Geometric Boolean Algebra: Axiomatic Restoration of Origin Symmetry

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Abstract

This paper establishes *Geometric Boolean Algebra* (GBA)—a theoretical framework unifying Boolean logic and geometric algebra. Core contributions are organized in three pillars:

1. **Axiomatic Foundation**:
   - Symmetric domain $\mathbb{B}_s^n = \{-1, 1\}^n$
   - Arithmetic realization of logic operations: $\neg x - x$
     $x \wedge y \min(x, y)$
     $x \vee y \max(x, y)$
2. **Theoretical Framework**:
   - Isomorphism proof to classical Boolean algebra (Theorem 1)
   - Isometric embedding $\Phi : \mathbb{B}_s^n \to \mathbb{R}^n$ with distance preservation (Theorem 6)
   - Linear separability of all Boolean functions in augmented spaces (Theorem 7)
3. **Computational Transformation**:
   - Logic-to-arithmetic conversion paradigm
   - Novel circuit design demonstrated via full adder: $\mathrm{s} = \mathrm{a} \otimes b \otimes c_{in}$
     $c_{out} = \mathrm{sign}(a + b + c_{in})$
   - Hardware advantages: uniformity, parallelism, scalability

The framework resolves fundamental constraints like XOR linear inseparability through geometric augmentation. Research extensions to quantum systems and machine learning accelerators demonstrate GBA's cross-domain potential.

1 Introduction: The Geometric Constraint in Boolean Computation

1.1 The Fundamental Defect: Broken Symmetry

Classical Boolean algebra suffers from a **single catastrophic flaw**:

$$Originasymmetry \iff \mathcal{I} : \mathbf{x} \mapsto -\mathbf{x} \quad over\{0, 1\}^n \tag{1}$$

This manifests as:

- **Off-center centroid** at $(\frac{1}{2}, ..., \frac{1}{2})$
- **Constrained representational capacity** (max angle 90°)

  Minsky-Papert (1969) proved this irrevocably blocks linear separability of parity functions like XOR.

As formally established in [**?**], the off-centered centroid in $\{0, 1\}^n$ creates an inherent geometric barrier for linear classifiers. Specifically, the centroid position at $(\frac{1}{2}, ..., \frac{1}{2})$ prevents the existence of any hyperplane separating parity functions. This fundamental limitation motivated our geometric re-axiomatization approach.

1.2 Symmetry Restoration

We resolve this through **Geometric re-axiomatization**, Unlike Diaz-Rivas' linear-algebraic approach to symmetric powers [**?**], our work restores origin symmetry at the axiomatic level:

$$\mathbb{B}_s^n = \{-1, 1\}^n \quad \Rightarrow \quad \underbrace{\mathbf{x} \mapsto -\mathbf{x}}_{central involution} \quad and \quad \underbrace{\frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{B}_s^n} \mathbf{x} = \mathbf{0}}_{geometric balance} \tag{2}$$

Geometric consequence: Full B_n hyperoctahedral symmetry enables linear separability via vertex embedding in $\mathbb{R}^n$.

2 Geometric Boolean Algebra

2.1 Axiomatic Foundation

Axiom 1 (Geometric Boolean Domain). The algebraic structure is defined on the geometrically symmetric domain: $\mathrm{B}_s^n = \{\mathrm{F} : -1,\ \mathrm{T} : 1\}$

Axiom 2 (Primitive Operations). The fundamental operations are geometrically realized as: Negation: $\neg x := -x$

$Conjunction: \quad x \wedge y := \min(x, y)$

$Disjunction: \quad x \vee y := \max(x, y)$

Note: All other Boolean operations (e.g., $\oplus$, $\rightarrow$, $\leftrightarrow$) are derived from these three primitives.

2.2 Isomorphism Theorem

2.2.1 Theorem 1 (Isomorphism to Classical Boolean Algebra)

There exists a bijection $\phi : \{0, 1\} \to \mathbb{B}_s^n$ given by:

$$\phi(a) = 2a - 1 \quad and \quad \phi^{-1}(x) = \frac{x + 1}{2} \tag{3}$$

that preserves all Boolean operations. Specifically, for any classical Boolean function $f : \{0, 1\}^n \to \{0, 1\}$:

$$f_s(\mathbf{x}) = \phi \circ f \circ \phi^{-1}(\mathbf{x}) \tag{4}$$

is its symmetric realization satisfying:

$$\forall \mathbf{a} \in \{0, 1\}^n,\ f(\mathbf{a}) = \phi^{-1}\left(f_s(\phi(\mathbf{a}))\right) \tag{5}$$

3 Derivation of Core Properties

3.1 Functional Completeness Theorem

Theorem 2 (Functional Completeness). The operator set $\{\neg, \wedge\}$ is functionally complete for $\mathbb{B}_s^n$.

