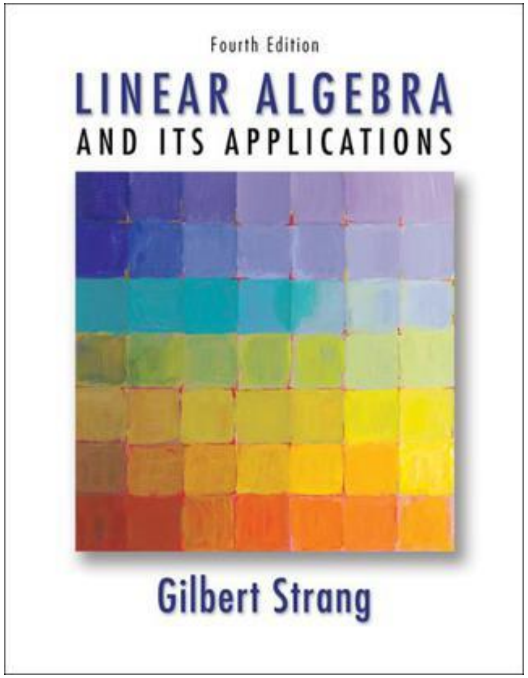
Linear Algebra



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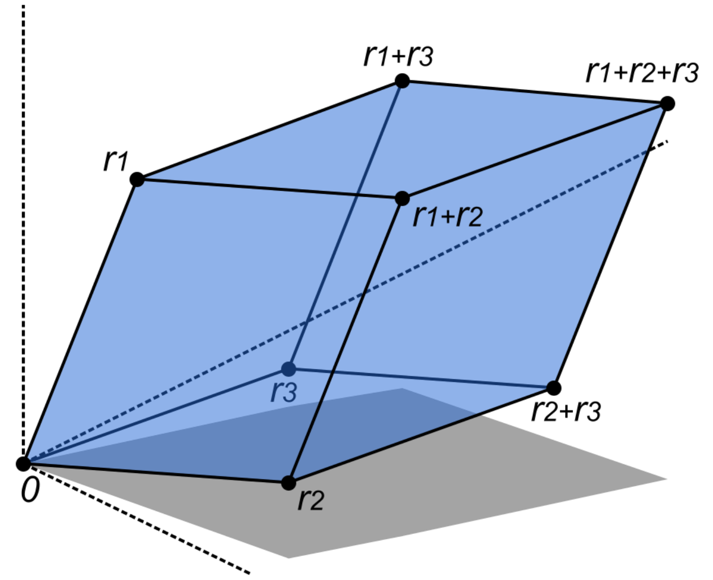
4 Determinants (行列式)

4.3

FORMULAS FOR THE DETERMINANT

- Formula from pivots
- A Big formula
- Expansion Rule
- *Calculations (Some Tricks)*



I. Formula from Pivots (行列式的计算公式：主元)

Theorem 1 *If* $\boldsymbol{A}$ *is invertible, then* $\boldsymbol{PA} = \boldsymbol{LDU}$ *and* $|\boldsymbol{P}| = \pm 1$. *The product rule gives*

$$|\boldsymbol{A}| = \pm\, |\boldsymbol{L}|\, |\boldsymbol{D}|\, |\boldsymbol{U}| = \pm\, |\boldsymbol{D}| = \pm(\textit{product of the pivots})$$

The sign ± 1 depends on whether the number of row exchanges is even (偶) or odd (奇). ($+1$ for even; -1 for odd)

For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & (ad-bc)/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix},$$

the product of the pivots is $ad - bc = |\boldsymbol{A}| = |\boldsymbol{D}|$.

If there is a row exchange, then

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}, \text{ and } |\boldsymbol{A}| = -|\boldsymbol{D}|.$$

Example 1 The $-1, 2, -1$ second difference matrix

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \frac{2}{1} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

Its determinant is the product of its pivots.

$$|A_4| = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right) = 5.$$

In general, for A_n in this pattern, we have $|A_n| = n + 1$.

II. A Big formula – An equivalent definition (行列式的等价定义)

For $n = 2$, we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c+0 & 0+d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0+d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad - bc.$$

To get nonzero terms: Suppose 1^{st} row has a nonzero term in column α, 2^{nd} row is nonzero in column β, …, and finally the n-th row in column υ.

Then the column numbers $\alpha, \beta, \ldots, \upsilon$ are all different.

