Geometric Boolean Algebra: Axiomatic Restoration of Origin Symmetry

QingYuan Qu
Independent Researcher
Tengzhou City, Shandong Province, China
992883600@qq.com

July 10, 2025

Abstract

This paper establishes *Geometric Boolean Algebra* (GBA)—a theoretical framework unifying Boolean logic and geometric algebra. Core contributions are organized in three pillars:

1. **Axiomatic Foundation**:
   - Symmetric domain $\mathbb{B}_s^n = \{-1, 1\}^n$
   - Arithmetic realization of logic operations: $\neg x - x$
     $x \wedge y \min(x, y)$
     $x \vee y \max(x, y)$
2. **Theoretical Framework**:
   - Isomorphism proof to classical Boolean algebra (Theorem 1)
   - Isometric embedding $\Phi : \mathbb{B}_s^n \to \mathbb{R}^n$ with distance preservation (Theorem 6)
   - Linear separability of all Boolean functions in augmented spaces (Theorem 7)
3. **Computational Transformation**:
   - Logic-to-arithmetic conversion paradigm
   - Novel circuit design demonstrated via full adder: $\mathrm{s} = \mathrm{a} \otimes b \otimes c_{in}$
     $c_{out} = \mathrm{sign}(a + b + c_{in})$
   - Hardware advantages: uniformity, parallelism, scalability

The framework resolves fundamental constraints like XOR linear inseparability through geometric augmentation. Research extensions to quantum systems and machine learning accelerators demonstrate GBA's cross-domain potential.

1 Introduction: The Geometric Constraint in Boolean Computation

1.1 The Fundamental Defect: Broken Symmetry

Classical Boolean algebra suffers from a **single catastrophic flaw**:

$$Originasymmetry \iff \mathcal{I} : \mathbf{x} \mapsto -\mathbf{x} \quad over\{0, 1\}^n \tag{1}$$

This manifests as:

- **Off-center centroid** at $(\frac{1}{2}, ..., \frac{1}{2})$
- **Constrained representational capacity** (max angle 90°)

  Minsky-Papert (1969) proved this irrevocably blocks linear separability of parity functions like XOR.

As formally established in [**?**], the off-centered centroid in $\{0, 1\}^n$ creates an inherent geometric barrier for linear classifiers. Specifically, the centroid position at $(\frac{1}{2}, ..., \frac{1}{2})$ prevents the existence of any hyperplane separating parity functions. This fundamental limitation motivated our geometric re-axiomatization approach.

1.2 Symmetry Restoration

We resolve this through **Geometric re-axiomatization**, Unlike Diaz-Rivas' linear-algebraic approach to symmetric powers [?], our work restores origin symmetry at the axiomatic level:

$$\mathbb{B}_s^n = \{-1,1\}^n \quad \Rightarrow \quad \underbrace{\mathbf{x} \mapsto -\mathbf{x}}_{central involution} \quad and \quad \underbrace{\frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{B}_s^n} \mathbf{x} = \mathbf{0}}_{geometric balance} \tag{2}$$

Geometric consequence: Full B_n hyperoctahedral symmetry enables linear separability via vertex embedding in $\mathbb{R}^n$.

2 Geometric Boolean Algebra

2.1 Axiomatic Foundation

Axiom 1 (Geometric Boolean Domain). The algebraic structure is defined on the geometrically symmetric domain: $\mathrm{B}_s^n = \{\mathrm{F} : -1,\ \mathrm{T} : 1\}$

Axiom 2 (Primitive Operations). The fundamental operations are geometrically realized as: Negation: $\neg x := -x$
$Conjunction: \quad x \wedge y := \min(x, y)$
$Disjunction: \quad x \vee y := \max(x, y)$

Note: All other Boolean operations (e.g., $\oplus$, $\rightarrow$, $\leftrightarrow$) are derived from these three primitives.

2.2 Isomorphism Theorem

2.2.1 Theorem 1 (Isomorphism to Classical Boolean Algebra)

There exists a bijection $\phi : \{0,1\} \to \mathbb{B}_s^n$ given by:

$$\phi(a) = 2a - 1 \quad and \quad \phi^{-1}(x) = \frac{x+1}{2} \tag{3}$$

that preserves all Boolean operations. Specifically, for any classical Boolean function $f : \{0,1\}^n \to \{0,1\}$:

$$f_s(\mathbf{x}) = \phi \circ f \circ \phi^{-1}(\mathbf{x}) \tag{4}$$

is its symmetric realization satisfying:

$$\forall \mathbf{a} \in \{0,1\}^n,\ f(\mathbf{a}) = \phi^{-1}\left(f_s(\phi(\mathbf{a}))\right) \tag{5}$$

3 Derivation of Core Properties

3.1 Functional Completeness Theorem

Theorem 2 (Functional Completeness). The operator set $\{\neg, \wedge\}$ is functionally complete for $\mathbb{B}_s^n$.

