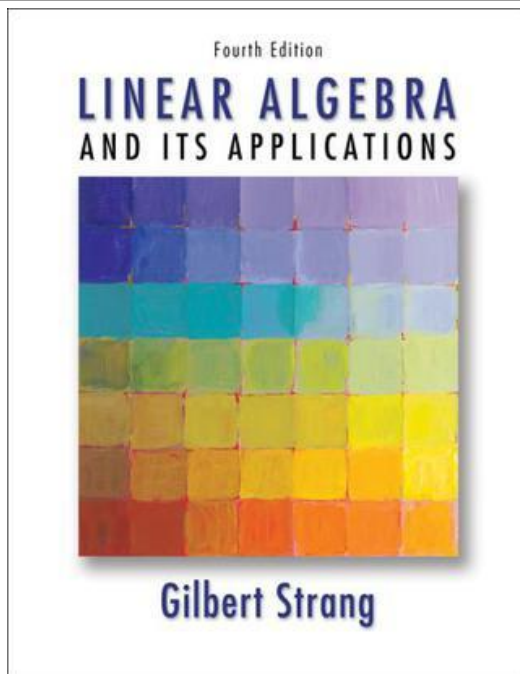


Linear Algebra



Instructor: Jing YAO

1

Matrices and Gaussian Elimination

1.4

MATRIX OPERATIONS

(矩阵运算)

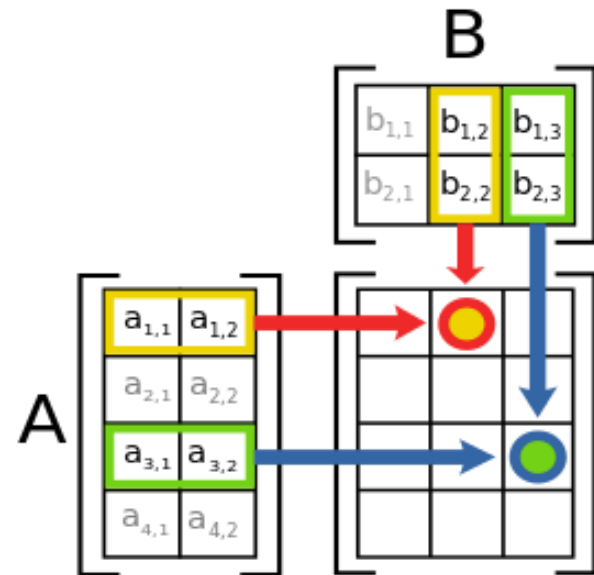
Addition

Scalar multiplication

Multiplication

Power

Elementary Matrices



MATRIX NOTATION

A matrix is an arrangement of mn elements with m rows and n columns, denoted by

$$\begin{array}{c} \text{Column } j \\ \text{Row } i \end{array} \left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array} \right] = A$$

$$A = [a_{ij}]_{m \times n}$$

If A is an $m \times n$ matrix, that is, a matrix with m rows and n columns, then the scalar entry in the i th row and j th column of A is denoted by a_{ij} and is called the (i, j) -entry of A .

THE ORDER IS IMPORTANT: rows \times columns

- If two matrices have the same number of rows and the same number of columns, then they are called **matrices of the same size (同型矩阵)**.

For example, $\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 3 & 7 \end{bmatrix}$ and $\begin{bmatrix} 14 & 3 \\ -8 & 4 \\ 3i & 9 \end{bmatrix}$.

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, and the corresponding entries are the same, i.e.,

$$a_{ij} = b_{ij} \quad (i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n),$$

then A and B are **equal (相等)**, denoted by $A = B$.

Attention! *equal* vs *equivalent* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = B$

Some special matrices

(1) 行数与列数都为 n 的矩阵称为 n 阶**方阵**(**square matrix**), 可记作 A_n .

例如,
$$\begin{bmatrix} 13 & 6 & 2i \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$
 是一个3阶复方阵.

(2) $1 \times n$ 矩阵 $A = (a_1, a_2, \dots, a_n)$ 称为**行矩阵**(或**行向量**, **row vector**).

$m \times 1$ 矩阵 $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ 称为**列矩阵**(或**列向量**, **column vector**).

(3) 方阵 $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ 称为**对角矩阵**(或**对角阵**),
 (主) 对角线
 (main diagonal)
 记作 $\text{diag}(a_{11}, a_{22}, \dots, a_{nn})$.