They are a reordering, or *permutation* (排列), of the numbers $1, 2, \ldots, n$.

For $n = 3$, we have

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

There are $n!$ ways to permute numbers $1,2,\ldots,n$.

Column numbers: α, β, υ= (1,2,3), (2,3,1), (3,1,2), (3,2,1), (2,1,3), (1,3,2).

Formulas for the Determinant

$|\boldsymbol{A}|$

$$= a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

- Every term is a product of $n = 3$ entries a_{ij}, with ***each row and column represented once***.
- If the columns come in the order $(\alpha, \dots, \upsilon)$, that term is the product $a_{1\alpha} \dots a_{n\upsilon}$ *times* the determinant of a permutation matrix $\boldsymbol{P}$.
- The determinant of the whole matrix $\boldsymbol{A}$ is the sum of these $n!$ terms.

*This leads a **'big formula'** for computing the determinant of* $\boldsymbol{A} = [a_{ij}]_{n\times n}$.

Theorem 2 (Big formula)

$$|\boldsymbol{A}| = \sum_{all\ \boldsymbol{P}'s} (a_{1\alpha} a_{2\beta} \cdots a_{n\upsilon})\ |\boldsymbol{P}|.$$

Notes:

$|\boldsymbol{P}| = 1\ or\ -1$ for an *even* or *odd* number of row exchanges.

For example,

$$\boldsymbol{P} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix},$$

with determinants equal to $1, 1, -1$, respectively.

Equivalently, it depends on the permutation of the n numbers being an even permutation (偶排列) *or odd permutation* (奇排列).

(2,3,1), (3,1,2): even; (1,3,2): odd.

(1,3,2) requires one exchange to recover (1,2,3);

(2,3,1), (3,1,2) requires two exchanges to recover (1,2,3).

Example 2 Let $f(x)=\begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 2x & 1 \end{vmatrix}$.

Find the coefficient of x^3 .

Solution

$$f(x)=\begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 2x & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} a_{11}a_{22}a_{33}a_{44}=x^3,$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{vmatrix} a_{11}a_{22}a_{34}a_{43}=-2x^3.$$

The coefficient of x^3 is -1.

思考

设

$$f(x)=\begin{vmatrix} a_{11}+x & a_{12}+x & a_{13}+x & a_{14}+x \\ a_{21}+x & a_{22}+x & a_{23}+x & a_{24}+x \\ a_{31}+x & a_{32}+x & a_{33}+x & a_{34}+x \\ a_{41}+x & a_{42}+x & a_{43}+x & a_{44}+x \end{vmatrix},$$

则多项式 $f(x)$ 可能的最高次数为_______.

III. Expansion of det$A$ in cofactors (使用代数余子式展开行列式)

No row or column can be used twice in the same term.

For $n = 3$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

Theorem 3 *The determinant of* $\boldsymbol{A}$ *is a combination of any row* i *times its cofactors:*

$$|\boldsymbol{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor C_{ij} is the determinant of $\boldsymbol{M}_{ij}$ with the correct sign:

$$C_{ij} = (-1)^{i+j}\left|\boldsymbol{M}_{ij}\right|.$$

The submatrix $\boldsymbol{M}_{ij}$ is formed by *throwing away row* i *and column* j.

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \quad \boldsymbol{M}_{23}=\begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix},$$

$$C_{23}=(-1)^{2+3}|\boldsymbol{M}_{23}|=-|\boldsymbol{M}_{23}|.$$

注意： 一个元素的代数余子式只与该元素所处位置有关，而与该元素等于多少无关.

思考 设 $D=\begin{vmatrix} 3 & -5 & 2 & 1 \\ 1 & 1 & 0 & -5 \\ -1 & 3 & 1 & 3 \\ 2 & -4 & -1 & -3 \end{vmatrix}$, 求 $2C_{11}-4C_{12}-C_{13}-3C_{14}$.

结论：某一行元素依次乘以另一行元素的代数余子式再求和，其结果等于0.

