
RSA in Real World

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1 Introduction

For centuries, the method of encrypting information has invariably been using a traditional cryptosystem, which is a single-cipher key system that requires group members participating in the encrypted communication to either obtain the key in advance, or to find a way to distribute the key through a secure channel. However, as more and more information is transmitted over the Internet, this traditional way may not provide a good trade-off between security and the cost of protecting the key, therefore, the need to encrypt and protect information becomes a stimulus to develop a new type of cryptosystem. While official messages can be distributed among users using secure channels through dedicated messengers with keys, making up the larger majority that ordinary users who need to achieve secure interactions in insecure networks. A technique that completely separates encryption from decryption, a.k.a. *Public Key Cryptography (PKC)*, has emerged as the best solution to this problem nowadays ^[1].

The RSA algorithm is considered as the most influential PKC algorithm currently, which is resistant to all known cryptographic attacks to date, and has been recommended by ISO as the standard for public key data encryption. In addition, by using the RSA algorithm, one can attach a *digital signature* to the end of the message, which can be easily verified by anyone, but cannot be forged ^[2]. In this project, we will focus on the application of RSA algorithm which can always successfully help people to protect their information in the Internet, a simple attempt to attack RSA is also included.

2 Theorem

2.1 Number Theory: Fundamentals

Throughout the lecture in which we learnt about RSA, the professor emphasized the difficulty of factoring large integers. Actually, the reliability of RSA algorithm just depends on this fact, in other words, the more difficult it is to factor a very large integer, the more reliable the RSA algorithm is. If someone were to find a fast factorization algorithm, the reliability of RSA would be extremely reduced, but it is very unlikely that such an algorithm

would be found.

Though it is easy to check whether a number is a composite number (basically, the $O(\sqrt{n})$ way is quick enough), according to CLRS^[2], up to the date of 2009, it was still not feasible to factorize any 1024-bit number using the best supercomputers and the best algorithms available at the time. It is commonly considered that the 1024-bit RSA key is *basically safe*, and the 2048-bit RSA key is *extremely safe*.

2.2 Attack

When the RSA algorithm uses the well selected p and q as mentioned in the next section, attacking it is really a hard problem. However, we can attempt to attack some "weak" RSAs which may not obey the ISO standard, for example, when the $n = pq$ is too small, or $|p - q|$ is too large or small.

2.2.1 Brute Force

Our first step is to factorize n . ① Applying Miller–Rabin primality test^[3] and quick power, in each term we need to check whether a number is a prime, we use $O(k \log^3 n)$, and even if we check the sumterm (say, p, q), we can get an $O(n \cdot k^2 \log^{1.5} n)$, still acceptable for a "rather small" n ; ② when n is less than 768-bit long (all the possible combinations within this range have been calculated and are saved into FactorDB^[4]), we can simply check the database and thus get p and q .

```
1 int64_t quickPow(int a, int b, int r) {
2     int64_t ans = 1, buff = a;
3     while (b) {
4         if (b & 1) ans = (ans * buff) % r;
5         buff = (buff * buff) % r;
6         b >>= 1;
7     }
8     return ans;
9 }
10
11 bool millerRabbin(int n, int a) {
12     int r = 0, s = n - 1;
13     if (n % a == 0) return false;
```

```

14     while (!(s & 1)) {
15         s >>= 1;
16         r++;
17     }
18     int64_t k = quickPow(a, s, n);
19     if (k == 1) return true;
20     for (int i = 0; i < r; i++, k = k * k % n)
21         if (k == n - 1) return true;
22     return false;
23 }
24
25 bool isPrime(int n) {
26     int tester[] = {2, 3, 5, 7};
27     if (n == 1) return false;
28     for (int i = 0; i < 4; i++) {
29         if (n == tester[i]) return true;
30         else if (!millerRabbin(n, tester[i])) return false;
31     }
32     return true;
33 }

```

Trivially, in the next step, with $d \cdot e \equiv 1 \pmod{\phi(n)}$ (here $\phi(n) = p \cdot q$), we can use the Extended Euclidean Algorithm to find the modular inverse of e , a.k.a. d , which only costs $O(\log(\min(p, q)))$.

```

1 void exgcd(int a, int b, int &g, int &x, int &y) {
2     if (!b) g = a, x = 1, y = 0;
3     else exgcd(b, a % b, g, y, x), y -= x * (a / b);
4 }
5
6 inline int modinv(int num, int64_t mod) {
7     int g, x, y;
8     exgcd(num, mod, g, x, y);
9     return ((x % mod) + mod) % mod;
10 }