对角线上元素全相等的对角阵称为**数量矩阵**(**scalar matrix**).

对角线上元素全为 1 的对角阵称为**单位矩阵**
 (**identity matrix**), 记为 \mathbf{I}_n 或 \mathbf{I} , 即

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(4) 元素全为零的矩阵称为**零矩阵(zero matrix)**,记作 **0**.

不同型的零矩阵是不同的.

(5) 对角线左下(右上)方的元素都为 0 的方阵称为
上(下)三角矩阵 (upper/lower triangular matrix).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & \\ \vdots & \vdots & \ddots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

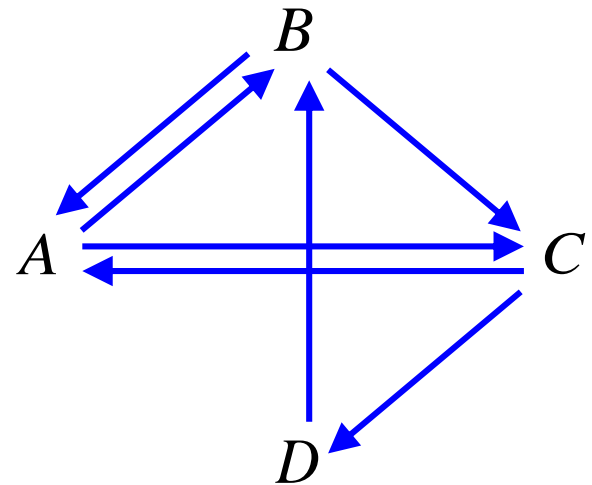
引例(introductory example):

城市间的航班图

如果从 A 到 B 有航班, 则
用带箭头的线连接 A 与 B .

航班图可用矩阵来表示:

		到达城市			
		A	B	C	D
出发城市	A	0	1	1	0
	B	1	0	1	0
	C	1	0	0	1
	D	0	1	0	0



引例(introductory example):

城市间的航班图

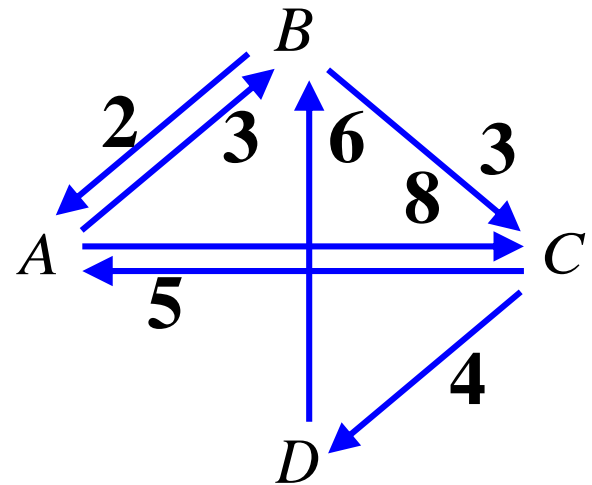
如果从 A 到 B 有航班, 则
用带箭头的线连接 A 与 B .

客流量也可用矩阵来表示:

$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第一天

第二天



城市间的航班图

如果从 A 到 B 有航班, 则用带箭头的线连接 A 与 B .

客流量也可用矩阵来表示:

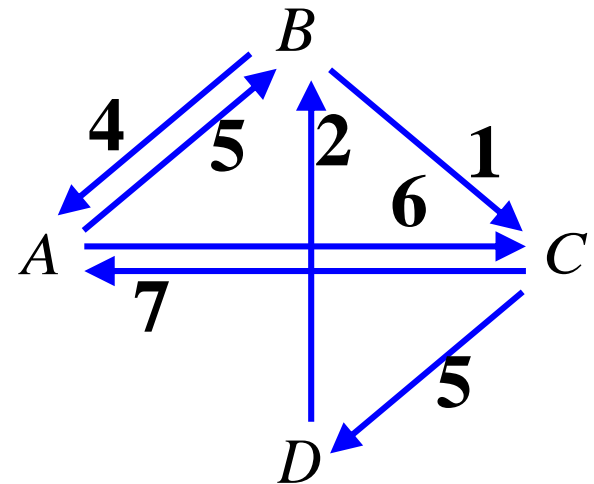
$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第一天

$$\begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

第二天

第三天...



问题: 各城市2天内发送的客流量?

