CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

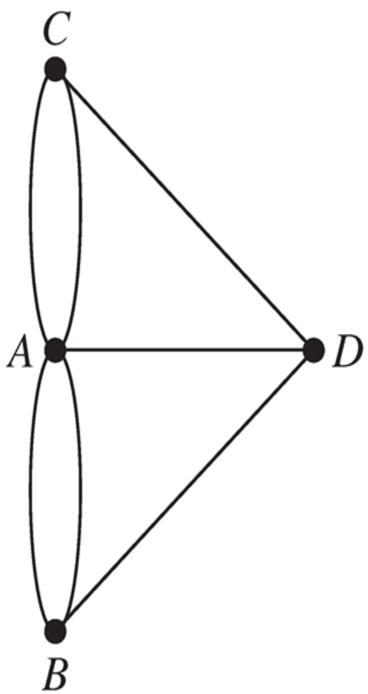
Dr. Qi Wang

Department of Computer Science and Engineering
Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

Euler Circuits and Euler Paths

- **Theorem** A connected multigraph with at least two vertices has an *Euler circuit* if and only if each of its vertices has **even** degree.

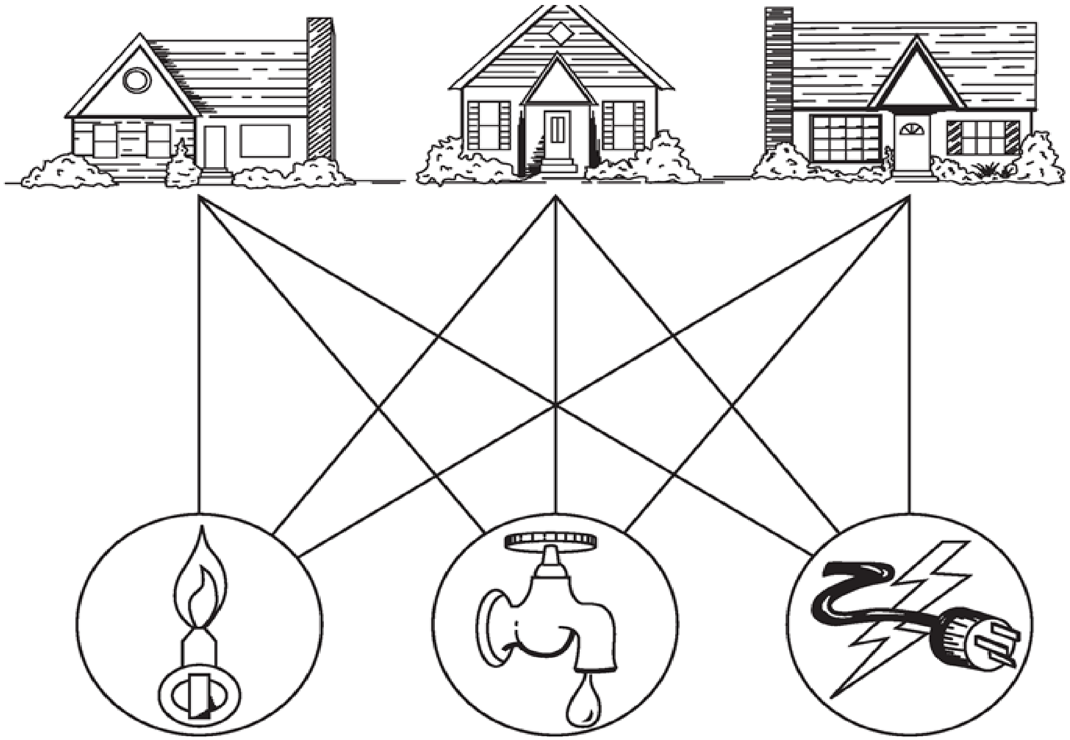
  Theorem A connected multigraph has an *Euler path* but not an *Euler circuit* if and only if it has exactly two vertices of odd degree.



No Euler circuit

Planar Graphs

- Join three houses to each of three seperate utilities.



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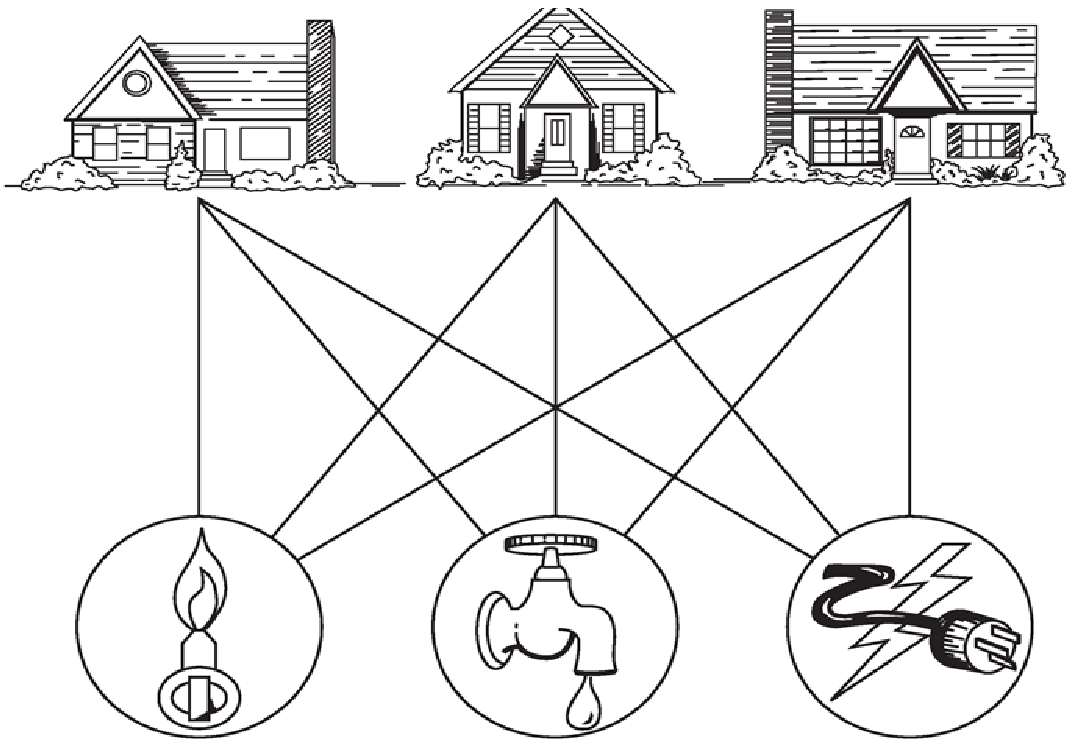

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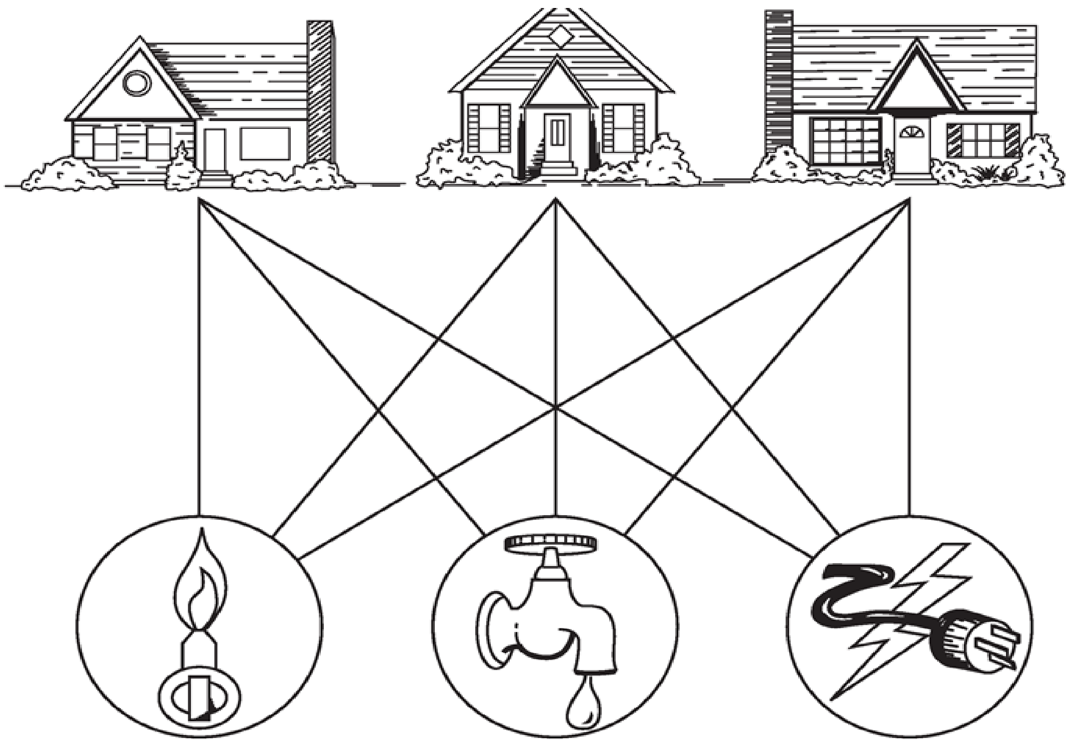

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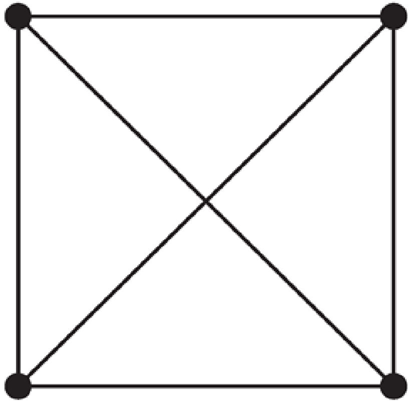

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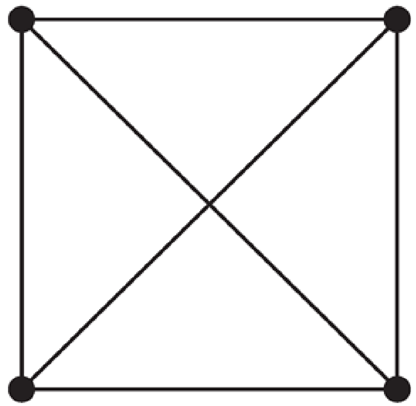

