

# Geometric Boolean Algebra: Axiomatic Restoration of Origin Symmetry

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## Abstract

This paper establishes *Geometric Boolean Algebra* (GBA)—a theoretical framework unifying Boolean logic and geometric algebra. Core contributions are organized in three pillars:

### 1. Axiomatic Foundation:

- Symmetric domain  $\mathbb{B}_s^n = \{-1, 1\}^n$
- Arithmetic realization of logic operations:  $\neg x = -x$   
 $x \wedge y = \min(x, y)$   
 $x \vee y = \max(x, y)$

### 2. Theoretical Framework:

- Isomorphism proof to classical Boolean algebra (Theorem 1)
- Isometric embedding  $\Phi : \mathbb{B}_s^n \rightarrow \mathbb{R}^n$  with distance preservation (Theorem 6)
- Linear separability of all Boolean functions in augmented spaces (Theorem 7)

### 3. Computational Transformation:

- Logic-to-arithmetic conversion paradigm
- Novel circuit design demonstrated via full adder:  $s = a \otimes b \otimes c_{in}$   
 $c_{out} = \text{sign}(a + b + c_{in})$
- Hardware advantages: uniformity, parallelism, scalability

The framework resolves fundamental constraints like XOR linear inseparability through geometric augmentation. Research extensions to quantum systems and machine learning accelerators demonstrate GBA's cross-domain potential.

## 1 Introduction: The Geometric Constraint in Boolean Computation

### 1.1 The Fundamental Defect: Broken Symmetry

Classical Boolean algebra suffers from a **single catastrophic flaw**:

$$\text{Originasymmetry} \iff \mathcal{I} : \mathbf{x} \mapsto -\mathbf{x} \text{ over } \{0, 1\}^n \quad (1)$$

This manifests as:

- **Off-center centroid** at  $(\frac{1}{2}, \dots, \frac{1}{2})$
- **Constrained representational capacity** (max angle  $90^\circ$ )

Minsky-Papert (1969) proved this irrevocably blocks linear separability of parity functions like XOR.

As formally established in [?], the off-centered centroid in  $\{0, 1\}^n$  creates an inherent geometric barrier for linear classifiers. Specifically, the centroid position at  $(\frac{1}{2}, \dots, \frac{1}{2})$  prevents the existence of any hyperplane separating parity functions. This fundamental limitation motivated our geometric re-axiomatization approach.

## 1.2 Symmetry Restoration

We resolve this through **Geometric re-axiomatization**. Unlike Diaz-Rivas' linear-algebraic approach to symmetric powers [?], our work restores origin symmetry at the axiomatic level:

$$\mathbb{B}_s^n = \{-1, 1\}^n \Rightarrow \underbrace{\mathbf{x} \mapsto -\mathbf{x}}_{\text{central involution}} \quad \text{and} \quad \underbrace{\frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{B}_s^n} \mathbf{x}}_{\text{geometric balance}} = \mathbf{0} \quad (2)$$

**Geometric consequence:** Full  $B_n$  hyperoctahedral symmetry enables linear separability via vertex embedding in  $\mathbb{R}^n$ .

## 2 Geometric Boolean Algebra

### 2.1 Axiomatic Foundation

**Axiom 1** (Geometric Boolean Domain). The algebraic structure is defined on the geometrically symmetric domain:  $\mathbb{B}_s^n = \{F : -1, T : 1\}$

**Axiom 2** (Primitive Operations). The fundamental operations are geometrically realized as: Negation:  $\neg x := -x$

Conjunction:  $x \wedge y := \min(x, y)$

Disjunction:  $x \vee y := \max(x, y)$

**Note:** All other Boolean operations (e.g.,  $\oplus$ ,  $\rightarrow$ ,  $\leftrightarrow$ ) are derived from these three primitives.

### 2.2 Isomorphism Theorem

#### 2.2.1 Theorem 1 (Isomorphism to Classical Boolean Algebra)

There exists a bijection  $\phi : \{0, 1\} \rightarrow \mathbb{B}_s^n$  given by:

$$\phi(a) = 2a - 1 \quad \text{and} \quad \phi^{-1}(x) = \frac{x + 1}{2} \quad (3)$$

that preserves all Boolean operations. Specifically, for any classical Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ :

$$f_s(\mathbf{x}) = \phi \circ f \circ \phi^{-1}(\mathbf{x}) \quad (4)$$

is its symmetric realization satisfying:

$$\forall \mathbf{a} \in \{0, 1\}^n, \quad f(\mathbf{a}) = \phi^{-1}(f_s(\phi(\mathbf{a}))) \quad (5)$$

## 3 Derivation of Core Properties

### 3.1 Functional Completeness Theorem

**Theorem 2** (Functional Completeness). The operator set  $\{\neg, \wedge\}$  is functionally complete for  $\mathbb{B}_s^n$ .

