Tilability of Platonic Solid Nets in Multiple Dimensions

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Abstract

For which dimensions, d, do all nets of each platonic solid tile $\mathbb{R}^{d-1}$? For the dodecahedron and icosahedron, we give counterexamples in $d = 2$, whereas for the octahedron and tetrahedron, we provide them in $d = 3$. For the cube, the question remains open, although we prove that every cube net in arbitrary dimension is a simple shape. We also suggest several sufficient criteria for determining tilability in $d = 2$.

Acknowledgments

I would like to thank Professor Emeritus Joseph O'Rourke, whose Introduction to Discrete and Computational Geometry course sparked my interest in tilings and unfoldings. It remains to this day the most informative and interesting class I've ever taken.

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Chapter 1

Introduction

For which dimensions, d, do all edge-unfoldings of each Platonic solid into the plane, $\mathbb{R}^{d-1}$, tile $\mathbb{R}^{d-1}$? Though previous work has already identified the 3- and 4-dimensional Platonic solids whose every edge-unfolding tiles, we give a new proof for some of these tilings. We do so by referencing criteria that are sufficient to show a shape's ability to tile; we also suggest several new, sufficient criteria for determining whether a shape tiles $\mathbb{R}^2$.

Ultimately, in answering the above question, we show that the nets of the dodecahedron and icosahedron do not all tile $\mathbb{R}^2$ and that the nets of the octahedron and tetrahedron do not tile $\mathbb{R}^3$. For the cube, the question remains open, although we prove that every cube net in arbitrary dimension is a simple shape.

The contents of the paper are organized as follows: Chapter 2 gives an overview of previous work on the subjects of unfoldings and tilings, including several existing tiling criteria; Chapter 3 expands upon the criteria discussed in the previous chapter; Chapter 4 describes the tilability of the 3-dimensional Platonic solids' nets; Chapter 5 describes the tilability of the higher-dimensional Platonic solids' nets; and Chapter 6 provides a conclusion and suggestions for future work.

Chapter 2

Background

This chapter provides a brief overview of necessary concepts to understand this paper, with particular focus on nets and tilings. These concepts will be used frequently throughout the following chapters.

2.1 Unfoldings and Nets

Folding and unfolding as a mathematical concept has both fascinated and stumped mathematicians for centuries. The study of nets has been traced back to Albrecht Dürer, who gave edge unfoldings, which he named “nets,” for several Platonic and Archimedean solids in his 1525 work “Unterweysung der Messung mit dem Zirkel un Richtscheyt in Linien Ebnen uhnd Gantzen Corporen” (“On Teaching Measurements with a Compass and Straightedge”)[1, 2]. His work inspired Dürer’s Problem, posed by Shephard in 1975, which asks whether every convex polyhedron possesses a net and which remains open to this day [3]. A full definition for nets is given later in this section.

This study is motivated, in part, by a common problem that arises in analyzing the geometric properties of a three-dimensional shape: many of the tools we use to communicate are only capable of providing two-dimensional representations of shapes.

A two-dimensional illustration of a three-dimensional shape necessarily cuts off part of that shape, unless it merely depicts the shape's skeleton. As such, we often rely on an alternate visual representation for a three-dimensional shape in a two-dimensional space: an unfolding, which is defined as a "simple polygon obtained by cutting the surface of the (shape) and unfolding it into a plane" [2]. Notably, this means an unfolding must always be connected, non-self-overlapping, and planar[4].

In the same way that we can use unfoldings to better illustrate three-dimensional shapes, we can use them for higher-dimensional shapes. The resulting nets lay 'flat' on the hyperplane $\mathbb{R}^{d-1}$ just as the nets of a three-dimensional solid lay flat on $\mathbb{R}^2$.

If the surface cuts of an unfolding are made solely along the $[d-2]$-facets[1] of a given shape, the unfolding is called a net. Nets allow us to see all of a shape's $[d-1]$- and $[d-2]$- in $\mathbb{R}^{d-2}$, while still preserving some information about the shape's original structure. For these reasons, nets can prove useful in a variety of contexts; nevertheless, it is worth noting that generating nets for a given shape is no computationally insignificant task. Firstly, we must determine which facets to cut. As "no one has discovered necessary and sufficient conditions for a collection of cut [facets] to unfold to a net," we must perform a lengthy combinatorial analysis of possible cuttings [4]. We know that the cut facets must form a spanning tree of the graph in which $[d-3]$-facets are nodes and $[d-2]$-facets are edges (Theorem 2.1.1). Thus, the number of spanning trees gives the total number of ways to form a net; however, multiple different spanning trees may be congruent under various isometries [5]. As such, there are usually far fewer nets than spanning trees for a given shape. Only once we have chosen which facets to cut can we actually generate a shape's nets, which can still be a substantial task, given that a three-dimensional shape with as few as 30 edges can have over 40,000 unfoldings.

Theorem 2.1.1. *Any edge-cutting of a shape which produces a valid net for that*

[1] A 0-facet is a point, a 1-facet is a line, a 2-facet is a face, and so on.

shape must form a spanning tree of the graph in which $[d-3]$-facets are nodes and $[d-2]$-facets are edges.

Proof. Assume that a given edge-cutting does not span the $[d-3]$-facets of the shape to which it corresponds. Then, there is at least one $[d-3]$-facet on the shape for which the surrounding curvature is preserved in $\mathbb{R}^d$, meaning the cutting does not produce a valid net. As such, a cutting must span the $[d-3]$-facets in order to produce a net of that shape.

Now assume that a given cutting contains a cycle. Then, at least one $[d-1]$-facet is disconnected from the shape, meaning the cutting does not form a valid (connected) net. As such, a cutting must be acyclic in order to produce a net.

Thus, any edge-cutting that produces a net of a given shape must be a spanning tree of the graph in which the shape's $[d-3]$-facets are nodes and $[d-2]$-facets are edges. □

2.2 Platonic Solid Nets

Platonic Solids, or regular polyhedra, are a special category of convex polyhedron defined by the congruence of all their faces, edges, and vertices. They consist of the tetrahedron, cube, octahedron, dodecahedron, and icosahedron; notably, these are the only possible Platonic solids (Theorem 2.2.1).

Theorem 2.2.1. *There are only 5 Platonic Solids: the regular tetrahedron, the regular octahedron, the regular icosahedron, the cube, and the regular dodecahedron.*

Proof. The Gauss-Bonnet Theorem states that $\sum \omega i = 4\pi$, or in other words, that the sum curvature of any 3-dimensional shape (and thus any Platonic solid) will be 4π [6]. Because all vertices of a Platonic solid must be identical, we know the curvature of any vertex on a Platonic solid must evenly divide 4π.

Let us begin by examining the case of the 'smallest' possible regular convex polyhedron. A regular polyhedron must have at least three two-dimensional faces incident to each vertex, each of which must have at least three sides of equal length. Thus, the 'smallest' possible regular convex polyhedron has vertices composed of 3 equilateral triangles. Such a vertex gives a curvature of $2\pi - \frac{3\pi}{3} = \pi$ radians. This allows for $\frac{4\pi}{\pi} = 4$ vertices. So, the first possible Platonic solid is composed of 4 vertices, each with three incident equilateral triangles. This is the tetrahedron. Now, let us examine the case of 4 equilateral triangles incident to each vertex. In this case, each vertex has a curvature of $2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$ radians, allowing for $\frac{4\pi}{\frac{2\pi}{3}} = 6$ vertices. This is the octahedron. Similarly, for a Platonic solid with 5 equilateral triangles incident to each vertex, every vertex would have curvature $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$, yielding $\frac{4\pi}{\frac{\pi}{3}} = 12$ vertices, which is the icosahedron. By contrast, if we try to create a Platonic solid with 6 equilateral triangles incident to each vertex, every vertex would have curvature $2\pi - \frac{6\pi}{3} = 0$ radians, which cannot form a shape. We must thus try with a 'larger' face.

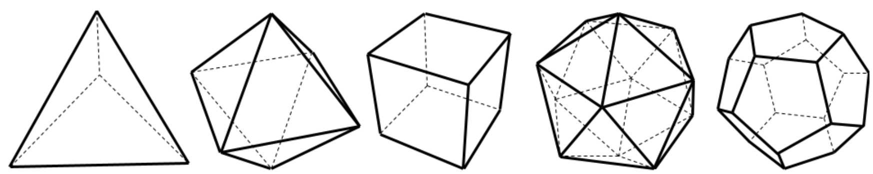
The next possible such face is the square, with 4 sides of equal length. For a Platonic solid with three squares incident to every vertex, each vertex would have curvature $2\pi - \frac{3\pi}{2} = \frac{\pi}{2}$ radians, yielding $\frac{4\pi}{\frac{\pi}{2}} = 8$ vertices. This is the cube. For a Platonic solid with 4 squares incident to each vertex, every vertex would have curvature $2\pi - \frac{4\pi}{2} = 0$ radians, which cannot form a shape. Again, we mus move to a 'larger' face.

The next possible regular face is the regular pentagon, with five sides of equal length. For a Platonic solid with three pentagons incident to every vertex, each vertex would have curvature $2\pi - \frac{3*3\pi}{5} = \frac{\pi}{5}$ radians, yielding $\frac{4\pi}{\frac{\pi}{5}} = 20$ vertices. This is the dodecahedron. For a Platonic solid with 4 pentagons incident to each vertex, every vertex would have curvature $2\pi - \frac{4*3\pi}{5} = \frac{-2\pi}{5}$ radians, which cannot form a shape.

Next, for a Platonic solid with three hexagons incident to every vertex, each vertex would have curvature $2\pi - \frac{3*2\pi}{3} = 0$ radians, which cannot form a shape. Because of this, we know that no Platonic solids can be formed with faces of 6 or more sides.

This puts a clear bound on the number of Platonic solids it is possible to have, so we know that there are only 5. $\square$

Figure 2.1: The 5 Platonic Solids



(a) The Tetrahedron, Octahedron, Cube, Icosahedron, and Dodecahedron, respectively

For clarity and conciseness, we will occasionally refer to these solids using their Schläfli Symbols. A Schläfli Symbol takes the form $\{x_1, x_2\}$, where x_1 gives the number of sides incident to a face and x_2 gives the number of faces incident to a vertex. So, the tetrahedron is given by $\{3, 3\}$, the octahedron by $\{3, 4\}$, the cube by $\{4, 3\}$, the icosahedron by $\{3, 5\}$, and the dodecahedron by $\{5, 3\}$.

We can refer to convex, regular polytopes in higher dimensions as "Platonic hypersolids." In $\mathbb{R}^4$, there are six Platonic hypersolids, whereas for $d \geq 5$, there are three. As with the three-dimensional Platonic solids, we can give a Platonic hypersolid by its Schläfli Symbol. For each additional dimension, the Schläfli Symbol gains an extra value, so that rather than $\{x_1, x_2\}$, we have $\{x_1, x_2, x_3, ..., x_{d-1}\}$. The symbol functions recursively, so that the last value of the symbol tells us how many of the (hyper)solids given by $\{x_1, x_2, x_3, ..., x_{d-2}\}$ are incident to each $[d-3]$-facet. For example, $\{5,3,3\}$ describes a shape with 3 dodecahedra ($\{5,3\}$) incident to each edge. Thus, the 4-dimensional Platonic hypersolids can be given by $\{3,3,3\}$ (the 5-cell), $\{3,3,4\}$ (the 16-cell), $\{3,3,5\}$ (the tetraplex), $\{3,4,3\}$ (the octaplex), $\{4,3,3\}$ (the tesseract), and $\{5,3,3\}$ (the dodecaplex). In $d \geq 5$, the only Platonic hypersolids are the d-simplex, given by the Schläfli Symbol $\{3,3,3,...,3\}$; the d-orthoplex, given by $\{3,3,3,...,3,4\}$; and the d-hypercube, given by $\{4,3,3,...,3\}$. These hypersolids follow a naming scheme that we will use occasionally throughout the paper, wherein we refer to a Platonic hypersolid in $\mathbb{R}^d$ as a d-hypersolid.

Platonic (hyper)solids have many fascinating and useful properties: each obeys Euler's Theorem ($Vertices - Edges + Faces = 2$), meaning every Platonic solid can be represented as a planar graph in $\mathbb{R}^{d-1}$; the tetrahedron/simplex is dual[2] to itself, while the cube/hypercube and octahedron/orthoplex are dual to each other, as are the dodecahedron and icosahedron[3]; and each has Prince Rupert's Property, meaning we can cut a tunnel through each Platonic solid large enough to permit the passage of an identical copy of that solid [6, 7].

In the study of unfoldings, as well, we find an interesting property, discovered by Takashi Horiyama and Wataru Shoji in 2011: "if we unfold a Platonic solid by cutting along its edges, we always obtain a flat nonoverlapping simple polygon" [2]. This, though, does not apply to all of the Platonic hypersolids. In fact, it is known that some ridge-unfoldings of d-orthoplexes for $d > 5$ self-overlap [8]. We will discuss this concept further in Chapter 5.

Shoji and Horiyama also created a catalog of all of the Platonic solids' nets (their methodology will be discussed in section 4.1), which can be viewed at `https://art.ist.hokudai.ac.jp/~horiyama/research/unfolding/` [2]. The tetrahedron

[2]The dual of a solid is a graph in which the solid's d facets are nodes and its $[d-1]$-facets are edges.

[3]Note that solids that are dual to each other have inverse Schläfli Symbols.

has 2 nets, the octahedron and cube have 11 nets each, and the icosahedron and dodecahedron have 43,380 nets each. The nets of the tetrahedron, octahedron, and cube are shown below.

