



CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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First-Order Linear Recurrences

- A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a *first-order linear recurrence*.
 - ◇ **First Order** because it only depends upon going back one step, i.e., $T(n-1)$
If it depends upon $T(n-2)$, it would be a **second-order** recurrence, e.g., $T(n) = T(n-1) + 2T(n-2)$.
 - ◇ **Linear** because $T(n-1)$ only appears to the **first power**.
Something like $T(n) = (T(n-1))^2 + 3$ would be a **non-linear** first-order recurrence relation.



First-Order Linear Recurrences

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First-Order Linear Recurrences

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When $f(n)$ is a constant, say r , the general solution is almost as easy as we derived before. Iterating the recurrence gives

$$\begin{aligned}T(n) &= rT(n-1) + g(n) \\&= r(rT(n-2) + g(n-1)) + g(n) \\&= r^2T(n-2) + rg(n-1) + g(n) \\&= r^3T(n-3) + r^2g(n-2) + rg(n-1) + g(n) \\&\vdots \\&= r^nT(0) + \sum_{i=0}^{n-1} r^i g(n-i)\end{aligned}$$



First-Order Linear Recurrences

- **Theorem** For any positive constants a and r , and any function g defined on nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^n r^{n-i} g(i).$$



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Proof by induction



Examples

- Solve $T(n) = 4T(n-1) + 2^n$ with $T(0) = 6$



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$$\begin{aligned}T(n) &= 6 \cdot 4^n + \sum_{i=1}^n 4^{n-i} \cdot 2^i \\&= 6 \cdot 4^n + 4^n \sum_{i=1}^n 4^{-i} \cdot 2^i \\&= 6 \cdot 4^n + 4^n \sum_{i=1}^n \left(\frac{1}{2}\right)^i \\&= 6 \cdot 4^n + \left(1 - \frac{1}{2^n}\right) \cdot 4^n \\&= 7 \cdot 4^n - 2^n.\end{aligned}$$



Examples

- Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$



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$$\begin{aligned} T(n) &= 10 \cdot 3^n + \sum_{i=1}^n 3^{n-i} \cdot i \\ &= 10 \cdot 3^n + 3^n \sum_{i=1}^n i \cdot 3^{-i} \end{aligned}$$



Examples

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Theorem. For any real number $x \neq 1$,

$$\sum_{i=1}^n ix^i = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.$$



Examples

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$$\begin{aligned}T(n) &= 10 \cdot 3^n + \sum_{i=1}^n 3^{n-i} \cdot i \\&= 10 \cdot 3^n + 3^n \sum_{i=1}^n i \cdot 3^{-i} \\&= 10 \cdot 3^n + 3^n \left(-\frac{3}{2}(n+1)3^{-(n+1)} - \frac{3}{4}3^{-(n+1)} + \frac{3}{4} \right) \\&= \frac{43}{4}3^n - \frac{n+1}{2} - \frac{1}{4}.\end{aligned}$$



Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iterating recurrences
- Three different behaviors



Divide and conquer algorithms

- We just analyzed recurrences of the form

$$T(n) = \begin{cases} b & \text{if } n = 0 \\ r \cdot T(n-1) + a & \text{if } n > 0 \end{cases}$$



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$$T(n) = r^n b + a \frac{1 - r^n}{1 - r}$$



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$$T(n) = r^n b + a \frac{1 - r^n}{1 - r}$$

We will now look at recurrences of the form

$$T(n) = \begin{cases} \text{something given} & \text{if } n \leq n_0 \\ r \cdot T(n/m) + a & \text{if } n > n_0 \end{cases}$$



Binary Search

- Someone has chosen a number x between 1 and n .
We need to discover x .



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Our strategy will be to always ask greater than questions, at each step halving our search range, until the range only contains one number, when we ask a final equal to question.



Binary Search Example

1

32

48

64

11 - 1



Binary Search Example

1

32

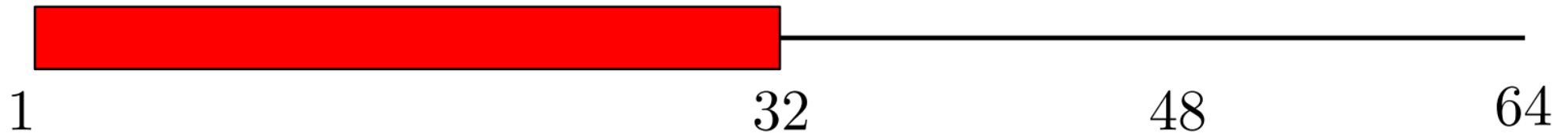
48

64

Is $x > 32$?

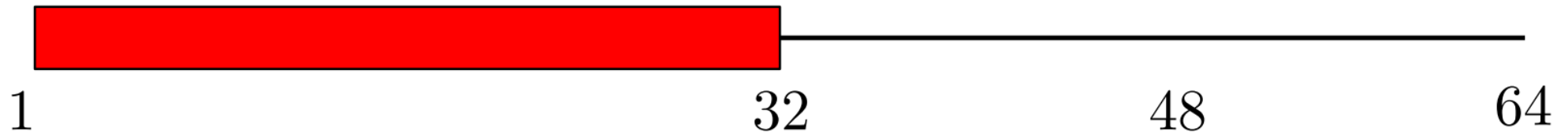


Binary Search Example



Is $x > 32$? Answer: Yes

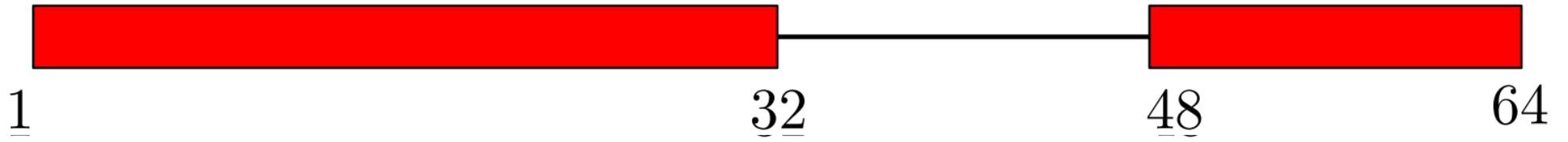
Binary Search Example



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Binary Search Example

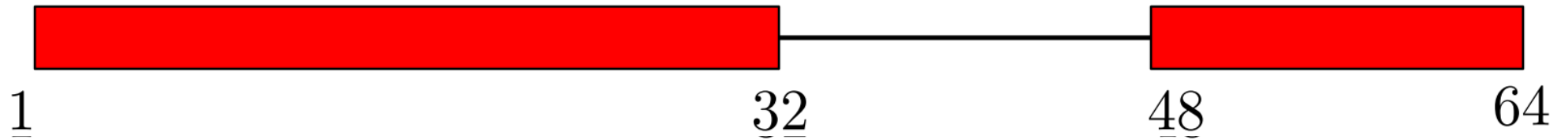


Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No



Binary Search Example



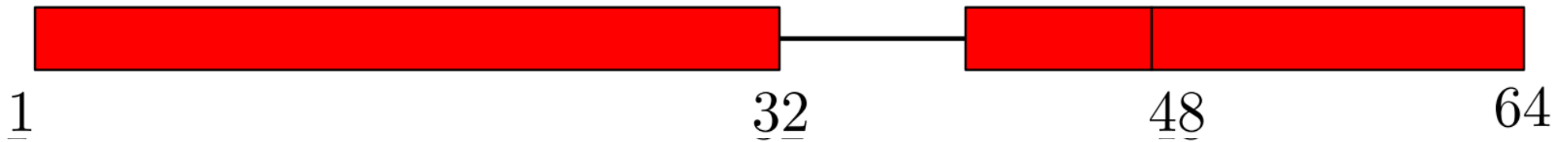
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Is $x > 40$?



Binary Search Example

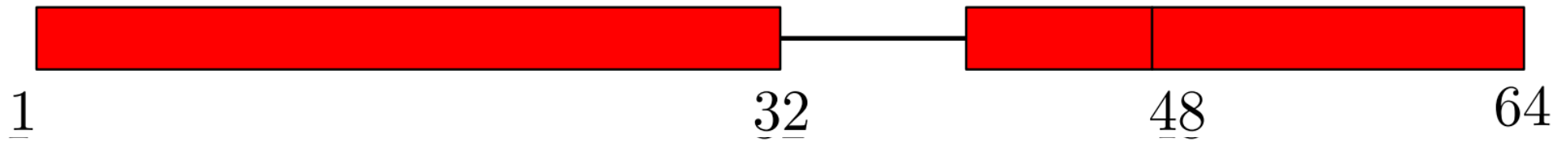


Is $x > 32$? Answer: Yes

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Is $x > 40$? Answer: No

Binary Search Example



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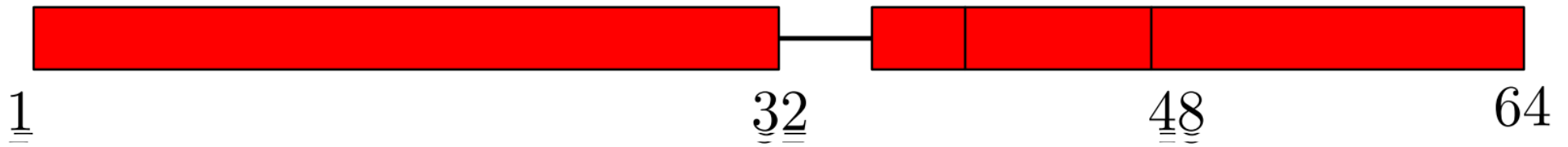
Is $x > 48$? Answer: No

Is $x > 40$? Answer: No

Is $x > 36$?