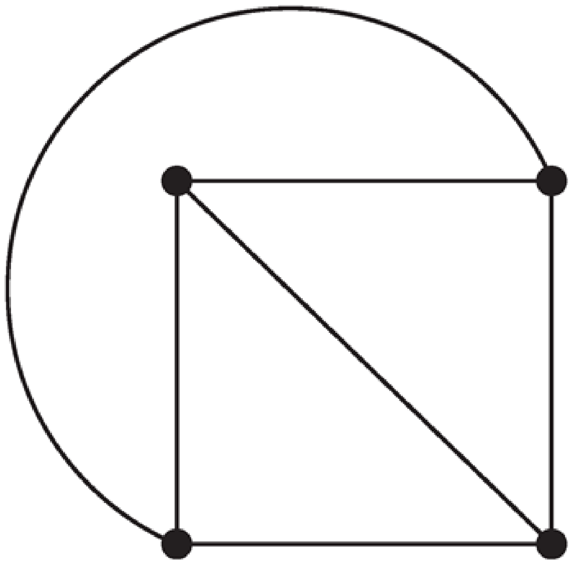

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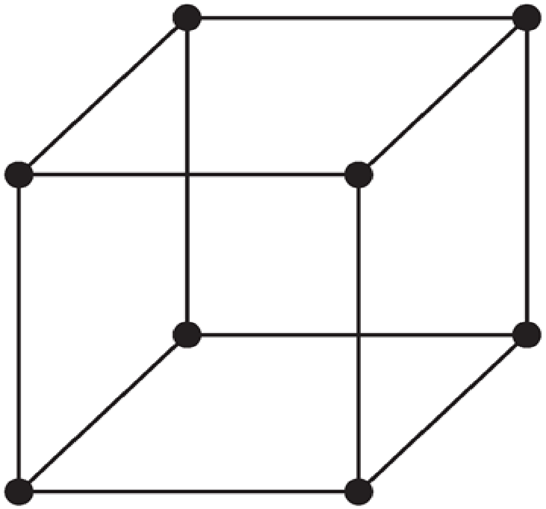

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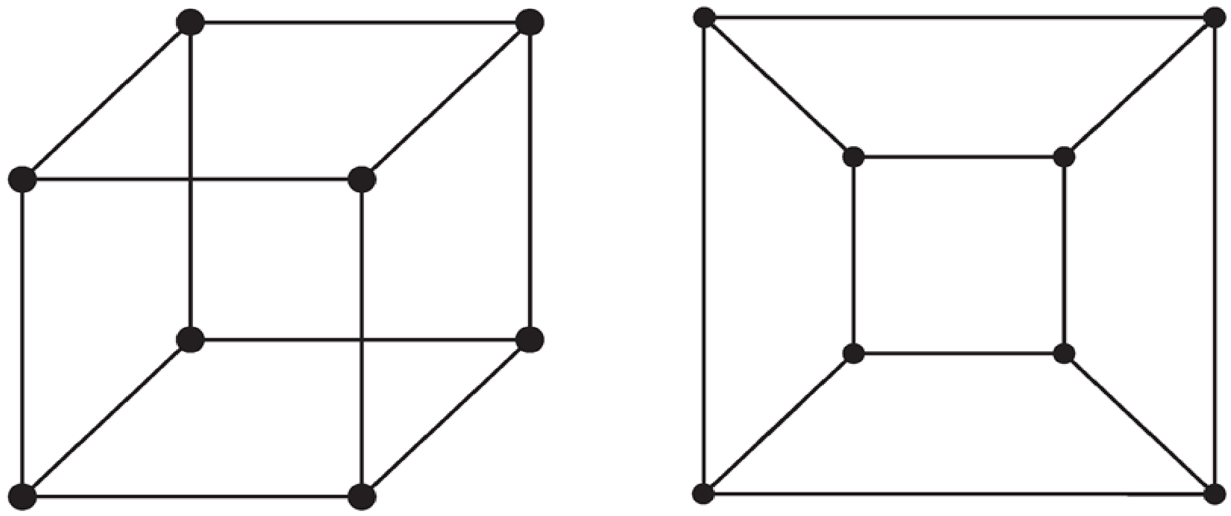

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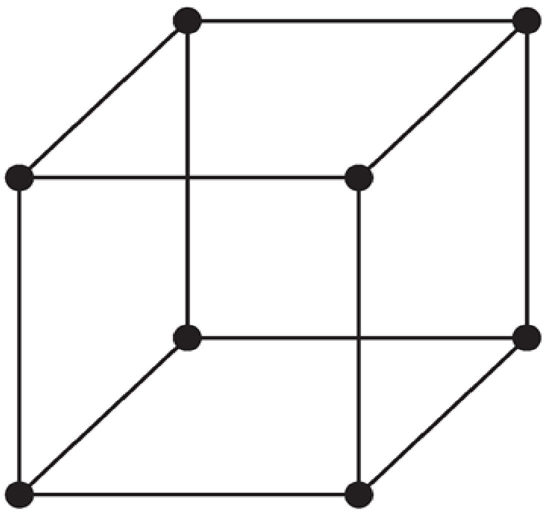

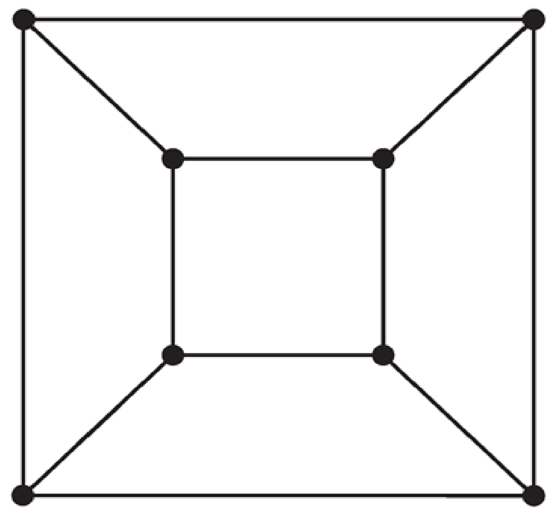

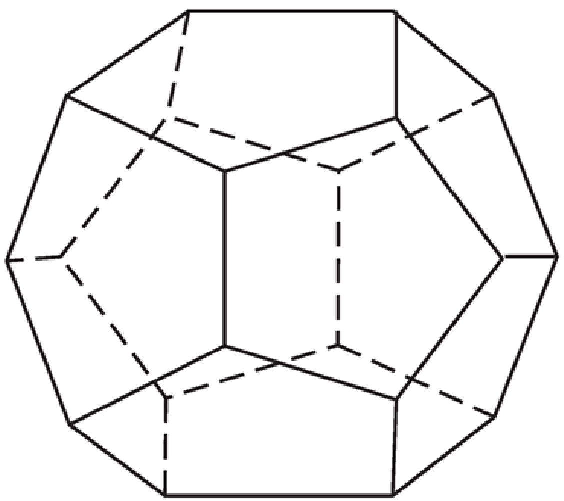

(a)

Planar Graphs

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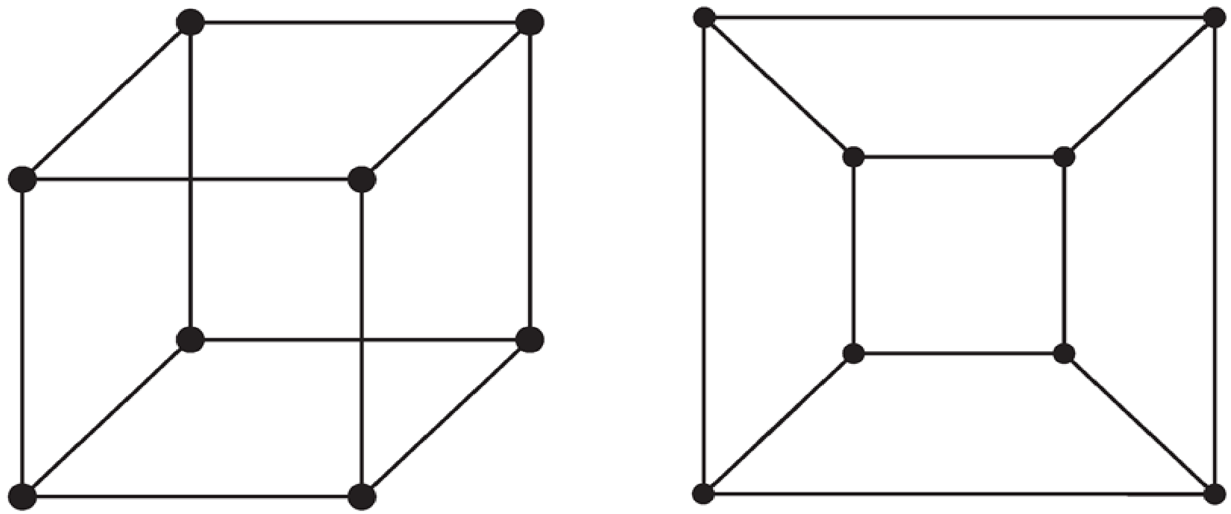

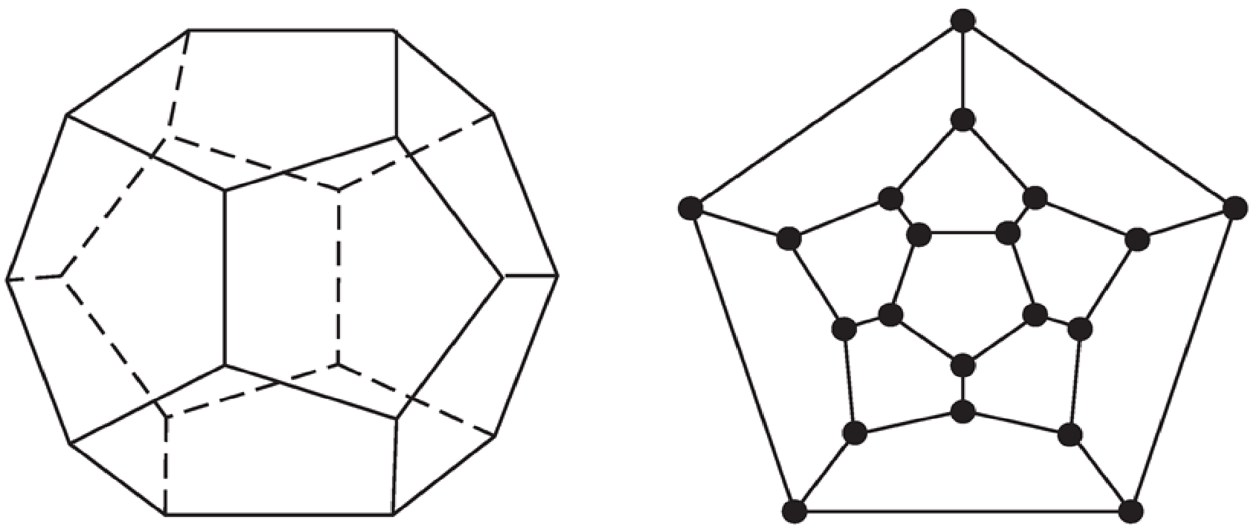

(a) (b)

