

GLOBAL
EDITION



Thomas' CALCULUS

Thirteenth Edition In SI Units

Chapter 12

Vectors and the Geometry of Space 向量和空间几何

12.1

Three-Dimensional Coordinate Systems

三维坐标系

rectangular coordinates

xy -plane,

yz -plane,

xz -plane,

$$P(x, y, z)$$

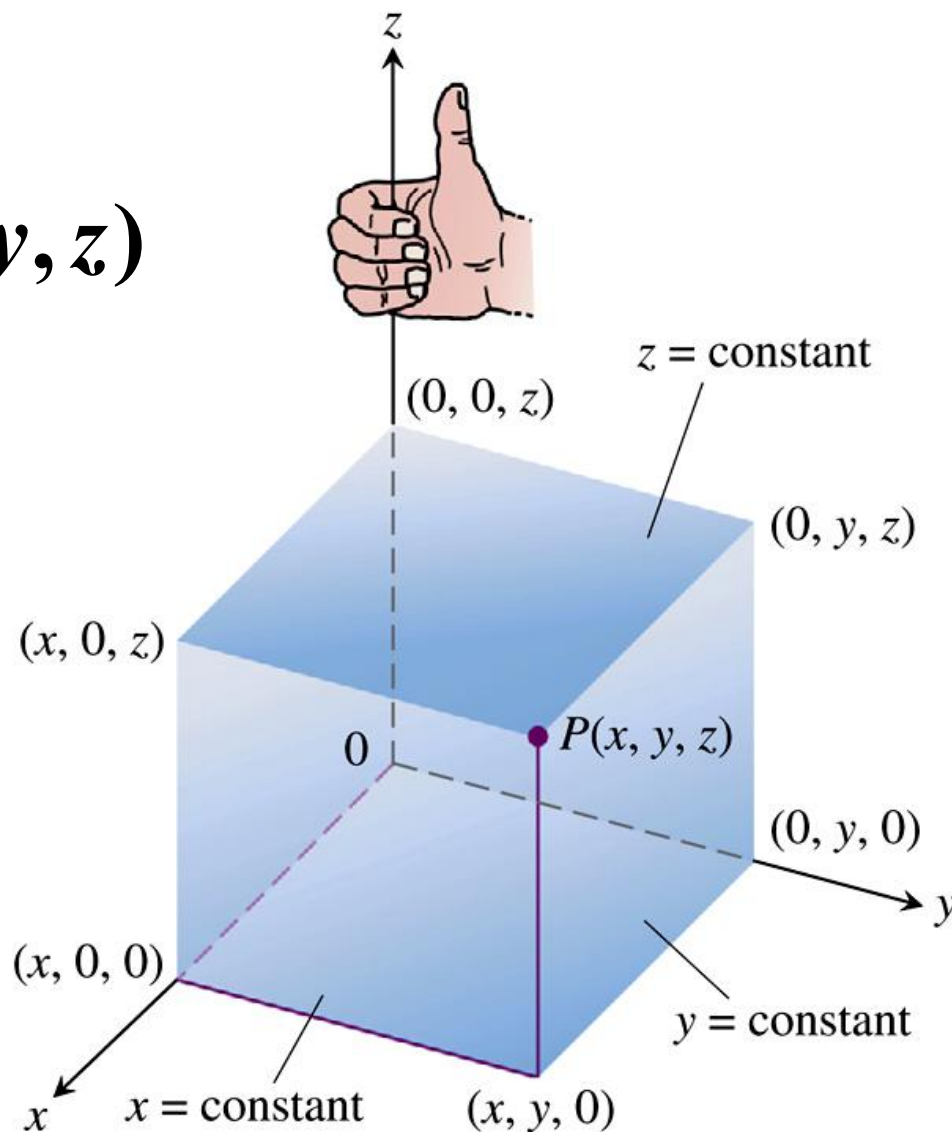


FIGURE 12.1 The Cartesian coordinate system is right-handed.

eight octants.

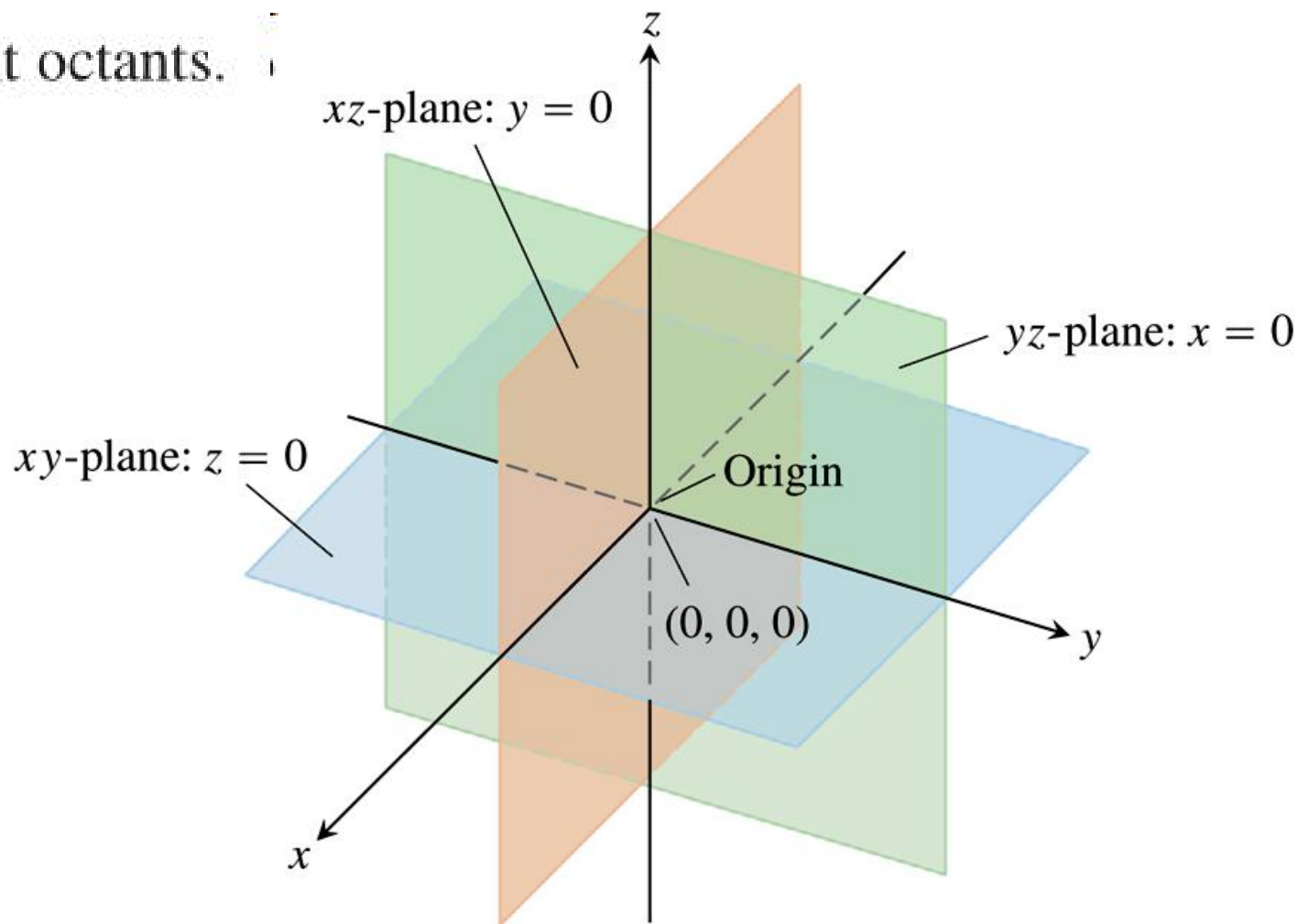


FIGURE 12.2 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

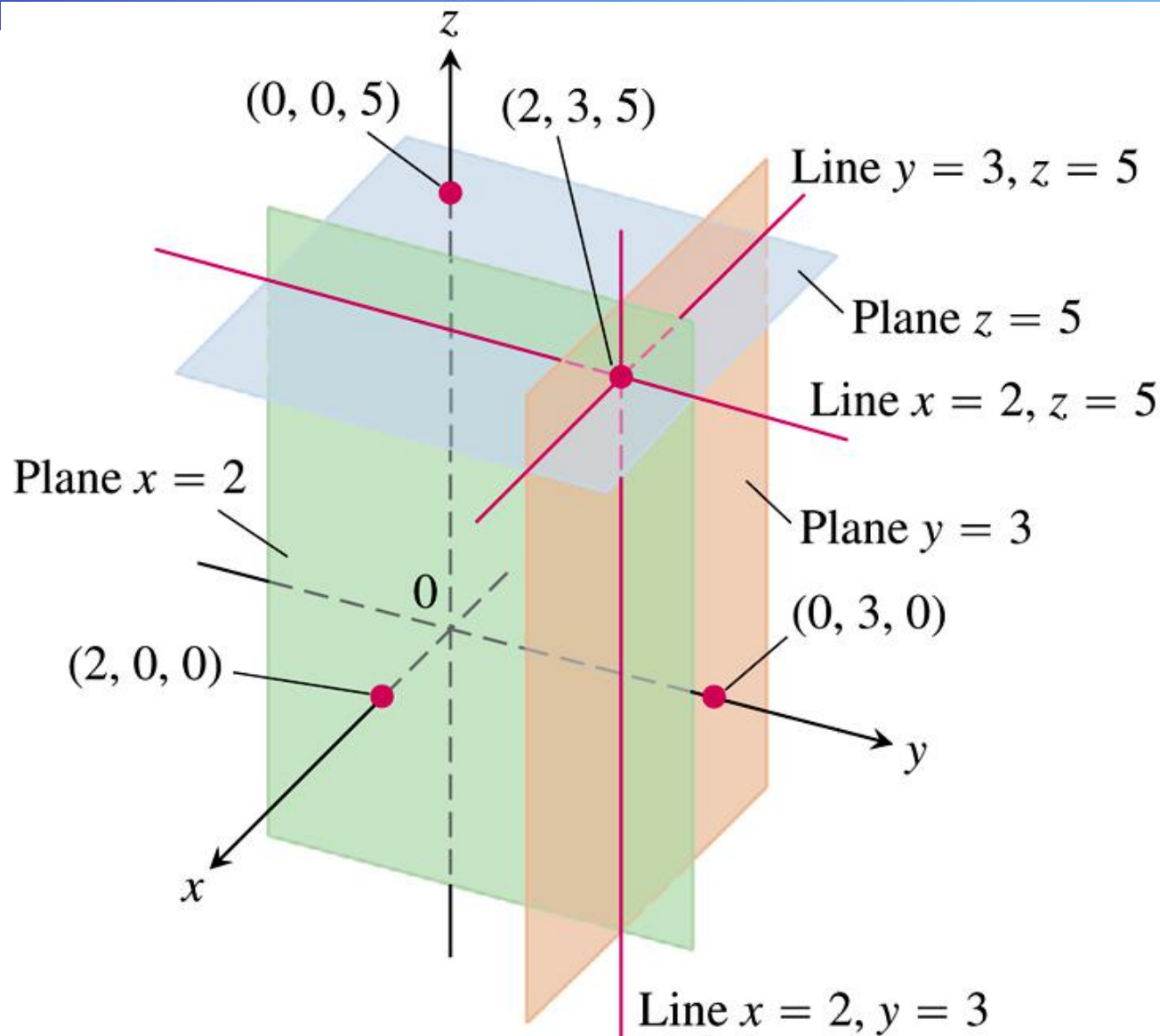


FIGURE 12.3 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.

EXAMPLE 1

We interpret these equations and inequalities geometrically.

(a) $z \geq 0$ 上半空间

(b) $x = -3$ 一个平面

(c) $z = 0, x \leq 0, y \geq 0$ 坐标面上的一部分

(d) $x \geq 0, y \geq 0, z \geq 0$ 第一卦限空间

(e) $-1 \leq y \leq 1$ 部分空间

(f) $y = -2, z = 2$ 空间一条直线

EXAMPLE 2 What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \quad \text{and} \quad z = 3?$$

Solution the circle $x^2 + y^2 = 4$ in the plane $z = 3$

Ex. Graphing points $P(x, y, z)$ satisfy the equation $y = x^2$.

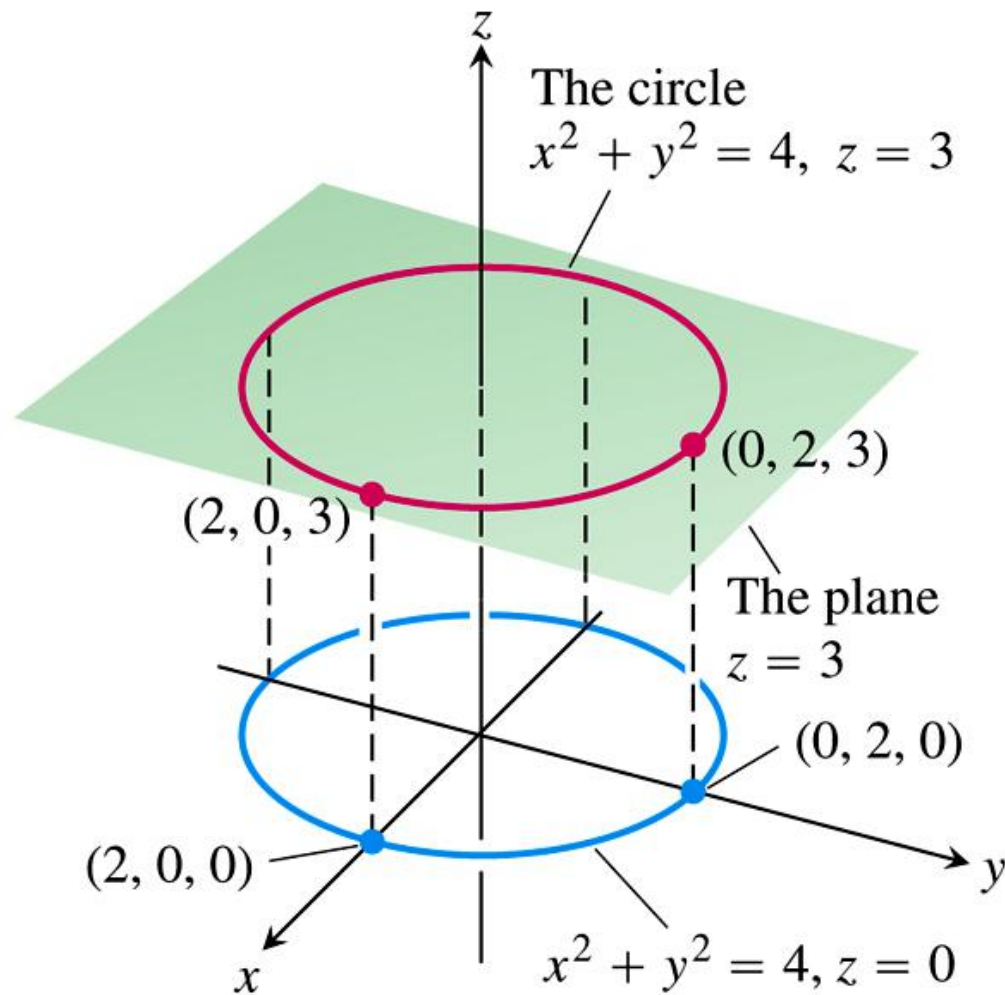
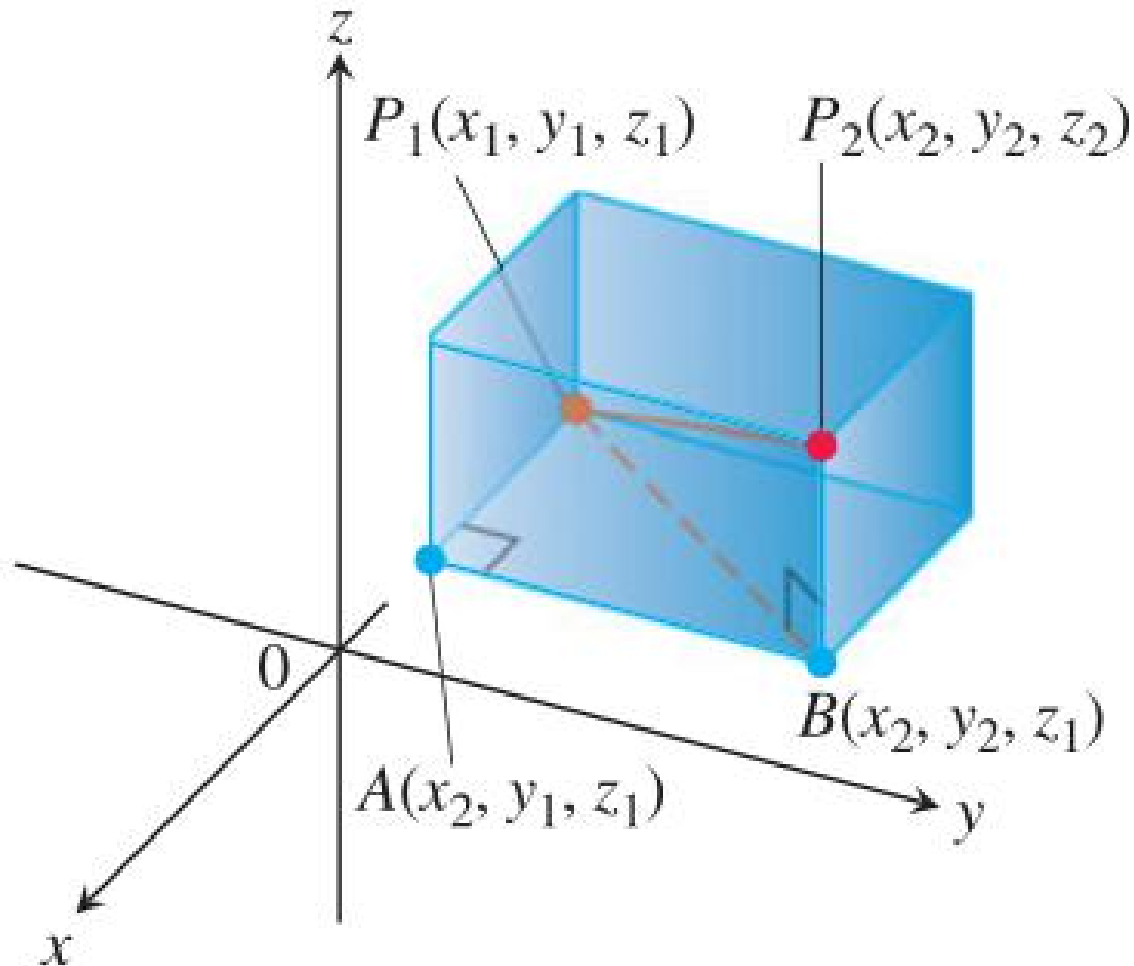


