

CS201: Discrete Math for Computer Science
2021 Fall Semester Written Assignment # 5
Due: Dec. 15th, 2021, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive*, *irreflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b) | a \text{ and } b \text{ have no letters in common}\}$

(2) $R_2 = \{(a, b) | a \text{ and } b \text{ are not the same length}\}$

(3) $R_3 = \{(a, b) | a \text{ is longer than } b\}$

Solution:

(1) Irreflexive, symmetric

(2) Irreflexive, symmetric

(3) Irreflexive, antisymmetric, transitive

□

Q.2 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

- (a) $2^{n(n+1)/2}$
- (b) $2^n 3^{n(n-1)/2}$
- (c) $2^{n(n-1)}$
- (d) $2^{n(n-1)/2}$
- (e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$
- (f) $3^{n(n-1)/2}$
- (g) 2^n

□

Q.3 Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

Solution: R^2 might not be irreflexive. For example, $R = \{(1, 2), (2, 1)\}$.

□

Q.4 Give an example of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.

Solution: We fix the ground set $S = \{a, b, c, d\}$, and we consider the relation $R = \{(a, b), (b, c), (c, d)\}$. Then the transitive closure of R equals $R^* = \{(a, b), (b, c), (c, d), (a, c), (b, d), (a, d)\}$. On the other hand, $R^2 = \{(a, c), (b, d)\}$, and $R^3 = \{(a, d)\}$. Hence, R^3 is necessary to get R^* .

□

Q.5 Suppose that R_1 and R_2 are both *reflexive* relations on a set A .

- (1) Show that $R_1 \oplus R_2$ is *irreflexive*.
- (2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.
- (3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Solution:

- (1) Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \oplus R_2$ for all $a \in A$. Thus, $R_1 \oplus R_2$ is irreflexive.
- (2) Yes. Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \cap R_2$.
- (3) Yes. Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \cup R_2$.

□

Q.6 Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$.

- (a) Show that R is an equivalence relation.
- (b) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation R ?
- (c) Give an interpretation of the equivalence classes for the equivalence relation R .

Solution:

- (a) For reflexivity, $((a, b), (a, b)) \in R$ because $a \cdot b = b \cdot a$. If $((a, b), (c, d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c, d), (a, b)) \in R$; this tells us that R is symmetric. Finally, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a, b), (e, f)) \in R$; this tells us that R is transitive.
- (b) The equivalence classes of $(1, 2)$ is the set of all pairs (a, b) such that the fraction a/b equals $1/2$.
- (c) The equivalence classes are the positive rational numbers.

□

Q.7 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a, b)R(c, d)$ if and only if $a + d = b + c$ is an *equivalence* relation.

Solution: $((a, b), (a, b)) \in R$ because $a + b = a + b$. Hence R is reflexive.

If $((a, b), (c, d)) \in R$ then $a + d = b + c$, so that $c + b = d + a$. It then follows that $((c, d), (a, b)) \in R$. Hence R is symmetric.

Suppose that $((a, b), (c, d))$ and $((c, d), (e, f))$ belong to R . Then $a + d = b + c$ and $c + f = d + e$. Adding these two equations and subtracting $c + d$ from both sides gives $a + f = b + e$. Hence $((a, b), (e, f))$ belongs to R . Hence, R is transitive.

□

Q.8 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?

Solution: 25. There are two possibilities to form exactly three different equivalence classes with 5 elements. One is 3, 1, 1 elements for each equivalence class, and the other is 2, 2, 1 elements for each equivalence class. By counting techniques, there are $\binom{5}{3} + \binom{5}{1} \cdot \binom{4}{2} / 2 = 25$.

□

Q.9 Let A be a set, let R and S be relations on the set A . Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both *equivalence relations*, then T is also an equivalence relation.

Solution: We need to show that T is reflexive, symmetric, and transitive.

Reflexive: For any x , we have $(x, x) \in R$ and $(x, x) \in S$, then $(x, x) \in T$.

Symmetric: Suppose that $(x, y) \in T$. This means $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both symmetric, we have $(y, x) \in R$ and $(y, x) \in S$. Then $(y, x) \in T$.

Transitive: Suppose that $(x, y) \in T$ and $(y, z) \in T$. Then $(x, y) \in R$ and $(y, x) \in R$ imply that $(x, z) \in R$. Similarly, we have $(x, z) \in S$. This will imply that $(x, z) \in T$.