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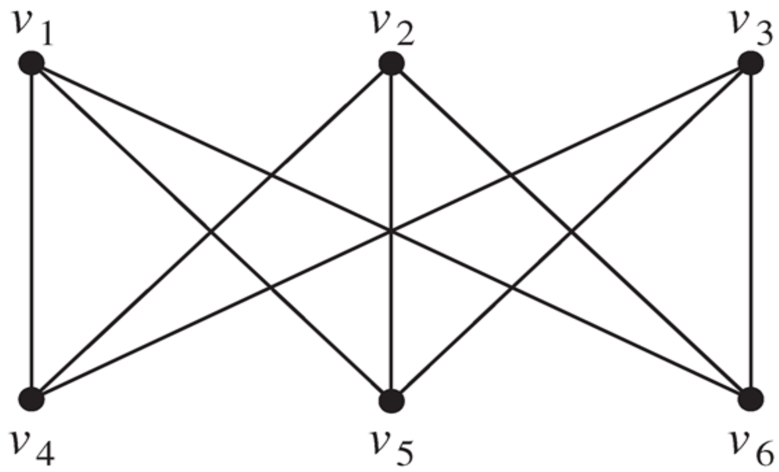

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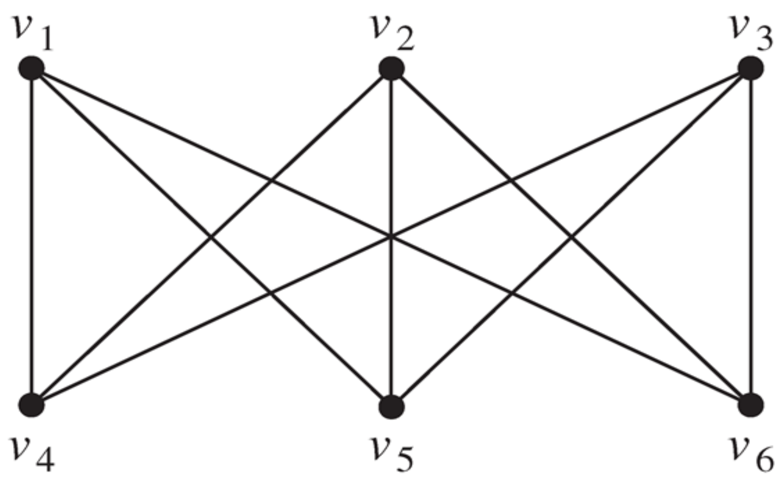

Applications

- ◇ IC design
- ◇ design of road networks

Euler's Formula

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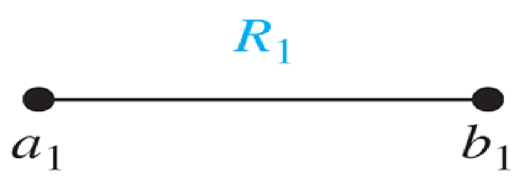

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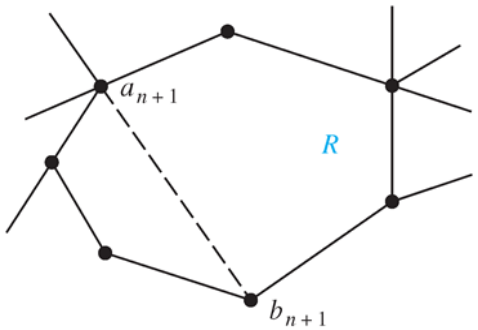

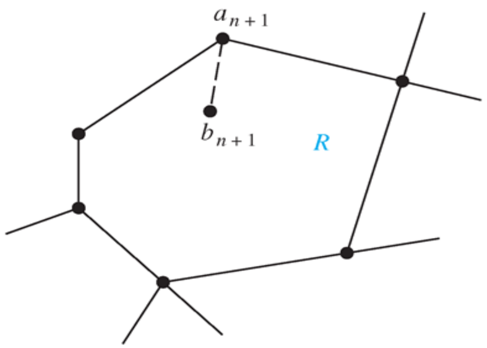

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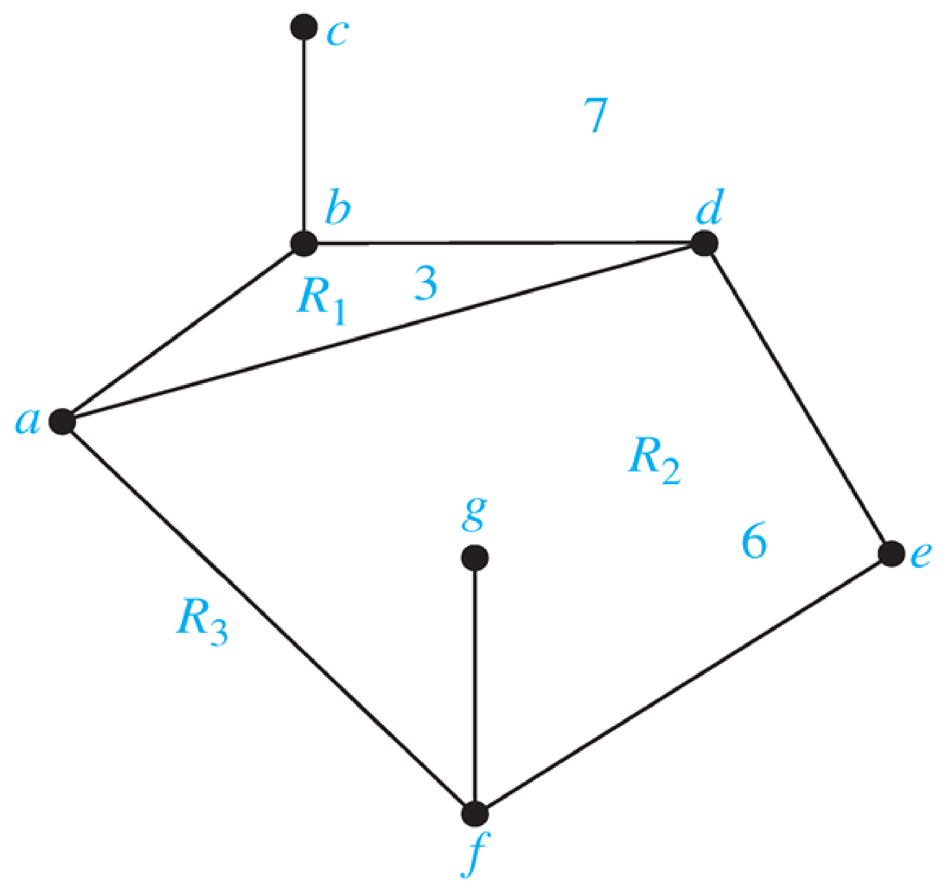

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  By Euler's formula, the proof is completed.

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Proof similar to that of Corollary 1.

Examples

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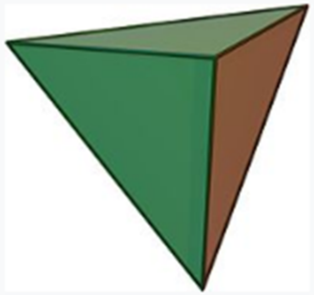
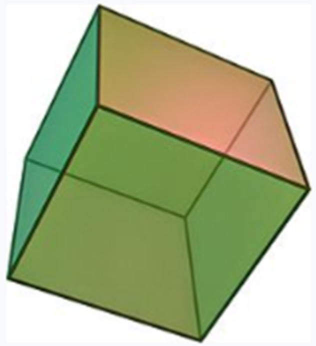
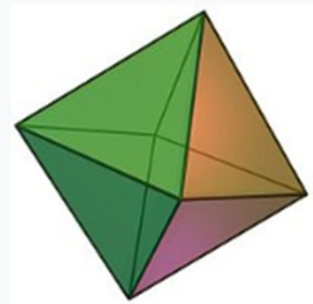
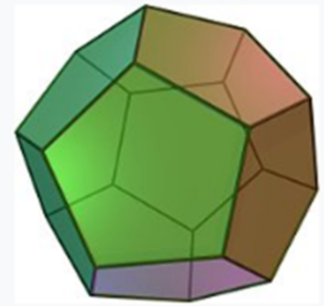
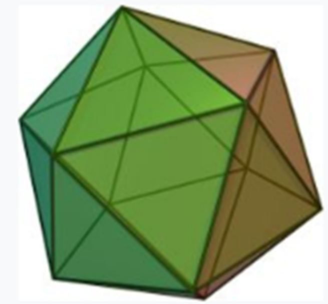
  Corollary 2 is used in the proof of Five Color Theorem.