FIGURE 12.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



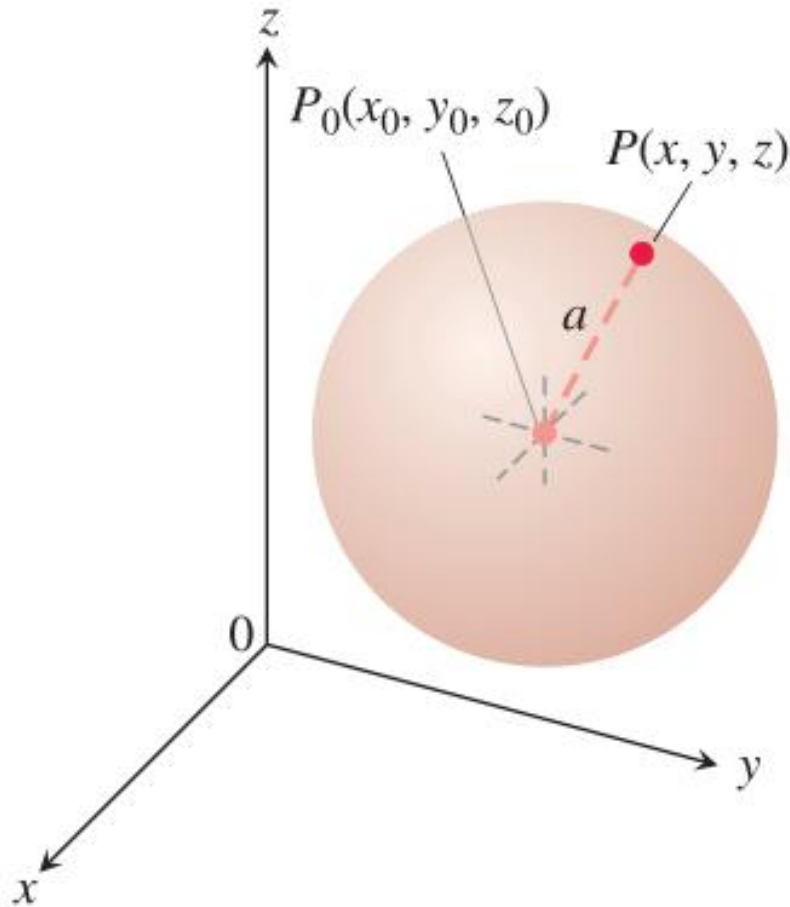
EXAMPLE 3

The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned}|P_1P_2| &= \sqrt{(-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\&= \sqrt{16 + 4 + 25} \\&= \sqrt{45} \approx 6.708.\end{aligned}$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$



$$|PP_0| = a$$

EXAMPLE 4 Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

EXAMPLE 5

Here are some geometric interpretations of inequalities and equations

(a) $x^2 + y^2 + z^2 < 4$

(b) $x^2 + y^2 + z^2 \leq 4$

(c) $x^2 + y^2 + z^2 > 4$

(d) $x^2 + y^2 + z^2 = 4, z \leq 0$

12.2

Vectors

向量

vector such as force, displacement, or velocity

represented by **directed line segment**

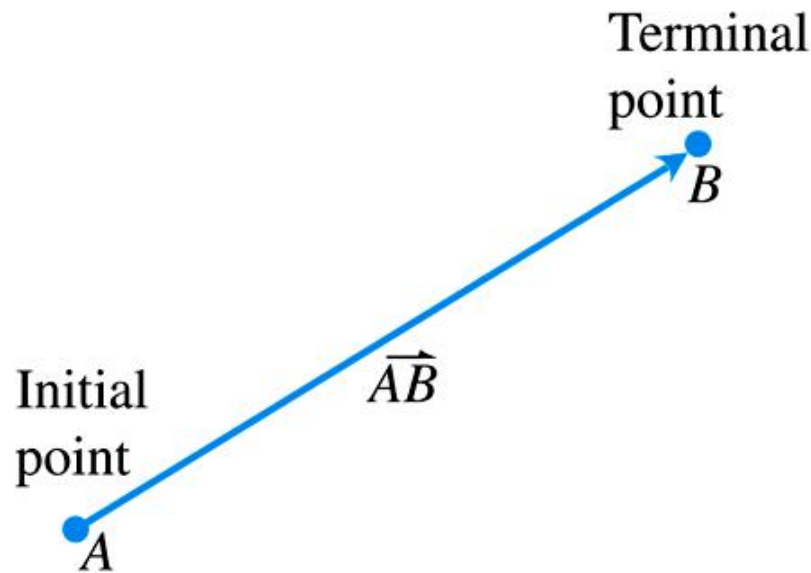


FIGURE 12.7 The directed line segment \overrightarrow{AB} is called a vector.

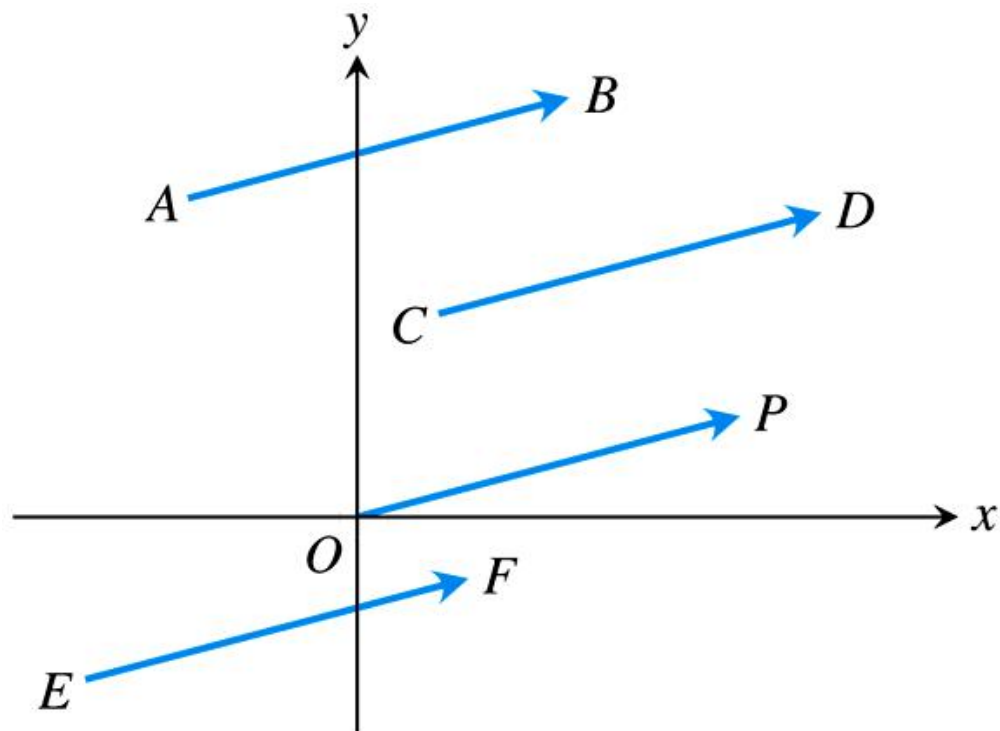
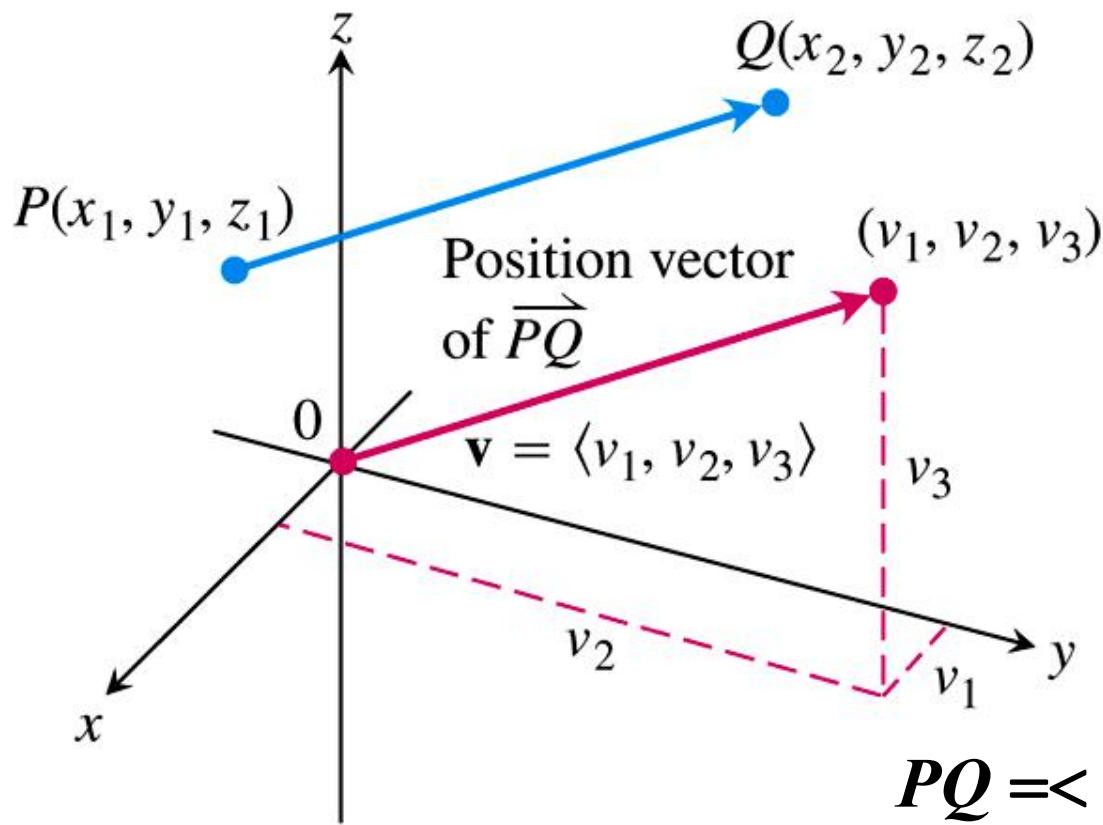


FIGURE 12.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

DEFINITIONS The vector represented by the directed line segment \overrightarrow{AB} has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.



$$\vec{PQ} = \mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

$$v_1 = x_2 - x_1$$

$$v_2 = y_2 - y_1$$

$$v_3 = z_2 - z_1$$

$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

FIGURE 12.10 A vector \vec{PQ} in standard position has its initial point at the origin. The directed line segments \vec{PQ} and \mathbf{v} are parallel and have the same length.

DEFINITION If \mathbf{v} is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If \mathbf{v} is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$,
the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to \overrightarrow{PQ} is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\mathbf{0} = \langle 0, 0, 0 \rangle.$$

EXAMPLE 1

Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector \mathbf{v} representing \overrightarrow{PQ} has components

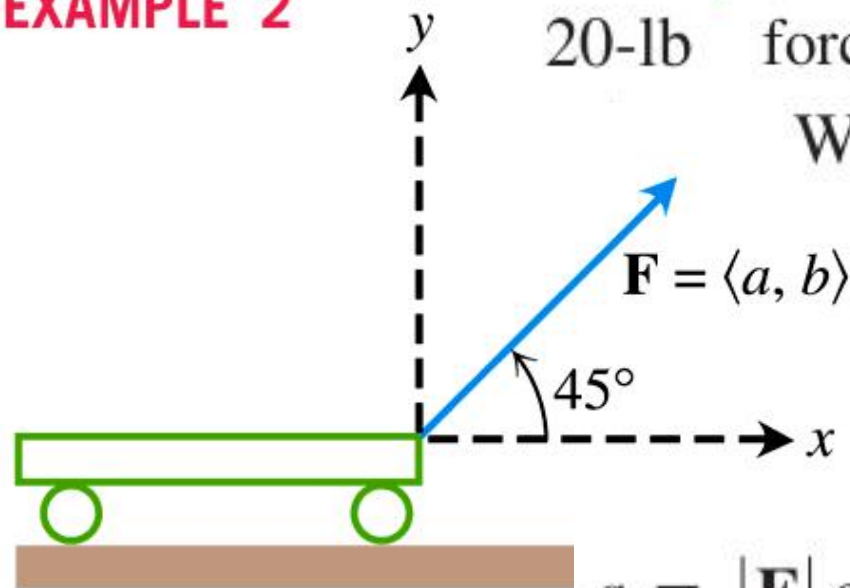
$$\begin{aligned}v_1 &= x_2 - x_1 = -5 - (-3) = -2, \\v_2 &= y_2 - y_1 = 2 - 4 = -2, \\v_3 &= z_2 - z_1 = 2 - 1 = 1.\end{aligned}\quad \mathbf{v} = \langle -2, -2, 1 \rangle.$$

(b) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

EXAMPLE 2

20-lb force \mathbf{F} making a 45° angle to the floor
What is the *effective* force



$$\mathbf{F} = \langle a, b \rangle$$

$$a = |\mathbf{F}| \cos 45^\circ = (20) \left(\frac{\sqrt{2}}{2} \right) \approx 14.14 \text{ lb.}$$

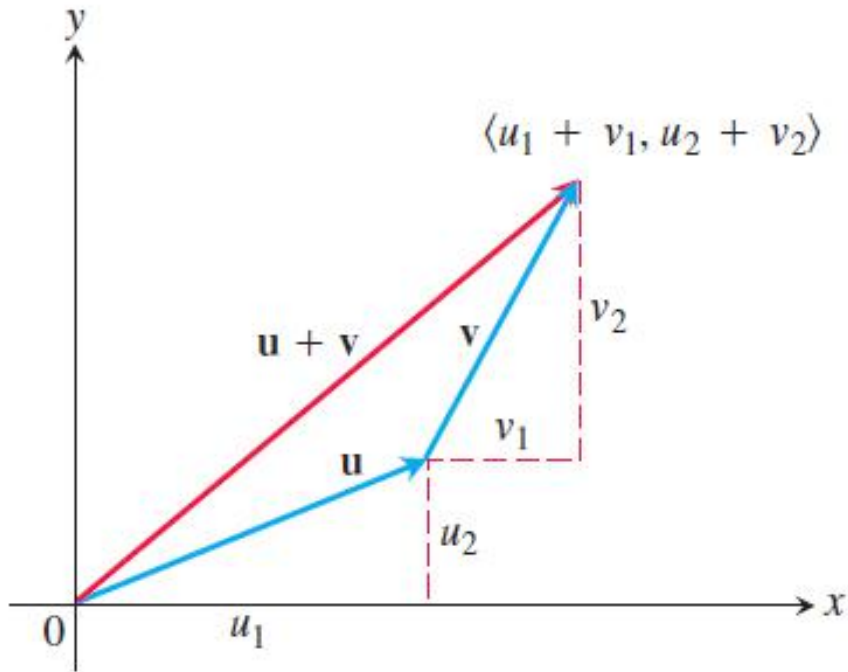
FIGURE 12.11 The force pulling the cart forward is represented by the vector \mathbf{F} whose horizontal component is the effective force (Example 2).

