Probability and Statistics
Tutorial 5

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Outline

Review

1. Classification of Random Variable
   - Discrete r.v.: T finite or countable.
   - (Generalized) Continuous r.v.
     - (Absolute) Continuous r.v. (This is the Continuous r.v. in this course)
     - Singular Continuous r.v.
     - Mixture
   - Mixed Type r.v.
2. Continuous r.v. X
   - (Definition) Continuous r.v. is the r.v. that admits the PDF $f_X(x)$ such that $F(x) = \int_{-\infty}^{x} f_X(x)dx$.
   - (Property) The CDF of continuous r.v. is continuous.
   - (Property) $P(X = x) = 0$ for any $x \in \mathrm{R}$.
   - (Property) The PDF $f_X(x) = F'_X(x)$.
   - (Property) $\int_{-\infty}^{+\infty} f_X(x)dx = 1$.
   - (Property) The PDF $f_X(x) \geq 0$.
   - (Property) $P(x_2 < x \leq x_1) = P(x_2 \leq x \leq x_1) = P(x_2 < x < x_1) = P(x_2 \leq x < x_1) = \int_{x_2}^{x_1} f_X(x)dx$.

Review

3. Uniform Distribution $U \sim Uniform(a, b)$
   - $f_U(u) = \frac{1}{b-a}$, $u \in (a, b)$.
   - $P(U \in (c, d)) = \frac{d-c}{b-a}$, for $(c, d) \subset (a, b)$.
4. Exponential Distribution $X \sim Exp(\lambda)$
   - $f_X(x) = \lambda e^{-\lambda x}$, $x > 0$.
   - $F_X(x) = 1 - e^{-\lambda x}$, $x > 0$.
   - (Memoryless Property) $P(X > t + s | X > s) = P(X > t)$.

例 假定自动取款机对每位顾客的服务时间（单位：分钟）服从 $\lambda=1/3$ 的指数分布. 如果有一顾客恰好在你前头走到空闲的取款机，求：

(1) 你至少等候3分钟的概率；

(2) 你等候时间在3分钟至6分钟之间的概率.

注 如果你到达时取款机正在为一名顾客服务，同时没有其他人在排队等候，问题的答案又如何？

由指数分布的无记忆性，取款机还需要花在你前面顾客身上的服务时间，与他刚到取款机相同，从而问题的答案不变.

Review

5. Normal Distribution $X \sim N(\mu, \sigma^2)$
   - $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathrm{R}$.
   - $\Phi(x) = F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$
   - $\Phi(-x) = 1 - \Phi(x)$
   - $\frac{X-\mu}{\sigma} \sim N(0, 1)$.
   - (3σ) $P(-\sigma < X - \mu < \sigma) \approx 0.68$, $P(-2\sigma < X - \mu < 2\sigma) \approx 0.95$, $P(-3\sigma < X - \mu < 3\sigma) \approx 0.997$.

Review

6. $Y = g(X)$, X is a discrete r.v.
   - $P(Y = k) = P(X \in g^{-1}(k)) = \sum_{g(j)=k} P(X = j)$
7. $Y = h(X)$, X is a continuous r.v.

基本流程: 求r.v $Y = g(X)$的概率密度函数.

① 求r.v Y的分布函数 $F_Y(y) = P\{Y \leq y\}$

② 转化为关于r.v X 的概率计算问题

需用到函数 $y = g(x)$的性质!

③ 求导 $f_Y(y) = F_Y'(y)$

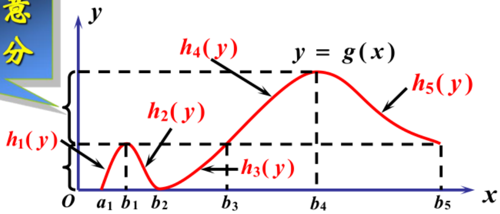
7. $Y = h(X)$, X is a continuous r.v.