Only 5 Platonic Solids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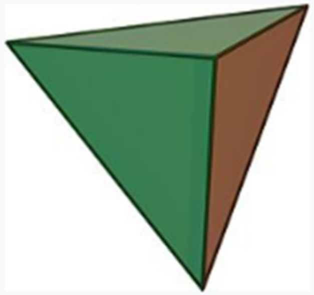	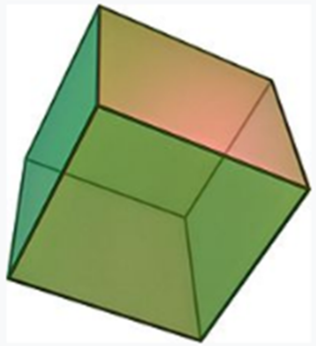	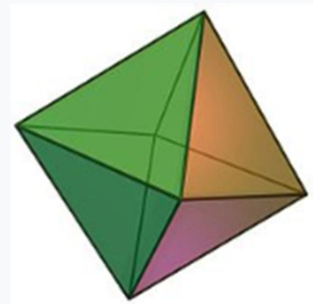	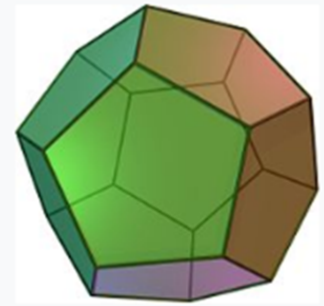	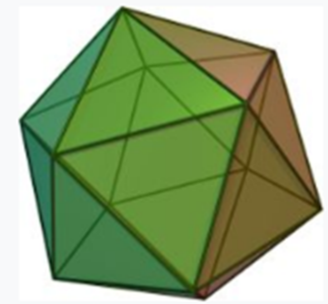
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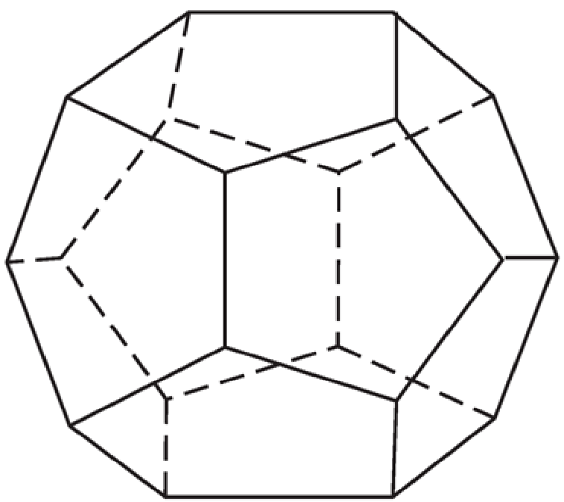

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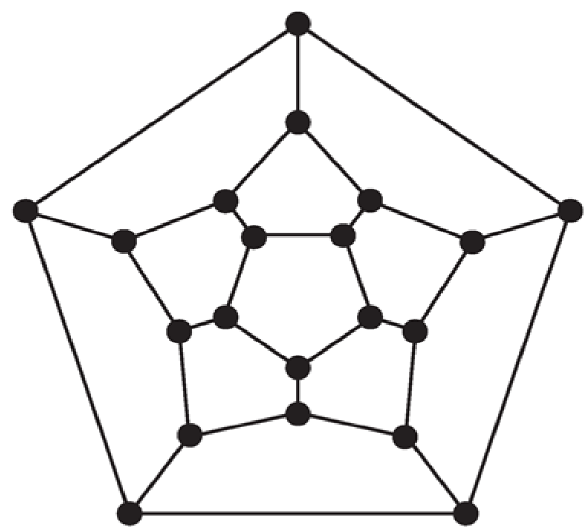

(b)

Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

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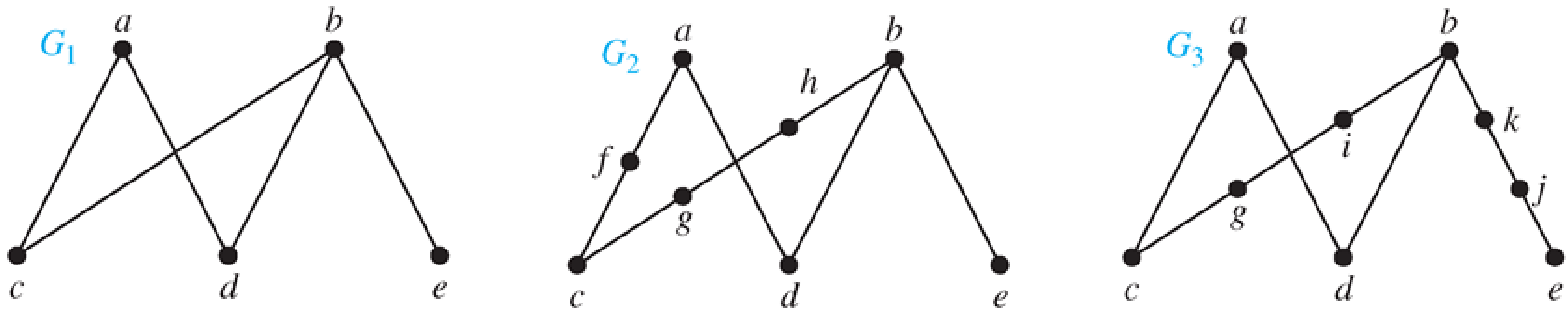

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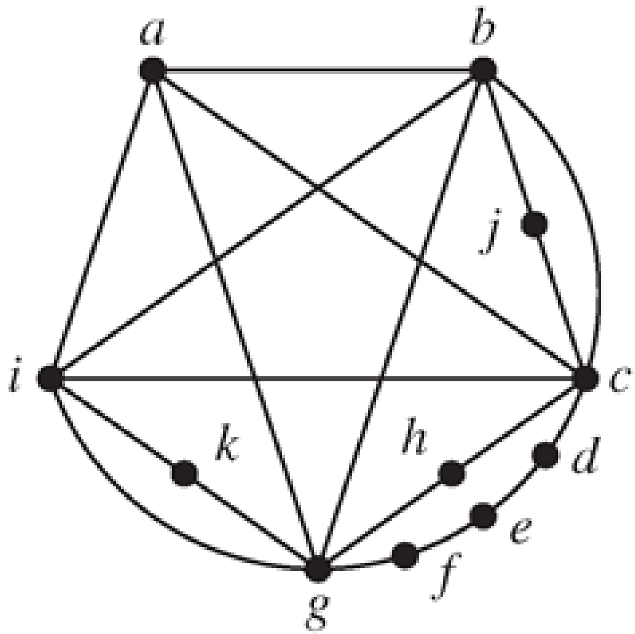
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



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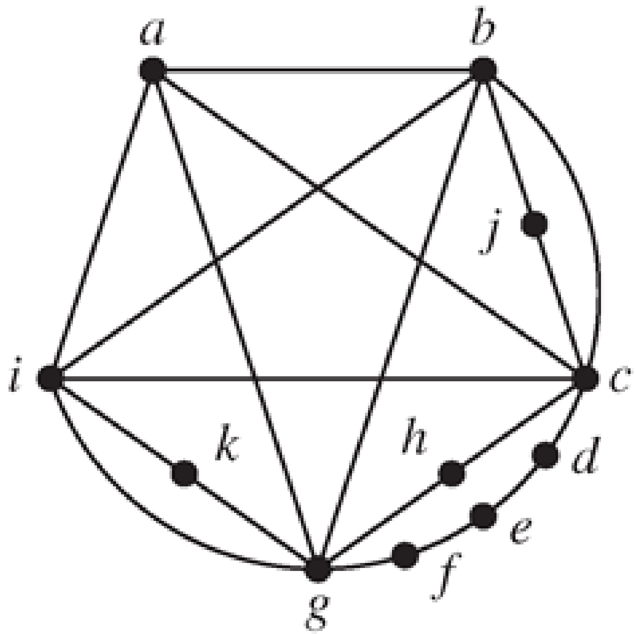
Examples



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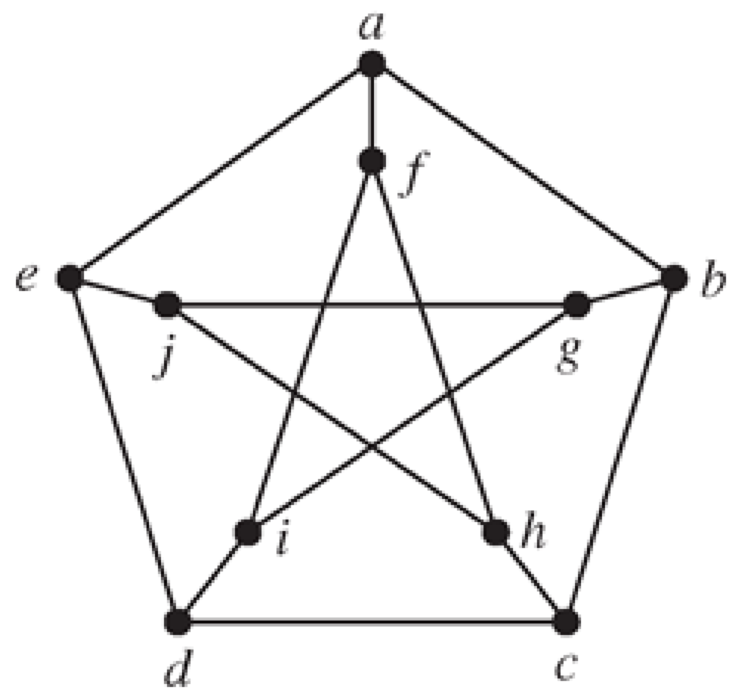

