Chapter 4: Expected Values (期望值)

➤ The Expected Value of a Random Variable(随机变量的期望)

➤ Variance and Standard Deviation(方差和标准差)

➤ Covariance and Correlation Coefficient (协方差和相关系数)

➤ Conditional Expectation (条件期望)

Chapter 4: Expected Values (期望值)

Relationship between r.v. X and r.v. Y

For r.v.s X and Y

$$D(X + Y) = D(X) + D(Y) + 2E\big[(X - E(X))(Y - E(Y))\big]$$

When X and Y are independent,

$$D(X + Y) = D(X) + D(Y)$$

Analysis: If X and Y are independent, then

$$E\big[(X - E(X))(Y - E(Y))\big] = 0.$$

Therefore, if

$$E\big[(X - E(X))(Y - E(Y))\big] \neq 0,$$

then X and Y cannot be independent. They must be related.

If X and Y are not independent, how to describe their relationship

Definition: If the variances of X and Y both exists, denote

$$Cov(X, Y) \triangleq E\big[(X - E(X))(Y - E(Y))\big]$$

Then $Cov(X, Y)$ is called the Covariance (协方差) of X and Y.

Basic properties of Covariance （协方差的基本特性）

$$Cov(X,Y) \triangleq E\big[(X - E(X))(Y - E(Y))\big]$$

1. If X and Y are independent $\Rightarrow$ $Cov(X,Y) = 0$

2. $Cov(X,Y) = Cov(Y,X)$

3. $D(X) = E\big(X - E(X)\big)^2 = Cov(X,X)$

✓ 4. $Cov(X,Y) = E(XY) - E(X)E(Y)$

> It is an important formula that is used often for calculation.

Proof:
$$Cov(X,Y) = E\big[(X - E(X))(Y - E(Y))\big]$$
$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$
$$= E(XY) - E[XE(Y)] - E[YE(X)] + E(X)E(Y)$$
$$= E(XY) - E(X)E(Y)$$

✓ 5. $D(X + Y) = D(X) + D(Y) + 2E\big[(X - E(X))(Y - E(Y))\big]$
$$= D(X) + D(Y) + 2Cov(X,Y)$$

Basic properties of Covariance（协方差的基本特性）

$$Cov(X, Y) \triangleq E\big[(X - E(X))(Y - E(Y))\big]$$

6. For any constant a and b

$$Cov(aX, bY) = E\big[(aX - E(aX))(bY - E(bY))\big]$$
$$= abE\big[(X - E(X))(Y - E(Y))\big]$$
$$= abCov(X, Y)$$

✓ 7. $Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$

Theorem: Bilinear（双线性）property

Suppose that $U = a + \sum_{i=1}^{n} b_i X_i$ and $V = c + \sum_{j=1}^{m} d_j Y_j$, then

$$Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i d_j Cov(X_i, Y_j)$$

The meaning of Covariance

$\because X, Y$ are independent $\quad\Longrightarrow\quad$ $Cov(X,Y) = 0$

$\therefore Cov(X,Y) \neq 0$ $\quad\Longrightarrow\quad$ X, Y are not independent

$\quad\Longrightarrow\quad$ X, Y must be related in some way

$Cov(X,Y) = 0$ $\quad\boxed{?}\quad$ X, Y are independent

No!

Example: Assume that r.v. (X, Y) follows the Uniform distribution in the circular region $G: x^2 + y^2 \leq 1$. Compute $Cov(X, Y)$, are X, Y independent?

Independence: X and Y are independent $\Longleftrightarrow$ $f(x, y) =_{a.e.} f_X(x)f_Y(y)$

Answer: The density function of (X, Y) is

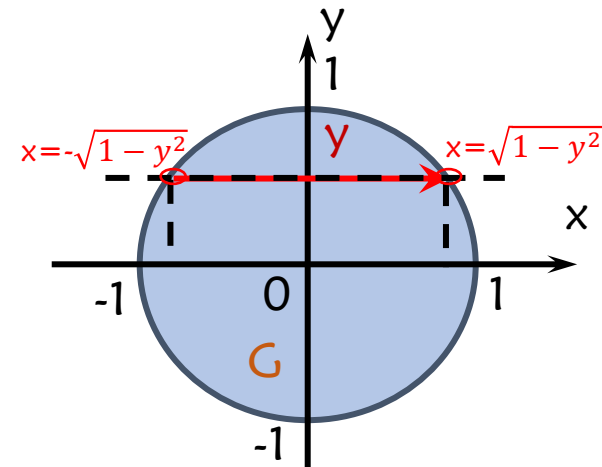$$f(x, y) = \begin{cases} 1/\pi, & x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal density functions of X and Y are:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2}, & |y| < 1 \\ 0, & |y| \geq 1 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$\because f(x, y) \neq f_X(x)f_Y(y)$

$\therefore X, Y$ are not independent.

x=-$\sqrt{1-y^2}$　y　x=$\sqrt{1-y^2}$

G

Example: Assume that r.v. (X, Y) follows the Uniform distribution in the circular region $G: x^2 + y^2 \leq 1$. Compute $Cov(X, Y)$, are X, Y independent?