*Proof:* By Axiom 2, we have:

$$x \vee y = \neg(\neg x \wedge \neg y) \quad (6)$$

Thus  $\vee$  is derivable. For any Boolean function  $f : \mathbb{B}_s^n \rightarrow \mathbb{B}_s^n$ , its algebraic normal form:

$$f(\mathbf{x}) = \bigvee_{\mathbf{a} \in f^{-1}(1)} \left( \bigwedge_{i: a_i=1} x_i \wedge \bigwedge_{j: a_j=-1} \neg x_j \right) \quad (7)$$

is constructible using only  $\neg$  and  $\wedge$ .  $\square$

### 3.2 Duality Principle

**Theorem 3** (Duality Principle). Let  $P = Q$  be an identity in  $\mathbb{B}_s^n$ . Under the duality transformation:

Operation	Dual
$\wedge$	$\vee$
$\vee$	$\wedge$
$\perp$	$\top$
$\top$	$\perp$

the dual identity  $P^d = Q^d$  holds.

*Proof sketch:* Duality arises from the geometric polarity of  $\mathbb{B}_s^n$ 's hypercube under central symmetry:

$$\min(x, y) \leftrightarrow \max(-x, -y) \quad \text{via} \quad \mathbb{B}_s^n \phi_n \{0, 1\}^n \quad (8)$$

**Example:**

- Original:  $x \wedge \perp = \perp \rightarrow \min(x, -1) = -1$
- Dual:  $x \vee \top = \top \rightarrow \max(x, 1) = 1$

### 3.3 Verification of Algebraic Laws

Law	$\mathbb{B}_s^n$ Verification
<b>De Morgan</b>	$-\min(x, y) = \max(-x, -y)$
<b>Distributive</b>	$\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$
<b>Absorption</b>	$\max(x, \min(x, y)) = x$

## 4 Geometric-Algebraic Isomorphism

### 4.1 Intrinsic Geometric Embedding

**Theorem 6** (Intrinsic Distance Preservation). The canonical embedding  $\iota : \mathbb{B}_s^n \hookrightarrow \mathbb{R}^n$  preserves the intrinsic distance structure:

$$|\iota(\mathbf{x}) - \iota(\mathbf{y})| = \sqrt{2n(1 - \cos \theta_{\mathbf{xy}})} = \sqrt{4d_H(\mathbf{x}, \mathbf{y})} \quad (9)$$

where  $d_H$  denotes the Hamming distance. This embedding satisfies:  $\mathbf{x} \cdot \mathbf{y} = n \cos \theta_{\mathbf{xy}} \quad \cos \theta_{\mathbf{xy}} = \frac{1}{n} \sum_{i=1}^n x_i y_i$

*Proof:* By direct computation:  $|\mathbf{x} - \mathbf{y}|^2 = \sum_{i=1}^n (x_i - y_i)^2 = \sum_{i=1}^n (2 - 2x_i y_i) \quad (\text{since } x_i^2 = y_i^2 = 1) = 2n - 2\mathbf{x} \cdot \mathbf{y}$ . The result follows from  $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \theta = n \cos \theta$ .  $\square$

**Corollary 6.1** (Spectral Property). The origin symmetry manifests as:

$$\frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{B}_s^n} \mathbf{x} = \mathbf{0} \quad (10)$$

enabling efficient harmonic analysis on the Boolean hypercube via Fourier-Walsh transform.

### 4.2 Linear Separability through Minimal Augmentation

**Theorem 7** (Optimal Linear Separability). For any Boolean function  $f : \mathbb{B}_s^n \rightarrow -1, 1$ , there exists an embedding  $\Psi_f : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$  with  $k \leq n2$  such that  $f$  is linearly separable under  $\Psi_f$ . Specifically for XOR:

$$\Psi_{\text{XOR}}(x_1, x_2) = (x_1, x_2, x_1 x_2), \quad \mathbf{w} = (0, 0, -1) \quad (11)$$

achieves separation with minimal dimension increase.

*Geometric Interpretation:* The augmented coordinates correspond to polynomial basis terms, forming a  $\mathbb{Z}_2$ -graded algebra:

$$\mathcal{A} = \bigoplus_k \Lambda^k(\mathbb{R}^n) \quad (12)$$

where  $\Lambda^k$  denotes the  $k$ -th exterior power.

### 4.3 Neural Representation Theorem

**Theorem 8** (Resolution of Linear Inseparability). The geometric limitation in neural representation learning [?] is resolved through coordinate augmentation:

- **Minimal augmentation:**  $\dim(\Psi_f) \leq n + n2$
- **Topological preservation:**  $\Psi_f$  maintains adjacency relations
- **Complexity separation:**  $\text{VCdim}(\Psi_f \circ \mathcal{H}_{\text{lin}}) = O(n^2)$

The separation hyperplane in augmented space  $\mathbb{R}^{n+k}$  admits geometric realization:

$$f(\mathbf{x}) = \text{sign} \left( \sum_{S \subseteq [n]} w_S \prod_{i \in S} x_i \right) \quad (13)$$

where  $w_S$  are spectral coefficients determined by the Fourier-Walsh expansion.

