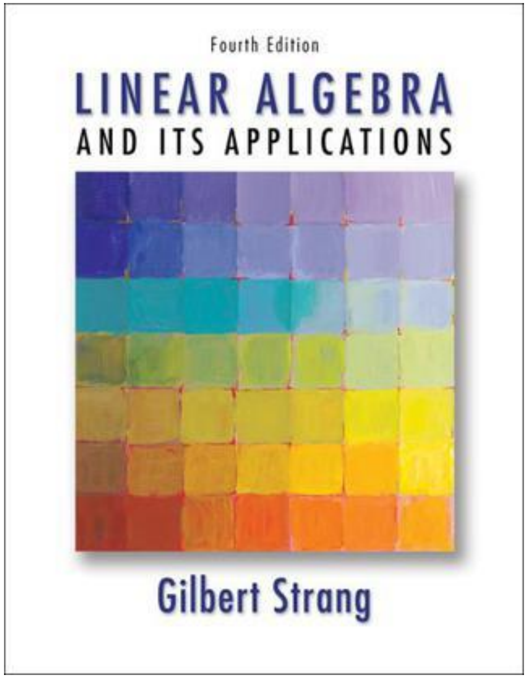
Linear Algebra



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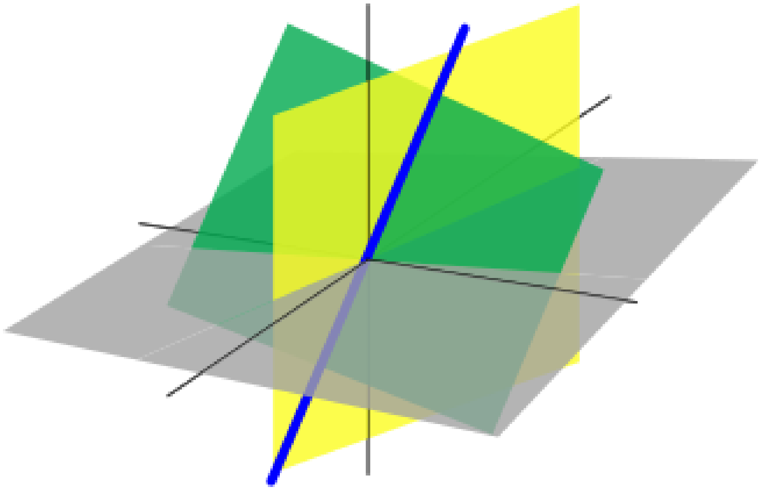
2 Vector Spaces (向量空间)

2.2

SOLVING ***Ax*=*0*** AND ***Ax*=*b***
(线性方程组的解)

- Solving ***Ax*=*0***
- Solving ***Ax*=*b***
- Rank (秩)



Introduction:

A system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Solution

一般线性方程组的矩阵表示

$$\boxed{\boldsymbol{Ax} = \boldsymbol{b}.}$$

齐次线性方程组的矩阵表示

homogeneous $\boxed{\boldsymbol{Ax} = \boldsymbol{0}.}$

解的结构:
在多解情况下，讨论
解与解之间的关系

Introduction: Simple cases and more …

- For a square invertible matrix $\boldsymbol{A}$, there is only one solution to $\boldsymbol{Ax} = \boldsymbol{b}$, and it is $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

- For a rectangular matrix $\boldsymbol{A}_{m\times n}$ $(m \neq n)$ or a square matrix without an inverse, there are new possibilities

  ($\boldsymbol{A}$ → echelon form $\boldsymbol{U}$ → reduced echelon form $\boldsymbol{R}$)

Introduction: Simple cases and more …

- For a square invertible matrix $\boldsymbol{A}$:
  - The nullspace contains only $\boldsymbol{x} = \boldsymbol{0}$; This zero solution is usually called the trivial solution (平凡解).

    (multiply $\boldsymbol{Ax} = \boldsymbol{0}$ by $\boldsymbol{A}^{-1}$)
  - The column space is the whole space.

    ($\boldsymbol{Ax} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$)
- The new questions appear when the nullspace contains *more than the zero vector* (i.e., there exists a nontrivial solution) and/or the column space contains *less than all vectors.*

For example,

The matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible.

$y + z = b_1$ and $2y + 2z = b_2$ usually have no solution.