Figure 2.2: The 2 nets of the tetrahedron

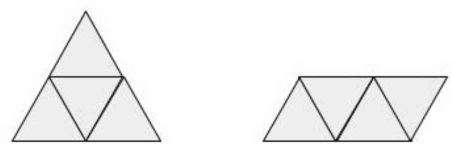
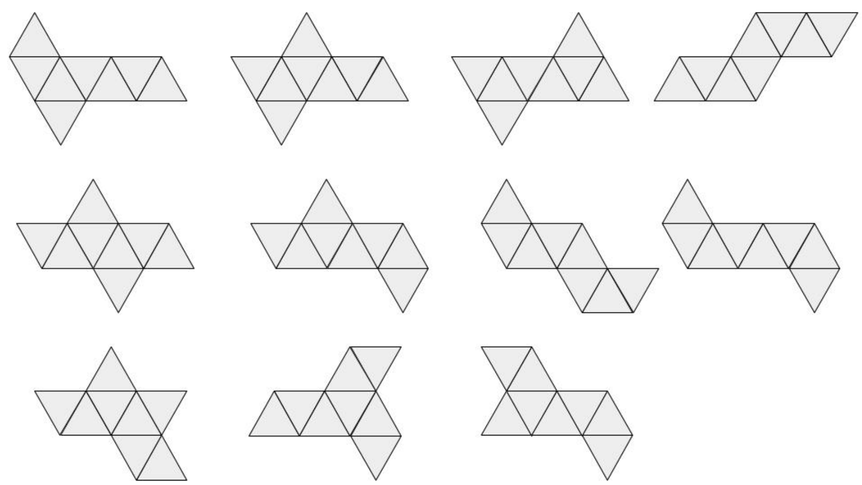


Figure 2.3: The 11 nets of the octahedron

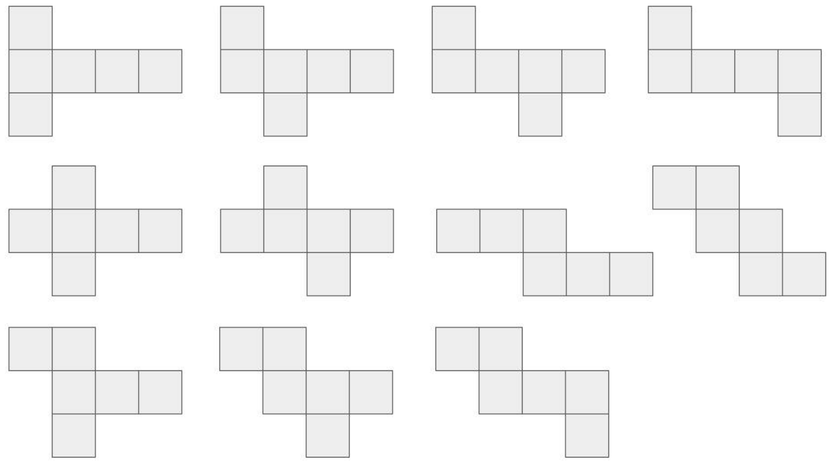


It is of note here that the (hyper)solids that are dual to each other have the same number of nets. This is because a valid net requires both cutting a spanning tree as described in Theorem 2.1.1 and leaving an uncut spanning tree on the shape's dual (see Theorem 2.2.2).

Theorem 2.2.2. *A valid net for a shape can be represented as a spanning tree of uncut edges on that shape's dual.*

Proof. Let there be a cycle of uncut edges in the net's dual graph. Then, the net has curvature in $\mathbb{R}^d$, and is thus not a valid net. Now let there be a node in the dual

Figure 2.4: The 11 nets of the cube



graph for which every edge is cut. Then, there is a $[d-1]$-facet that is disconnected from the remainder of the net, meaning the net is not valid. Thus, there must be a spanning tree of uncut edges on net's dual graph for a net to be valid. $\square$

2.3 Tiling

Tilings, also known as tessellations, are complete, overlap-free coverings of $\mathbb{R}^n$. In other words, tilings are formed from shapes glued together in such a way that their surfaces cover the plane (or space, for higher dimensions) without gap or overlap.

Similarly to unfolding, tilings have interested artists, philosophers, and mathematicians for thousands of years. Indeed, archaeological evidence shows the existence of tiling as an art form dating to before the Common Era [9]. Even in the book of Exodus, there is mention of work with bricks/tile. The continuation of this fascination is evident in the facades of the Great Mosque at Qairouan (9th Century), the Mudéjar buildings of Aragon (12th Century), The Alhambra Palace (9th and 13th Centuries), and the Taj Mahal (17th Century). Why has humanity remained so captivated by tilings through the centuries? As Ian Gabmini and Laurent Vuillon put it, tilings "reveal an incredible amount of simple to state problems that translate into very complex combinatorial ones" [10]. In practice, this means that despite the best efforts of thousands of years of thinkers to uncover the rules that dictate tilings, they remain a fairly mysterious subject. In fact, the problem of whether one can create a tiling from a set of two or more tiles is unsolvable. [11, 12].

One of the main areas of intrigue in the study of tilings surrounds various shapes' tilability, which we will define as follows: "A plane figure is said to tile the plane $\mathbb{R}^2$ if copies of the figure cover the plane with no gaps nor overlaps when placed end to end" [13].

In 2007, Jin Akiyama showed with example tilings that all cube, tetrahedron, and octahedron unfoldings tile the plane [13]. Furthermore, he gave an icosahedron unfolding that does not tile the plane. This paper, however, attempts to provide a more mathematically rigorous form of proof that these Platonic solid nets do or do not tile the plane.

In examining a shape's tilability, we wish to identify characteristics that may be necessary or sufficient to show that the shape tiles. Though there is no known set of characteristics that is both necessary and sufficient to show that any shape tiles the plane, necessary and/or sufficient tiling criteria have been discovered for some problems of narrower scope. For instance, knowing that a shape is a triangle or quadrilateral is sufficient to show that it tiles the plane.

Theorem 2.3.1. *All quadrilaterals tile* $\mathbb{R}^2$.

Proof. Any given point in $\mathbb{R}^2$ is surrounded by 360°. The sum interior angle of any quadrilateral is also 360°. Thus, we may tile with a given quadrilateral simply by placing all its angles incident to some point in $\mathbb{R}^2$ and repeating this process at all the quadrilateral's vertices throughout $\mathbb{R}^2$. In practice, this can also be accomplished by reflecting the quadrilateral over the midpoint of each of its sides. □

Theorem 2.3.2. *All triangles tile* $\mathbb{R}^2$.

Proof. Rotating any triangle 180° over the midpoint of one of its sides yields a parallelogram. By Theorem 2.3.1, any parallelogram tiles $\mathbb{R}^2$. Thus, any triangle tiles $\mathbb{R}^2$. □

2.3.1 Convex Polygon Criteria

For convex polygons specifically, there exists a handful of criteria related to tilability, which are outlined below. For these theorems, we will name the points of the given polygon $A, B, C, ...$ in the counterclockwise order. We will name its edges based on the vertices to which they are incident, where order is inconsequential.

Theorem 2.3.3. *The regular pentagon does not tile* $\mathbb{R}^2$.

Proof. Any given point in $\mathbb{R}^2$ is surrounded by 360°, whereas each interior angle of the regular pentagon measures 108°. 360° is not divisible by 108°, nor is 180° (in the event that some angles are incident to a straight line). Thus, there is no way to surround a point in the plane with regular pentagons without gap or overlap. □

Here, we must note that no net of the dodecahedron can possibly tile $\mathbb{R}^2$, as the faces of the dodecahedron are regular polygons.

Theorem 2.3.4. *[14, 15, 16, 17, 18] A given pentagon tiles* $\mathbb{R}^2$ *if and only if it satisfies one or more of the following conditions:*

1. $A + E = 180°$.
2. $A + D = 180°$; $|AB| = |DE|$.
3. $B = D = E = 120°$; $|AB| = |BC|$; $|DE| = |CD| + |EA|$.
4. $B = D = 90°$; $|AB| = |BC|$; $|CD| = |DE|$.
5. $C = 2A = 120°$; $|EA| = |AB|$; $|BC| = |CD|$.

6. $A + D = 180°$; $B = 2D$; $|EA| = |AB| = |BC|$; $|CD| = |DE|$.

7. $2E + B = 360°$; $2C + D = 360°$; $|AB| = |BC| = |CD| = |DE|$.

8. $2B + A = 360°$; $2D + E = 360°$; $|EA| = |AB| = |CD| = |DE|$.

9. $2A + E = 360°$; $2B + D = 360°$; $|EA| = |BC| = |CD| = |DE|$.

10. $A = 90°$; $B + E = 180°$; $2C - E = 180°$; $2D + E = 360°$; $|EA| = |AB| = |BC| + |DE|$.

11. $A = 90°$; $B + D = 180°$; $2E + D = 360°$; $|BC| = |CD| = 2|AB| + |DE|$.

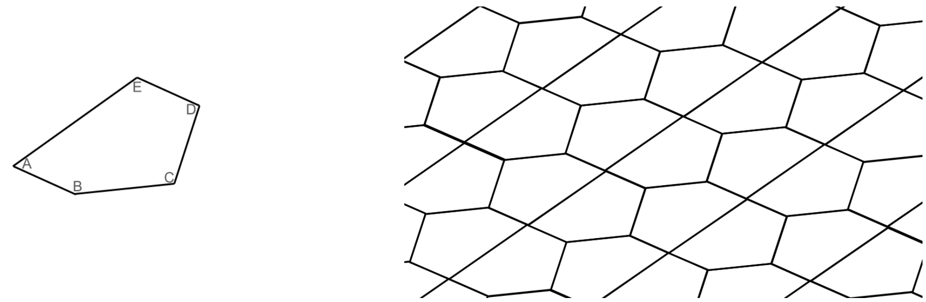
12. $A = 90°$; $C + E = 180°$; $2B + C = 360°$; $2|EA| = |BC| + |DE| = |CD|$.

13. $A = C = 90°$; $2B = 2E = 360° - D$; $2|BC| = 2|CD| = |DE|$.

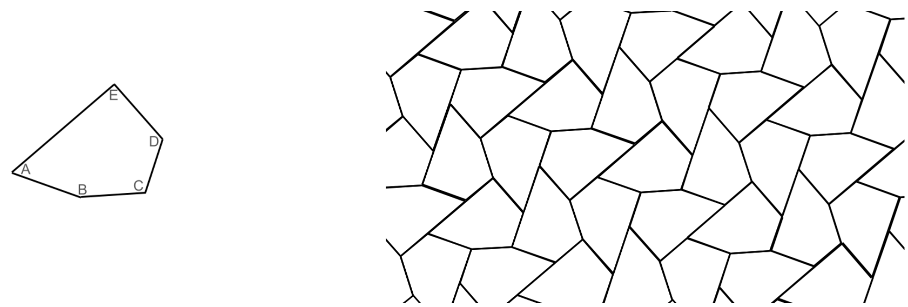
14. $D = 90°$; $2C + B = 360°$; $E + B = 180°$; $A + C + D = 360°$; $|EA| = |AB| = 2|DE| = 2|BC|$.

15. $A = 60°$, $B = 135°$, $C = 105°$, $D = 90°$, *and* $E = 150°$; $2|AB| = 2|CD| = 2|DE| = EA$.

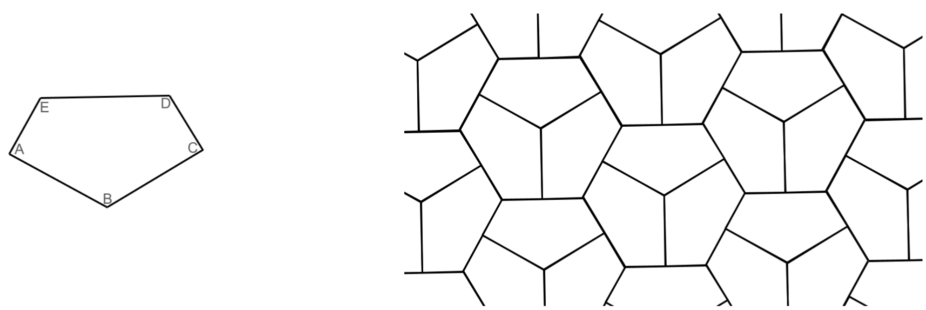
Figure 2.5: Pentagons Satisfying Theorem 2.3.4 and Corresponding Tilings



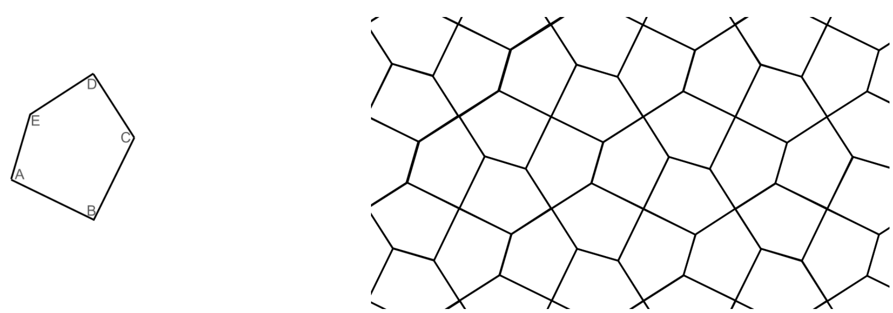
(a) Theorem 2.3.4 (1)



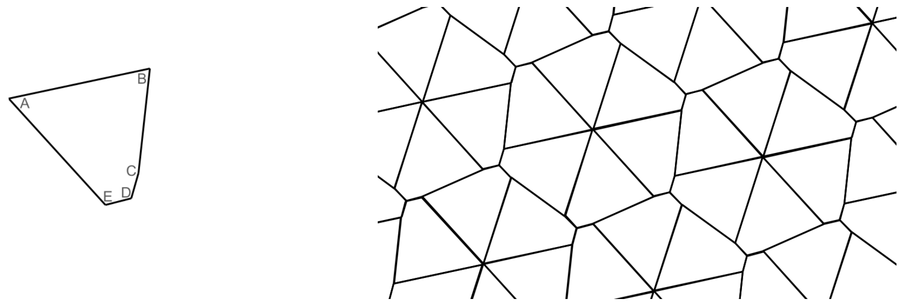
(b) Theorem 2.3.4 (2)



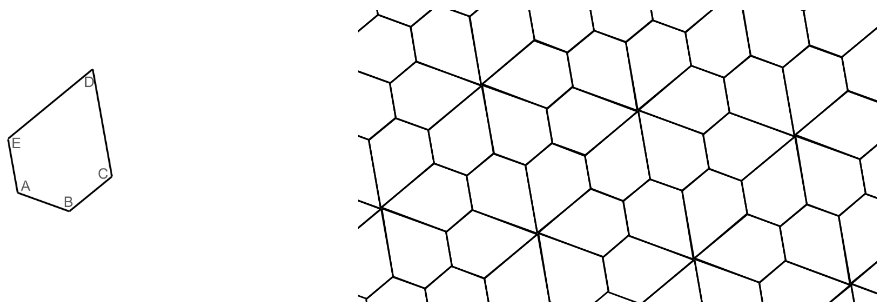
(c) Theorem 2.3.4 (3)



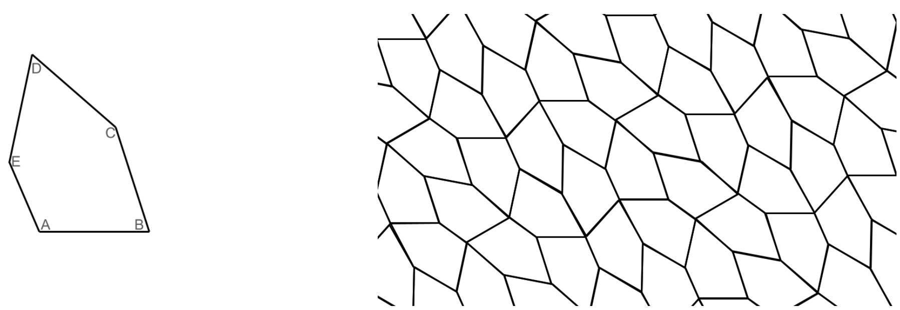
(d) Theorem 2.3.4 (4)