Matrix operations (矩阵的运算)

Matrix operations

(矩阵的运算)

I. Addition (矩阵的加法)

II. Scalar multiplication (数与矩阵相乘)

III. Multiplication (矩阵的乘法)

IV. Power (方阵的幂)

Elementary Matrices (初等矩阵)

I. 矩阵的加法(Addition)

1. 定义 (Definition)

each entry in $A+B$ is the sum of the corresponding entries in A and B .

设有两个 $m \times n$ 矩阵 $A = [a_{ij}]$ 和 $B = [b_{ij}]$, 那么矩阵 A 与 B 的**和(sum)**记作 $A+B$, 规定为

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

注 只有两个矩阵同型时, 才能进行加法运算;

(The sum $A+B$ is defined only when A and B are the same size.)

例如

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & -9 & 0 \\ 3 & 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 9 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+1 & 3+8 & -5+9 \\ 1+6 & -9+5 & 0+4 \\ 3+3 & 6+2 & 8+1 \end{bmatrix} = \begin{bmatrix} 13 & 11 & 4 \\ 7 & -4 & 4 \\ 6 & 8 & 9 \end{bmatrix}.$$

2. 矩阵加法的运算规律

Let A , B , and C be matrices of the same size, then

$$(1) \quad A + B = B + A;$$

$$(2) \quad (A + B) + C = A + (B + C);$$

$$(3) \quad -A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m1} & \cdots & -a_{mn} \end{bmatrix} = [-a_{ij}];$$

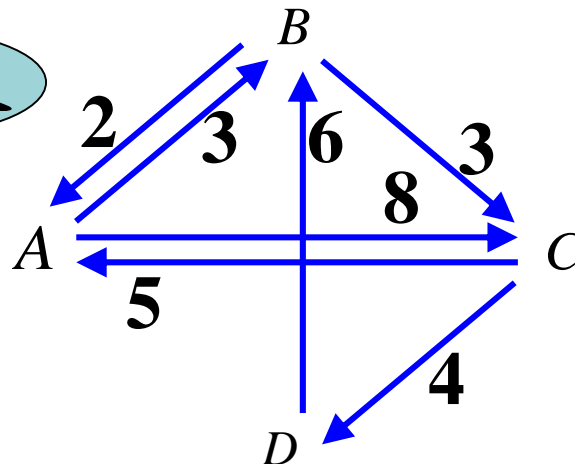
$$(4) \quad A + (-A) = \mathbf{0}, \quad A - B = A + (-B).$$

定义矩阵的减法(subtraction)

思考 城市间航班客流量

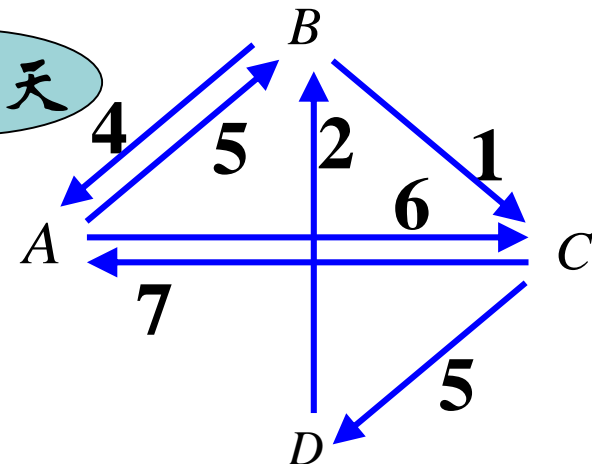


第一天



$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第二天



$$\begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

问题1: 各城市2天内发送的客流量?

问题2: 收取的机场建设费有多少?

II. 数与矩阵相乘(Scalar multiplication)

1. 定义

数 λ 与矩阵 A 的乘积 (简称为数乘, scalar multiple)

记作 λA 或 $A\lambda$, 规定为

$$\lambda A = A\lambda = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}.$$

If λ is a scalar and A is a matrix, then the **scalar multiple** λA is the matrix whose columns are λ times the corresponding columns in A .

2. 数乘矩阵的运算规律

Let A , B , and C be matrices of the same size ($m \times n$), and let λ and μ be scalars, then

$$(1) (\lambda\mu)A = \lambda(\mu A);$$

$$(2) (\lambda + \mu)A = \lambda A + \mu A;$$

$$(3) \lambda(A + B) = \lambda A + \lambda B.$$

矩阵的加法与数乘统称为矩阵的线性运算.