Binary Search Example



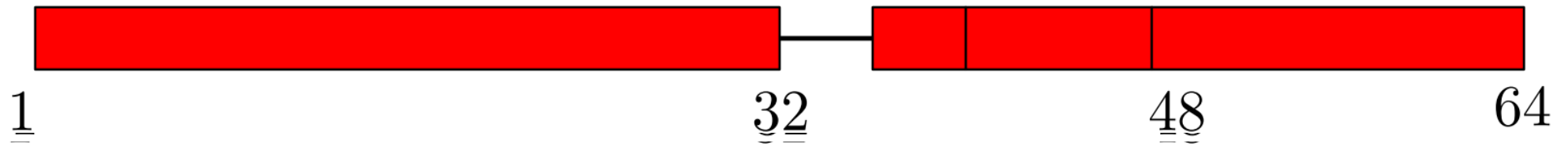
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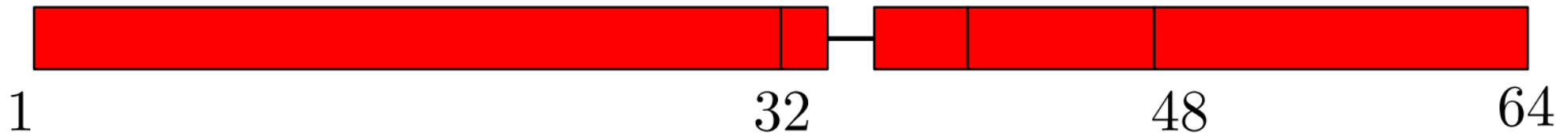
Is $x > 40$? Answer: No

Is $x > 36$? Answer: No

Is $x > 34$?



Binary Search Example



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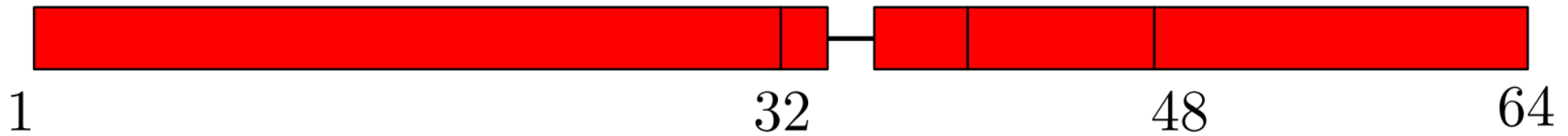
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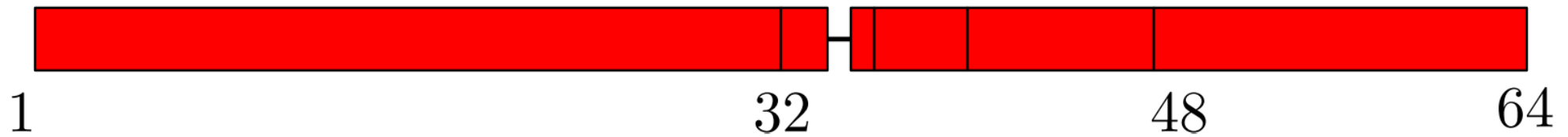
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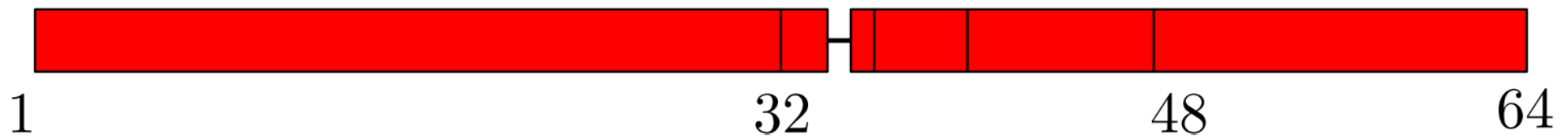
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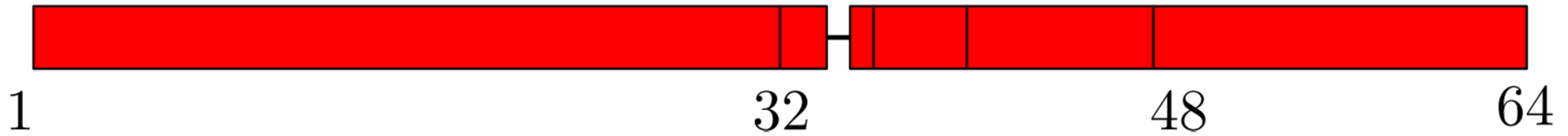
Is $x > 36$? Answer: No

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Is $x > 35$? Answer: No

Is $x = 35$?

Binary Search Example



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Is $x > 40$?	Answer: No
Is $x > 36$?	Answer: No
Is $x > 34$?	Answer: Yes
Is $x > 35$?	Answer: No
Is $x = 35$?	Answer: BINGO!



Binary Search Example

- **Method**: Each guess **reduces** the problem to one in which the range is only **half** as big.



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Note: When n is a power of 2, $T(n)$, the number of questions in a binary search on $[1, n]$, satisfies

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$



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This can also be proved **inductively**, similar to the tower of Hanoi recurrence.



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Base case (1 item): $T(1) = 1$ to ask: “**Is the number k ?**”



Binary Search Example

$$(*) \quad T(n) = \begin{cases} C_1 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + C_2 & \text{if } n \geq 2 \end{cases}$$

For simplicity, we will (usually) assume that n is a power of 2 (or sometimes 3 or 4) and also often that constants such as C_1, C_2 are 1. This will let us replace a recurrence such as (*) by one such as (**).



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In practice, the solution of (*) will be very close to that of (**) (this can be proved mathematically). Hence, we can restrict attention to (**).



Growth Rates of Solutions to Recurrences

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- Three different behaviors



Iterating Recurrences: Example 1

■

$$(*) \quad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$



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This corresponds to solving a problem of size n , by

- (i) solving 2 subproblems of size $n/2$ and
- (ii) doing n units of additional work

or using $T(1)$ work for “bottom” case of $n = 1$



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In the course “Analysis of Algorithms”, this is exactly how **Mergesort** works.



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We now see how to solve $(*)$ by algebraically iterating the recurrence.



Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

Assume that n is a power of 2

$$T(n) = 2T\left(\frac{n}{2}\right) + n = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$



Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

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$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + n &= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n \\ &= 4T\left(\frac{n}{4}\right) + 2n &= 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n \end{aligned}$$



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Iterating Recurrences: Example 1

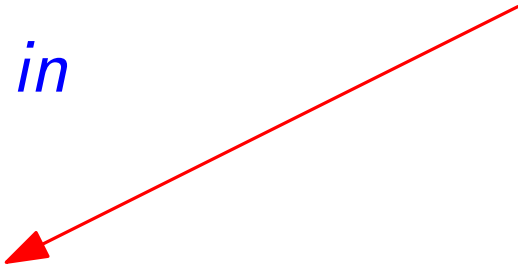
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$$\begin{aligned}&\vdots \quad \vdots \\&= 2^i T\left(\frac{n}{2^i}\right) + in \\&\vdots \quad \vdots \\&= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + (\log_2 n)n\end{aligned}$$

End when $i = \log_2 n$





Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

Assume that n is a power of 2

$$T(n) = 2T\left(\frac{n}{2}\right) + n = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

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$$= 8T\left(\frac{n}{8}\right) + 3n$$

$$\vdots \quad \vdots$$
$$= 2^i T\left(\frac{n}{2^i}\right) + in$$

End when $i = \log_2 n$

$$\vdots \quad \vdots$$
$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + (\log_2 n)n$$

$$= nT(1) + n\log_2 n$$



Iterating Recurrences: Example 1

- We just iterated the recurrence to derive that the solution to

$$(*) \quad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

is $nT(1) + n \log_2 n$.



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is $nT(1) + n \log_2 n$.

Note: Technically, we still need to use **induction** to prove that our solution is correct. Practically, we **never** explicitly perform this step, since it is obvious how the induction would work.