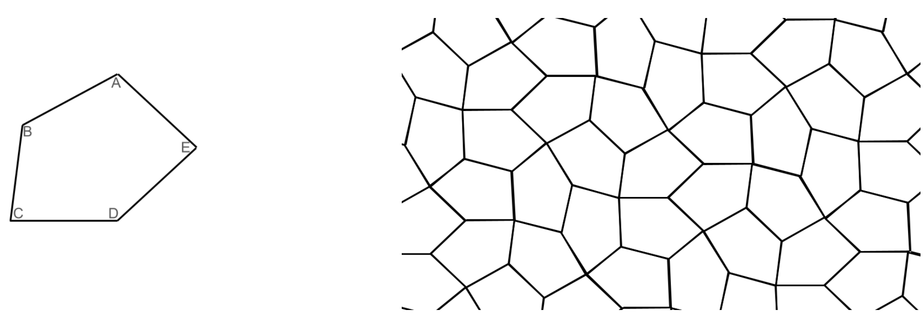
(e) Theorem 2.3.4 (5)



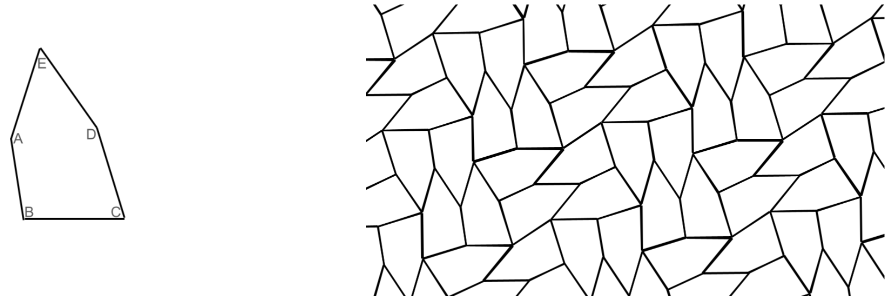
(f) Theorem 2.3.4 (6)



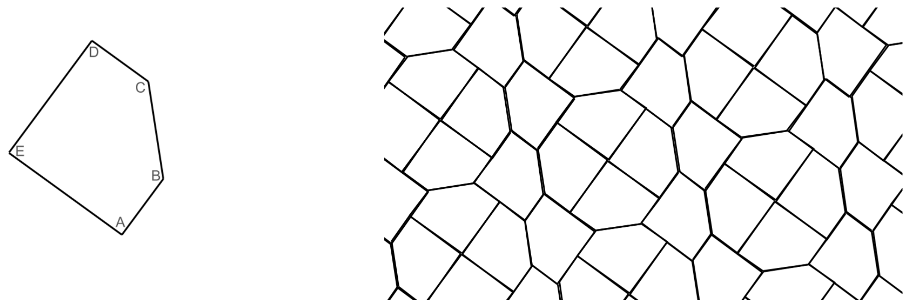
(g) Theorem 2.3.4 (7)



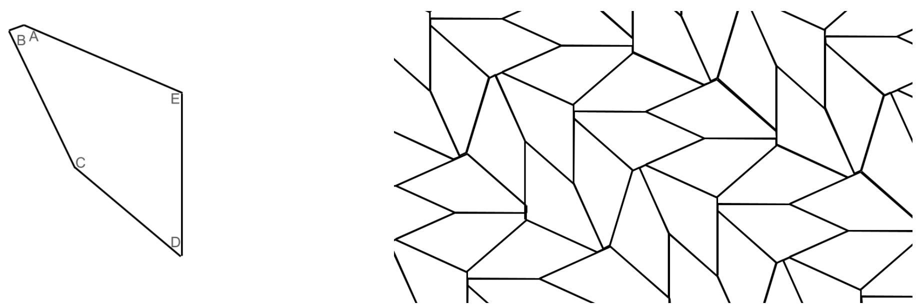
(h) Theorem 2.3.4 (8)



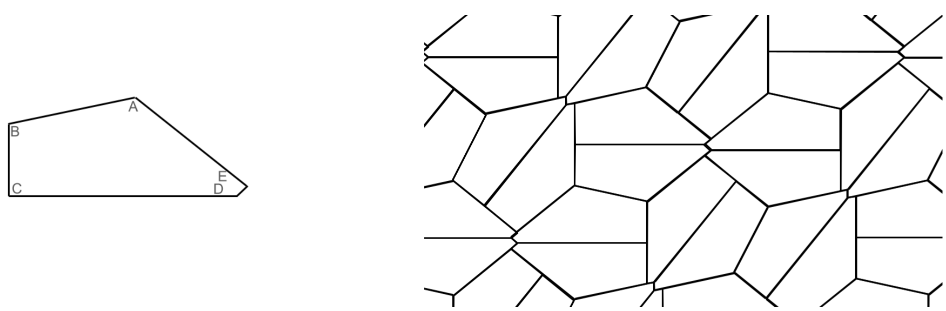
(i) Theorem 2.3.4 (9)



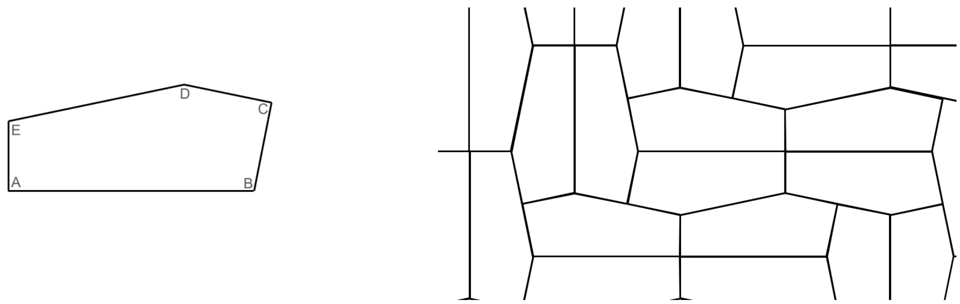
(j) Theorem 2.3.4 (10)



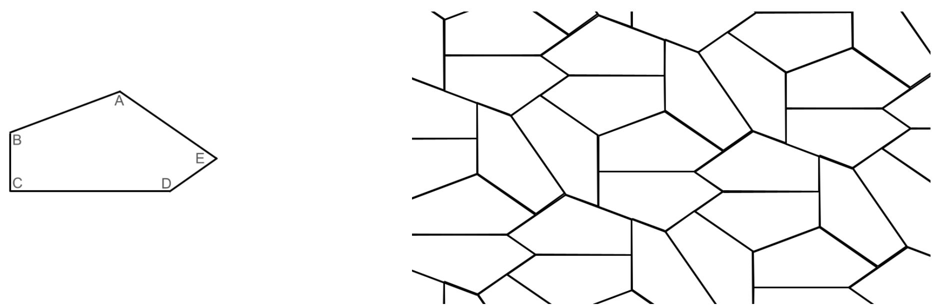
(k) Theorem 2.3.4 (11)



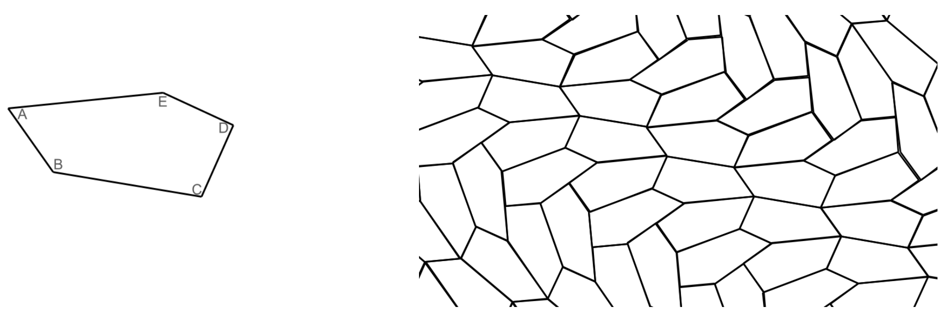
(l) Theorem 2.3.4 (12)



(m) Theorem 2.3.4 (13)



(n) Theorem 2.3.4 (14)



(o) Theorem 2.3.4 (15)

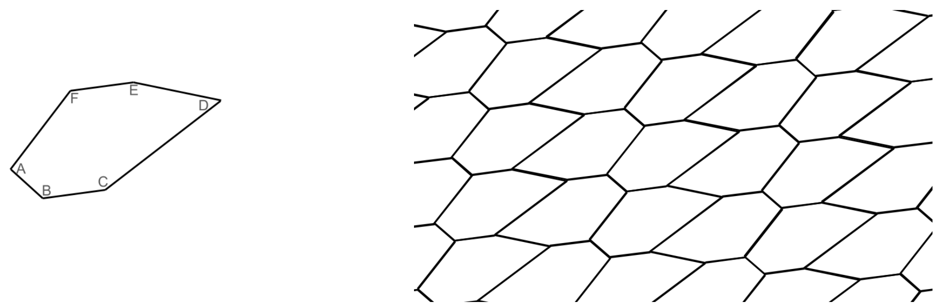
Theorem 2.3.5. *[14] A given hexagon tiles $\mathbb{R}^2$ if and only if it satisfies one or more of the following conditions:*

1. $F + A + B = 360°$; $C + D + E = 360°$; $|BC| = |EF|$.

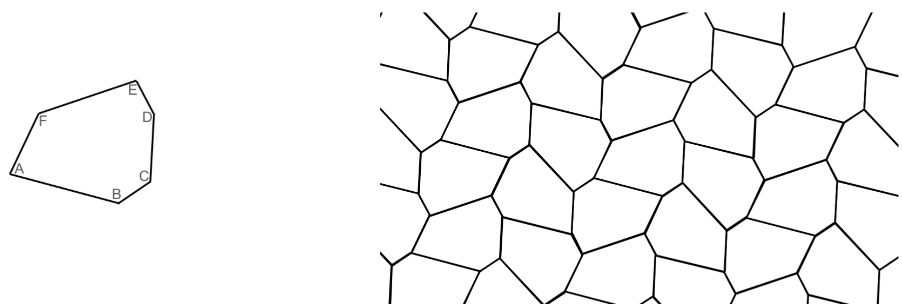
2. $A + B + D = 360°$; $C + E + F = 360°$; $|BC| = |DE|$; $|FA| = |CD|$.

3. $A = C = E = 120°$; $|FA| = |AB|$; $|BC| = |CD|$; $|DE| = |EF|$.

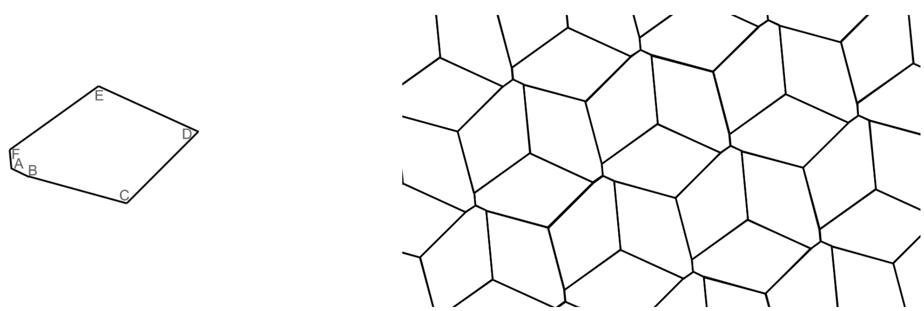
Figure 2.6: Hexagons Satisfying Theorem 2.3.5 and Corresponding Tilings



(a) Theorem 2.3.5(1)



(b) Theorem 2.3.5(2)



(c) Theorem 2.3.5(3)

Theorem 2.3.6. *No convex polygon with seven or more sides can tile* $\mathbb{R}^2$.

Proof. A convex polygon has interior vertex angles measuring exclusively less than 180 degrees, by definition. Thus, for every point where a vertex of a convex polygon meets other tiles, there will be at least three tiles incident to that point. As such, the average interior angle measure of the polygon must be less than or equal to 120°. However, a convex polygon with n vertices has a sum interior angle of $180(n-2)$, which yields an average interior angle of $\frac{180(n-2)}{n} = 180 - \frac{360}{n}$. For $n > 6$, we can see that $\frac{360}{n} \leq 52$, so the average interior angle is always greater than 120°. $\square$

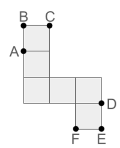
Still, what of nonconvex polygons with more than four sides? For these too, we have identified some criteria that allow us to analyze a shape's ability to tile. The rest of this chapter will concern itself with these criteria. Notably, all of these criteria rely not necessarily on a shape's vertices to identify boundary segments (edges), but instead on a series of manually selected points along the shape's boundary. As such, for the following criteria, we will use the following naming conventions: we choose six consecutive points along the boundary of the given polygon – $A, B, C, ...$ – and we name the segments between these points based on the points to which they are incident with respect to order (meaning BA is a 180° rotation of AB around its midpoint).

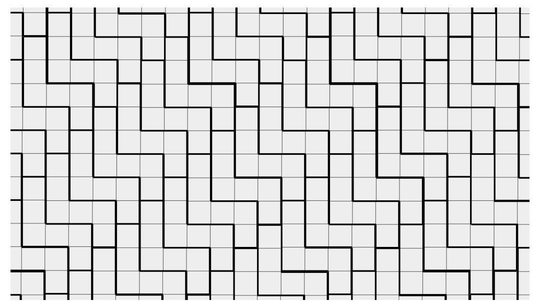
2.3.2 The Beauquier-Nivat Criterion

The Beauquier-Nivat Criterion (stated in Theorem 2.3.7) is necessary and sufficient to show that a polyomino (a polygon which is an edge-gluing of congruent squares) tiles by translation [19, 10]. It does not, however, hold for tilings with rotation and/or reflection, nor does it hold for polygons other than polyominoes. An example of a polyomino that satisfies the Beauquier-Nivat Criterion, along with its associated tiling, is given in Figure 2.7.

Theorem 2.3.7. *[19] A polyomino tiles $\mathbb{R}^2$ by translations if and only if we can choose six consecutive dividing points on its boundary such that AB is a translation of ED, BC is a translation of FE, and CD is a translation of AF; one of these segment pairs may be empty.*

Figure 2.7: Polyomino Satisfying the Beauquier-Nivat Criterion (Theorem 2.3.7) and Corresponding Tiling





2.3.3 The Conway Criterion

An earlier criterion, discovered by John H. Conway, applies to any polygon, but is only sufficient; furthermore, it applies only for tilings by translation and 180° rotation [20, 21]. It can be stated as follows:

Theorem 2.3.8. *[21] A polygon tiles $\mathbb{R}^2$ if we can choose six consecutive dividing points along its boundary in such a way that all of the following conditions are satisfied:*

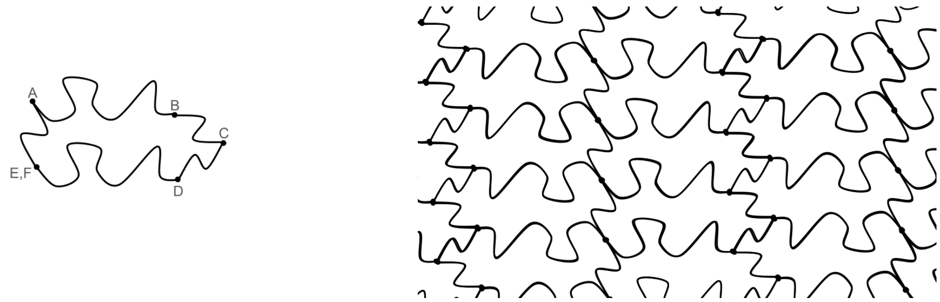
1. *At least three of the chosen points are distinct.*

2. *AB is a translation of ED. Note that the endpoints of AB and ED can be collinear.*

3. *BC, CD, EF, and FA are centrosymmetric.* [4]

[4] A centrosymmetric segment is unaltered by a 180° rotation about its midpoint; in other words XY is centrosymmetric if $XY = YX$.