Example 1 (Continued) The $-1, 2, -1$ second difference matrix (4×4)

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

$$C_{11} = |A_3| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = |A_2|.$$

$$|A_4| = 2C_{11} - C_{12} = 2|A_3| - |A_2| = 2(4) - 3 = 5.$$

The same idea applies to every A_n: $|A_n| = 2|A_{n-1}| - |A_{n-2}|.$

By recursion (递推), we can get $|A_n| = 2(n) - (n-1) = n+1.$

IV. Some Tricks for Computing Determinants

例1 计算 $D=\begin{vmatrix} a_1+\lambda & a_2 & \cdots & a_n \\ a_1 & a_2+\lambda & \cdots & a_n \\ & \cdots & \cdots & \\ a_1 & a_2 & \cdots & a_n+\lambda \end{vmatrix}$.

解1 D 的每个列写成两个子列之和:

$$D=\begin{vmatrix} a_1+\lambda & a_2+0 & \cdots & a_n+0 \\ a_1+0 & a_2+\lambda & \cdots & a_n+0 \\ \vdots & \vdots & & \vdots \\ a_1+0 & a_2+0 & \cdots & a_n+\lambda \end{vmatrix},$$

D 可以分解为 2^n 个行列式的和.

这些行列式中只有 $n+1$ 个非零, 它们的和为

$$D=\begin{vmatrix} a_1 & & & \\ a_1 & \lambda & & \\ \vdots & & \ddots & \\ a_1 & & & \lambda \end{vmatrix}+\begin{vmatrix} \lambda & a_2 & & \\ & a_2 & & \\ & \vdots & \ddots & \\ & a_2 & & \lambda \end{vmatrix}+\cdots$$

分拆法

$$+\begin{vmatrix} \lambda & & & a_n \\ & \lambda & & a_n \\ & & \ddots & \vdots \\ & & & a_n \end{vmatrix}+\begin{vmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{vmatrix},$$

故 $$D=\lambda^{n-1}(a_1+a_2+\cdots+a_n+\lambda).$$

解2 将 D 的每列都加到第一列，得

$$D=\begin{vmatrix} a_1+a_2+\cdots+a_n+\lambda & a_2 & \cdots & a_n \\ a_1+a_2+\cdots+a_n+\lambda & a_2+\lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1+a_2+\cdots+a_n+\lambda & a_2 & \cdots & a_n+\lambda \end{vmatrix}$$

$$=(a_1+a_2+\cdots+a_n+\lambda)\begin{vmatrix} 1 & a_2 & \cdots & a_n \\ 1 & a_2+\lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ 1 & a_2 & \cdots & a_n+\lambda \end{vmatrix}.$$

将右端行列式的第i列减去第一列的a_i倍$(2\le i\le n)$, 得

$$D=(a_1+a_2+\cdots+a_n+\lambda)\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & \lambda \end{vmatrix}$$

$$=(a_1+a_2+\cdots+a_n+\lambda)\lambda^{n-1}.$$

$\boxed{\text{三角化法}}$

解3 先把第一行乘以(−1)加到以下各行，再把后面各列加到第一列.

$$D=\begin{vmatrix} a_1+\lambda & a_2 & \cdots & a_n \\ a_1 & a_2+\lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n+\lambda \end{vmatrix}=\begin{vmatrix} a_1+\lambda & a_2 & \cdots & a_n \\ -\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\lambda & 0 & \cdots & \lambda \end{vmatrix}$$

$$=\begin{vmatrix} a_1+a_2+\cdots+a_n+\lambda & a_2 & \cdots & a_n \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda \end{vmatrix}$$

三角化法

$$=(a_1+a_2+\cdots+a_n+\lambda)\lambda^{n-1}.$$

解4 先加上一行一列, 化为一个 $n+1$ 阶行列式:

$$D=\begin{vmatrix} a_1+\lambda & a_2 & \cdots & a_n \\ a_1 & a_2+\lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n+\lambda \end{vmatrix}=\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1+\lambda & a_2 & \cdots & a_n \\ 0 & a_1 & a_2+\lambda & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & \cdots & a_n+\lambda \end{vmatrix}_{n+1}$$

$$=\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n+1}.$$

当 $\lambda \neq 0$时, 把后n列的 $1/\lambda$倍都加到第一列, 得

$$D=\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n+1} = \begin{vmatrix} 1+\sum_{i=1}^{n}\dfrac{a_i}{\lambda} & a_1 & a_2 & \cdots & a_n \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n+1}$$

$$=\left(1+\sum_{i=1}^{n}\frac{a_i}{\lambda}\right)\lambda^n = (a_1+a_2+\cdots+a_n+\lambda)\lambda^{n-1}.$$

当 $\lambda = 0$时, 上式显然成立.