Example 1 Let $\mathbf{A} - 3\mathbf{B} = 4\mathbf{A} - \mathbf{C}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Find \mathbf{C} .

Solution From $\mathbf{A} - 3\mathbf{B} = 4\mathbf{A} - \mathbf{C}$, we have

$$\mathbf{C} = 4\mathbf{A} - \mathbf{A} + 3\mathbf{B} = 3(\mathbf{A} + \mathbf{B}),$$

$$\text{so } \mathbf{C} = \begin{bmatrix} 3(1+1) & 3(-1-1) \\ 3(0+0) & 3(2+1) \\ 3(3-1) & 3(1+0) \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 9 \\ 6 & 3 \end{bmatrix}.$$

III. 矩阵乘法(Multiplication)

引例1 超市购物

=>Product of Matrices

同样的商品在不同的超市内的售价是不尽相同的. 这样, 在一次需要购买多种商品时, 就有到哪一家超市去买花费最少的问题.

这就要用到价格矩阵, 如

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \quad \begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$$

可用来表示3家超市里4种商品的“价目表”
第1行的元依次表示超市1里4种商品的售价

III. 矩阵乘法(Multiplication)

引例1 超市购物

x_{11}	x_{12}	x_{13}	x_{14}	1.7	1.1	21	7	超市1
x_{21}	x_{22}	x_{23}	x_{24}	1.5	1.4	26	9	超市2
x_{31}	x_{32}	x_{33}	x_{34}	1.8	1.3	28	8	超市3

购物者1对4种商品的需求分别为 $a_{11}, a_{21}, a_{31}, a_{41}$ ，
则在不同超市去购买所需花费总额为？

若有 n 名购物者，则可将他们的需求构成需求矩阵

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$$

那么这 n 名购物者的采购
方案可以用一个数表来表示：
总价矩阵 $\equiv ?$

$$\text{价格矩阵} \times \text{需求矩阵} = \text{总价矩阵}$$

	商品1	商品2	商品3	商品4	购物者1	购物者2	购物者 n	
超市1	$\begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$	需求1						
超市2				需求2					
超市3					需求3				
				需求4					

	购物者1	购物者2	购物者 n
超市1	$\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \end{bmatrix}$			
超市2				
超市3				

矩阵乘法(Multiplication)

引例2 数学例子

设 x_1, x_2, x_3 和 y_1, y_2 是两组变量, 它们之间的关系为

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{aligned} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = [a_{ik}]_{2 \times 3}$$

又设 t_1, t_2 是另一组变量, 它们与 x_1, x_2, x_3 的关系为

$$\begin{aligned} x_1 &= b_{11}t_1 + b_{12}t_2 \\ x_2 &= b_{21}t_1 + b_{22}t_2 \\ x_3 &= b_{31}t_1 + b_{32}t_2 \end{aligned} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = [b_{kj}]_{3 \times 2}$$

则

c_{11}

$$y_1 = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})t_1 + (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})t_2$$

$$y_2 = (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})t_1 + (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})t_2$$

矩阵 $C = [c_{ij}]_{2 \times 2}$ 是矩阵 A 与 B 的一个运算, 定义为矩阵的乘积.

矩阵乘法(Multiplication)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

其中

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj},$$

Row-column rule for computing AB

上式右边 (i, j) 元素 c_{ij} 等于左边的第一个矩阵的
第 i 行与第二个矩阵的第 j 列对应元素乘积之和。

矩阵运算中具有的特殊规律, 主要产生于矩阵的乘法运算。

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

Each entry of \mathbf{AB} is the product (乗積) of a *row* and a *column*:
 $(\mathbf{AB})_{ij} = (\text{row } i \text{ of } \mathbf{A}) \text{ times } (\text{column } j \text{ of } \mathbf{B})$

Each column of \mathbf{AB} is the product of a *matrix* and a *column*:
 $\text{column } j \text{ of } \mathbf{AB} = \mathbf{A} \text{ times } (\text{column } j \text{ of } \mathbf{B})$

Each row of \mathbf{AB} is the product of a *row* and a *matrix*:
 $\text{row } i \text{ of } \mathbf{AB} = (\text{row } i \text{ of } \mathbf{A}) \text{ times } \mathbf{B}$

矩阵乘法(Multiplication)

