

Probability and Statistics

Tutorial 1

Siyi Wang

Southern University of Science and Technology

11951002@mail.sustech.edu.cn

September 16, 2020

Outline

- 1 General Information
- 2 Review
- 3 Homework
- 4 Supplement Exercises
- 5 Further Reading

Quiz Arrangement

- 4th, 7th, 11th, 15th week
- Quiz length: 50 minutes

Homework

- Deadline: Every Mon 18:30
- All homework last week

Midterm

- Time: Nov 8th 10:00-12:00 a.m.

1. Probability Theory

- A branch of Mathematics.
- Take it as a mathematical model of chance phenomenon.
(Motivation)

2. Probability Space

- A probability space is a triple (Ω, \mathcal{F}, P) .
- Ω : Sample space. (The **set** of all possible outcomes.)
- \mathcal{F} : Event space. (A **collection** of some **subsets** of Ω .) (' σ -field').
- P : Probability measure. (Defined on \mathcal{F}).

3. Set/Event Operation

- Commutative Law: $A \cap B = B \cap A$, $A \cup B = B \cup A$.
- Associative Law: $A \cap (B \cap C) = (A \cap B) \cap C$,
 $A \cup (B \cup C) = (A \cup B) \cup C$.
(Hence, $A \cap B \cap C$ and $A \cup B \cup C$ are well-defined.)
- Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- De Morgan's Law: $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$. (Then, we have
 $\overline{\bigcup_{i=1}^n A_i} = \bigcap_{i=1}^n \bar{A}_i$ and $\overline{\bigcap_{i=1}^n A_i} = \bigcup_{i=1}^n \bar{A}_i$.)
- $A = B \iff A \subset B, B \subset A$.
- $A \subset B \iff$ For any $x \in A$, we have $x \in B$.
- $A - B = A \cap \bar{B}$ and $(A - B) \cup B = A \cup B$.

4. Definition of Probability Measure P in (Ω, \mathcal{F}, P)

- $P : \mathcal{F} \rightarrow [0, 1]$.
- $P(\Omega) = 1$.
- $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$, where $A_i \cap A_j = \emptyset$, for any $i \neq j$. (Countable Additivity)

5. Countable Additivity and Finite Additivity.

- $P(\bigcup_{i=1}^N A_i) = \sum_{i=1}^N P(A_i)$, where $A_i \cap A_j = \emptyset$, for any $i \neq j$. (Finite Additivity)
- Countable Additivity implies Finite Additivity.

Homework

P20, 5

Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

Solution

We have $\{A \text{ or } B \text{ occurs}\} = A \cup B$ and $\{\text{Both } A \text{ and } B \text{ occur}\} = A \cap B$. Then, $C = (A \cup B) \cap \overline{(A \cap B)}$.

P20, 6

Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

- List the sample space.
- List the elements that make up the following events: (1) A = the sum of the two values is at least 5, (2) B = the value of the first die is higher than the value of the second, (3) C = the first value is 4.
- List the elements of the following events: (1) $A \cap C$, (2) $B \cup C$, (3) $A \cap (B \cup C)$.

Homework

Solution

a. $\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

b. (1) $A = \{(1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

(2) $B = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$.

(3) $C = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$.

c. (1) $A \cap C = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$.

(2) $B \cup C = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$.

(3) $A \cap (B \cup C) = \{(3,2), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$.

Exercise 1

1. 写出下列随机试验的样本空间：

(1) 抛三枚硬币；

(2) 抛三颗骰子；

(3) 连续抛一枚硬币，直至出现正面为止；

(4) 口袋中有黑、白、红球各一个，从中任取两个球；先从中取出一个，放回后再取出一个；

(5) 口袋中有黑、白、红球各一个，从中任取两个球；先从中取出一个，不放回后再取出一个。

Solution

解 (1) $\Omega = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$, 共含有 $2^3 = 8$ 个样本点, 其中 0 表示反面, 1 表示正面. (3) 中的 0 与 1 也是此意.

(2) $\Omega = \{(x,y,z) : x,y,z = 1,2,3,4,5,6\}$, 共含有 $6^3 = 216$ 个样本点.

(3) $\Omega = \{(1), (0,1), (0,0,1), (0,0,0,1), \dots\}$, 共含有可列个样本点.

(4) $\Omega = \{\text{黑黑}, \text{黑白}, \text{黑红}, \text{白黑}, \text{白白}, \text{白红}, \text{红黑}, \text{红白}, \text{红红}\}$.

(5) $\Omega = \{\text{黑白}, \text{黑红}, \text{白黑}, \text{白红}, \text{红黑}, \text{红白}\}$.

Exercise 2

3. 设 A, B, C 为三事件, 试表示下列事件:

(1) A, B, C 都发生或都不发生;

(2) A, B, C 中不多于一个发生;

(3) A, B, C 中不多于两个发生;

(4) A, B, C 中至少有两个发生.