$\boxed{\text{加边法}}$

All roads lead to Rome

例2 证明**Vandermonde行列式**

$$D_n=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}=\prod_{1\le j<i\le n}(x_i-x_j).$$

证 对行列式的阶数n用数学归纳法．因为

$$D_2=\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}=x_2-x_1=\prod_{1\le j<i\le 2}(x_i-x_j),$$

所以 $n=2$ 时，等式成立．

假设等式对于 $n-1$ 阶 **Vandermonde** 行列式成立.

从第 n 行开始, 每行减去上一行的 x_1 倍, 有

$$D_n=\begin{vmatrix}1&1&\cdots&1\\x_1&x_2&\cdots&x_n\\x_1^2&x_2^2&\cdots&x_n^2\\\vdots&\vdots&&\vdots\\x_1^{n-2}&x_2^{n-2}&\cdots&x_n^{n-2}\\x_1^{n-1}&x_2^{n-1}&\cdots&x_n^{n-1}\end{vmatrix}=\begin{vmatrix}1&1&1&\cdots&1\\0&x_2-x_1&x_3-x_1&\cdots&x_n-x_1\\0&x_2(x_2-x_1)&x_3(x_3-x_1)&\cdots&x_n(x_n-x_1)\\\vdots&\vdots&\vdots&&\vdots\\0&x_2^{n-3}(x_2-x_1)&x_3^{n-3}(x_3-x_1)&\cdots&x_n^{n-3}(x_n-x_1)\\0&x_2^{n-2}(x_2-x_1)&x_3^{n-2}(x_3-x_1)&\cdots&x_n^{n-2}(x_n-x_1)\end{vmatrix}$$

按第一列展开, 并提出每列的公因子, 就有

$$D_n=(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\begin{vmatrix}1 & 1 & \cdots & 1\\ x_2 & x_3 & \cdots & x_n\\ \vdots & \vdots & & \vdots\\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2}\end{vmatrix}.$$

n-1阶**Vandermonde**行列式

因此由归纳法假设得

$$D_n=(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\prod_{2\le j<i\le n}(x_i-x_j)$$

$$=\prod_{1\le j<i\le n}(x_i-x_j).$$

归纳法

应用 设曲线 $y = a_0 + a_1x + a_2x^2 + a_3x^3$ 通过四点 $(1, 3)$、$(2, 4)$、$(3, 3)$、$(4, -3)$, 求系数 a_0, a_1, a_2, a_3 .