1. Definition

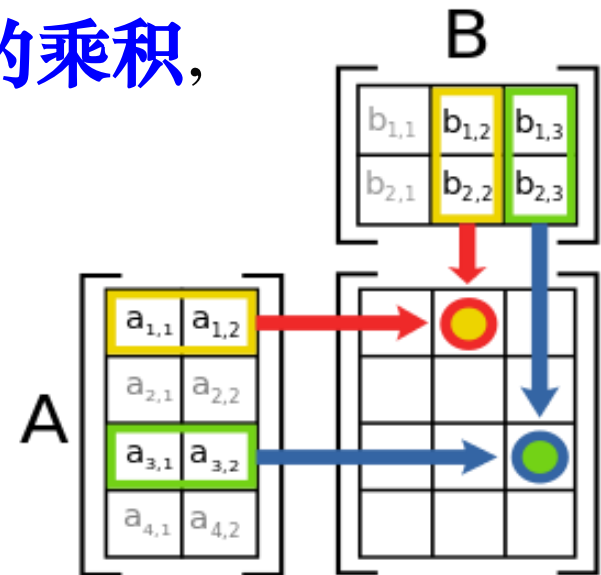
设 $A = [a_{ik}]_{m \times p}$, $B = [b_{kj}]_{p \times n}$ 为两个矩阵, 令

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj},$$

$$i = 1, 2, \dots, m; j = 1, 2, \dots, n,$$

称 $m \times n$ 矩阵 $C = [c_{ij}]_{m \times n}$ 为 **A 与 B 的乘积**,
记为 **$C = AB$** .

注 只有当第一个矩阵的列数等于第二个矩阵的行数时, 两个矩阵才能相乘.



Example 2 Find AB , where

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & 0 \\ 0 & 5 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Solution:

$$C = AB = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & 0 \\ 0 & 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 7 \\ 10 & 2 & -6 \\ -2 & 17 & 10 \end{bmatrix}.$$

3×4 4×3 3×3

Match
Size of AB

Exercises

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 & 8 \\ 6 & 0 & 1 \end{bmatrix} = ?$$

注 只有当第一个矩阵的列数等于第二个矩阵的行数时，两个矩阵才能相乘.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = [1 \times 3 + 2 \times 2 + 3 \times 1] = 10$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

□ **Example** Let A and B be $n \times 1$ and $1 \times n$ matrices, and

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, B = [b_1 \quad b_2 \quad \cdots \quad b_n]$$


Compute AB and BA .

□ **Solution**

$$AB = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{bmatrix}$$

$$BA = [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]$$

A system of
linear equations



[illegible]

The i -th equation:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$(a_{i1}, a_{i2}, \dots, a_{in}) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_i \quad (i = 1, 2, \dots, m)$$

$$\text{Let } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$Ax=b$$

System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

Vector Equation

Matrix Equation

$$Ax = b.$$

Coefficient Matrix


$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$$

Augmented Matrix

$$(A, b) = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \ b]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$


Solution
(解向量)

System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$



Coefficient Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

Vector Equation



Matrix Equation


$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$



Augmented Matrix

$$(\mathbf{A}, \mathbf{b}) = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \ \mathbf{b}]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$


Solution
(解向量)

2. Rules for Matrix Multiplication

(1) Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$, $C = [c_{ij}]_{p \times r}$, then

$$(AB)C = A(BC), \quad k(AB) = (kA)B = A(kB).$$

associative law of multiplication

(2) Let $A = [a_{ij}]_{m \times p}$, $B = [b_{ij}]_{p \times n}$, $C = [c_{ij}]_{p \times n}$, $D = [d_{ij}]_{n \times s}$, then

$$A(B + C) = AB + AC, \quad (B + C)D = BD + CD.$$

left distributive law

right distributive law

(3) Let $A = [a_{ij}]_{m \times n}$, I_m , I_n are identity matrices of degree m and n respectively, then

$$A = I_m A = A I_n; \quad kA = (kI_m)A = A(kI_n).$$

identity for matrix multiplication

证明: $(AB)C=A(BC)$.

设 $A=(a_{ij})_{m \times n}$, $B=(b_{ij})_{n \times p}$, $C=(c_{ij})_{p \times r}$, 则 $(AB)C$ 与 $A(BC)$ 都是 $m \times r$ 矩阵.

