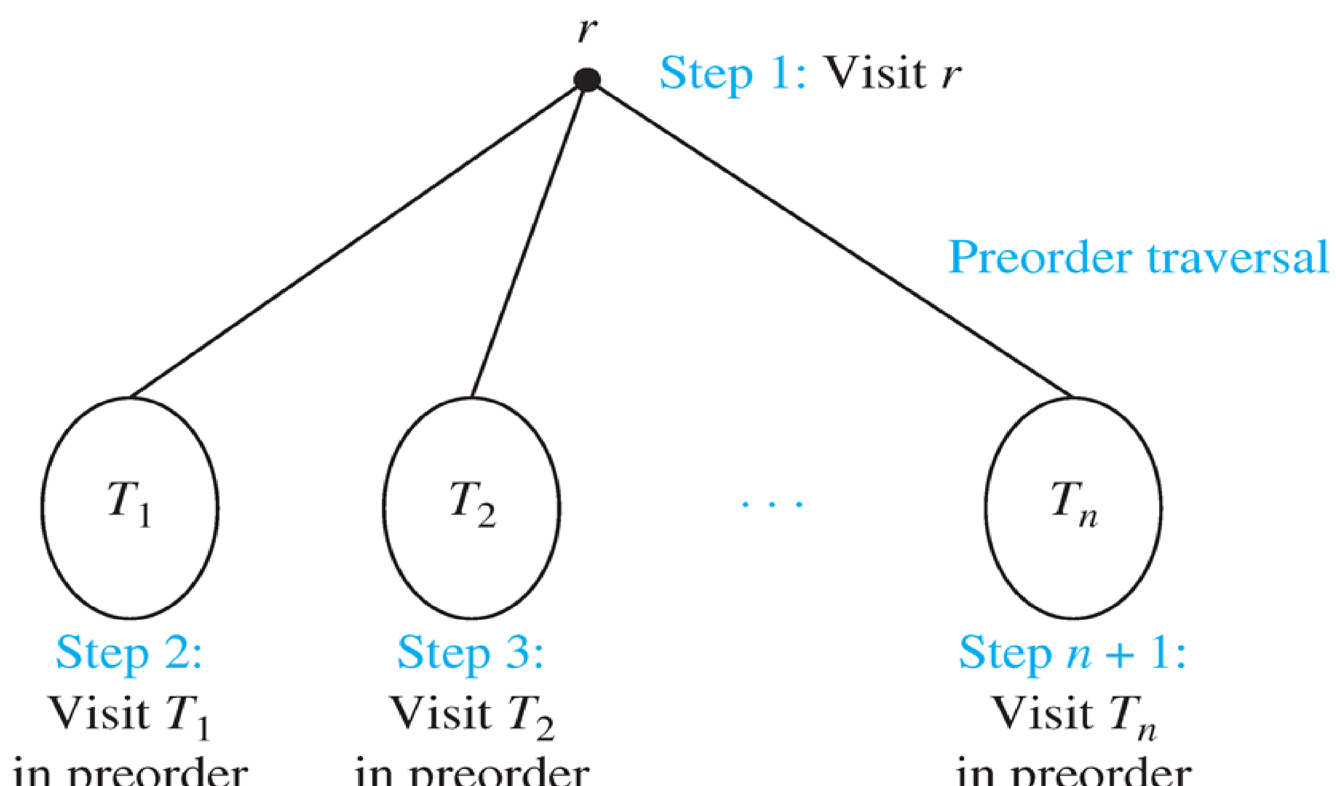
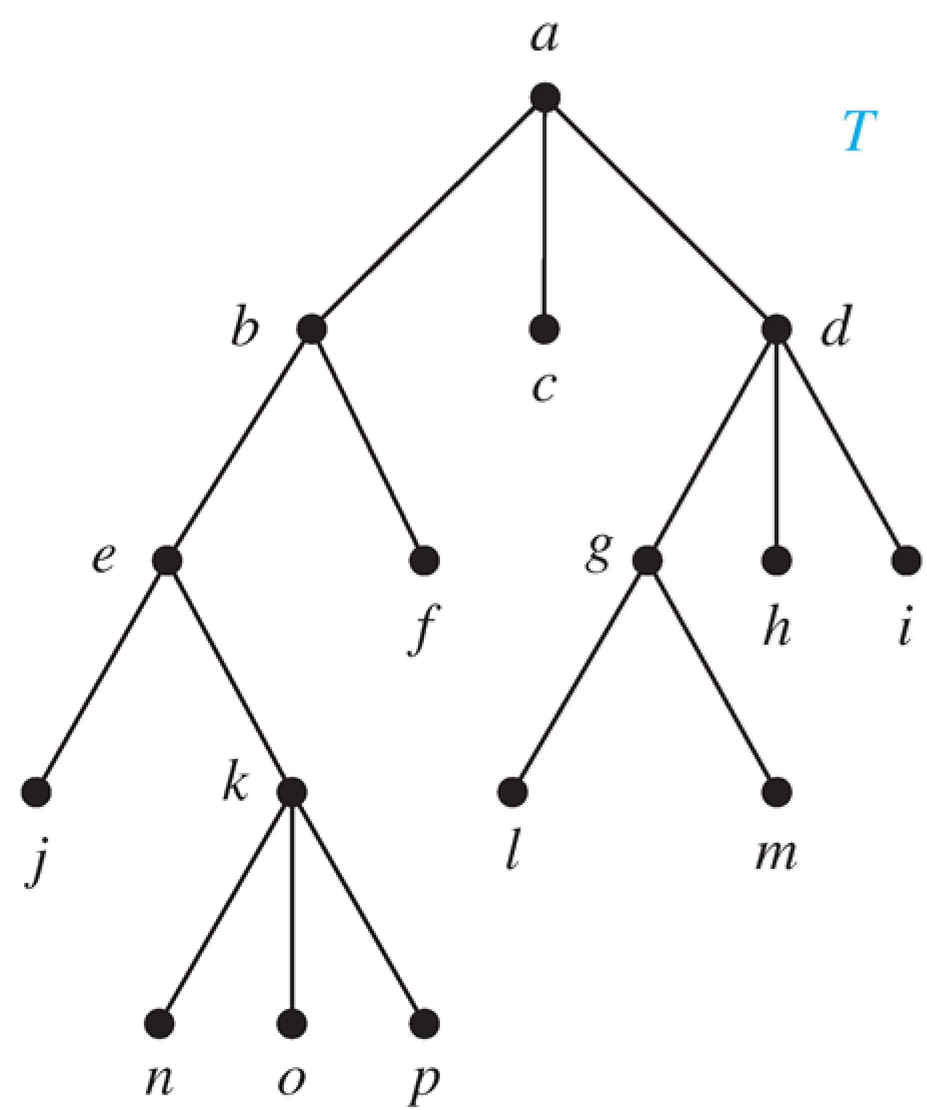