There is ***no solution*** unless $b_2 = 2b_1$. The column space of $\boldsymbol{A}$ contains only those $\boldsymbol{b}$'s, the multiples of $(1, 2)^T$.

When $b_2 = 2b_1$ there are ***infinitely many solutions***.

A particular solution to $y + z = 2$ and $2y + 2z = 4$ is $\boldsymbol{x}_p = (1, 1)^T$.

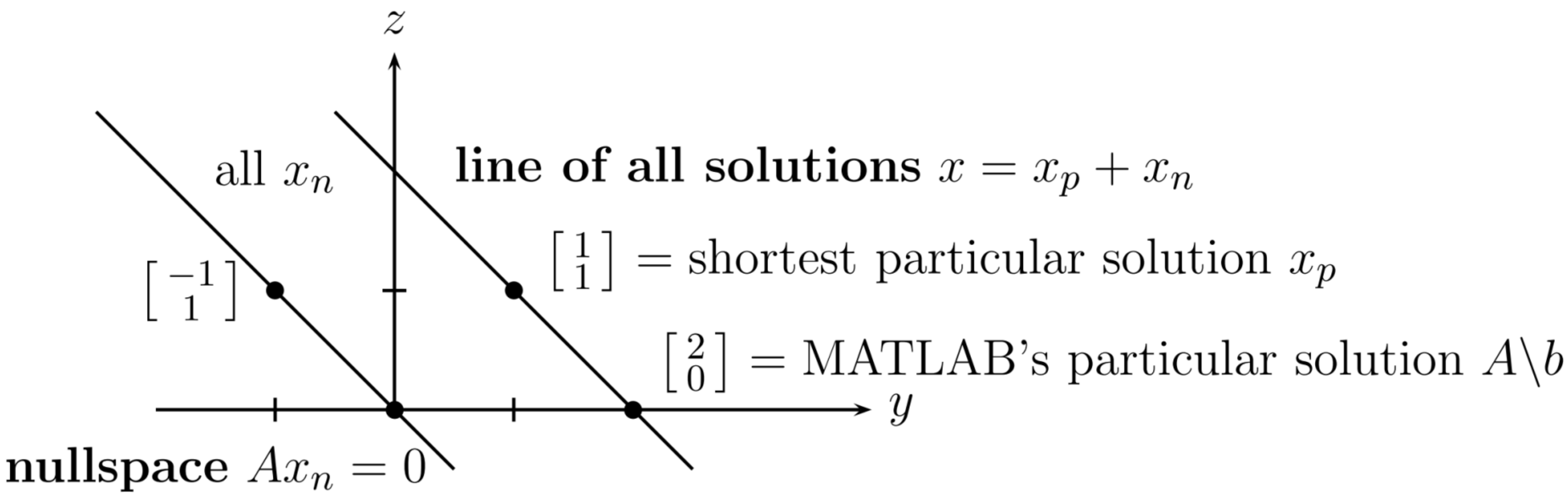
The nullspace of $\boldsymbol{A}$ contains $(-1, 1)^T$ and all its multiples $\boldsymbol{x}_n = (-c, c)^T$, $c \in \mathbf{R}$.

Complete solution to

$$\begin{aligned} y + z &= 2 \\ 2y + 2z &= 4 \end{aligned}$$

is solved by $\boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - c \\ 1 + c \end{bmatrix}$

Complete solution: $\boldsymbol{Ax}_p = \boldsymbol{b}$ and $\boldsymbol{Ax}_n = \boldsymbol{0}$ produce $\boldsymbol{A}(\boldsymbol{x}_p + \boldsymbol{x}_n) = \boldsymbol{b}$.



Complete solution to

$$y + z = 2$$
$$2y + 2z = 4$$

is solved by $\mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1-c\\1+c\end{bmatrix}$

Complete solution: $\boldsymbol{A}\boldsymbol{x}_p = \boldsymbol{b}$ and $\boldsymbol{A}\boldsymbol{x}_n = \boldsymbol{0}$ produce $\boldsymbol{A}(\boldsymbol{x}_p + \boldsymbol{x}_n) = \boldsymbol{b}$.

I. Solving *Ax = 0*

Example 1 Find a spanning set for the nullspace of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution: The first step is to find the general solution of ***Ax*=*0*** in terms of free variables.