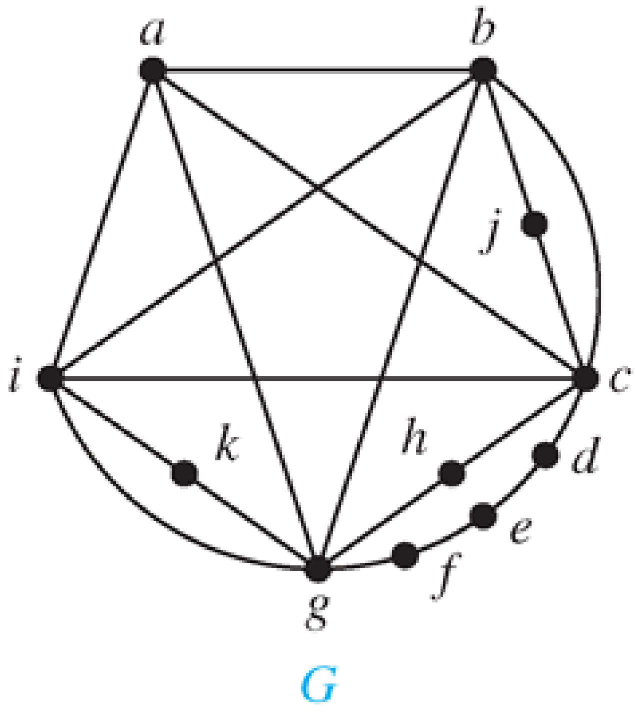
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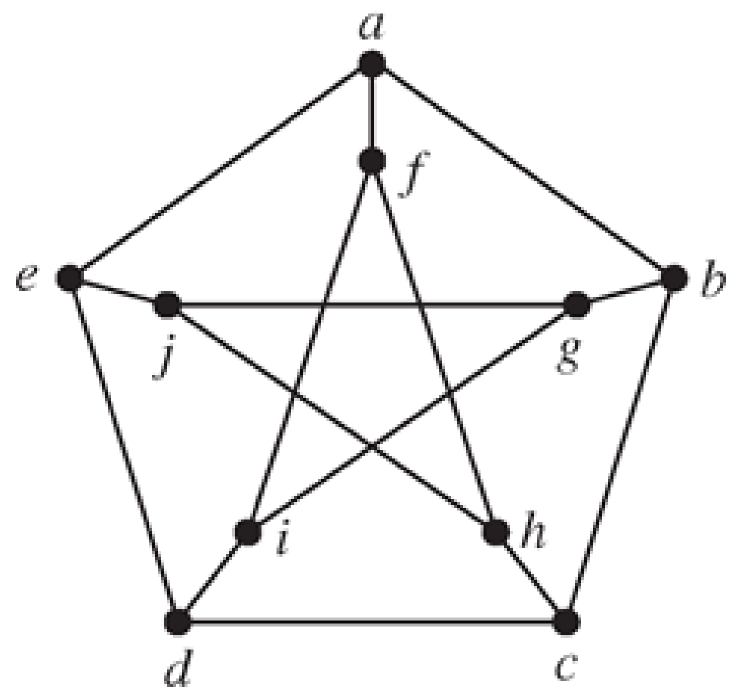
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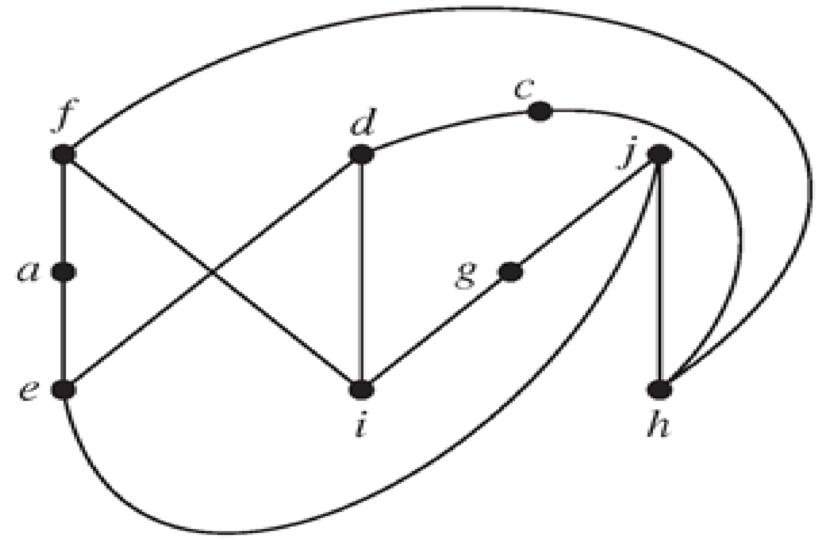
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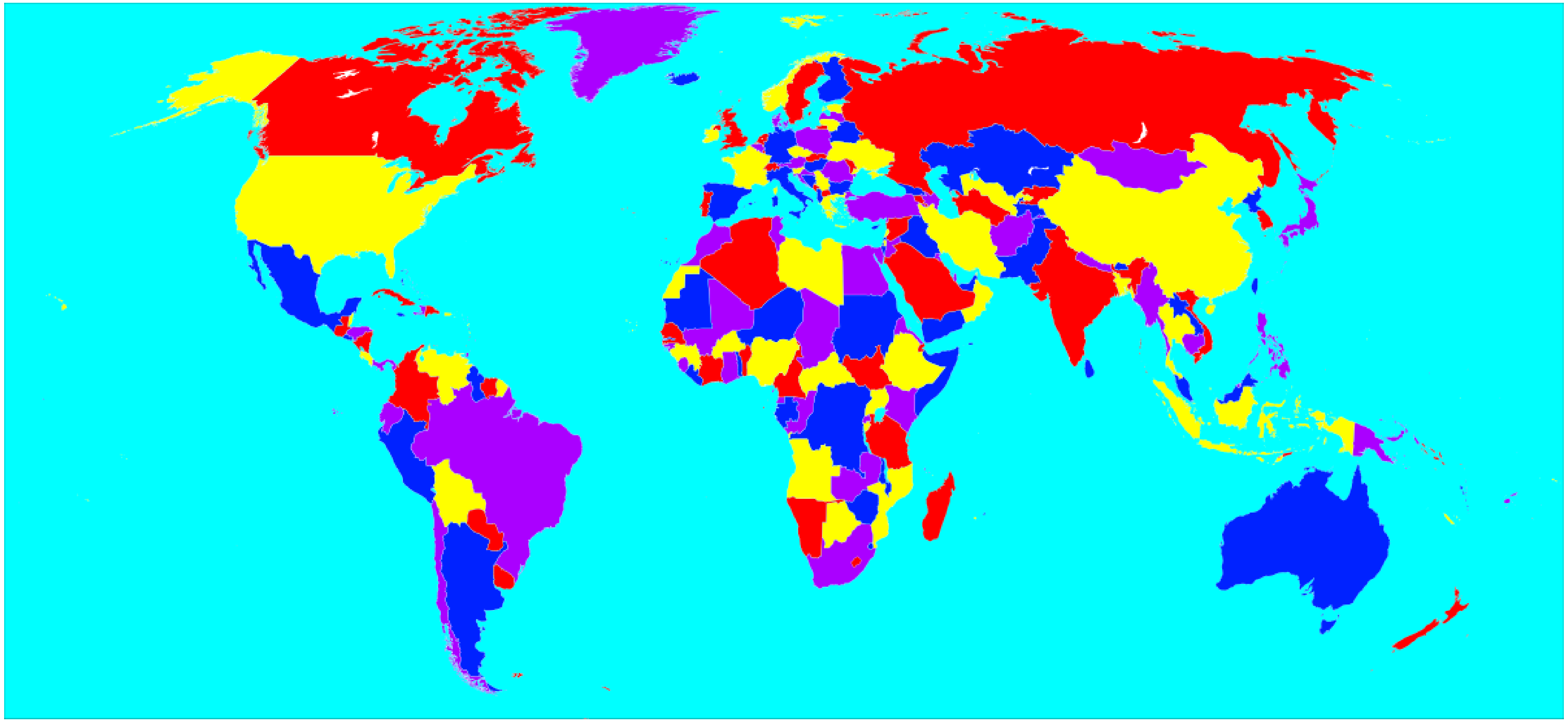
Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



- **Four-color theorem**
  - ◇ first proposed by Francis Guthrie in 1852
  - ◇ his brother Frederick Guthrie told Augustus De Morgan
  - ◇ De Morgan wrote to William Hamilton
  - ◇ Alfred Kempe proved it incorrectly in 1879
  - ◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*
  - ◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)
  - ◇ Kempe's incorrect proof serves as a basis

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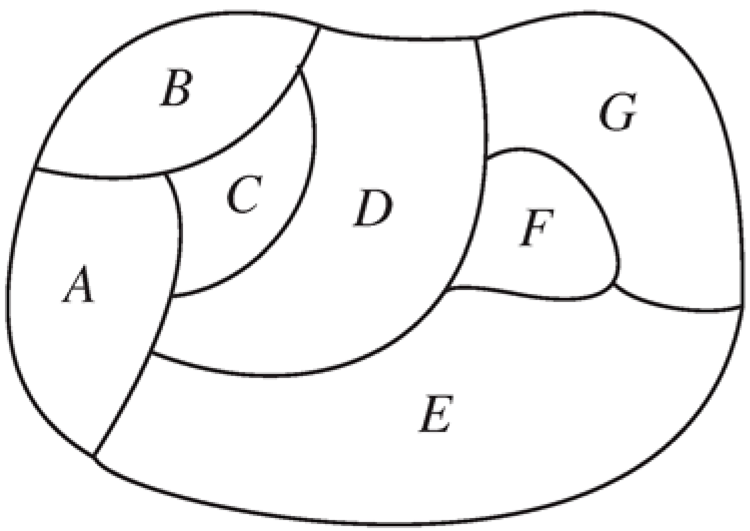

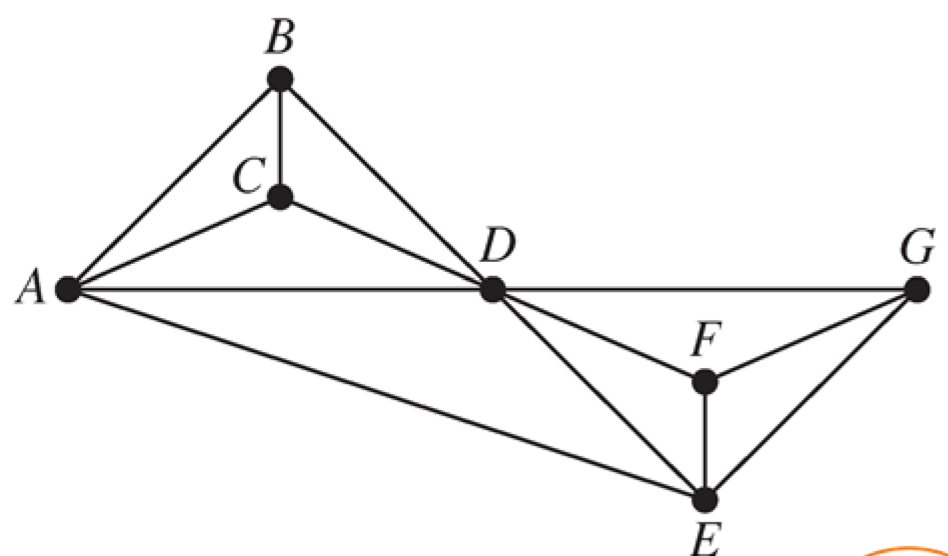

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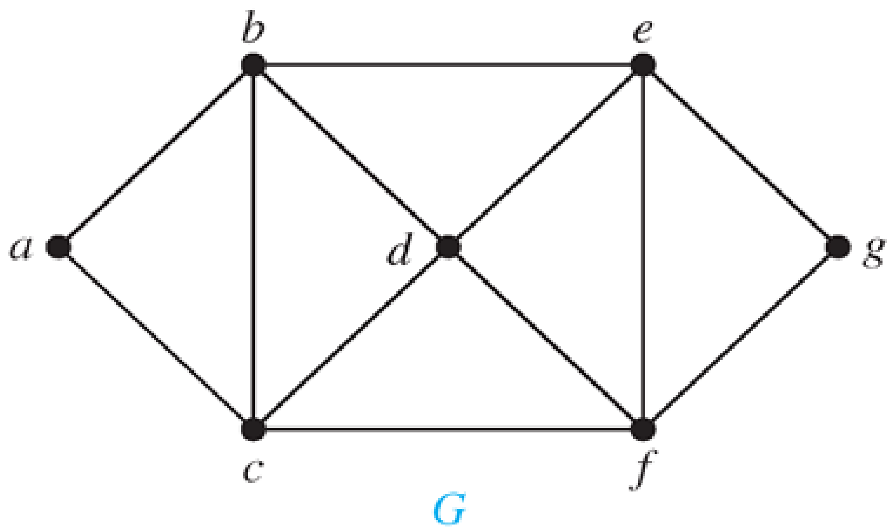