Proof: By Axiom 2, we have:

$$x \vee y = \neg(\neg x \wedge \neg y) \tag{6}$$

Thus $\vee$ is derivable. For any Boolean function $f : \mathbb{B}_s^n \to \mathbb{B}_s^n$, its algebraic normal form:

$$f(\mathbf{x}) = \bigvee_{\mathbf{a} \in f^{-1}(1)} \left(\bigwedge_{i : a_i = 1} x_i \wedge \bigwedge_{j : a_j = -1} \neg x_j \right) \tag{7}$$

is constructible using only $\neg$ and $\wedge$. $\square$

3.2 Duality Principle

Theorem 3 (Duality Principle). Let $P = Q$ be an identity in $\mathbb{B}_s^n$. Under the duality transformation:

Operation	Dual
$\wedge$	$\vee$
$\vee$	$\wedge$
$\perp$	$\top$
$\top$	$\perp$

the dual identity $P^d = Q^d$ holds.

Proof sketch: Duality arises from the geometric polarity of $\mathbb{B}_s^n$'s hypercube under central symmetry:

$$\min(x, y) \leftrightarrow \max(-x, -y) \quad via \quad \mathbb{B}_s^n \phi_n \{0, 1\}^n \tag{8}$$

Example:

- Original: $x \wedge \perp = \perp \to \min(x, -1) = -1$
- Dual: $x \vee \top = \top \to \max(x, 1) = 1$

3.3 Verification of Algebraic Laws

Law	$\mathbb{B}_s^n$ Verification
De Morgan	$-\min(x, y) = \max(-x, -y)$
Distributive	$\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$
Absorption	$\max(x, \min(x, y)) = x$

4 Geometric-Algebraic Isomorphism

4.1 Intrinsic Geometric Embedding

Theorem 6 (Intrinsic Distance Preservation). The canonical embedding $\iota : \mathbb{B}s^n \hookrightarrow \mathbb{R}^n$ preserves the intrinsic distance structure:

$$|\iota(\mathbf{x}) - \iota(\mathbf{y})| = \sqrt{2n(1 - \cos\theta\mathbf{xy})} = \sqrt{4d_H(\mathbf{x}, \mathbf{y})} \tag{9}$$

where d_H denotes the Hamming distance. This embedding satisfies: x $\cdot\mathbf{y} = n\cos\theta_{\mathbf{xy}}$ $\cos\theta_{\mathbf{xy}} = \frac{1}{n}\sum_{i=1}^n x_i y_i$

Proof: By direct computation: $|$x-y$|^2 = \sum_{i=1}^n (x_i - y_i)^2 = \sum_{i=1}^n (2 - 2x_i y_i) \quad (since \mathrm{x}_i^2 = y_i^2 = 1)$ = 2n - 2x$\cdot\mathbf{y}$ The result follows from $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta = n\cos\theta$. □

Corollary 6.1 (Spectral Property). The origin symmetry manifests as:

$$\frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{B}_s^n} \mathbf{x} = \mathbf{0} \tag{10}$$

enabling efficient harmonic analysis on the Boolean hypercube via Fourier-Walsh transform.

4.2 Linear Separability through Minimal Augmentation

Theorem 7 (Optimal Linear Separability). For any Boolean function $f : \mathbb{B}s^n \to -1, 1$, there exists an embedding $\Psi_f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ with $k \leq n2$ such that f is linearly separable under Ψ_f. Specifically for XOR:

$$\Psi\mathrm{XOR}(x_1, x_2) = (x_1, x_2, x_1 x_2), \quad \mathbf{w} = (0, 0, -1) \tag{11}$$

achieves separation with minimal dimension increase.

Geometric Interpretation: The augmented coordinates correspond to polynomial basis terms, forming a $\mathbb{Z}2$-graded algebra:

$$\mathcal{A} = \bigoplus k = 0^n \Lambda^k(\mathbb{R}^n) \tag{12}$$

where Λ^k denotes the k-th exterior power.

4.3 Neural Representation Theorem

Theorem 8 (Resolution of Linear Inseparability). The geometric limitation in neural representation learning [?] is resolved through coordinate augmentation:

- **Minimal augmentation**: $\dim(\Psi_f) \leq n + n2$
- **Topological preservation**: Ψ_f maintains adjacency relations
- **Complexity separation**: $\mathrm{VCdim}(\Psi_f \circ \mathcal{H}_{\mathrm{lin}}) = O(n^2)$

The separation hyperplane in augmented space $\mathbb{R}^{n+k}$ admits geometric realization:

$$f(\mathbf{x}) = \mathrm{sign}\left(\sum_{S \subseteq [n]} w_S \prod_{i \in S} x_i \right) \tag{13}$$

where w_S are spectral coefficients determined by the Fourier-Walsh expansion.