Answer: Since

$$f_X(x) = \begin{cases} \dfrac{2}{\pi}\sqrt{1-x^2}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

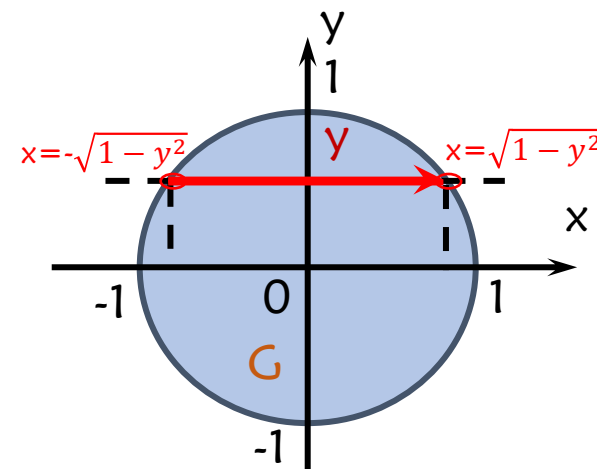$$f_Y(y) = \begin{cases} \dfrac{2}{\pi}\sqrt{1-y^2}, & |y| < 1 \\ 0, & |y| \geq 1 \end{cases}$$

$$\therefore E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = 0$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = 0$$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y)\, dx\, dy = \frac{1}{\pi} \int_{x^2+y^2 \leq 1} xy\, dx\, dy = 0$$

$$\therefore Cov(X, Y) = E(XY) - E(X)E(Y) = 0$$

Therefore, X and Y are not independent, but $Cov(X, Y) = 0$.

Correlation Coefficient

Consider r.v.s after "standardization(单位化/标准化)":

$$X^* \triangleq \frac{X - E(X)}{\sqrt{D(X)}}, Y^* \triangleq \frac{Y - E(Y)}{\sqrt{D(Y)}}$$

Then

$$Cov(X^*, Y^*) = \frac{Cov(X, Y)}{\sqrt{D(X)D(Y)}}$$

Definition:

$$\rho_{XY} \triangleq Cov(X^*, Y^*) = \frac{Cov(X, Y)}{\sqrt{D(X)D(Y)}}$$

$\boldsymbol{\rho_{XY}}$ is called the **correlation coefficient (相关系数)** of X and Y, which is a dimensionless quantity (无量纲数值).

What the relationship between X and Y does ρ_{XY} represent ❓

Answer: Consider the linear relation between X and Y. Use r.v.

$$\hat{Y} = a + bX \ (a, b \text{ are constants})$$

to represent r.v. Y approximately, and consider the Mean Squared Error (均方误差)

$$e = E\left[\left(Y - \hat{Y}\right)^2\right] = E\left[\left(Y - (a + bX)\right)^2\right]$$

$$= E\left(Y^2\right) + b^2 E\left(X^2\right) + a^2 - 2bE(XY) + 2abE(X) - 2aE(Y)$$

Let

$$\begin{cases} \dfrac{\partial e}{\partial a} = 2a + 2bE(X) - 2E(Y) = 0 \\[2mm] \dfrac{\partial e}{\partial b} = 2bE\left(X^2\right) - 2E(XY) + 2aE(X) = 0 \end{cases}$$

Then get

$$\begin{cases} b_0 = \dfrac{Cov(X, Y)}{D(X)} \\[4mm] a_0 = E(Y) - b_0 E(X) = E(Y) - E(X)\dfrac{Cov(X, Y)}{D(X)} \end{cases}$$

$$\begin{cases} b_0 = \dfrac{Cov(X,Y)}{D(X)} \\ a_0 = E(Y) - b_0 E(X) = E(Y) - E(X)\dfrac{Cov(X,Y)}{D(X)} \end{cases}$$

