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Signals and Systems

October 15, 2019

Homework #2

1. Q1.27 (a,f)
   a. $y(t) = x(t-2) + x(2-t)$
      i. The **Linear (3)** and **Stable (5)** properties hold for this Continuous-Time System. To determine this:
      First, we consider the output at t=0:
      $$y(0) = x(-2) + x(2)$$
      The output y(0) is dependent upon the past value, x(-2), and the future value, x(2). Therefore, by definition, this system cannot be **Causal** and it has **Memory**.

      Next, we consider a shift t_0 in the output y(t):
      $$y(t-t_0) = x(t-t_0) + x(2-t+t_0)$$
      If the input is shifted to t_0 and passed through the system, then the output is:
      $$x(t-t_0) \to y(t) = x(t-t_0) + x(2-t-t_0)$$
      From this, it is clear that a shift of t_0 in the output doesn't have a corresponding shift in the input. Therefore, this system is **Time-Variant**.

      Then, we apply the superposition principle to verify the linearity of the system. Let's consider:
      $$y_1(t) = x_1(t-2) + x_1(2-t)$$
      $$y_2(t) = x_2(t-2) + x_2(2-t)$$
      We then consider a third input $x_3(t)$ such that $x_3(t)$ is a linear combination of $x_1(t)$ and $x_2(t)$:
      $$x_3(t) = ax_1(t) + bx_2(t)$$
      Thus, the output $y_3(t)$ is given as:
      $$y_3(t) = x_3(t-2) + x_3(2-t)$$
      $$y_3(t) = ax_1(t-2) + bx_2(t-2) + ax_1(2-t) + bx_2(2-t)$$
      $$y_3(t) = ax_1(t-2) + ax_1(2-t) + bx_2(t-2) + bx_2(2-t)$$
      $$y_3(t) = a[x_1(t-2) + x_1(2-t)] + b[x_2(t-2) + x_2(2-t)]$$
      $$y_3(t) = ay_1(t) + by_2(t)$$
      From this, it is clear that this system satisfies both Additivity and Homogeneity properties, therefore this system is **Linear**.

      Finally, let's consider $|x(t)| < \infty$ for all t, then:
      $$|y(t)| = |x(t-2) + x(2-t)|$$
      $$|y(t)| \le |x(t-2)| + |x(2-t)| \qquad (\because |a+b| \le |a| + |b|)$$
      $$|y(t)| < \infty \qquad (\because |x(t)| < \infty)$$
      From this, it is clear that this system is **Stable**.

f. $y(t) = x\left(\frac{t}{3}\right)$

i. The **Linear (3)** and **Stable (5)** properties hold for this Continuous-Time System. To determine this:
First, we consider the output at t=0:

$$y(0) = x\left(\frac{0}{3}\right) = x(0)$$

The output, y(3) at t=3, is dependent upon the past value, x(1). The output, y(-3) at t=-3, is dependent upon the future value, x(-1). Therefore, by definition, this system cannot be **Causal** and has **Memory**.

Next, we consider a shift t_0 in the output y(t):

$$y\left(\frac{t-t_0}{3}\right) = x\left(\frac{t-t_0}{3}\right)$$

If the input is shifted to t_0 and passed through the system, then the output is

$$x(t-t_0) \rightarrow y(t) = x\left(\frac{t}{3} - t_0\right)$$

From this, it is clear that a shift of t_0 in the output does not have a corresponding shift in the input, which implies that the system is **Time-variant**.

Then, we apply the superposition principle to verify the linearity of the system. Let's consider:

$$y_1(t) = x_1\left(\frac{t}{3}\right)$$
$$y_2(t) = x_2\left(\frac{t}{3}\right)$$

Let's now consider a third input $x_3(t)$ such that $x_3(t)$ is a linear combination of $x_1(t)$ and $x_2(t)$:

$$x_3(t) = ax_1(t) + bx_2(t)$$

Thus, the output $y_3(t)$ would be given as:

$$y_3(t) = x_3\left(\frac{t}{3}\right)$$
$$y_3(t) = ax_1\left(\frac{t}{3}\right) + bx_2\left(\frac{t}{3}\right)$$
$$y_3(t) = ay_1(t) + by_2(t)$$

From this, it is clear that this system satisfies both Additivity and Homogeneity properties. Therefore, this system is **Linear**.