Vector Algebra Operations

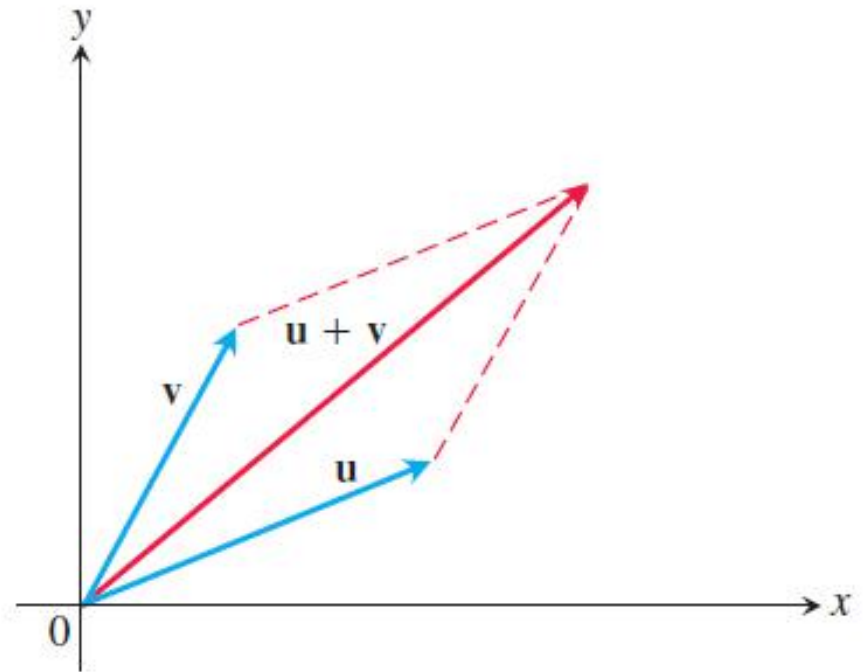
DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$



(a)



(b)

FIGURE 12.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition in which both vectors are in standard position.

$$\begin{aligned} |k\mathbf{u}| &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2} \sqrt{u_1^2 + u_2^2 + u_3^2} = |k| |\mathbf{u}|. \end{aligned}$$

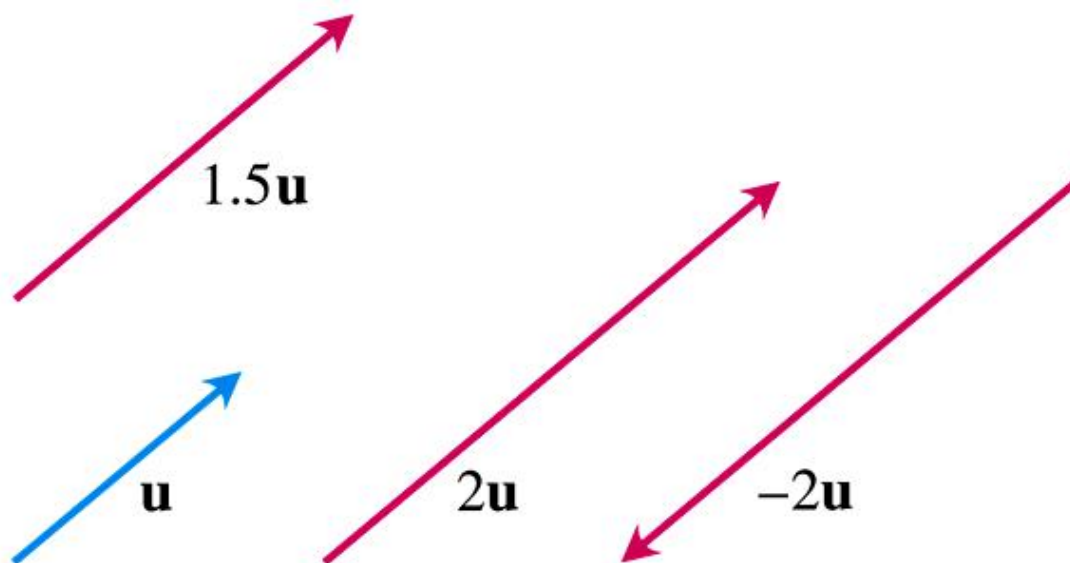
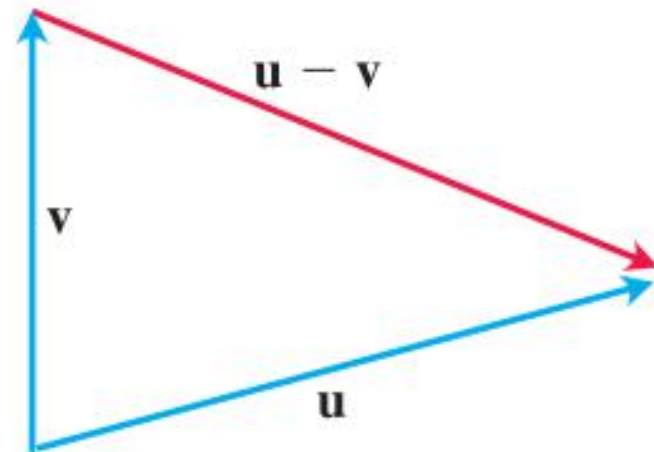
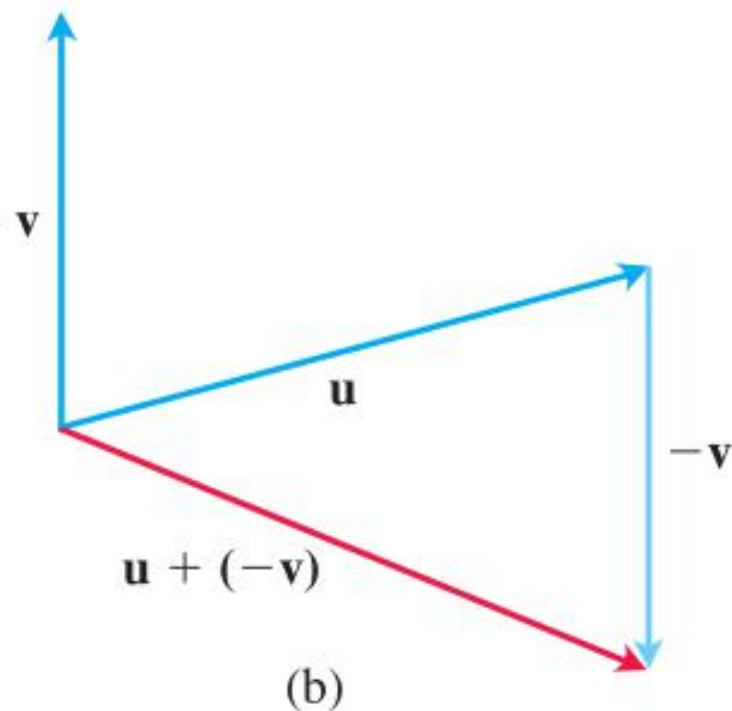


FIGURE 12.13 Scalar multiples of \mathbf{u} .

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$



EXAMPLE 3

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

(a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left| \frac{1}{2}\mathbf{u} \right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} , \mathbf{w} be vectors and a , b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5. $0\mathbf{u} = \mathbf{0}$

6. $1\mathbf{u} = \mathbf{u}$

7. $a(b\mathbf{u}) = (ab)\mathbf{u}$

8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

The **standard unit vectors**

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.\end{aligned}$$

$$P_1(x_1, y_1, z_1) \quad P_2(x_2, y_2, z_2)$$

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

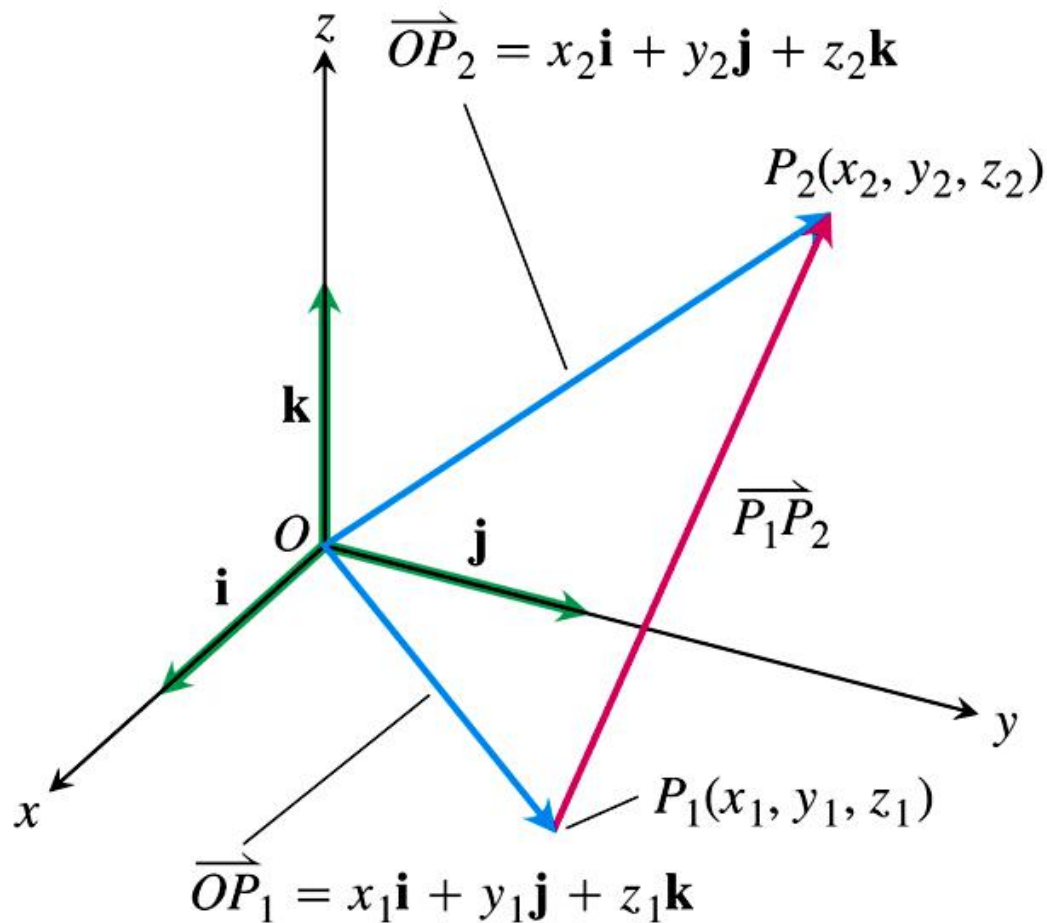


FIGURE 12.15 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

Whenever $\mathbf{v} \neq \mathbf{0}$,

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

EXAMPLE 4 Find a unit vector \mathbf{u} in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 6

A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force \mathbf{F} as a product of its magnitude and direction.

Solution
$$\mathbf{F} = 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

Midpoint of a Line Segment

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

$$\text{let } M(x, y, z) \quad \overrightarrow{P_1M} = \frac{1}{2} \overrightarrow{P_1P_2}$$

$$\langle x - x_1, y - y_1, z - z_1 \rangle = \left\langle \frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2}, \frac{z_2 - z_1}{2} \right\rangle$$

$$x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}, z = \frac{z_1 + z_2}{2}$$

EXAMPLE 7

The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2} \right) = (5, 1, 2).$$

Applications

EXAMPLE 8

A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Solution \mathbf{u} is the velocity of the airplane alone $|\mathbf{u}| = 500$

\mathbf{v} is the velocity of the tailwind, $|\mathbf{v}| = 70$

$$\mathbf{u} = \langle 500, 0 \rangle$$

$$\mathbf{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle.$$

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle = 535\mathbf{i} + 35\sqrt{3}\mathbf{j}$$

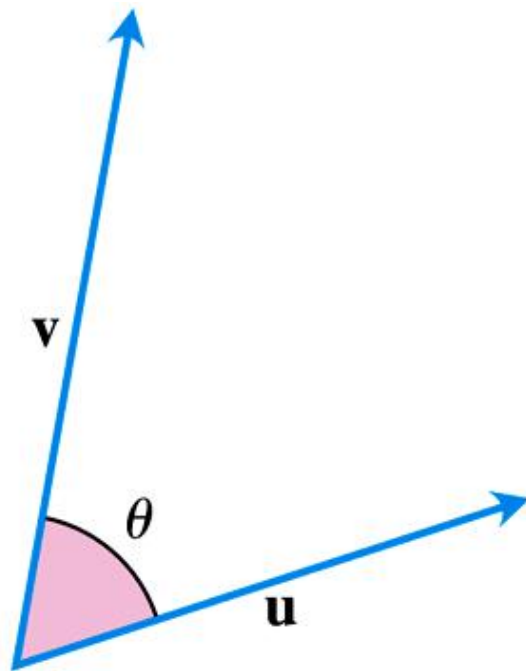
$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

$$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5^\circ.$$

12.3

The Dot Product

点积、内积



$$0 \leq \theta \leq \pi$$

FIGURE 12.20 The angle between \mathbf{u} and \mathbf{v} .

the projection of one vector onto another

When is the projection positive? negative?

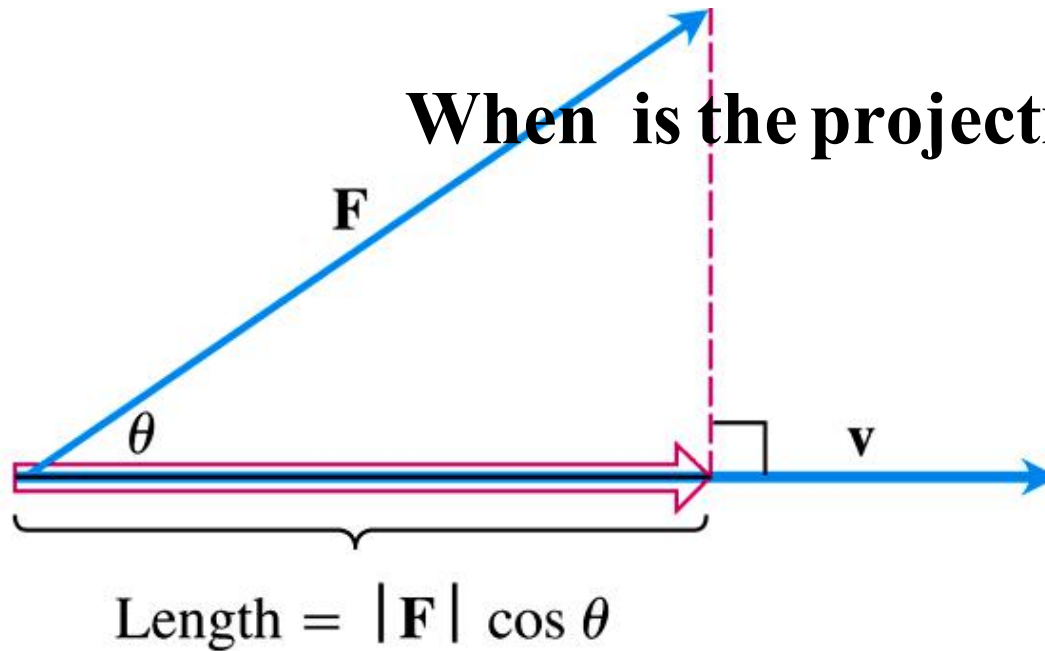


FIGURE 12.19 The magnitude of the force \mathbf{F} in the direction of vector \mathbf{v} is the length $|\mathbf{F}| \cos \theta$ of the projection of \mathbf{F} onto \mathbf{v} .

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right).$$

DEFINITION The dot product $\mathbf{u} \cdot \mathbf{v}$ (“ \mathbf{u} dot \mathbf{v} ”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

EXAMPLE 1

$$\begin{aligned} \text{(a)} \quad \langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle &= (1)(-6) + (-2)(2) + (-1)(-3) \\ &= -6 - 4 + 3 = -7 \end{aligned}$$

$$\text{(b)} \quad \left(\frac{1}{2} \mathbf{i} + 3\mathbf{j} + \mathbf{k} \right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1$$

Proof of Theorem 1

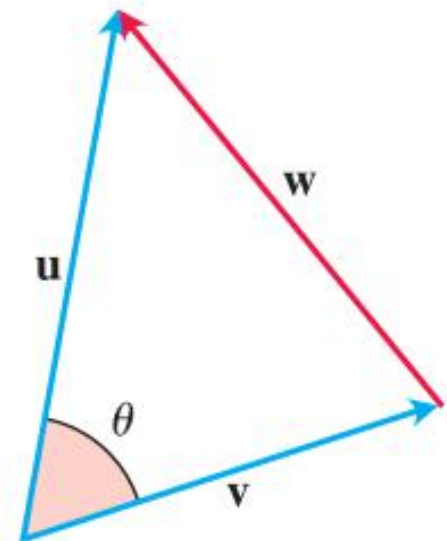
Applying the law of cosines

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle \quad \mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

$$\mathbf{w} \text{ is } \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle,$$



$$|\mathbf{u}|^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2} \right)^2 = u_1^2 + u_2^2 + u_3^2$$

$$|\mathbf{v}|^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2} \right)^2 = v_1^2 + v_2^2 + v_3^2$$

$$\begin{aligned} |\mathbf{w}|^2 &= \left(\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2} \right)^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \end{aligned}$$

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

$$|\mathbf{u}||\mathbf{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3 \quad \mathbf{u \cdot v = |u||v|\cos\theta}$$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}.$$

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

The Angle Between Two Nonzero Vectors \mathbf{u} and \mathbf{v}

EXAMPLE 2

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution $\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{-4}{(3)(7)} \right)$$

$$\approx 1.76 \text{ radians or } 100.98^\circ.$$

EXAMPLE 3

Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (Figure 12.22).

Solution

The angle θ is the angle between the vectors \overrightarrow{CA} and \overrightarrow{CB} .