推广的定理

定理 设 r.v X 的密度函数为 $f(x)$，又函数 $g(x)$ 在互不相交的区间 $(a_1, b_1), (a_2, b_2), \cdots$ 上逐段严格单调，且其反函数 $h_1(y), h_2(y), \cdots$ 均连续可导，则 $Y = g(X)$ 的密度函数为

$$f_Y(y) = \begin{cases} \sum_{i=1} |h_i'(y)| \cdot f(h_i(y)), & h_1(y), h_2(y), \cdots \text{ 有意义} \\ 0, & \text{其它} \end{cases}$$

使得反函数有意义的 y 有两部分



8. Using Uniform Distribution to generate some other distributions.
   - $U \sim Uniform(0, 1)$, The CDF of X is $F_X(x)$.
   - $Z = F_X(X)$. Then, $P(Z \leq z) = P(X \leq F^{-1}(z)) = F(F^{-1}(z)) = z$. That is, $Z \sim Uniform(0, 1)$.
   - $Y = F_X^{-1}(U)$. Then, $P(Y \leq y) = P(U \leq F_X(y)) = F_X(y)$. That is, $Y \sim X$.

Homework

33. 令 $F(x)=1-\exp(-\alpha x^{\beta})$，$x\geqslant 0$，$\alpha>0$，$\beta>0$，且当 $x<0$ 时 $F(x)=0$. 证明：F 是 cdf. 并计算其相应的密度函数.

Solution

Since $F(x)$ is continuous and nondecreasing, $F(-\infty) = 0$ and $F(\infty) = 1$, then $F(x)$ is a CDF.
The pdf of $F(x)$ is:

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \alpha\beta x^{\beta-1} exp(-\alpha x^{\beta}), & x > 0 \end{cases} \tag{1}$$

Homework

40. 假设 X 的密度函数在 $0 \leqslant x \leqslant 1$ 时，$f(x) = cx^2$，否则，$f(x) = 0$.

a. 计算 c.

b. 计算 cdf.

c. $P(0.1 \leqslant X \leqslant 0.5)$ 是多少？

Homework

Solution

a. Since $\int_0^1 cx^2 = 1$*, then* $\frac{c}{3} = 1$*. Hence,* $c = 3$*.*

b. CDF:

$$F(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases} \tag{2}$$

c. $P(0.1 \le X \le 0.5) = F(0.5) - F(0.1) = 0.124$

Homework

45. 假设电子元件的寿命服从 $\lambda = 0.1$ 的指数分布.

a. 计算寿命小于 10 的概率.

b. 计算寿命在 5 到 15 之间的概率.

c. 若寿命大于 t 的概率为 0.01, 计算 t.

Homework

Solution

a. $P(L < 10) = \int_0^{10} \lambda e^{-\lambda t} dt = 1 - e^{-0.1*10} = 1 - e^{-1}$.
b. $P(5 < L < 15) = (1 - e^{-0.1*15}) - (1 - e^{-0.1*5}) = e^{-0.5} - e^{-1.5}$.
c. *Since* $0.01 = P(L > t) = e^{-0.1t}$, *then* $t = 20 \ln 10$.

Homework

52. 假设在某个总体中，个体身高近似服从参数为 $\mu = 70$ 英寸和 $\sigma = 3$ 英寸的正态分布.

a. 身高超过 6 英尺的总体比例是多少?

b. 如果我们用厘米表示身高，那么身高的分布是什么? 用米表示呢?

Homework

Solution

a. 6 foot =72 inch.

$P(H > 72) = P(\frac{H-70}{3} > \frac{2}{3}) = 1 - \Phi(\frac{2}{3}) = 0.2154.$

b. 1 foot=0.0254 m=2.54 cm.

Then, $\mu = 177.8cm = 1.778m$ *and* $\sigma = 7.62cm = 0.0762m$. *That is,* $H \sim N(177.8, (7.62)^2)$ *cm.*

Homework

53. 令 X 是具有 $\mu = 5$ 和 $\sigma = 10$ 的正态随机变量. 计算 (a)$P(X > 10)$，(b)$P(-20 < X < 15)$，(c) 满足 $P(X > x) = 0.05$ 的 x 值.