Finally, let's consider $|x(t)| < \infty$ for all t, then:

$$|y(t)| = \left|x\left(\frac{t}{3}\right)\right|$$

As $|x(t)| < \infty$, the time scaled version of x(t) will only spread the signal over time, but it is still stable. Therefore, this system is **Stable**.

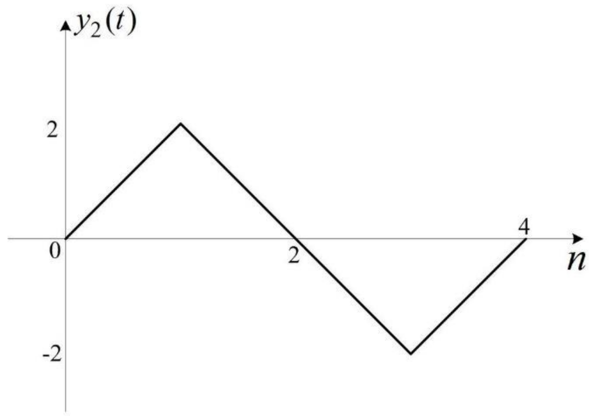
2. Q1.31
   a. Given a Linear Time-Invariant (LTI) system whose response to an input signal $x_1(t)$ is $y_1(t)$. That is, $x_1(t) \xrightarrow{LTI} y_1(t)$. From the input $x_2(t)$ depicted in P1.31c, the signal $x_2(t)$ can be written in terms of $x_1(t)$:

$$x_2(t) = x(t) - x(t-2)$$

Because $x_1(t)$ is linear and time-invariant, the output $y_2(t)$ of signal $x_2(t)$ is:

$$x_2(t) = x_1(t) - x_1(t-2) \xrightarrow{LTI} y_2(t) = y_1(t) - y_1(t-2)$$

Thus, the signal $y_2(t)$ can be depicted below:



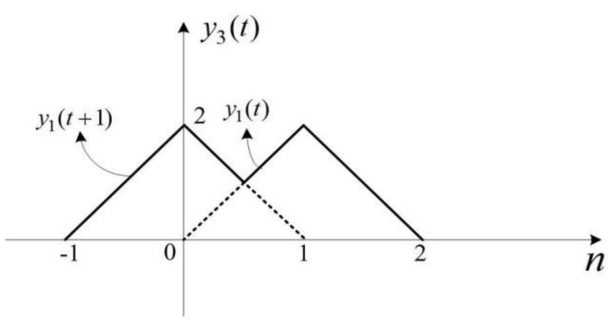
   b. From the input $x_3(t)$ depicted in P1.31d, the signal $x_3(t)$ can be written in terms of $x_1(t)$:

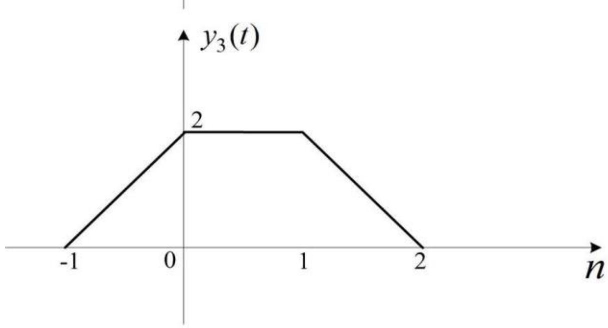
$$x_3(t) = x(t) + x(t+1)$$

Because $x_2(t)$ is linear and time-invariant, the output $y_3(t)$ of signal $x_3(t)$ is:

$$x_3(t) = x_1(t) + x_1(t+1) \xrightarrow{LTI} y_3(t) = y_1(t) + y_1(t+1)$$

Thus, the signal $y_3(t)$ can be depicted below:





3. Q2.8

a. $x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \delta(t+2) + 2\delta(t+1)$

Given this information, we can determine y(t) using the convolution y(t)=x(t)*h(t):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\{\delta(t-\tau+2) + 2\delta(t-\tau+1)\}d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t+2-\tau)d\tau + 2\int_{-\infty}^{\infty} x(\tau)\delta(t+1-\tau)d\tau$$

$$y(t) = x(t+2) + 2x(t+1)$$

Now, let's consider x(t+2):

$$x(t+2) = \begin{cases} t+3, & -2 \le t \le -1 \\ -t, & -1 < t \le 0 \\ 0, & \text{otherwise} \end{cases}$$

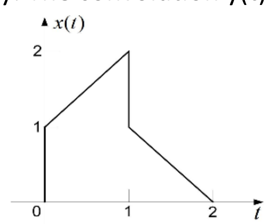
Similarly, let's consider x(t+1):

$$x(t+1) = \begin{cases} t+2, & -1 \le t \le 0 \\ 1-t, & 0 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

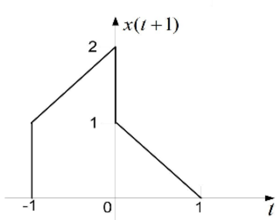
Therefore, $y(t) = x(t+2) + 2x(t+1)$ is given as:

$$y(t) = \begin{cases} t+3, & -2 \le t \le -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

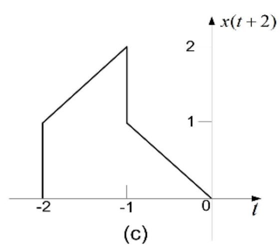
x(t) is plotted below in graph (a). x(t+2) and x(t+1) are plotted below in graphs (b) and (c) respectively. The convolution y(t) is plotted below in graph (d):



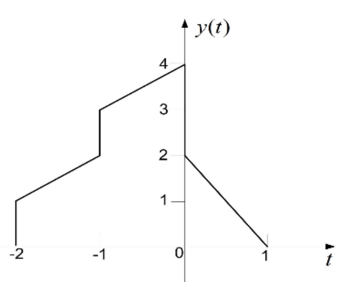
(a)



(b)



(c)



(d)

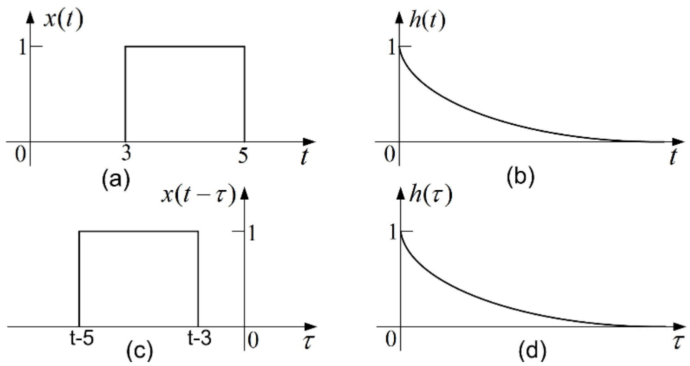
4. Q2.11
   a. Given:

$$x(t) = u(t-3) - u(t-5) = \begin{cases} 1, & 3 \le t < 5 \\ 0, & \text{otherwise} \end{cases}$$
$$h(t) = e^{-3t}u(t)$$

The convolution is given by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

We then plot x(t), h(t), x(t-τ), h(τ) below in graphs (a), (b), (c), and (d) respectively:



For t-3<0, y(t)=0. That is $y(t) = 0$, if $t < 3$.