Solution

解 (1) $ABC \cup \bar{A} \bar{B} \bar{C}.$

(2) $\bar{A} \bar{B} \bar{C} \cup A \bar{B} \bar{C} \cup \bar{A} B \bar{C} \cup \bar{A} \bar{B} C.$

(3) $\Omega - ABC = \overline{ABC} = \bar{A} \cup \bar{B} \cup \bar{C}.$

(4) $AB \cup AC \cup BC.$

Exercise 3

1. 设随机事件 A, B 满足条件 $AB = \bar{A}\bar{B}$. 试求 $A \cup B$. ←

Solution

Since $A \cap B = \overline{A} \cap \overline{B}$, then we have $\overline{A} \cap \overline{B} = A \cap B \cap \overline{B} = \emptyset$. Hence, $A \cup B = \overline{(\overline{A} \cap \overline{B})} = \overline{\emptyset} = \Omega$.

Exercise 4

2. 试把事件 $A_1 \cup A_2 \cup \Lambda \cup A_n$ 表示成 n 个两两互不相容事件之并. \leftarrow

Supplement Exercises

Solution

Let $B_1 = A_1$ and $B_k = A_k - \bigcup_{i=1}^{k-1} A_i$ for $k = 2, \dots, n$. Then, we will prove the following by induction: $\bigcup_{i=1}^k B_i = \bigcup_{i=1}^k A_i$, for $k = 1, \dots, n$.

First, we have $B_1 = A_1$.

Then, assume we have $\bigcup_{i=1}^k B_i = \bigcup_{i=1}^k A_i$, then we consider $\bigcup_{i=1}^{k+1} B_i$. We have

$$\bigcup_{i=1}^{k+1} B_i = B_{k+1} \cup \left(\bigcup_{i=1}^k B_i \right) = \left(A_{k+1} - \bigcup_{i=1}^k A_i \right) \cup \left(\bigcup_{i=1}^k A_i \right) = \bigcup_{i=1}^{k+1} A_i.$$

Hence, by induction, we have $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$.

Now we need to prove $B_i \cap B_j = \emptyset$, for $i \neq j$.

Assume $i < j$, then we have

$$B_i \cap B_j = B_i \cap \left(A_j - \bigcup_{l=1}^{j-1} A_l \right) = B_i \cap \left(A_j - \bigcup_{l=1}^{j-1} B_l \right) = \emptyset.$$

Exercise 5

4. 指出下列事件等式成立的条件.

(1) $A \cup B = A;$

(2) $AB = A.$

Solution

解 (1) $A \supset B$;
(2) $A \subset B$.

Exercise 6

5. 设 X 为随机变量, 其样本空间为 $\Omega = \{0 \leq X \leq 2\}$, 记事件 $A = \{0.5 < X \leq 1\}$, $B = \{0.25 \leq X < 1.5\}$, 写出下列各事件:

(1) $\bar{A}B$; (2) $\bar{A} \cup B$; (3) \overline{AB} ; (4) $\overline{A \cup B}$.

Solution

$$(1) \overline{AB} = (\{0 \leq X \leq 0.5\} \cup \{1 < X \leq 2\}) \cap \{0.25 \leq X < 1.5\} \\ = \{0.25 \leq X \leq 0.5\} \cup \{1 < X < 1.5\}.$$

$$(2) \overline{A} \cup B = \{0 \leq x \leq 2\} = \Omega.$$

(3) 由于 $A \subset B$, 所以 $AB = A$, 故

$$\overline{AB} = \overline{A} = \{0 \leq X \leq 0.5\} \cup \{1 < X \leq 2\}.$$

(4) 由于 $A \subset B$, 所以 $A \cup B = B$, 故

$$\overline{A \cup B} = \overline{B} = \{0 \leq X < 0.25\} \cup \{1.5 \leq X \leq 2\}.$$

1. Countability

- (Definition) For a set A , if there exists a bijection $f : A \rightarrow \mathbb{N}$, then we say A is **countable/countable infinite**.
- Finite set: $\{1\}, \{1, 2, 3\}$.
- Countable set: $\mathbb{N}, \{-2, -1, 0, 1, 2, \dots\}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}^2$.
- Uncountable infinite set: $\mathbb{R}, [0, 1], \mathbb{R} \setminus \mathbb{Q}$.

2. Self-study Resources and Methods

- Stack Exchange: Math Stack Exchange, Mathoverflow, Stat Stack Exchange, Quant Stack Exchange, ...
- If you want to learn some topic by yourself, you can google 'topic+lecture notes' to find some notes from top university.
- Wikipedia
- Self-study is essential in university.

3. Some Interesting Math Channel.

- 3Blue1Brown
- Numberphile

Thank you!