Homework

Solution

a. $P(X > 10) = P(\frac{X-5}{10} > \frac{1}{2}) = 1 - \Phi(\frac{1}{2}) = 0.3085.$
b. $P(-20 < X < 15) = P(-2.5 < \frac{X-5}{10} < 1) = \Phi(1) - \Phi(-2.5) = \Phi(1) + \Phi(2.5) - 1 = 0.8351.$
c. $0.05 = P(X > x) = P(\frac{X-5}{10} > \frac{x-5}{10}) = 1 - \Phi(\frac{x-5}{10}),$ *then* $x \approx 21.5.$

Homework

1.已知随机变量 X 的密度函数为

$$f(x) = Ae^{-|x|}, \quad -\infty < x < +\infty,$$

求：（1）A 值；（2）$P\{0<X<1\}$;　(3)　$F(x)$.

Homework

Solution

(1) $\frac{1}{2} = \int_0^{\infty} Ae^{-x}dx = A$. Hence, $A = \frac{1}{2}$.
(2) $P(0 < X < 1) = \frac{1}{2}\int_0^1 e^{-x}dx = \frac{1}{2}(1 - e^{-1})$.
(3)

$$F(x) = \begin{cases} \frac{1}{2}e^x, & x < 0 \\ \frac{1}{2}(2 - e^{-x}), & x \geq 0. \end{cases} \tag{3}$$

Homework

2.设顾客在某银行的窗口等待服务的时间X（以分钟计）服从指数分布$E(\frac{1}{5})$.某顾客在窗口等待服务，若超过 10 分钟他就离开.他一个月要到银行 5 次，以Y表示一个月内他未等到服务而离开窗口的次数，试写出Y的分布律，并求 $P\{Y \geq 1\}$.

Homework

Solution

$P(X > 10) = \int_{10}^{\infty} \frac{1}{5} e^{-\frac{1}{5}x} dx = e^{-2}.$
$Y \sim Bin(5, e^{-2}),$ *that is,* $P(Y = k) = C_5^k e^{-2k}(1 - e^{-2})^{5-k},\ k = 0, 1, .., 5.$
$P(Y \geq 1) = 1 - P(Y = 0) = 1 - (1 - e^{-2})^5.$

54. 如果 $X \sim N(0, \sigma^2)$，计算 $Y = |X|$ 的密度.

Homework

Solution

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, $x \in \mathrm{R}$

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}, & y \geq 0 \end{cases} \tag{4}$$

59. 如果 U 是 $[-1,1]$ 上的均匀分布，计算 U^2 的密度函数.

Homework

Solution

First, the pdf of U is $f_U(x) = \frac{1}{2}$, *for* $u \in [-1, 1]$.
Then, for $u \in [0, 1]$, $P(U^2 \leq u) = P(-\sqrt{u} \leq U \leq \sqrt{u}) = \sqrt{u}$. *Hence, the pdf of* U^2 *is*

$$f_{U^2}(u) = \begin{cases} \frac{1}{2}u^{-\frac{1}{2}}, & u \in [0, 1] \\ 0, & otherwise \end{cases} \tag{5}$$

64. 证明：如果 X 具有密度函数 f_X，且 $Y = aX + b$，那么有

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Homework

Solution

Since $h(x) = ax + b$, *then* $h^{-1}(y) = \frac{y-b}{a}$, *then* $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$.

Homework

补充题1 设随机变量 X 的频率函数为

X	-2	-1	0	1	2
P	$1/5$	$1/6$	$1/5$	$1/15$	$11/30$

求 $Y=X^2$ 的频率函数.

Homework

Solution

$P(Y=0)=\frac{1}{5}$, $P(Y=1)=\frac{7}{30}$, $P(Y=4)=\frac{17}{30}$.