解 把四个点的坐标代入曲线方程, 得线性方程组

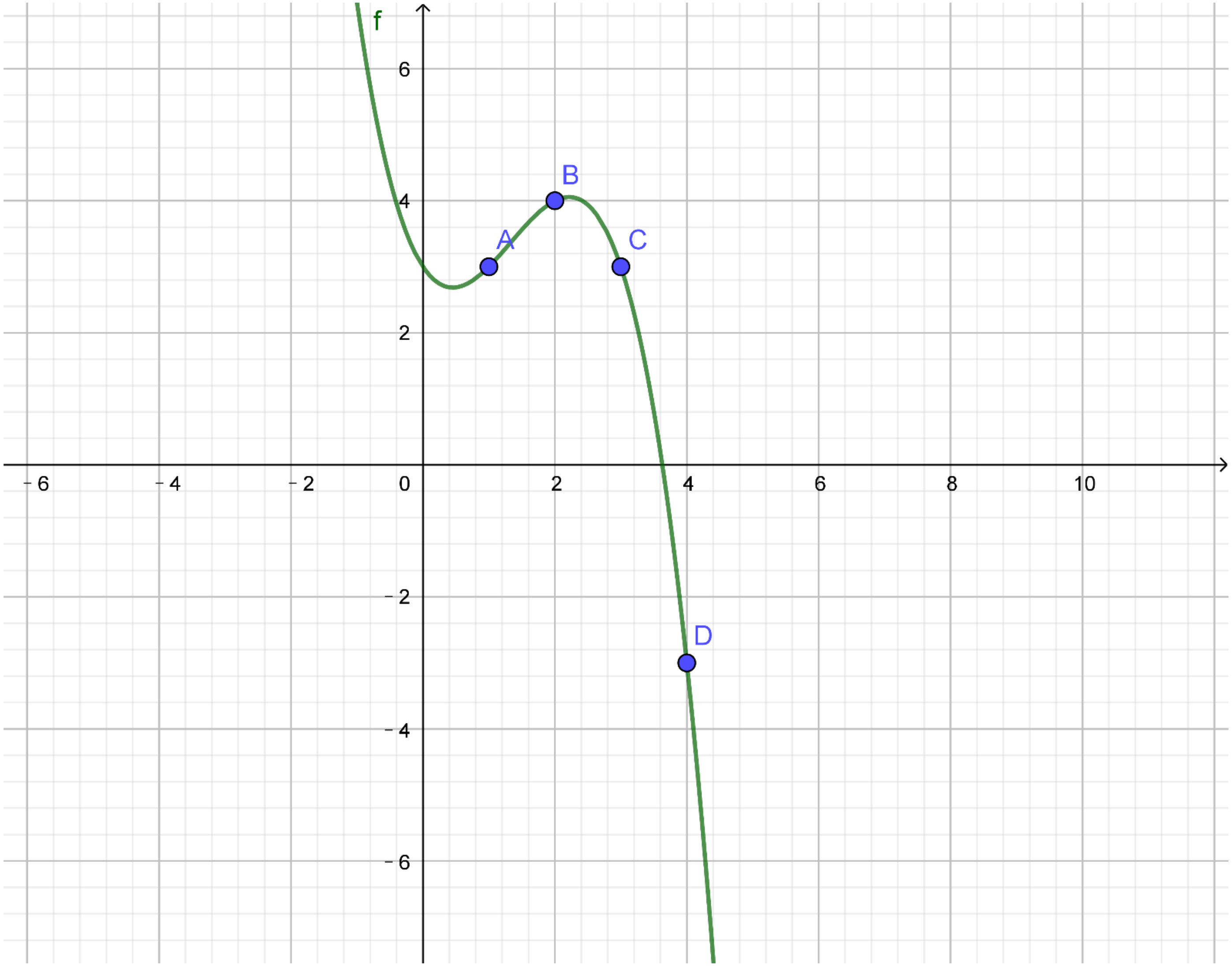
$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 3, \\ a_0 + 2a_1 + 4a_2 + 8a_3 = 4, \\ a_0 + 3a_1 + 9a_2 + 27a_3 = 3, \\ a_0 + 4a_1 + 16a_2 + 64a_3 = -3. \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix} = (4-1)(3-1)(2-1)(4-2)(3-2)(4-3) = 12,$$

Vandermonde行列式

所以方程有唯一解 $a_0 = 3, \quad a_1 = -\frac{3}{2},$
$a_2 = 2, \quad a_3 = -\frac{1}{2}$

即曲线方程为 $y = 3 - \frac{3}{2}x + 2x^2 - \frac{1}{2}x^3.$



一般地，过$n+1$个x坐标不同的点$(x_i, y_i), i = 1, \ldots, n+1$，可唯一地确定一个$n$次曲线的方程$y = a_0 + a_1 x + \cdots + a_n x^n$.

例3 计算

$$D_n = \begin{vmatrix} \alpha+\beta & \alpha\beta & & & \\ 1 & \alpha+\beta & \alpha\beta & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha+\beta & \alpha\beta \\ & & & 1 & \alpha+\beta \end{vmatrix}.$$

解 按照第一行展开, 得

$$D_n = (\alpha+\beta)\begin{vmatrix} \alpha+\beta & \alpha\beta & & \\ 1 & \alpha+\beta & \alpha\beta & \\ & \ddots & \ddots & \ddots \\ & & 1 & \alpha+\beta \end{vmatrix} - \alpha\beta\begin{vmatrix} 1 & \alpha\beta & & \\ 0 & \alpha+\beta & \alpha\beta & \\ & \ddots & \ddots & \ddots \\ & & 1 & \alpha+\beta \end{vmatrix}.$$

再把第二个行列式按照第一列展开, 得

$$D_n = (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2},$$

从而 $D_n - \alpha D_{n-1} = \beta(D_{n-1} - \alpha D_{n-2})$

$$= \beta^2(D_{n-2} - \alpha D_{n-3}) = \cdots = \beta^{n-2}(D_2 - \alpha D_1),$$