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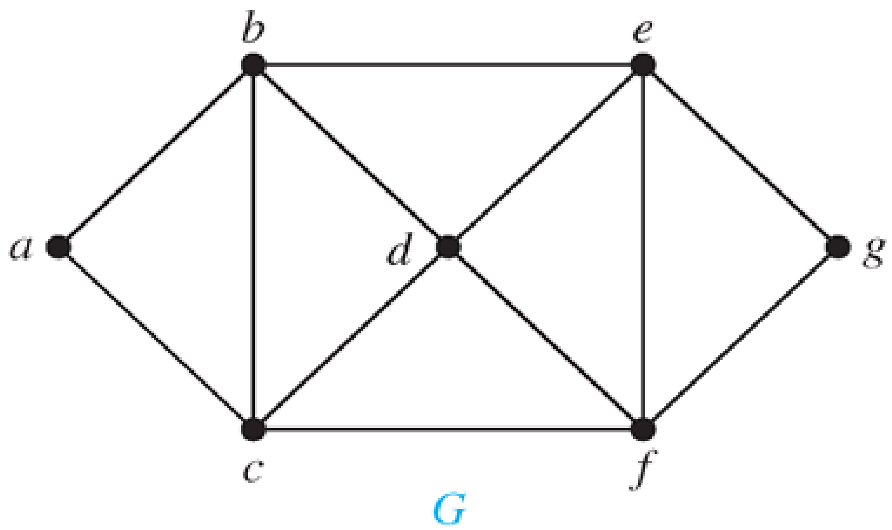

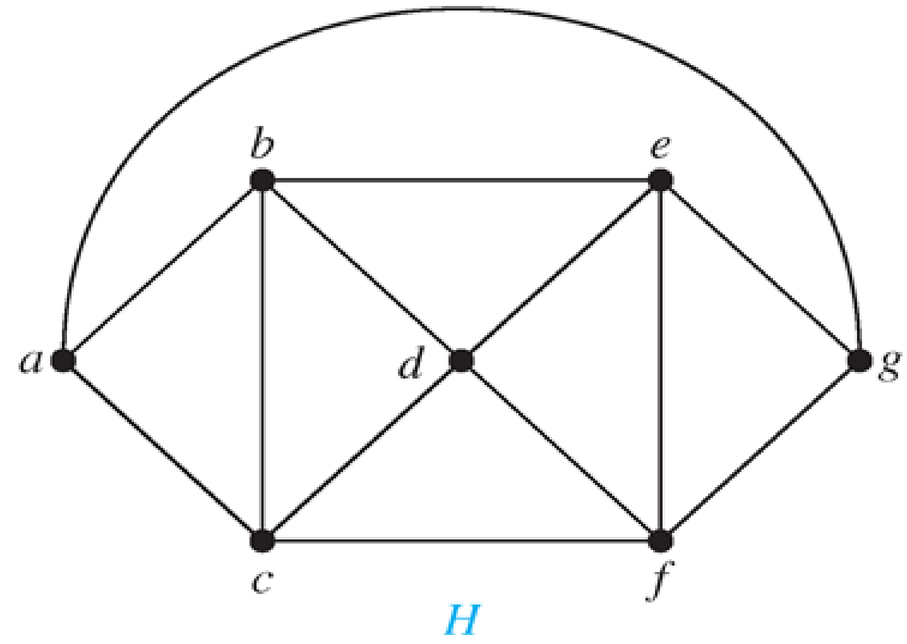

Graph Coloring

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

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  Inductive step: Consider a planar graph with $k+1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.

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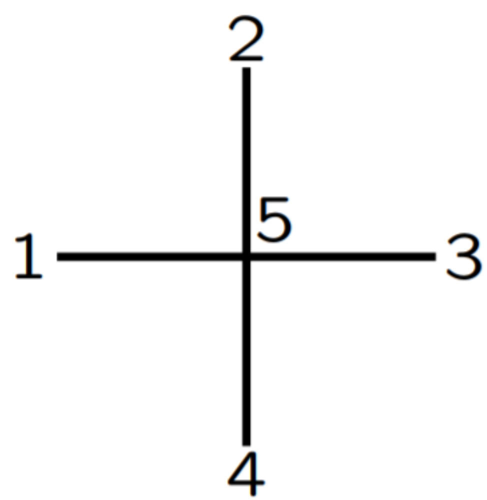
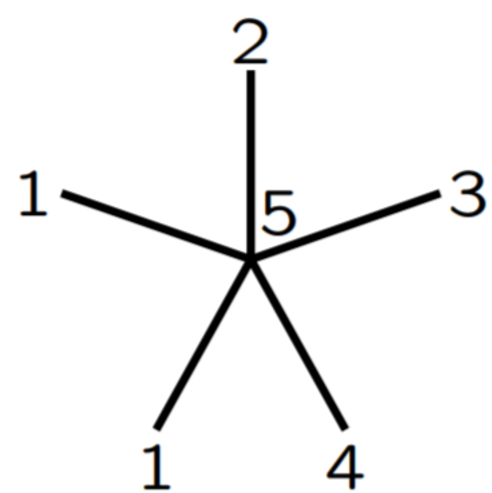
  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.





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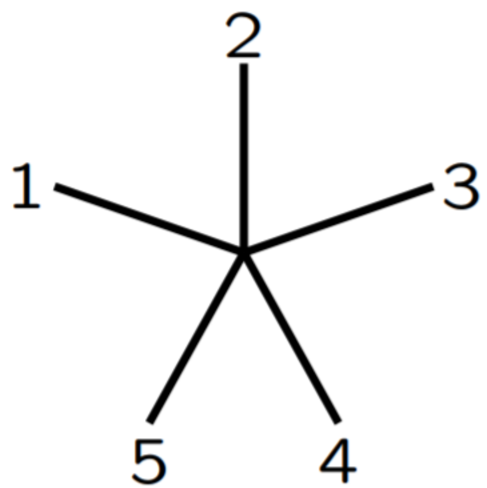
  If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).



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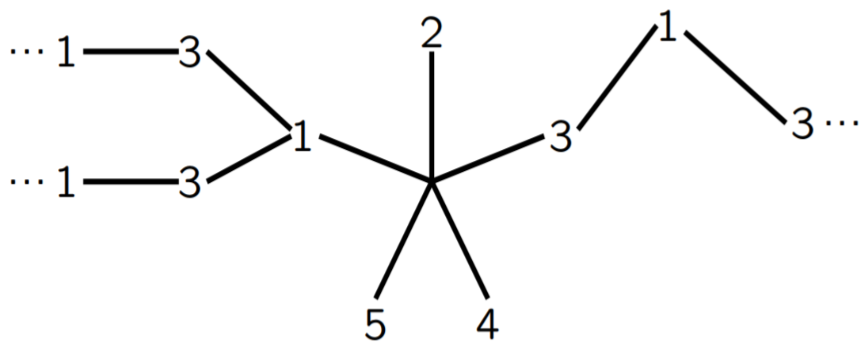

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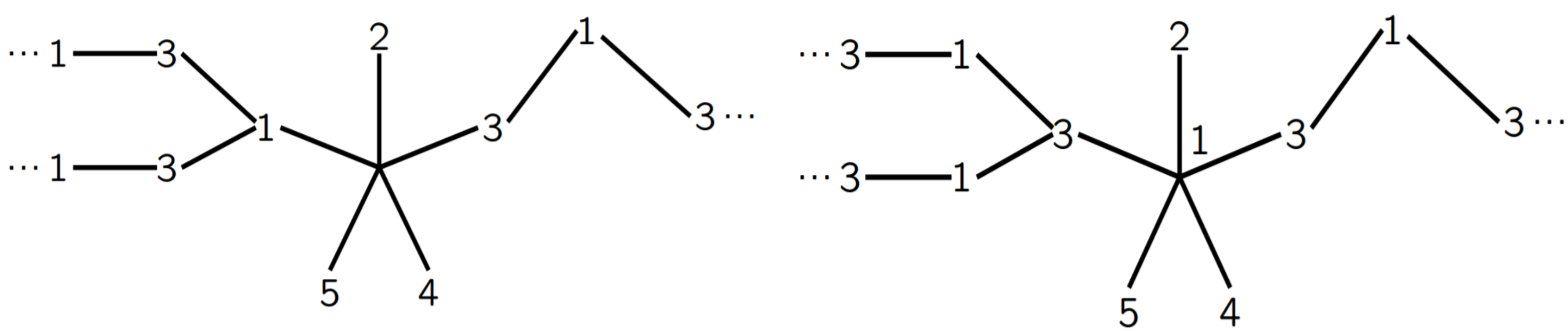

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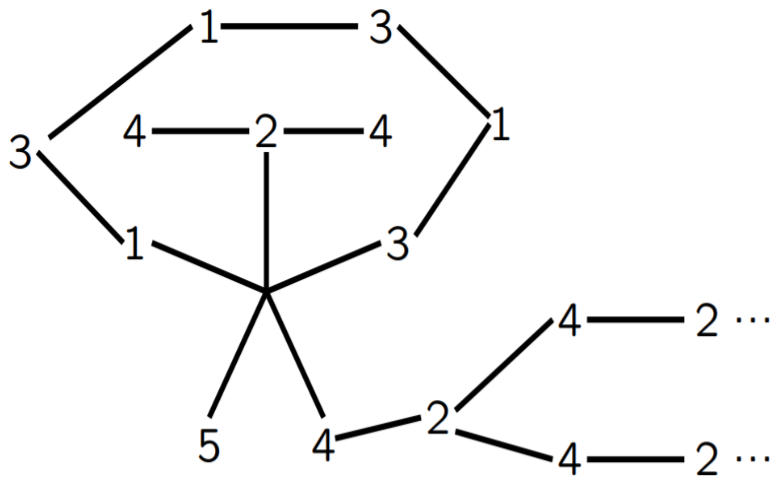