Homework

补充题2 设随机变量 X 的概率密度为

$$f(x)=\begin{cases}\dfrac{2x}{\pi^2} & 0<x<\pi\\ 0 & \text{其它}\end{cases}$$

求 $Y=\sin X$ 的概率密度.

Homework

Solution

$x = h_1(y) = \arcsin y$ *and* $x = h_2(y) = \pi - \arcsin y$, *for* $y \in [0, 1]$. *Then,* $f_Y(y) = \frac{1}{\sqrt{1-y^2}} \frac{2}{\pi^2} \arcsin y + \frac{1}{\sqrt{1-y^2}} \frac{2}{\pi^2}(\pi - \arcsin y) = \frac{1}{\sqrt{1-y^2}} \frac{2}{\pi}$, *for* $y \in [0, 1]$.

Homework

补充题3

设$P\{X=k\}=\left(\frac{1}{2}\right)^k, k=1,2,\ldots,$ 令

$$Y=\begin{cases} 1, & \text{当}X\text{取偶数时} \\ -1, & \text{当}X\text{取奇数时}. \end{cases}$$

求随机变量 X 的函数 Y 的分布律.

Homework

Solution

$$P(Y=1)=\sum_{k=1}^{\infty}(\tfrac{1}{2})^{2k}=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3};$$

$$P(Y=1)=\sum_{k=0}^{\infty}(\tfrac{1}{2})^{2k+1}=\frac{\frac{1}{2}}{1-\frac{1}{4}}=\frac{2}{3}.$$

Homework

补充题4

设随机变量 X 在区间（1，2）上服从均匀分布，试求随机变量$Y = e^{2x}$,的概率密度$f_Y(y)$.

Solution

$x = h(y) = \frac{1}{2} \ln y$ *and*

$$f_Y(y) = \begin{cases} \frac{1}{2y}, & y \in (e^2, e^4) \\ 0, & otherwise \end{cases} \tag{6}$$

Supplement Exercises

Exercise 1

4. 若随机变量 $K \sim N(\mu,\sigma^2)$，而方程 $x^2+4x+K=0$ 无实根的概率为 0.5，试求 μ.

Supplement Exercises

Solution

解 方程 $x^2+4x+K=0$ 无实根等价于 $16-4K<0$,所以由题意知

$$0.5=P(16-4K<0)=P(K>4)=1-\Phi\left(\frac{4-\mu}{\sigma}\right).$$

由此得知 $\mu=4$.

Exercise 2

27. 设随机变量 X 服从正态分布 $N(0,\sigma^2)$，若 $P(|X| > k) = 0.1$，试求 $P(X < k)$.

Supplement Exercises

Solution

解　由题设条件知

$$0.9 = P(-k \leqslant X \leqslant k) = \Phi(k/\sigma) - \Phi(-k/\sigma) = 2\Phi(k/\sigma) - 1,$$

由此得 $\Phi(k/\sigma) = 0.95$. 所以 $P(X < k) = \Phi(k/\sigma) = 0.95$.

Exercise 3

24. 某单位招聘员工，共有 10 000 人报考. 假设考试成绩服从正态分布，且已知 90 分以上有 359 人，60 分以下有 1 151 人. 现按考试成绩从高分到低分依次录用 2 500 人，试问被录用者中最低分为多少?

Solution

解 记 X 为考试成绩,则 $X \sim N(\mu,\sigma^2)$. 由频率估计概率知

$$0.0359 = P(X > 90) = 1 - \Phi\left(\frac{90-\mu}{\sigma}\right),$$

$$0.1151 = P(X < 60) = \Phi\left(\frac{60-\mu}{\sigma}\right),$$

上面两式可改写为

$$0.9641 = \Phi\left(\frac{90-\mu}{\sigma}\right), \qquad 0.8849 = \Phi\left(\frac{\mu-60}{\sigma}\right),$$

再查表得

$$\frac{90-\mu}{\sigma} = 1.8, \qquad \frac{\mu-60}{\sigma} = 1.2,$$

由此解得 $\mu = 72, \sigma = 10$. 设被录用者中最低分为 k,则由

$$0.25 = P(X \geqslant k) = 1 - \Phi\left(\frac{k-72}{10}\right), \quad 或 \quad 0.75 = \Phi\left(\frac{k-72}{10}\right),$$

查表得 $(k-72)/10 \geqslant 0.675$,从中解得 $k \geqslant 78.75$,因此取被录用者中最低分为 78.75 分即可.