Iterating Recurrences: Example 2

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$



Iterating Recurrences: Example 2



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$$T(n) = T\left(\frac{n}{2}\right) + 1$$



Iterating Recurrences: Example 2



$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + 1 = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$



Iterating Recurrences: Example 2



$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 &= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 \end{aligned}$$



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$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 &&= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 &&= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \end{aligned}$$



Iterating Recurrences: Example 2



$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 &&= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 &&= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \\ &= T\left(\frac{n}{2^3}\right) + 3 \end{aligned}$$



Iterating Recurrences: Example 2

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 &&= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 &&= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \\ &= T\left(\frac{n}{2^3}\right) + 3 \\ &\quad \vdots \quad \quad \quad \vdots \\ &= T\left(\frac{n}{2^i}\right) + i \end{aligned}$$



Iterating Recurrences: Example 2

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + 1 = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

$$= T\left(\frac{n}{2^2}\right) + 2 = \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2$$

$$= T\left(\frac{n}{2^3}\right) + 3$$

$$\begin{array}{c} \vdots \\ \vdots \\ = T\left(\frac{n}{2^i}\right) + i \end{array}$$

$$\begin{array}{c} \vdots \\ \vdots \\ = T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \end{array}$$



Iterating Recurrences: Example 2

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + 1 = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

$$= T\left(\frac{n}{2^2}\right) + 2 = \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2$$

$$= T\left(\frac{n}{2^3}\right) + 3$$

$$\vdots \quad \vdots$$
$$= T\left(\frac{n}{2^i}\right) + i$$

$$\vdots \quad \vdots$$
$$= T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n = 1 + \log_2 n$$



Iterating Recurrences: Example 3

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$



Iterating Recurrences: Example 3



$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \end{aligned}$$



Iterating Recurrences: Example 3



$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n$$

$$= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n$$



Iterating Recurrences: Example 3

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\ &= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\ &\quad \vdots \\ &= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \end{aligned}$$



Iterating Recurrences: Example 3

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n$$

$$= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n$$

$$\begin{array}{c} \vdots \\ \vdots \\ = T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \end{array}$$

$$\begin{array}{c} \vdots \\ \vdots \\ = T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \end{array}$$



Iterating Recurrences: Example 3

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n$$

$$= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n$$

$$\vdots \quad \vdots$$

$$= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n$$

$$\vdots \quad \vdots$$

$$= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n$$

$$= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n$$



Iterating Recurrences: Example 3

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n$$

$$= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n$$

$$\vdots \quad \vdots$$

$$= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n$$

$$\vdots \quad \vdots$$

$$= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n$$

$$= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n = \Theta(n)$$



Iterating Recurrences: Example 4

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$



Iterating Recurrences: Example 4

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + n$$



Iterating Recurrences: Example 4

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + n = 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n$$



Iterating Recurrences: Example 4

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{3}\right) + n &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\ &= 3^2 T\left(\frac{n}{3^2}\right) + 2n \end{aligned}$$



Iterating Recurrences: Example 4

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{3}\right) + n &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\ &= 3^2 T\left(\frac{n}{3^2}\right) + 2n &= 3^2\left(3T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}\right) + 2n \end{aligned}$$



Iterating Recurrences: Example 4

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{3}\right) + n &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\ &= 3^2 T\left(\frac{n}{3^2}\right) + 2n &= 3^2\left(3T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}\right) + 2n \\ &= 3^3 T\left(\frac{n}{3^3}\right) + 3n \end{aligned}$$



Iterating Recurrences: Example 4

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

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Iterating Recurrences: Example 4

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{3}\right) + n &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\ &= 3^2 T\left(\frac{n}{3^2}\right) + 2n &= 3^2\left(3T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}\right) + 2n \\ &= 3^3 T\left(\frac{n}{3^3}\right) + 3n \\ &\quad \vdots \quad \vdots \\ &= 3^i T\left(\frac{n}{3^i}\right) + in \\ &\quad \vdots \quad \vdots \\ &= 3^{\log_3 n} T\left(\frac{n}{3^{\log_3 n}}\right) + n \log_3 n \end{aligned}$$



Iterating Recurrences: Example 4

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{3}\right) + n &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\ &= 3^2 T\left(\frac{n}{3^2}\right) + 2n &= 3^2\left(3T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}\right) + 2n \\ &= 3^3 T\left(\frac{n}{3^3}\right) + 3n \\ &\quad \vdots \quad \vdots \\ &= 3^i T\left(\frac{n}{3^i}\right) + in \\ &\quad \vdots \quad \vdots \\ &= 3^{\log_3 n} T\left(\frac{n}{3^{\log_3 n}}\right) + n \log_3 n = n + n \log_3 n \end{aligned}$$



Iterating Recurrences: Example 5

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$



Iterating Recurrences: Example 5

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$



Iterating Recurrences: Example 5

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n = 4\left(4T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n$$



Iterating Recurrences: Example 5

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Iterating Recurrences: Example 5

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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Iterating Recurrences: Example 5

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n &&= 4\left(4T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n \\ &= 4^2 T\left(\frac{n}{2^2}\right) + \frac{4}{2}n + n &&= 4^2\left(4T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + \frac{4}{2}n + n \\ &= 4^3 T\left(\frac{n}{2^3}\right) + \frac{4^2}{2^2}n + \frac{4}{2}n + n \end{aligned}$$



Iterating Recurrences: Example 5

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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Iterating Recurrences: Example 5

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n &&= 4\left(4T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n \\ &= 4^2 T\left(\frac{n}{2^2}\right) + \frac{4}{2}n + n &&= 4^2\left(4T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + \frac{4}{2}n + n \\ &= 4^3 T\left(\frac{n}{2^3}\right) + \frac{4^2}{2^2}n + \frac{4}{2}n + n \\ &\quad \vdots \quad \vdots \\ &= 4^i T\left(\frac{n}{2^i}\right) + \frac{4^{i-1}}{2^{i-1}}n + \cdots + \frac{4^2}{2^2}n + n \\ &\quad \vdots \quad \vdots \\ &= 4^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{4^{\log_2 n - 1}}{2^{\log_2 n - 1}}n + \cdots + \frac{4}{2}n + n \end{aligned}$$



Iterating Recurrences: Example 5

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n &&= 4\left(4T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n \\ &= 4^2 T\left(\frac{n}{2^2}\right) + \frac{4}{2}n + n &&= 4^2\left(4T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + \frac{4}{2}n + n \\ &= 4^3 T\left(\frac{n}{2^3}\right) + \frac{4^2}{2^2}n + \frac{4}{2}n + n \\ &\quad \vdots \quad \vdots \\ &= 4^i T\left(\frac{n}{2^i}\right) + \frac{4^{i-1}}{2^{i-1}}n + \cdots + \frac{4^2}{2^2}n + n \\ &\quad \vdots \quad \vdots \\ &= 4^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{4^{\log_2 n - 1}}{2^{\log_2 n - 1}}n + \cdots + \frac{4}{2}n + n \\ &= 2n^2 - n \end{aligned}$$



Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iteration recurrences
- Three different behaviors



Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$



Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

- ◇ all three recurrences iterate $\log_2 n$ times
- ◇ in each case, size of subproblem in next iteration is **half** the size in the preceding iteration level



Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form

$$T(n) = aT(n/2) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following **big Θ** bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$



Three Different Behaviors

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3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

Proof

We already proved Case 1 when $a = 1$ in Example 3.

(will not prove it for $1 < a < 2$)

We already proved Case 2 in Example 1.

We will now prove Case 3.



Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.



Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

Iterating as in Example 5 gives

$$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots + \frac{a}{2} + 1\right) n$$



Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

Iterating as in Example 5 gives

$$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots + \frac{a}{2} + 1\right) n$$

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Work at
“bottom”

Iterated
Work



Total work

- The total work is

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$



Total work

- The total work is

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Since $a > 2$, the geometric series is Θ of the largest term.

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1 - (a/2)^{\log_2 n}}{1 - a/2} = n\Theta((a/2)^{\log_2 n - 1})$$



Total work

- n times the largest term in the geometric series is

$$n \left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$



Total work

- n times the largest term in the geometric series is

$$n \left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$

Notice that

$$a^{\log_2 n} = (2^{\log_2 a})^{\log_2 n} = (2^{\log_2 n})^{\log_2 a} = n^{\log_2 a}$$



Total work

- n times the largest term in the geometric series is

$$n \left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$

Notice that

$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

So the total work is

$$a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$



Total work

- n times the largest term in the geometric series is

$$n \left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$

Notice that

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So the total work is

$$a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

$$\Theta(n^{\log_2 a})$$

$$\Theta(n^{\log_2 a})$$



Example 5 Recap

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$



Example 5 Recap

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$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$a = 4$, so the Theorem says that

$$T(n) = \Theta(n^{\log_2 a}) = \Theta(n^{\log_2 4}) = \Theta(n^2)$$



Example 5 Recap

■

$$(*) \quad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$a = 4$, so the Theorem says that

$$T(n) = \Theta(n^{\log_2 a}) = \Theta(n^{\log_2 4}) = \Theta(n^2)$$

This matches with the exact answer of $2n^2 - n$.



Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form

$$T(n) = aT(n/2) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.



The Master Theorem

- **Theorem** Suppose that we have a recurrence of the form

$$T(n) = aT(n/b) + cn^d,$$

where a is a positive integer, $b \geq 1$, c, d are real numbers with c positive and d nonnegative, and $T(1)$ is nonnegative. Then we have the following **big Θ** bounds on the solution:

1. If $a < b^d$, then $T(n) = \Theta(n^d)$.
2. If $a = b^d$, then $T(n) = \Theta(n^d \log n)$.
3. If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$



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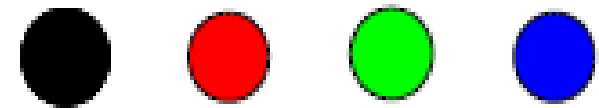
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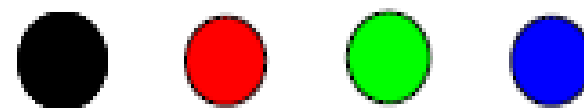


What about when order counts?

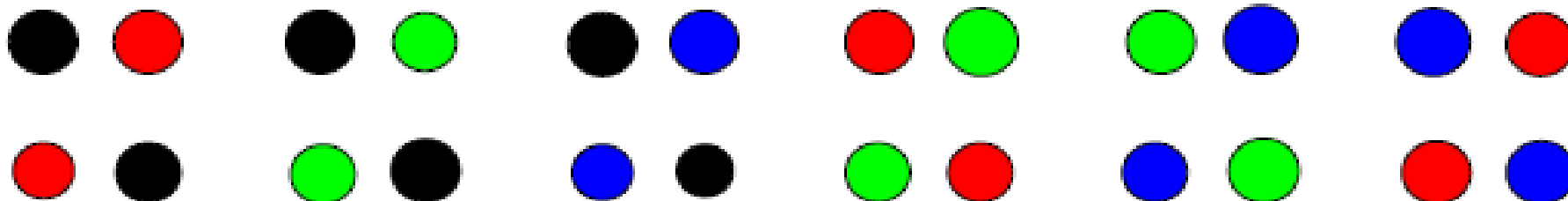
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Counting may be very hard, not trivial.