因 $D_1 = \alpha+\beta,\ D_2 = \alpha^2+\alpha\beta+\beta^2$，故

$$D_n - \alpha D_{n-1} = \beta^n.$$

于是

$$D_n = \alpha D_{n-1} + \beta^n = \alpha(\alpha D_{n-2} + \beta^{n-1}) + \beta^n$$

$$= \alpha^2 D_{n-2} + \alpha\beta^{n-1} + \beta^n = \alpha^3 D_{n-3} + \alpha^2\beta^{n-2} + \alpha\beta^{n-1} + \beta^n$$

$$= \cdots = \alpha^{n-1}D_1 + \alpha^{n-2}\beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n$$

$$= \alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n.$$

递推法

或者由 $D_n - \alpha D_{n-1} = \beta^n.$

类似可得出 $D_n - \beta D_{n-1} = \alpha^n.$

联立两式，可解得

$$D_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

$$= \alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n.$$

当 $n = 1, 2$ 时, 上式也成立.

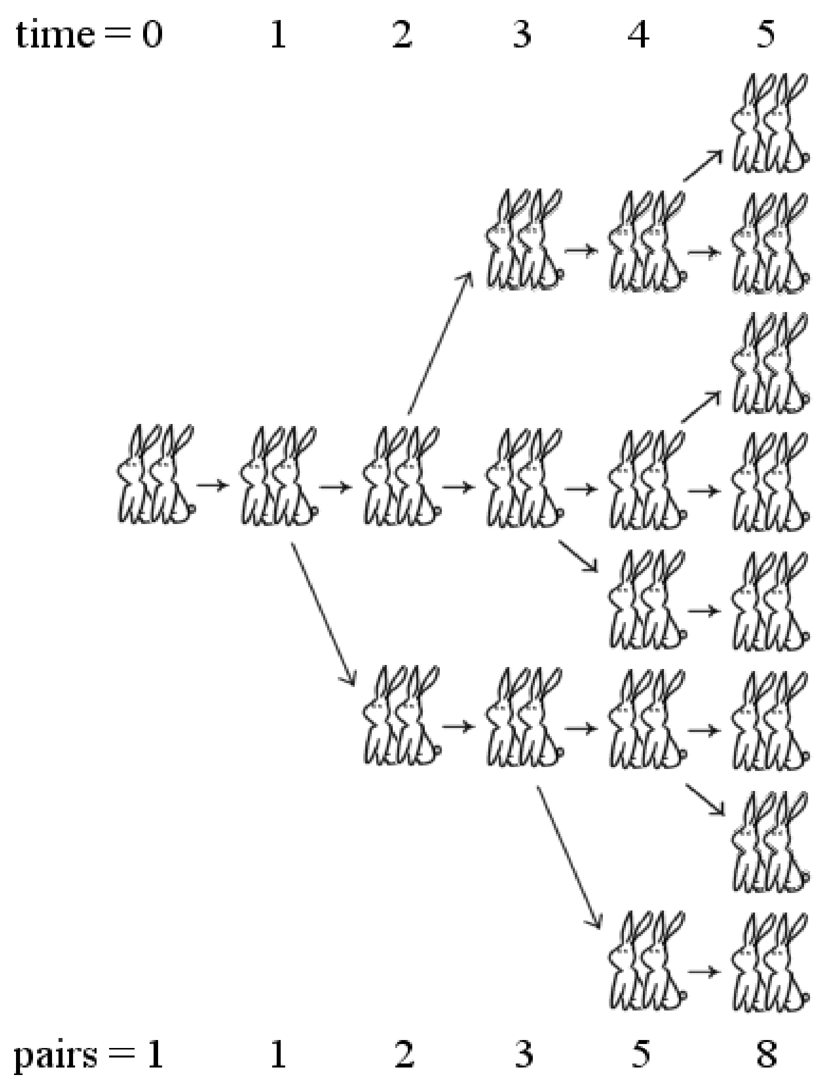
$\boxed{\text{递推法}}$

拓展应用： Fibonacci数列

Fibonacci gave this sequence as an answer to the following mathematical puzzle:

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The answer is the sequence **1, 1, 2, 3, 5, 8, 13, 21, . . .**, as illustrated below:



Fibonacci数列：1, 2, 3, 5, 8, 13, 21, 35, …

满足： $F_n = F_{n-1} + F_{n-2} (n \geq 3),\quad F_1 = 1,\quad F_2 = 2.$

(1) 证明：Fibonacci数列的通项可由下述行列式表示：

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}.$$

(2) 求Fibonacci数列的通项公式.