Figure 2.8: Shape Satisfying the Conway Criterion (Theorem 2.3.8) and Corresponding Tiling



Note that Theorems 2.3.1 and 2.3.2, as well as item 2 of Theorem 2.3.4 and item 1 of Theorem 2.3.5 are given by the Conway Criterion. This, along with the fact that it applies to all polygons, illustrates why the Conway Criterion is perhaps the most useful known tiling criterion and why most of chapter 4 will be concerned with it.

Importantly, all of the above criteria not only show that a shape can tile, but also show us how to construct a tiling with that shape. These constructions are shown in Figures 2.5-2.8. This concept is central to the ideas presented in Chapter 3.

Chapter 3

Expanding Tiling Criteria

The first contribution of the paper is presented in this chapter, consisting of 17 new sufficient tiling criteria. The relevant background concepts have been introduced in Chapter 2, and the results will be used in Chapter 4.

Showing that a Platonic solid net satisfies one or more of the criteria given in Chapter 2 is sufficient to prove that the given net tiles. Of these criteria, however, only the Conway Criterion (Theorem 2.3.8) is applicable to all Platonic solid nets, as the vast majority of these nets are neither convex nor polyominoes. Thus, if we wish to be able to test all of the Platonic solid nets against these criteria, we must first find ways to expand the criteria so that they become applicable to other types of shapes.

To understand what this means, let us begin by examining Theorem 2.3.2 and its associated tiling construction. Here, we see that a tiling of a triangle can be constructed simply by rotating that triangle 180° over the midpoint of each of its sides. Thus, if we replace every side of the triangle with a centrosymmetric segment, we may still obtain a tiling in the same fashion (see Figures 3.1 and 3.2). The same may be said for Theorem 2.3.1 (see Figures 3.3 and 3.4).

While this result is not new – it is, in fact, corollary to the Conway Criterion – it gives us a new way to approach tiling criteria. By examining the tiling constructions

Figure 3.1: Composition of a Shape Using 3 Centrosymmetric Segments, Based on a Triangle

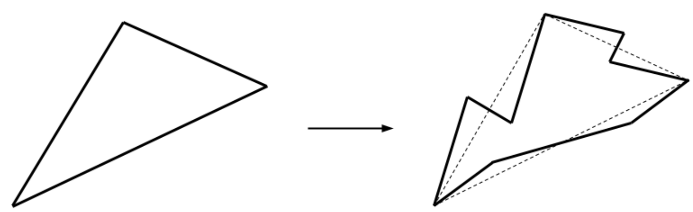


Figure 3.2: A Shape Composed of 3 Centrosymmetric Segments Tiles Equivalently to a Triangle

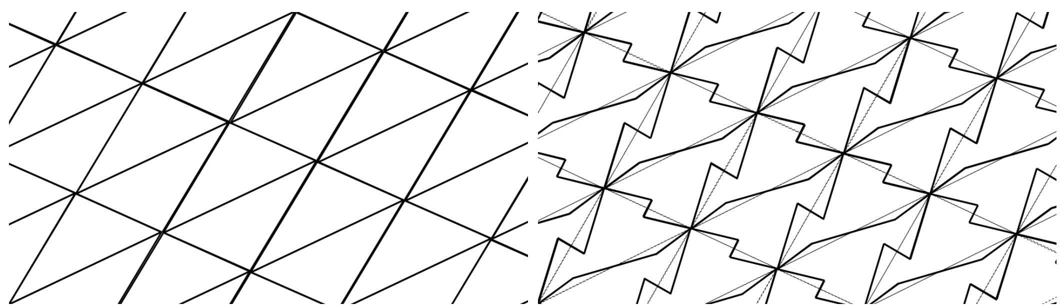
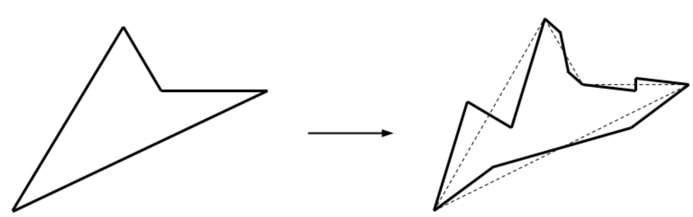
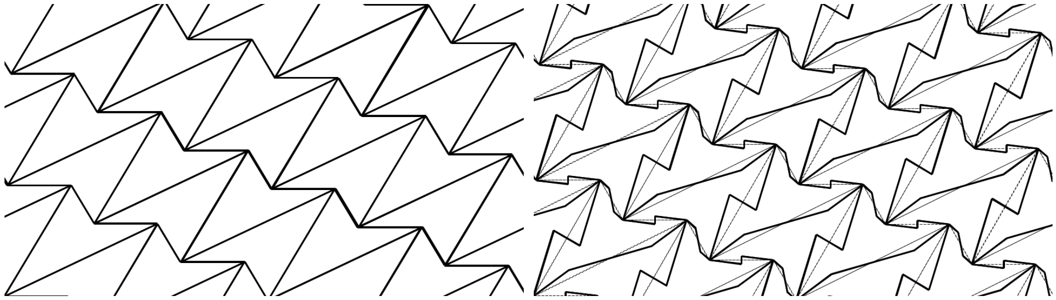


Figure 3.3: Composition of a Shape Using 4 Centrosymmetric Segments, Based on a Quadrilateral



associated with these criteria, we have found a way to expand them to become applicable to any simple shape, as opposed to only for triangles/rectangles. For any of the convex polygon tiling criteria stated in Chapter 2, we may do the same. Where

Figure 3.4: A Shape Composed of 4 Centrosymmetric Segments Tiles Equivalently to a Quadrilateral



a shape is rotated 180° about an edge in its tiling, we can replace that edge with a centrosymmetric segment; where a shape is reflected across an edge in its tiling, we can replace that edge with a shape that is symmetric by reflection; where two edges are placed against each other in a tiling, we can replace those edges with segments that are congruent by translation and/or rotation; etc.

Corollary 3.0.1. *(See Theorem 2.3.8): Any two-dimensional shape whose boundary is composed of exactly three or four centrosymmetric segments* $\mathbb{R}^2$*.*

For the following theorems, we will name the dividing points that define the boundary segments of the given polygon $A, B, C, ...$ in the counterclockwise order. We will name its boundary segments based on the vertices to which they are incident, with respect to order (meaning segment BA is a 180° rotation of segment AB around its midpoint). Angles will be determined by segment endpoints. We will refer to the reflection of a segment across its midpoint as $\mathcal{R}$. Lastly, we will use $\equiv$ to refer to two segments that are congruent only by translation and/or rotation.

Theorem 3.0.1. *A polygon tiles* $\mathbb{R}^2$ *if we can choose five consecutive dividing points along its boundary such that one or more of the following conditions are satisfied:*

1. $E + A = 180°$*;* EA*,* BC*,* CD*, and* $DE + AB$ *are centrosymmetric.*

2. $A + D = 180°$*;* $AB \equiv \mathcal{R}(DE)$*;* BC *and* $CD + \mathcal{R}(AE)$ *are centrosymmetric.*

3. $B = D = E = 120°$*;* $AB \equiv CB$*;* $DE \equiv DC + AE$*.*

4. $B = D = 90°$; $AB \equiv CB$; $CD \equiv ED$; AE *is centrosymmetric.*

5. $C = 2A = 120°$; $AB \equiv AE$; $BC \equiv DC$; DE *is centrosymmetric.*

6. $A + D = 180°$; $B = 2D$; $AB \equiv AE \equiv BC$; $CD \equiv ED$; *all segments are centrosymmetric.*

7. $2E + B = 360°$; $2C + D = 360°$; $\mathcal{R}(BA) \equiv BC \equiv DC \equiv ED$; AB, BC, CD, *and* DE *are centrosymmetric;* EA *is a line.*

8. $2B + A = 360°$; $2D + E = 360°$; $EA \equiv \mathcal{R}(AB) \equiv DC \equiv DE$; AB, CD, DE, *and* EA *are centrosymmetric;* BC *is a line.*

9. $2A + E = 360°$; $2B + D = 360°$; $EA \equiv BC \equiv DC \equiv \mathcal{R}(DE)$; EA, BC, CD, *and* DE *are centrosymmetric;* AB *is a line.*

10. $E = 90°$; $A+D = 180°$; $2B-D = 180°$; $2C+D = 360°$; $EA \equiv DE \equiv AB+CD$; BC, CD, *and* DE *are centrosymmetric.*

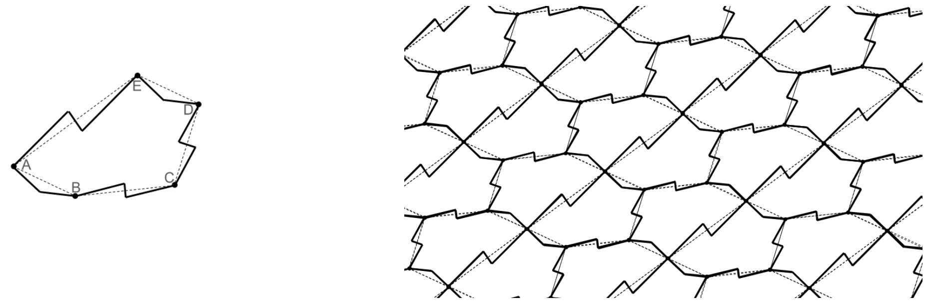
11. $A = 90°$; $B + D = 180°$; $2E + D = 360°$; $BC \equiv DC \equiv \mathcal{R}(ED) + \mathcal{R}(BA) + AB$; EA, BC, *and* DE *are centrosymmetric.*

12. $A = 90°$; $C+E = 180°$; $2B+C = 360°$; $\mathcal{R}(CD) \equiv BC+\mathcal{R}(ED) \equiv EA+\mathcal{R}(AE)$; AB *is a line;* BC *and* DE *are centrosymmetric.*

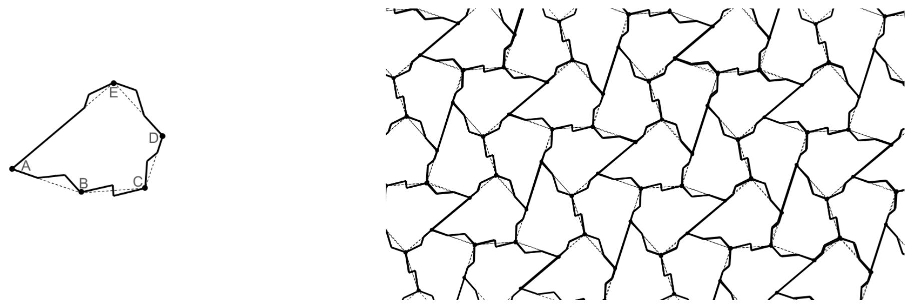
13. $A = C = 90°$; $2B = 2E = 360° - D$; $CD \equiv CB$; $DE \equiv \mathcal{R}(BC) + \mathcal{R}(CD)$; AB *and* EA *are lines;* DE *is centrosymmetric.*

14. $D = 90°$; $2C + B = 360°$; $E + B = 180°$; $A + D + C = 360°$; $AB \equiv AE \equiv ED + \mathcal{R}(DE)$; $\mathcal{R}(AB) \equiv DE + BC$; CD *is a line;* EA, BC, *and* $\mathcal{R}(EA) + BC$ *are centrosymmetric.*

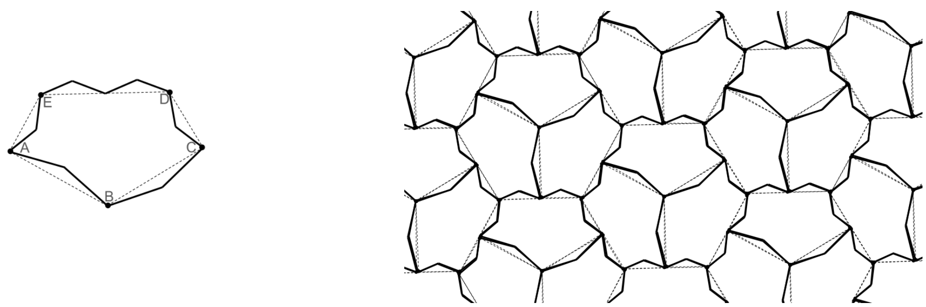
Figure 3.5: Shapes Satisfying Theorem 3.0.1 and Corresponding Tilings



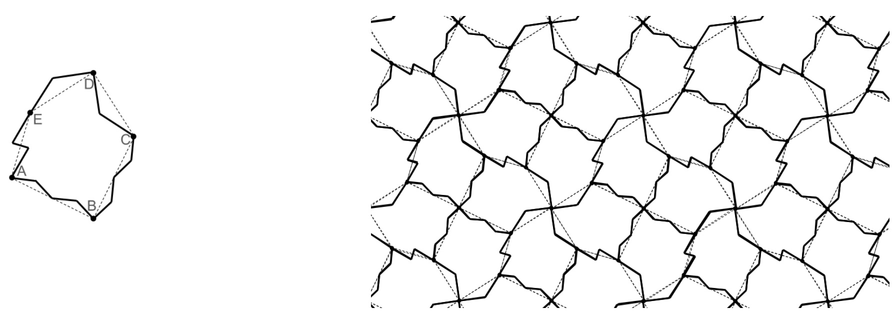
(a) Theorem 3.0.1(1)



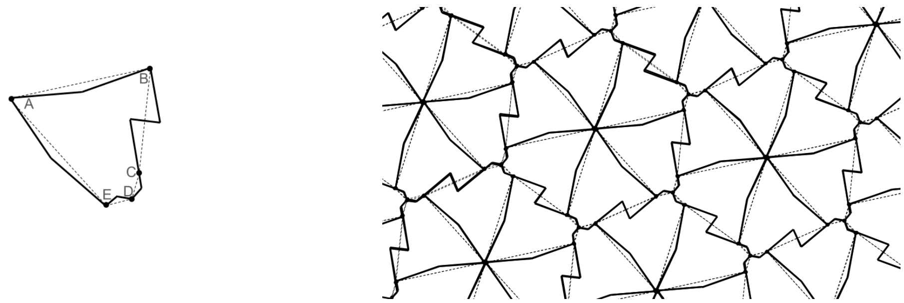
(b) Theorem 3.0.1(2)