- simplify the solution by decomposing the problem



Basic Counting Rules

- *the Product Rule*

- *the Sum Rule*



Basic Counting Rules

■ *the Product Rule*

- ◇ A count decomposes into a sequence of **dependent** counts
(each element in the first count is associated with all elements of the second count)

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(elements of counts are alternatives)



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In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., A23). What is the total number of seats?



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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., A23). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$



The Product Rule

- **Product Rule:** If a count of elements can be broken down into a **sequence of dependent counts** where the first count yields n_1 elements, the second n_2 elements, and k th count n_k elements, then the total number of elements is

$$n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$



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How many **onto** functions?



The Product Rule

- The following loop is a part of program computing the product of two matrices.

```
(1) for i = 1 to r
(2)   for j = 1 to m
(3)     S = 0
(4)     for k = 1 to n
(5)       S = S + A[i,k] * B[k,j]
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How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?



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Example

You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. **How many options do you have to get from A to B?**



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Example

You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. **How many options do you have to get from A to B?**

We may **use the sum rule.**

$$12 + 5 + 10$$



The Sum Rule

- **Sum Rule:** If a count of elements can be broken down into a **set of independent counts** where the first count yields n_1 elements, the second n_2 elements, and k th count n_k elements, then the total number of elements is

$$n = n_1 + n_2 + \cdots + n_k$$



The Sum Rule

- The following loop is from [selection sort](#).