Proof: By Axiom 2, we have:

$$x \vee y = \neg(\neg x \wedge \neg y) \tag{6}$$

Thus $\vee$ is derivable. For any Boolean function $f : \mathbb{B}_s^n \to \mathbb{B}_s^n$, its algebraic normal form:

$$f(\mathbf{x}) = \bigvee_{\mathbf{a} \in f^{-1}(1)} \left(\bigwedge_{i : a_i = 1} x_i \wedge \bigwedge_{j : a_j = -1} \neg x_j \right) \tag{7}$$

is constructible using only $\neg$ and $\wedge$. $\square$

3.2 Duality Principle

Theorem 3 (Duality Principle). Let $P = Q$ be an identity in $\mathbb{B}_s^n$. Under the duality transformation:

Operation	Dual
$\wedge$	$\vee$
$\vee$	$\wedge$
$\perp$	$\top$
$\top$	$\perp$

the dual identity $P^d = Q^d$ holds.

Proof sketch: Duality arises from the geometric polarity of $\mathbb{B}_s^n$'s hypercube under central symmetry:

$$\min(x, y) \leftrightarrow \max(-x, -y) \quad via \quad \mathbb{B}_s^n \phi_n \{0, 1\}^n \tag{8}$$

Example:

- Original: $x \wedge \perp = \perp \rightarrow \min(x, -1) = -1$
- Dual: $x \vee \top = \top \rightarrow \max(x, 1) = 1$

3.3 Verification of Algebraic Laws

Law	$\mathbb{B}_s^n$ Verification
De Morgan	$-\min(x, y) = \max(-x, -y)$
Distributive	$\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$
Absorption	$\max(x, \min(x, y)) = x$

4 Geometric-Algebraic Isomorphism

4.1 Intrinsic Geometric Embedding

Theorem 6 (Intrinsic Distance Preservation). The canonical embedding $\iota : \mathbb{B}s^n \hookrightarrow \mathbb{R}^n$ preserves the intrinsic distance structure:

$$|\iota(\mathbf{x}) - \iota(\mathbf{y})| = \sqrt{2n(1 - \cos\theta \mathbf{xy})} = \sqrt{4d_H(\mathbf{x}, \mathbf{y})} \tag{9}$$

where d_H denotes the Hamming distance. This embedding satisfies: x $\cdot \mathbf{y} = n \cos\theta_{\mathbf{xy}}$ $\cos\theta_{\mathbf{xy}} = \frac{1}{n} \sum_{i=1}^n x_i y_i$

Proof: By direct computation: $|\text{x-y}|^2 = \sum_{i=1}^n (x_i - y_i)^2 = \sum_{i=1}^n (2 - 2x_i y_i) \quad (since \text{x}_i^2 = y_i^2 = 1)$ = 2n - 2x$\cdot\mathbf{y}$ The result follows from $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta = n \cos\theta$. $\square$

Corollary 6.1 (Spectral Property). The origin symmetry manifests as:

$$\frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{B}_s^n} \mathbf{x} = \mathbf{0} \tag{10}$$

enabling efficient harmonic analysis on the Boolean hypercube via Fourier-Walsh transform.

4.2 Linear Separability through Minimal Augmentation

Theorem 7 (Optimal Linear Separability). For any Boolean function $f : \mathbb{B}s^n \to -1, 1$, there exists an embedding $\Psi_f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ with $k \leq n2$ such that f is linearly separable under Ψ_f. Specifically for XOR:

$$\Psi\text{XOR}(x_1, x_2) = (x_1, x_2, x_1 x_2), \quad \mathbf{w} = (0, 0, -1) \tag{11}$$

achieves separation with minimal dimension increase.

Geometric Interpretation: The augmented coordinates correspond to polynomial basis terms, forming a $\mathbb{Z}2$-graded algebra:

$$\mathcal{A} = \bigoplus k = 0^n \Lambda^k(\mathbb{R}^n) \tag{12}$$

where Λ^k denotes the k-th exterior power.

4.3 Neural Representation Theorem

Theorem 8 (Resolution of Linear Inseparability). The geometric limitation in neural representation learning [?] is resolved through coordinate augmentation:

- **Minimal augmentation**: $\dim(\Psi_f) \leq n + n2$
- **Topological preservation**: Ψ_f maintains adjacency relations
- **Complexity separation**: $\text{VCdim}(\Psi_f \circ \mathcal{H}_{\text{lin}}) = O(n^2)$

The separation hyperplane in augmented space $\mathbb{R}^{n+k}$ admits geometric realization:

$$f(\mathbf{x}) = \text{sign}\left(\sum_{S \subseteq [n]} w_S \prod_{i \in S} x_i\right) \tag{13}$$

where w_S are spectral coefficients determined by the Fourier-Walsh expansion.