```

2.2.2 Low Cryptographic Index Attack

Suppose that when e is extreme small (certainly this does not obey the RSA standard), attacker should manually attempt the case of $e = 1$ and $e = 2$, also $e = 3$ can be tried: ① if the original message is not large enough, a.k.a. $m^3 < n$, we can straightly get $m = \sqrt[3]{n}$; ② if $\sqrt[3]{n}$

makes no sense, one can try several small integer k , s.t. $\sqrt[3]{C \pm k \cdot n}$ can be identified as the plaintext.

Another similar case is *broadcast attack*^[5]. When the chosen encryption index is low and the same message is sent to a group of receivers using the same encryption index, then a broadcast attack can be performed to get the plaintext. In this case, we get different n (modulo) and c (ciphertext) for different messages, but they share the same m (plaintext) and e (encryption index). In this case, the CRT could be applied to solve the only solution. Suppose we got k groups of m and e :

$$\begin{cases} C_1 \equiv m^e \pmod{n_1} \\ C_2 \equiv m^e \pmod{n_2} \\ \dots \\ C_k \equiv m^e \pmod{n_k} \end{cases}$$

Since $n_1 \cdots n_k$ are pairwise prime, the congruence equation group has the only solution:

$$m^e \equiv \sum_{i=1}^k C_i M_i y_i \pmod{M}$$

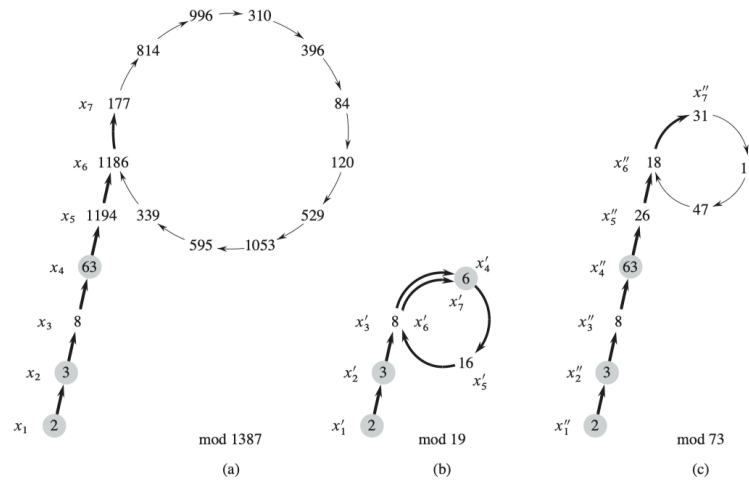
where $M = \prod_{j=1}^k n_j$, $M_i = \frac{M}{n_i}$ and $M_i y_i \equiv 1 \pmod{n_i}$. Then we can almost get the m^e , only small integer k need to be applied that $m^e = \sum_{i=1}^k C_i M_i y_i + kM$.

A similar case is *low decryption index attack*, where e is large and n seems difficult to be factorized, formally we say $d < \frac{1}{3}n^{1/4}$ and $q < p < 2q$. Wiener's attack^[6] can find the solution in polynomial time.

2.2.3 Pollard's rho heuristic

When the *improper* paragrams p and q are selected, say, $|p - q|$ is either too large or too small, we can apply *the Format method* or *the Pollard rho method* to decompose n quickly. The Pollard's rho heuristic is an effective algorithm to factorize large numbers, consider an integer R , the traditional way of trial division by all integers up to R is able to factorize any number less than R^2 , while Pollard's rho heuristic can (probably) find the solution for integers up to R^4 (for our target n , it seems Pollard's rho takes $O(n^{1/4})$, however, considering the input length, a.k.a. $\ell = \log_2 n$, we say it takes $O(2^{\ell/4})$, which is exponent), using the same

time complexity ^[2,7]. This algorithm first randomly choose a small integer x_1 as a seed, then run a function (s.t. $f(x_1) = x_1^2 + a$) to find x_2 . If $\gcd(x_1 - x_2, n) \neq 1$, the result is recorded as one factor of n , otherwise, we back the first step and take x_2 as the previous n .