- **Example**



Preorder Traversal

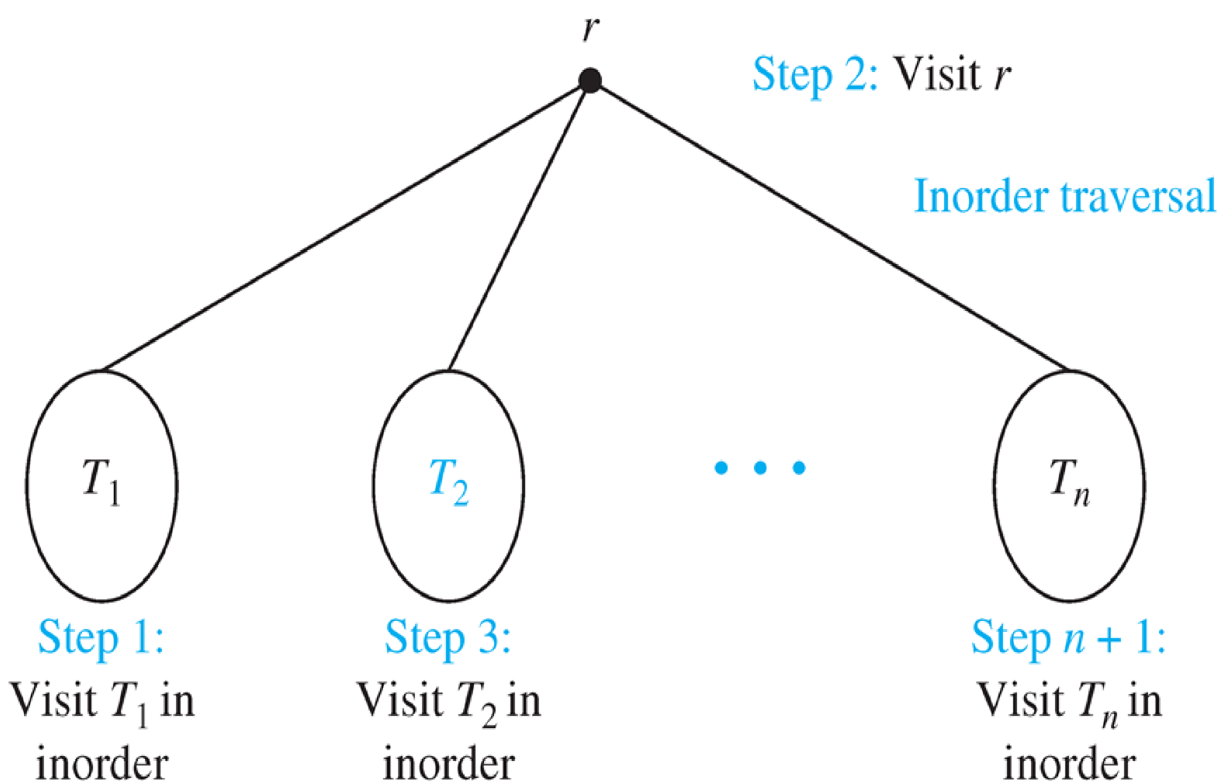
procedure *preorder* (T: ordered rooted tree)
r := root of T
list r
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    preorder($T(c)$)

Inorder Traversal

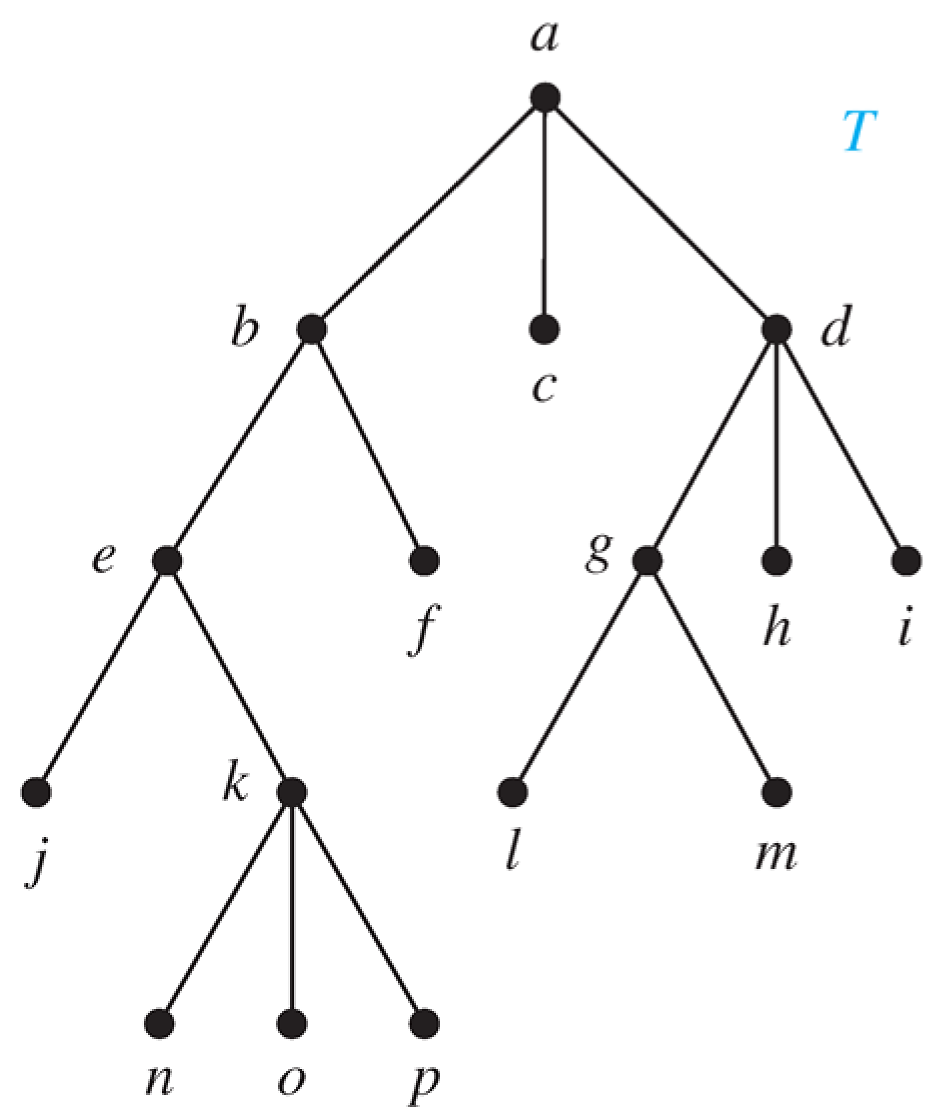
- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.

Inorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.



- **Example**



Inorder Traversal

procedure *inorder* (T: ordered rooted tree)
r := root of T
if r is a leaf **then** list r
else
    l := first child of r from left to right
    $T(l)$:= subtree with l as its root
    inorder($T(l)$)
    list(r)
    for each child c of r from left to right
        $T(c)$:= subtree with c as root
        inorder($T(c)$)

Postorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.

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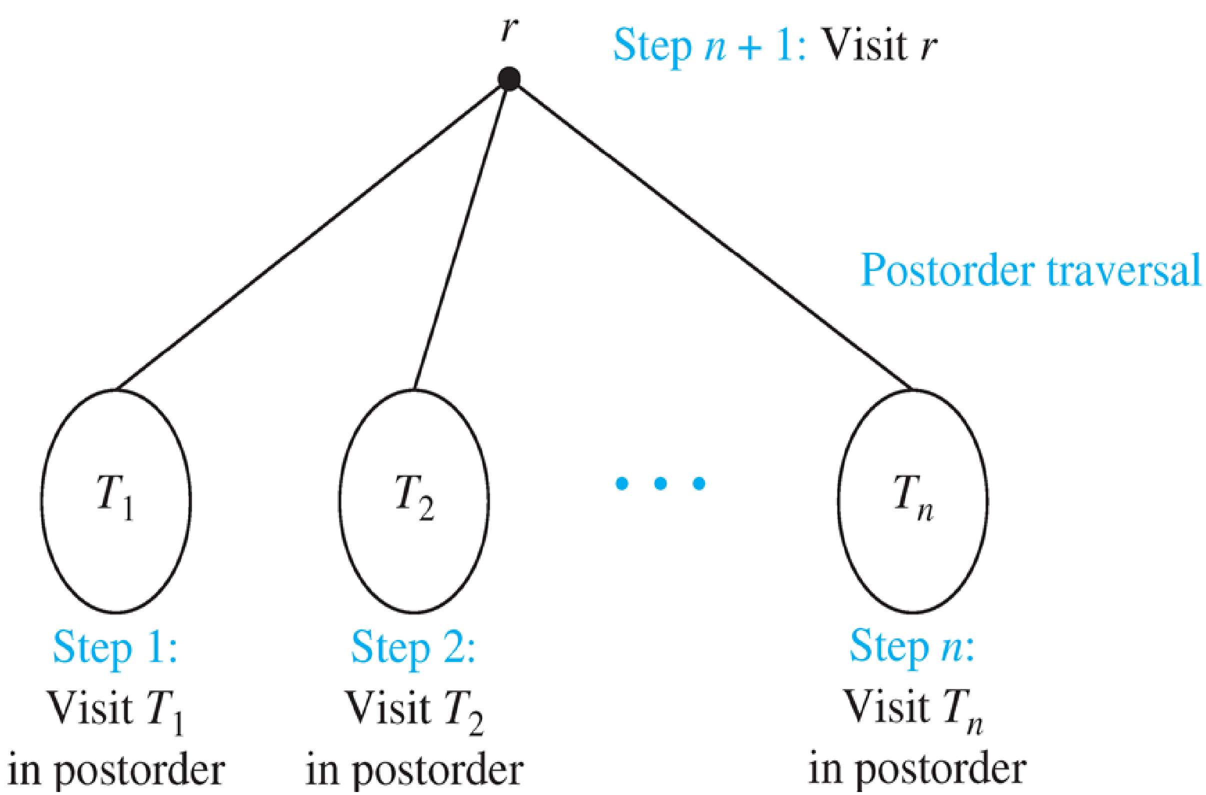