Row reduce the matrix ***A*** to *reduced echelon form* in order to write the basic variables in terms of the free variables:

$$\left[\begin{array}{ccccc:c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free. (x_1, x_3: basic variables, also called pivot variables)

The general solution is $x_1 = 2x_2+x_4-3x_5$, $x_3 = -2x_4+2x_5$, with x_2, x_4, and x_5 free (called free variables).

Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{w}} = x_2\boldsymbol{u} + x_4\boldsymbol{v} + x_5\boldsymbol{w}$$

Every linear combination of ***u***, ***v***, and ***w*** is an element of N(***A***). Thus {***u***, ***v***, ***w***} is a spanning set for N(***A***).

u, ***v***, ***w*** : special solutions

The *nullspace* of ***A*** can be spanned by a few ***special solutions***, where a solution of ***Ax* = *0*** is called *special* if it is a solution for which each free variable takes value 1 or 0.

The best way to find all solutions to ***Ax* = *0*** *is from the special solutions:*

Step 1. After reaching ***Rx* = *0***, identify the pivot variables (i.e., basic variables) and free variables. (***R***: reduced echelon form of ***A***)

Step 2. Give one free variable the value 1, set the other free variables to 0, and solve ***Rx* = *0*** for the basic variables. This ***x*** is a special solution.

Step 3. Every free variable produces its own “special solution” by step 2. The combinations of special solutions form the nullspace—all solutions to ***Ax* = *0*** .

II. Solving $Ax = b$

- As observed above, all solutions of a homogeneous system of linear equations form a vector space (齐次线性方程组的解集构成一个向量空间). This enables us to write down the solutions in a nice way.
- However, solutions of a non-homogeneous system do not have such a nice property (非齐次线性方程组的解集不能构成向量空间) .
- A natural question is what we can say about solutions of a general system of linear equations.
- Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$,

  and the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Lemma (引理) *If* $\mathbf{u}$ *and* $\mathbf{w}$ *are two solutions of* $A\mathbf{x} = \mathbf{b}$, *then* $\mathbf{u}-\mathbf{w}$ *is a solution of* $A\mathbf{x} = \mathbf{0}$.

Let $\mathbf{x}_p$ be a ***particular* solution** of $A\mathbf{x} = \mathbf{b}$. Then any solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has the form

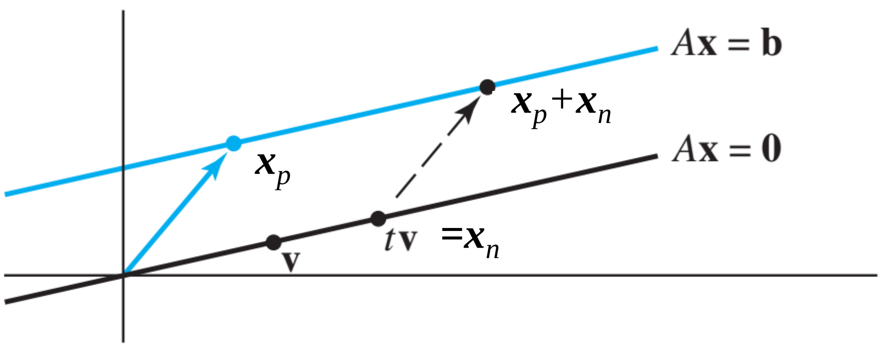
$$\mathbf{x}_{\text{complete}} = \mathbf{x}_{\text{nullspace}} + \mathbf{x}_{\text{particular}}.$$

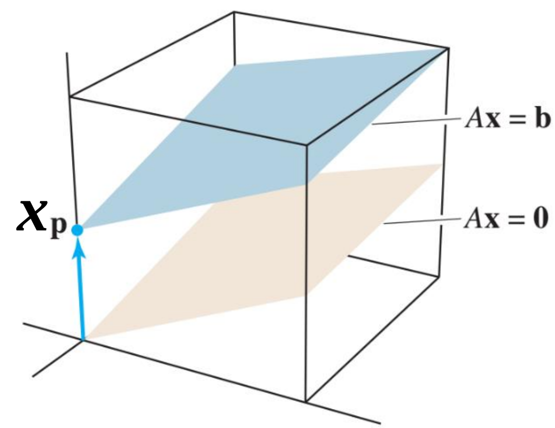
Theorem *The solutions of a homogenous system* $\boldsymbol{Ax} = \boldsymbol{0}$ *form a subspace* $\mathrm{N}(\boldsymbol{A})$, *and each solution* $\boldsymbol{x}$ *of* $\boldsymbol{Ax} = \boldsymbol{b}$ *has the form*