注:当 $p < 0.5$ 时,满足等式 $\Phi(x) = p$ 的 x 在标准正态分布函数表上不易查

Exercise 4

5. 设随机变量 X 服从$(-\pi/2,\pi/2)$上的均匀分布,求随机变量 $Y=\cos X$ 的密度函数 $p_Y(y)$.

Solution

解 X 的密度函数为

$$p_X(x) = \begin{cases} 1/\pi, & -\pi/2 < x < \pi/2, \\ 0, & \text{其他}. \end{cases}$$

由于 X 在 $(-\pi/2, \pi/2)$ 内取值，所以 $Y = \cos X$ 的可能取值区间为 $(0,1)$. 在 Y 的可能取值区间外，$p_Y(y) = 0$.

当 $0 < y < 1$ 时，使 $\{Y \leq y\}$ 的 x 取值范围为两个互不相交的区间 Δ_1 和 Δ_2，其中 $\Delta_1 = (-\pi/2, -\arccos y)$，$\Delta_2 = (\arccos y, \pi/2)$. 如图 2.18.

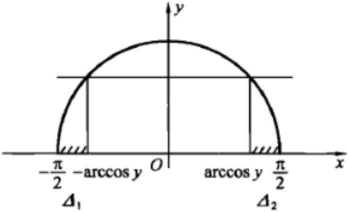


图 2.18

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Exercise 5

12. 设 $X \sim N(0,\sigma^2)$,求 $Y = X^2$ 的分布.

Supplement Exercises

Solution

解 因为 $Y = X^2$ 的可能取值区间为 $(0, +\infty)$，所以当 $y \leqslant 0$ 时，Y 的密度函数为 $p_Y(y) = 0$. 而当 $y > 0$ 时，Y 的分布函数为

$$F_Y(y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}),$$

对上式两边关于 y 求导，得

$$p_Y(y) = p_X(\sqrt{y})\frac{1}{2\sqrt{y}} + p_X(-\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{y}}\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{y}{2\sigma^2}\right\}.$$

即

$$p_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}\,\sigma}\exp\left\{-\dfrac{y}{2\sigma^2}\right\}, & y > 0, \\ 0, & \text{其他}. \end{cases}$$

这是伽玛分布 $Ga\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$.

Supplement Exercises

Exercise 6

3. 求以下分布的中位数：

(1) 区间(a,b)上的均匀分布；

(2) 正态分布 $N(\mu,\sigma^2)$；

(3) 对数正态分布 $LN(\mu,\sigma^2)$.

Supplement Exercises

Solution

解　(1) 从 $0.5=\int_a^{x_{0.5}}\frac{1}{b-a}\mathrm{d}x$ 中解得 $x_{0.5}=\frac{a+b}{2}$.

(2) 记 $X\sim N(\mu,\sigma^2)$,由 $P(X\leqslant\mu)=\Phi\left(\frac{\mu-\mu}{\sigma}\right)=0.5$ 可得 $x_{0.5}=\mu$.

(3) 记 $Y\sim LN(\mu,\sigma^2)$,令 $X=\ln Y$,则 $X\sim N(\mu,\sigma^2)$. 又记 $x_{0.5}$ 为 X 的中位数,$y_{0.5}$ 为 Y 的中位数. 则由(2)知 $x_{0.5}=\mu$,即

$$0.5=P(X\leqslant\mu)=P(\ln Y\leqslant\mu)=P(Y\leqslant e^{\mu})$$

由此得 $y_{0.5}=e^{\mu}$.