The **YAFU** project is a practical tool which implements several algorithms to factor large integers. The simple test demonstrates decrypting a RSA-512 encrypted message:

```

1  -----BEGIN RSA PRIVATE KEY-----
2  MIIBOQIBAAJAbq4Z0mGxWeeJPaCcWlNNxZV1SJVQ10IxAAAnASG7Zk1B38q3X3Uj+
3  tUYdsU2PcHWJvxu1E0K9+ETHTMqk/Zye1wIDAQABAKBcndG7y8Y13lto5K+vwDQ
4  ex3WrCQmzRv1lUMTGbd13n+8ExvmvWokSNbK+wSuhE8oSTkBeALgo4AzuZ+mVY1B
5  AiEAvNqMqFiVdrMfQWCze0IkFCDEupOpVa4qEyXqxXVwArcCIQCWCDGUc5FvvGqB
6  056ydSpDcGBfuL7NKt5iUL0Bffuk4QIhAKK+Q1AfZk2v9lNEneauDKE7y8xsyxQG
7  zkNJ/ZLDrQ7pAiBnd90hGRYi1fEpkQFgF3d/L0f5+8HyYocdjIrcLZLPYQIgDbTD
8  czwDs36QYNnV/hheqxaHPsYLBjeXW8MfzWa09ZE=
9  -----END RSA PRIVATE KEY-----
10
11  -----BEGIN PUBLIC KEY-----
12  MFswDQYJKoZIhvcNAQEBBQADSwAwRwJAbq4Z0mGxWeeJPaCcWlNNxZV1SJVQ10Ix
13  AAAnASG7Zk1B38q3X3Uj+tUYdsU2PcHWJvxu1E0K9+ETHTMqk/Zye1wIDAQAB
14  -----END PUBLIC KEY-----
15
16  SUSTech CS201 ==Encrypted==> MHo0LqFCa3eRlSc+BZvf5ZPlE7Nu4ge1Xef/7VnfgewLD0+kLkmc6gJfGs6+
    M2601JKpBHrRCyvKvSau7JCrmg==

```

```

1  def pollard_rho(self, n, seed=2, p=2, c=1):
2      if n & 1 == 0:
3          return 2
4      if n % 3 == 0:
5          return 3

```

```

6  if n % 5 == 0:
7      return 5
8  if is_prime(n):
9      return n
10 f = lambda x: x ** p + c
11 x, y = seed, seed
12 while True:
13     x = f(x) % n
14     y = f(f(y)) % n
15     d = gcd((x - y), n)
16     if d > 1:
17         return d

```

The code is borrowed from RsaCtfTool^[8], the slow version uses the above python code, and the fast version uses YAFU.

2.3 Standards

As the several possible attacks we discussed above, we learn that the RSA algorithm highly depends on a pair of properly selected p and q . They should not only long enough, but also different properly in their values. The *PKCS #1* (Public-Key Cryptography Standards) defined in RFC 3447^[9] specified the mathematical definitions and properties that RSA public and private keys must have, which should basically avoid the cases discussed above that may make attacking it easier.

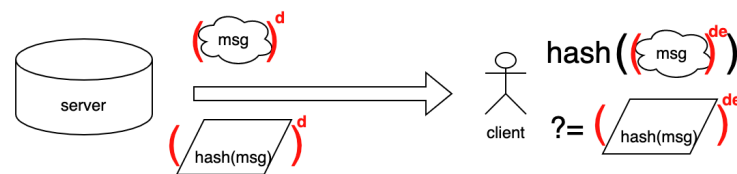
Basically, a standard RSA key should include at least there features: ① the n id at a length of $2^{8(k-1)} \leq n < 2^{8k}$, where the length k of the modulus must be at least 12 octets to accommodate the block formats in PKCS #1; ② multi-prime keys are allowed since version 2.1, which lets $n = \prod_{i=1}^r p_i$ thus making n harder to be cracked.