$$\boldsymbol{x} = \boldsymbol{x}_n + \boldsymbol{x}_p$$

where $\boldsymbol{x}_p$ *is a particular solution of* $\boldsymbol{Ax} = \boldsymbol{b}$, *and* $\boldsymbol{x}_n \in \mathrm{N}(\boldsymbol{A})$.

$\boldsymbol{Ax}=\boldsymbol{0}$ 的一般解$\boldsymbol{x}_n$ + $\boldsymbol{Ax}=\boldsymbol{b}$ 的特解$\boldsymbol{x}_p$ = $\boldsymbol{Ax}=\boldsymbol{b}$ 的一般解$\boldsymbol{x}$





parallel solution sets of $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{Ax} = \boldsymbol{b}$
(Left: 1 free variable; Right: 2 free variables)

Example 2 For the matrix $\boldsymbol{A}=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\end{bmatrix}$,

the nullspace of $\boldsymbol{A}$ is the space spanned by the solutions of $\boldsymbol{Ax}=\boldsymbol{0}$, which is

$$x = z,\ y = -2z.$$

So a solution vector is of the form

$$(x, y, z)^{\mathrm{T}} = (z, -2z, z)^{\mathrm{T}} = (1, -2, 1)^{\mathrm{T}} z,$$

where $z \in \mathbf{R}$.

Consider the system of linear equations $\boldsymbol{Ax}=\boldsymbol{b}$, where $\boldsymbol{b}=\begin{bmatrix}1\\ 2\end{bmatrix}$.

We observe that $\boldsymbol{Ax}=\boldsymbol{b}$ has a *particular solution* $(0,1,0)^{\mathrm{T}}$.

Therefore, the solution set for $\boldsymbol{Ax}=\boldsymbol{b}$ is

$$\{(0, 1, 0)^{\mathrm{T}} + (1, -2, 1)^{\mathrm{T}} z \mid z \in \mathbf{R}\}.$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 2\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 1\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 2 & 1\end{array}\right]$$

$$[\boldsymbol{A} \ \vdots\ \boldsymbol{b}] \rightarrow [\boldsymbol{U} \ \vdots\ \boldsymbol{c}] \rightarrow [\boldsymbol{R} \ \vdots\ \boldsymbol{d}]$$

$$\begin{aligned} x \quad - z &= 0\\ y + 2z &= 1\end{aligned}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ -2\\ 1\end{bmatrix} z + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

The best way to write all solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ *:*

Step 1. Row reduce $\boldsymbol{Ax} = \boldsymbol{b}$ to $\boldsymbol{Ux} = \boldsymbol{c}$ or $\boldsymbol{Rx} = \boldsymbol{d}$.

Step 2. With free variables = 0, find a particular solution to $\boldsymbol{Ax}_p = \boldsymbol{b}$ (or $\boldsymbol{Ux}_p = \boldsymbol{c}$ or $\boldsymbol{Rx}_p = \boldsymbol{d}$).

Step 3. Find the special solutions to $\boldsymbol{Ax} = \boldsymbol{0}$ (or $\boldsymbol{Ux} = \boldsymbol{0}$ or $\boldsymbol{Rx} = \boldsymbol{0}$). Each free variable, in turn, is 1. Then

$$\boldsymbol{x} = \boldsymbol{x}_p + (\text{any combination } \boldsymbol{x}_n \text{ of special solutions}).$$

$\boldsymbol{A}$: coefficient matrix
$[\boldsymbol{A}\ \ \boldsymbol{b}]$: augmented matrix
$\boldsymbol{U}$: the echelon form of $\boldsymbol{A}$
$\boldsymbol{R}$: the reduced echelon form of $\boldsymbol{A}$

Example 3 Find the condition on b_1, b_2, b_3 to have a solution; Solve $\boldsymbol{Ax} = \boldsymbol{b}$ when $\boldsymbol{b} = (0,6,-6)^T$.