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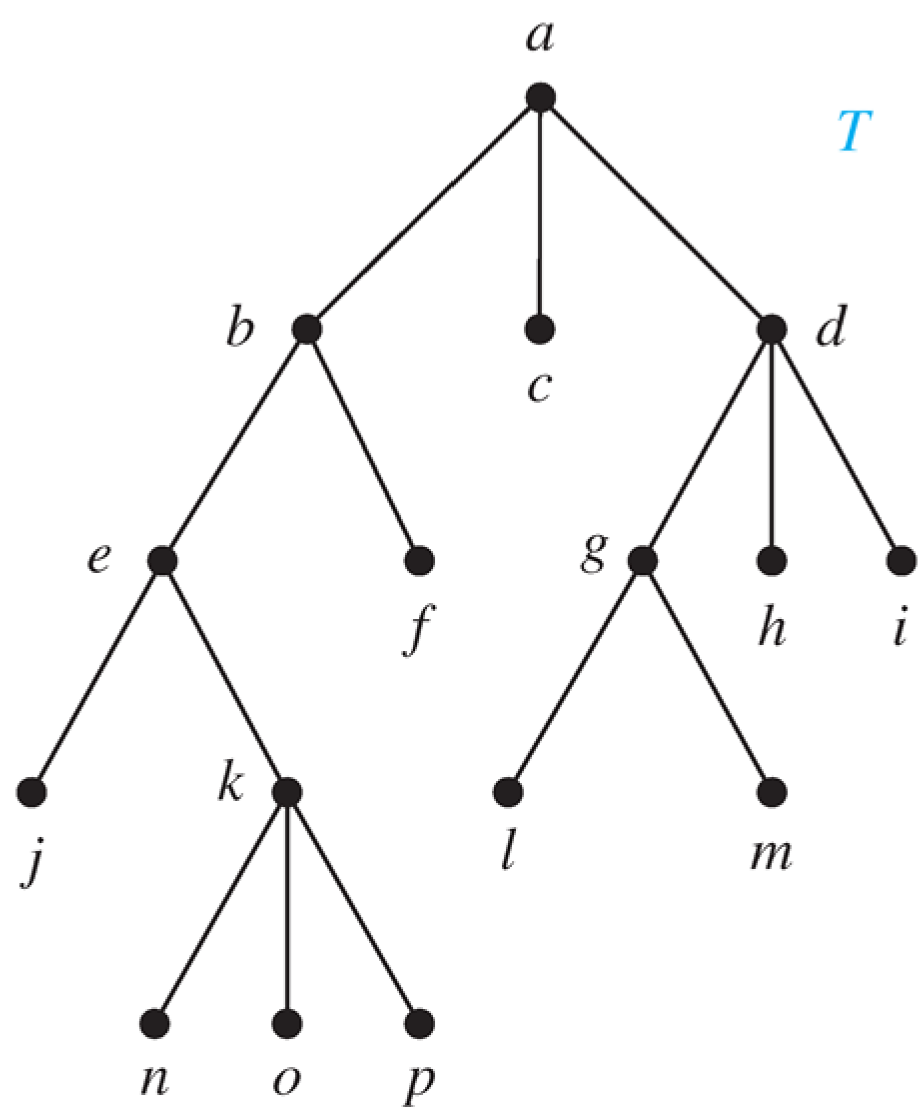
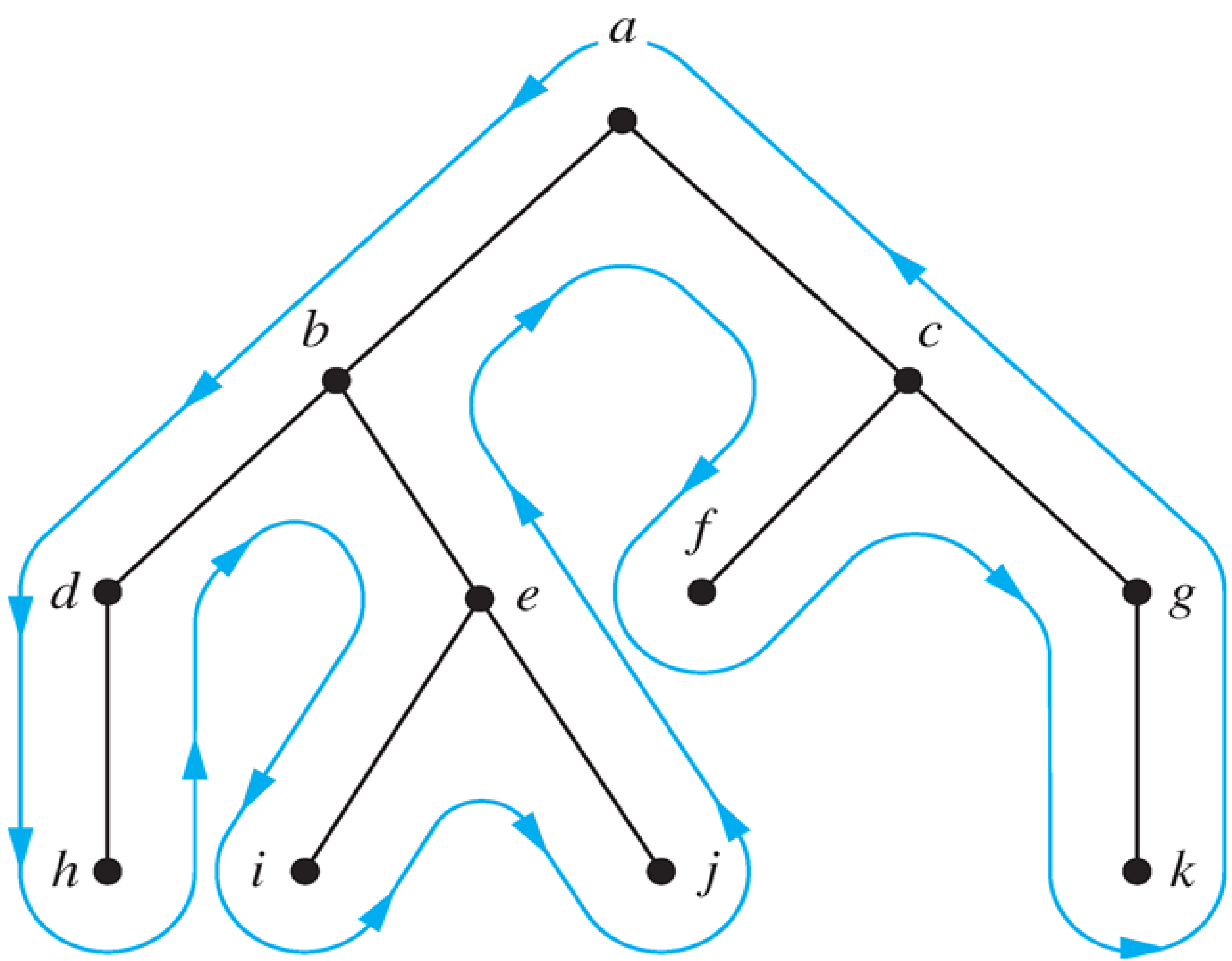


Postorder Traversal

procedure *postordered* (T: ordered rooted tree)
r := root of T
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    postorder($T(c)$)
list r

Preorder, Inorder, Postorder Traversal



Expression Trees

- Complex expressions can be represented using ordered rooted trees

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Example

consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$

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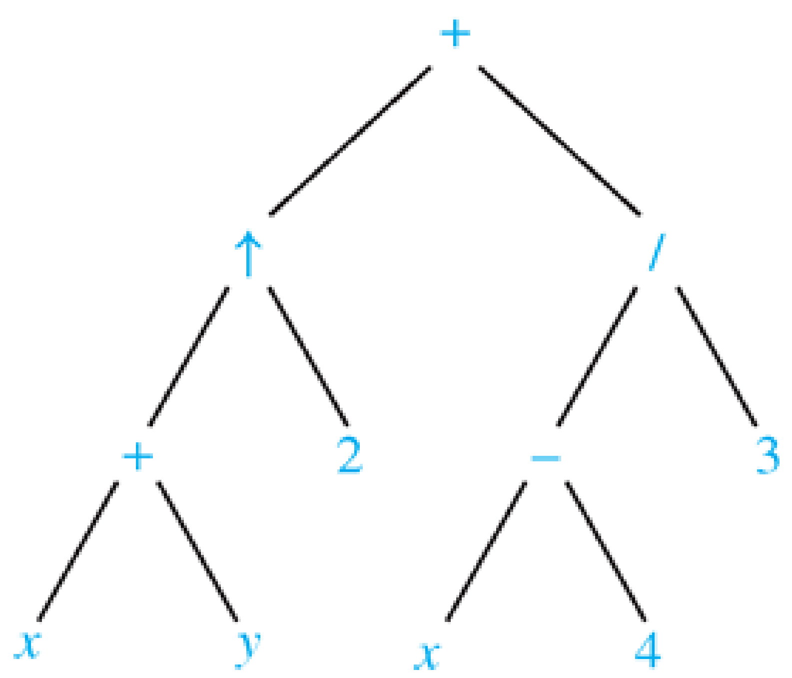