只需证明: $\forall i=1, \dots, m, \forall j=1, \dots, r$, 有

$$\begin{aligned}
 [(AB)C]_{ij} &= \sum_{k=1}^p (AB)_{ik} C_{kj} = \sum_{k=1}^p \left(\sum_{l=1}^n a_{il} b_{lk} \right) c_{kj} \\
 &= \sum_{l=1}^n a_{il} \left(\sum_{k=1}^p b_{lk} c_{kj} \right) \quad \text{交换和号顺序} \\
 &= \sum_{l=1}^n a_{il} (BC)_{lj} = [A(BC)]_{ij}
 \end{aligned}$$

所以 $(AB)C=A(BC)$.

问题： 矩阵乘法是否满足**交换律(commutative law)**,
即 $AB = BA$?

$$\begin{bmatrix} 1 & 6 & 8 \\ 6 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

不一定同
时有意义

不一定
同型

不
相等

例外：
可交换

The product of two matrices is not commutative:

AB is **not necessarily** equal to BA .

For example, if $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix},$$

therefore $AB \neq BA$.

Warnings:

1. In general, $AB \neq BA$.
2. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = \mathbf{0}$ or $B = \mathbf{0}$.
3. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.

投票： 两个上(下)三角阵 A 与 B 的乘积 AB 是否仍是上(下)三角阵？

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \quad C = AB = (c_{ij})_{n \times n}$$

其主对角元 $(AB)_{ii} = ?$

证明： 两个上(下)三角阵 A 与 B 的乘积 AB 仍是上(下)三角阵, 且其主对角元 $(AB)_{ii}=a_{ii}b_{ii}$.

证 设 $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \quad C=AB=(c_{ij})_{n \times n}$

$i > j$ 时, $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^n a_{ik} b_{kj}$

$$a_{ik} = 0 \quad (k = 1, 2, \dots, i-1)$$

$$b_{kj} = 0 \quad (k = i, i+1, \dots, n)$$

即 $i > j$ 时, $c_{ij} = 0$ C 为上三角阵

而 $c_{ii} = \sum_{k=1}^n a_{ik} b_{ki} = \sum_{k=1}^{i-1} a_{ik} b_{ki} + \sum_{k=i}^n a_{ik} b_{ki}$
 $= 0 + a_{ii} b_{ii} = a_{ii} b_{ii}$.

$$a_{ik} = 0 \quad (k = 1, 2, \dots, i-1)$$

$$b_{ki} = 0 \quad (k = i+1, \dots, n)$$

IV. 方阵的幂(Power)

1. 定义

设 A 是 n 阶方阵, 定义 A 的幂为

$$A^0 = I, \quad A^1 = A, \quad A^2 = A^1 A^1, \quad \dots, \quad A^{k+1} = A^k A^1.$$

注 只有方阵, 它的幂才有意义.

2. 矩阵的幂的运算规律

设 k, l 为非负整数, 则

$$(1) \quad A^{k+l} = A^k A^l;$$

$$(2) \quad (A^k)^l = A^{kl}.$$

由矩阵乘法不满足交换律, 一般地 $(AB)^k \neq A^k B^k$.

但也有例外, 如设 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$, 则有

$$AB = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, BA = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, \Rightarrow AB = BA.$$

注 当 A 与 B 可交换时, 有

$$(AB)^k = A^k B^k,$$

$$(A + B)^2 = A^2 + 2AB + B^2,$$

$$(A - B)(A + B) = A^2 - B^2.$$

设

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

为一元多项式, 其中 a_0, a_1, \dots, a_n 为多项式的系数.

设 A 为方阵, 则称

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I$$

为**方阵 A 的多项式**.

(3) 设 $f(x), g(x), h(x)$ 为一元多项式, A 是方阵.

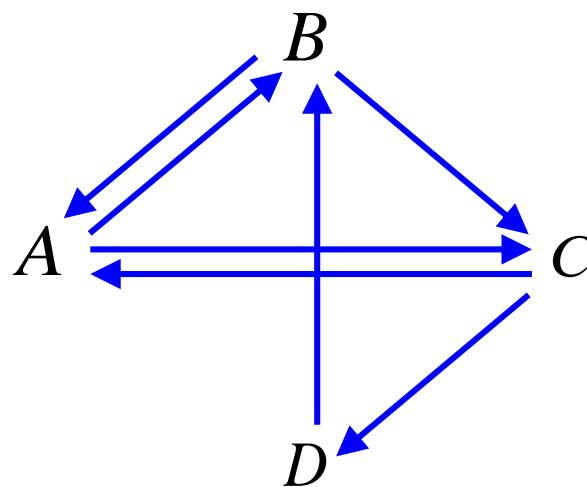
如果 $f(x) = g(x) h(x)$, 则 $f(A) = g(A) h(A)$.