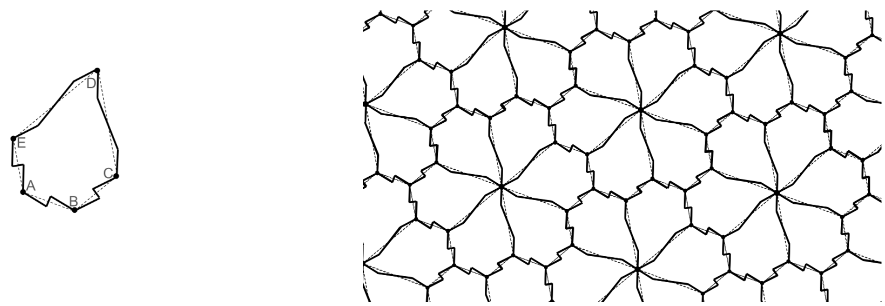
(c) Theorem 3.0.1(3)



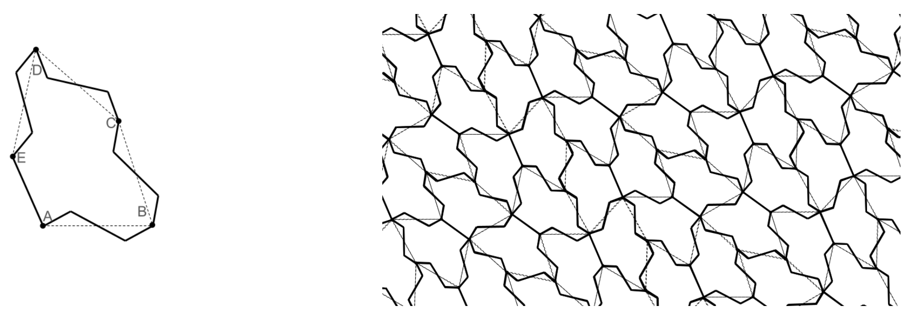
(d) Theorem 3.0.1(4)



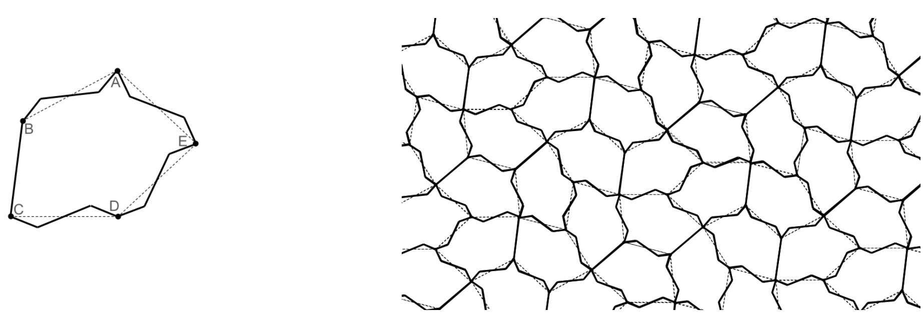
(e) Theorem 3.0.1(5)



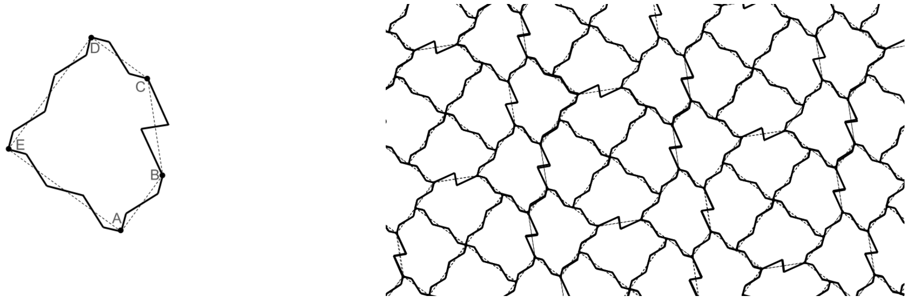
(f) Theorem 3.0.1(6)



(g) Theorem 3.0.1(7)



(h) Theorem 3.0.1(8)



(i) Theorem 3.0.1(10)

Theorem 3.0.2. *A polygon tiles $\mathbb{R}^2$ if we can choose six consecutive dividing points along its boundary such that one or more of the following conditions are satisfied:*

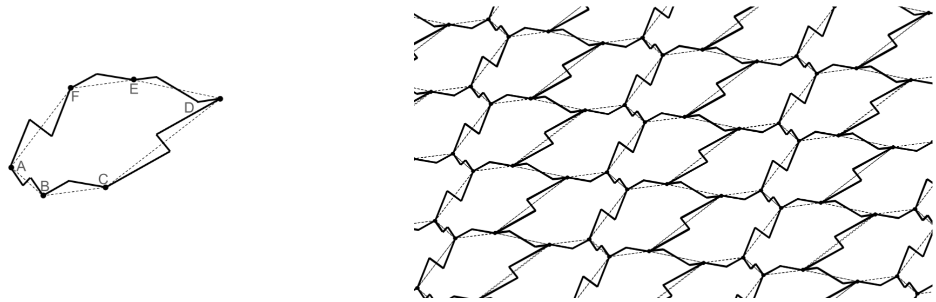
1. $C + D + E = 360°$; $F + A + B = 360°$; $BC \equiv FE$; AB, CD, DE, *and* FA *are centrosymmetric.*[1]

2. $A + B + D = 360°$; $C + E + F = 360°$; $BC \equiv \mathcal{R}(DE)$; $CD \equiv \mathcal{R}(FA)$; AB *and* EF *are centrosymmetric.*

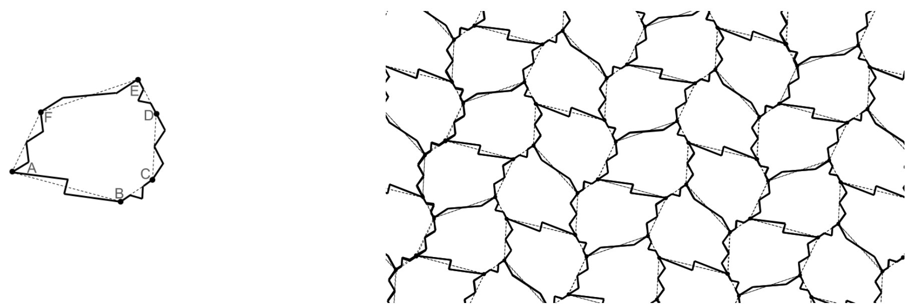
3. $A = C = E = 120°$; $AB \equiv AF$; $BC \equiv DC$; $DE \equiv FE$.

[1]This given by the Conway Criterion, or Theorem 2.3.8.

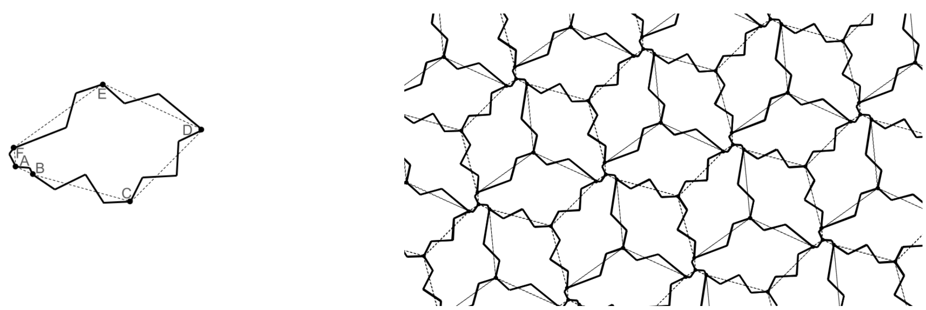
Figure 3.6: Shapes Satisfying Theorem 3.0.2 and Corresponding Tilings



(a) Theorem 3.0.2(1)



(b) Theorem 3.0.2(2)



(c) Theorem 3.0.2(3)

Chapter 4

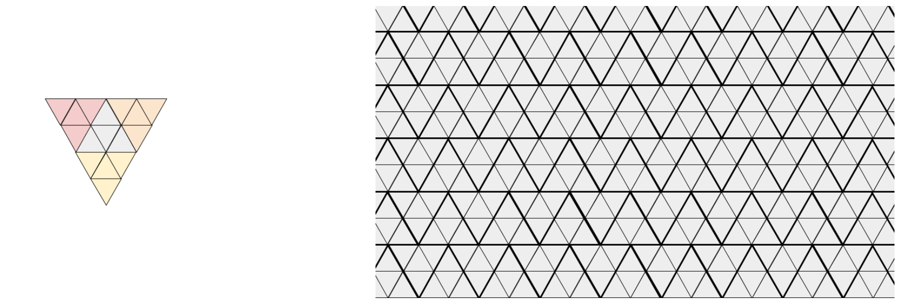
Tilability of Tetrahedron, Octahedron, Cube, and Icosahedron Nets

The second contribution of the paper is presented in this chapter, consisting of a new proof that all nets of the tetrahedron, octahedron, and cube tile. The relevant background concepts have been introduced in Chapters 2 and 3. Some concepts from this chapter will be used in Chapter 5.

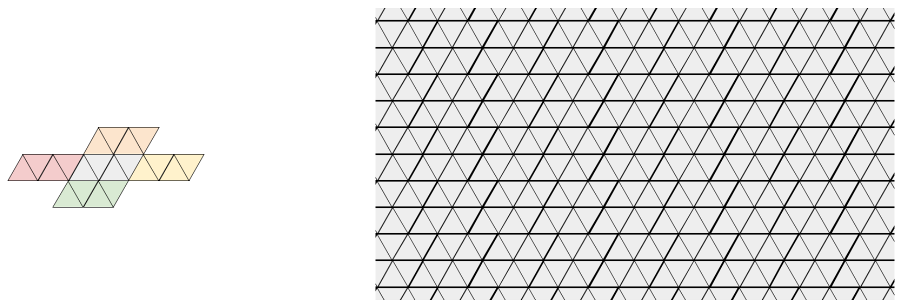
4.1 Tetrahedron Nets

As shown in Figure 2.2, the tetrahedron has only two nets. One is triangular in shape, while the other is a quadrilateral. Thus, by Theorems 2.3.2 and 2.3.1, respectively, both nets tile $\mathbb{R}^2$. The nets' respective surroundings and tilings are shown below:

Figure 4.1: Surroundings and Tilings for the Tetrahedron Nets



(a) Net 1



(b) Net 2

4.2 Octahedron Nets

We will begin, as it is the most broadly-applicable of our criteria, by testing each of the octahedron's nets against the Conway Criterion (Theorem 2.3.8). As shown in Figure 4.2, all but one of the octahedron's nets satisfies this criterion. The corresponding surroundings and tilings for these are shown in Figure 4.3.

Figure 4.2: Satisfaction of the Conway Criterion by All Octahedron Nets Except No. 9

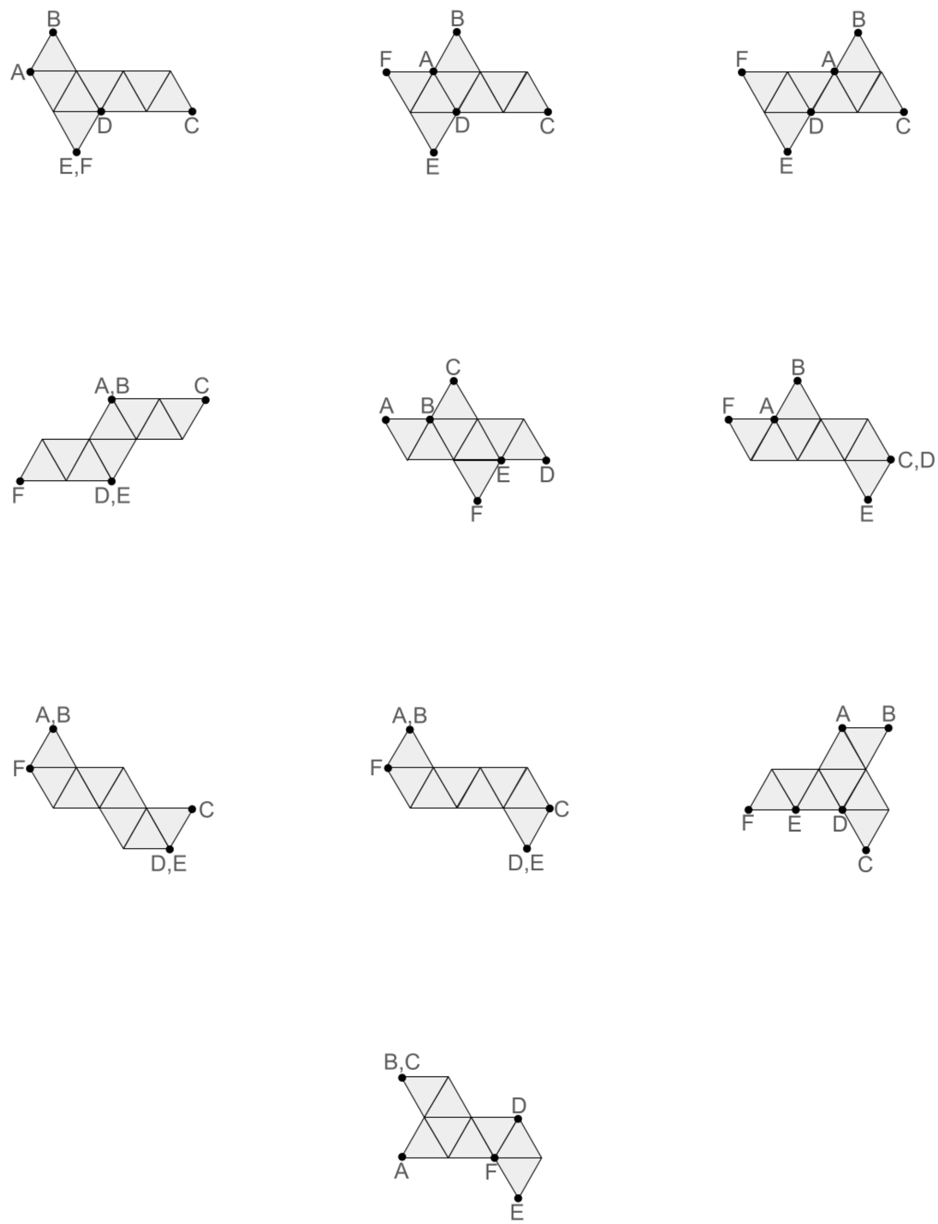
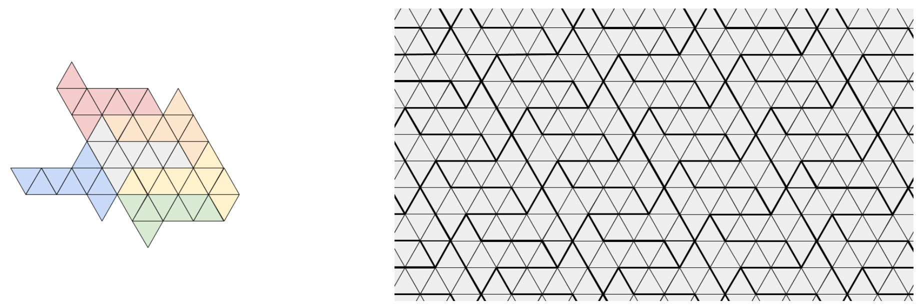
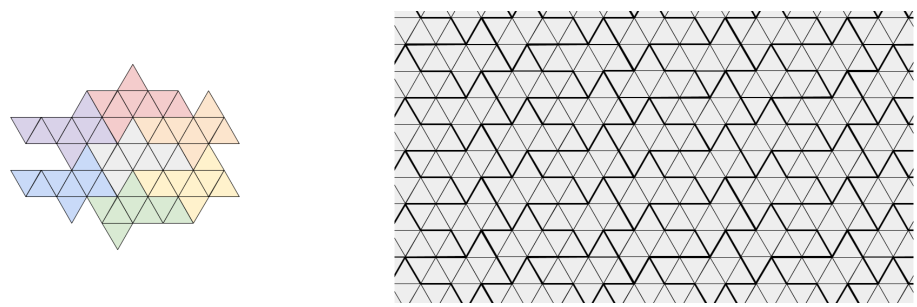