Infix Notation

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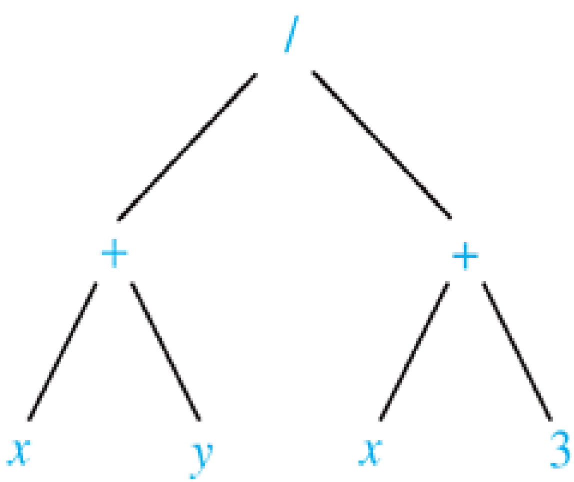

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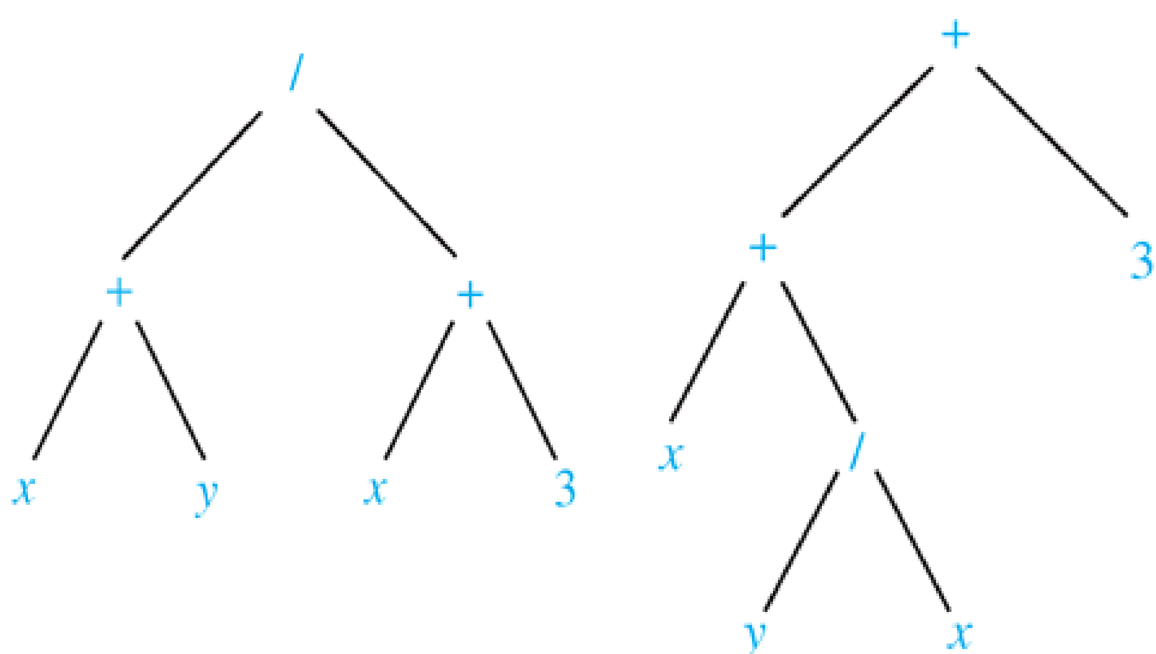

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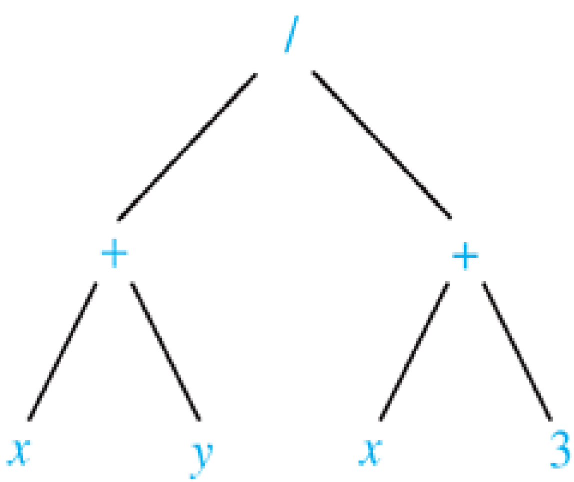

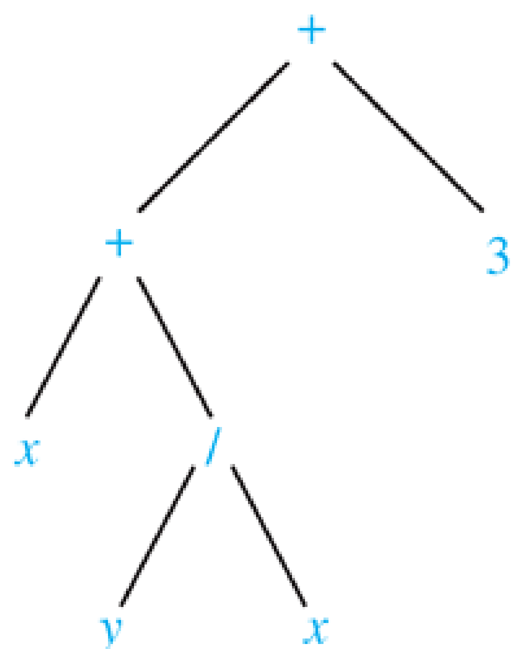

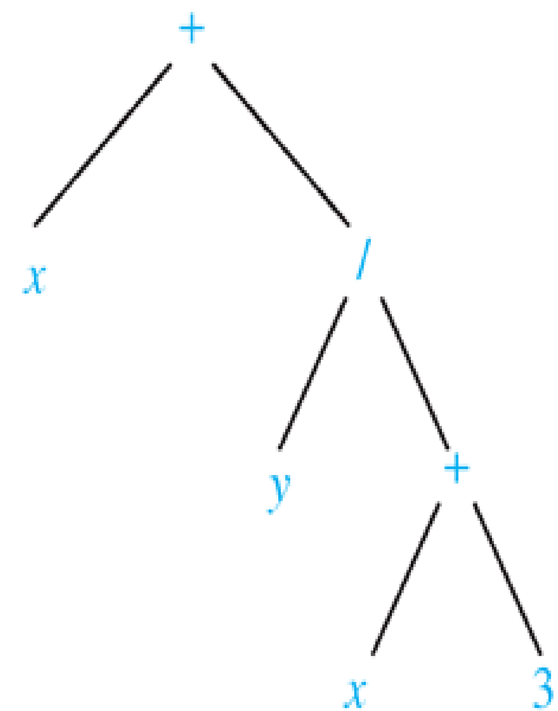

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  Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

Prefix Notation

- **Example**

  $+ \ - \ * \, 2 \ 3 \ 5 \ / \ \uparrow \ 2 \ 3 \ 4$

Prefix Notation

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$$+ \; - \; * \; 2 \; 3 \; 5 \; / \; \uparrow \; 2 \; 3 \; 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8 / 4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

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  Postfix expressions are evaluated by working from left to right. When we encounter an operator, we perform the operation with the two operands to the left.