$$\begin{aligned} \mathbf{1}x_1 + \mathbf{2}x_2 + \mathbf{3}x_3 + \mathbf{5}x_4 &= b_1 \\ \mathbf{2}x_1 + \mathbf{4}x_2 + \mathbf{8}x_3 + \mathbf{12}x_4 &= b_2 \\ \mathbf{3}x_1 + \mathbf{6}x_2 + \mathbf{7}x_3 + \mathbf{13}x_4 &= b_3 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 2 & 4 & 8 & 12 & \boldsymbol{b_2} \\ 3 & 6 & 7 & 13 & \boldsymbol{b_3} \end{bmatrix} \to \left[\begin{array}{cccc|l} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \boldsymbol{b_3 - 3b_1} \end{array}\right]$$

$$\to \left[\begin{array}{cccc|l} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \boldsymbol{b_3 + b_2 - 5b_1} \end{array}\right]$$

The last row shows the solvability condition $b_3 + b_2 - 5b_1 = 0$.(有解条件)

For $\boldsymbol{b} = (0,6,-6)^T$,

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$[U \quad c] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow [R \quad d] = \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & \mathbf{-9} \\ 0 & 0 & \mathbf{1} & 1 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns

Everything is revealed by Rx = d

The special solutions to ***Ax*=*0*** have free variables $x_2 = 1, x_4 = 0$ and $x_2 = 0, x_4 = 1$.
The particular solution to ***Ax*=*b*** has free variables $x_2 = 0, x_4 = 0$.

The matrix with columns being special solutions is called the *nullspace matrix*:

$$N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

Particular solution to $\boldsymbol{Ax}_p = \boldsymbol{b}$:

$$x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

The complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$:

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

where $k_1, k_2 \in \mathbf{R}$.

Example 4 设线性方程组 $\begin{cases} ax_1 + x_2 + x_3 = 4 \\ x_1 + bx_2 + x_3 = 3 \\ x_1 + 2bx_2 + x_3 = 4 \end{cases}$

就参数 $a,\ b$ 讨论方程组的解的情况, 有解时并求出解.

解 用初等行变换将增广矩阵化为阶梯阵.

$$\left[\begin{array}{ccc|c} a & 1 & 1 & 4 \\ 1 & b & 1 & 3 \\ 1 & 2b & 1 & 4 \end{array}\right] \xrightarrow{\text{row exchange}} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 1 & 2b & 1 & 4 \\ a & 1 & 1 & 4 \end{array}\right] \xrightarrow[r_3-ar_1]{r_2-r_1} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & b & 0 & 1 \\ 0 & 1-ab & 1-a & 4-3a \end{array}\right]$$

$$\xrightarrow{r_3+ar_2} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & b & 0 & 1 \\ 0 & 1 & 1-a & 4-2a \end{array}\right] \xrightarrow[r_2\leftrightarrow r_3]{r_2-br_3} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & 1 & 1-a & 4-2a \\ 0 & 0 & (a-1)b & 1-4b+2ab \end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc:c}1 & b & 1 & 3\\ 0 & 1 & 1-a & 4-2a\\ 0 & 0 & (a-1)b & 1-4b+2ab\end{array}\right]$$

(1) 当 $(a-1)\,b\neq 0$时, 有唯一解

$$x_1=\frac{2b-1}{(a-1)b},\qquad x_2=\frac{1}{b},\qquad x_3=\frac{1-4b+2ab}{(a-1)b}$$

(2) 当a=1, 且$1-4b+2ab=1-2b=0$, 即 $b=1/2$ 时, 有无穷多解.

化为 $\left[\begin{array}{ccc:c}1 & 1/2 & 1 & 3\\ 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 0\end{array}\right]\rightarrow\left[\begin{array}{ccc:c}1 & 0 & 1 & 2\\ 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 0\end{array}\right]$, 于是方程组的一般解为 $\boldsymbol{x}=(2,2,0)^{\mathrm{T}}+k(-1,0,1)^{\mathrm{T}}$ (k为任意常数）.

(3) 当a=1, $b\neq 1/2$ 时, $1-4b+2ab\neq 0$, 方程组无解.

(4) 当b=0 时, $1-4b+2ab=1\neq 0$ 时, 方程组无解.