### 4.4 Linearization of XOR

**Theorem 7** (Linear Representation of XOR). Define the quadratic embedding  $\Psi : \mathbb{B}_s^2 \rightarrow \mathbb{R}^3$ :

$$\Psi(x_1, x_2) = (x_1, x_2, x_1 x_2) \quad (14)$$

**XOR is linearly separable** under  $\Psi$  with weight  $\mathbf{w} = (0, 0, -1)$ :

$$\text{XOR}(x_1, x_2) = \text{sign}(\mathbf{w}^T \Psi(\mathbf{x})) = -\text{sign}(x_1 x_2) \quad (15)$$

*Proof via Geometric Verification:*

$\mathbf{x}$	$\mathbf{w}^T \Psi(\mathbf{x})$	Output	XOR
$(-1, -1)$	$-(1) = -1$	F	F
$(-1, 1)$	$-(-1) = 1$	T	T
$(1, -1)$	$-(-1) = 1$	T	T
$(1, 1)$	$-(1) = -1$	F	F

**Geometric Insight:** Embedded points  $\{\Psi(\mathbf{x})\}$  form a tetrahedron in  $\mathbb{R}^3$ . The hyperplane  $z = 0$  (where  $\mathbf{w} = (0, 0, -1)$ ) separates:

- $\text{XOR} = T \iff z < 0$
- $\text{XOR} = F \iff z > 0$

## 5 Application: Arithmetized Logic Circuit Design

Our symmetry restoration extends Shannon's seminal Boolean circuit model [?] by enabling linear separability in geometric embeddings. While Shannon established the foundation for digital circuit design using classical Boolean algebra, our geometric approach provides:

- Unified arithmetic-logic operations
- Native support for linear separability
- Direct mapping to continuous computation

## 5.1 Arithmetized Design Paradigm

Under geometric Boolean algebra, logic gates map to arithmetic operations: NOT :  $-x$

AND :  $\min(x, y)$

OR :  $\max(x, y)$

XOR :  $-x \cdot y$  (*Theorem7*) This transforms discrete logic into continuous arithmetic, enabling novel circuit design methodologies.

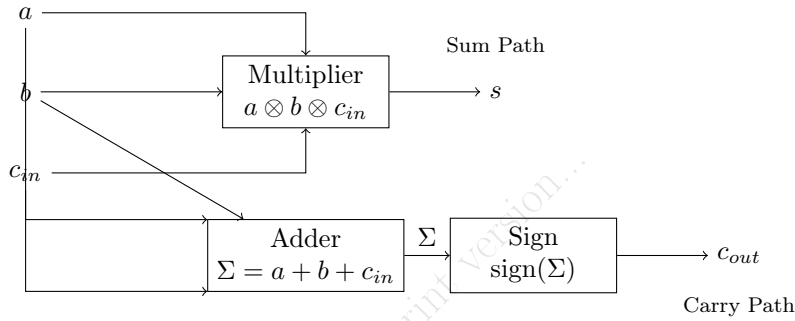
## 5.2 Full Adder Arithmetic Implementation

For inputs  $a, b, c_{in} \in \mathbb{B}_s^1$ :  $s = a \cdot b \cdot c_{in}$  ( $3-input XOR$ )

$c_{out} = \text{sign}(a + b + c_{in})$  (*majority function*)

**Circuit Realization:**

- **Sum  $s$ :** Direct implementation via multiplicative operator:  $s = a \otimes b \otimes c_{in}$
- **Carry  $c_{out}$ :** Linear threshold function:  $c_{out} = \text{sign}(\Sigma)$  where  $\Sigma = a + b + c_{in}$



## 5.3 Comparison with Traditional Implementation

Classical full adders require 5 gates (2×XOR, 2×AND, 1×OR). The arithmetic implementation uses:

- 1 three-input multiplier (equivalent to 2×AND + 1×XOR)
- 1 sign comparator (simple threshold circuit)

**Advantages:**

1. **Parallel processing:** Independent computation of  $s$  and  $c_{out}$  eliminates gate propagation delay
2. **Hardware homogenization:** Uniform arithmetic units replace heterogeneous logic gates
3. **Dimensional scalability:**  $n$ -bit adders extend naturally via vector operations

**Verification:** Inputs  $(a, b, c_{in}) = (1, -1, 1)$ :  $s = 1 \cdot (-1) \cdot 1 = -1$  ( $\leftrightarrow False$ )

$\Sigma = 1 + (-1) + 1 = 1 > 0 \Rightarrow c_{out} = 1$  ( $\leftrightarrow True$ ) Matches full adder specification: sum=0, carry=1

## 6 Conclusion

This work establishes Geometric Boolean Algebra through:

- Symmetric domain  $\mathbb{B}_s^n = \{-1, 1\}$  with natural negation
- Arithmetic realization of logic operations
- Guaranteed linear separability via geometric embedding

The framework suggests new possibilities for:

- Hardware design simplification
- Quantum computation interfaces
- Machine learning acceleration

“The most profound symmetries often emerge from the simplest observations.”

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## References

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