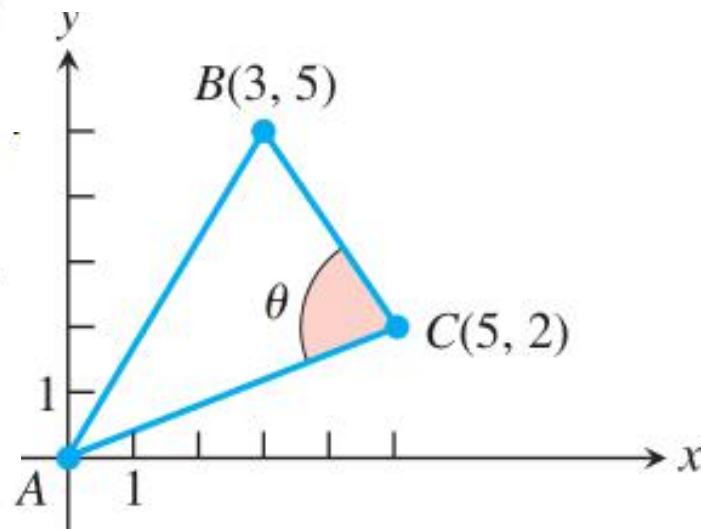
$$\overrightarrow{CA} = \langle -5, -2 \rangle \quad \overrightarrow{CB} = \langle -2, 3 \rangle.$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

$$\theta = \cos^{-1} \left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}| |\overrightarrow{CB}|} \right) \approx 78.1^\circ \text{ or } 1.36 \text{ radians.}$$



Orthogonal Vectors

DEFINITION Vectors \mathbf{u} and \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE 4 if two vectors are orthogonal, calculate their dot product.:

(a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0. \quad ; \quad \text{orthogonal}$$

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$

$$\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0. \quad \text{orthogonal}$$

(c) $\mathbf{0}$ is orthogonal to every vector \mathbf{u} since

$$\mathbf{0} \cdot \mathbf{u} = \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle = 0.$$

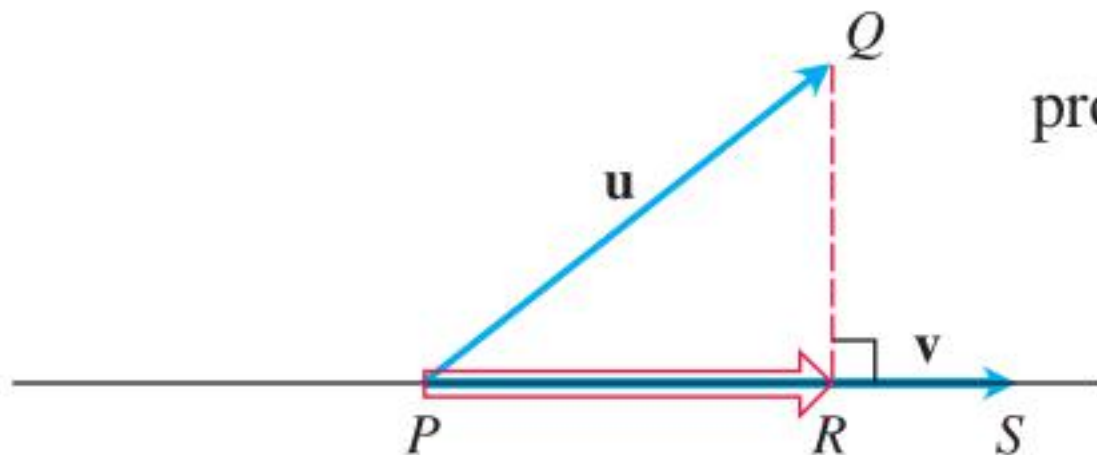
Dot Product Properties and Vector Projections

Properties of the Dot Product

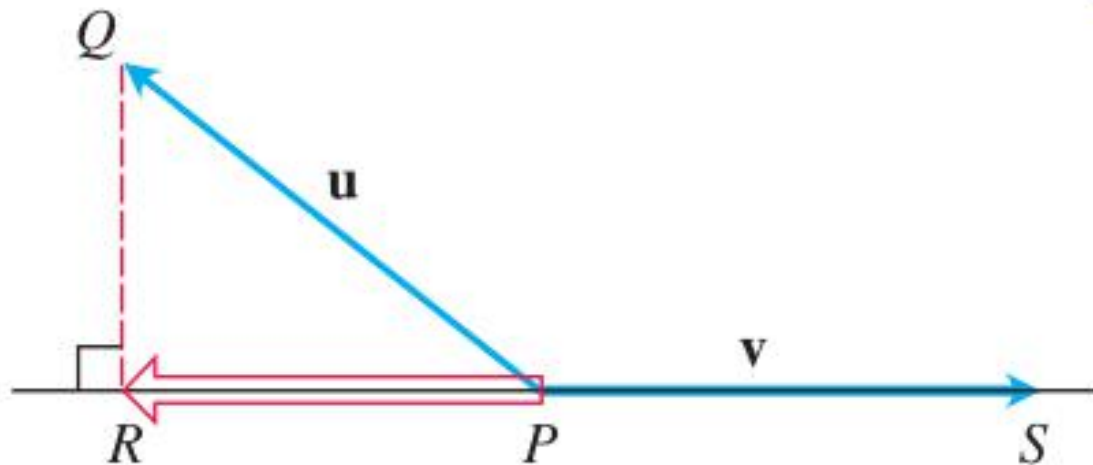
If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

$\text{proj}_{\mathbf{v}} \mathbf{u}$ (“the vector projection of \mathbf{u} onto \mathbf{v} ”).



$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.\end{aligned}$$



The vector projection of \mathbf{u} onto \mathbf{v} is the vector

$$\mathbf{proj}_{\mathbf{v}} \mathbf{u} = (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$

The scalar component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}.$$

EXAMPLE 5

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of \mathbf{u} in the direction of \mathbf{v} .

Solution

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}.\end{aligned}$$

$$|\mathbf{u}| \cos \theta = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} - \frac{2}{3} \mathbf{k} \right) = -\frac{4}{3}.$$

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{4}{3} \cdot \frac{1}{3} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

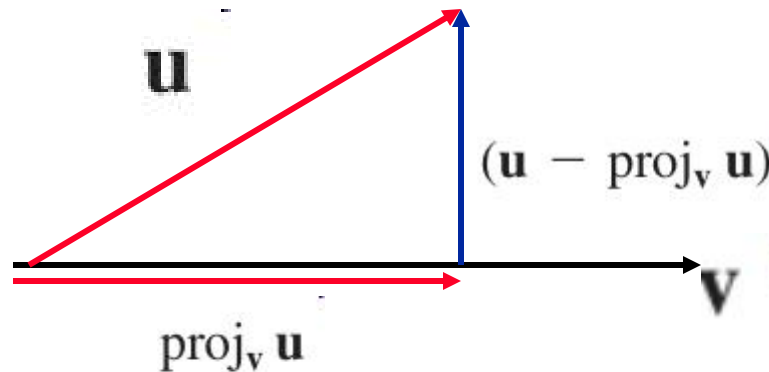
EXAMPLE 6

Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of \mathbf{F} in the direction of \mathbf{v} .

Solution $|\mathbf{F}| \cos \theta = \frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{5 - 6}{\sqrt{1 + 9}} = -\frac{1}{\sqrt{10}}.$

$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{F} &= (|\mathbf{F}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{1}{\sqrt{10}} \frac{1}{\sqrt{10}} (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10} (\mathbf{i} - 3\mathbf{j}) \end{aligned}$$

$$\mathbf{u} = \text{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) = \underbrace{\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}}_{\text{Parallel to } \mathbf{v}} + \underbrace{\left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \right)}_{\text{Orthogonal to } \mathbf{v}}$$



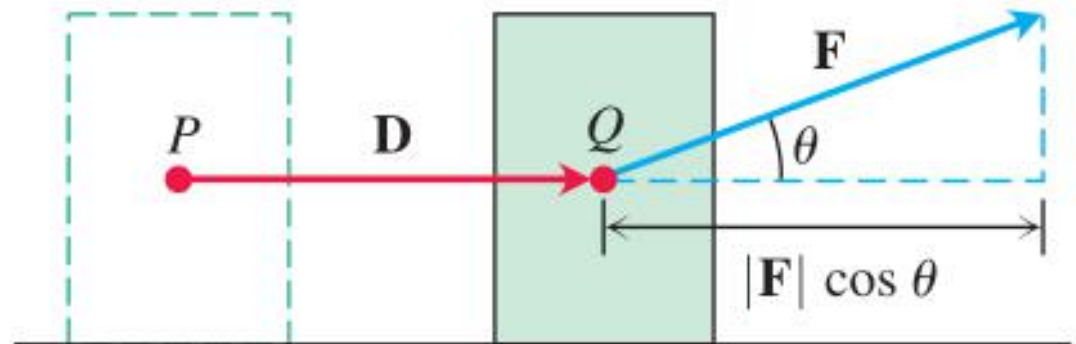
Work

If a force \mathbf{F} moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of \mathbf{F} in the direction of \mathbf{D} . If θ is the angle between \mathbf{F} and \mathbf{D}

$$\text{Work} = \left(\begin{array}{l} \text{scalar component of } \mathbf{F} \\ \text{in the direction of } \mathbf{D} \end{array} \right) (\text{length of } \mathbf{D})$$

$$= (|\mathbf{F}| \cos \theta) |\mathbf{D}|$$

$$= \mathbf{F} \cdot \mathbf{D}.$$



DEFINITION

The **work** done by a constant force \mathbf{F} acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is $W = \mathbf{F} \cdot \mathbf{D}$.

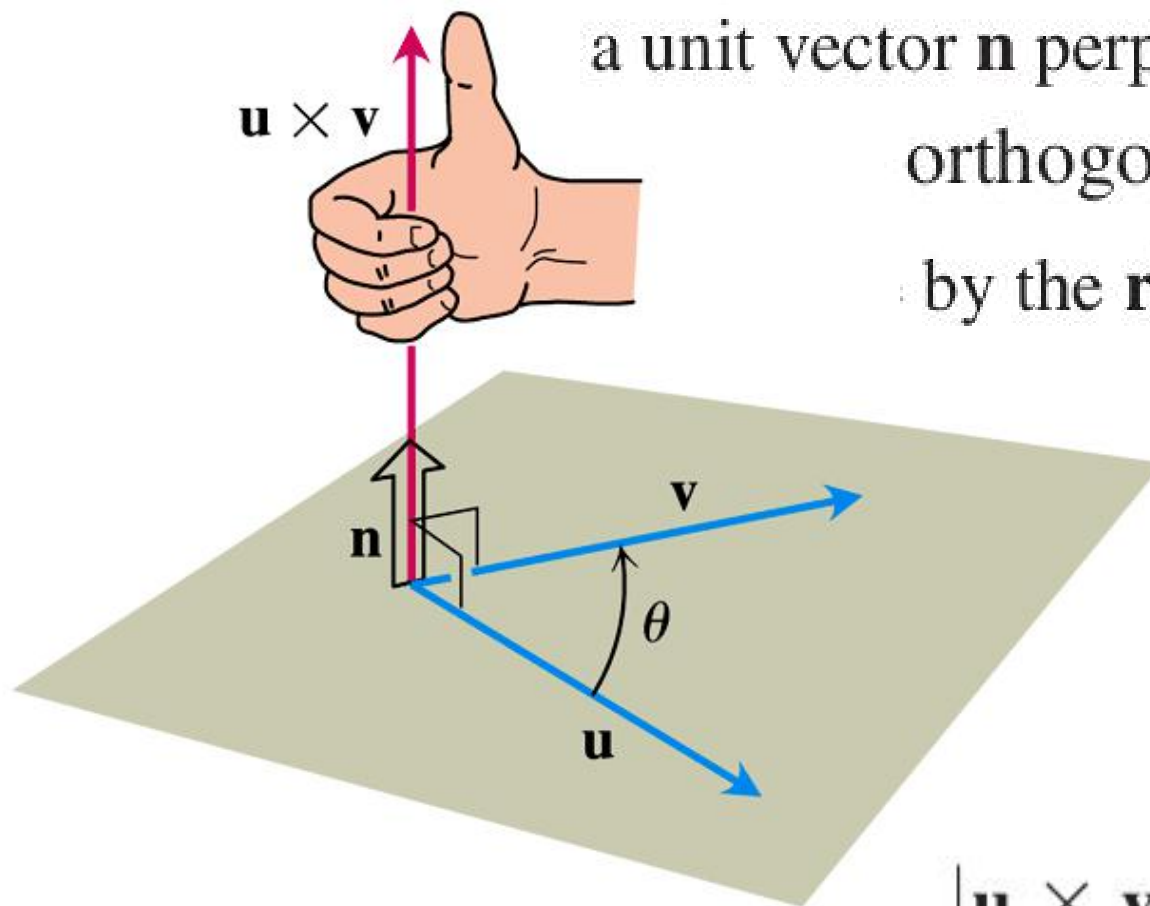
EXAMPLE 7 If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60^\circ$,

$$\begin{aligned}\text{Work} &= \mathbf{F} \cdot \mathbf{D} \\ &= |\mathbf{F}| |\mathbf{D}| \cos \theta \\ &= (40)(3) \cos 60^\circ \\ &= (120)(1/2) = 60 \text{ J (joules)}.\end{aligned}$$

12.4

The Cross Product

向量的叉积



a unit vector \mathbf{n} perpendicular to the plane
orthogonal to both \mathbf{u} and \mathbf{v}
by the **right-hand rule**.

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

FIGURE 12.27 The construction of
 $\mathbf{u} \times \mathbf{v}$.

DEFINITION

The **cross product** $\mathbf{u} \times \mathbf{v}$ (“**u cross v**”) is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}| |\mathbf{v}| \sin \theta) \mathbf{n}.$$

If one or both of \mathbf{u} and \mathbf{v} are zero, we also define $\mathbf{u} \times \mathbf{v}$ to be zero.

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r , s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$

6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

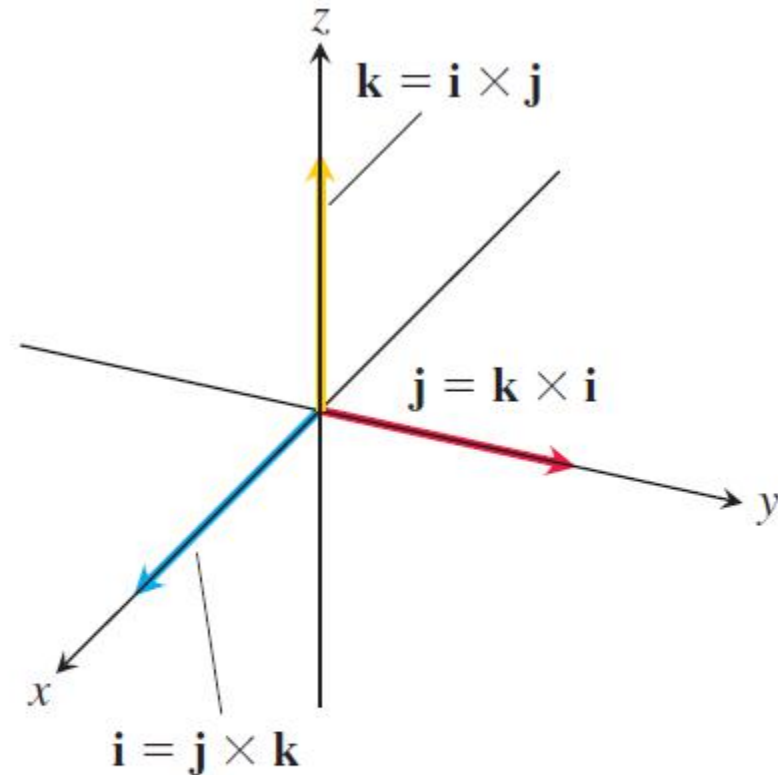
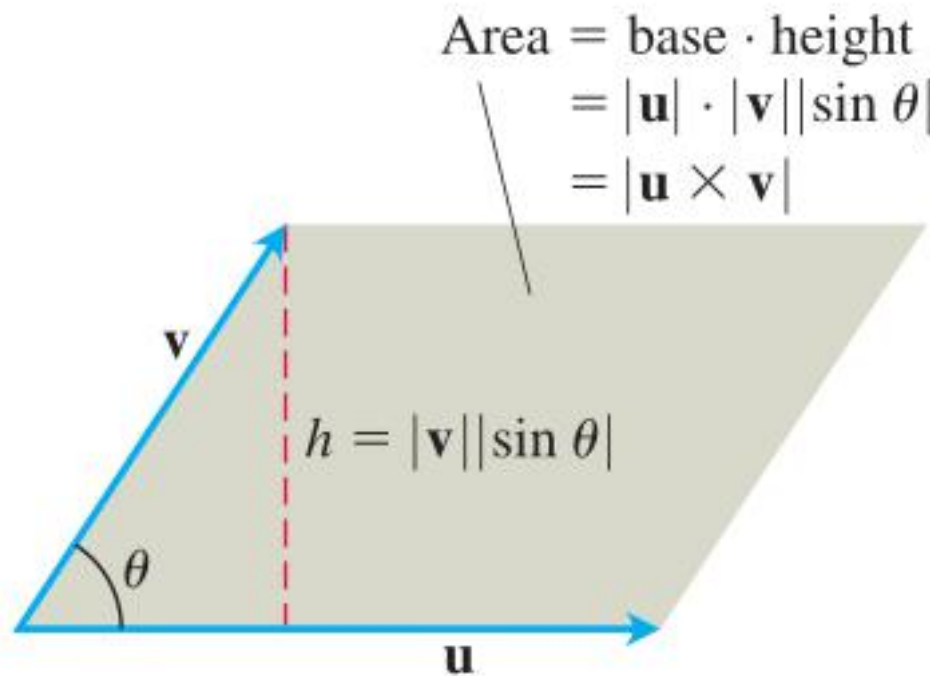


FIGURE 12.29 The pairwise cross products of \mathbf{i} , \mathbf{j} , and \mathbf{k} .