4.4 Linearization of XOR

Theorem 7 (Linear Representation of XOR). Define the quadratic embedding $\Psi : \mathbb{B}_s^2 \to \mathbb{R}^3$:

$$\Psi(x_1, x_2) = (x_1, x_2, x_1 x_2) \tag{14}$$

XOR is linearly separable under Ψ with weight $\mathbf{w} = (0, 0, -1)$:

$$\text{XOR}(x_1, x_2) = \text{sign}(\mathbf{w}^T \Psi(\mathbf{x})) = -\text{sign}(x_1 x_2) \tag{15}$$

Proof via Geometric Verification:

$\mathbf{x}$	$\mathbf{w}^T \Psi(\mathbf{x})$	Output	XOR
$(-1, -1)$	$-(1) = -1$	F	F
$(-1, 1)$	$-(-1) = 1$	T	T
$(1, -1)$	$-(-1) = 1$	T	T
$(1, 1)$	$-(1) = -1$	F	F

Geometric Insight: Embedded points $\{\Psi(\mathbf{x})\}$ form a tetrahedron in $\mathbb{R}^3$. The hyperplane $z = 0$ (where $\mathbf{w} = (0, 0, -1)$) separates:

- $XOR = T \iff z < 0$
- $XOR = F \iff z > 0$

5 Application: Arithmetized Logic Circuit Design

Our symmetry restoration extends Shannon's seminal Boolean circuit model [?] by enabling linear separability in geometric embeddings. While Shannon established the foundation for digital circuit design using classical Boolean algebra, our geometric approach provides:

- Unified arithmetic-logic operations
- Native support for linear separability
- Direct mapping to continuous computation

5.1 Arithmetized Design Paradigm

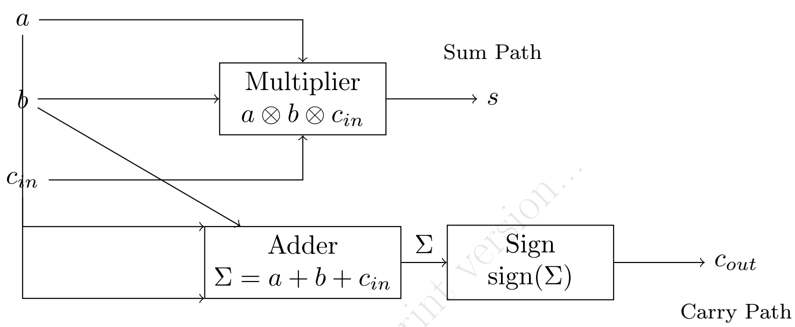
Under geometric Boolean algebra, logic gates map to arithmetic operations: NOT : -x
AND : $\min(x, y)$
OR : $\max(x, y)$
XOR : $-x \cdot y$ $(Theorem 7)$ This transforms discrete logic into continuous arithmetic, enabling novel circuit design methodologies.

5.2 Full Adder Arithmetic Implementation

For inputs $a, b, c_{in} \in \mathbb{B}_s^1$: s = a $\cdot b \cdot c_{in}$ $(3 - inputXOR)$
$c_{out} = \text{sign}(a + b + c_{in})$ $(majorityfunction)$

Circuit Realization:

- **Sum** s: Direct implementation via multiplicative operator: $s = a \otimes b \otimes c_{in}$
- **Carry** c_{out}: Linear threshold function: $c_{out} = \text{sign}(\Sigma)$ $where$ $\Sigma = a + b + c_{in}$



5.3 Comparison with Traditional Implementation

Classical full adders require 5 gates (2×XOR, 2×AND, 1×OR). The arithmetic implementation uses:

- 1 three-input multiplier (equivalent to 2×AND + 1×XOR)
- 1 sign comparator (simple threshold circuit)

Advantages:

1. **Parallel processing**: Independent computation of s and c_{out} eliminates gate propagation delay
2. **Hardware homogenization**: Uniform arithmetic units replace heterogeneous logic gates
3. **Dimensional scalability**: n-bit adders extend naturally via vector operations

Verification: Inputs $(a, b, c_{in}) = (1, -1, 1)$: s = 1 $\cdot(-1) \cdot 1 = -1$ $(\leftrightarrow False)$
$\Sigma = 1 + (-1) + 1 = 1 > 0 \Rightarrow c_{out} = 1$ $(\leftrightarrow True)$ Matches full adder specification: sum=0, carry=1

6 Conclusion

This work establishes Geometric Boolean Algebra through:

- Symmetric domain $\mathbb{B}_s^n = \{-1, 1\}$ with natural negation
- Arithmetic realization of logic operations
- Guaranteed linear separability via geometric embedding

The framework suggests new possibilities for:

- Hardware design simplification
- Quantum computation interfaces
- Machine learning acceleration

"The most profound symmetries often emerge from the simplest observations."

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