Postfix Notation

- **Example**

  $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

- **Example**

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$$\underbrace{7\ \ 6\ \ -}_{7 - 6 = 1}\ 4\ \uparrow\ 9\ \ 3\ \ /\ \ +$$

$$\underbrace{1\ \ 4\ \ \uparrow}_{1^4 = 1}\ 9\ \ 3\ \ /\ \ +$$

$$1\ \underbrace{9\ \ 3\ \ /}_{9 / 3 = 3}\ +$$

$$\underbrace{1\ \ 3\ \ +}_{1 + 3 = 4}$$

Spanning Trees

- **Definition** Let G be a simple graph. A *spanning tree* of G is a subgraph of G that is a tree containing every vertex of G.

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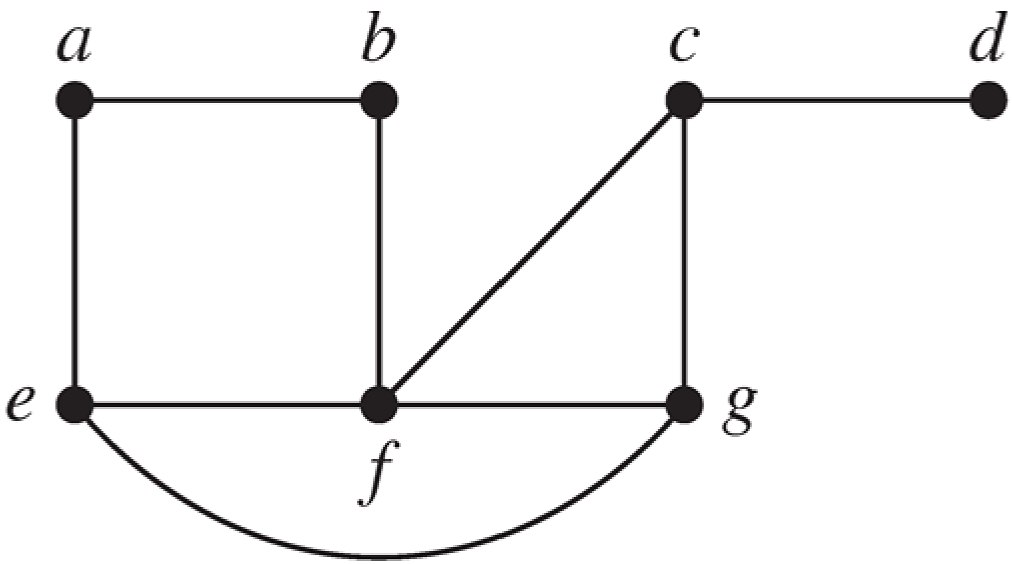

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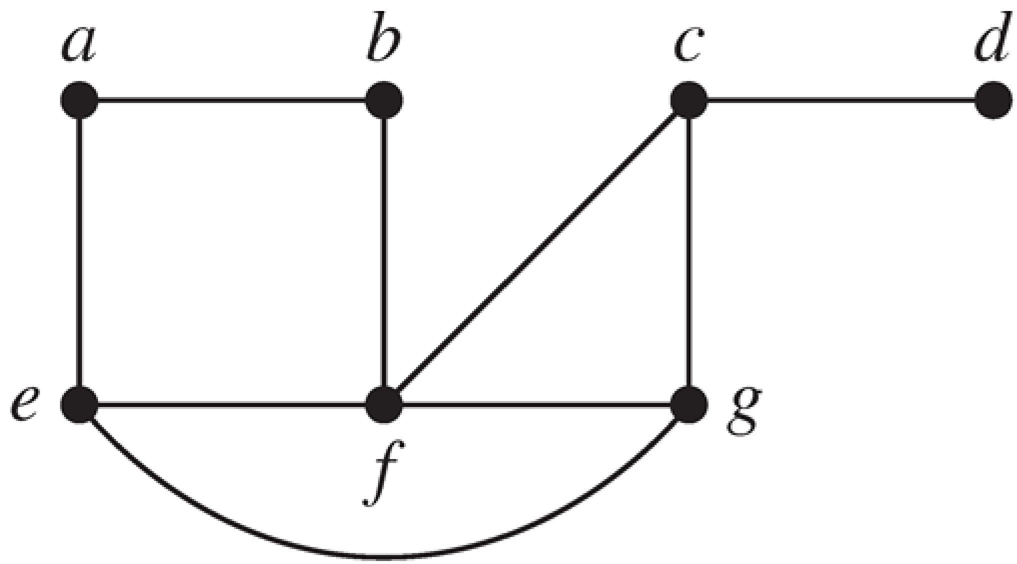

remove edges to avoid circuits

Spanning Trees

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easy

Depth-First Search

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  Continue adding to this path as long as possible.

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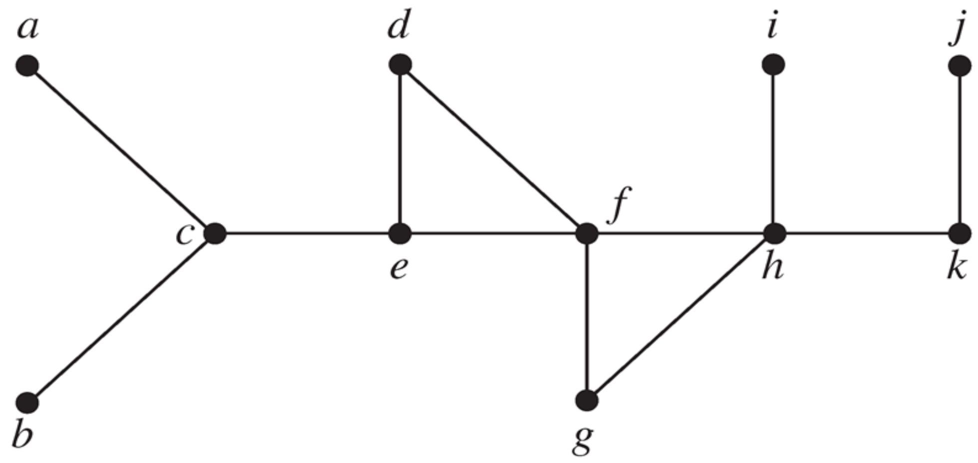
  ◇ Otherwise, move back to some vertex to repeat this procedure (*backtracking*)

Depth-First Search

- **Example**



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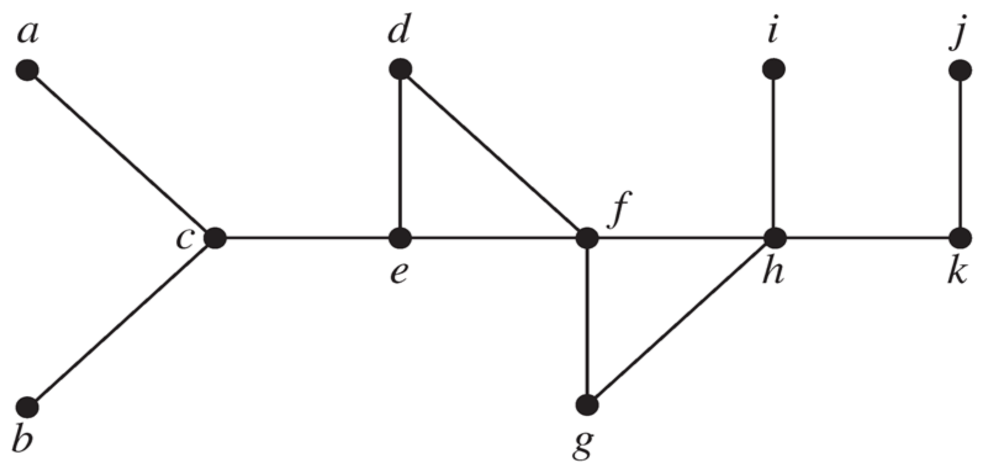