$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$



Determinant Formula for $\mathbf{u} \times \mathbf{v}$

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.$$

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\ &= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\ &\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\ &\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}. \end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE 1

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}\end{aligned}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

EXAMPLE 2

Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$

Solution $\overrightarrow{PQ} \times \overrightarrow{PR}$

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}.\end{aligned}$$

EXAMPLE 3

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$

Solution $|\vec{PQ} \times \vec{PR}| = |6\mathbf{i} + 6\mathbf{k}|$
 $= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE 4

Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane,

$$\mathbf{n} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

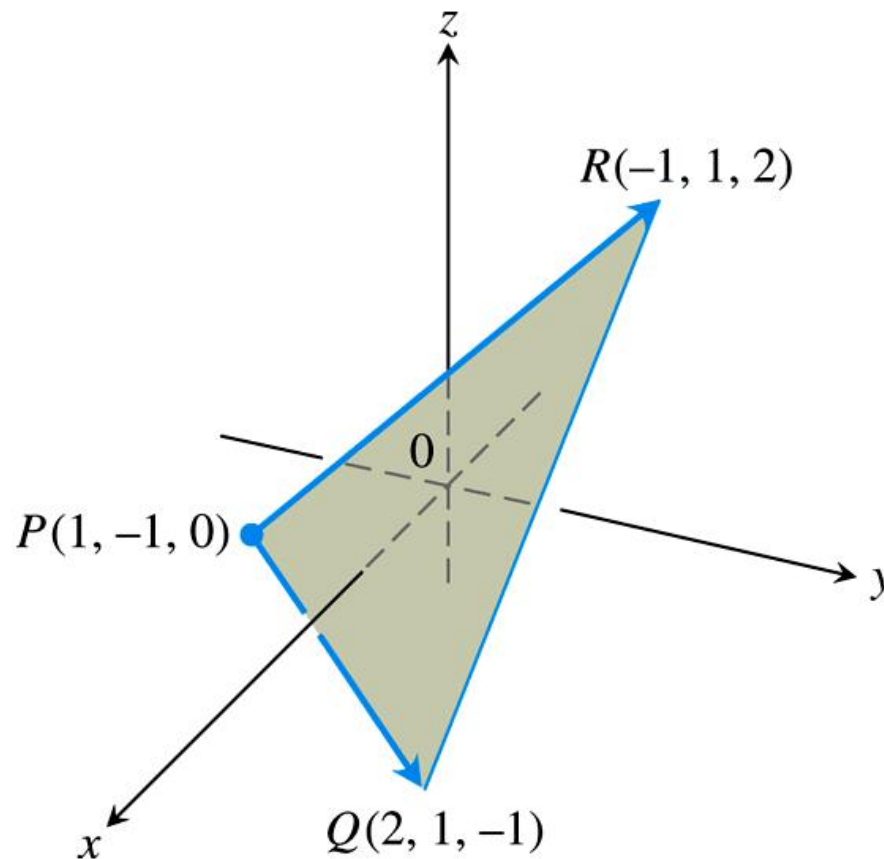
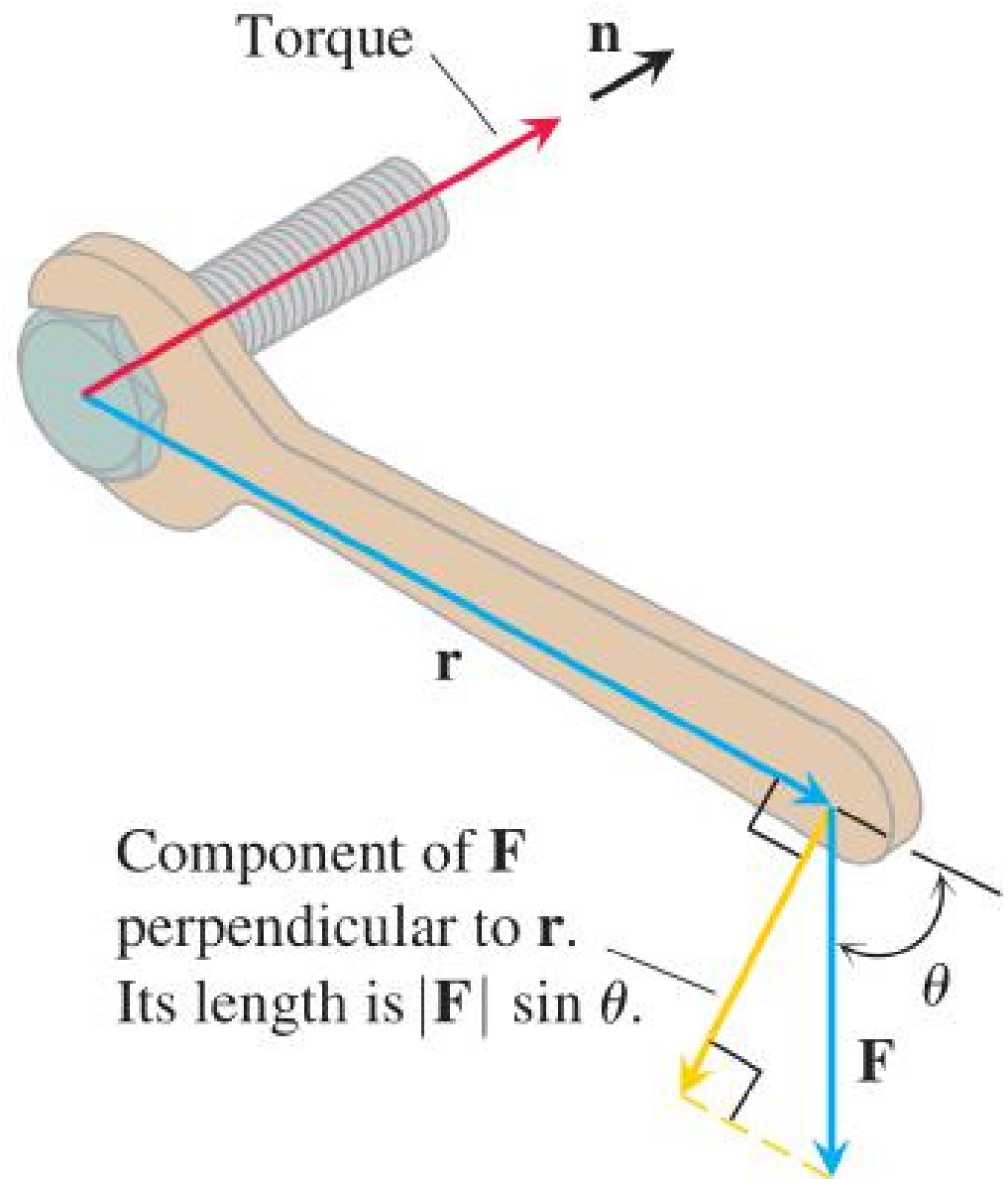


FIGURE 12.31 The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane of triangle PQR (Example 2). The area of triangle PQR is half of $|\vec{PQ} \times \vec{PR}|$ (Example 3).

Torque

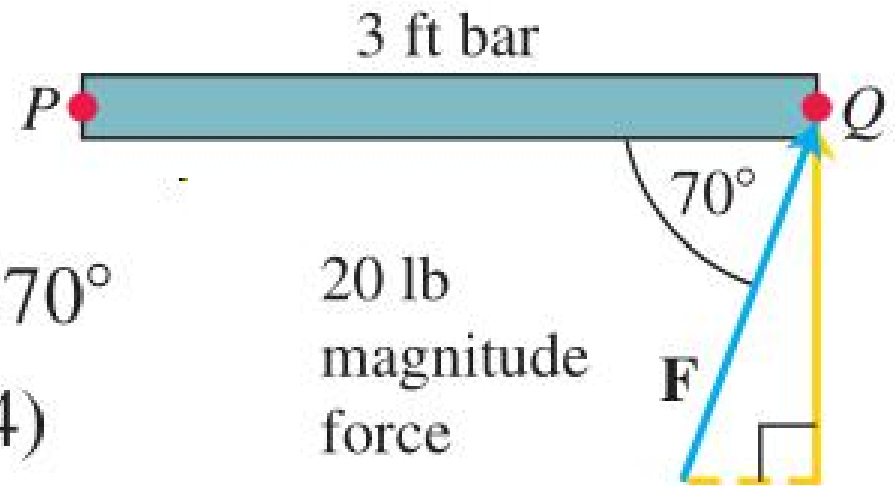
$$\mathbf{r} \times \mathbf{F},$$



EXAMPLE 5

The magnitude of the torque generated by force \mathbf{F} at the pivot point P in Figure 12.33 is

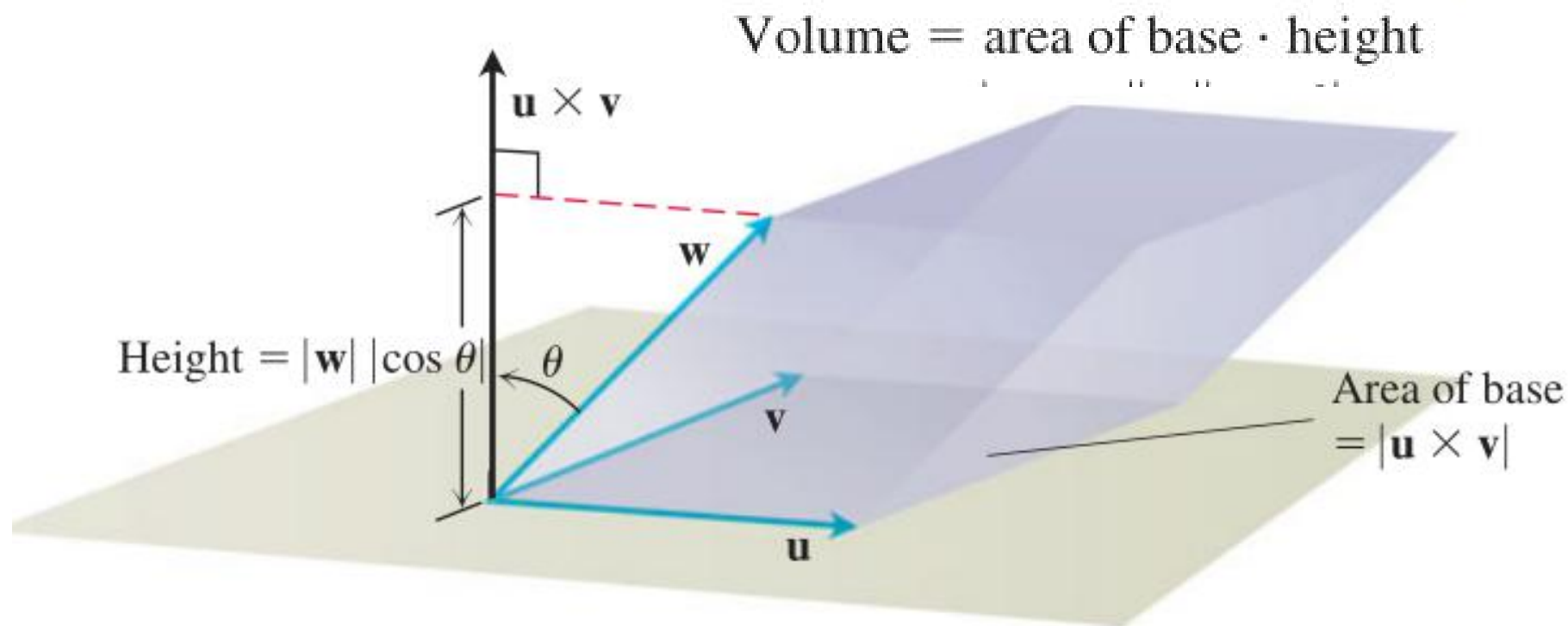
$$\begin{aligned} |\vec{PQ} \times \mathbf{F}| &= |\vec{PQ}| |\mathbf{F}| \sin 70^\circ \\ &\approx (3)(20)(0.94) \\ &\approx 56.4 \text{ ft-lb.} \end{aligned}$$



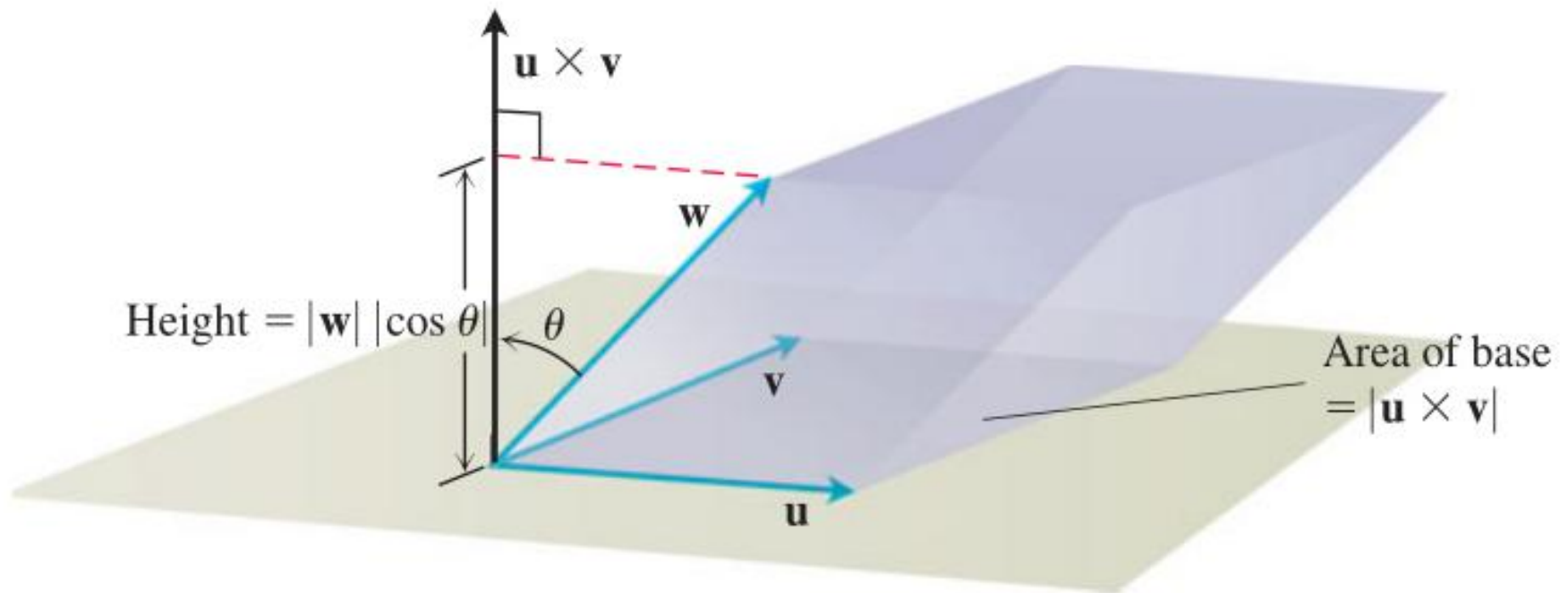
Triple Scalar or Box Product

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of \mathbf{u} , \mathbf{v} , and \mathbf{w}

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta|,$$



$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$



$$\begin{aligned}
 (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right] \cdot \mathbf{w} \\
 &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
 &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
 \end{aligned}$$

Calculating the Triple Scalar Product as a Determinant

EXAMPLE 6

Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} =$$

$$(1) \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - (2) \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