Exercise 7

36. 如果 U 是 $[0,1]$ 上的均匀随机变量，那么随机变量 $X=[nU]$ 的分布是什么？其中 $[t]$ 表示不超过 t 的最大整数.

Supplement Exercises

Solution

$P(X = k) = P(U \in [\frac{k}{n}, \frac{k+1}{n})) = \frac{1}{n},\ k = 0, 1, .., n-1.$

Exercise 8

Problem 5. Consider a triangle and a point chosen within the triangle according to the uniform probability law. Let X be the distance from the point to the base of the triangle. Given the height of the triangle, find the CDF and the PDF of X.

Solution

Solution to Problem 3.5. Let $A = bh/2$ be the area of the given triangle, where b is the length of the base, and h is the height of the triangle. From the randomly chosen point, draw a line parallel to the base, and let A_x be the area of the triangle thus formed. The height of this triangle is $h - x$ and its base has length $b(h - x)/h$. Thus $A_x = b(h - x)^2/(2h)$. For $x \in [0, h]$, we have

$$F_X(x) = 1 - \mathbf{P}(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h - x)^2/(2h)}{bh/2} = 1 - \left(\frac{h - x}{h}\right)^2,$$

while $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > h$.

The PDF is obtained by differentiating the CDF. We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \dfrac{2(h - x)}{h^2}, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 9

Problem 8. Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1 - p$.

(a) Show that the PDF of X is given by

$$f_X(x) = pf_Y(x) + (1 - p)f_Z(x).$$

(b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0, \\ (1-p)\lambda e^{-\lambda x}, & \text{if } x \geq 0, \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

Solution

Solution to Problem 3.8. (a) By the total probability theorem, we have

$$F_X(x) = \mathbf{P}(X \leq x) = p\mathbf{P}(Y \leq x) + (1-p)\mathbf{P}(Z \leq x) = pF_Y(x) + (1-p)F_Z(x).$$

By differentiating, we obtain

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Consider the random variable Y that has PDF

$$f_Y(y) = \begin{cases} \lambda e^{\lambda y}, & \text{if } y < 0 \\ 0, & \text{otherwise,} \end{cases}$$

and the random variable Z that has PDF

$$f_Z(z) = \begin{cases} \lambda e^{-\lambda z}, & \text{if } y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

We note that the random variables $-Y$ and Z are exponential. Using the CDF of the

Supplement Exercises

Exercise 10

Problem 1. If X is a random variable that is uniformly distributed between -1 and 1. find the PDF of $\sqrt{|X|}$ and the PDF of $-\ln|X|$.

Solution

Solution to Problem 4.1. Let $Y = \sqrt{|X|}$. We have, for $0 \le y \le 1$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\sqrt{|X|} \le y) = \mathbf{P}(-y^2 \le X \le y^2) = y^2,$$

and therefore by differentiation,

$$f_Y(y) = 2y, \qquad \text{for } 0 \le y \le 1.$$

Let $Y = -\ln|X|$. We have, for $y \ge 0$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\ln|X| \ge -y) = \mathbf{P}(X \ge e^{-y}) + \mathbf{P}(X \le -e^{-y}) = 1 - e^{-y},$$

and therefore by differentiation

$$f_Y(y) = e^{-y}, \qquad \text{for } y \ge 0,$$

so Y is an exponential random variable with parameter 1. This exercise provides a method for simulating an exponential random variable using a sample of a uniform random variable.

Supplement Exercises

Exercise 11

EXAMPLE 1.6(A) Consider a post office having two clerks, and suppose that when A enters the system he discovers that B is being served by one of the clerks and C by the other. Suppose also that A is told that his service will begin as soon as either B or C leaves. If the amount of time a clerk spends with a customer is exponentially distributed with mean $1/\lambda$, what is the probability that, of the three customers, A is the last to leave the post office?