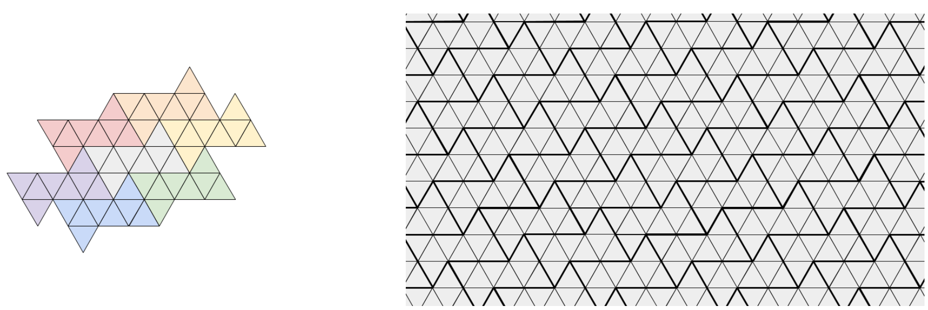
Figure 4.3: Surroundings and Tilings for All Octahedron Nets Except No. 9



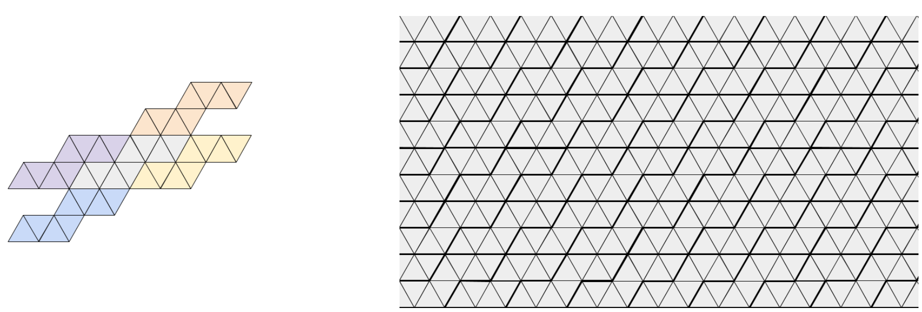
(a) Net 1



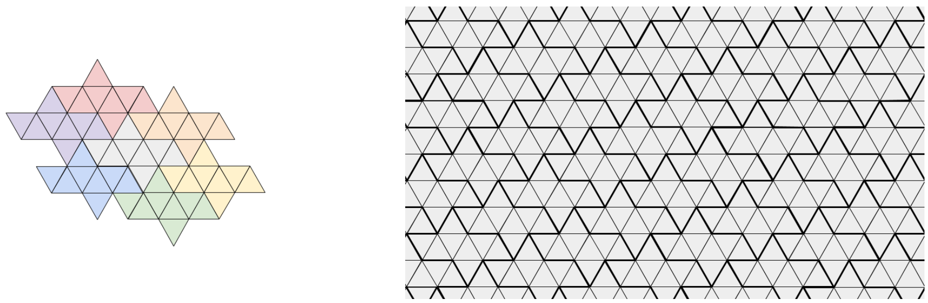
(b) Net 2



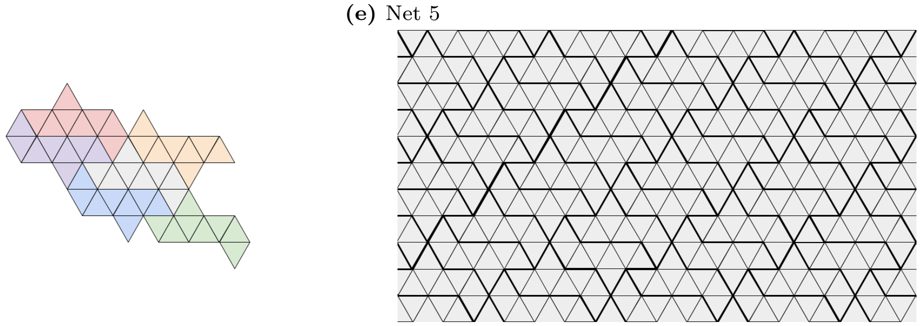
(c) Net 3



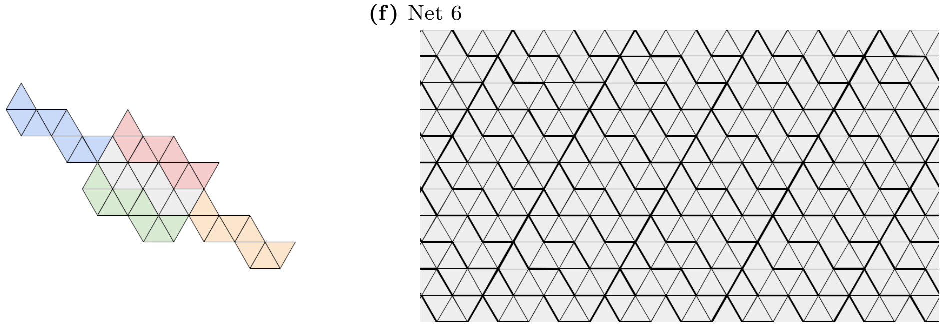
(d) Net 4



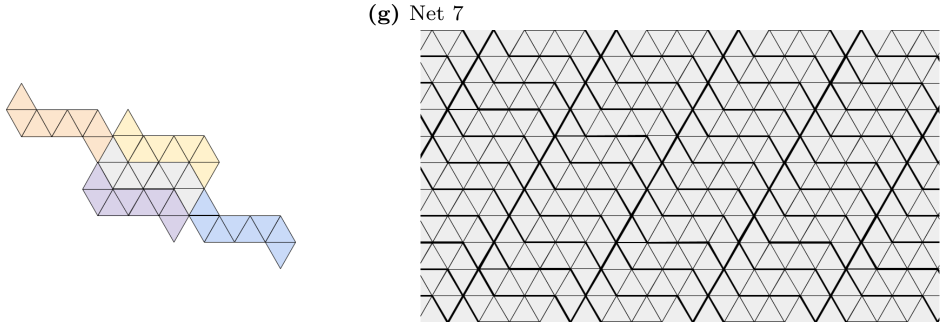
(e) Net 5



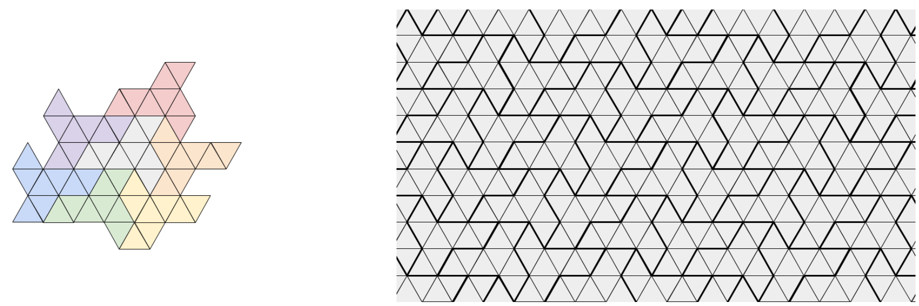
(f) Net 6



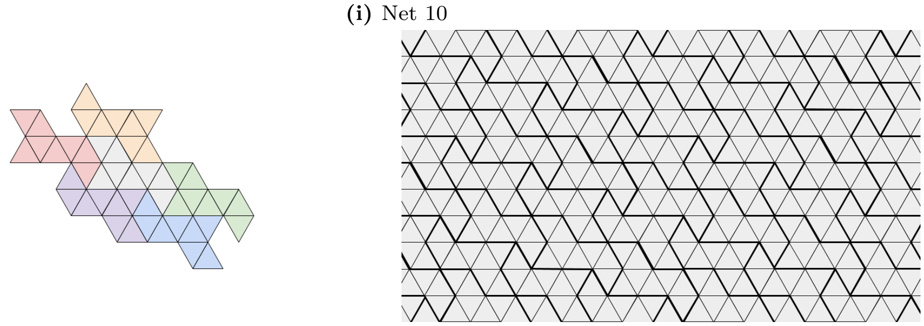
(g) Net 7



(h) Net 8



(i) Net 10



(j) Net 11

The ninth net, by contrast, provides us with a problem. It not only fails to satisfy the Conway Criterion, but it furthermore fails to satisfy any criterion we have put forth thus far. Careful examination of it, though, shows that two copies glued to each other *do* satisfy the Conway Criterion. This is shown in Figure 4.4, and the corresponding surrounding and tiling is shown in Figure 4.5.

Figure 4.4: Satisfaction of the Conway Criterion by Edge-Glued Copies of Octahedron Net 9

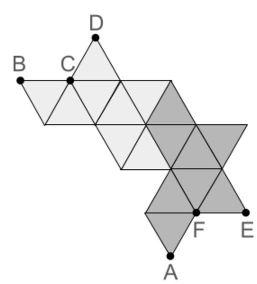
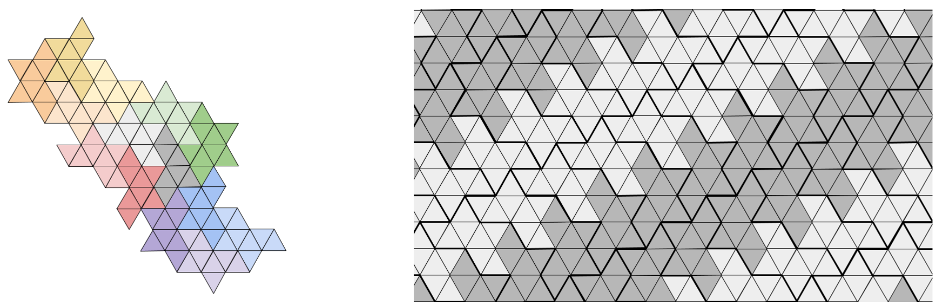


Figure 4.5: Surrounding and Tiling for Octahedron Net 9



So, by Theorem 2.3.8, all octahedron nets tile $\mathbb{R}^2$.

4.3 Cube Nets

Again, we will begin our analysis of the tilability of the cube's nets by testing each against the Conway Criterion (Theorem 2.3.8). As shown in Figure 4.6, all of the cube's nets satisfy the Conway Criterion. Thus, by Theorem 2.3.8, all nets of the cube tile $\mathbb{R}^2$. The corresponding surroundings and tilings for each are shown in Figure 4.7.

Figure 4.6: Satisfaction of the Conway Criterion by All 11 Cube Nets

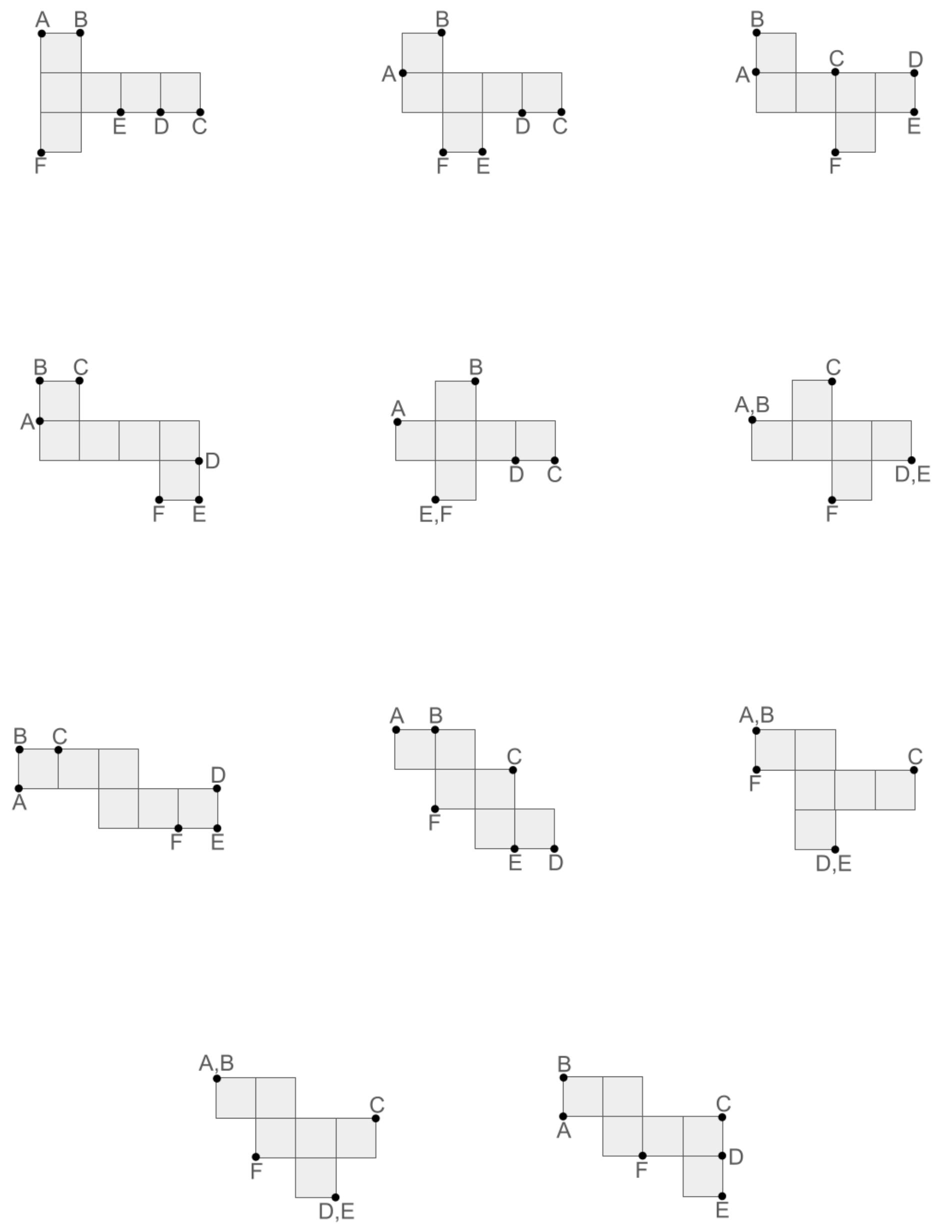
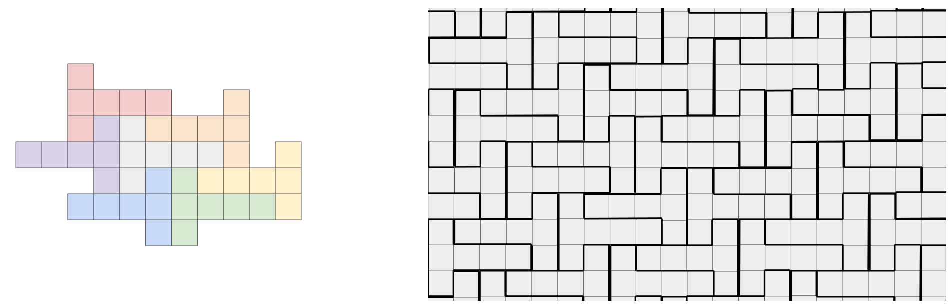
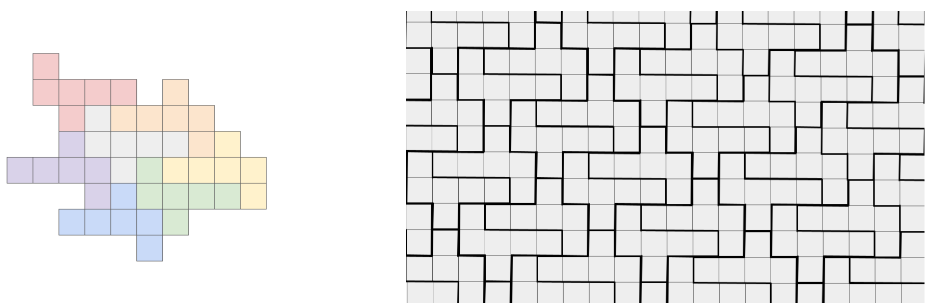