12.5

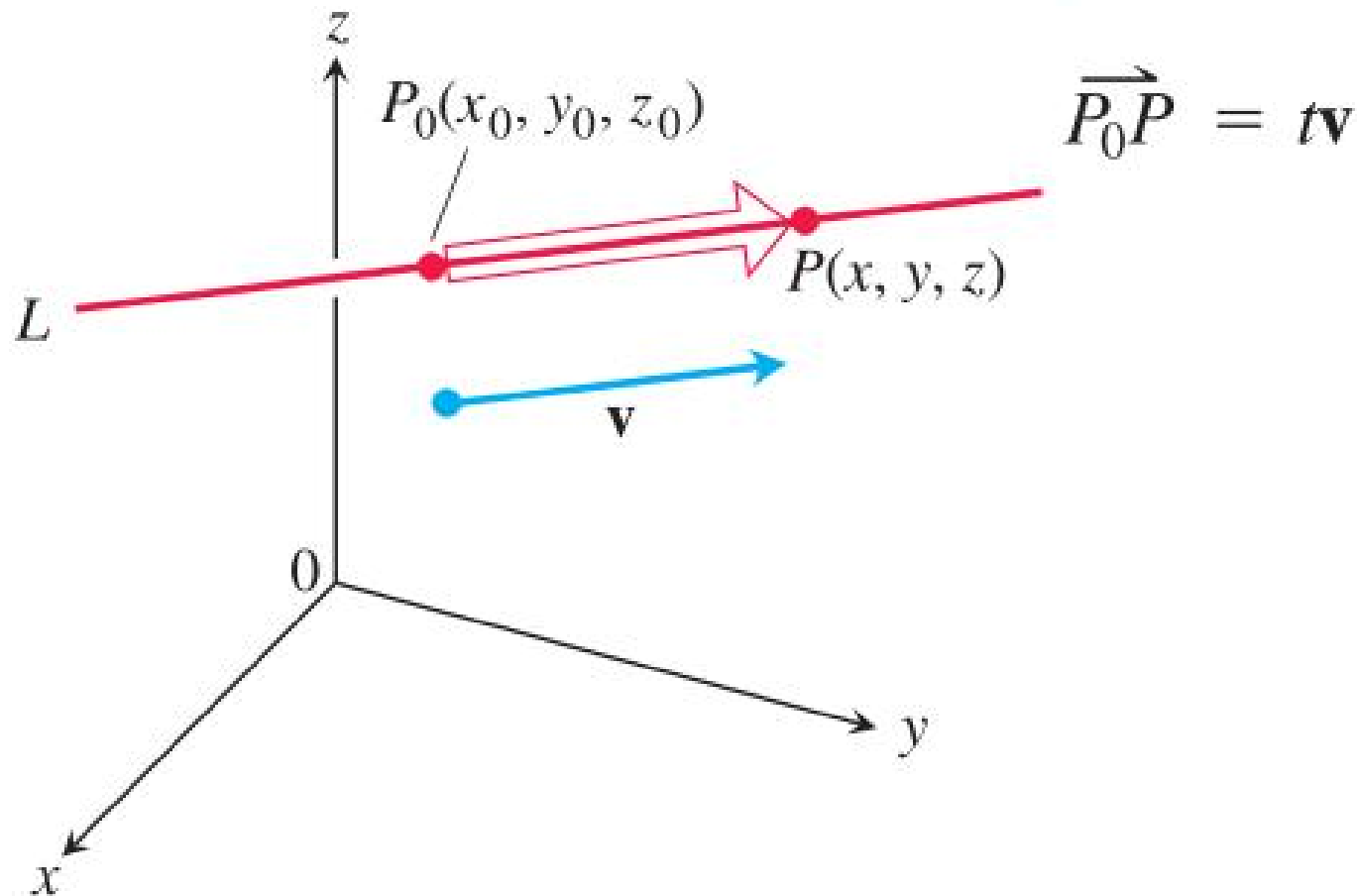
Lines and Planes in Space

直线和平面

$$P_0(x_0, y_0, z_0)$$

$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.$$

$$\forall P(x, y, z) \in L$$



$$\overrightarrow{P_0P} = t\mathbf{v}$$

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}).$$

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},$$

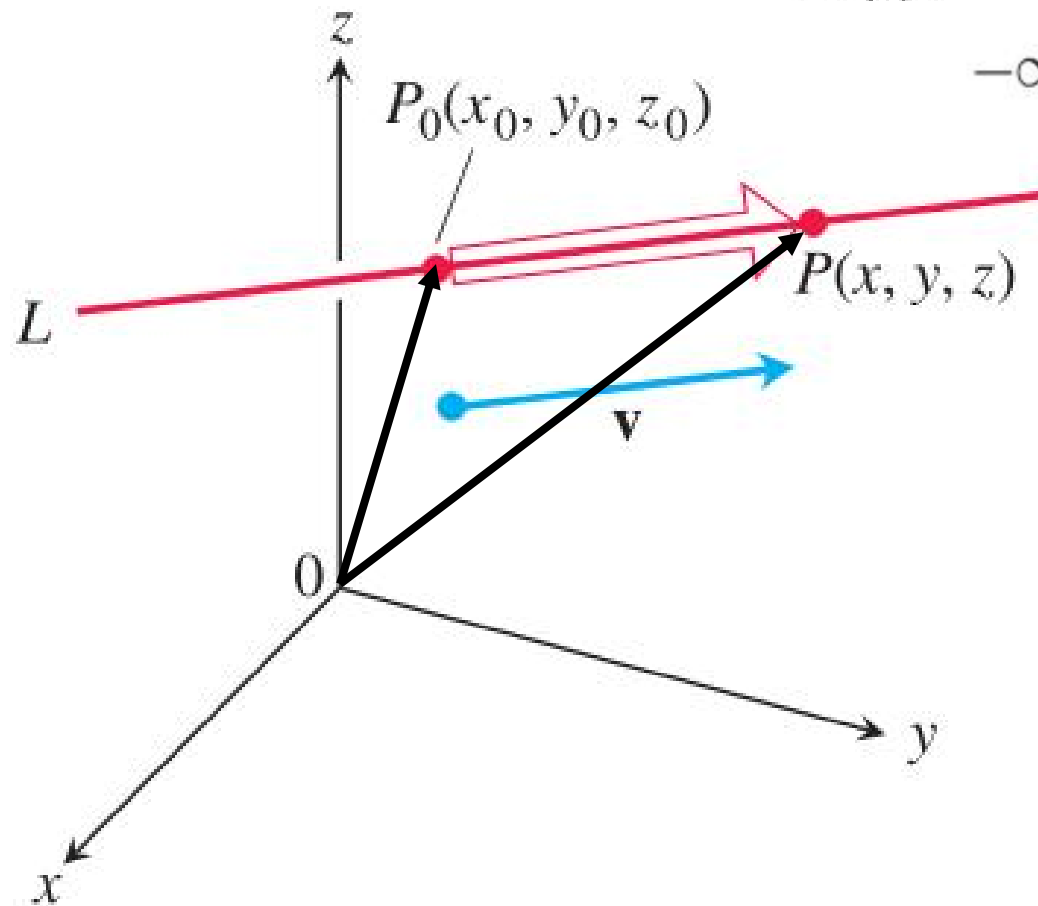
Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to \mathbf{v} is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty,$$

空间直线的向量式方程

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},$$
$$-\infty < t < \infty,$$



Parametric Equations for a Line

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},$$

the line through $P_0(x_0, y_0, z_0)$ parallel to

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

$$-\infty < t < \infty$$

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

直线的对称式方程

EXAMPLE 1

Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Solution $x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t.$

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution $\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k}$
 $= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Solution $x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$
 $0 \leq t \leq 1.$

$P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$.

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= \mathbf{r}_0 + t|\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Initial
position

Time

Speed

Direction

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

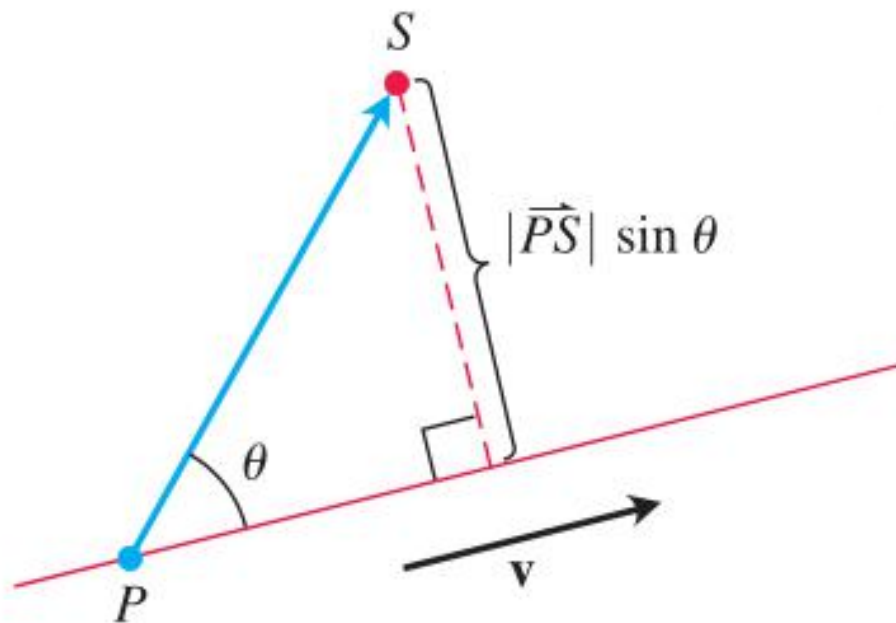
Solution $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}). \quad \mathbf{r}(10) = 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle.\end{aligned}$$

The Distance from a Point to a Line in Space

S to a line that passes through a point P parallel to \mathbf{v} ,



$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution L passes through $P(1, 3, 0)$ $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

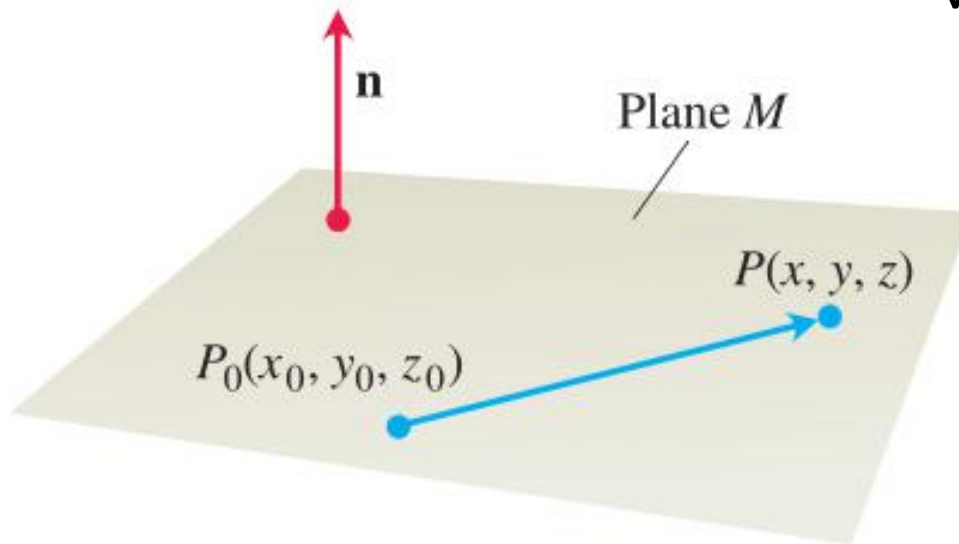
$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation for a Plane in Space

that plane M passes through a point $P_0(x_0, y_0, z_0)$
perpendicular to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

$$\forall P(x, y, z) \in M$$

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$



$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0,$$

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Component equation simplified: $Ax + By + Cz = D,$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

EXAMPLE 6

Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$
$$5x + 2y - z = -22.$$

EXAMPLE 7

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution

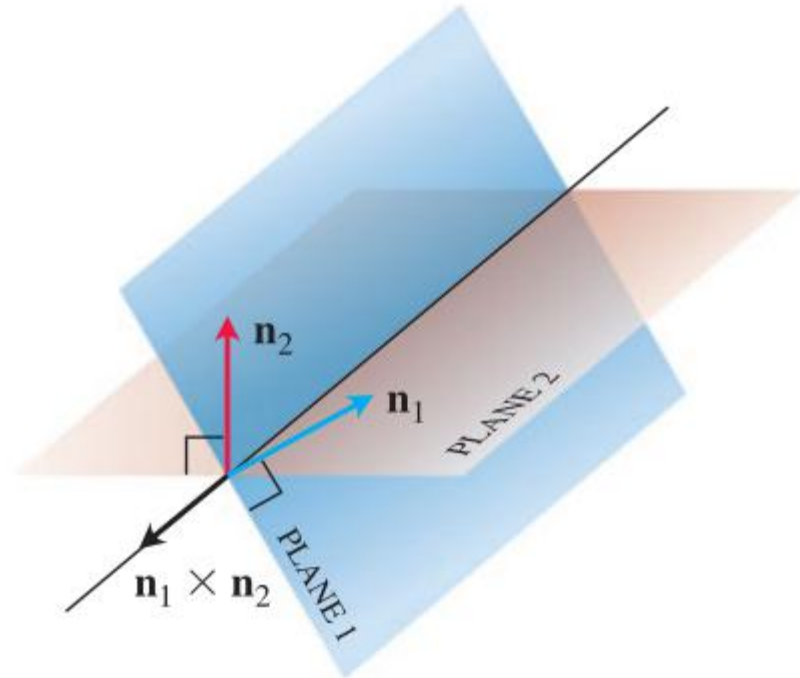
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$
$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$
$$3x + 2y + 6z = 6.$$

EXAMPLE 8

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$



EXAMPLE 9

Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$

Substituting $z = 0$ $(3, -1, 0)$.

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

EXAMPLE 10 Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$t = -1.$$

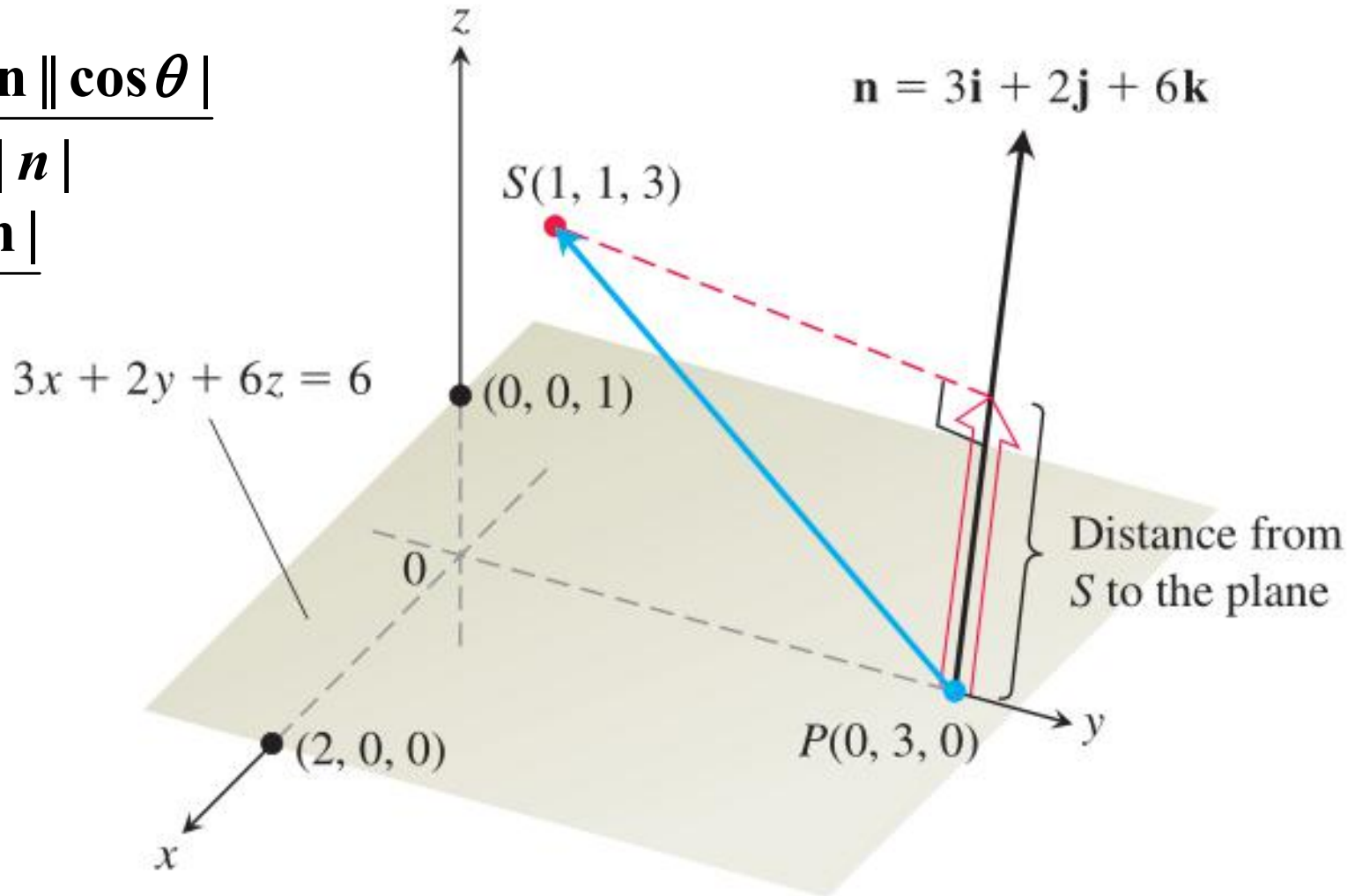
$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, -2, 1 - 1\right) = \left(\frac{2}{3}, -2, 0\right).$$

The Distance from a Point to a Plane

$$d = |PS| \cos \theta$$

$$= \frac{|PS| |n| \cos \theta}{|n|}$$

$$= \frac{|PS \cdot n|}{|n|}$$



EXAMPLE 11

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

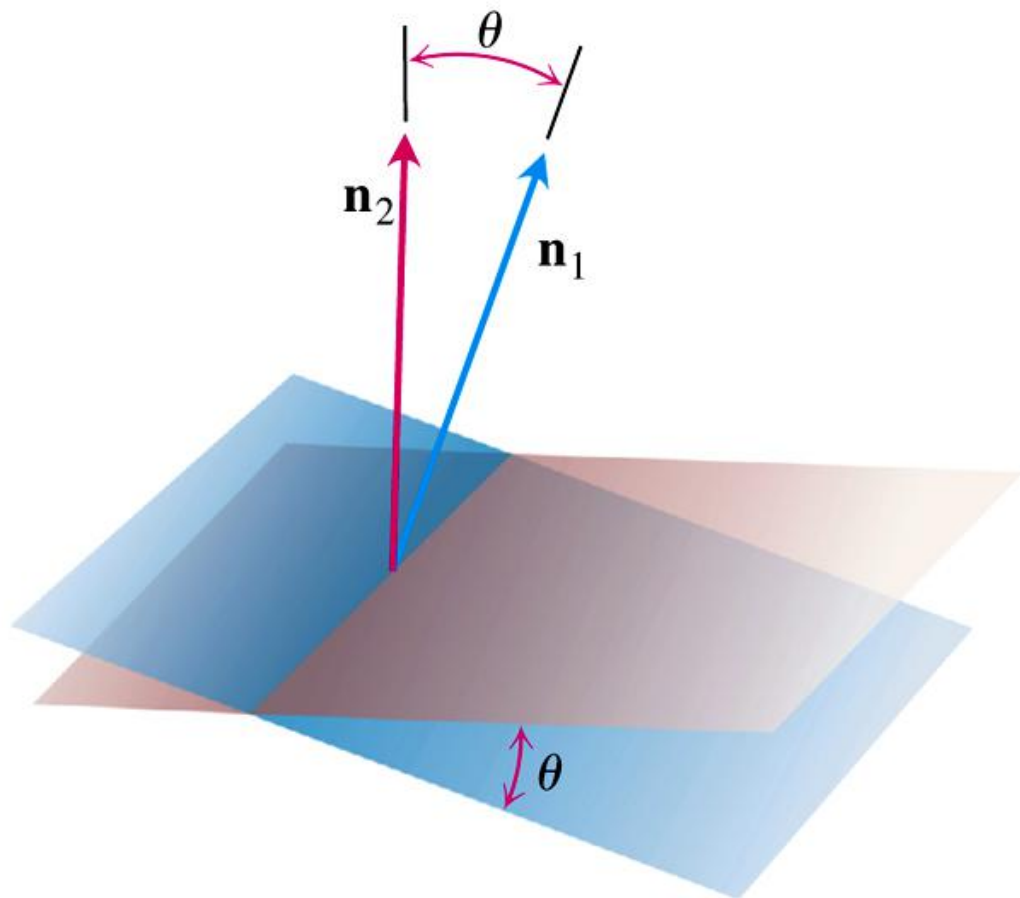
Solution $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. $P(0,3,0)$