3 Application: HTTPS

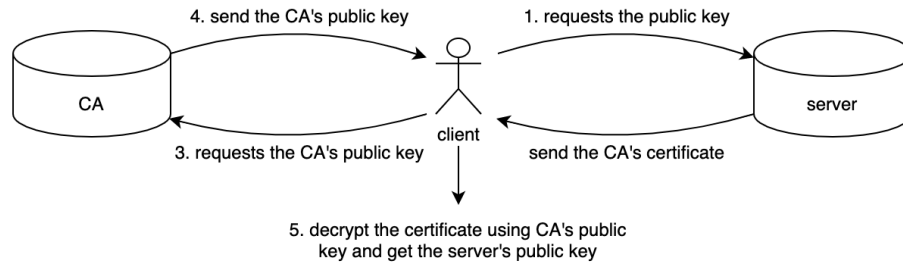
When it comes to the Internet, it is natural to mention the *HTTPS* protocol. The HTTP protocol uses plaintext, which means that when a packet is transmitted using the HTTP protocol and is unfortunately cut off in the middle, the data inside will be completely exposed. Therefore, if the packet contains the user's account and password, it could be assumed that

the user's account number and password have been leaked. The HTTPS protocol uses SSL to encrypt the data, so even if the data is intercepted, the user's data cannot be known without the decryption (private) key.

When clients are communicating with the server, each client uses the same public key of the server, which should make only the server that can decrypt the information. However, when the server sending data to the client, it is also necessary to let the client know the data it received keeps unchanged from the server, in this case, we should introduce the digital signatures. In short, the digital signature is to form a digest of the sent data using a hash function and then encrypt it with the server's private key; when the client gets the data, it apply the same hash function to the data it receives and compares it with the digest decrypted with the public key, if the two results have no difference, it means that the data transmitted in the middle is not tampered.



Still there exists one quistion: how to make sure that we are communicating with the "real" server (that is, to prevent an attacker from intercepting a client's request at the proxy server level, redirecting it to his own server, and giving the client the public key of that server, so that the client mistakenly thinks he is interacting with the correct server)? For this problem, the HTTPS protocol built a concept: *authoritative third-party authority (CA)*. When a server sends its public key and some private information to the CA, the CA encrypts these data with its own private key, this is called the digital certificate (*SSL certificate*). When the server sends data to a client, it also sends the certificate downloaded from CA to the local, when the client gets the certificate, again it decrypts the certificate with CA's public key to get the real server's public key. The client-server interaction process under the https protocol can be summarized as follows:



As for the SSL part, another famous tool is OpenSSL, which of course supports several operations about RSA.

```

[I] ~ λ openssl genrsa -out private.pem
Generating RSA private key, 2048 bit long modulus
..+++
.....+++
e is 65537 (0x10001)
[I] ~ λ openssl rsa -pubout -in private.pem -out public.pem
writing RSA key
  
```

(a) Using OpenSSL to generate RSA public and private keys

```

File: private.pem
1 -----BEGIN RSA PRIVATE KEY-----
2 MIIEowIBAAKCAQEAoPb80cLoP10qBPN715Bv9atvjTD09chxlv1ayGkrVEJD
3 TkVdPhZtT8B/Hag7vvyYNNd0kSt1osM0Jk3bqrfSwednGf+/t2m/lg179L
4 6G1Sh1x0a1MYB0zyUR9rX0J1E0BJ50a8mr+LzySutkj9vdlUu6qCaJFcpGSY
5 1b5QXB/2he123TMBG6cGr1821udUyVqAU3Ygt1AcjZT7K9MT0Hm+ZTeerW
6 H06G+qLr1Vopngz2SE155+HxHv8BP9QZ8R7Fc4VH3H1507JMSQ118pFFns
7 8WdA1445jPFW0v7KChCt+xyBwepd1j97420vID0AB8aQ7B8AS1532rY/Vm2E/
8 J840mrgw4dL/5c1r5N3SN/Vkpd19Dacw/KoXq9NG8qVXUHy779jegr7X
9 T0a4953J2oaQCKOE58gsxz1PZD1w9xzE0AEQ6RtM1/1uXhsGwM/N5Q
10 B1CPM67AL3dPM6Gfs1W6R0dW011UK3CfD6zw7Feg5W6L7R3VZeJ1rY3YQp1f
11 4/LrjQZ+SPR7qAqnsM41VACZ/adMlrn9F48cMnYgwbE17+JH6Gm2e7Ku5
12 P0AungrFLm0WQ/dH8m123Y0vLKE0f2ePw4N4CA48R09m2L00Rk1mFH0Z0U
13 K87MPKcYEA1R0Mtdr9p9X0b+3hG1R0b0drR0X090kS1U0m0a7+Kf0X1n
14 qmLupM4C3gT0PTC3JULFR1xKq08K21t05/Noq07qK0K5ZCSP1X+3x1Mkt
15 1RdT48056w+z1X51sPg3aSPH030vgn0r0a+61vev19b10U0r1ZBUCgYAwBjM
16 JJD1AR48VPha5E67+SF3205r78aBukFp/KP7eJ1E/XZLaPKLjL3LucZKKG
17 8AuFT+q5vCG5AH9BqbnZ17LAmYteJknMEBRvz0Pq/w0G6Pnhy19HMeCuzc
18 v5tzmYKvq+75Yqtb5eWxuy6TubHn78c7S06Cg1Bje/DJKC3/ez0M78aUAA
19 PmH7Fmujef4Dux21r0Z0uHt1Q0m1X0p1J0q0R0A7Zcc2Dq1K0R0M0WS00
20 D120K1R0S0F/Rq+C0x0Z0pVYn0K0Wf0wZ2a2+4fC5y1E5482EYAN0p0f0g0v
21 c/DUVf0e1A0U0EG7p0I0z0o0K0B0B0c0B81J6P0c1W0xc1YFxB7T5E6XfKf0B0X
22 u+X0a93btKVMUdK4B0UB0NAYB+Yf/gb4F092JMP0M0X+GR1G2b3y0NA/68P+
23 I20a5k03c0vFKce/uU/CmKq5ma81e08e6f9aK3d0Hd0B0aH0E1X0eR0xJ01
24 11X0A0G0AK1PLp0v03D0TLwJAS0B2QJ2VXKSF+0x0BMYdF0G0B80L0cc1JUBJ
25 Ky6Kx1LgR0R0W1L03g1Yf0u0C1B0W0p3P0B0p0G0z0u0B20Igc1Dx0Q0W7
26 1Yh0v0H0w0d0K7Z0TFFC4Z04uJH7K2g0L1r0u0r0B0M0E0K0V0D
  