4.4 Linearization of XOR

Theorem 7 (Linear Representation of XOR). Define the quadratic embedding $\Psi : \mathbb{B}_s^2 \to \mathbb{R}^3$:

$$\Psi(x_1, x_2) = (x_1, x_2, x_1 x_2) \tag{14}$$

XOR is linearly separable under Ψ with weight $\mathbf{w} = (0, 0, -1)$:

$$\mathrm{XOR}(x_1, x_2) = \mathrm{sign}(\mathbf{w}^T \Psi(\mathbf{x})) = -\mathrm{sign}(x_1 x_2) \tag{15}$$

Proof via Geometric Verification:

$\mathbf{x}$	$\mathbf{w}^T\Psi(\mathbf{x})$	Output	XOR
$(-1,-1)$	$-(1) = -1$	F	F
$(-1,1)$	$-(-1) = 1$	T	T
$(1,-1)$	$-(-1) = 1$	T	T
$(1,1)$	$-(1) = -1$	F	F

Geometric Insight: Embedded points $\{\Psi(\mathbf{x})\}$ form a tetrahedron in $\mathbb{R}^3$. The hyperplane $z = 0$ (where $\mathbf{w} = (0, 0, -1)$) separates:

- $XOR = T \iff z < 0$
- $XOR = F \iff z > 0$

5 Application: Arithmetized Logic Circuit Design

Our symmetry restoration extends Shannon's seminal Boolean circuit model [?] by enabling linear separability in geometric embeddings. While Shannon established the foundation for digital circuit design using classical Boolean algebra, our geometric approach provides:

- Unified arithmetic-logic operations
- Native support for linear separability
- Direct mapping to continuous computation

5.1 Arithmetized Design Paradigm

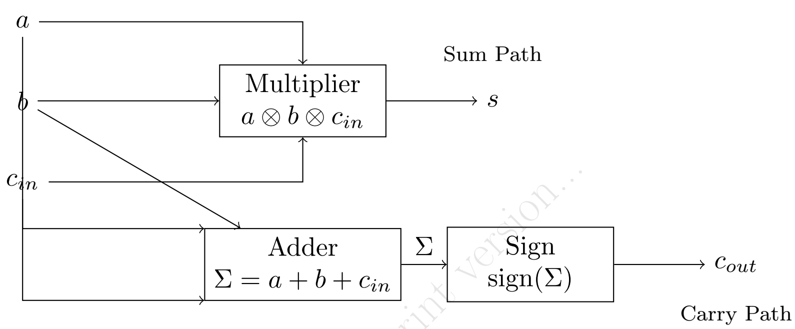
Under geometric Boolean algebra, logic gates map to arithmetic operations: NOT : -x
AND : $\min(x, y)$
OR : $\max(x, y)$
XOR : $-x \cdot y \quad (Theorem 7)$ This transforms discrete logic into continuous arithmetic, enabling novel circuit design methodologies.

5.2 Full Adder Arithmetic Implementation

For inputs $a, b, c_{in} \in \mathbb{B}_s^1$: s = a $\cdot b \cdot c_{in} \quad (3 - input XOR)$
$c_{out} = \text{sign}(a + b + c_{in}) \quad (majority function)$

Circuit Realization:

- **Sum** s: Direct implementation via multiplicative operator: $s = a \otimes b \otimes c_{in}$
- **Carry** c_{out}: Linear threshold function: $c_{out} = \text{sign}(\Sigma) \quad where \quad \Sigma = a + b + c_{in}$



5.3 Comparison with Traditional Implementation

Classical full adders require 5 gates (2×XOR, 2×AND, 1×OR). The arithmetic implementation uses:

- 1 three-input multiplier (equivalent to 2×AND + 1×XOR)
- 1 sign comparator (simple threshold circuit)

Advantages:

1. **Parallel processing**: Independent computation of s and c_{out} eliminates gate propagation delay
2. **Hardware homogenization**: Uniform arithmetic units replace heterogeneous logic gates
3. **Dimensional scalability**: n-bit adders extend naturally via vector operations

Verification: Inputs $(a, b, c_{in}) = (1, -1, 1)$: s = 1 $\cdot(-1) \cdot 1 = -1 \quad (\leftrightarrow False)$
$\Sigma = 1 + (-1) + 1 = 1 > 0 \Rightarrow c_{out} = 1 \quad (\leftrightarrow True)$ Matches full adder specification: sum=0, carry=1

6 Conclusion

This work establishes Geometric Boolean Algebra through:

- Symmetric domain $\mathbb{B}_s^n = \{-1, 1\}$ with natural negation
- Arithmetic realization of logic operations
- Guaranteed linear separability via geometric embedding

The framework suggests new possibilities for:

- Hardware design simplification
- Quantum computation interfaces
- Machine learning acceleration

"The most profound symmetries often emerge from the simplest observations."

The author gratefully acknowledges DeepSeek for its invaluable assistance in theorem derivation and formal verification. This work benefited significantly from human-AI collaborative exploration of geometric-algebraic duality.

References

[1] Rafael Diaz and Mariolys Rivas. Symmetric boolean algebras, 2006.

[2] Marvin Minsky and Seymour Papert. *Perceptrons: An Introduction to Computational Geometry*. MIT Press, Cambridge, MA, USA, 1969.

[3] David E. Rumelhart, Geoffrey E. Hinton, and Ronald J. Williams. Learning representations by back-propagating errors. *Nature*, 323:533–536, 1986.

[4] Claude E Shannon. A symbolic analysis of relay and switching circuits. *Electrical Engineering*, 57(12):713–723, 1938.