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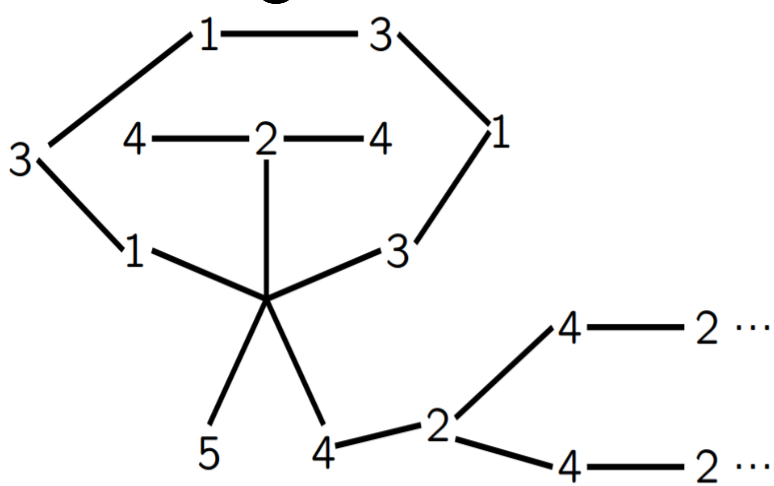

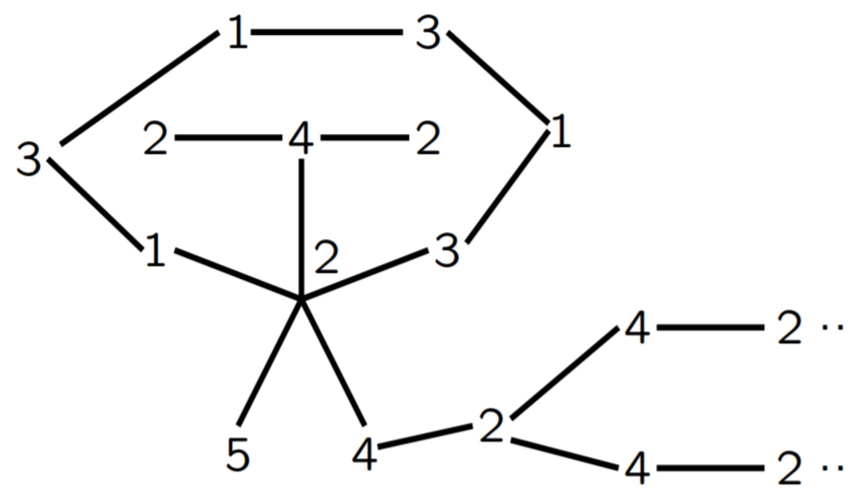

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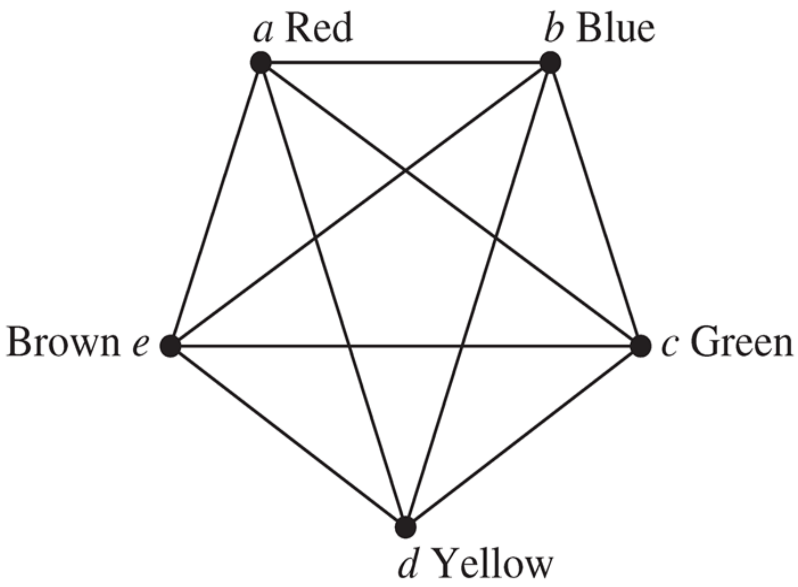

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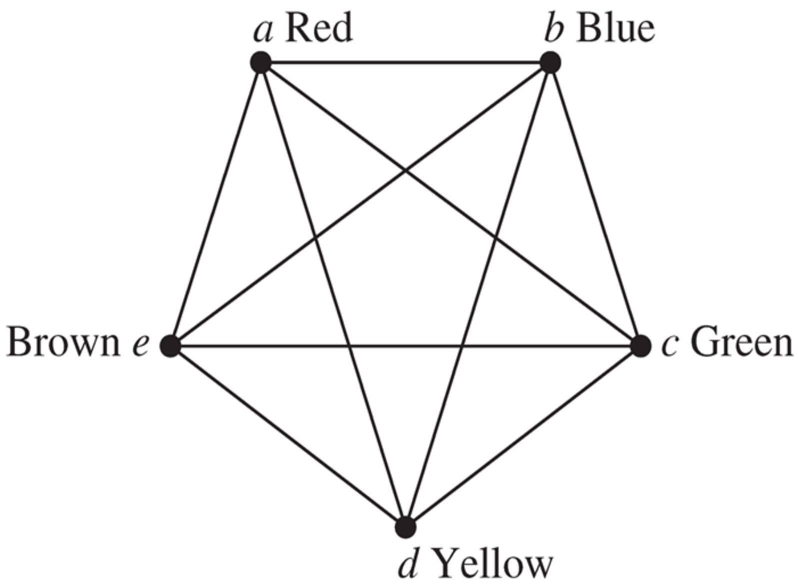

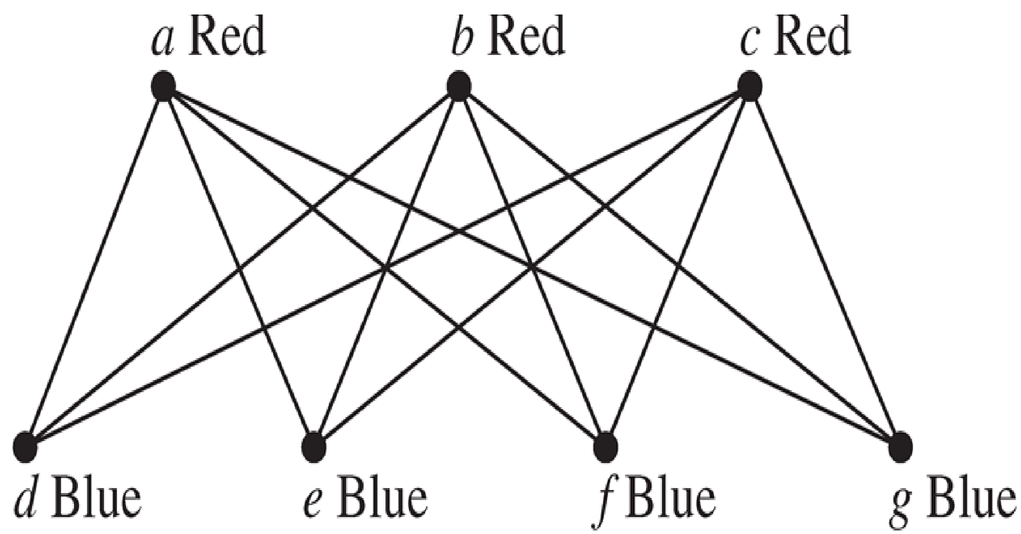

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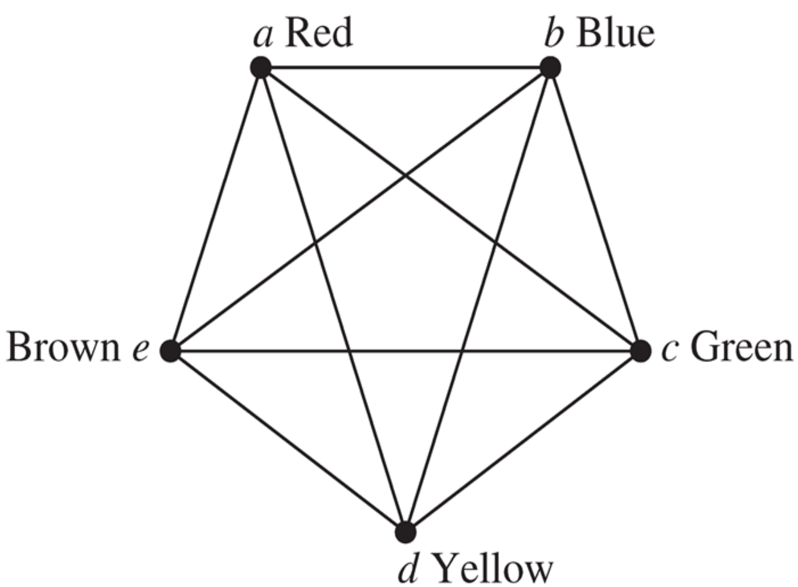