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

$$\begin{aligned}\overrightarrow{PS} &= (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},\end{aligned}$$

$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| = \frac{17}{7}.$$



Angles Between Planes

$$0 \leq \theta < \pi$$

FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

EXAMPLE 12

Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{4}{21} \right) \approx 1.38 \text{ radians.}$$

12.6

Cylinders and Quadric Surfaces

柱面和二次曲面

Cylinders

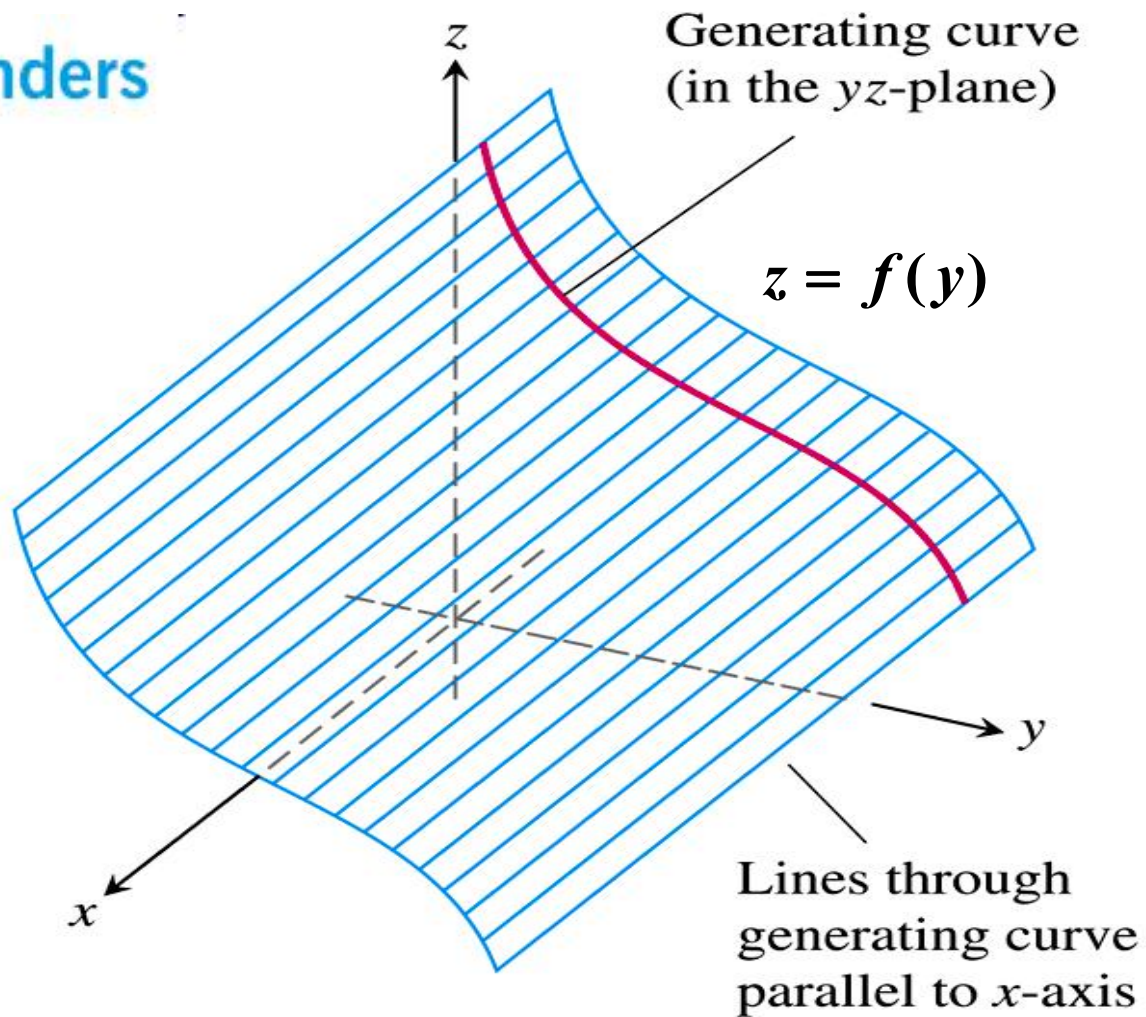
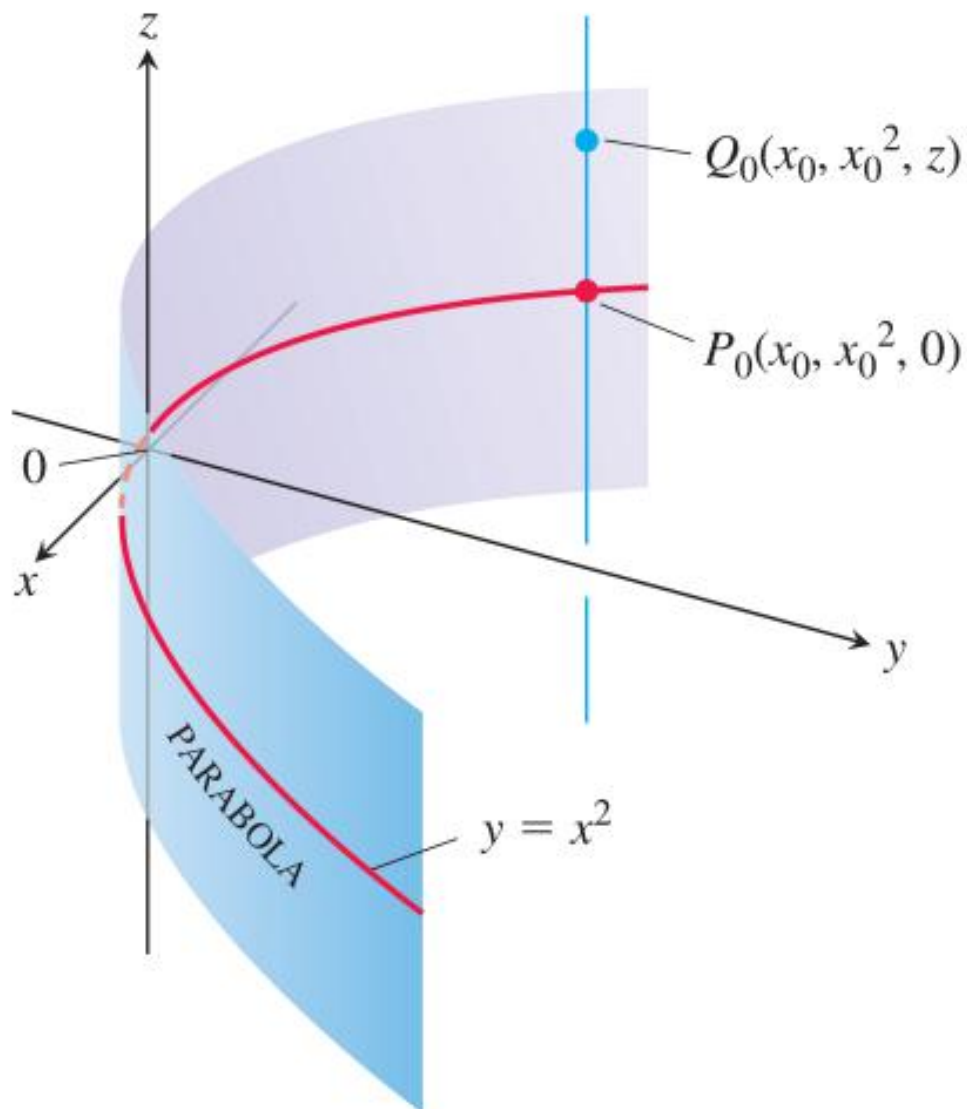


FIGURE 12.43 A cylinder and generating curve.

EXAMPLE 1

the cylinder $y = x^2$



Quadric Surfaces

a second-degree equation in x , y , and z .

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz = G,$$

EXAMPLE 2 The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

伸缩变换 $\frac{x}{a} = X, \frac{y}{b} = Y, \frac{z}{c} = Z$

$$X^2 + Y^2 + Z^2 = 1$$

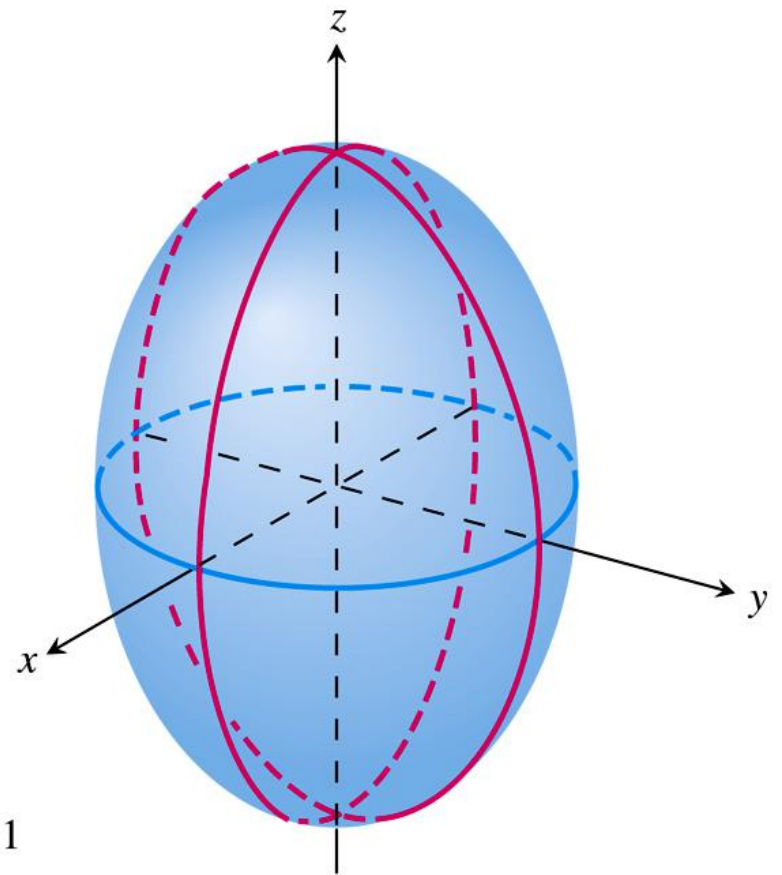
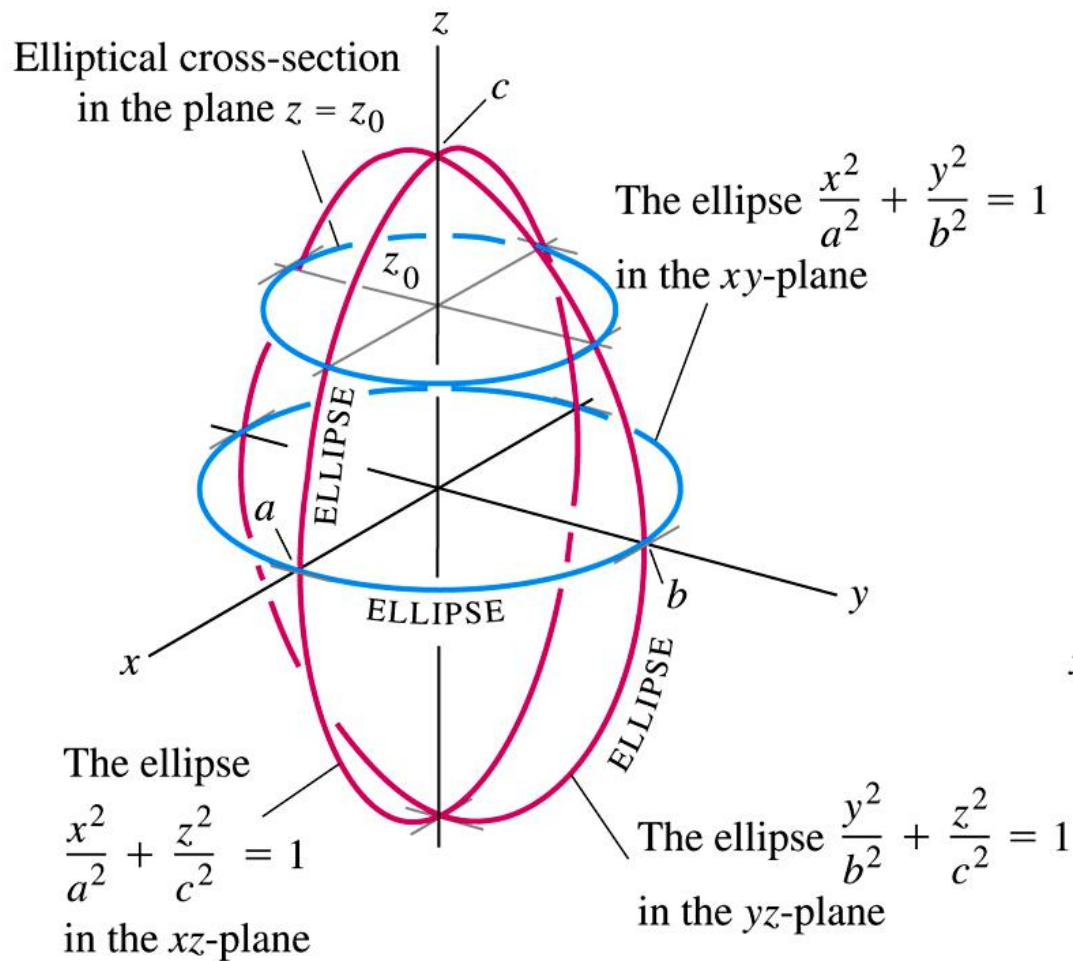


FIGURE 12.45 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

Table 12.1 Graphs of Quadric Surfaces (cont)