Therefore:

$$\min_{a,b} e = \min_{a,b} E\left[\left(Y-(a+bY)\right)^2\right] = E\left[\left(Y-(a_0+b_0X)\right)^2\right]$$

$$= E\left[\left(Y-(E(Y)-b_0E(X)+b_0X)\right)^2\right]$$

$$= E\left[\left(Y-E(Y)\right)^2\right] + b_0^2 E\left[\left(X-E(X)\right)^2\right] - 2b_0 Cov(X,Y)$$

$$= D(Y) + b_0^2 D(X) - 2b_0 Cov(X,Y)$$

$$= D(Y) + b_0 Cov(X,Y) - 2b_0 Cov(X,Y) = D(Y) - b_0 Cov(X,Y)$$

$$= D(Y)\left[1 - \frac{[Cov(X,Y)]^2}{D(X)D(Y)}\right]$$

$$= D(Y)(1 - \rho_{XY}^2)$$

Important relations (重要关系式)

In conclusion,
$$\min_{a,b} E\left[\left(Y-(a+bY)\right)^2\right] = E\left[\left(Y-(a_0+b_0X)\right)^2\right]$$
$$= D(Y)(1-\rho_{XY}^2)$$

Theorem ① $|\rho_{XY}| \le 1$

② $|\rho_{XY}| = 1 \iff Y =_{a.e.} a + bX$ (a, b are constants)

The real meaning of ρ_{XY}

$$\because \min_{a,b} e = \min_{a,b} E\left[\left(Y - (a + bX)\right)^2\right] = D(Y)(1 - \rho_{XY}^2)$$

$\therefore$ When $|\rho_{XY}|$ is large, the mean square error e is small.

The relation between X and Y is closer to linear

Specifically, when $|\rho_{XY}| = 1$, the relation between X and Y is almost perfectly linear.

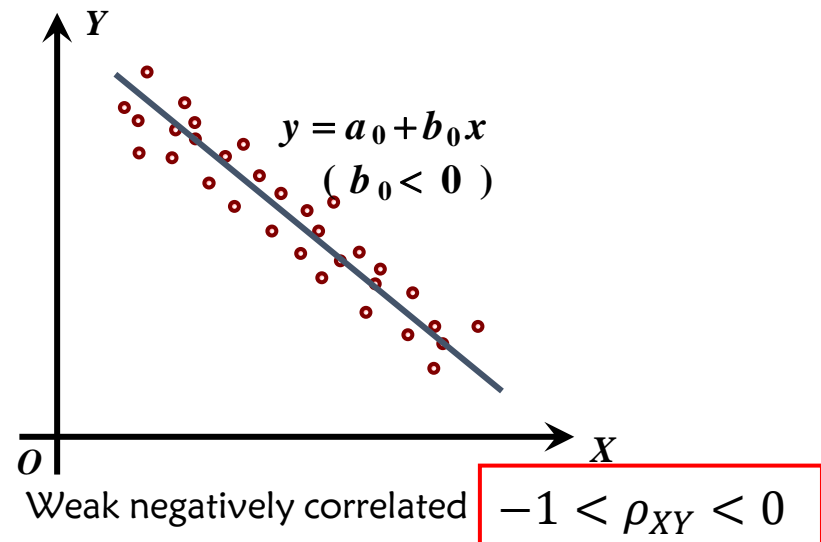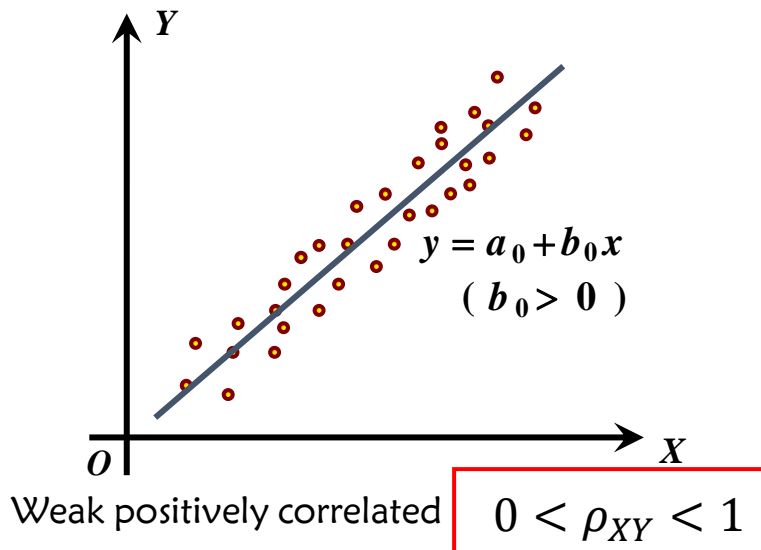Otherwise, when $|\rho_{XY}|$ is small, the linear relation between X and Y is weak.