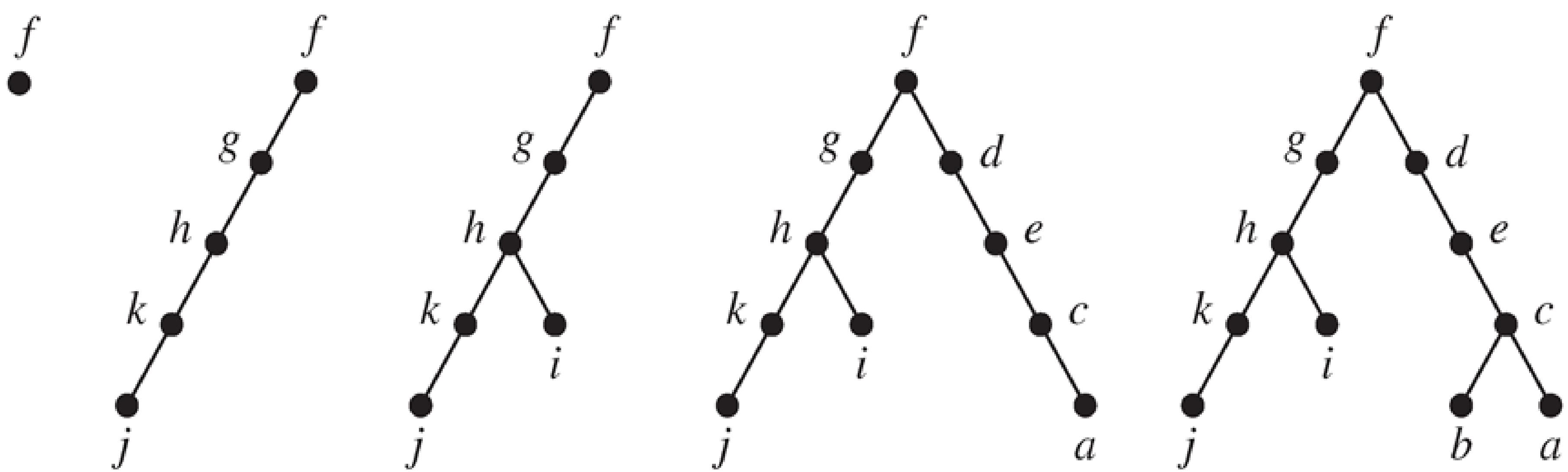

Depth-First Search Algorithm

procedure *DFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
visit(v_1)

procedure *visit*(v: vertex of G)
for each vertex w adjacent to v and not yet in T
  add vertex w and edge $\{v, w\}$ to T
  visit(w)

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time complexity: $O(e)$

Breadth-First Search

- This is the second algorithm that we build up spanning trees by successively adding edges.

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  ◇ First arbitrarily choose a vertex of the graph as the root.

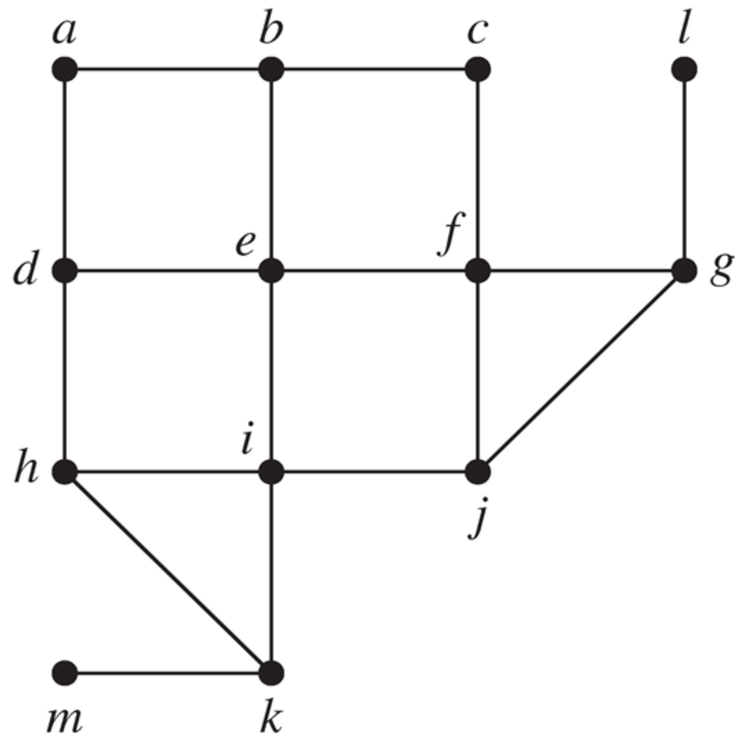
  ◇ Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges

  ◇ For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.

  ◇ Continue in this manner until all vertices have been added.

Breadth-First Search

- **Example**



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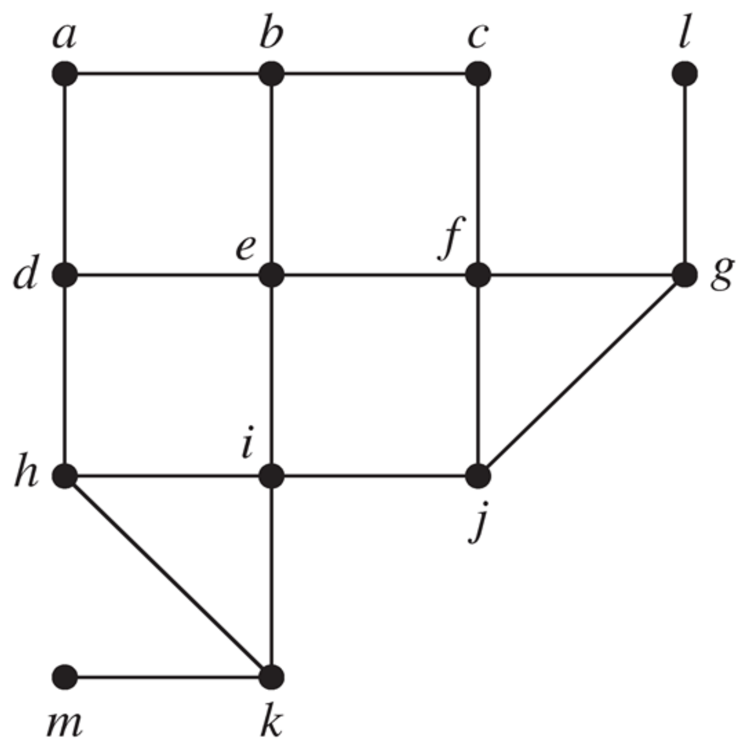