（原方程组中后两个方程是矛盾方程）

III. The Rank of a Matrix (矩阵的秩)

Recall the method for solving systems of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ we learnt before:

Covert the *augmented matrix* $[\boldsymbol{A} \mid \boldsymbol{b}]$

into *row echelon form* $[\boldsymbol{U} \mid \boldsymbol{c}]$

or further convert it into *reduced row echelon form* $[\boldsymbol{R} \mid \boldsymbol{d}]$

Since elementary row operations do not change the solutions of the system (初等行变换不改变方程组的解), the three systems of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{Ux} = \boldsymbol{c},\ \boldsymbol{Rx} = \boldsymbol{d}$$

have same solutions (同解).

We have also noticed that the number of free variables for $\boldsymbol{Ax} = \boldsymbol{b}$ depends on the number of <u>non-zero rows of the row echelon form $\boldsymbol{U}$</u>.

This leads to an important parameter (参数) for a matrix.

Definition 1 For a matrix $\boldsymbol{A}$, let $\boldsymbol{U}$ be the row echelon form. Then the **rank** of $\boldsymbol{A}$ ($\boldsymbol{A}$的秩), denoted by rank($\boldsymbol{A}$), is the number of non-zero rows of $\boldsymbol{U}$ (行阶梯形矩阵$\boldsymbol{U}$的非零行的行数).

Obviously, the following properties hold.

• The rank is not bigger than the number of rows.

• Suppose elimination reduces $\boldsymbol{Ax} = \boldsymbol{b}$ to $\boldsymbol{Ux} = \boldsymbol{c}$ and $\boldsymbol{Rx} = \boldsymbol{d}$, with r pivot rows and r pivot columns. **The rank of those matrices is** r.

The last $m-r$ rows of $\boldsymbol{U}$ and $\boldsymbol{R}$ are zero, so there is a solution only if the last $m- r$ entries of $\boldsymbol{c}$ and $\boldsymbol{d}$ are also zero.

• The rank r is crucial. It counts the pivot rows in the “row space” and the pivot columns in the “column space”.

There are $n - r$ special solutions in the nullspace.

There are $m- r$ solvability conditions on $\boldsymbol{b}$ or $\boldsymbol{c}$ or $\boldsymbol{d}$.

Therefore, determining the rank of a given matrix is interesting for understanding the matrix.

To do this, we only need to *use elementary row operations to covert the matrix into row echelon form*.

Example 5 Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}.$$

To find the rank of $\boldsymbol{A}$, we convert $\boldsymbol{A}$ into row echelon form

$$\boldsymbol{U} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -4 \end{bmatrix}.$$

Thus $\boldsymbol{A}$ has rank 3, called **full rank (满秩)**.

It follows that for any $\boldsymbol{b}$, the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.

More properties hold for the rank of matrices:

$$\operatorname{rank}(\boldsymbol{AB}) \le \min\{\operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{B})\}.$$

Proof (hints) Let ***A***, ***B*** be $m\times n$ and $n\times s$ matrices respectively. If we partition ***A by columns***:

$$\boldsymbol{AB} = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n]\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix} = \left[\sum_{i=1}^{n} b_{i1}\boldsymbol{\alpha}_i, \sum_{i=1}^{n} b_{i2}\boldsymbol{\alpha}_i, \cdots, \sum_{i=1}^{n} b_{in}\boldsymbol{\alpha}_i\right]$$

Then every column of ***AB*** is a combination of the column of ***A***.

Then the dimensions of the column spaces give: $\operatorname{rank}(\boldsymbol{AB}) \le \operatorname{rank}(\boldsymbol{A})$.

Similarly, by partitioning ***B by rows***, we can prove that $(\boldsymbol{AB}) \le \mathrm{r}(\boldsymbol{B})$.

对于线性方程组 $\boldsymbol{Ax}=\boldsymbol{b}$, 下列命题等价:

(1) 方程组有解(或相容);

(2) $\boldsymbol{b}$可由$\boldsymbol{A}$的列向量组线性表示, 即 $\boldsymbol{b}\in C(\boldsymbol{A})$;

(3) $\mathrm{rank}([\boldsymbol{A} \mid \boldsymbol{b}])= \mathrm{rank}(\boldsymbol{A})$, 即增广矩阵的秩等于系数矩阵的秩.

Key words: *free variables, basic variables (pivot variables), special solutions, particular solution, complete solution, rank*

Homework

See Blackboard