利用**递推法**进行计算还可推广到：

一般的n阶三对角行列式

$$D_n = \begin{vmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & c & \ddots & \ddots & \\ & & & \ddots & a & b \\ & & & & c & a \end{vmatrix}$$

$$D_n = \begin{vmatrix} \alpha+\beta & \alpha\beta & & & \\ 1 & \alpha+\beta & \alpha\beta & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha+\beta & \alpha\beta \\ & & & 1 & \alpha+\beta \end{vmatrix}.$$

观察 下面行列式的元素有何特点?

$$D=\begin{vmatrix} \alpha & \beta & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 & 0 \\ x_1 & y_1 & b & c & c \\ x_2 & y_2 & c & b & c \\ x_3 & y_3 & c & c & b \end{vmatrix}.$$

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定义 在 n 阶行列式 D 中, 任取 k 行 $i_1, i_2, \ldots, i_k$ 与 k 列 $j_1, j_2, \ldots, j_k$, $k \le n$, 将这些行与列交叉处的元素按原来相对位置构成的 k 阶子矩阵, 记为$\boldsymbol{N}$. 其行列式, 称为 D 的一个 **k 阶子式**, 记为 $|\boldsymbol{N}|$.

划去这些行和列后所剩下的元素依原次序构成的一个 $n-k$ 阶子矩阵, 记为 $\boldsymbol{M}$.

称

$$C = (-1)^{i_1+i_2+\cdots+i_k+j_1+j_2+\cdots+j_k} \, |\boldsymbol{M}|$$

为 $|\boldsymbol{N}|$ 的**代数余子式**.

例如,
$$D=\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ -3 & 3 & -7 & 9 & -5 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{vmatrix},$$

二阶子式

$$\begin{vmatrix} -1 & 2 \\ -5 & 7 \end{vmatrix},$$

其代数余子式

$$C=(-1)^{1+4+2+3}\begin{vmatrix} -3 & 9 & -5 \\ 2 & -2 & 1 \\ 4 & -10 & 2 \end{vmatrix}.$$

定理 (Laplace) 在 n 阶行列式 D 中, 任取 k 行(列), 由这 k 行(列)所组成的一切 k 阶子式与它们的代数余子式的乘积的和等于行列式 D.

例如,
$$D=\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ -3 & 3 & -7 & 9 & -5 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{vmatrix},$$

取定 1, 3, 4 行后, D 可以表示为 $C_5^3=10$ 个乘积的和.

例4 计算$D=\begin{vmatrix}1&2&1&4\\0&-1&2&1\\1&0&1&3\\0&1&3&1\end{vmatrix}$.

在行列式中取定第一、二行, 得到6个子式:

$$\begin{vmatrix}1&2\\0&-1\end{vmatrix},\begin{vmatrix}1&1\\0&2\end{vmatrix},\begin{vmatrix}1&4\\0&1\end{vmatrix},\begin{vmatrix}2&1\\-1&2\end{vmatrix},\begin{vmatrix}2&4\\-1&1\end{vmatrix},\begin{vmatrix}1&4\\2&1\end{vmatrix}.$$

它们对应的代数余子式为

$$(-1)^{(1+2)+(1+2)}\begin{vmatrix}1&3\\3&1\end{vmatrix}=\begin{vmatrix}1&3\\3&1\end{vmatrix},$$

$$(-1)^{(1+2)+(1+3)}\begin{vmatrix}0&3\\1&1\end{vmatrix}=-\begin{vmatrix}0&3\\1&1\end{vmatrix},\cdots\cdots$$

根据Laplace定理, 得

$$\begin{aligned}D&=(-1)\times(-8)-2\times(-3)+1\times(-1)\\&\quad+5\times1-6\times3+(-7)\times1\\&=-7.\end{aligned}$$

例5 设

$$D = \begin{vmatrix} \boxed{\begin{matrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{matrix}} & \mathbf{0} \\ \begin{matrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nk} \end{matrix} & \boxed{\begin{matrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{matrix}} \end{vmatrix},$$

$\boxed{\text{分块法}}$

$$D_1 = \det[a_{ij}] = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}, \quad D_2 = \det[b_{ij}] = \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix},$$

证明 $D = D_1 D_2$.