Supplement Exercises

Solution

The answer is obtained by reasoning as follows: Consider the time at which A first finds a free clerk. At this point either B or C would have just left and the other one would still be in service. However, by the lack of memory of the exponential, it follows that the amount of additional time that this other person has to spend in the post office is exponentially distributed with mean $1/\lambda$. That is, it is the same as if he was just starting his service at this point Hence, by symmetry, the probability that he finishes before A must equal $\frac{1}{2}$

Supplement Exercises

Exercise 12

2.13 Consider a sequence of independent coin flips, each of which has probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trial. (For example, $X = 3$ if either TTTH or HHHT is observed.) Find the distribution of X, and find $\mathrm{E}\,X$.

Supplement Exercises

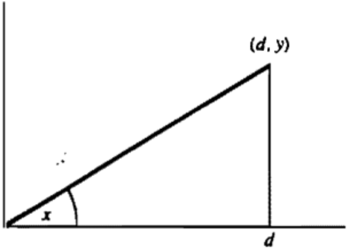
Solution

2.13 $P(X=k) = (1-p)^k p + p^k(1-p)$, $k = 1, 2, \ldots$. Therefore,

$$\begin{aligned} \mathrm{E}X &= \sum_{k=1}^{\infty} k[(1-p)^k p + p^k(1-p)] &= (1-p)p\left[\sum_{k=1}^{\infty} k(1-p)^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}\right] \\ &= (1-p)p\left[\frac{1}{p^2} + \frac{1}{(1-p)^2}\right] &= \frac{1-2p+2p^2}{p(1-p)}. \end{aligned}$$

Exercise 13

2.12 A random right triangle can be constructed in the following manner. Let X be a random angle whose distribution is uniform on $(0, \pi/2)$. For each X, construct a triangle as pictured below. Here, Y = height of the random triangle. For a fixed constant d, find the distribution of Y and $\mathrm{E}\,Y$.



Supplement Exercises

Solution

2.12 We have $\tan x = y/d$, therefore $\tan^{-1}(y/d) = x$ and $\frac{d}{dy}\tan^{-1}(y/d) = \frac{1}{1+(y/d)^2}\,\frac{1}{d}dy = dx$. Thus,

$$f_Y(y) = \frac{2}{\pi d}\,\frac{1}{1+(y/d)^2}, \quad 0 < y < \infty.$$

This is the Cauchy distribution restricted to $(0, \infty)$, and the mean is infinite.

Exercise 14

2.7 Let X have pdf $f_X(x) = \frac{2}{9}(x+1), \quad -1 \leq x \leq 2.$

(a) Find the pdf of $Y = X^2$. Note that Theorem 2.1.8 is not directly applicable in this problem.

Solution

a. Theorem 2.1.8 does not directly apply. Instead write

$$\begin{aligned} P(Y \le y) &= P(X^2 \le y) \\ &= \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}) & \text{if } |x| \le 1 \\ P(1 \le X \le \sqrt{y}) & \text{if } x \ge 1 \end{cases} \\ &= \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx & \text{if } |x| \le 1 \\ \int_1^{\sqrt{y}} f_X(x)dx & \text{if } x \ge 1 \end{cases}. \end{aligned}$$

Differentiation gives

$$f_y(y) = \begin{cases} \frac{2}{9}\frac{1}{\sqrt{y}} & \text{if } y \le 1 \\ \frac{1}{9} + \frac{1}{9}\frac{1}{\sqrt{y}} & \text{if } y \ge 1 \end{cases}.$$

Supplement Exercises

Exercise 15

definition of $F_X^{-1}(y)$.

2.9 If the random variable X has pdf

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise,} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a uniform(0, 1) distribution.

Solution

2.9 From the probability integral transformation, Theorem 2.1.10, we know that if $u(x) = F_x(x)$, then $F_x(X) \sim \text{uniform}(0,1)$. Therefore, for the given pdf, calculate

$$u(x) = F_x(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ (x-1)^2/4 & \text{if } 1 < x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

Thank you!