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Signals and Systems

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Final Exam

1.

a. The given system is non-causal because it does not satisfy the condition for causality:

$$h(n) = 0 \qquad \forall n < 0$$

For the given system, we can see that h(-1) = -1 and that h(-2) = 1.

b. Any system is said to be BIBO stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

For the given system, we have:

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-3}^{3} |h(n)|$$
$$= |h(-3)| + |h(-2)| + |h(-1)| + |h(0)| + |h(1)| + |h(2)| + |h(3)|$$
$$= |0| + |1| + |-1| + |-1| + |-1| + |1| + |0|$$
$$= 4 < \infty$$

Therefore, the given system is BIBO stable.

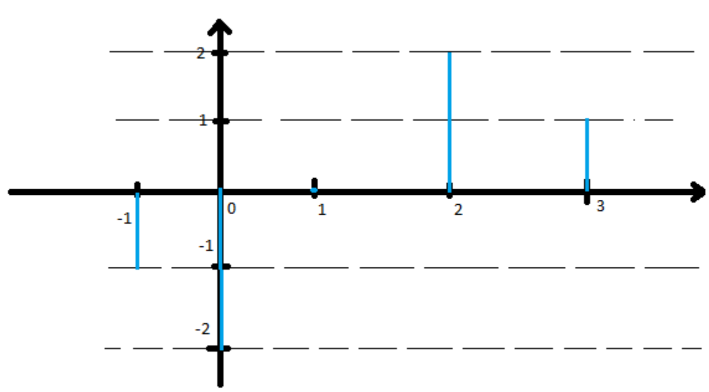
c. To find the output y(n), we can write:

$$y(n) = x(n) + h(n)$$
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

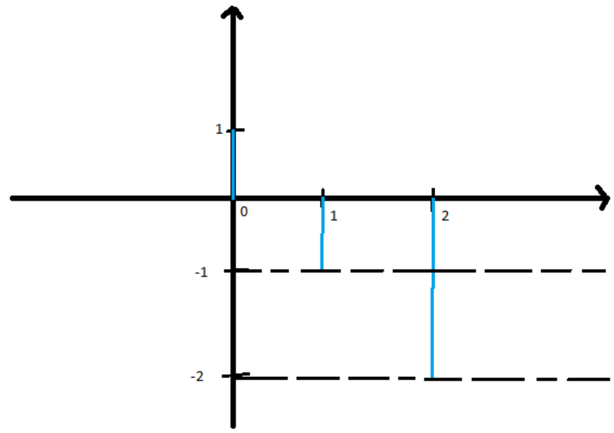
Therefore:

$$y(-1) = x(0)h(-1) + x(2)h(-3) = -1 + 0 = -1$$
$$y(0) = x(0)h(0) + x(2)h(-2) = -1 + (-1) = -2$$
$$y(1) = x(0)h(1) + x(2)h(-1) = -1 + 1 = 0$$
$$y(2) = x(0)h(2) + x(2)h(0) = 1 + 1 = 2$$
$$y(3) = x(0)h(3) + x(2)h(1) = 0 + 1 = 1$$

The output graph of y(n) is as follows:



2. First, let's plot the graph of x(n):



Next, let's consider the equation

$$y(n) = x(n) + h(n)$$
$$= \sum_{k=0}^{2} x(k)h(n-k)$$

From this, we can say:

$$y(0) = x(0) * h(0) = 3 = (1)\big(h(0)\big)$$
$$\rightarrow h(0) = 3$$

$$y(1) = x(0)h(1) + x(1)h(0) = -2 = h(1) - 3$$
$$\rightarrow h(1) = 1$$

$$y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = -2$$
$$\rightarrow h(2) = 5$$

$$y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = -5$$
$$\rightarrow h(3) = 2$$

$$y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = -12$$
$$\rightarrow h(4) = 0$$

$$y(5) = x(0)h(5) + x(1)h(4) + x(2)h(3) = -4$$
$$\rightarrow h(5) = 0$$

$$h(n) = 0 \ \ \forall n > 5$$

Thus, we can write our impulse response h(n) as:

$$h(n) = \begin{cases} [3, 1, 5, 2] & \text{where } n = 0{,}1{,}2{,}3 \text{ respectively} \\ 0 & \text{where n} > 3 \end{cases}$$

3. Given $h(t) = 2W\,\mathrm{sinc}(2Wt)$

   If we take Laplace transform of this equation, we get:

$$H(W) = \frac{2W}{2W}\mathrm{rect}\left(\frac{f}{2W}\right) \qquad \left[\because \mathrm{sinc}(Bt) \leftrightarrow \frac{1}{B}\mathrm{rect}\left(\frac{f}{B}\right)\right]$$
$$= \mathrm{rect}\left(\frac{f}{2W}\right)$$
$$= 1 \text{ for} - W \leq f \leq W$$

   a. To determine output when $f_0 = 100\text{Hz},\ W = 1000\text{Hz}$, we know that:

$$x(t) = A + B\cos(2\pi f_0 t) = A + B\cos(2\pi(100)t)$$

   Because $f_0 < W$, it allows the original signal

$$\therefore y(t) = x(t) = A + B\cos(200\pi t)$$

   b. To determine output when $f_0 = 1000\text{Hz},\ W = 100\text{Hz}$, we know that:

$$x(t) = A + B\cos(2\pi f_0 t) = A + B\cos(2\pi(1000)t)$$

   Because $f_0 > W$, the signal will not be allowed into the circuit

$$\therefore y(t) = A + B\cos(2\pi(1000)t)$$
$$= A + B\cos(2000\pi t)$$
$$= A$$

4. From the diagram, we can state the following:

$$q(n) = x(n) + kq(n-1) + kq(n-2)$$
$$y(n) = q(n) + kq(n-1)$$

   Next, we can apply the z-transform:

$$1)\quad Q(z) = X(z) + kz^{-1}Q(z) + kz^{-2}Q(z)$$
$$2)\quad Y(z) = Q(z) + kz^{-1}Q(z)$$

   We can rewrite equation 1) to say

$$X(z) = Q(z)[1 - kz^{-1} - kz^{-2}]$$
$$\rightarrow Q(z) = \frac{X(z)}{1 - kz^{-1} - kz^{-2}}$$

   Similarly, we can rewrite equation 2) to say

$$Y(z) = Q(z)[1 + kz^{-1}]$$

   We can then say:

$$\frac{Y(z)}{X(z)} = \frac{1 + kz^{-1}}{1 - kz^{-1} - kz^{-2}}$$

   Next, we calculate our transfer function for k=2:

$$H(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} - 2z^{-2}} = \frac{z(z+2)}{z^2 - 2z - 2}$$

   We can finally calculate the roots of our denominator for our transfer function at k=2:

$$z = -0.732, \qquad 2.732$$

   We can see that the pole lies outside the unit circle of the z-plane, therefore the system given is not BIBO stable at k=2.

5.

a. If the system is both causal and stable, then all the poles of H(z) must lie inside the unit circle of the z-plane because the ROC is of the form $|z| > r_{max}$ (For right-handed signal). Since the unit circle is included in the ROC, then we must have $r_{max} < 1$. Given:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{10 - 2z^{-1}}{(a + z^{-1})^2}$$

Next, we can calculate for the ROC of our system:

$$\text{ROC:}\ \ z^{-1} > -a$$
$$\rightarrow z < -\frac{1}{a}$$
$$\rightarrow |z| < \left|\frac{1}{a}\right|$$
$$\rightarrow 1 < \left|\frac{1}{a}\right|$$
$$\rightarrow 1 < |a|$$
$$\rightarrow -1 < a < 1$$

Therefore, the given system is both causal and stable for -1 < a < 1.

b. Given:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{10 - 2z^{-1}}{a^2 + 2az^{-1} + z^{-2}}$$
$$\rightarrow Y(z)(a^2 + 2az^{-1} + z^{-2}) = X(z)(10 - 2z^{-1})$$
$$\rightarrow a^2Y(z) + 2az^{-1}Y(z) + z^{-2}Y(z) = 10X(z) - 2z^{-1}X(z)$$

If we take the inverse z-transform of the above equation, we will get:

$$a^2y(n) + 2ay(n-1) + y(n-2) = 10x(n) - 2x(n-1)$$
$$\therefore y(n) = \frac{1}{a^2}[(10x(n) - 2x(n-1)) - 2ay(n-1) - y(n-2)]$$

6.

a. As given, the transfer function for a causal, first order, finite impulse response filter in the z domain is of the form

$$H(z) = a_0 + a_1z^{-1}$$

To analyze this in the frequency domain, we can say

$$H(\Omega) = a_0 + a_1e^{-j\Omega}$$

At $\Omega = 0$:

$$H(\Omega) = 1$$
$$\rightarrow a_0 + a_1e^{-j(0)} = 1$$
$$\rightarrow 1) \qquad a_0 + a_1 = 1$$

At $\Omega = \pi$:

$$H(\Omega) = \frac{1}{2}$$
$$\rightarrow a_0 + a_1e^{-j\pi} = \frac{1}{2}$$
$$\rightarrow 2) \qquad a_0 - a_1 = \frac{1}{2}$$

Solving equations 1) and 2) above, we get

$$[a_0 + a_1 = 1] + \left[a_0 - a_1 = \frac{1}{2}\right]$$
$$2a_0 = \frac{3}{2}$$
$$a_0 = \frac{3}{4}$$