```
(1) for i = 1 to n-1
(2)     for j = i+1 to n
(3)         if (A[i] > A[j])
(4)             exchange A[i] and A[j]
```



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- The following loop is from **selection sort**.

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```

How many **comparisons** (in terms of n) does this program carry out in total among all iterations of line 3?



More Complex Counting

- Typically requires a combination of the sum and product rules.



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Example

Each password is **6 to 8 characters** long, where each character is an lowercase letter or a digit. Each password must contain **at least one digit**. How many possible passwords are there?



More Complex Counting

- Typically requires a combination of the sum and product rules.

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Each password is 6 to 8 characters long, where each character is an lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

$$P = P_6 + P_7 + P_8$$



Tree Diagrams

- A *tree* is a structure that consists of a *root*, *branches* and *leaves*.



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Example

What is the number of bit strings of length 4 that **do not have two consecutive 1's**?



Tree Diagrams

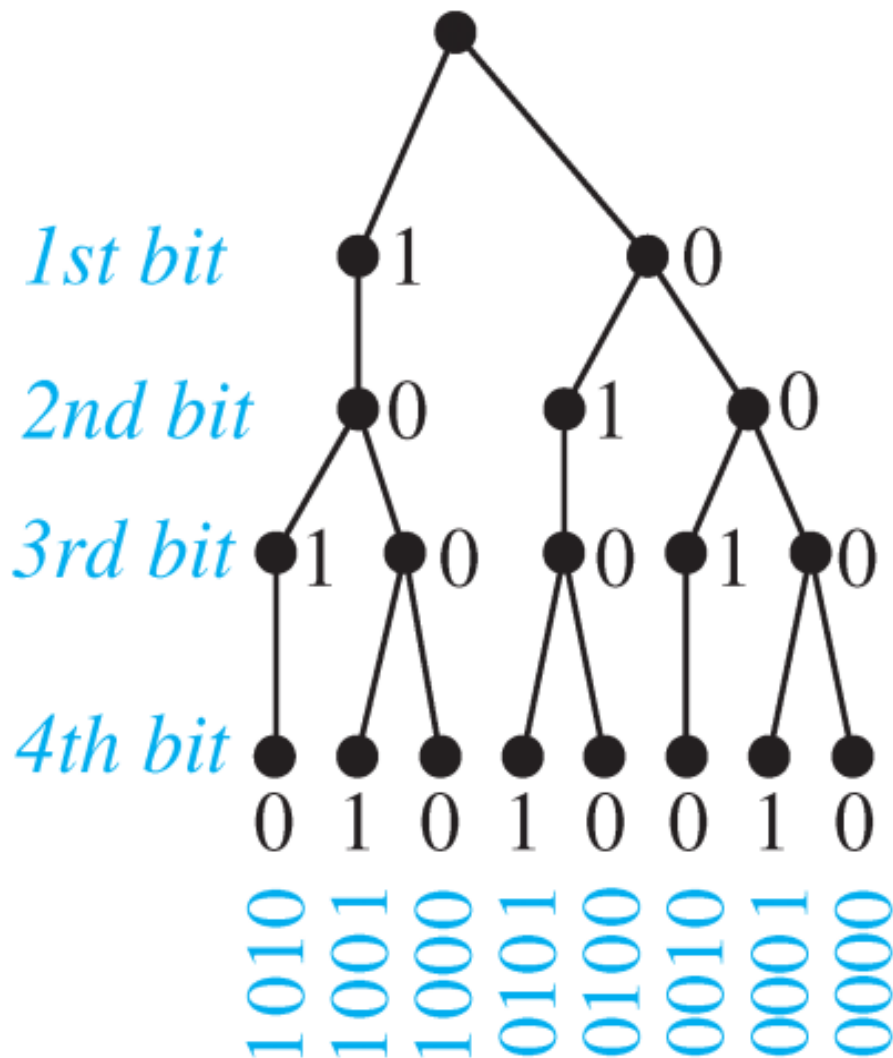
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Problem and record count appears on

Example

What is the probability of having two consecutive 1s in a 4-bit sequence?



h 4 that do not

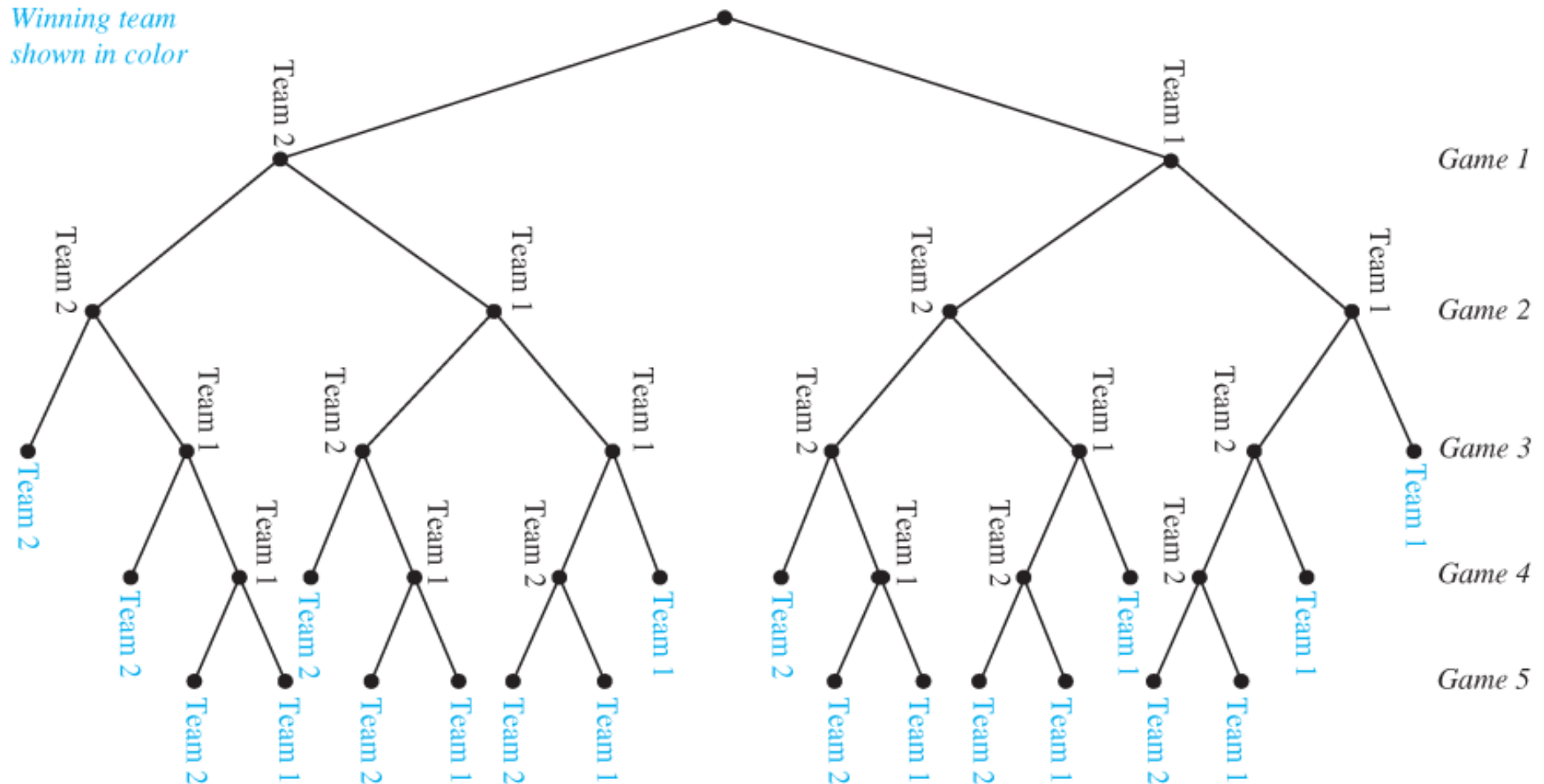
Tree Diagram