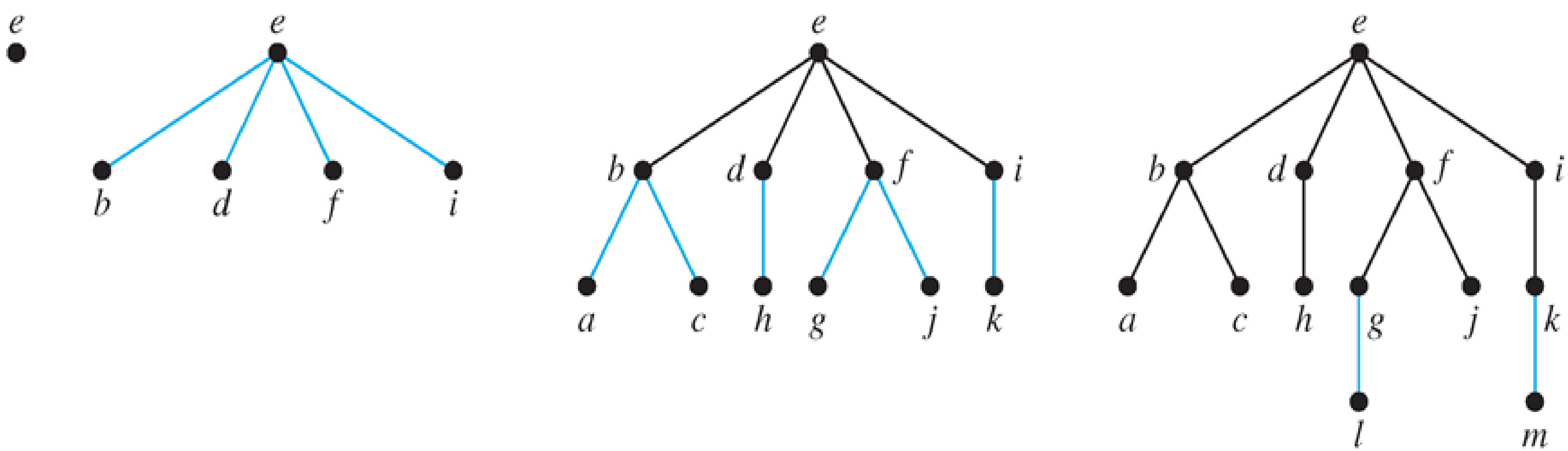

Breadth-First Search

```
procedure BFS(G: connected graph with vertices v_1, v_2, ..., v_n)
T := tree consisting only of the vertex v_1
L := empty list  visit(v_1)
put v_1 in the list L of unprocessed vertices
while L is not empty
    remove the first vertex, v, from L
    for each neighbor w of v
        if w is not in L and not in T then
            add  w to the end of the list L
            add  w and edge {v,w} to T
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Breadth-First Search

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Applications of DFS, BFS

- find paths, circuits, connected components, cut vertices, ...

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  find shortest paths, determine whether bipartite, ...

Applications of DFS, BFS

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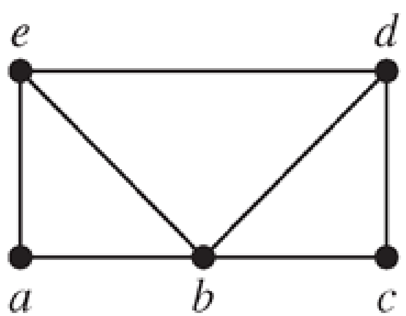
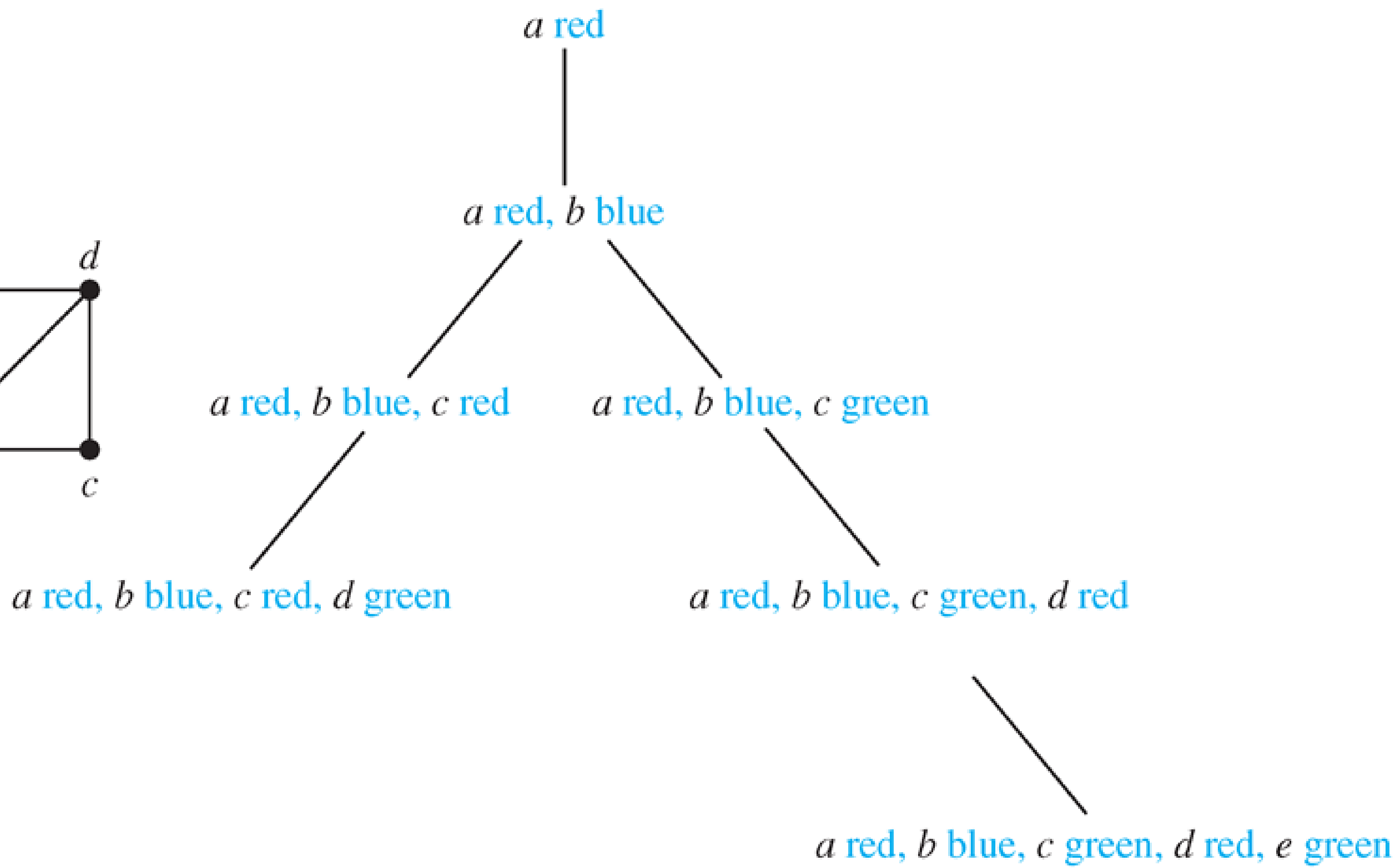
  graph coloring, sums of subsets, ...

Applications of DFS, BFS





Applications of DFS, BFS