城市间航班图



航班图可用矩阵来表示:

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$



练习

1: 计算 M^2 ?

2: 思考 M^2 中元素的实际含义.

$$M^2 = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Example 3 Compute $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n$.

Solution Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then \mathbf{A} is a scalar matrix, and $\mathbf{AB} = \mathbf{BA}$, Therefore,

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n &= (\mathbf{A} + \mathbf{B})^n = \sum_{k=0}^n \mathbf{C}_n^k \mathbf{A}^{n-k} \mathbf{B}^k \\ &= \mathbf{C}_n^0 \mathbf{A}^n \mathbf{B}^0 + \mathbf{C}_n^1 \mathbf{A}^{n-1} \mathbf{B} + \mathbf{C}_n^2 \mathbf{A}^{n-2} \mathbf{B}^2 + \cdots \mathbf{C}_n^n \mathbf{B}^n. \end{aligned}$$

Since $\mathbf{B}^2 = \mathbf{0}$, we have $\mathbf{B}^2 = \mathbf{B}^3 = \dots = \mathbf{B}^n = \mathbf{0}$. And

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n = \mathbf{A}^n + n\mathbf{A}^{n-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} n3^{n-1} & 0 \\ 0 & n3^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} 0 & n3^{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{bmatrix}.$$

V. Elementary Matrices

定义 由单位矩阵 I 经过一次初等变换得到的方阵称为**初等矩阵**. (An **elementary matrix** is one that is obtained by performing a single elementary operation on an identity matrix.)

三种初等变换对应着三种初等矩阵:

- (1) 对调两行或两列——**初等对换矩阵**;
- (2) 以数 $k \neq 0$ 乘某行或某列——**初等倍乘矩阵**;
- (3) 以数 k 乘某行(列)加到另一行(列)上去——**初等倍加矩阵**.

(1)初等对换矩阵:

将单位矩阵的第 i, j 行(或列)对换

$$P_{ij} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ \leftarrow \text{第 } i \text{ 行} & & 0 & \cdots & 1 & \\ & & \vdots & & \vdots & \\ & & & 1 & & \\ \leftarrow \text{第 } j \text{ 行} & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

第 i 列 第 j 列

(2)初等倍乘矩阵:

将单位矩阵第 i 行(或列)乘 $k \neq 0$

$$D_i(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & k & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{第 } i \text{ 行} \\ \text{第 } i \text{ 列} \end{array}$$

(3)初等倍加矩阵:

将单位矩阵第 i 行乘 k 加到第 j 行,
 或将第 j 列乘 k 加到第 i 列

$$E_{ij}(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & k & \cdots & 1 \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

← 第 i 行

← 第 j 行

第 i 列 第 j 列

■ **Example 4** Let

$$\mathbf{K}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Compute $\mathbf{K}_1\mathbf{A}$, $\mathbf{K}_2\mathbf{A}$, and $\mathbf{K}_3\mathbf{A}$, and describe how these products can be obtained by elementary row operations on \mathbf{A} .

Solution:

Addition of -4 times **row** 1 of **A** to **row** 3 produces $\mathbf{K}_1\mathbf{A}$.

$$\mathbf{K}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}.$$

$$\mathbf{K}_2\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

An interchange of **rows** 1 and 2 of **A** produces $\mathbf{K}_2\mathbf{A}$.

$$\mathbf{K}_3\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Multiplication of **row** 3 of **A** by 5 produces $\mathbf{K}_3\mathbf{A}$.

- **Left-multiplication** (左乘, that is, multiplication on the left) by K_1 in Example 4 has the same effect on any $3 \times n$ matrix.
- Since $K_1 I = K_1$, we see that K_1 *itself* is produced by this same row operation on the identity. (注：由单位矩阵 I 经过一次初等变换得到的方阵称为**初等矩阵**)
- Example 4 illustrates the following general fact about elementary matrices.
- If an elementary row operation is performed on an $m \times n$ matrix A , the resulting matrix can be written as KA , where the $m \times m$ matrix K is created by performing the same row operation on I_m .

What if -- right-multiplication?