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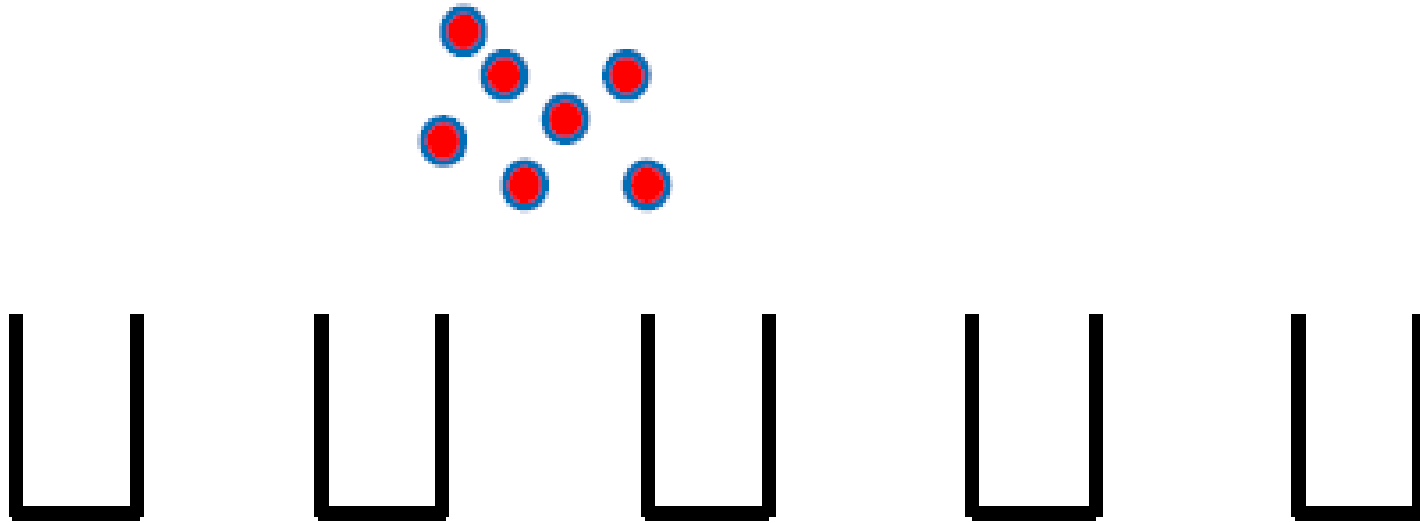


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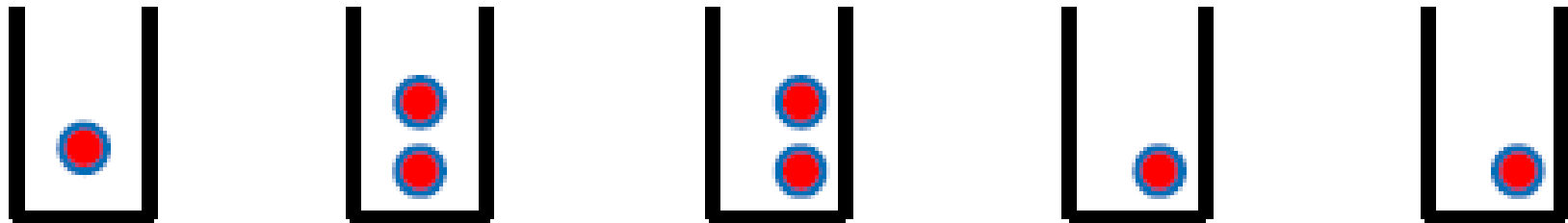


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Proof by contradiction



Pigeonhole Principle

- **Theorem** If there are $k + 1$ objects and k bins, then there is **at least** one bin with two or more objects.

Proof by contradiction

Example

Assume that there are 367 students. Are there any two people who have the same birthday?

There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?



Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.



Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example

Assume there are 100 students. How many of them were born in the same month?



Bijections and Permutations

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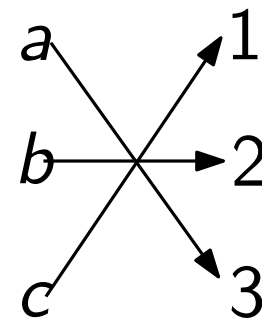


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$f : \{a, b, c\} \rightarrow \{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$ is a bijection.

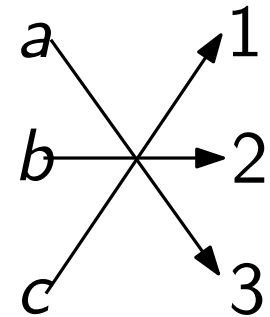


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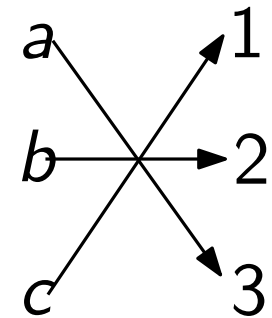
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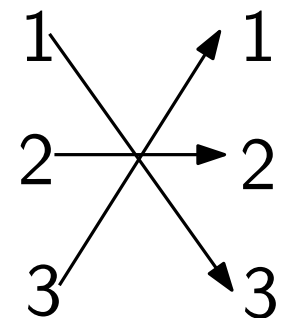
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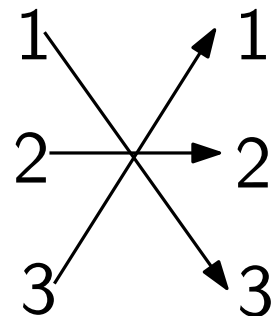
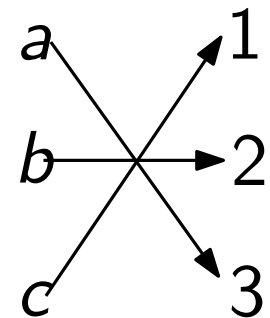
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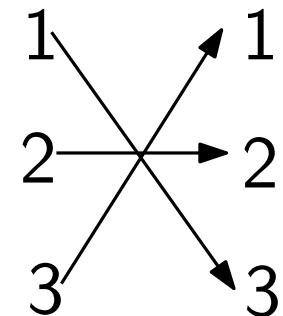
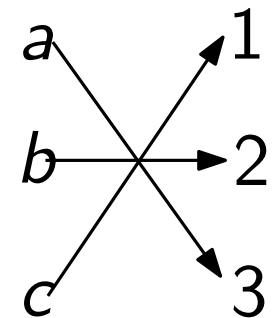
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exactly one arrow leaves each item on the left and exactly one arrow arrives at each item on the right.