Definition:

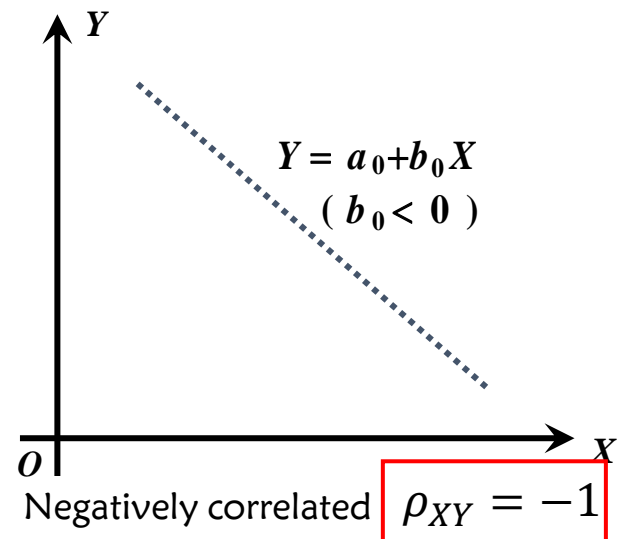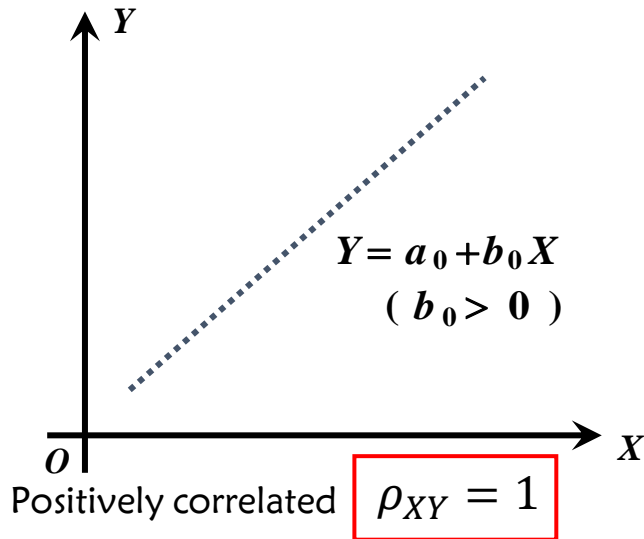When $\rho_{XY} = 1$, X and Y are positively correlated $\left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\}$ Correlated

When $\rho_{XY} = -1$, X and Y are negatively correlated

When $\rho_{XY} = 0$, X and Y are not correlated

Figures showing the linear relation between X and Y

$Y = a_0 + b_0 X$
$(b_0 > 0)$

Positively correlated　$\boxed{\rho_{XY} = 1}$

$Y = a_0 + b_0 X$
$(b_0 < 0)$

Negatively correlated　$\boxed{\rho_{XY} = -1}$

$y = a_0 + b_0 x$
$(b_0 > 0)$

Weak positively correlated　$\boxed{0 < \rho_{XY} < 1}$

$y = a_0 + b_0 x$
$(b_0 < 0)$

Weak negatively correlated　$\boxed{-1 < \rho_{XY} < 0}$

Figures showing the linear relation between X and Y

Not correlated $\boxed{\rho_{XY} = 0}$
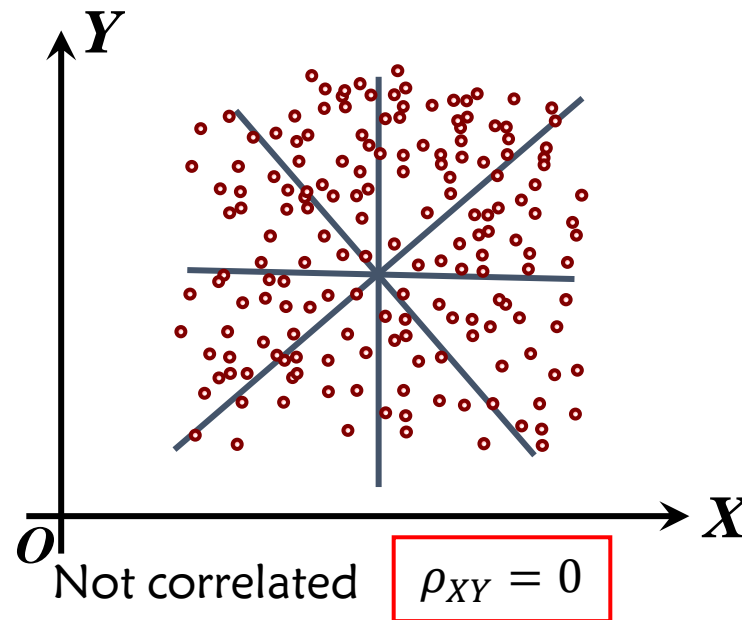
Example: Assume that $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, what is the correlation coefficient of X and Y?