Addition of -4 times **column 3** of **A** to **column 1** produces AK_1 .

$$AK_1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a-4c & b & c \\ d-4f & e & f \\ g-4i & h & i \end{bmatrix}.$$

$$AK_2 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}.$$

An interchange of **columns** 1 and 2 of **A** produces AK_2 .

$$AK_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & b & 5c \\ d & e & 5f \\ g & h & 5i \end{bmatrix}.$$

Multiplication of **column** 3 of **A** by 5 produces AK_3 .

初等矩阵与矩阵的乘积

1) 用 m 阶初等矩阵 P_{ij} **左乘** 矩阵 $A = [a_{ij}]_{m \times n}$, 得

$$P_{ij}A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \end{matrix}$$

相当于把矩阵 A 第 i **行** 与第 j **行** 对调 ($r_i \leftrightarrow r_j$).

2) 用 m 阶初等矩阵 $D_i(k)$ 左乘矩阵 $A = [a_{ij}]_{m \times n}$, 得

$$D_i(k)A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{第 } i \text{ 行}$$

相当于以数 k 乘矩阵 A 的第 i 行 (kr_i) ($k \neq 0$).

3) 用 $E_{ij}(k)$ 左乘矩阵 $A = [a_{ij}]_{m \times n}$, 得

$$E_{ij}(k)A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} + ka_{i1} & a_{j2} + ka_{i2} & \cdots & a_{jn} + ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \end{matrix}$$

相当于把 A 的第 i 行乘 k 加到第 j 行 ($r_j + kr_i$).

- 4) 用 n 阶初等矩阵 P_{ij} 右乘矩阵 $A = [a_{ij}]_{m \times n}$, 相当于把矩阵 A 第 i 列与第 j 列对调 ($c_i \leftrightarrow c_j$).
- 5) 用 n 阶初等矩阵 $D_i(k)$ 右乘矩阵 $A = [a_{ij}]_{m \times n}$, 相当于以数 k 乘矩阵 A 的第 i 列 (kc_i).
- 6) 用 n 阶初等矩阵 $E_{ij}(k)$ 右乘矩阵 $A = [a_{ij}]_{m \times n}$, 相当于将矩阵 A 的第 j 列乘数 k 加到第 i 列 ($c_i + kc_j$).

三种初等矩阵左乘矩阵 A 是对 A 作相应的初等行变换
 三种初等矩阵右乘矩阵 B 是对 B 作相应的初等列变换

Note:

$$\mathbf{K}_1 = \mathbf{P}_{13}, \mathbf{K}_2 = \mathbf{E}_{14}(c), \mathbf{K}_3 = \mathbf{D}_2(k)$$

Example 5 Let

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Find $\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3$.**Solution** We note that $\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 = \mathbf{P}_{13} \mathbf{E}_{14}(c) \mathbf{D}_2(k)$,

so

$$\mathbf{K}_2 \mathbf{K}_3 = \mathbf{E}_{14}(c) \mathbf{D}_2(k) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1 \end{bmatrix}$$

Note:

$$\mathbf{K}_1 = \mathbf{P}_{13}, \mathbf{K}_2 = \mathbf{E}_{14}(c), \mathbf{K}_3 = \mathbf{D}_2(k)$$

$$\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 = \mathbf{P}_{13} \mathbf{K}_2 \mathbf{K}_3, \text{ and}$$

$$\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 = \mathbf{P}_{13} \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & k & 0 & 0 \\ 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 \end{bmatrix}$$

Remark We can also use right multiplication and the corresponding elementary column operations to do the calculation. It leads to the same result.

Example 6 (将初等矩阵概念用于消元: Elimination)

For 3 equations in 3 unknowns:

Suppose E subtracts twice the first equation from the second.

Suppose F is the matrix for the next step, *to add row 1 to row 3*.

$$E = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}.$$

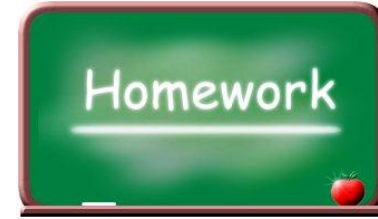
*These two matrices **do commute** and the product does both steps at once:*

$$EF = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 1 & & 1 \end{bmatrix} = FE.$$

In either order, EF or FE , this changes rows 2 and 3 using row 1.

What if : E is the same but G add row 2 to row 3? **$EG \neq GE$.**

Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**
- Pay attention to the notation

In the textbook, the *elementary matrix* E_{ij} subtracts l times row j from row i .

$$E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -l & & 1 \end{bmatrix},$$

Deadline (DDL):

- Next tutorial class