Thus,

the left and right sides must have the same size.



The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?



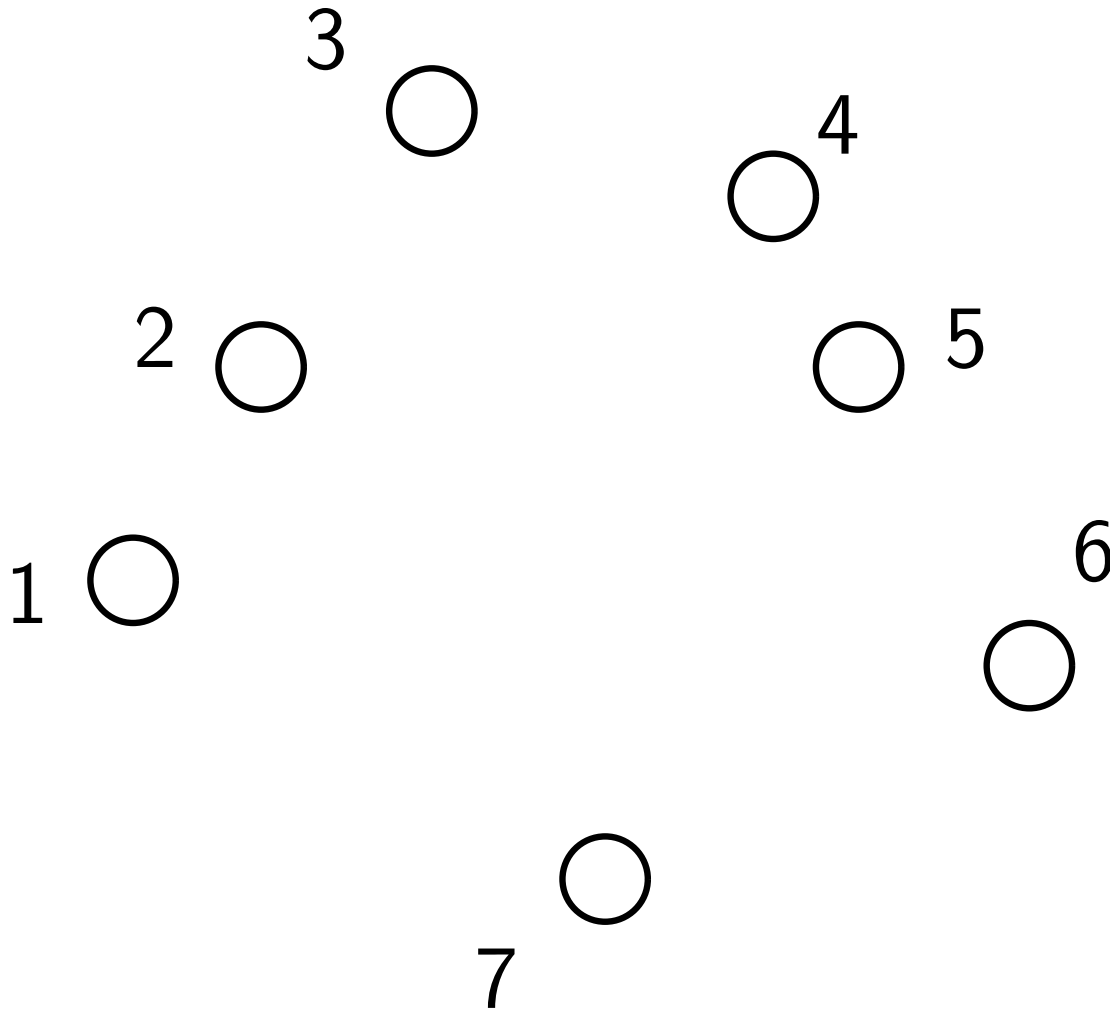
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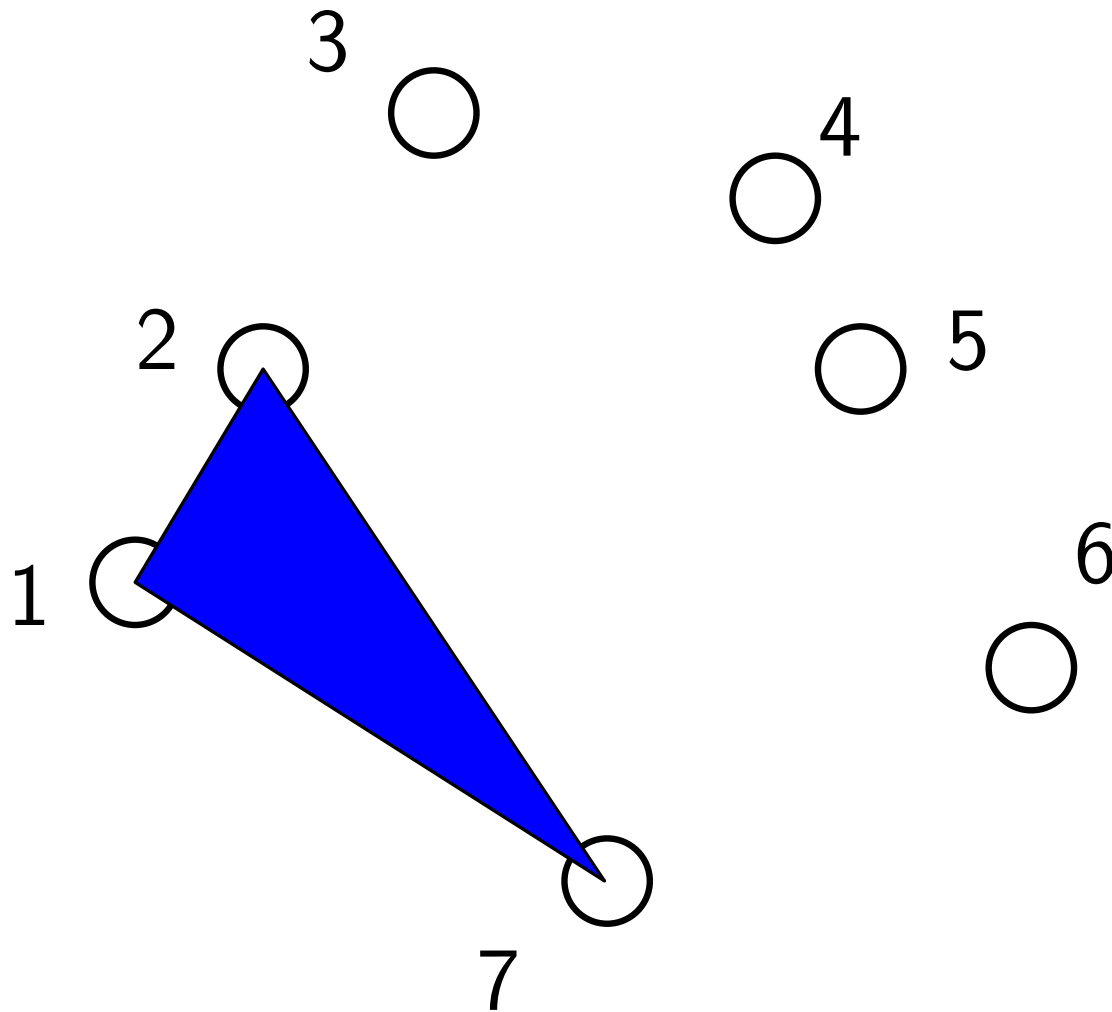


50 - 2



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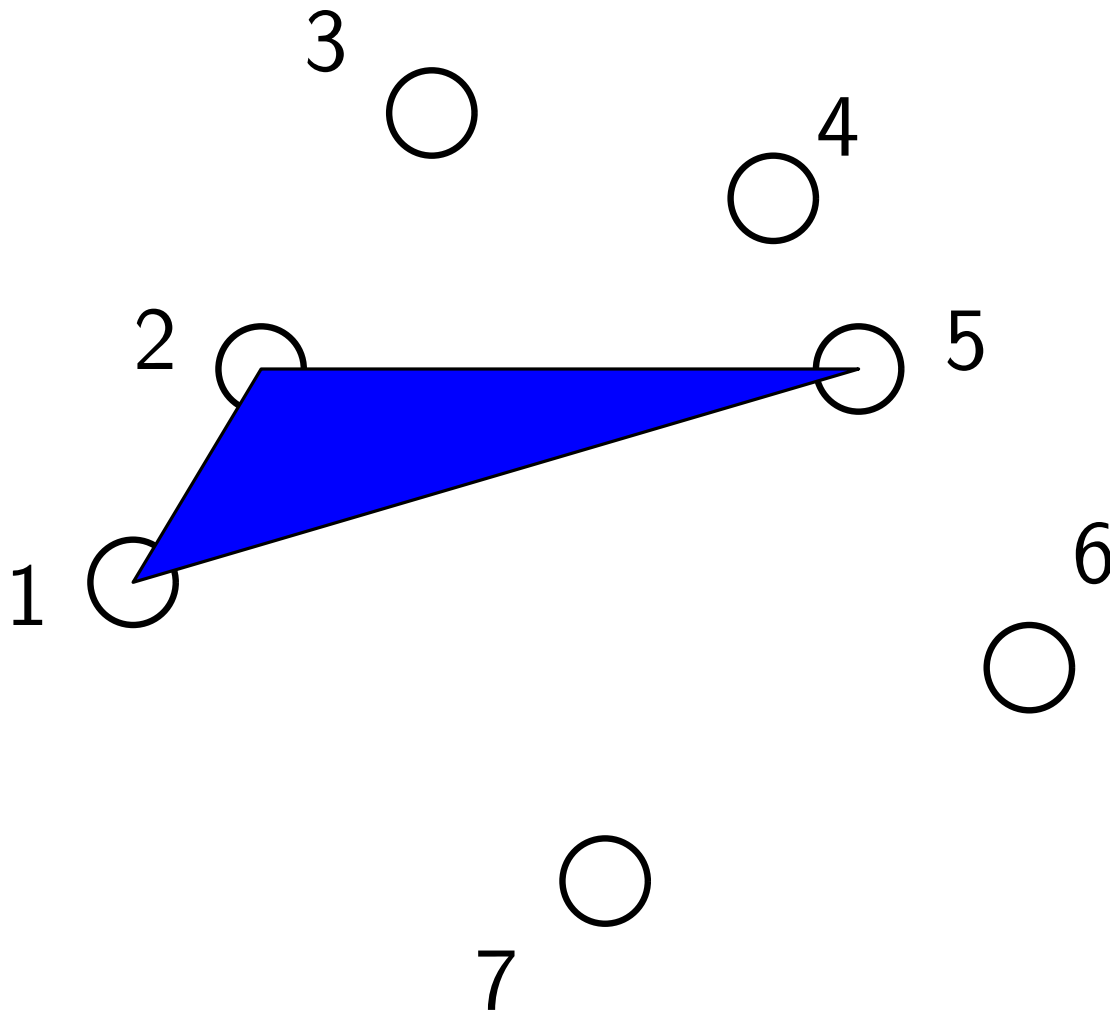
1 – 2 – 7: yes

50 - 3



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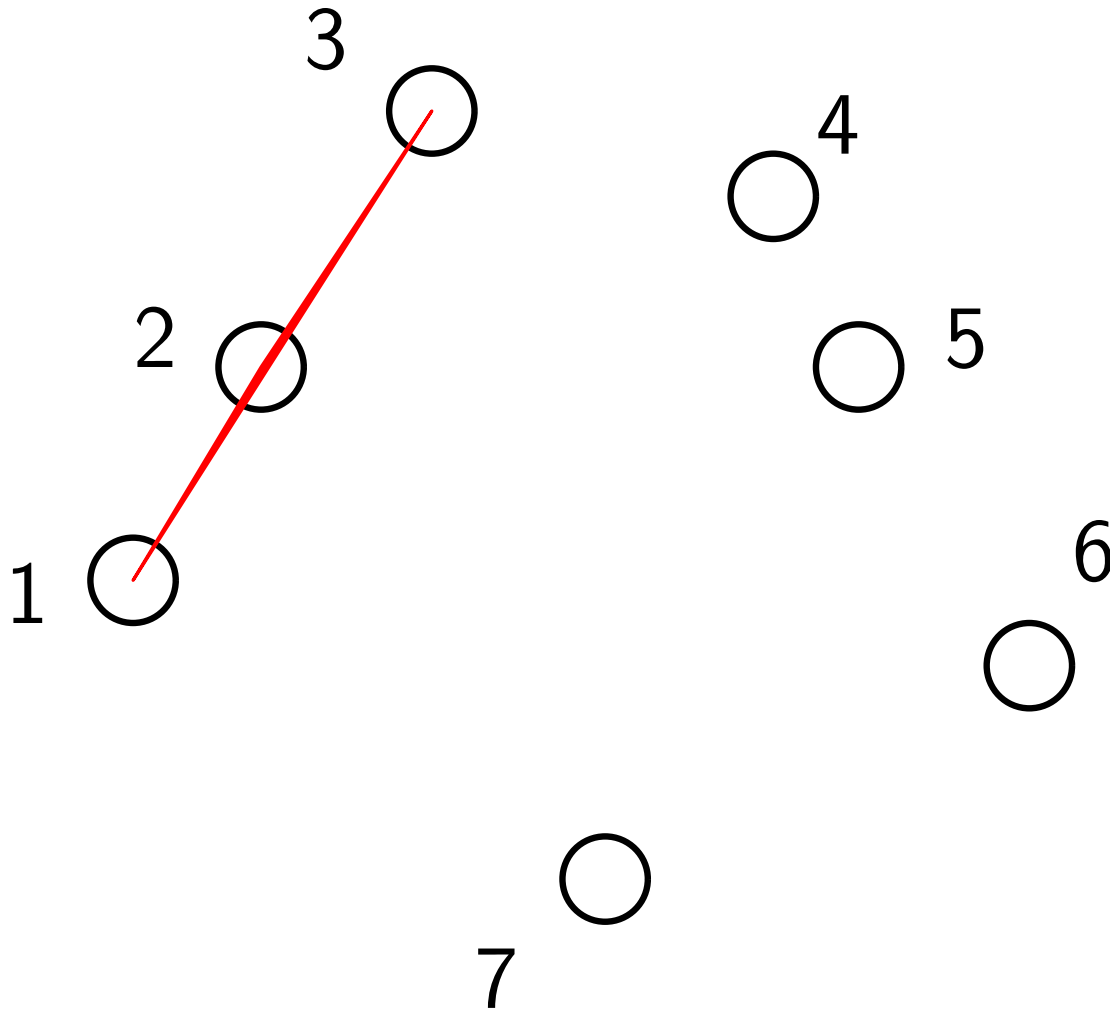


1 – 2 – 7: yes

1 – 2 – 5: yes

Counting Triangles

- 3 points form a **triangle** if and only if **they are non collinear**



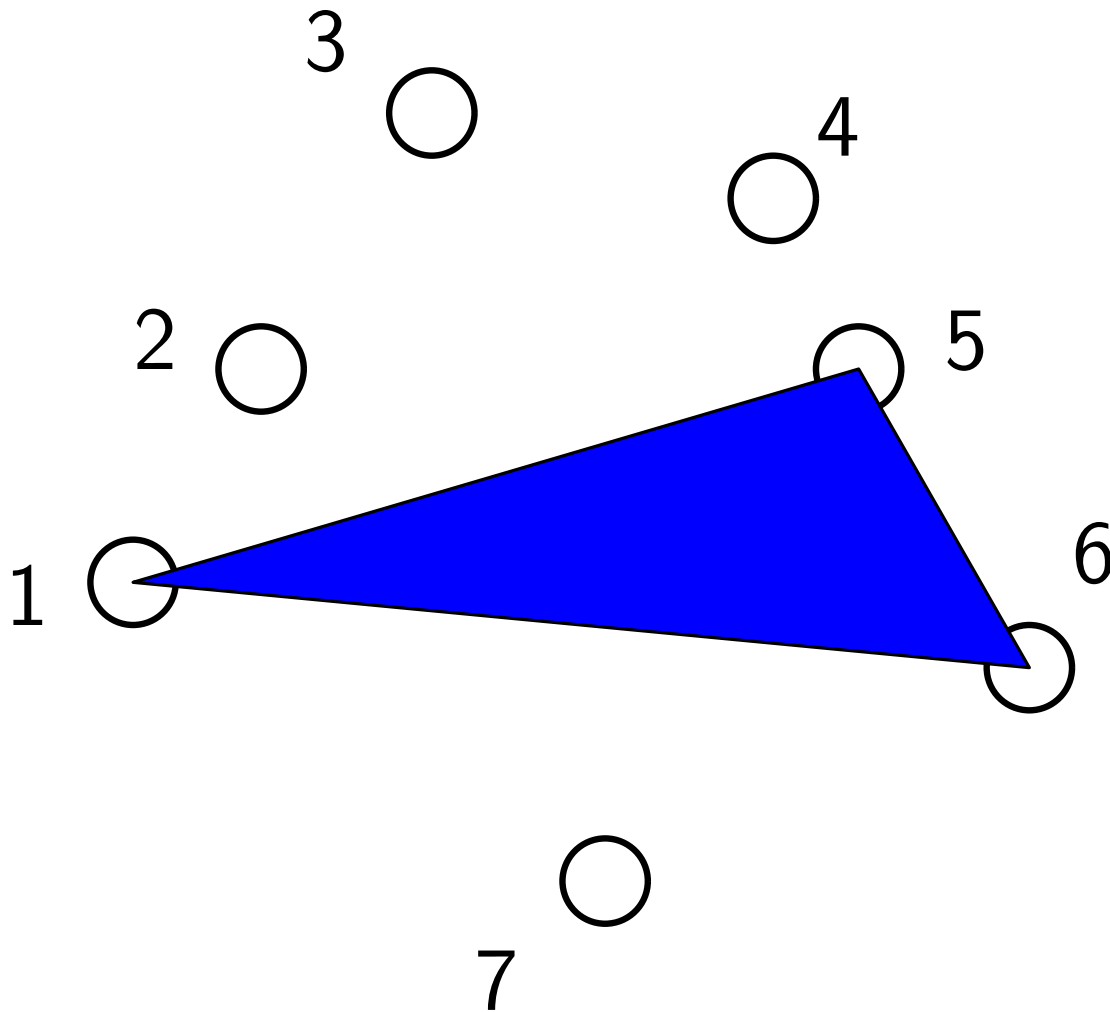
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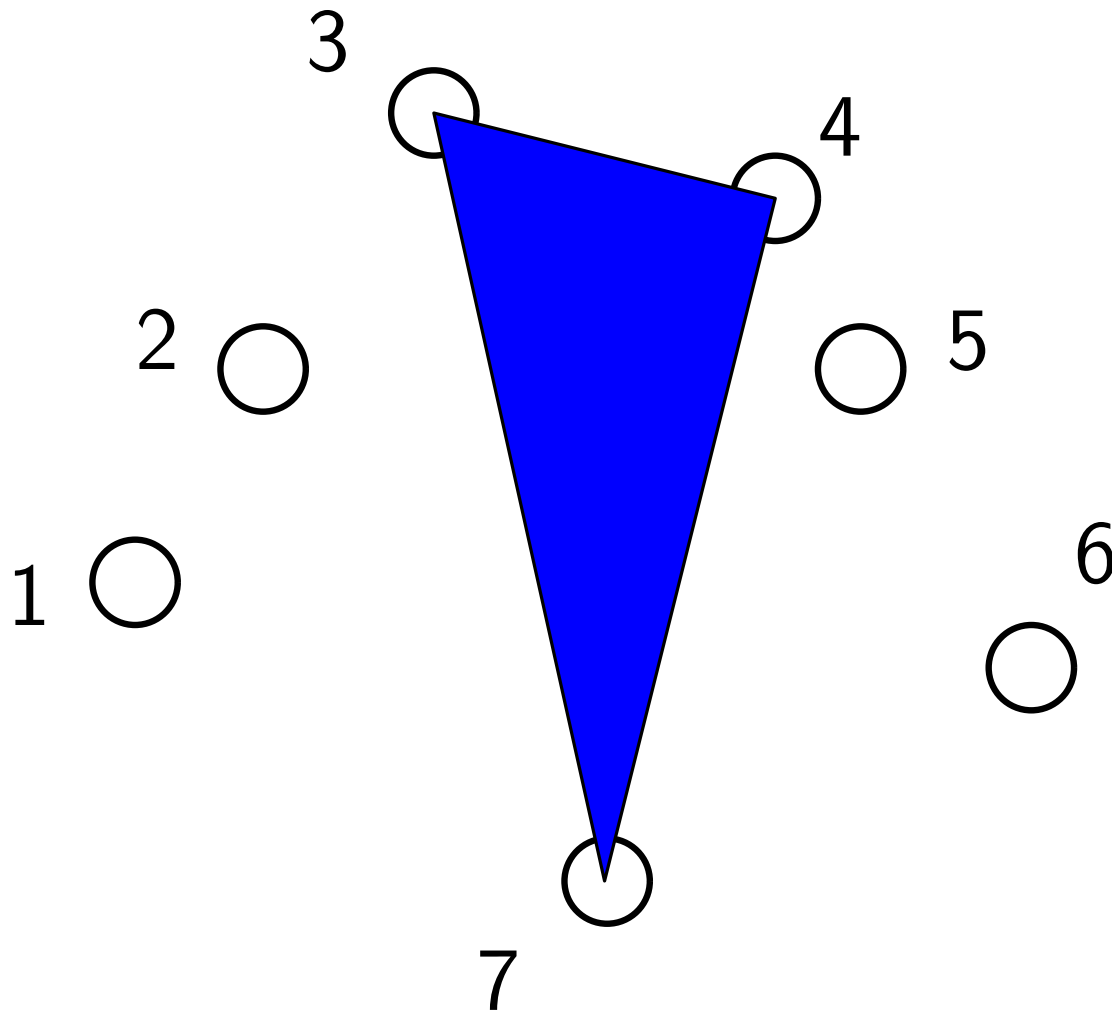
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1 – 5 – 6: yes

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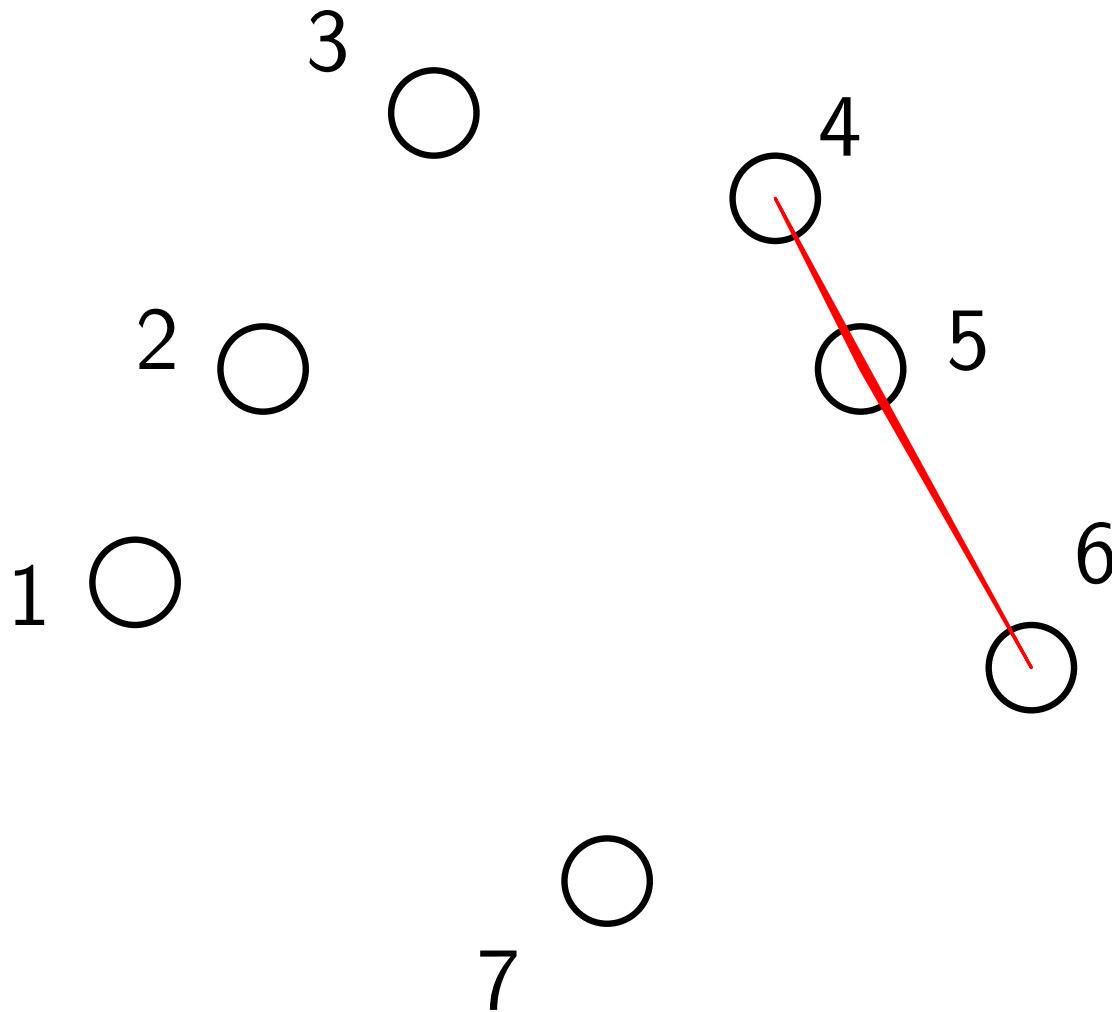
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3 – 4 – 7: yes

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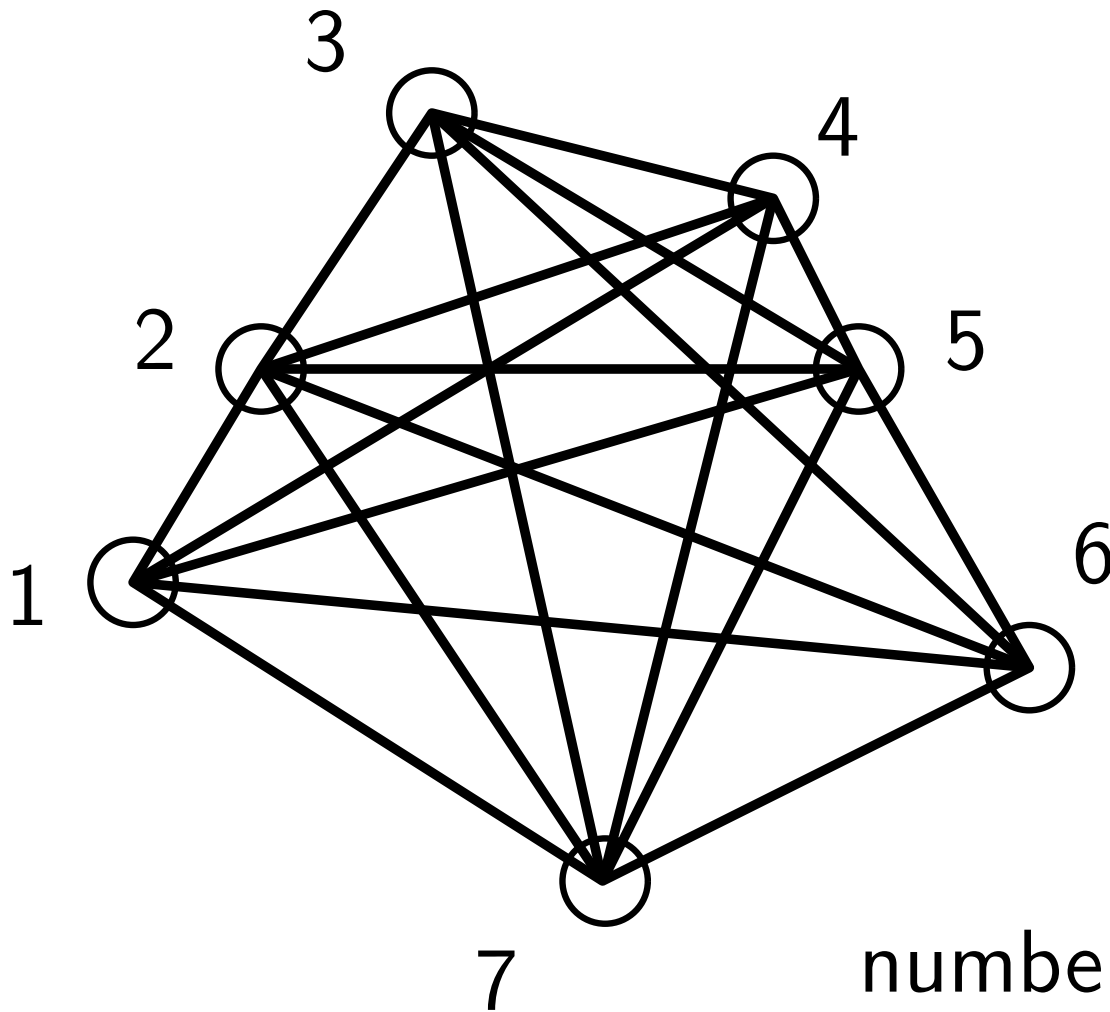
1 – 5 – 6: yes

3 – 4 – 7: yes

4 – 5 – 6: **no**

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1 – 2 – 5: yes

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1 – 5 – 6: yes

3 – 4 – 7: yes

4 – 5 – 6: no

number of triangles: 33

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For example, if $n = 4$, then triples (i, j, k) used by algorithm are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$.

Counting Triangles

- Want to compute the number of *increasing triples* (i, j, k) with $1 \leq i < j < k \leq n$.