Review

7 $f(x, y)$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} \right. \right.$$

$$\left. \left. - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right] \right\}$$

2 $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$

3 $E(X) = \mu_1, E(Y) = \mu_2$

      $D(X) = \sigma_1^2, D(Y) = \sigma_2^2$

Guess $\rho_{XY} = \rho$?   We can prove it

Example: Assume that $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, what is the correlation coefficient of X and Y?

Proof:

$$Cov(X, Y) = E[(X - \mu_1)(Y - \mu_2)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_1)(y - \mu_2) f(x, y) dx dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\sigma_1 \sigma_2 \sqrt{1 - \rho^2} tu + \rho \sigma_1 \sigma_2 u^2 \right) e^{-(t^2 + u^2)/2} dt du$$

$$f(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} \left[\left(\frac{y - \mu_2}{\sigma_2} - \rho \frac{x - \mu_1}{\sigma_1} \right)^2 + (1 - \rho^2) \frac{(x - \mu_1)^2}{\sigma_1^2} \right] \right\}$$

Let $\quad t = \frac{1}{\sqrt{1 - \rho^2}} \left(\frac{y - \mu_2}{\sigma_2} - \rho \frac{x - \mu_1}{\sigma_1} \right) \quad$ and $\quad u = \frac{x - \mu_1}{\sigma_1}$

Example: Assume that $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, what is the correlation coefficient of X and Y?

Proof:

$$Cov(X, Y) = E[(X - \mu_1)(Y - \mu_2)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\sigma_1 \sigma_2 \sqrt{1 - \rho^2} tu + \rho \sigma_1 \sigma_2 u^2 \right) e^{-(t^2 + u^2)/2} dt du$$

$$= \frac{\rho \sigma_1 \sigma_2}{2\pi} \int_{-\infty}^{\infty} e^{-t^2/2} dt \cdot \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du$$

$$+ \frac{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}}{2\pi} \int_{-\infty}^{\infty} t e^{-t^2/2} dt \cdot \int_{-\infty}^{\infty} u e^{-u^2/2} du$$

$$= \frac{\rho \sigma_1 \sigma_2}{2\pi} \sqrt{2\pi} \cdot \sqrt{2\pi} = \rho \sigma_1 \sigma_2$$

$$\therefore \rho_{XY} = \frac{Cov(X, Y)}{\sqrt{D(X)D(Y)}} = \frac{\rho \sigma_1 \sigma_2}{\sigma_1 \sigma_2} = \rho$$

Relationship between independence and correlation

X and Y are independent ? X and Y have no correlation

X and Y have no relationship at all, including linear relation.

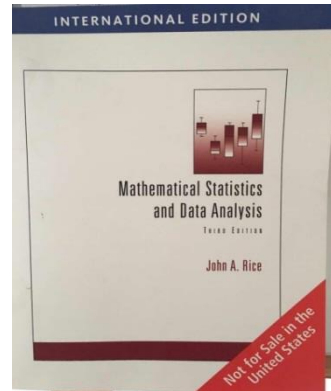X and Y don't have linear relation but may have other relation.

Special case Assume that $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then

X and Y are independent $\Longleftrightarrow$ $\rho = 0$ $\Longleftrightarrow$ $\rho_{XY} = 0$

$\Longleftrightarrow$ X and Y have no correlation

Homework

INTERNATIONAL EDITION

Mathematical Statistics and Data Analysis

Third Edition

John A. Rice

Not for Sale in the United States

P171：54、60 and the extra question

Extra： Assume that the density function for X is

$$f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$$

(1) Compute $E(X)$ and $D(X)$.

(2) Are X and $|X|$ independent or not? State your reason.

(3) Are X and $|X|$ correlated or not? State your reason

Supplementary Questions

1. Suppose that the joint density function of (X, Y) is
$$f(x,y) = \begin{cases} \dfrac{x+y}{8}, & 0 \le x \le 2, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$
Compute $E(X),\ E(Y),\ Cov(X,Y),\ \rho_{XY},\ D(X+Y)$.

2. X and Y are independent random variables which both follow the normal distribution $N(\mu, \sigma^2)$. If $Z = \alpha X + \beta Y, W = \alpha X - \beta Y$, compute $Cov(Z, W)$ and ρ_{ZW}.