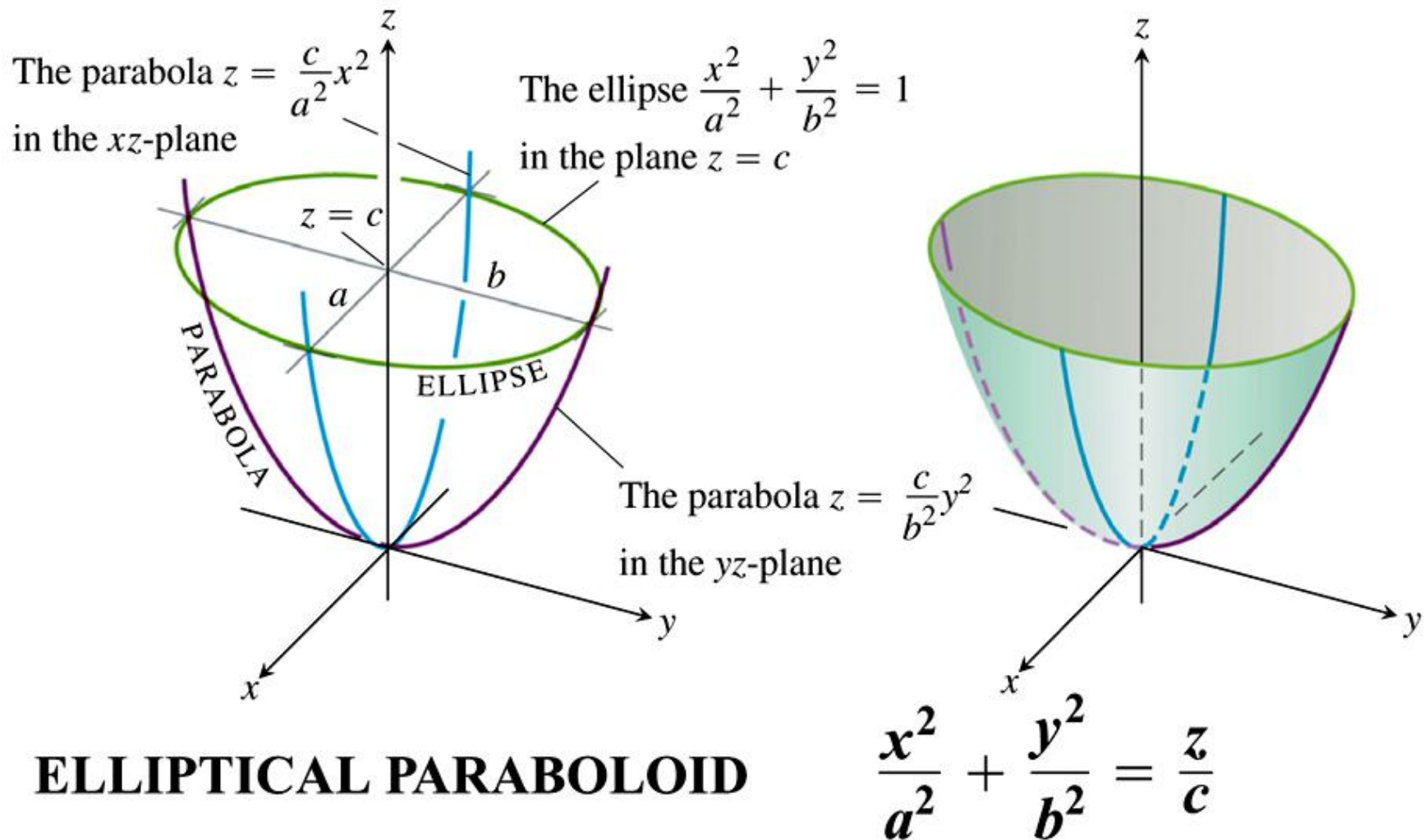
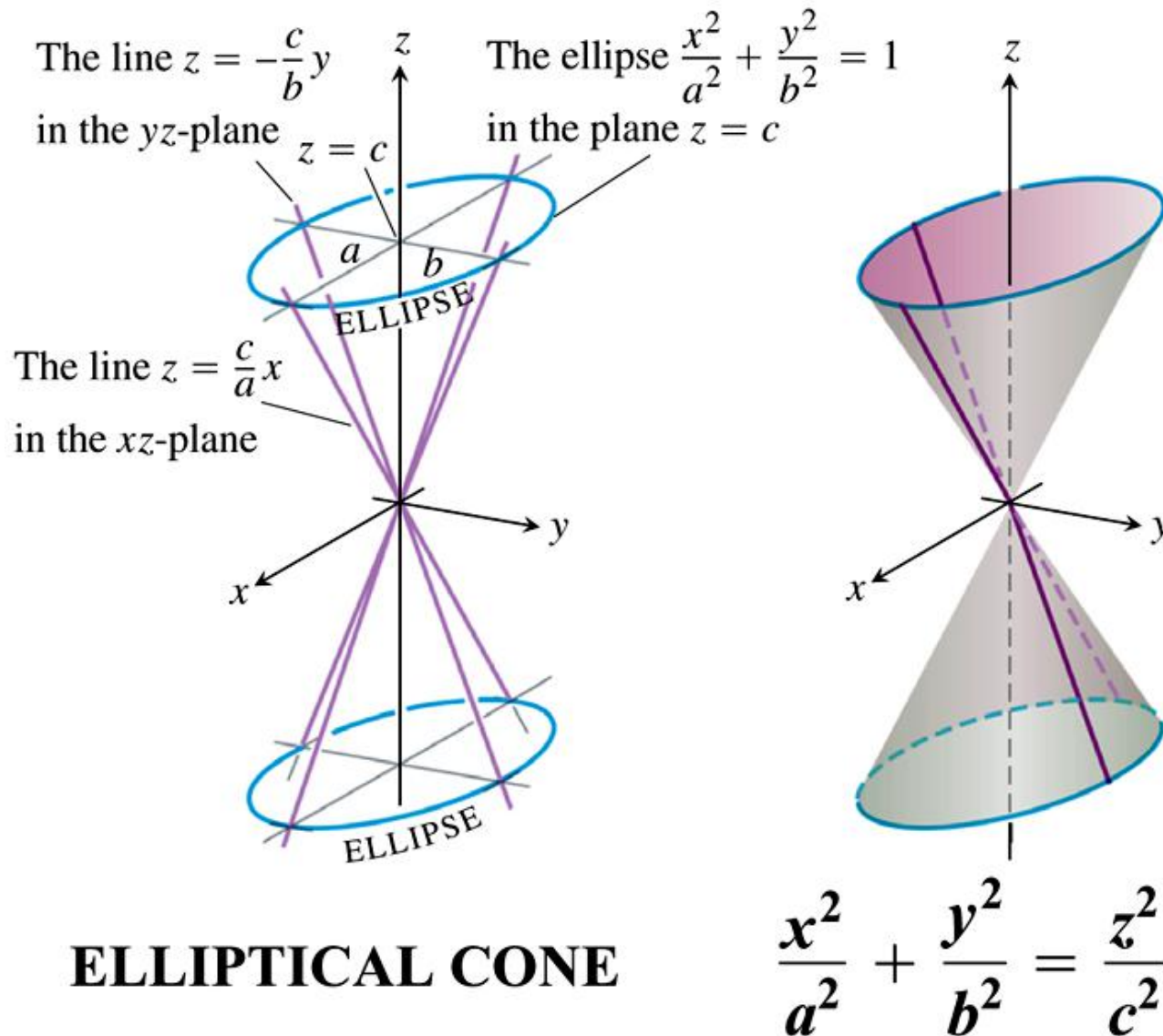
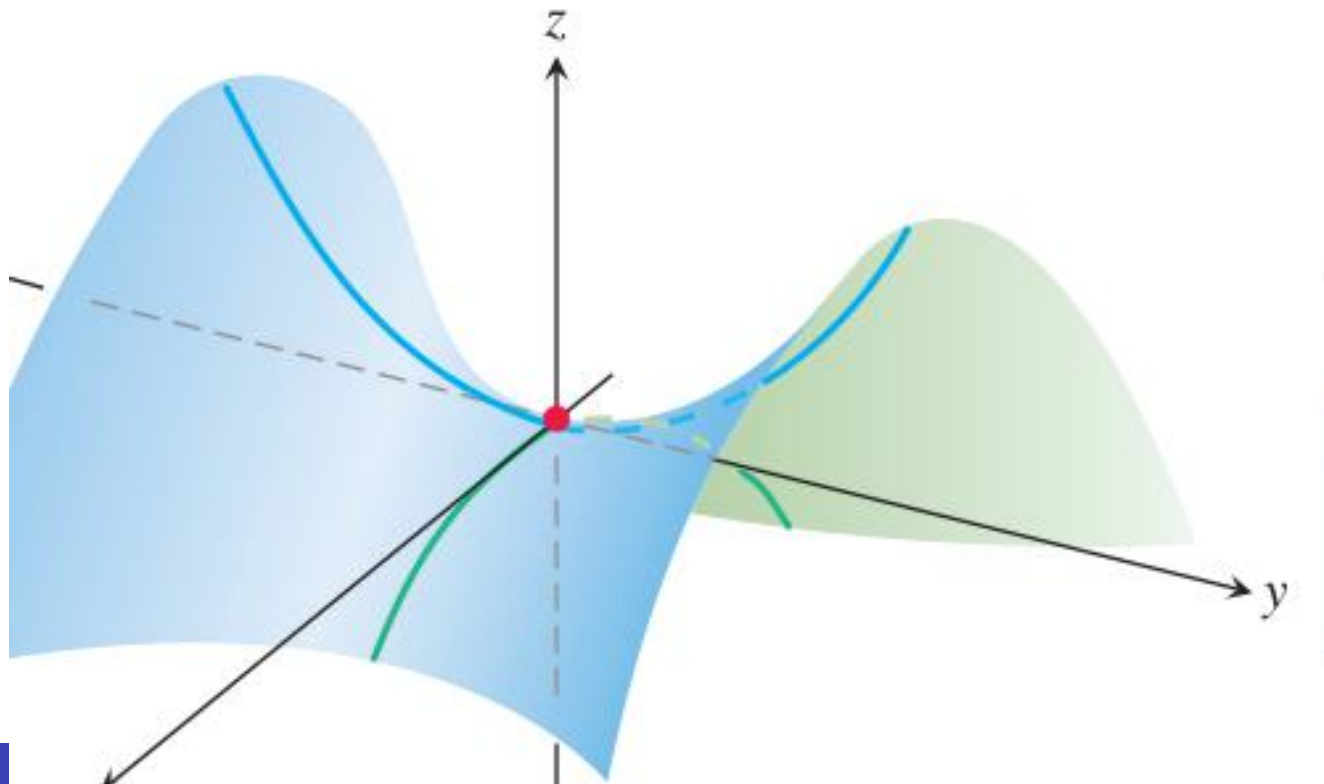


Table 12.1 Graphs of Quadric Surfaces (cont)



EXAMPLE 3 The hyperbolic paraboloid

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$$



The parabola $z = \frac{c}{b^2}y^2$ in the yz -plane

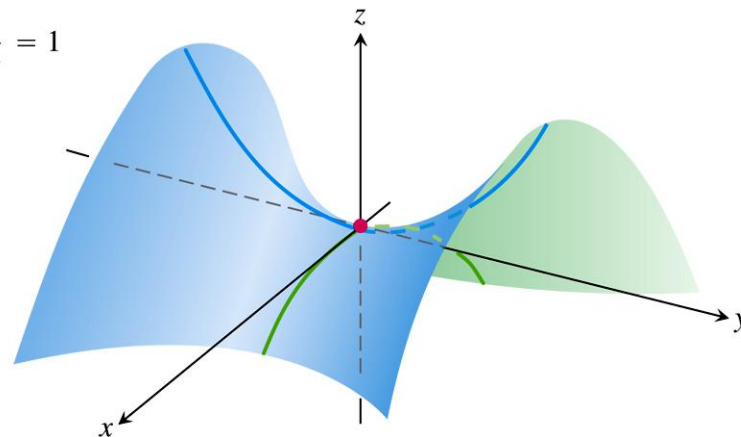
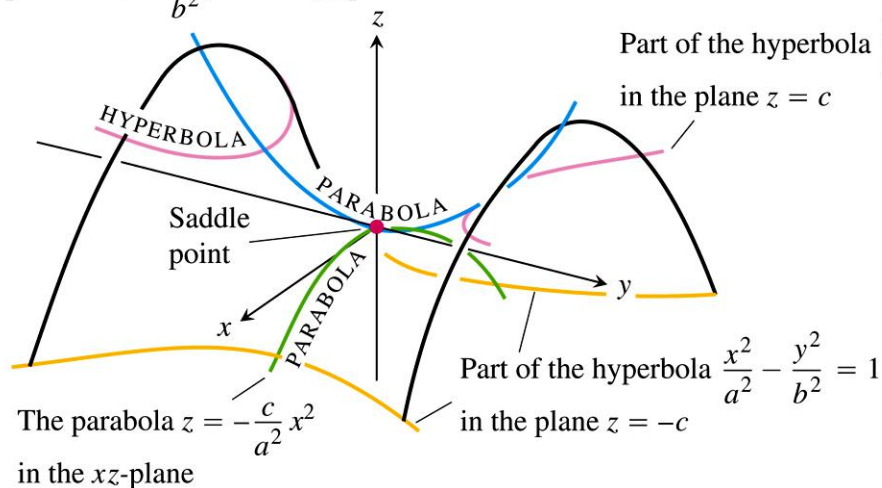


FIGURE 12.46 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c, c > 0$. The cross-sections in planes perpendicular to the z -axis above and below the xy -plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

Table 12.1 Graphs of Quadric Surfaces (cont)

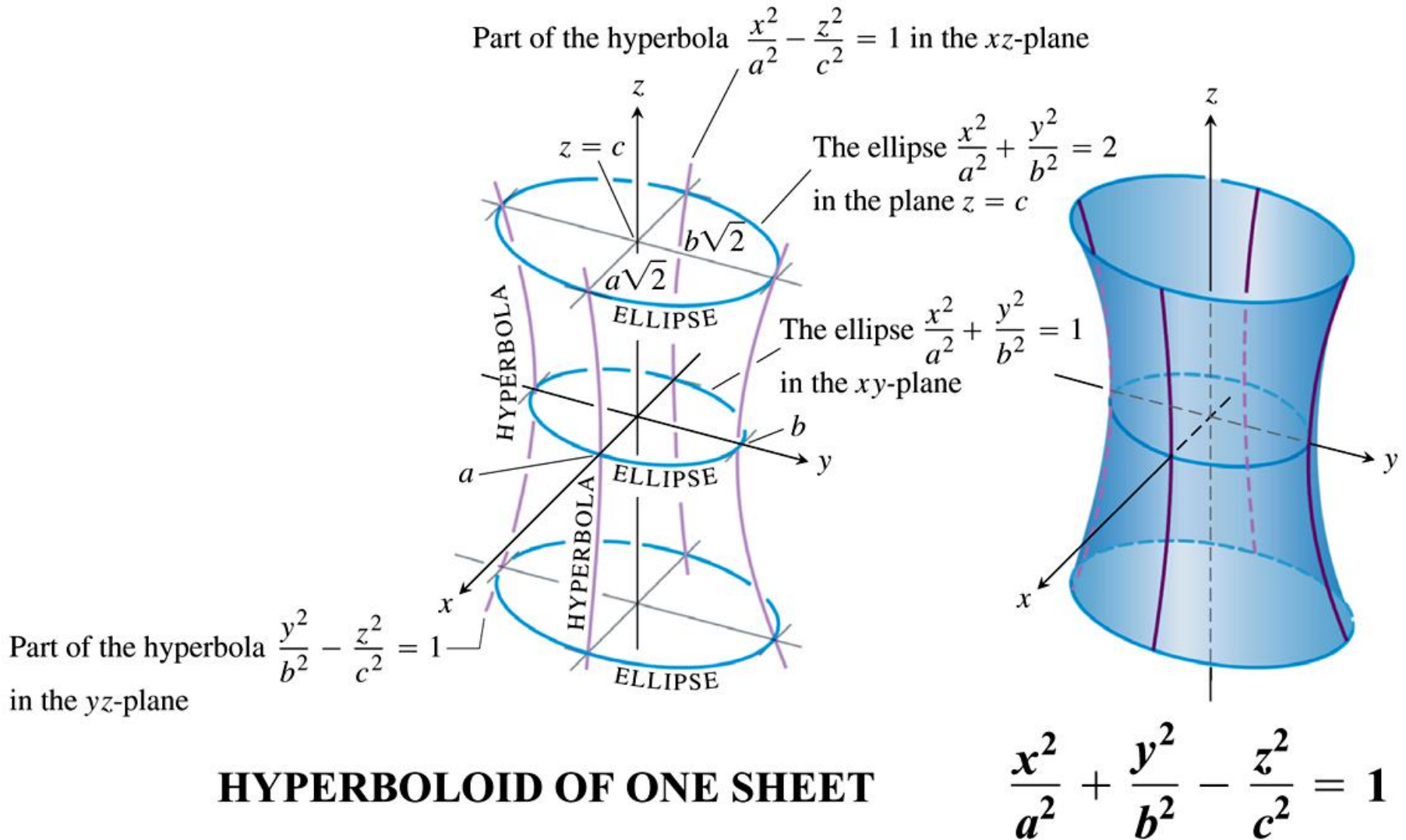
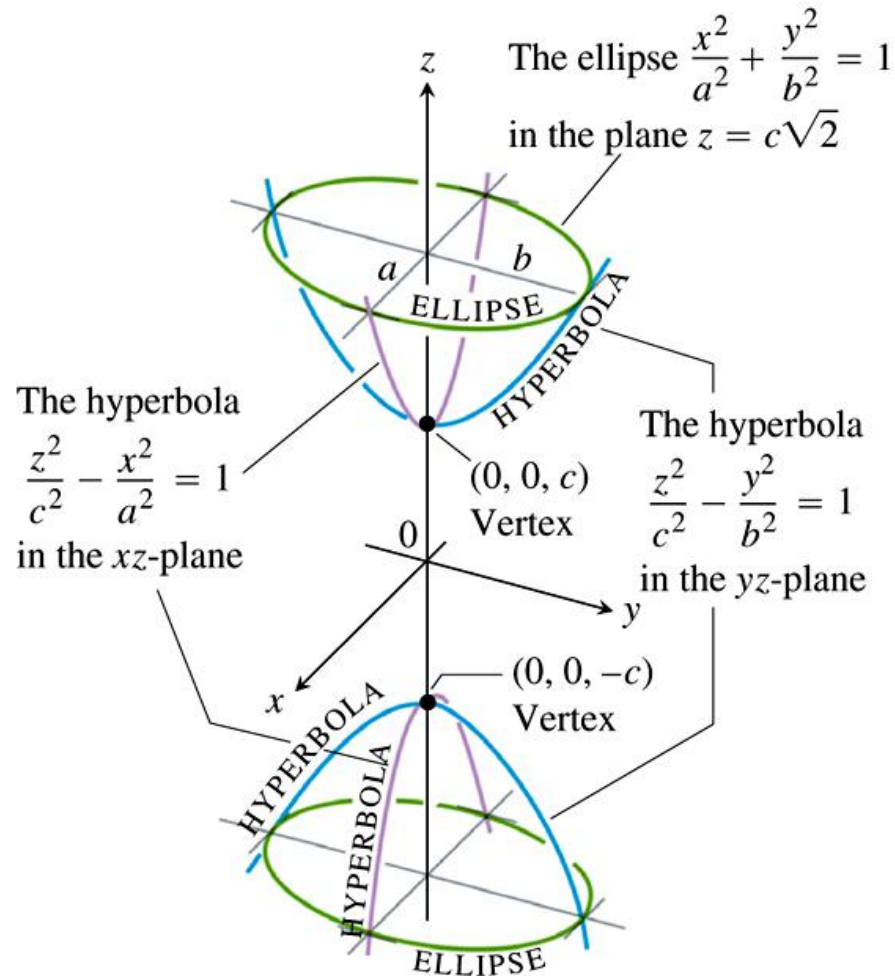
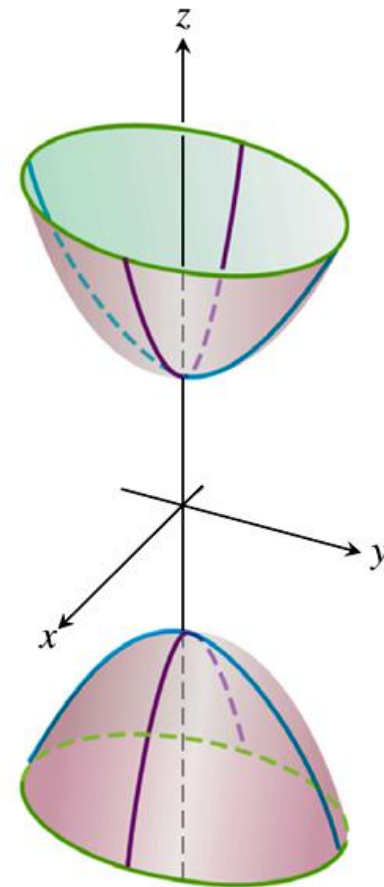


Table 12.1 Graphs of Quadric Surfaces (cont)



HYPERBOLOID OF TWO SHEETS



$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$