思考 设 $D=\begin{vmatrix} c_{11} & \cdots & c_{1n} & a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{kn} & a_{k1} & \cdots & a_{kk} \\ b_{11} & \cdots & b_{1n} & & & \\ \vdots & \ddots & \vdots & & \mathbf{0} & \\ b_{n1} & \cdots & b_{nn} & & & \end{vmatrix}$,

$$D_1=\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}, \quad D_2=\begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix},$$

那么 $D=-D_1D_2$?

$$\boxed{D=(-1)^{kn}D_1D_2.}$$

例6 证明

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & 0 & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{vmatrix}=0$$

证 选定第4, 5, 6行, 在这三行中再任意选定三列一共可以构成 $C_6^3=20$ 个不同的三阶行列式，但每个行列式中至少有一列的元素全部为0, 故 $D=0$.

例7 设A, B, C, D 都是 n 阶矩阵，$|A| \neq 0, AC = CA$，证明:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$$

证 做分块初等行变换化矩阵为上三角分块阵，

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & -CA^{-1}B + D \end{bmatrix}$$

两边再取行列式

$$\boxed{\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||-CA^{-1}B + D|}$$

$$= |-ACA^{-1}B + AD| = |-CB + AD| = |AD - CB|$$

这里用到了 $AC = CA$.

例8 已知 A, B 为 n 阶矩阵, 证明 $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|$.

证 利用左乘和右乘块初等矩阵构造出$A+B$, $A-B$ 和上三角块阵. 如第1行加到第2行; 再第2列乘$(-I)$加到第1列.

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} A & B \\ A+B & A+B \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} = \begin{bmatrix} A & B \\ A+B & A+B \end{bmatrix}\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$$
$$= \begin{bmatrix} A-B & B \\ 0 & A+B \end{bmatrix}$$

上式两边取行列式

因为 $\begin{vmatrix} I & 0 \\ I & I \end{vmatrix} = \begin{vmatrix} I & 0 \\ -I & I \end{vmatrix} = 1$, 所以 $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|$.

例9 (抽象型行列式)

设4阶矩阵$\boldsymbol{A}=[\boldsymbol{\alpha},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3]$, $\boldsymbol{B}=[\boldsymbol{\beta},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3]$, 其中$\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3$是4维列向量, 且$|\boldsymbol{A}|=3$, $|\boldsymbol{B}|=-1$, 求$|\boldsymbol{A}+2\boldsymbol{B}|$.

解 $\boldsymbol{A}+2\boldsymbol{B}=[\boldsymbol{\alpha}+2\boldsymbol{\beta},3\boldsymbol{\gamma}_1,3\boldsymbol{\gamma}_2,3\boldsymbol{\gamma}_3]$.

因此

$$|\boldsymbol{A}+2\boldsymbol{B}|=|\boldsymbol{\alpha}+2\boldsymbol{\beta},3\boldsymbol{\gamma}_1,3\boldsymbol{\gamma}_2,3\boldsymbol{\gamma}_3|$$

$$=3^3|\boldsymbol{\alpha}+2\boldsymbol{\beta},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3| \quad \text{因子能提}$$

$$=3^3(|\boldsymbol{\alpha},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3|+2|\boldsymbol{\beta},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3|) \quad \text{行列可拆}$$

$$=27.$$

例10 (抽象型行列式)

已知$\boldsymbol{A}$是3阶矩阵，$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$是3维线性无关的列向量，若$\boldsymbol{A\alpha}_1=\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2, \boldsymbol{A\alpha}_2=\boldsymbol{\alpha}_2+\boldsymbol{\alpha}_3, \boldsymbol{A\alpha}_3=\boldsymbol{\alpha}_3+\boldsymbol{\alpha}_1$，求$|\boldsymbol{A}|$.

解 $\boldsymbol{A}[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3]=[\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2+\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_3+\boldsymbol{\alpha}_1]$.

即 $\boldsymbol{A}[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3]=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3]\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}$.

记$\boldsymbol{P}=[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3]$，则$\boldsymbol{P}$可逆，从而

$$|\boldsymbol{A}|=\begin{vmatrix}1&0&1\\1&1&0\\0&1&1\end{vmatrix}=2.$$

Key words:

Formula from pivots

The Big Formula

Expansion

Calculations (Tricks)

Homework

See Blackboard