Substituting back in, we get

$$\frac{3}{4} + a_1 = 1$$
$$a_1 = \frac{1}{4}$$

Therefore, we can say that a causal, first-order, finite impulse response filter has been designed with

$$H(z) = \frac{3}{4} + \frac{1}{4}z^{-1}$$

b. Taking the inverse z transform of the H(z) we solved above, we get

$$y(n) = \frac{3}{4}x(n) + \frac{1}{4}x(n-1)$$

Additionally, we know that our input $x(n) = \{1,1,1,1,1,1,1\}$, all other values 0. We now find output y(n) when x(n) has the above values:

$$y(0) = \frac{3}{4}x(0) + \frac{1}{4}x(-1) = \frac{3}{4}$$

$$y(1) = \frac{3}{4}x(1) + \frac{1}{4}x(0) = 1$$

$$y(2) = \frac{3}{4}x(2) + \frac{1}{4}x(1) = 1$$

$$y(3) = \frac{3}{4}x(3) + \frac{1}{4}x(2) = 1$$

$$y(4) = \frac{3}{4}x(4) + \frac{1}{4}x(3) = 1$$

$$y(5) = \frac{3}{4}x(5) + \frac{1}{4}x(4) = 1$$

$$y(6) = \frac{3}{4}x(6) + \frac{1}{4}x(5) = 1$$

$$y(7) = \frac{3}{4}x(7) + \frac{1}{4}x(6) = \frac{1}{4}$$

We can say applying the filter obtained in part a) to the given input x(n) gives us the following output $y(n) = \left\{\frac{3}{4}, 1,1,1,1,1,1, \frac{1}{4}\right\}$

7. Given:

$$H(z) = \frac{b_0}{1 + a_1 z^{-1}}$$
$$\rightarrow \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1}}$$
$$Y(z)(1 + a_1 z^{-1}) = b_0 X(z)$$

We can rewrite the above to say:

$$y(n) + a_1 y(n-1) = b_0 x(n)$$

a. We now ensure requirement 1 of the question is met.

$$y(n) + a_1 y(n-1) = 0$$
$$\rightarrow y(n) = -a_1 y(n-1)$$

For the 2 samples below, we will use the following:

$$y(n) = h(n) = b_0 (a_1)^n u(n)$$

At n=0:

$$y(0) = b_0 (a_1)^0 = k$$
$$\rightarrow b_0 = k$$

At time of 0.02 seconds (because of y(n) 36% decay), we have our second sample:

$$y(2) = (a_1)^2 y(0) = (a_1)^2 k = 0.36k$$
$$\rightarrow (a_1)^2 = 0.36$$
$$\rightarrow a_1 = 0.6$$

b. We now ensure requirement 2 of the question is met.

$$y(n) = y(n-1) = 1$$

Substituting our above condition in, we get:

$$1 + a_1 = b_0$$
$$\rightarrow b_0 = 1 + 0.6$$
$$\rightarrow b_0 = 1.6$$

Therefore, our system response is:

$$y(n) + 0.6y(n-1) = 1.6x(n)$$

or

$$H(z) = \frac{1.6}{1 + 0.6z^{-1}}$$

8. Given:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1}}$$

We can replace with:

$$H(\Omega) = \frac{b_0}{1 + a_1 e^{-j\Omega}}$$

At $\Omega = 0$:

$$H(\Omega) = \frac{b_0}{1 + a_1} = 1$$
$$\rightarrow b_0 = 1 + a_1$$

At $\Omega = \frac{\pi}{2}$:

$$H(\Omega) = \frac{b_0}{1 + a_1 e^{-j\frac{\pi}{2}}} = 1$$

But we know that $e^{-j\frac{\pi}{2}} = -j$:

$$H(\Omega) = \frac{b_0}{1 - a_1 j}$$
$$|H(\Omega)| = \frac{b_0}{\sqrt{1 + (a_1)^2}} = \frac{1}{\sqrt{2}}$$
$$\rightarrow (b_0)^2(2) = 1 + (a_1)^2$$

But we know that $b_0 = 1 + a_1$:

$$\rightarrow (1 + a_1)^2(2) = 1 + (a_1)^2$$
$$\rightarrow a_1 = -2 \pm \sqrt{3}$$
$$\rightarrow b_0 = 1 + a_1 = -1 \pm \sqrt{3}$$

For given filter to be stable, our ROC must include unit circle. For given filter to be causal, our ROC should be outer of our outermost pole. The pole is at $z = -a_1$. Our ROC would therefore be $|z| > |a_1|$. For our ROC to include unit circle, $|a_1| < 1$.

Let's analyze at $a_1 = -2 \pm \sqrt{3}$ to see stability:

At $a_1 = -2 + \sqrt{3}$:

$$a_1 = -2 + \sqrt{3}$$
$$a_1 = -2 + 1.732$$
$$a_1 = -0.268$$

Here, we have $|z| > |a_1| \rightarrow |z| > |-0.268|$. This ROC is stable because it includes the unit circle.

At $a_1 = -2 - \sqrt{3}$:

$$a_1 = -2 - \sqrt{3}$$
$$a_1 = -3.732$$

Here, our ROC is $|z| > |-3.732|$ and does not include the unit circle, therefore it is not stable.

The only stable option for a_1 is $a_1 = -2 + \sqrt{3} = -0.268$.
Thus, we have $b_0 = 1 + a_1 = -1 + \sqrt{3} = 0.732$

We have calculated the coefficients such that our filter is stable and causal and fulfills the two criteria:

$$a_1 = -2 + \sqrt{3}$$
$$b_0 = -1 + \sqrt{3}$$
$$H(z) = \frac{\left(-1 + \sqrt{3}\right)}{1 + \left(-2 + \sqrt{3}\right)z^{-1}}$$