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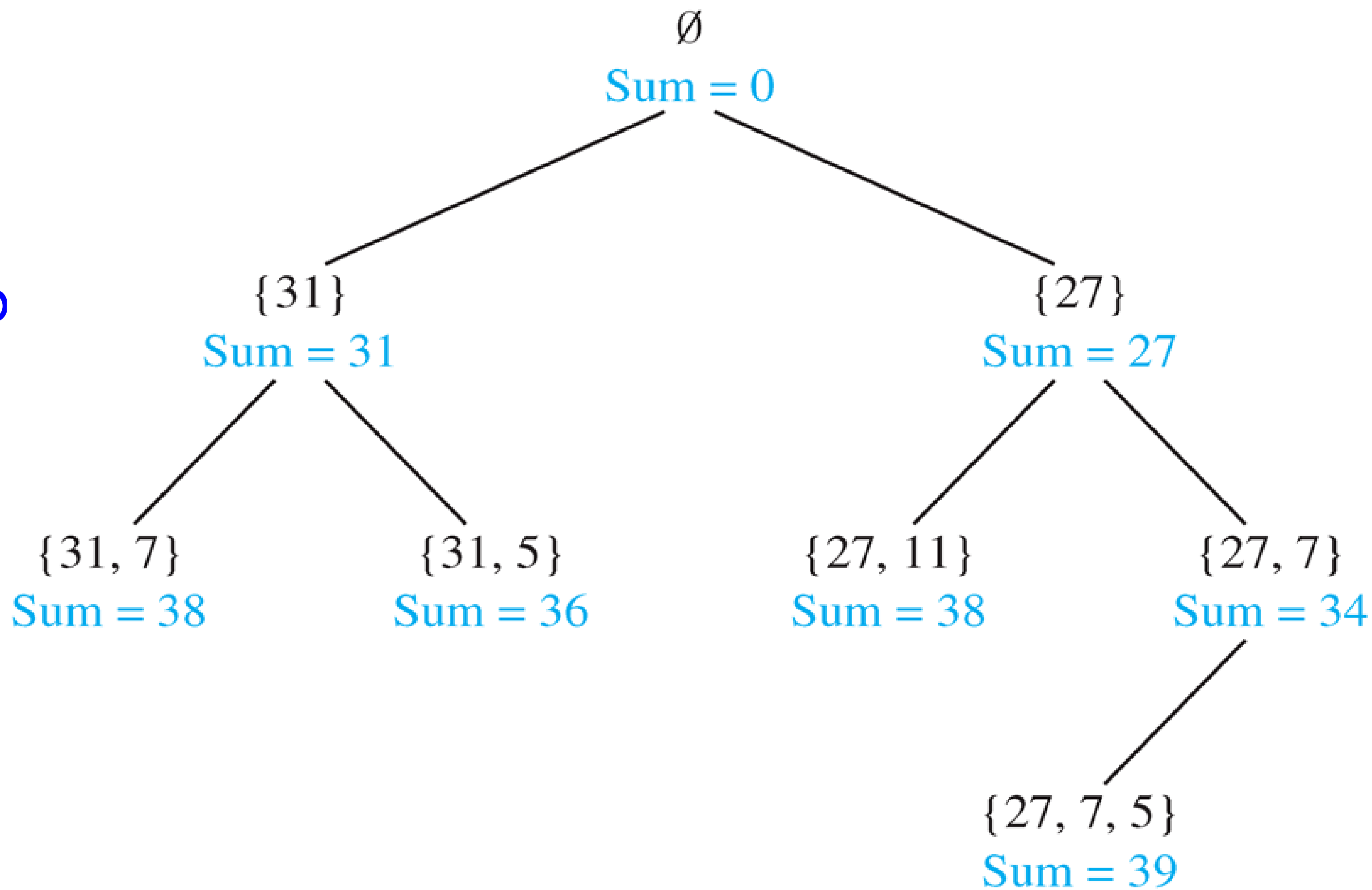
  find

  grap



find a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum 39

Minimum Spanning Trees

- **Definition** A *minimum spanning tree* in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

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  two greedy algorithms: Prim's Algorithm, Kruscal's Algorithm

Prim's Algorithm

ALGORITHM 1 Prim's Algorithm.

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for $i := 1$ **to** $n - 2$
    e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T
    $T := T$ with e added
return T {T is a minimum spanning tree of G}

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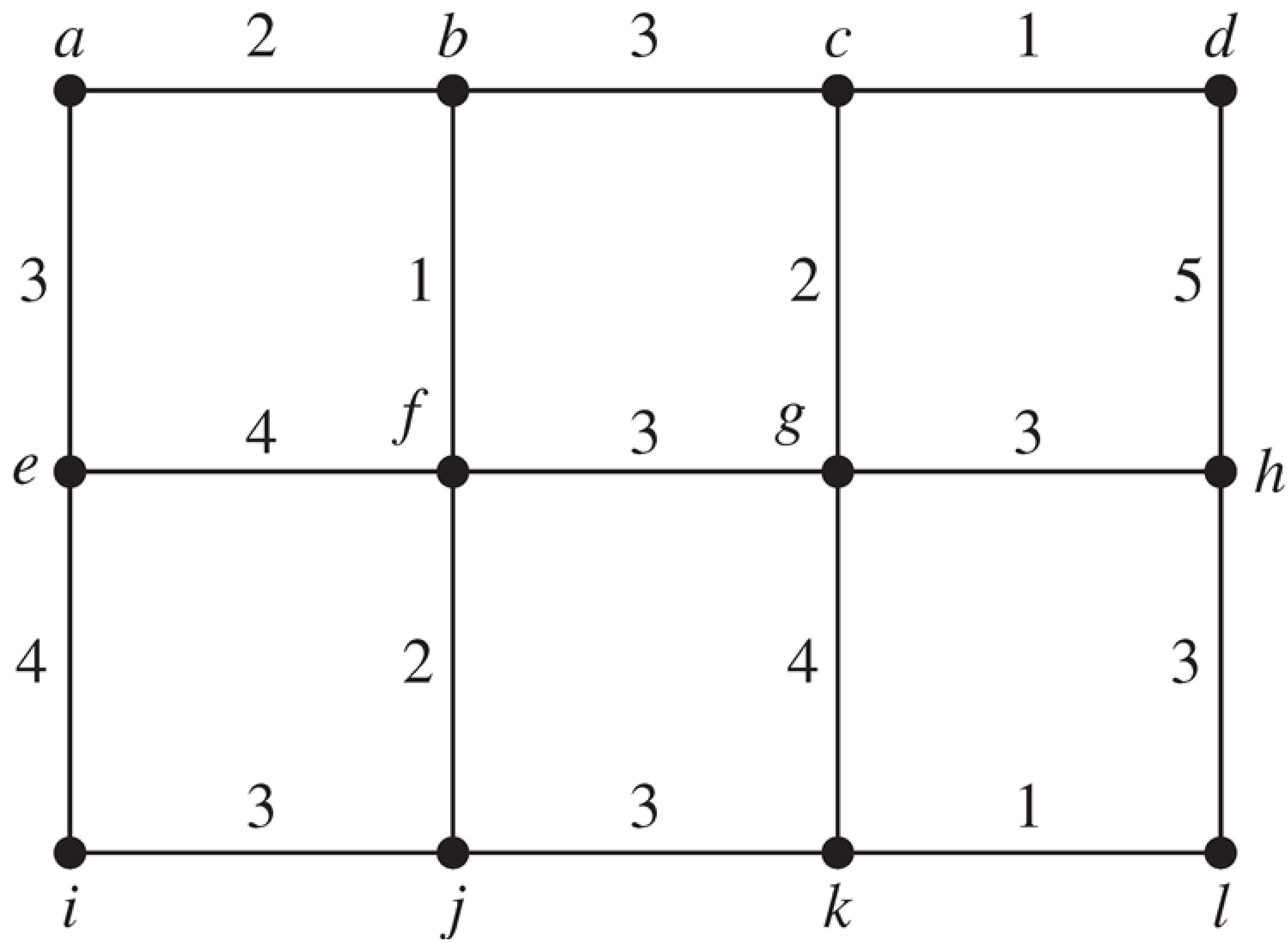
time complexity: $e \log v$

- **Example**



Kruscal's Algorithm

ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
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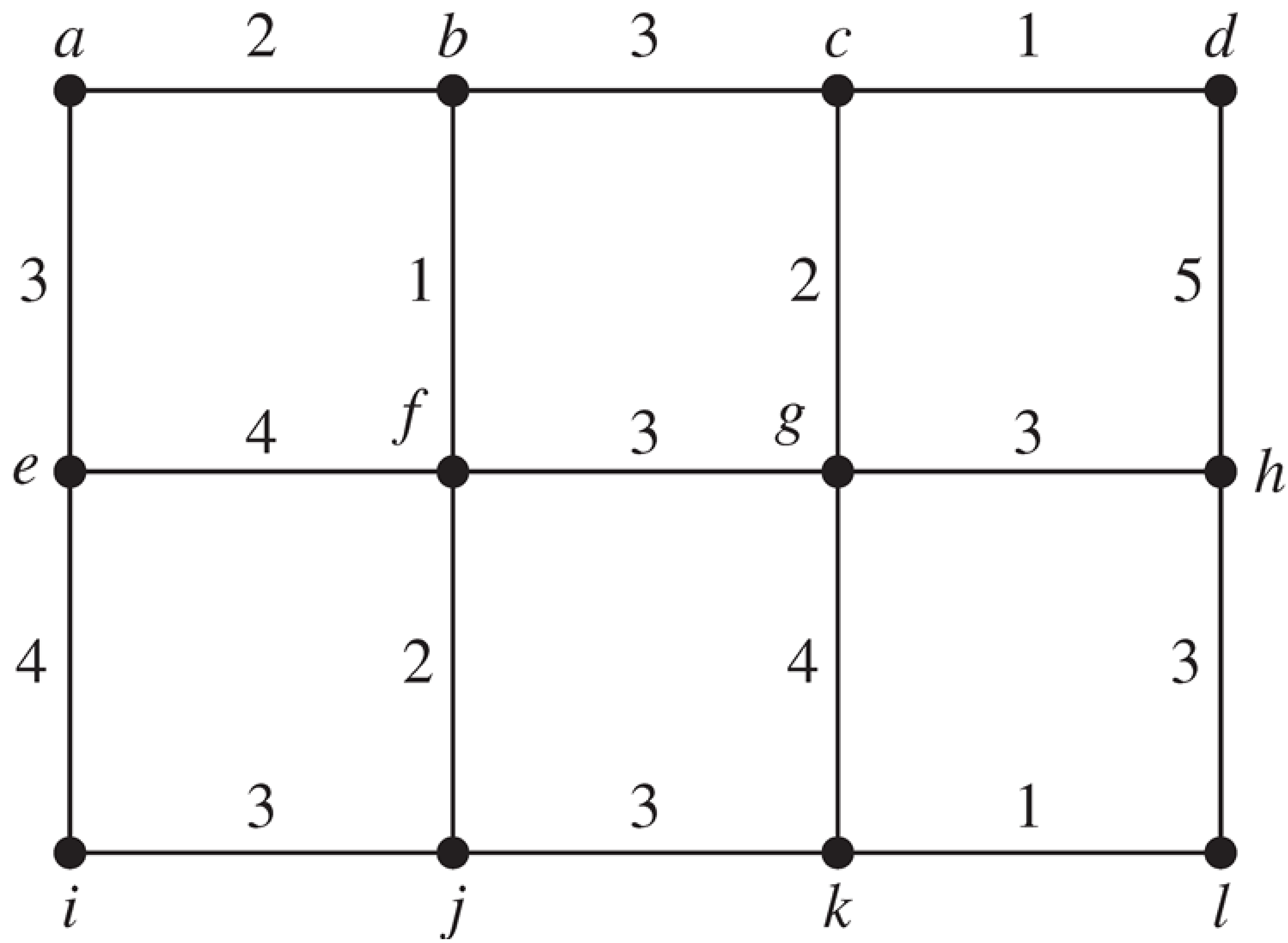
see *CLRS* / *Algorithm Design*, J. Kleinberg, E. Tardos

Kruscal's Algorithm

- **Example**



Next Lecture

- course review ...