```

(b) Private key generated in last step

```

[I] ~ λ bat demo.plaintext
File: demo.plaintext
1 SUSTech
2 CS201
3 离散数学

[I] ~ λ openssl rsautl -pubin -encrypt -in demo.plaintext -inkey public.pem -out demo.encrypted
[I] ~ λ hexdump demo.encrypted
00000000 61 a5 25 5a a5 a1 a6 b6 29 68 aa 77 c5 c2 8f fb
00000010 65 34 19 ed ad cf eb ee e5 b8 70 c6 53 a5 0b 29
00000020 8c 35 d0 15 df 3d cb 82 4d fb 27 e1 6a 8d fd 21
00000030 b9 19 43 f5 c7 fa 46 ef 6c 84 a7 96 46 18 e5 5c
00000040 8a 3b 64 66 3b 3d 10 7a 87 54 8b ae de 2a ed b8
00000050 ac f3 ad 45 5a ac c3 4a 0f 83 17 38 a7 0d d6 31
00000060 71 87 b2 39 e9 96 d2 d5 da d8 12 d8 e1 91 e6 e7
00000070 67 e6 92 73 6e 63 65 a2 cc 94 55 5e eb 9f 20 4f
00000080 7f 19 9a f5 2e 65 eb 92 72 a3 ef 95 c7 1f 74 9c
00000090 a7 85 8b 7c 6b 36 15 bc ed 90 24 35 6b c0 af 86
000000a0 78 b0 3b 1d 16 f8 ef af 64 ef 7b b9 0d 69 77 cb
000000b0 7d 89 d3 0c 9e 5a f3 aa 47 4e 4b 56 17 a6 20 04
000000c0 a6 6e 6b bd c9 03 3f b1 d5 3d 56 5d 26 bd 82 d6
000000d0 a3 71 2e ed 63 8c 21 f1 3f fd 3b 5f 48 11 2b 75
000000e0 a3 c5 fc 53 6c f9 1a 7e 7b 51 87 51 a0 ab 78 1e
000000f0 a1 49 a8 6b 0d 88 6e c9 04 74 6d e1 ca 17 48 ac
00001000
  
```

(c) Applying the public key to encrypt the plaintext, then we can view the result in hex form

```

[I] ~ λ openssl rsautl -decrypt -in demo.encrypted -inkey private.pem -out demo.decrypted
[I] ~ λ bat demo.decrypted
File: demo.decrypted
1 SUSTech
2 CS201
3 离散数学
  
```

(d) Use the private key to decrypt the encrypted file

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