□

Q.10 Let \sim be a relation defined on \mathbb{N} by the rule that $x \sim y$ if $x = 2^k y$ or $y = 2^k x$ for some $k \in \mathbb{N}$. Show that \sim is an equivalence relation.

Solution: We first show the following lemma.

Lemma For any $x, y \in \mathbb{N}$, $x \sim y$ if and only if there exists some $k \in \mathbb{Z}$ such that $x = 2^k y$ in \mathbb{Q} .

Proof. Suppose that $x \sim y$. Then either $x = 2^k y$ for some $k \in \mathbb{N} \subseteq \mathbb{Z}$ and we are done, or $y = 2^{k'} x$ for some $k' \in \mathbb{N}$. In the latter case, solve for $x = 2^{-k'} y$ and let $k = -k'$. In the other direction, if $x = 2^k y$, and $k \geq 0$, then $x = 2^k y$ for some $k \in \mathbb{N}$, giving $x \sim y$. If instead $k < 0$, then $y = 2^{-k} x$, again giving $x \sim y$.

To show \sim is an equivalence relation, we show the following three properties.

Reflexive For any $x \in \mathbb{N}$, $x = 2^0 x$ so $x \sim x$.

Symmetric If $x \sim y$, then from **Lemma** there exists $k \in \mathbb{Z}$ such that $x = 2^k y$. But then $y = 2^{-k} x$, so applying the lemma again, gives $y \sim x$.

Transitive If $x \sim y \sim z$, then $x = 2^k y$ and $y = 2^\ell z$ for some $k, \ell \in \mathbb{Z}$ by **Lemma**. Solve to get $x = 2^{k+\ell} z$, which gives $x \sim z$.

□

Q.11 Given functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g .

(a) Prove that \preceq is a partial ordering.

(b) Prove or disprove: \preceq is a total ordering.

Solution:

(a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq f(x)$ and thus $f(x) = g(x)$. Since this holds for all x , we have $f = g$.

Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$, giving $f(x) \leq h(x)$. So, $f \preceq h$.

(b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x , $f(x) \leq g(x)$, and it is not the case that for all x , $g(x) \leq f(x)$. That is, these two functions are incomparable.

□

Q.12 Which of these are posets?

- (a) $(\mathbf{R}, =)$
- (b) $(\mathbf{R}, <)$
- (c) (\mathbf{R}, \leq)
- (d) (\mathbf{R}, \neq)

Solution:

- (a) Yes. (It is the smallest partial order: reflexivity ensures that every partial order contains at least all pairs (a, a) .)
- (b) No. It is not reflexive.
- (c) Yes.
- (d) No. The relation is not reflexive, not antisymmetric, not transitive.

□

Q.13 Consider a relation \propto on the set of functions from \mathbb{N}^+ to \mathbb{R} , such that $f \propto g$ if and only if $f = O(g)$.

- (a) Is \propto an equivalence relation?
- (b) Is \propto a partial ordering?
- (c) Is \propto a total ordering?

Solution:

- (a) No. \propto is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.
- (b) No. \propto is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

- (c) No. It is not partial ordering, then not a total ordering.

□

Q.14 Answer these questions for the partial order represented by this Hasse diagram.

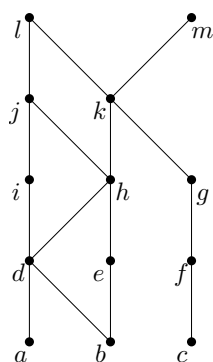


Figure 1: Q.14

- Find the maximal elements.
- Find the minimal elements.
- Is there a greatest element?
- Is there a least element?
- Find all upper bounds of $\{a, b, c\}$.
- Find the least upper bound of $\{a, b, c\}$, if it exists.
- Find all lower bounds of $\{f, g, h\}$.
- Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

- The maximal elements are the ones with no other elements above them, namely l and m .

- (b) The minimal elements are the ones with no other elements below them, namely a, b and c .
- (c) There is no greatest element, since neither l nor m is greater than the other.
- (d) There is no least elements, since neither a nor b is less than the other.
- (e) We need to find elements from which we can find downward paths to all of a, b , and c . It is clear that k, l and m are the elements fitting this description.
- (f) Since k is less than both l and m , it is the least upper bound of a, b and c .
- (g) No element is less than both f and h , so there are no lower bounds.
- (h) Since there is no lower bound, there cannot be greatest lower bound.

□

□