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Claim: Number of increasing triples is **exactly** the same as number of 3-element subsets from $\{1, 2, \dots, n\}$

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Claim: Number of increasing triples is **exactly** the same as number of 3-element subsets from $\{1, 2, \dots, n\}$

Why? Let $X =$ set of increasing triples and
 $Y =$ set of 3-element subsets from $\{1, 2, \dots, n\}$

Counting Triangles

- Want to compute the number of *increasing triples* (i, j, k) with $1 \leq i < j < k \leq n$.

Claim: Number of increasing triples is **exactly** the same as number of 3-element subsets from $\{1, 2, \dots, n\}$

Why? Let $X =$ set of increasing triples and
 $Y =$ set of 3-element subsets from $\{1, 2, \dots, n\}$

Define: $f : X \rightarrow Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a **bijection** (why) so $|X| = |Y|$

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f is a bijection because

f is one-to-one

if $(i, j, k) \neq (i', j', k') \Rightarrow f((i, j, k)) \neq f((i', j', k'))$

f is onto

if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$

where $i < j < k$ so $f((i, j, k)) = \gamma$.

Counting Pairs

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We actually already saw that $|X| = |Y| = \binom{n}{2}$



The Bijection Principle

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Currently, we started with the problem of counting the **# of increasing triples** and changed it to the problem of counting the **# of 3-element sets from $\{1, 2, \dots, n\}$**



Next Lecture

■ recurrence ...