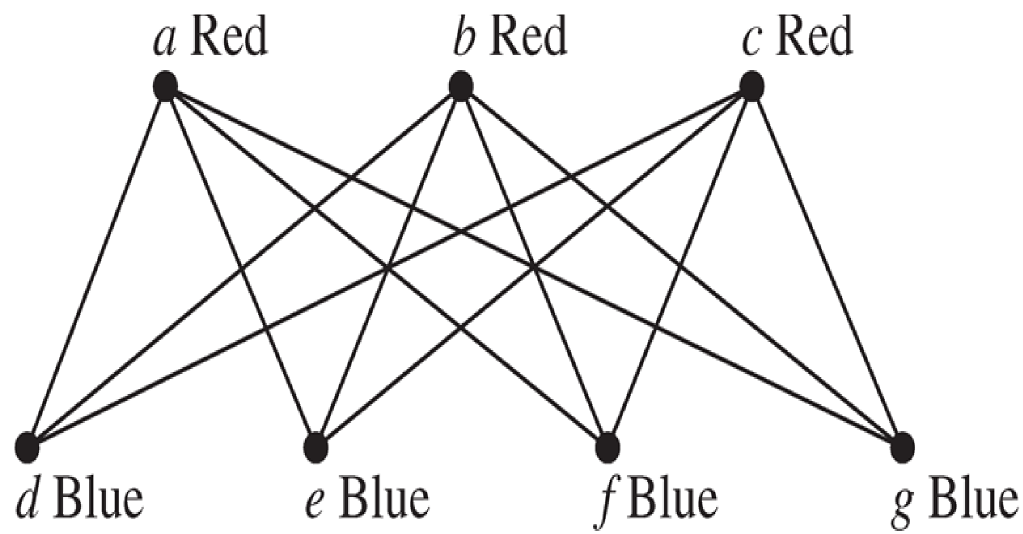

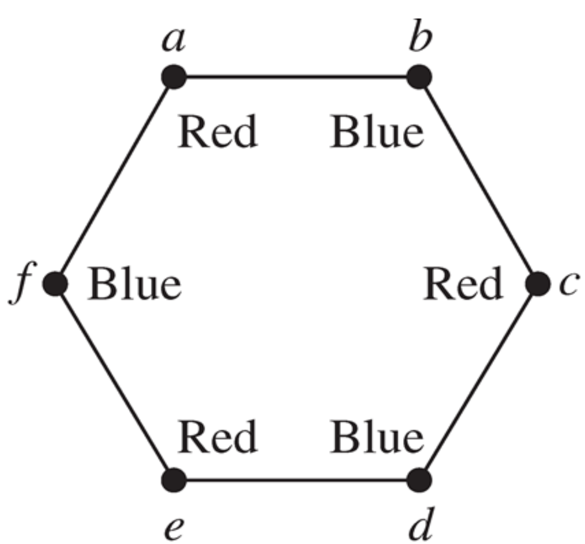

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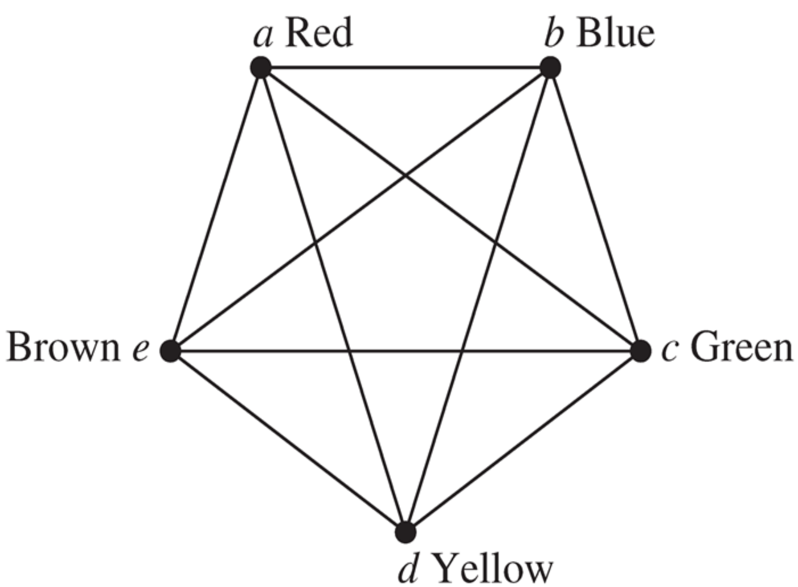
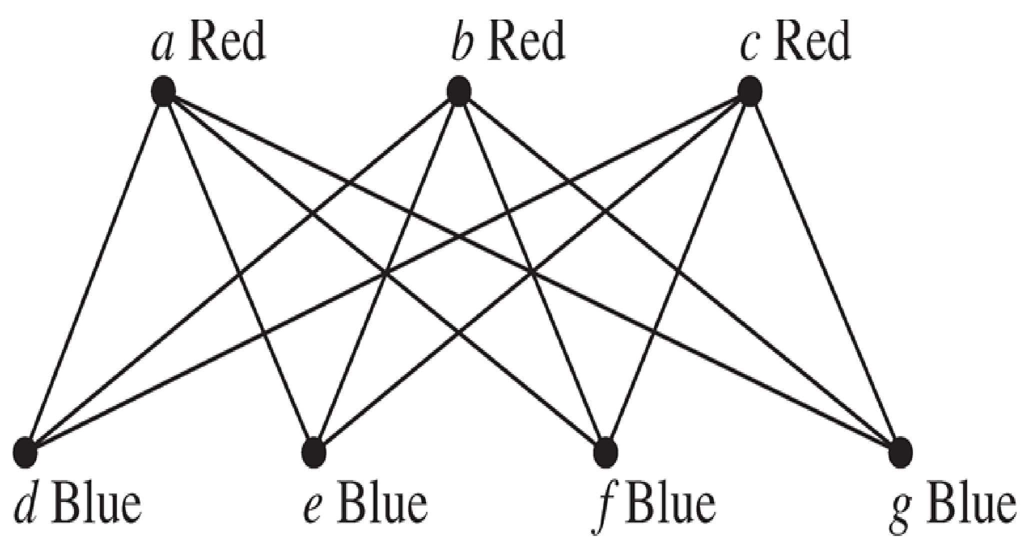
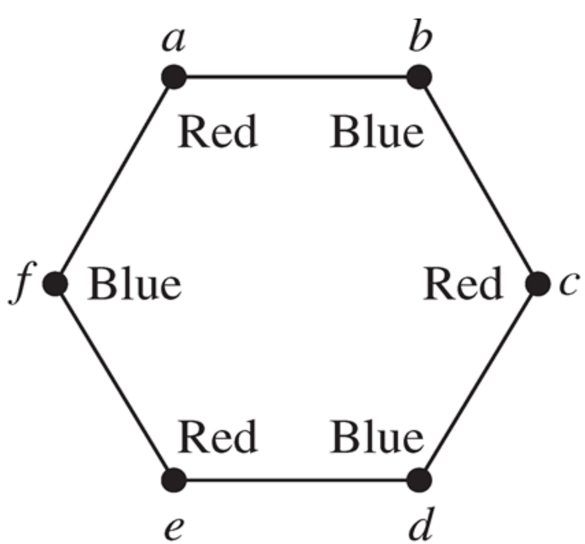
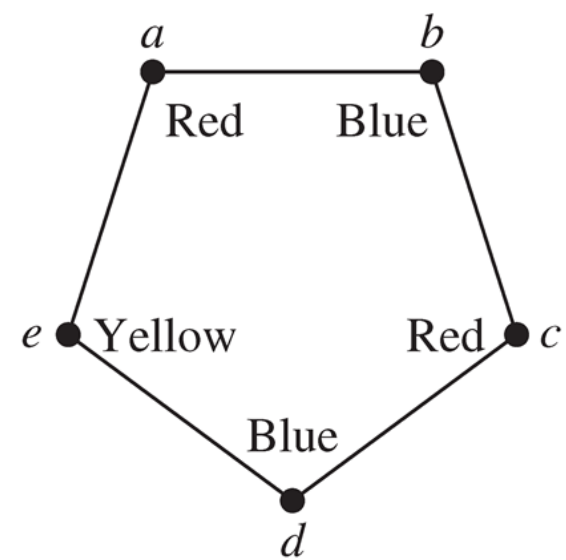




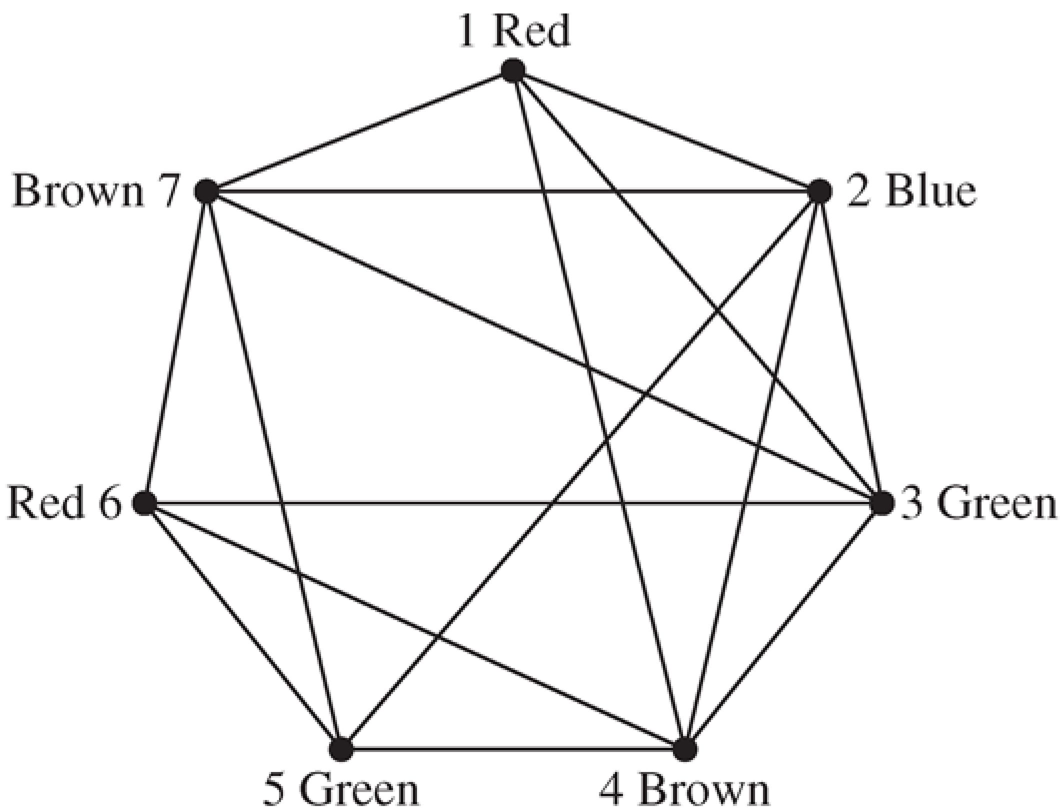




Applications of Graph Coloring

- Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

- Channel Assignments

Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Next Lecture

- tree ...