For t-3≥0 & t-5<0:

$$y(t) = \int_{0}^{t-3} (1)e^{-3\tau}d\tau$$
$$y(t) = -\frac{1}{3}e^{-3\tau}\Big|_{0}^{t-3} \qquad ; 3 \le t < 5$$
$$y(t) = \frac{1 - e^{-3(t-3)}}{3}, \quad \text{if } 3 \le t < 5$$

For t-5≥0:

$$y(t) = \int_{t-5}^{t-3} (1)e^{-3\tau}d\tau$$
$$y(t) = -\frac{1}{3}e^{-3\tau}\Big|_{t-5}^{t-3} \qquad ; t \ge 5$$
$$y(t) = \frac{1}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right) \quad ; t \ge 5$$
$$y(t) = \frac{(1 - e^{-6})e^{-3(t-5)}}{3}, \quad \text{if } t \ge 5$$

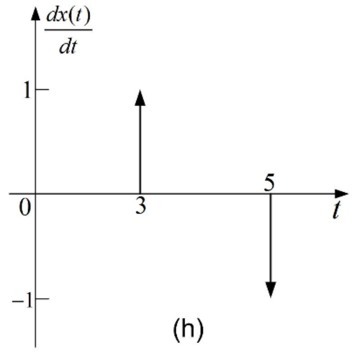
That is, convolution y(t) is as follows:

$$y(t) = \begin{cases} 0, & \text{if } -\infty < t < 3 \\ \dfrac{1 - e^{-3(t-3)}}{3}, & \text{if } 3 \le t < 5 \\ \dfrac{(1 - e^{-6})e^{-3(t-5)}}{3}, & \text{if } t \ge 5 \end{cases}$$

b. To compute g(t), we first consider $\frac{dx(t)}{dt}$:

$$\frac{dx(t)}{dt} = \frac{d}{dt}\{u(t-3) - u(t-5)\} = \delta(t-3) - \delta(t-5)$$

This signal, $\frac{dx(t)}{dt}$, is plotted below:



(h)

Next, we plug back into g(t):

$$g(t) = \big(\delta(t-3)h(t)\big) - \big(\delta(t-5)h(t)\big)$$
$$g(t) = h(t-3) - h(t-5)$$
$$g(t) = \big(e^{-3(t-3)}u(t-3)\big) - \big(e^{-3(t-5)}u(t-5)\big)$$

Therefore, we can write g(t) as:

$$g(t) = \begin{cases} 0, & -\infty < t < 3 \\ e^{-3(t-3)}, & 3 \le t < 5 \\ (e^{-6}-1)e^{-3(t-5)}, & t \ge 5 \end{cases}$$

c. We can see that y(t) calculated in part (a) is related to g(t) calculated in part b as:

$$g(t) = \frac{dy(t)}{dt} = \left(\frac{dx(t)}{dt}\right)h(t)$$

5. Q2.19 (a)

a. Given S_1 is Causal LTI with output response:

$$w[n] = \frac{1}{2}w[n-1] + x[n]$$

and S_2 is Causal LTI with output response:

$$y[n] = \alpha y[n-1] + \mathrm{B}w[n]$$

Let's rewrite:

$$w[n] = \frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1]$$

Next:

$$w[n-1] = \frac{1}{\beta}y[n-1] - \frac{\alpha}{\beta}y[n-2]$$

Substituting, we get:

$$w[n] - \frac{1}{2}w[n-1] = \frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1] - \frac{1}{2\beta}y[n-1] - \frac{\alpha}{2\beta}y[n-2] = x[n]$$

$$y[n] = \left(\alpha + \frac{1}{2}\right)y[n-1] - \left(\frac{\alpha}{2}\right)y[n-2] + \beta \mathrm{x[n]}$$

$$y[n] = -\frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] + x[n]$$

On comparing the above 2 equations, we solve for α and β:

$$\alpha = \frac{1}{4}, \beta = 1$$