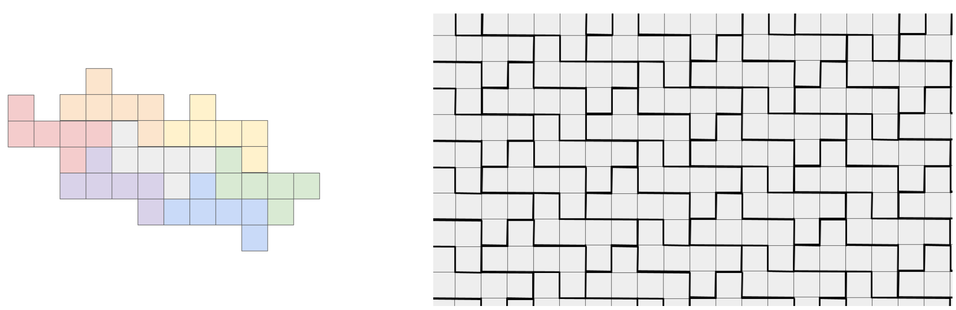
Figure 4.7: Surroundings and Tilings for All Cube Nets



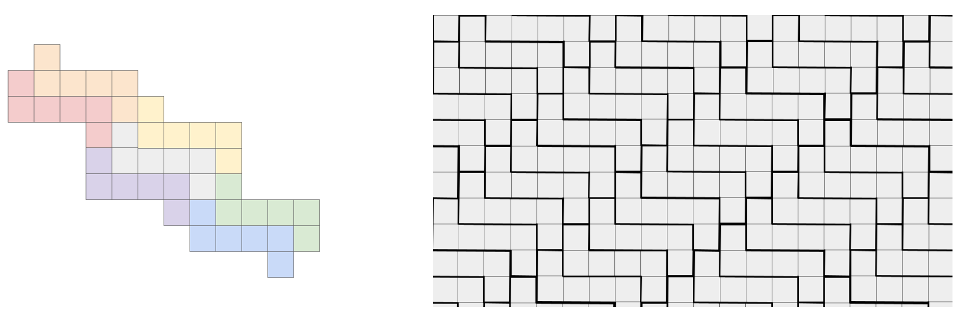
(a) Net 1



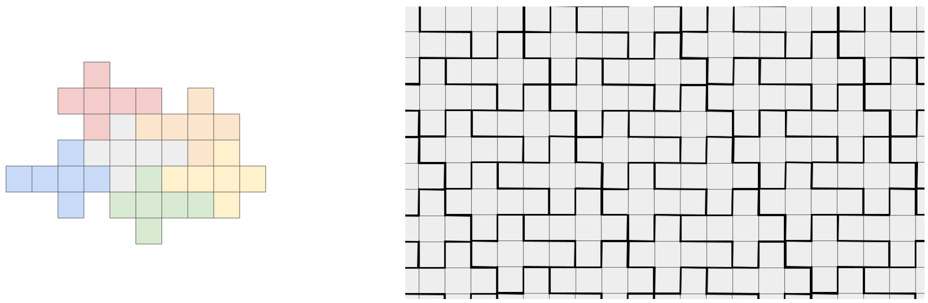
(b) Net 2



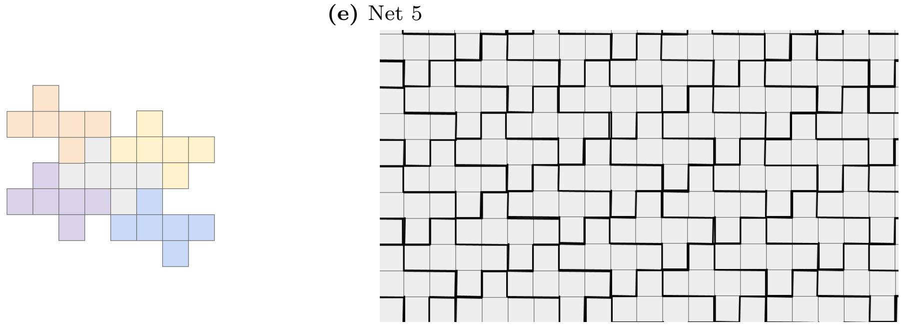
(c) Net 3



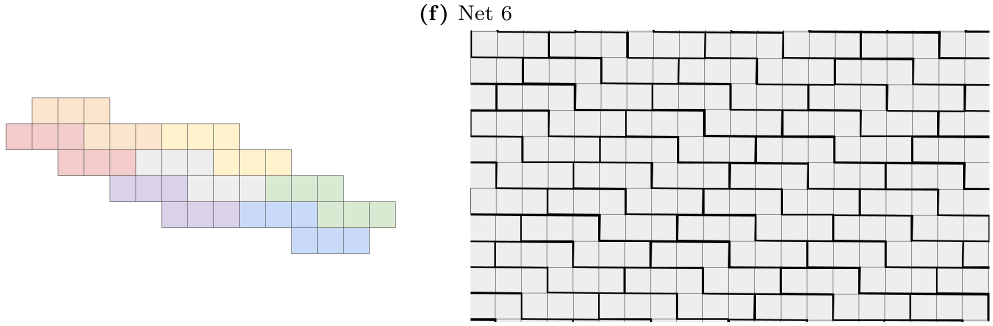
(d) Net 4



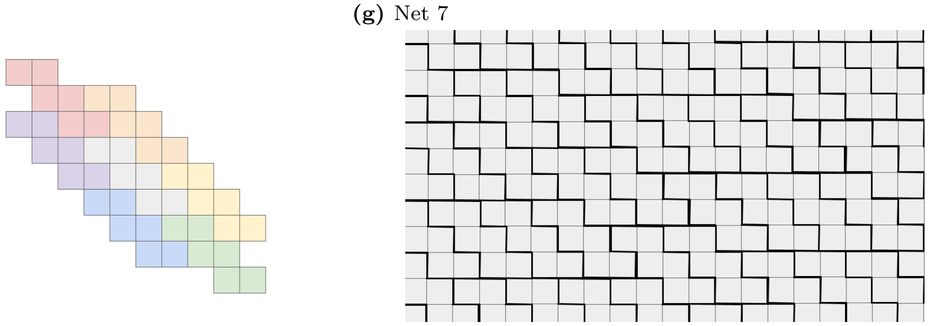
(e) Net 5



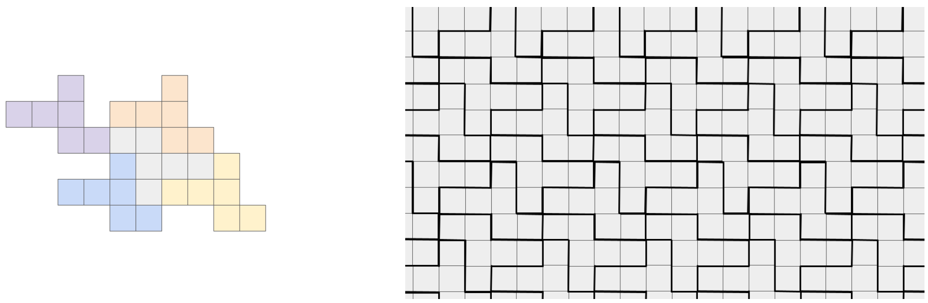
(f) Net 6



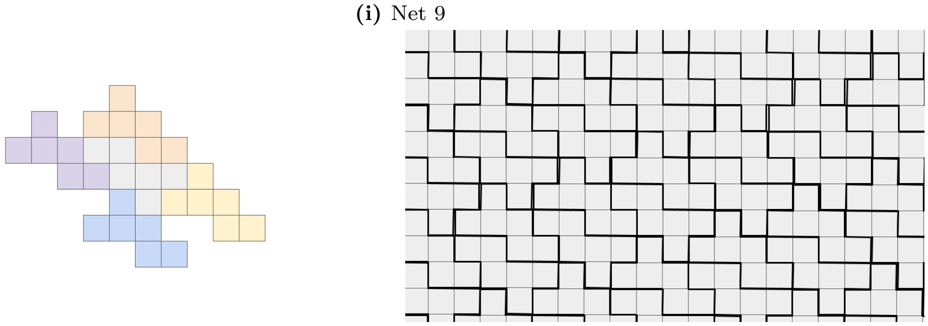
(g) Net 7



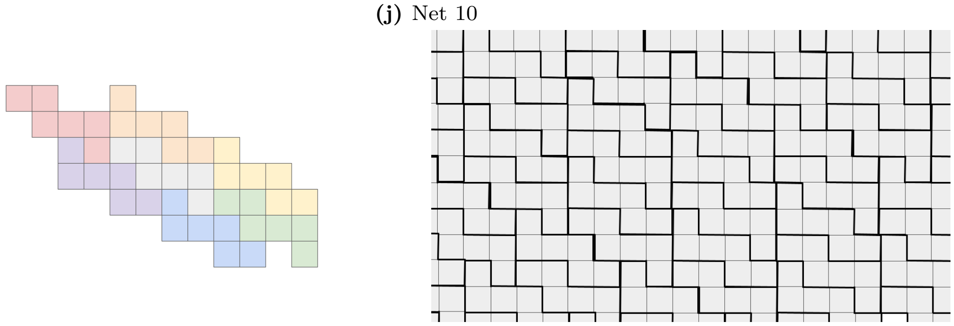
(h) Net 8



(i) Net 9



(j) Net 10



(k) Net 11

4.4 Icosahedron Nets

Though some of the icosahedron's nets tile (see Figures 4.8 and 4.9), not all do. Figure 4.10 provides an example of an icosahedron net that does not tile. There are no edge-gluings to the highlighted edge that do not cause gaps or overlaps, as illustrated in Figure 4.11.

Figure 4.8: Satisfaction of the Conway Criterion by an Icosahedron Net

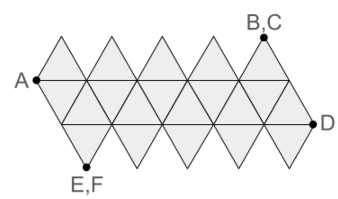


Figure 4.9: Surrounding and Tiling for an Icosahedron Net

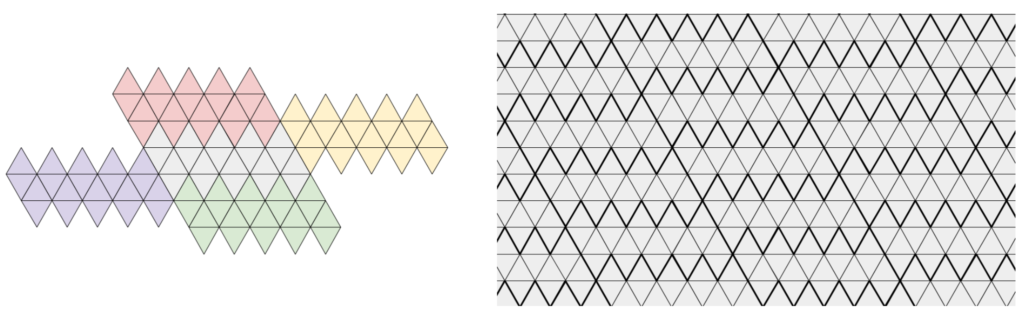


Figure 4.10: An Icosahedron Net That Does Not Tile

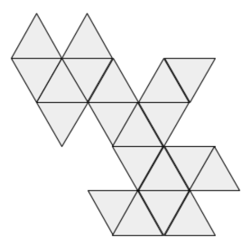
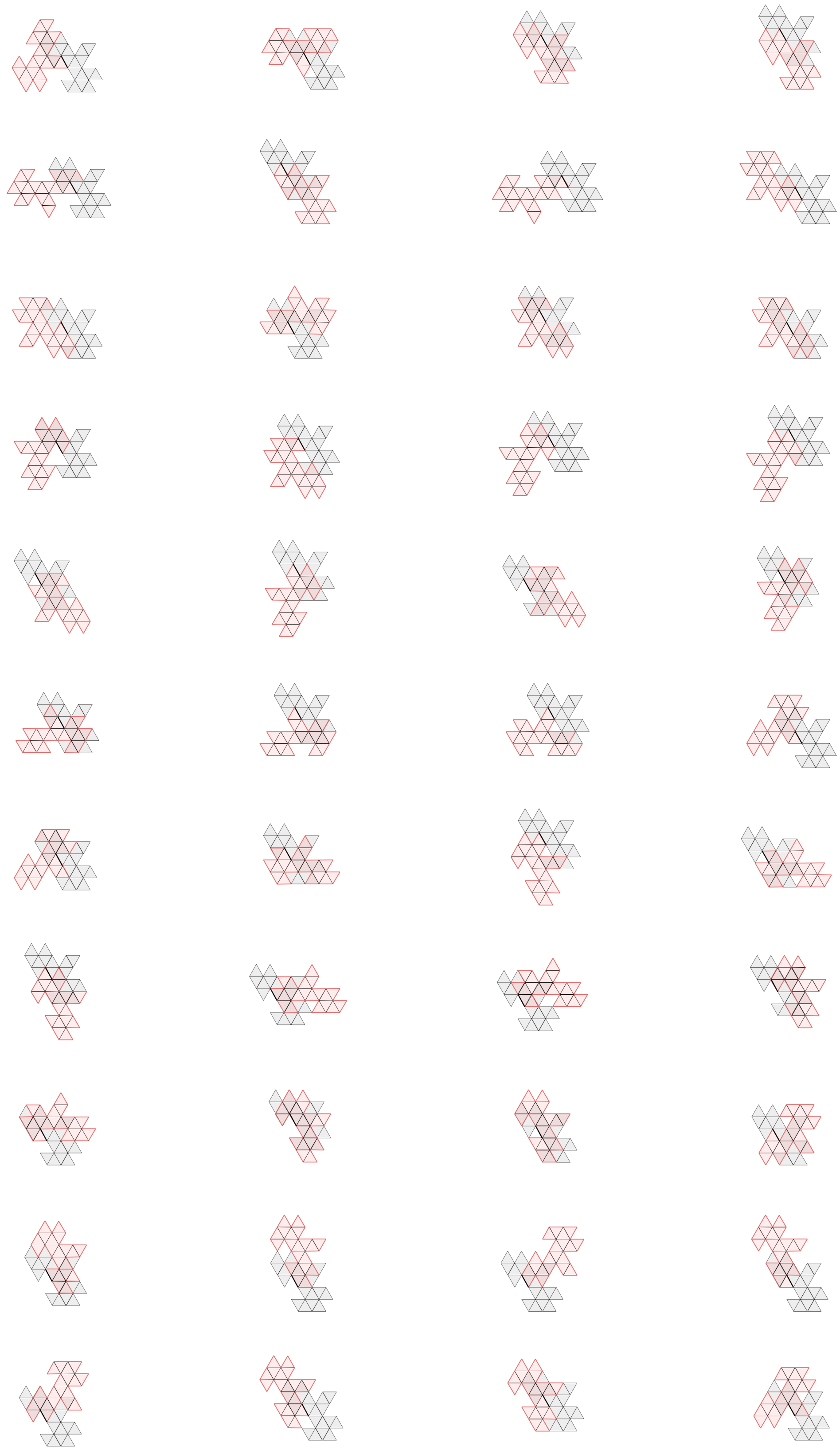


Figure 4.11: All Edge-Gluings of the Icosahedron Net Shown in Figure 4.10 Create Self-Overlap or Holes



Chapter 5

Tilability of Higher-Dimensional Platonic Solid Nets

The final contribution of the paper is presented in this chapter, consisting of a proof that all nets of the d-hypercube are simple polytopes, regardless of d. The relevant background concepts have been introduced in Chapters 2 and 4.

Do all nets for each of the d-dimensional Platonic solids tile $\mathbb{R}^{d-1}$ for arbitrary d? We have already answered this question in the negative for the dodecahedron and icosahedron; notably, neither of these solids has an analogous polytope for $d \geq 5$. For the tetrahedron, octahedron, and cube, we have yet to answer this question. Luckily, each of these solids corresponds to a Platonic hypersolid that is present in arbitrary dimension: the tetrahedron corresponds to the simplex, the cube to the hypercube, and the octahedron to the orthoplex.

Before answering the above question, it is important to acknowledge that tilability, though already complex in $\mathbb{R}^2$, becomes far more complex for $d > 2$. In $\mathbb{R}^2$, we saw that the interior angle measure of regular polygon adding to or dividing 2π was sufficient to prove that a shape tiles (see Theorems 2.3.1 and 2.3.2). In $\mathbb{R}^3$, the analagous characteristic would be the interior curvature of a regular polyhedron adding to or dividing 4π.[1] However, this characteristic is not sufficient to show that a polyhedron tiles. In order to prove tilability in $\mathbb{R}^3$, we must also show that the dihedral angle (the angle between two faces) of a regular polyhedron adds to or divides 2π. For each additional dimension, more constraints are added, making the problem of tilability more difficult.

As mentioned in Chapter 2, the nets of the Platonic solids are edge-gluings of regular polygons, the nets of Platonic d-hypersolids will be edge-gluings of Platonic $[d-1]$-(hyper)solids. So, much as the nets of the dodecahedron do not tile $\mathbb{R}^2$ because the regular pentagon does not tile $\mathbb{R}^2$, the analagous $[d-1]$-(hyper)solid must tile for any net of a given d-hypersolid to tile $\mathbb{R}^{d-1}$.

Thus, in analyzing whether the nets of the Platonic 4-hypersolids tile $\mathbb{R}^3$, we must first examine whether the Platonic solids tile $\mathbb{R}^3$. We have already stated (in the proof of Theorem 2.2.1) that all Platonic solids have an interior curvature that sums to or evenly divides 4π. However, as aforementioned, this property alone is not sufficient to prove that a polytope tiles $\mathbb{R}^3$. We also need to know that the dihedral angle of the given polytope sums to or divides 2π. As such, we obtain the following:

Theorem 5.0.1. *The cube is the only Platonic solid that tiles* $\mathbb{R}^3$.

Proof. The dihedral angle of the tetrahedron is given by $\arccos(\frac{1}{3})$, which does not divide 2π. Similarly, the dihedral angle of the octahedron is given by $\arccos(\frac{-1}{3})$, the dihedral angle of the icosahedron is given by $\arccos(\frac{-\sqrt{5}}{3})$, and the dihedral angle of the dodecahedron is given by $\arccos(\frac{-\sqrt{5}}{5})$ – none of which evenly divide 2π.
By contrast, the dihedral angle of the cube is given by $\arccos(0) = \frac{\pi}{2}$, which evenly divides 2π. This, paired with the fact that angle curvature of a cube's vertex evenly divides 4π, shows that the cube tiles. $\square$

[1] We can think of the 'curvature' of a point in $\mathbb{R}^d$ as the surface area of a unit sphere in $\mathbb{R}^d$, which is given by $\frac{d\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$. In order to tile, a shape must be able to completely cover the 'curvature' around this point.

This means that the nets of the d-simplex and d-orthoplex do not tile for arbitrary d. By contrast, we can continue to any d, and the d-hypercube will still tile $\mathbb{R}^d$ (see Theorem 5.0.2). As such, we cannot, for any d, use the tilability of the $[d-1]$-hypercube as grounds to rule out the possibility that all nets of the d-hypercube tile $\mathbb{R}^{d-1}$.

Theorem 5.0.2. *The d-hypercube tiles $\mathbb{R}^d$ for arbitrary d.*

Proof. The d-hypercube can be given as a convex hull around the set of all d-permutations of the alphabet 0,1 (the 2-hypercube, for example, is given as the convex hull around (0,0), (0,1), (1,0), (1,1)). As such, the translation ($x_1 \pm 1$, $x_2 \pm 1$, ... , $x_d \pm 1$) gives the position for an adjacent, although not overlapping, d-hypercube. Thus, this translation defines a tiling by the d-hypercube. □

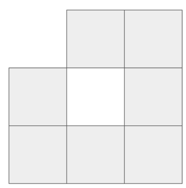
5.1 Hypercube Nets

The next step in analyzing whether a hypercube's nets tile is to ensure that none of its nets self-overlap or have self-contained holes, which would automatically preclude them from tiling. In [22], it was shown that if direction x occurs in a path along the spanning tree of an orthoplex, direction $-x$ does not. As, by theorem 2.2.2, the nets of the d-hypercube ($\{4, 3^{d-1}\}$) can be represented as spanning trees of the d-orthoplex ($\{3^{d-1}, 4\}$) mapped onto the lattice $\mathbb{Z}^{d-1}$, this implies that all hypercube nets are without self-overlap. Now, we show in Theorem 5.1.1 that no net of the d-hypercube has a fully self-contained hole. This does not preclude the existence of rings (holes fully enclosed in $\mathbb{R}^{d-1}$ or below); however, the presence of a ring within a net would not prevent it from tiling.

Theorem 5.1.1. *No net of the d-hypercube contains a fully-enclosed hole.*

Proof. A connected net composed of squares in $\mathbb{R}^2$ must consist of at least 7 squares in order to contain a fully self-contained hole (see Figure 5.1). If we examine $\mathbb{R}^3$ instead, we see that 7 cubes is not sufficient. The hole now has two additional sides, and thus needs two extra cubes to be fully enclosed. An additional two cubes are then needed to connect these to the initial shape (as a net must be fully connected). Each time we increase in dimension, we must enclose the hole on two new sides while preserving the connectivity of the net, meaning an extra 4 $[d-1]$-hypercubes are needed. As such, a net of the d-hypercube must be composed of at least $4d-1$ $[d-1]$-hypercubes in order to have a self-contained hole. By contrast, a d-hypercube is composed of $2d$ $[d-1]$-facets. For all $d \geq 0$, then, the number of $[d-1]$-facets in the net of the $d-\textit{hypercube}$ is insufficient to create a hole that is fully contained by the net. $\square$

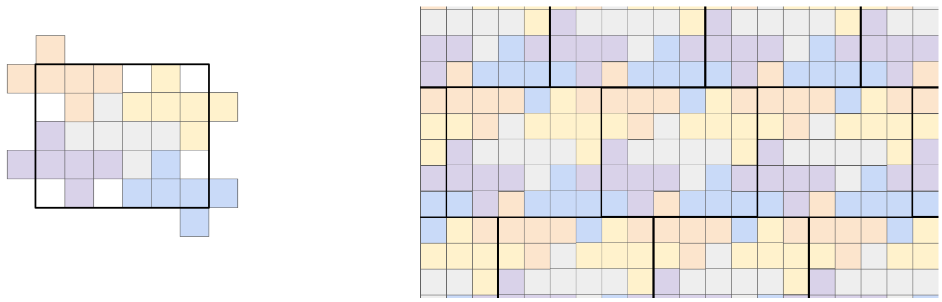
Figure 5.1: A Hole in $\mathbb{R}^2$, Fully-Enclosed by a Connected Set of Edge-Glued Squares



In 2021, Moritz Firsching showed that all 261 nets of the tesseract ($\{4, 3, 3\}$) tile $\mathbb{R}^3$. He did so by examining all ways in which copies of each given unfolding can be arranged in an $x \times y \times z$ box, where $8|xyz$. The copies do not need to fit perfectly within the box; there can be portions that are unfilled and others that extend beyond the box. If, for any of these arrangements, all portions extending beyond the box on one side are congruent by translation to an unfilled portion on the opposite side, the net tiles (see Figure 5.2 for an example in two dimensions). A catalog of these nets and their respective tiling arrangements is available at `https://firsching.ch/mo/198722/unfoldings`.

Despite years of effort, including by the authorship of this paper, no one has yet shown that all nets of the d-hypercube tile $\mathbb{R}^{d-1}$ for arbitrary d or established a d for

Figure 5.2: Copies of Cube Net 6 in a *num*x*num* Box and Corresponding Tiling



which some net does not tile. This is in part due to difficulties in characterizing nets, which will be discussed further in the next chapter.

Chapter 6

Conclusion

Though it may seem that these problems are fairly abstract, applications of unfoldings and tilings extend well beyond computational theory and art. Manufacturers are interested in both unfoldings and tilings as means of minimizing offcut waste in order to render the manufacturing process more environmentally and economically sustainable. NASA has used origami-based folding and unfolding techniques for the Webb telescope, and other space agencies are exploring the applications of these mathematical concepts to atmospheric re-entry heat shields. Additionally, for chemists and physicists alike, both folding and tiling offer useful insights into chemical structure. All of these potential applications are driving the study of tilings and unfoldings to new depths.

In this context, it becomes evident how tiling criteria – and an expanded knowledge of tiling and unfolding in general – can have practical, real-world use. In light of this, we provide below a brief summary of the contributions presented in this thesis, along with a discussion of challenges and suggestions for future work.

6.1 Summary of Contributions

This paper consists primarily of three contributions: In response to the question "for which dimensions, d, do all nets of each platonic solid tile $\mathbb{R}^{d-1}$?" we give a mathematically rigorous proof that all tetrahedron, octahedron, and cube nets tile $\mathbb{R}^2$, but that only the hypercube could potentially tile $\mathbb{R}^d$ for all d. We also give an original proof that every cube net in arbitrary dimension is a simple shape. Additionally, we suggest several sufficient criteria for determining tilability in $d = 2$.

6.2 Discussion

The initial aim of this paper was to prove/disprove that all nets of the d-hypercube tile $\mathbb{R}^{d-1}$ for arbitrary d. Throughout the first few months of examination, though, we ran into a variety of difficulties that placed this beyond the scope of what could be accomplished in this project.

One such difficulty arises in that the method used to show that all tesseract nets tile (outlined in Chapter 5), though advantageous due to its relative computational simplicity and generalizability to higher dimensions, is limited in its utility given that it requires us to generate all nets and then test them individually for each dimension. Others have continued along this path, analyzing hypercube nets' tilability one dimension at a time. These methods, though, are incapable of showing that all nets of the d-hypercube tile for arbitrary d.

In order to respond to our question, then, we must instead look to traits shared by all hypercube nets that might allow them to tile. Here too, though, we run into an issue. As mentioned in Chapter 2, "no one has discovered necessary and sufficient conditions for a collection of cut edges to unfold to a net," a fact which deeply limits our ability to characterize nets [4].

Another challenge arises in that tilability is potentially unsolvable. As stated in Chapter 2, the problem of tilability for a set of two or more tiles (dubbed the Domino Problem) is undecidable. Robert Berger's 1966 proof of this also showed that consequently, there exists an aperiodic tileset, a set of shapes that tile only aperiodically [12]. The recent discovery of an aperiodic monotile, then, suggests that the question of tilability could be undecidable even for a single tile [23]. This would not necessarily mean the problem of hypercube net tilability is unsolvable, but it would imply the lack of any set of conditions both necessary and sufficient for a shape to tile.

6.3 Open Problems and Future Work

In response to the central question posed in this paper, only one item is left open:

Open Problem 6.3.1. *Do all nets of the d-hypercube tile $\mathbb{R}^{d-1}$ for arbitrary d?*

However, based on the difficulties that arose in the course of researching the contents of this paper, future work on the subject might also be forced to take into account the following two problems:

Open Problem 6.3.2. *Is tilability for a single shape unsolvable?*

Open Problem 6.3.3. *Is there a set of conditions that is both necessary and sufficient for a set of cut edges/facets to form a net?*

The responses to these problems could well dictate whether Open Problem 6.2.1, and thus the central question posed in this paper, can actually be solved feasibly. As such, an understanding of these questions is likely to be central to any future work in this area.

If tilability is solvable in general, though, continuing to define new tiling criteria could aid in the search for a single set of conditions that is both necessary and sufficient for a shape to tile, a discovery that could have immense impacts on applied and theoretical computation alike.

In either case, the studies of tilings and nets remain full of open problems, such as Dürer's Problem, that have existed for decades, and sometimes centuries or millennia. In the midst of such an excess of unknowns, this field has no shortage of potential avenues for future work.

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